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Existence and uniqueness of global weak solutions to a Cahn–Hilliard–Stokes–Darcy system for two phase incompressible flows in karstic geometry

Daozhi Han [a], Xiaoming Wang [a,*], Hao Wu [b]

[a] *Department of Mathematics, Florida State University, Tallahassee, FL 32306-4510, USA*
[b] *School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, China*

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Abstract

We study the well-posedness of a coupled Cahn–Hilliard–Stokes–Darcy system which is a diffuse-interface model for essentially immiscible two phase incompressible flows with matched density in a karstic geometry. Existence of finite energy weak solution that is global in time is established in both 2D and 3D. Weak–strong uniqueness property of the weak solutions is provided as well.

MSC: 35K61; 76T99; 76S05; 76D07

Keywords: Cahn–Hilliard–Stokes–Darcy system; Two phase flow; Karstic geometry; Interface boundary conditions; Diffuse-interface model; Well-posedness

* Corresponding author. Fax: +1 850 644 4053.
E-mail addresses: dhan@math.fsu.edu (D. Han), wxm@math.fsu.edu (X. Wang), haowufd@yahoo.com (H. Wu).

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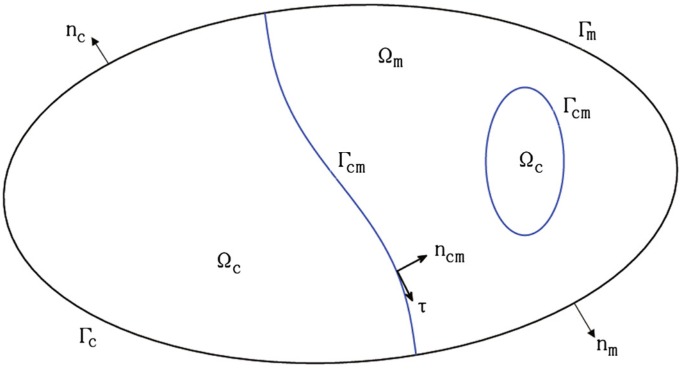

Fig. 1. Schematic illustration of the domain in 2D.

1. Introduction

Applications such as contaminant transport in karst aquifer, oil recovery in karst oil reservoir, proton exchange membrane fuel cell technology and cardiovascular modelling require the coupling of flows in conduits with those in the surrounding porous media. Geometric configurations that contain both conduit (or vug) and porous media are termed karstic geometry. Moreover, many flows are naturally multi-phase and hence multi-phase flows in the karstic geometry are of interest. Despite the importance of the subject, little work has been done in this area. Our main goal here is to analyze a diffuse-interface model for two phase incompressible flows with matched densities in the karstic geometry that was recently derived in [1] via Onsager's extremum principle.

To fix the notation, let us assume that the two-phase flows are confined in a bounded connected domain $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ of $C^{2,1}$ boundary $\partial\Omega$. The unit outer normal at $\partial\Omega$ is denoted by $\mathbf{n}$. The domain Ω is partitioned into two non-overlapping regions such that $\overline{\Omega} = \overline{\Omega}_c \cup \overline{\Omega}_m$ and $\Omega_c \cap \Omega_m = \emptyset$, where Ω_c and Ω_m represent the underground conduit (or vug) and the porous matrix region, respectively. We denote by $\partial\Omega_c$ and $\partial\Omega_m$ the boundaries of the conduit and the matrix part, respectively. Both $\partial\Omega_c$ and $\partial\Omega_m$ are assumed to be Lipschitz continuous. The interface between the two parts (i.e., $\partial\Omega_c \cap \partial\Omega_m$) is denoted by Γ_{cm}, on which $\mathbf{n}_{cm}$ denotes the unit normal to Γ_{cm} pointing from the conduit part to the matrix part. Then we denote $\Gamma_c = \partial\Omega_c \backslash \Gamma_{cm}$ and $\Gamma_m = \partial\Omega_m \backslash \Gamma_{cm}$ with $\mathbf{n}_c, \mathbf{n}_m$ being the unit outer normals to Γ_c and Γ_m. We assume that both Γ_m and Γ_{cm} have positive measure (namely, $|\Gamma_m| > 0$, $|\Gamma_{cm}| > 0$) but allow $\Gamma_c = \emptyset$, i.e. Ω_c can be enclosed completely by Ω_m. A two-dimensional geometry is illustrated in Fig. 1. When $d=3$, we also assume that the surfaces Γ_c, Γ_m, Γ_{cm} have Lipschitz continuous boundaries. On the conduit/matrix interface Γ_{cm}, we denote by $\{\boldsymbol{\tau}_i\}$ $(i=1,\dots,d-1)$ a local orthonormal basis for the tangent plane to Γ_{cm}.

In the sequel, the subscript m (or c) emphasizes that the variables are for the matrix part (or the conduit part). We denote by $\mathbf{u}$ the mean velocity of the fluid mixture and by φ the phase function related to the concentration of the fluid (volume fraction). The following convention will be assumed throughout the paper

$$\mathbf{u}|_{\Omega_m} = \mathbf{u}_m, \qquad \mathbf{u}|_{\Omega_c} = \mathbf{u}_c, \qquad \varphi|_{\Omega_m} = \varphi_m, \qquad \varphi|_{\Omega_c} = \varphi_c.$$

Governing PDE system. To the best of our knowledge, the first diffuse-interface model for incompressible two-phase flows in karstic geometry with matched densities was recently derived in [1] by utilizing Onsager's extremal principle (see references therein). Our aim in this paper is to study its well-posedness. Indeed, we can perform the analysis for a more general system, in which the Stokes equation can also be time-dependent. Thus, we shall consider the following Cahn–Hilliard–Stokes–Darcy system (CHSD for brevity) coupled through a set of interface boundary conditions (see (1.16)–(1.22) below):

$$\rho_0 \varpi \partial_t \mathbf{u}_c = \nabla \cdot \mathbb{T}(\mathbf{u}_c, P_c) + \mu_c \nabla \varphi_c, \quad \text{in } \Omega_c, \tag{1.1}$$

$$\nabla \cdot \mathbf{u}_c = 0, \quad \text{in } \Omega_c, \tag{1.2}$$

$$\partial_t \varphi_c + \mathbf{u}_c \cdot \nabla \varphi_c = \operatorname{div}\big(\mathrm{M}(\varphi_c) \nabla \mu_c\big), \quad \text{in } \Omega_c, \tag{1.3}$$

$$\mathbf{u}_m = -\frac{\rho_0 g \Pi}{\nu(\varphi_m)} (\nabla P_m - \mu_m \nabla \varphi_m), \quad \text{in } \Omega_m, \tag{1.4}$$

$$\nabla \cdot \mathbf{u}_m = 0, \quad \text{in } \Omega_m, \tag{1.5}$$

$$\partial_t \varphi_m + \mathbf{u}_m \cdot \nabla \varphi_m = \operatorname{div}\big(\mathrm{M}(\varphi_m) \nabla \mu_m\big), \quad \text{in } \Omega_m, \tag{1.6}$$

where the chemical potentials μ_c, μ_m are given by

$$\mu_j = \gamma \left(-\epsilon \Delta \varphi_j + \frac{1}{\epsilon} \big(\varphi_j^3 - \varphi_j\big) \right), \quad j \in \{c, m\}. \tag{1.7}$$

Here, the parameter ϖ in (1.1) is a nonnegative constant. When $\varpi = 0$, the system (1.1)–(1.6) reduces to the CHSD system derived in [1]. ρ_0 represents the fluid density, and g is the gravitational constant. The parameter $\gamma > 0$ is related to the surface tension. We remark that the Stokes equation (1.1) can be viewed as low Reynolds number approximation of the Navier–Stokes equation, while the Darcy equation (1.4) can be viewed as the quasi-static approximation for the saturated flow model under the assumption that the porous media pressure adjusts instantly to changes in the fluid velocity [2,3].

In the diffuse-interface model of immiscible two phase flows, the chemical potential μ (see Eq. (1.7)) is given by the variational derivative of the following free energy functional

$$E(\varphi) := \gamma \int_\Omega \left(\frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon} F(\varphi) \right) dx, \tag{1.8}$$

where $F(\varphi)$ is the Helmholtz free energy and usually taken to be a non-convex function of φ for immiscible two phase flows, e.g., a double-well polynomial of Ginzburg–Landau type in our present case:

$$F(\varphi) = \frac{1}{4} \big(\varphi^2 - 1\big)^2. \tag{1.9}$$

Singular potential of Flory–Huggins type can be treated as well, see for instance [4]. The first term (i.e., the gradient part) of E is a diffusion term that represents the hydrophilic part of the free-energy, while the second term (i.e., the bulk part) expresses the hydrophobic part of the free-energy. The small constant ϵ in (1.8) is the capillary width of the binary mixture. As the constant $\epsilon \to 0$, φ will approach 1 and -1 almost everywhere, and the contribution due to the induced stress will converge to a measure-valued force term supported only on the interface between regions $\{\varphi = 1\}$ and $\{\varphi = -1\}$ (cf. [5,6]). The nonlinear terms $\mu_c \nabla\varphi_c$ and $\mu_m \nabla\varphi_m$ in the convective Cahn–Hilliard equations (1.3) and (1.6) can be interpreted as the "elastic" force (or Korteweg force) exerted by the diffusive interface of the two phase flow. This "elastic" force converges to the surface tension at sharp interface limit $\epsilon \to 0$ at least heuristically (cf. e.g., [5,7]). Since the value of γ does not affect the analysis, we simply set $\gamma = 1$ throughout the rest of the paper. Likewise, we set the fluid density ρ_0 and gravitational constant g to be 1 without loss of generality.

The two phase flow in the conduit part and matrix part is described by the Stokes equation (1.1) and the Darcy equation (1.4), respectively. In (1.1), the Cauchy stress tensor $\mathbb{T}$ is given by

$$\mathbb{T}(\mathbf{u}_c, p_c) = 2\nu(\varphi_c)\mathbb{D}(\mathbf{u}_c) - P_c\mathbb{I}$$

where $\mathbb{D}(\mathbf{u}_c) = \frac{1}{2}(\nabla\mathbf{u}_c + \nabla^T\mathbf{u}_c)$ is the symmetric rate of deformation tensor and $\mathbb{I}$ is the $d \times d$ identity matrix. Besides, P_c and P_m stand for the modified pressures that also absorb the effects due to gravitation. The viscosity and the mobility of the CHSD model are denoted by ν and M, respectively. They are assumed to be suitable functions that may depend on the phase function φ (see Section 2.3). $M(\varphi)$ is taken to be the same (function of the phase function) for the conduit and the matrix for simplicity. In Eq. (1.4), Π is a $d \times d$ matrix standing for permeability of the porous media. It is related to the hydraulic conductivity tensor of the porous medium $\mathbb{K}$ through the relation $\Pi = \frac{\nu\mathbb{K}}{\rho_0 g}$. In the literature, $\mathbb{K}$ is usually assumed to be a bounded, symmetric and uniformly positive definite matrix but could be heterogeneous [8].

Next, we describe the initial boundary (or interface) conditions of the CHSD system (1.1)–(1.6).

Initial conditions. The CHSD system (1.1)–(1.6) is subject to the initial conditions

$$\mathbf{u}_c|_{t=0} = \mathbf{u}_0(x), \quad \text{in } \Omega_c, \tag{1.10}$$

$$\varphi|_{t=0} = \varphi_0(x), \quad \text{in } \Omega. \tag{1.11}$$

In particular, when $\varpi = 0$, we do not need the initial condition (1.10) for $\mathbf{u}_c$.

Boundary conditions on Γ_c and Γ_m. The boundary conditions on Γ_c and Γ_m take the following form:

$$\mathbf{u}_c = \mathbf{0}, \quad \text{on } \Gamma_c, \tag{1.12}$$

$$\mathbf{u}_m \cdot \mathbf{n}_m = 0, \quad \text{on } \Gamma_m, \tag{1.13}$$

$$\frac{\partial\varphi_c}{\partial\mathbf{n}_c} = \frac{\partial\mu_c}{\partial\mathbf{n}_c} = 0, \quad \text{on } \Gamma_c, \tag{1.14}$$

$$\frac{\partial\varphi_m}{\partial\mathbf{n}_m} = \frac{\partial\mu_m}{\partial\mathbf{n}_m} = 0, \quad \text{on } \Gamma_m. \tag{1.15}$$

Interface conditions on Γ_{cm}. The CHSD system (1.1)–(1.6) is coupled through the following set of interface conditions:

$$\varphi_m = \varphi_c, \quad \text{on } \Gamma_{cm}, \tag{1.16}$$

$$\mu_m = \mu_c, \quad \text{on } \Gamma_{cm}, \tag{1.17}$$

$$\frac{\partial \varphi_m}{\partial \mathbf{n}_{cm}} = \frac{\partial \varphi_c}{\partial \mathbf{n}_{cm}}, \quad \text{on } \Gamma_{cm}, \tag{1.18}$$

$$\frac{\partial \mu_m}{\partial \mathbf{n}_{cm}} = \frac{\partial \mu_c}{\partial \mathbf{n}_{cm}}, \quad \text{on } \Gamma_{cm}, \tag{1.19}$$

$$\mathbf{u}_m \cdot \mathbf{n}_{cm} = \mathbf{u}_c \cdot \mathbf{n}_{cm}, \quad \text{on } \Gamma_{cm}, \tag{1.20}$$

$$-\mathbf{n}_{cm} \cdot \big(\mathbb{T}(\mathbf{u}_c, P_c)\mathbf{n}_{cm}\big) = P_m, \quad \text{on } \Gamma_{cm}, \tag{1.21}$$

$$-\boldsymbol{\tau}_i \cdot \big(\mathbb{T}(\mathbf{u}_c, P_c)\mathbf{n}_{cm}\big) = \alpha_{BJSJ} \frac{\nu(\varphi_m)}{\sqrt{\mathrm{trace}(\Pi)}} \boldsymbol{\tau}_i \cdot \mathbf{u}_c, \quad \text{on } \Gamma_{cm}, \tag{1.22}$$

for $i = 1, ..., d-1$.

The first four interface conditions (1.16)–(1.19) are simply the continuity conditions for the phase function, the chemical potential and their normal derivatives, respectively. Condition (1.20) indicates the continuity in normal velocity that guarantees the conservation of mass, i.e., the exchange of fluid between the two sub-domains is conservative. Condition (1.21) represents the balance of two driving forces, the pressure in the matrix and the normal component of the normal stress of the free flow in the conduit, in the normal direction along the interface. The last interface condition (1.22) is the so-called Beavers–Joseph–Saffman–Jones (BJSJ) condition (cf. [9,10]), where $\alpha_{BJSJ} \geq 0$ is an empirical constant determined by the geometry and the porous material. The BJSJ condition is a simplified variant of the well-known Beavers–Joseph (BJ) condition (cf. [11]) that addresses the important issue of how the porous media affects the conduit flow at the interface:

$$-\boldsymbol{\tau}_i \cdot \big(2\nu\mathbb{D}(\mathbf{u}_c)\big)\mathbf{n}_{cm} = \alpha_{BJ} \frac{\nu}{\sqrt{\mathrm{trace}(\Pi)}} \boldsymbol{\tau}_i \cdot (\mathbf{u}_c - \mathbf{u}_m), \quad \text{on } \Gamma_{cm},\ i = 1, ..., d-1.$$

This empirical condition essentially claims that the tangential component of the normal stress that the free flow incurs along the interface is proportional to the jump in the tangential velocity over the interface. To get the BJSJ condition, the term $-\boldsymbol{\tau}_i \cdot \mathbf{u}_m$ on the right-hand side is simply dropped from the corresponding BJ condition. Mathematically rigorous justification of this simplification under appropriate assumptions can be found in [12].

There is an abundant literature on mathematical studies of single component flows in karstic geometry [2,13–24]. Those aforementioned mathematical works on the flows in karst aquifers treat the case of confined saturated aquifer where only one type of fluid (e.g., water) occupies the whole region exclusively. The mathematical analysis is already a challenge due to the complicated coupling of the flows in the conduits and the surrounding matrix, which are governed by different physical processes, the complex geometry of the network of conduits as well as the strong heterogeneity.

The current work contributes to, to the authors' best knowledge, a first rigorous mathematical analysis of the diffuse-interface model for two phase incompressible flows in the karstic geometry. In particular, we prove the existence of global finite energy solutions in the sense of Definition 2.1 to the CHSD system (1.1)–(1.22) (see Theorem 2.1). The proof is based on a novel semi-implicit discretization in time numerical scheme (3.1)–(3.5) that satisfies a discrete version of the dissipative energy law (2.2) (see Proposition 3.2 below). One can thus establish the existence of weak solutions to the resulting nonlinear elliptic system via the Leray–Schauder degree theory (cf. [25,26]). Then the existence of global finite energy solutions to the original CHSD system follows from a suitable compactness argument. We point out that our numerical scheme (3.1)–(3.5) differs from the one proposed and studied by Feng and Wise [27] (for the Cahn–Hilliard–Darcy system in simple domain) in the sense that, among others, both the elastic forcing term $\mu\nabla\varphi$ in the Stokes/Darcy equations and the convection term $\mathbf{u}\cdot\nabla\varphi$ in the Cahn–Hilliard equation are treated in a fully implicit way. As a consequence, we are able to prove the existence of finite energy solutions by only imposing the initial data $\varphi_0 \in H^1(\Omega)$, whereas in [27] the authors have to assume $\varphi_0 \in H^2(\Omega)$ (or at least $H^1(\Omega)\cap L^\infty(\Omega)$), which is not natural in view of the basic energy law (2.2). On the other hand, this choice of discretization brings extra difficulties such that neither the variational approach in [28,27] nor the monotonicity method devised in [29] can be applied. Besides, the complexity of the domain geometry also motivates us to introduce an equivalent norm for the velocity field (Eq. (3.73)), which is necessary for the analysis in the case of stationary Stokes equation ($\varpi = 0$). After the existence result is obtained, a weak–strong uniqueness property of the weak solutions is shown via the energy method (cf. Theorem 2.2 for the precise statement). We note that existence and uniqueness of strong solutions to the coupled CHSD system (1.1)–(1.22) is beyond the scope of this manuscript and will be addressed in a forthcoming work.

It is worth mentioning that there are a lot of works on diffuse-interface models for immiscible two phase incompressible flow with matched densities in a single domain setting. For instance, concerning the Cahn–Hilliard–Navier–Stokes system (Model H), existence of weak solutions, existence and uniqueness of strong solutions and long time dynamics are established in [4,30–32] and references therein. As for the Cahn–Hilliard–Darcy (also referred to as Cahn–Hilliard–Hele–Shaw) system in porous media or in the Hele–Shaw cell, the readers are referred to [27,33–36] for latest results.

The rest of this paper is organized as follows. In Section 2, we first introduce the appropriate functional spaces and derive a dissipative energy law associated with the CHSD system (1.1)–(1.22). After that, we present the definition of suitable weak solutions and state the main results of this paper. Section 3 is devoted to the existence of global finite energy weak solutions. We first obtain the existence of weak solutions to an implicit time-discretized system by the Leray–Schauder degree theory. Then the existence of finite energy weak solutions to the original CHSD system follows from a compactness argument. Finally, in Section 4 we prove the weak–strong uniqueness property of the weak solutions.

2. Preliminaries and main results

2.1. *Functional spaces*

We first introduce some notations. If X is a Banach space and X' is its dual, then $\langle u, v\rangle \equiv \langle u, v\rangle_{X',X}$ for $u \in X'$, $v \in X$ denotes the duality product. The inner product on a Hilbert space H is denoted by $(\cdot,\cdot)_H$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, then $L^q(\Omega)$, $1 \le q \le \infty$ denotes the usual Lebesgue space and $\|\cdot\|_{L^q(\Omega)}$ denotes its norm. Similarly, $W^{m,q}(\Omega)$, $m \in \mathbb{N}$, $1 \le q \le \infty$, denotes the usual Sobolev space with norm $\|\cdot\|_{W^{m,p}(\Omega)}$. When $q = 2$, we simply denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. Besides, the fractional order Sobolev spaces $H^s(\Omega)$ ($s \in \mathbb{R}$) are defined as in [37, Section 4.2.1]. If I is an interval of $\mathbb{R}^+$ and X a Banach space, we use the function space $L^p(I; X)$, $1 \le p \le +\infty$, which consists of p-integrable functions with values in X. Moreover, $C_w(I; X)$ denotes the topological vector space of all bounded and weakly continuous functions from I to X, while $W^{1,p}(I,X)$ $(1 \le q < +\infty)$ stands for the space of all functions u such that $u, \frac{du}{dt} \in L^p(I;X)$, where $\frac{du}{dt}$ denotes the vector-valued distributional derivative of u. Bold characters are used to denote vector spaces.

Given $v \in L^1(\Omega)$, we denote by $\bar{v} = |\Omega|^{-1} \int_\Omega v(x)dx$ its mean value. Then we define the space $\dot{L}^2(\Omega) := \{v \in L^2(\Omega) : \bar{v} = 0\}$ and $\dot{v} = \mathrm{P}_0 v := v - \bar{v}$ the orthogonal projection onto $\dot{L}^2(\Omega)$. Furthermore, we denote $\dot{H}^1(\Omega) = H^1(\Omega) \cap \dot{L}^2(\Omega)$, which is a Hilbert space with inner product $(u,v)_{\dot{H}^1} = \int_\Omega \nabla u \cdot \nabla v dx$ due to the classical Poincaré inequality for functions with zero mean. Its dual space is simply denoted by $\dot{H}^{-1}(\Omega)$.

For our CHSD problem with domain decomposition, we introduce the following spaces

$$\begin{aligned}
\mathbf{H}(\mathrm{div}; \Omega_j) &:= \big\{\mathbf{w} \in \mathbf{L}^2(\Omega_j) \,\big|\, \nabla \cdot \mathbf{w} \in \mathbf{L}^2(\Omega_j)\big\}, \quad j \in \{c, m\}, \\
\mathbf{H}_{c,0} &:= \big\{\mathbf{w} \in \mathbf{H}^1(\Omega_c) \,\big|\, \mathbf{w} = \mathbf{0} \text{ on } \Gamma_c\big\}, \\
\mathbf{H}_{c,\mathrm{div}} &:= \{\mathbf{w} \in \mathbf{H}_{c,0} \mid \nabla \cdot \mathbf{w} = 0\}, \\
\mathbf{H}_{m,0} &:= \big\{\mathbf{w} \in \mathbf{H}(\mathrm{div}; \Omega_m) \,\big|\, \mathbf{w} \cdot \mathbf{n}_m = 0 \text{ on } \Gamma_m\big\}, \\
\mathbf{H}_{m,\mathrm{div}} &:= \{\mathbf{w} \in \mathbf{H}_{m,0} \mid \nabla \cdot \mathbf{w} = 0\}, \\
X_m &:= \dot{H}^1(\Omega_m).
\end{aligned}$$

We denote by $(\cdot,\cdot)_c$, $(\cdot,\cdot)_m$ the inner products on the spaces $L^2(\Omega_c)$, $L^2(\Omega_m)$, respectively (also for the corresponding vector spaces). The inner product on $L^2(\Omega)$ is simply denoted by $(\cdot,\cdot)$. Then it is clear that

$$(u,v) = (u_m, v_m)_m + (u_c, v_c)_c, \qquad \|u\|^2_{L^2(\Omega)} = \|u_m\|^2_{L^2(\Omega_m)} + \|u_c\|^2_{L^2(\Omega_c)},$$

where $u_m := u|_{\Omega_m}$ and $u_c := u|_{\Omega_c}$.

On the interface Γ_{cm}, we consider the fractional Sobolev spaces $H^{\frac{1}{2}}_{00}(\Gamma_{cm})$ and $H^{\frac{1}{2}}(\Gamma_{cm})$ for (Lipschitz) surface Γ_{cm} when $d = 3$ or curve when $d = 2$ with the following equivalent norms (see [38, p. 66], or [39]):

$$\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}(\Gamma_{cm})} &= \int_{\Gamma_{cm}} |u|^2 dS + \int_{\Gamma_{cm}} \int_{\Gamma_{cm}} \frac{|u(x) - u(y)|^2}{|x-y|^d} dx dy, \\
\|u\|^2_{H^{\frac{1}{2}}_{00}(\Gamma_{cm})} &= \|u\|^2_{H^{\frac{1}{2}}(\Gamma_{cm})} + \int_{\Gamma_{cm}} \frac{|u(x)|^2}{\rho(x, \partial\Gamma_{cm})} dx,
\end{aligned}$$

where $\rho(x, \partial\Gamma_{cm})$ denotes the distance from x to $\partial\Gamma_{cm}$. The above norms are not equivalent except when Γ_{cm} is a closed surface or curve (cf. [24]). If Γ_{cm} is a subset of $\partial\Omega_c$ with positive measure, then $H^{\frac{1}{2}}_{00}(\Gamma_{cm})$ is a trace space of functions of $H^1(\Omega_c)$ that vanish on $\partial\Omega_c \backslash \Gamma_{cm}$. Similarly in the vectorial case, we have $\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma_{cm}) = \mathbf{H}_{c,0}|_{\Gamma_{cm}}$. $H^{\frac{1}{2}}_{00}(\Gamma_{cm})$ is a non-closed subspace of $H^{\frac{1}{2}}(\Gamma_{cm})$ and has a continuous zero extension to $H^{\frac{1}{2}}(\partial\Omega_c)$. For $H^{\frac{1}{2}}_{00}(\Gamma_{cm})$, we have the following continuous embedding result (cf. [17]): $H^{\frac{1}{2}}_{00}(\Gamma_{cm}) \subsetneqq H^{\frac{1}{2}}(\Gamma_{cm}) \subsetneqq H^{-\frac{1}{2}}(\Gamma_{cm}) \subsetneqq (H^{\frac{1}{2}}_{00}(\Gamma_{cm}))'$.

We note that $H^{-\frac{1}{2}}(\partial\Omega_c)|_{\Gamma_{cm}} \nsubseteq H^{-\frac{1}{2}}(\Gamma_{cm})$ and $H^{-\frac{1}{2}}(\partial\Omega_c)|_{\Gamma_{cm}} \subset (H_{00}^{\frac{1}{2}}(\Gamma_{cm}))'$, where the space $H^{-\frac{1}{2}}(\partial\Omega_c)|_{\Gamma_{cm}}$ is defined in the following way: for all $f \in H^{-\frac{1}{2}}(\partial\Omega_c)|_{\Gamma_{cm}}$ and $g \in H_{00}^{\frac{1}{2}}(\Gamma_{cm})$, $\langle f, g\rangle_{H^{-\frac{1}{2}}(\partial\Omega_c)|_{\Gamma_{cm}},\, H_{00}^{\frac{1}{2}}(\Gamma_{cm})} := \langle f, \tilde{g}\rangle_{H^{-\frac{1}{2}}(\partial\Omega_c),\, H^{\frac{1}{2}}(\partial\Omega_c)}$ with $\tilde{g}$ being the zero extension of g to $\partial\Omega_c$.

For any $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega_c)$, its normal component $\mathbf{u}\cdot\mathbf{n}_{cm}$ is well defined in $(H_{00}^{\frac{1}{2}}(\Gamma_{cm}))'$, and for all $q \in H^1(\Omega_c)$ such that $q = 0$ on $\partial\Omega_c \backslash \Gamma_{cm}$, we have

$$(\nabla\cdot\mathbf{u}, q)_c = (\mathbf{u}, \nabla q)_c + \langle \mathbf{u}\cdot\mathbf{n}_{cm}, q\rangle_{(H_{00}^{\frac{1}{2}}(\Gamma_{cm}))',\, H_{00}^{\frac{1}{2}}(\Gamma_{cm})}.$$

Similar identity holds on the matrix domain Ω_m.

2.2. Basic energy law

An important feature of the CHSD system (1.1)–(1.22) is that it obeys a dissipative energy law. To this end, we first note that the total energy of the coupled system is given by:

$$\mathcal{E}(t) = \int_{\Omega_c} \frac{\varpi}{2}|\mathbf{u}_c|^2 dx + \int_{\Omega} \left[\frac{\epsilon}{2}|\nabla\varphi|^2 + \frac{1}{\epsilon}F(\varphi)\right] dx. \tag{2.1}$$

Then we have the following formal result:

Lemma 2.1 *(Basic energy law). Let $(\mathbf{u}_m, \mathbf{u}_c, \varphi)$ be a smooth solution to the initial boundary value problem (1.1)–(1.22). Then $(\mathbf{u}_m, \mathbf{u}_c, \varphi)$ satisfies the following basic energy law:*

$$\frac{d}{dt}\mathcal{E}(t) = -\mathcal{D}(t) \le 0, \quad \forall t \ge 0, \tag{2.2}$$

where the energy dissipation $\mathcal{D}$ is given by

$$\begin{aligned} \mathcal{D}(t) = {} & \int_{\Omega_m} \nu(\varphi_m)\Pi^{-1}|\mathbf{u}_m|^2 dx + \int_{\Omega_c} 2\nu(\varphi_c)\big|\mathbb{D}(\mathbf{u}_c)\big|^2 dx \\ & + \int_{\Omega} \mathrm{M}(\varphi)\big|\nabla\mu(\varphi)\big|^2 dx + \frac{\alpha_{BJSJ}}{\sqrt{\mathrm{trace}(\Pi)}} \sum_{i=1}^{d-1} \int_{\Gamma_{cm}} \nu(\varphi)|\mathbf{u}_c\cdot\boldsymbol{\tau}_i|^2 dS. \end{aligned} \tag{2.3}$$

Proof. For the conduit part, multiplying Eqs. (1.1), (1.3) by $\mathbf{u}_c$ and $\mu(\varphi_c)$, respectively, integrating over Ω_c, and adding the resultants together, we get

$$\begin{aligned} & \frac{d}{dt}\int_{\Omega_c} \frac{\varpi}{2}|\mathbf{u}_c|^2 dx + \int_{\Omega_c} \partial_t\varphi_c\mu(\varphi_c) dx \\ & = \int_{\Omega_c} \big[\nabla\cdot\mathbb{T}(\mathbf{u}_c, P_c)\big]\cdot\mathbf{u}_c dx + \int_{\Omega_c} \mu(\varphi_c)\mathrm{div}\big(\mathrm{M}(\varphi_c)\nabla\mu(\varphi_c)\big) dx. \end{aligned}$$

After integration by parts and using the boundary conditions, we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_c}\left[\frac{\varpi}{2}|\mathbf{u}_c|^2+\frac{\epsilon}{2}|\nabla\varphi_c|^2+\frac{1}{\epsilon}F(\varphi_c)\right]dx+\int_{\Omega_c}\mathrm{M}(\varphi_c)\left|\nabla\mu(\varphi_c)\right|^2dx\\
&\quad=\int_{\Omega_c}\left[\nabla\cdot\mathbb{T}(\mathbf{u}_c,P_c)\right]\cdot\mathbf{u}_c dx+\int_{\Gamma_{cm}}\mathrm{M}(\varphi_c)\mu(\varphi_c)\frac{\partial\mu(\varphi_c)}{\partial\mathbf{n}_{cm}}dS\\
&\qquad+\epsilon\int_{\Gamma_{cm}}\partial_t\varphi_c\frac{\partial\varphi_c}{\partial\mathbf{n}_{cm}}dS. \qquad (2.4)
\end{aligned}
$$

Applying the divergence theorem to the first term on the right-hand side of (2.4), we infer from the boundary conditions (1.12), (1.21), (1.22) and the incompressibility condition (1.2) that

$$
\begin{aligned}
&\int_{\Omega_c}\left[\nabla\cdot\mathbb{T}(\mathbf{u}_c,P_c)\right]\cdot\mathbf{u}_c dx\\
&\quad=\int_{\Gamma_{cm}}\left(\mathbb{T}(\mathbf{u}_c,P_c)\mathbf{n}_{cm}\right)\cdot\mathbf{u}_c dS-\int_{\Omega_c}\mathbb{T}(\mathbf{u}_c,P_c):\nabla\mathbf{u}_c dx\\
&\quad=\sum_{i=1}^{2}\int_{\Gamma_{cm}}\left(\boldsymbol{\tau}_i^T\mathbb{T}(\mathbf{u}_c,P_c)\mathbf{n}_{cm}\right)(\mathbf{u}_c\cdot\boldsymbol{\tau}_i)dS\\
&\qquad+\int_{\Gamma_{cm}}\left(\mathbf{n}_{cm}^T\mathbb{T}(\mathbf{u}_c,P_c)\mathbf{n}_{cm}\right)(\mathbf{u}_c\cdot\mathbf{n}_{cm})dS\\
&\qquad-\int_{\Omega_c}\left(2\nu(\varphi_c)\mathbb{D}(\mathbf{u}_c)-P_c\mathbb{I}\right):\nabla\mathbf{u}_c dx\\
&\quad=-\frac{\alpha_{BJSJ}}{\sqrt{\mathrm{trace}(\Pi)}}\sum_{i=1}^{d-1}\int_{\Gamma_{cm}}\nu(\varphi_m)|\mathbf{u}_c\cdot\boldsymbol{\tau}_i|^2dS-\int_{\Gamma_{cm}}P_m(\mathbf{u}_c\cdot\mathbf{n}_{cm})dS\\
&\qquad-\int_{\Omega_c}2\nu(\varphi_c)\left|\mathbb{D}(\mathbf{u}_c)\right|^2dx. \qquad (2.5)
\end{aligned}
$$

Next, we consider the matrix part. Multiplying Eq. (1.6) by $\mu(\varphi_m)$ and integrating over Ω_m, we get

$$
\int_{\Omega_m}\partial_t\varphi_m\mu(\varphi_m)+(\mathbf{u}_m\cdot\nabla\varphi_m)\mu(\varphi_m)dx=\int_{\Omega_m}\mu(\varphi_m)\mathrm{div}\left(\mathrm{M}(\varphi_m)\nabla\mu(\varphi_m)\right)dx. \qquad (2.6)
$$

On the other hand, we infer from the Darcy equation (1.1) that

$$
\mu(\varphi_m)\nabla\varphi_m=\nu(\varphi_m)\Pi^{-1}\mathbf{u}_m+\nabla P_m.
$$

Using this fact and integration by parts, we infer from the boundary condition (1.15) that

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_m}\left[\frac{\epsilon}{2}|\nabla\varphi_m|^2+\frac{1}{\epsilon}F(\varphi_m)\right]dx \\
&\quad+\epsilon\int_{\Gamma_{cm}}\partial_t\varphi_m\frac{\partial\varphi_m}{\partial\mathbf{n}_{cm}}dS+\int_{\Omega_m}\big(\nu(\varphi_m)\Pi^{-1}|\mathbf{u}_m|^2+\mathbf{u}_m\cdot\nabla P_m\big)dx \\
&=-\int_{\Gamma_{cm}}\mathrm{M}(\varphi_m)\mu(\varphi_m)\frac{\partial\mu(\varphi_m)}{\partial\mathbf{n}_{cm}}dS-\int_{\Omega_m}\mathrm{M}(\varphi_m)\big|\nabla\mu(\varphi_m)\big|^2dx,
\end{aligned}\tag{2.7}$$

where we recall that $\mathbf{n}_{cm}$ denotes the unit normal to interface Γ_{cm} pointing from the conduit to the matrix. By the divergence theorem and the incompressibility condition (1.5), we get

$$\begin{aligned}
\int_{\Omega_m}\mathbf{u}_m\cdot\nabla P_m dx&=\int_{\Omega_m}\big[\nabla\cdot(P_m\mathbf{u}_m)-P_m(\nabla\cdot\mathbf{u}_m)\big]dx \\
&=-\int_{\Gamma_{cm}}P_m\mathbf{u}_m\cdot\mathbf{n}_{cm}dS.
\end{aligned}\tag{2.8}$$

Then (2.7) becomes

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_m}\left[\frac{\epsilon}{2}|\nabla\varphi_m|^2+\frac{1}{\epsilon}F(\varphi_m)\right]dx \\
&\quad+\int_{\Omega_m}\big(\nu(\varphi_m)\Pi^{-1}|\mathbf{u}_m|^2+\mathrm{M}(\varphi_m)\big|\nabla\mu(\varphi_m)\big|^2\big)dx \\
&=-\int_{\Gamma_{cm}}\mathrm{M}(\varphi_m)\mu(\varphi_m)\frac{\partial\mu(\varphi_m)}{\partial\mathbf{n}_{cm}}dS-\epsilon\int_{\Gamma_{cm}}\partial_t\varphi_m\frac{\partial\varphi_m}{\partial\mathbf{n}_{cm}}dS \\
&\quad+\int_{\Gamma_{cm}}P_m\mathbf{u}_m\cdot\mathbf{n}_{cm}dS.
\end{aligned}\tag{2.9}$$

Finally, combining (2.4), (2.5) and (2.9), using the definition of φ as well as the continuity conditions (1.16)–(1.17) on interface Γ_{cm}, we can cancel the boundary terms and conclude the basic energy law (2.2). The proof is complete. □

2.3. *Weak formulation and main results*

We make the following assumptions on viscosity ν, mobility coefficient M as well as the permeability matrix Π:

(A1) $\nu \in C^1(\mathbb{R})$, $\underline{\nu} \le \nu(s) \le \bar{\nu}$ and $|\nu'(s)| \le \tilde{\nu}$ for $s \in \mathbb{R}$, where $\bar{\nu}$, $\underline{\nu}$ and $\tilde{\nu}$ are positive constants.
(A2) $\mathrm{M} \in C^1(\mathbb{R})$, $\underline{m} \le \mathrm{M}(s) \le \bar{m}$ and $|\mathrm{M}'(s)| \le \tilde{m}$ for $s \in \mathbb{R}$, where $\bar{m}$, $\underline{m}$ and $\tilde{m}$ are positive constants.
(A3) The permeability Π is isotropic, bounded from above and below (so is the hydraulic conductivity tensor $\mathbb{K}$), namely, $\Pi = \kappa(x)\mathbb{I}$ with $\mathbb{I}$ being the $d \times d$ identity matrix and $\kappa(x) \in L^\infty(\Omega)$ such that there exist $\bar{\kappa} > \underline{\kappa} > 0$, $\underline{\kappa} \le \kappa(x) \le \bar{\kappa}$ a.e. in Ω.

Below we introduce the notion of finite energy weak solution to the CHSD system (1.1)–(1.22) as well as its corresponding weak formulation.

Definition 2.1. Suppose that $d = 2, 3$ and $T > 0$ is arbitrary. Let $\alpha = \frac{8}{5}$ when $d = 3$ and $\alpha < 2$ being arbitrary close to 2 when $d = 2$.

Case 1: $\varpi > 0$. We consider the initial data $\mathbf{u}_0(x) \in \mathbf{L}^2(\Omega_c)$, $\varphi_0 \in H^1(\Omega)$. The functions $(\mathbf{u}_c, \mathbf{u}_m, P_m, \varphi, \mu)$ with the following properties

$$\mathbf{u}_c \in L^\infty\big(0, T; \mathbf{L}^2(\Omega_c)\big) \cap L^2(0, T; \mathbf{H}_{c,\mathrm{div}}) \cap W^{1,\alpha}\big(0, T; \big(\mathbf{H}^1(\Omega_c)\big)'\big), \tag{2.10}$$

$$\mathbf{u}_m \in L^2\big(0, T; \mathbf{L}^2(\Omega_m)\big), \tag{2.11}$$

$$P_m \in L^\alpha(0, T; X_m), \tag{2.12}$$

$$\varphi \in L^\infty\big(0, T; H^1(\Omega)\big) \cap L^2\big(0, T; H^3(\Omega)\big) \cap W^{1,\alpha}\big(0; T; \big(H^1(\Omega)\big)'\big), \tag{2.13}$$

$$\mu \in L^2\big(0, T; H^1(\Omega)\big), \tag{2.14}$$

is called a finite energy weak solution of the CHSD system (1.1)–(1.22), if the following conditions are satisfied:

(1) For any $\mathbf{v}_c \in C_0^\infty((0, T); \mathbf{H}_{c,\mathrm{div}})$ and $q_m \in C([0, T]; X_m)$,

$$\begin{aligned}
&-\varpi \int_0^T (\mathbf{u}_c, \partial_t \mathbf{v}_c)_c dt + 2\int_0^T \big(\nu(\varphi_c)\mathbb{D}(\mathbf{u}_c), \mathbb{D}(\mathbf{v}_c)\big)_c dt \\
&\quad + \int_0^T \left(\frac{\Pi}{\nu(\varphi_m)}\big[\nabla P_m - \mu(\varphi_m)\nabla\varphi_m\big], \nabla q_m\right)_m dt \\
&\quad + \sum_{i=1}^{d-1} \frac{\alpha_{BJSJ}}{\sqrt{\mathrm{trace}(\Pi)}} \int_0^T \int_{\Gamma_{cm}} \nu(\varphi_m)(\mathbf{u}_c \cdot \boldsymbol{\tau}_i)(\mathbf{v}_c \cdot \boldsymbol{\tau}_i) dS dt \\
&\quad + \int_0^T \int_{\Gamma_{cm}} P_m(\mathbf{v}_c \cdot \mathbf{n}_{cm}) dS dt - \int_0^T \int_{\Gamma_{cm}} (\mathbf{u}_c \cdot \mathbf{n}_{cm}) q_m dS dt \\
&= \int_0^T \big(\mu(\varphi_c)\nabla\varphi_c, \mathbf{v}_c\big)_c dt,
\end{aligned} \tag{2.15}$$

moreover, the velocity $\mathbf{u}_m$ in the matrix part satisfies

$$\int_0^T (\mathbf{u}_m, \mathbf{v}_m)_m dx = -\int_0^T \left(\frac{\Pi}{\nu(\varphi_m)} [\nabla P_m - \mu(\varphi_m)\nabla\varphi_m], \mathbf{v}_m \right)_m dt, \tag{2.16}$$

for any $\mathbf{v}_m \in C([0, T]; \mathbf{L}^2(\Omega_m))$.

(2) For any $\phi \in C_0^\infty((0, T); H^1(\Omega))$,

$$-\int_0^T (\varphi, \partial_t\phi)dt + \int_0^T \big(\mathrm{M}(\varphi)\nabla\mu(\varphi), \nabla\phi\big)dt = -\int_0^T (\mathbf{u}\cdot\nabla\varphi, \phi)dt, \tag{2.17}$$

$$\int_0^T \big(\mu(\varphi), \phi\big)dt = \int_0^T \left[\frac{1}{\epsilon}\big(f(\varphi), \phi\big) + \epsilon(\nabla\varphi, \nabla\phi)\right]dt. \tag{2.18}$$

(3) $\mathbf{u}_c|_{t=0} = \mathbf{u}_0(x)$, $\varphi|_{t=0} = \varphi_0(x)$.

(4) The finite energy solution satisfies the energy inequality

$$\mathcal{E}(t) + \int_s^t \mathcal{D}(\tau)d\tau \le \mathcal{E}(s), \tag{2.19}$$

for all $t \in [s, T)$ and almost all $s \in [0, T)$ (including $s = 0$), where the total energy $\mathcal{E}$ is given by (2.1).

Case 2: $\varpi = 0$. In this case, we do not need the initial condition for $\mathbf{u}_c$. The regularity property for $\mathbf{u}_c$ (cf. (2.10)) is simply replaced by

$$\mathbf{u}_c \in L^2(0, T; \mathbf{H}_{c,\mathrm{div}}). \tag{2.20}$$

The finite energy weak solution $(\mathbf{u}_c, \mathbf{u}_m, P_m, \varphi, \mu)$ still fulfills the above properties (1)–(4) with $\varpi = 0$ in corresponding formulations.

Remark 2.1. In the above weak formulation (2.15)–(2.16), the reason we do not break the force term $\nabla P_m - \mu(\varphi_m)\nabla\varphi_m$ is that this term (or equivalently, the velocity in the matrix part $\mathbf{u}_m$) has better regularity/integrability than its two components (see (2.11)–(2.12)).

Remark 2.2. We note that the interface boundary conditions (1.13)–(1.22) are enforced as a consequence of the weak formulation stated above. Note also that the pressure terms P_c and P_m are only uniquely determined up to an additive constant in the strong form (1.1)–(1.22), i.e., they satisfy the same set of equations with the same boundary conditions as well as interface conditions after being shifted by the same constant. As a consequence, it makes sense to seek P_m in the space $\dot{H}^1(\Omega_m)$ (i.e., X_m). The equivalence for smooth solutions between the weak formulation and the classical form can be verified in a straightforward way.

Now we are in a position to state the main results of this paper:

Theorem 2.1 *(Existence of finite energy weak solutions). Suppose that $d = 2, 3$ and the assumptions (A1)–(A3) are satisfied.*

(i) If $\varpi > 0$, for any $\mathbf{u}_0 \in \mathbf{L}^2(\Omega_c)$, $\varphi_0 \in H^1(\Omega)$ and $T > 0$ being arbitrary, the CHSD system (1.1)–(1.22) admits at least one global finite energy weak solution $\{\mathbf{u}_c, \mathbf{u}_m, P_m, \varphi, \mu\}$ in the sense of Definition 2.1.

(ii) If $\varpi = 0$, for any $\varphi_0 \in H^1(\Omega)$, the CHSD system (1.1)–(1.22) admits at least one global finite energy weak solution $\{\mathbf{u}_c, \mathbf{u}_m, P_m, \varphi, \mu\}$ in the sense of Definition 2.1.

Theorem 2.2 *(Weak–strong uniqueness). Let $d = 2, 3$, $\varpi \geq 0$ and the assumptions (A1)–(A3) be satisfied. Suppose that $\{\mathbf{u}_c, \mathbf{u}_m, P_m, \varphi\}$ is a finite energy weak solution to the CHSD system (1.1)–(1.22) in $(0, T) \times \Omega$ and $\{\tilde{\mathbf{u}}_c, \tilde{\mathbf{u}}_m, \tilde{P}_m, \tilde{\varphi}\}$ is a regular solution emanating from the same initial data with the following regularity conditions*

$$\tilde{\mathbf{u}}_c \in L^{\frac{8}{3}}(0, T; \mathbf{H}_{c,\mathrm{div}}), \qquad \tilde{\mathbf{u}}_m \in L^{\frac{8}{3}}(0, T; \mathbf{H}_{m,\mathrm{div}}), \qquad \tilde{\varphi} \in L^{\frac{8}{3}}\big(0, T; H^3(\Omega)\big),$$

then it holds

$$\mathbf{u}_c = \tilde{\mathbf{u}}_c, \qquad \mathbf{u}_m = \tilde{\mathbf{u}}_m, \qquad P_m = \tilde{P}_m, \qquad \varphi = \tilde{\varphi}.$$

3. Existence of weak solutions

We shall apply a semi-discretization approach (finite difference in time, cf. [40,41]) to prove the existence result Theorem 2.1. First, a discrete in time, continuous in space numerical scheme is proposed and shown to be mass-conservative and energy law preserving. Then, the existence of weak solutions to the discretized system is proved by the Leray–Schauder degree theory. Last, an approximate solution is constructed and its convergence to the weak solution of the original CHSD system (1.1)–(1.22) is established via a compactness argument.

3.1. A time discretization scheme

Here we propose a semi-implicit time discretization scheme to the weak formulation (2.15)–(2.18). Recall our convention

$$\varphi|_{\Omega_c} = \varphi_c, \qquad \varphi|_{\Omega_m} = \varphi_m, \qquad \mu|_{\Omega_c} = \mu_c, \qquad \mu|_{\Omega_m} = \mu_m.$$

For arbitrary but fixed $T > 0$ and positive integer $N \in \mathbb{N}$, we denote by $\delta = \Delta t = \frac{T}{N}$ the time step size. Given $(\mathbf{u}_c^k, \varphi_c^k, P_m^k, \varphi_m^k)$, $k = 0, 1, 2, ..., N-1$, we want to determine $(\mathbf{u}_c, \varphi_c, P_m, \varphi_m) = (\mathbf{u}_c^{k+1}, \varphi_c^{k+1}, P_m^{k+1}, \varphi_m^{k+1})$ as a solution of the following nonlinear elliptic system

$$\varpi\left(\frac{\mathbf{u}_c^{k+1} - \mathbf{u}_c^k}{\delta}, \mathbf{v}_c\right)_c + 2\big(\nu(\varphi_c^k)\mathbb{D}\big(\mathbf{u}_c^{k+1}\big), \mathbb{D}(\mathbf{v}_c)\big)_c$$
$$+ \left(\frac{\Pi}{\nu(\varphi_m^k)}\big[\nabla P_m^{k+1} - \mu_m^{k+1}\nabla\varphi_m^{k+1}\big], \nabla q_m\right)_m$$

$$
\begin{aligned}
&+\sum_{i=1}^{d-1}\frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}}\int_{\Gamma_{cm}}\nu(\varphi_m^k)\big(\mathbf{u}_c^{k+1}\cdot\boldsymbol{\tau}_i\big)(\mathbf{v}_c\cdot\boldsymbol{\tau}_i)dS\\
&+\int_{\Gamma_{cm}}P_m^{k+1}(\mathbf{v}_c\cdot\mathbf{n}_{cm})dS-\int_{\Gamma_{cm}}\big(\mathbf{u}_c^{k+1}\cdot\mathbf{n}_{cm}\big)q_m dS\\
&=\big(\mu_c^{k+1}\nabla\varphi_c^{k+1},\mathbf{v}_c\big)_c,
\end{aligned}
\tag{3.1}
$$

$$\left(\frac{\varphi^{k+1}-\varphi^k}{\delta},\phi\right)+\big(\mathbf{u}^{k+1}\cdot\nabla\varphi^{k+1},\phi\big)=-\big(\mathrm{M}(\varphi^k)\nabla\mu^{k+1},\nabla\phi\big),\tag{3.2}$$

$$\big(\mu^{k+1},\phi\big)=\frac{1}{\epsilon}\big(\tilde{f}\big(\varphi^{k+1},\varphi^k\big),\phi\big)+\epsilon\big(\nabla\varphi^{k+1},\nabla\phi\big),\tag{3.3}$$

for any $\mathbf{v}_c\in\mathbf{H}_{c,\mathrm{div}}$, $q_m\in X_m$ and $\phi\in H^1(\Omega)$. In the above formulation, the vector $\mathbf{u}^{k+1}$ satisfies $\mathbf{u}^{k+1}|_{\Omega_c}=\mathbf{u}_c^{k+1}$ and $\mathbf{u}^{k+1}|_{\Omega_m}=\mathbf{u}_m^{k+1}$, where

$$\mathbf{u}_m^{k+1}=-\frac{\Pi}{\nu(\varphi_m^k)}\big(\nabla P_m^{k+1}-\mu_m^{k+1}\nabla\varphi_m^{k+1}\big).\tag{3.4}$$

The function $\tilde{f}(\phi,\psi)$ in Eq. (3.3) is derived from a convex splitting approximation to the non-convex function $F(\varphi)$ (see (1.9)) and it takes the following form (cf. e.g., [42,28])

$$\tilde{f}(\phi,\psi)=\phi^3-\psi.\tag{3.5}$$

Remark 3.1. We note that Eqs. (3.1)–(3.4) are strongly coupled, which demands suitable choices on discretization schemes in order to prove the existence of weak solutions (see [28,27,29] for related diffuse-interface models). Here, the advective term in the Cahn–Hilliard equation (i.e., the second term $\mathbf{u}\cdot\nabla\varphi$ in Eq. (3.2)) and accordingly the elastic forcing term $\mu\nabla\varphi$ in Eqs. (3.1), (3.4) are discretized fully implicitly. Under this fully implicit discretization, it is possible to preserve a discrete energy law (see Lemma 3.2) in analogy to the continuous one (2.2), moreover it enables us to obtain the existence of weak solutions under the natural assumption on initial data such that $\varphi_0\in H^1(\Omega)$. In [28,27], a different semi-implicit treatment of the advective term and the elastic forcing term for the Cahn–Hilliard–Darcy system in a simple domain was proposed. The discretization therein still keeps a discrete energy law while one needs to assume $\varphi_0\in H^2(\Omega)$ (or at least $H^1(\Omega)\cap L^\infty(\Omega)$) to obtain the existence of weak solutions.

In the following content of this subsection, we will temporarily omit the superscript $k+1$ for $\mathbf{u}_c^{k+1}$, P_m^{k+1}, $\mathbf{u}_m^{k+1}$, φ^{k+1}, μ^{k+1} for the sake of simplicity. Besides, we just provide the proof for the case $\varpi>0$, while the argument can be easily adapted to the simpler case $\varpi=0$ with minor modifications.

A few *a priori* estimates can be readily derived. First, one can deduce that the above numerical scheme keeps the mass conservation property.

Lemma 3.1. *Suppose that* $\mathbf{u}_c^k \in \mathbf{L}^2(\Omega_c)$, $\varphi^k \in H^1(\Omega)$ *and* $\{\mathbf{u}_c, P_m, \varphi, \mu\} \in \mathbf{H}_{c,\mathrm{div}} \times X_m \times H^3(\Omega) \times H^1(\Omega)$ *solve the nonlinear system* (3.1)–(3.4). *Then* $\mathbf{u}_m$ *(given by* (3.4)*) satisfies*

$$\mathbf{u}_m \in \mathbf{H}_{m,\mathrm{div}}, \qquad \mathbf{u}_m \cdot \mathbf{n}_{cm} = \mathbf{u}_c \cdot \mathbf{n}_{cm} \in H^{\frac{1}{2}}(\Gamma_{cm}). \tag{3.6}$$

Moreover, the following mass-conservation holds

$$\int_\Omega \varphi\, dx = \int_\Omega \varphi^k\, dx. \tag{3.7}$$

Proof. It is clear from Eq. (3.4) and the Sobolev embedding theorem ($d \le 3$) that $\mathbf{u}_m \in \mathbf{L}^2(\Omega_m)$. Taking the test function $\mathbf{v}_c = 0$ in Eq. (3.1) and utilizing Eq. (3.4), one obtains

$$(\mathbf{u}_m, \nabla q_m)_m - \int_{\Gamma_{cm}} (\mathbf{u}_c \cdot \mathbf{n}_{cm}) q_m\, dS = 0, \quad \forall q_m \in X_m, \tag{3.8}$$

which easily yields that $\nabla \cdot \mathbf{u}_m = 0$ in the sense of distribution and then $\mathbf{u}_m \in \mathbf{H}(\mathrm{div}; \Omega_m)$. Thus, the normal component $\mathbf{u}_m \cdot \mathbf{n}$ is well-defined in $H^{-\frac{1}{2}}(\partial\Omega_m)$ ($\mathbf{n}$ denotes the unit outer normal on $\partial\Omega_m$ and it corresponds to $\mathbf{n}_m$ on Γ_m and to $-\mathbf{n}_{cm}$ on Γ_{cm}, respectively). Applying Green's formula to the first term in Eq. (3.8) gives that

$$\mathbf{u}_m \cdot \mathbf{n}_m = 0 \quad \text{in } H^{-\frac{1}{2}}(\Gamma_m) \quad \text{and} \quad \mathbf{u}_m \cdot \mathbf{n}_{cm} = \mathbf{u}_c \cdot \mathbf{n}_{cm} \quad \text{in } \big(H_{00}^{\frac{1}{2}}(\Gamma_{cm})\big)'.$$

Therefore, $\mathbf{u}_m \in \mathbf{H}_{m,\mathrm{div}}$. It follows from the trace theorem that $\mathbf{u}_c \cdot \mathbf{n}_{cm} \in H^{\frac{1}{2}}(\Gamma_{cm})$, then one further gets $\mathbf{u}_m \cdot \mathbf{n}_{cm} = \mathbf{u}_c \cdot \mathbf{n}_{cm}$ in $H^{\frac{1}{2}}(\Gamma_{cm})$.

The mass-conservation (3.7) now follows from taking the test function $\phi = 1$ in Eq. (3.2) and performing integration by parts. □

The next lemma shows that the numerical scheme (3.1)–(3.5) satisfies a discrete analogue of the basic energy law (2.1).

Lemma 3.2. *Suppose that* $\mathbf{u}_c^k \in \mathbf{L}^2(\Omega_c)$, $\varphi^k \in H^1(\Omega)$ *and* $\{\mathbf{u}_c, P_m, \varphi, \mu\} \in \mathbf{H}_{c,\mathrm{div}} \times X_m \times H^3(\Omega) \times H^1(\Omega)$ *solve the system* (3.1)–(3.4). *Then the following discrete energy inequality holds*

$$\begin{aligned}
&\mathcal{E}(\mathbf{u}_c, \varphi) + \delta\big(\nu(\varphi_m^k)\Pi^{-1}\mathbf{u}_m, \mathbf{u}_m\big)_m + 2\delta\big(\nu(\varphi_c^k)\mathbb{D}(\mathbf{u}_c), \mathbb{D}(\mathbf{u}_c)\big)_c \\
&\quad + \delta \int_\Omega \mathrm{M}(\varphi^k)|\nabla\mu|^2 dx + \frac{\delta\alpha_{BJSJ}}{\sqrt{\mathrm{trace}(\Pi)}} \sum_{i=1}^{d-1} \int_{\Gamma_{cm}} \nu(\varphi_m^k)|\mathbf{u}_c \cdot \boldsymbol{\tau}_i|^2 dS \\
&\quad + \frac{\varpi}{2}\big(\mathbf{u}_c - \mathbf{u}_c^k, \mathbf{u}_c - \mathbf{u}_c^k\big)_c + \frac{\epsilon}{2}\big\|\nabla(\varphi - \varphi^k)\big\|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon}\big\|\varphi - \varphi^k\big\|_{L^2(\Omega)}^2 \\
&\le \mathcal{E}\big(\mathbf{u}_c^k, \varphi^k\big),
\end{aligned} \tag{3.9}$$

where the energy functional $\mathcal{E}$ *is defined in* (2.1).

Proof. Taking $\mathbf{v}_c = \mathbf{u}_c$, $q_m = P_m$ in (3.1), using (3.4) and the elementary identity

$$a \cdot (a-b) = \frac{1}{2}\big(|a|^2 - |b|^2 + |a-b|^2\big), \quad \forall a, b \in \mathbb{R} \text{ or } \mathbb{R}^d, \tag{3.10}$$

we have

$$\begin{aligned}
&\frac{\varpi}{2\delta}(\mathbf{u}_c, \mathbf{u}_c)_c + \frac{\varpi}{2\delta}\big(\mathbf{u}_c - \mathbf{u}_c^k, \mathbf{u}_c - \mathbf{u}_c^k\big)_c + 2\big(\nu\big(\varphi_c^k\big)\mathbb{D}(\mathbf{u}_c), \mathbb{D}(\mathbf{u}_c)\big)_c \\
&\quad + \big(\nu\big(\varphi_m^k\big)\Pi^{-1}\mathbf{u}_m, \mathbf{u}_m\big)_m + \sum_{i=1}^{d-1} \frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}} \int_{\Gamma_{cm}} \nu\big(\varphi_m^k\big)|\mathbf{u}_c \cdot \boldsymbol{\tau}_i|^2 dS \\
&= \frac{\varpi}{2\delta}\big(\mathbf{u}_c^k, \mathbf{u}_c^k\big)_c + (\mu\nabla\varphi, \mathbf{u}).
\end{aligned} \tag{3.11}$$

By a direct calculation, we infer from the definition of the convex splitting function $\tilde{f}$ that

$$\begin{aligned}
\tilde{f}(\phi, \psi)(\phi - \psi) &= F(\phi) - F(\psi) + \frac{1}{4}\big(\phi^2 - \psi^2\big)^2 + \frac{1}{2}(\phi - \psi)^2 + \frac{1}{2}\phi^2(\phi - \psi)^2 \\
&\geq F(\phi) - F(\psi) + \frac{1}{2}(\phi - \psi)^2.
\end{aligned} \tag{3.12}$$

Then taking the test functions $\phi = \mu$ in (3.2) and $\phi = \varphi - \varphi^k$ in (3.3), after integration by parts, we infer from (3.10) and (3.12) that

$$\left(\frac{\varphi - \varphi^k}{\delta}, \mu\right) + (\mathbf{u} \cdot \nabla\varphi, \mu) + \int_\Omega \mathrm{M}\big(\varphi^k\big)|\nabla\mu|^2 dx = 0, \tag{3.13}$$

where

$$\begin{aligned}
\big(\varphi - \varphi^k, \mu\big) &= \frac{1}{\epsilon}\big(\tilde{f}\big(\varphi, \varphi^k\big), \varphi - \varphi^k\big) + \epsilon\big(\nabla\varphi, \nabla\big(\varphi - \varphi^k\big)\big) \\
&\geq \frac{\epsilon}{2}\|\nabla\varphi\|_{L^2(\Omega)}^2 + \frac{\epsilon}{2}\big\|\nabla\big(\varphi - \varphi^k\big)\big\|_{L^2(\Omega)}^2 - \frac{\epsilon}{2}\big\|\nabla\varphi^k\big\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{\epsilon}\big(F(\varphi) - F\big(\varphi^k\big), 1\big) + \frac{1}{2\epsilon}\big\|\varphi - \varphi^k\big\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.14}$$

Combining the above estimates (3.11)–(3.14) together, we easily conclude the discrete energy inequality (3.9). □

3.2. *Existence of weak solutions to the discrete problem*

In order to prove the existence of solutions to the discrete problem (3.1)–(3.4), we shall adapt an argument involving the Leray–Schauder degree theory (cf. e.g., [25]) that has been used in [26] to show the existence of weak solutions to a diffuse-interface model in simple domain with general densities. The idea is to rewrite the system (3.1)–(3.3) in terms of suitable "good" operator denoted by $\mathcal{T}_k$ and "bad" operator denoted by $\mathcal{G}_k$ such that

$$\mathcal{T}_k(\mathbf{w}) = \mathcal{G}_k(\mathbf{w}), \tag{3.15}$$

where $\mathbf{w} := \{\mathbf{u}_c, P_m, \varphi, \mu\}$ is the solution. More precisely, in the abstract equation (3.15) the operators $\mathcal{T}_k : \mathbf{X} \to \mathbf{Y}$ and $\mathcal{G}_k : \mathbf{X} \to \mathbf{Y}$ (see (3.34)–(3.35) for their detailed definition and the associated spaces $\mathbf{X}$ and $\mathbf{Y}$ will be specified in (3.33)) basically correspond to, respectively, the left-hand side and right-hand side of the following reformulation of the system (3.1)–(3.3) (dropping the superscript $k+1$ for simplicity as mentioned before)

$$\begin{aligned}
&(\mathbf{u}_c, \mathbf{v}_c)_c + 2\big(\nu\big(\varphi_c^k\big)\mathbb{D}(\mathbf{u}_c), \mathbb{D}(\mathbf{v}_c)\big)_c + \left(\frac{\Pi}{\nu(\varphi_m^k)}\nabla P_m, \nabla q_m\right)_m \\
&\quad + \sum_{i=1}^{d-1} \frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}} \int_{\Gamma_{cm}} \nu\big(\varphi_m^k\big)(\mathbf{u}_c\cdot\boldsymbol{\tau}_i)(\mathbf{v}_c\cdot\boldsymbol{\tau}_i)\,dS \\
&\quad + \int_{\Gamma_{cm}} P_m(\mathbf{v}_c\cdot\mathbf{n}_{cm})\,dS - \int_{\Gamma_{cm}} (\mathbf{u}_c\cdot\mathbf{n}_{cm})q_m\,dS \\
&= (\mu_c\nabla\varphi_c, \mathbf{v}_c)_c + (\mathbf{u}_c, \mathbf{v}_c)_c + \left(\frac{\Pi}{\nu(\varphi_m^k)}\mu_m\nabla\varphi_m, \nabla q_m\right)_m \\
&\quad - \left(\frac{\varpi}{\delta}\big(\mathbf{u}_c - \mathbf{u}_c^k\big), \mathbf{v}_c\right)_c,
\end{aligned} \tag{3.16}$$

$$-\big(\mathrm{M}\big(\varphi^k\big)\nabla\mu, \nabla\phi\big) = \left(\frac{\varphi-\varphi^k}{\delta}, \phi\right) + (\mathbf{u}\cdot\nabla\varphi, \phi), \tag{3.17}$$

$$\frac{1}{\epsilon}\big(\varphi^3, \phi\big) + \epsilon(\nabla\varphi, \nabla\phi) = \left(\mu + \frac{1}{\epsilon}\varphi^k, \phi\right). \tag{3.18}$$

As will be shown below, the operator $\mathcal{T}_k : \mathbf{X} \to \mathbf{Y}$ is continuous and invertible with $\mathcal{T}_k^{-1}(\mathbf{0}) = \mathbf{0}$, while the operator $\mathcal{G}_k : \mathbf{X} \to \mathbf{Y}$ is compact. Thus the abstract equation (3.15) can be recasted into

$$\big(\mathcal{I} - \mathcal{T}_k^{-1}\mathcal{G}_k\big)(\mathbf{w}) = \mathbf{0},$$

where $\mathcal{I} : \mathbf{X} \to \mathbf{X}$ is the identity operator. Then the existence of solutions can be shown by the Leray–Schauder degree theory.

Remark 3.2. Note that Eq. (3.16) is derived from an addition of a term $(\mathbf{u}_c, \mathbf{v}_c)_c$ on both sides of Eq. (3.1). This modification is necessary in proving the invertibility of the operator associated with the left-hand side of Eq. (3.16), especially under the circumstance $|\Gamma_c| = 0$ where only the version (3.21) of Korn's inequality can be applied.

We shall divide the proof for the existence of weak solutions to the approximate problem (3.1)–(3.4) into three steps.

Step 1. *Invertibility of operators associated with the left-hand sides of the reformulated system (3.16)–(3.18).*

First, we deal with the operator associated with the left-hand side of Eq. (3.16). Define the product space

$$\mathbf{V} := \mathbf{H}_{c,\mathrm{div}} \times X_m. \tag{3.19}$$

Then we introduce the operator $\mathcal{L}_k : \mathbf{V} \to \mathbf{V}'$ that can be associated with the following bilinear form $a(\cdot,\cdot) : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$:

$$\begin{aligned}
&\big\langle \mathcal{L}_k(\mathbf{u}_c, P_m), (\mathbf{v}_c, q_m)\big\rangle_{\mathbf{V}',\mathbf{V}} \\
&\quad = a\big((\mathbf{u}_c, P_m), (\mathbf{v}_c, q_m)\big) \\
&\quad = 2\big(\nu\big(\varphi_c^k\big)\mathbb{D}(\mathbf{u}_c), \mathbb{D}(\mathbf{v}_c)\big)_c + (\mathbf{u}_c, \mathbf{v}_c)_c + \left(\frac{\Pi}{\nu(\varphi_m^k)}\nabla P_m, \nabla q_m\right)_m \\
&\qquad + \sum_{i=1}^{d-1} \frac{\alpha_{BJSJ}}{\sqrt{\mathrm{trace}(\Pi)}} \int_{\Gamma_{cm}} \nu\big(\varphi_m^k\big)(\mathbf{u}_c \cdot \boldsymbol{\tau}_i)(\mathbf{v}_c \cdot \boldsymbol{\tau}_i)\,dS \\
&\qquad + \int_{\Gamma_{cm}} P_m(\mathbf{v}_c \cdot \mathbf{n}_{cm})\,dS - \int_{\Gamma_{cm}} (\mathbf{u}_c \cdot \mathbf{n}_{cm}) q_m\,dS,
\end{aligned} \tag{3.20}$$

for any $(\mathbf{u}_c, P_m), (\mathbf{v}_c, q_m) \in \mathbf{V}$.

Recall the following Korn's inequality (cf. e.g., [43]),

$$\|\mathbf{v}_c\|_{\mathbf{H}^1(\Omega_c)} \le C\big(\|\mathbf{v}_c\|_{\mathbf{L}^2(\Omega_c)} + \big\|\mathbb{D}(\mathbf{v}_c)\big\|_{\mathbf{L}^2(\Omega_c)}\big), \quad \forall \mathbf{v}_c \in \mathbf{H}_{c,\mathrm{div}}, \tag{3.21}$$

where the constant C depends only on Ω_c. Moreover, if the boundary Γ_c has non-zero measure, the Korn inequality can be simplified as (cf. e.g., [44])

$$\|\mathbf{v}_c\|_{\mathbf{H}^1(\Omega_c)} \le C\big\|\mathbb{D}(\mathbf{v}_c)\big\|_{\mathbf{L}^2(\Omega_c)}, \quad \forall \mathbf{v}_c \in \mathbf{H}_{c,\mathrm{div}}. \tag{3.22}$$

As a consequence, using the assumptions (A1), (A3) and the Poincaré inequality, we deduce that the above bilinear form $a(\cdot,\cdot)$ is coercive on $\mathbf{V}$, namely, for any $(\mathbf{u}_c, P_m) \in \mathbf{V}$,

$$\begin{aligned}
&a\big((\mathbf{u}_c, P_m), (\mathbf{u}_c, P_m)\big) \\
&\quad = 2\big(\nu\big(\varphi_c^k\big)\mathbb{D}(\mathbf{u}_c), \mathbb{D}(\mathbf{u}_c)\big)_c + (\mathbf{u}_c, \mathbf{u}_c)_c + \left(\frac{\Pi}{\nu(\varphi_m^k)}\nabla P_m, \nabla P_m\right)_m \\
&\qquad + \sum_{i=1}^{d-1} \frac{\alpha_{BJSJ}}{\sqrt{\mathrm{trace}(\Pi)}} \int_{\Gamma_{cm}} \nu\big(\varphi_m^k\big)|\mathbf{u}_c \cdot \boldsymbol{\tau}_i|^2\,dS \\
&\quad \ge C_1\|\mathbf{u}_c\|_{\mathbf{H}^1(\Omega_c)}^2 + C_2\|P_m\|_{H^1(\Omega_m)}^2,
\end{aligned}$$

for some constants C_1, C_2 independent of $\mathbf{u}_c$, P_m and φ^k.

Then by the Lax–Milgram lemma, we can easily deduce that

Lemma 3.3. *Assume that the assumptions (A1) and (A3) are satisfied. Then for any given $\varphi^k \in H^1(\Omega)$, the operator $\mathcal{L}_k : \mathbf{V} \to \mathbf{V}'$ is invertible and its inverse $\mathcal{L}_k^{-1} : \mathbf{V}' \to \mathbf{V}$ is continuous.*

Next, we state the invertibility of the operator induced by the left-hand side of Eq. (3.17). To this end, we recall the following simple facts in [26]. Define the operator $\mathrm{div}_N : \mathbf{L}^2(\Omega) \to \dot{H}^{-1}(\Omega)$ by

$$\langle \mathrm{div}_N \mathbf{v}, \phi\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)} = -(\mathbf{v}, \nabla\phi), \quad \forall \phi \in \dot{H}^1(\Omega).$$

The operator div_N acts on vector fields, which do not necessarily vanish on the boundary, and involves boundary conditions in a weak sense. Let $\mathrm{M} \in L^\infty(\Omega)$ such that $\mathrm{M}(x) \geq m_0 > 0$ almost every in Ω. We then introduce the operator $\mathrm{div}_N(\mathrm{M}(x)\nabla\cdot) : \dot{H}^1(\Omega) \to \dot{H}^{-1}(\Omega)$ defined as

$$\big\langle \mathrm{div}_N\big(\mathrm{M}(x)\nabla\varphi\big), \phi\big\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)} = -\big(\mathrm{M}(x)\nabla\varphi, \nabla\phi\big), \quad \forall \phi \in \dot{H}^1(\Omega).$$

Then the operator $\mathrm{div}_N(\mathrm{M}(x)\nabla\cdot)$ is an isomorphism due to an easy application of the Lax–Milgram lemma.

Hence, under the assumption (A2), it is easy to see that

Lemma 3.4. *Assume that the function* M *satisfies (A2). For any given $\varphi^k \in H^1(\Omega)$, the operator*

$$\mathcal{D}_k := \mathrm{div}_N\big(\mathrm{M}\big(\varphi^k\big)\nabla\cdot\big) : \dot{H}^1(\Omega) \to \dot{H}^{-1}(\Omega) \tag{3.23}$$

is invertible and its inverse $\mathcal{D}_k^{-1} : \dot{H}^{-1}(\Omega) \to \dot{H}^1(\Omega)$ is continuous.

We now proceed to the solvability of Eq. (3.18). For any given function $\varphi^k \in H^1(\Omega)$, we define the nonlinear operator $\mathcal{S}_k : \dot{H}^1(\Omega) \to \dot{H}^{-1}(\Omega)$ as follows

$$\big\langle \mathcal{S}_k(\psi), \phi\big\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)} = \epsilon(\nabla\psi, \nabla\phi) + \frac{1}{\epsilon}\big(\big(\psi + \overline{\varphi^k}\big)^3, \phi\big), \quad \forall \phi \in \dot{H}^1(\Omega), \tag{3.24}$$

where $\overline{\varphi^k} = |\Omega|^{-1}\int_\Omega \varphi^k dx$.

Then we have

Lemma 3.5. *Let $\varphi^k \in H^1(\Omega)$ be fixed. For any given function $\mu_0 \in \dot{H}^{-1}(\Omega)$, there exists a unique solution $\psi \in \dot{H}^1(\Omega)$ to the equation $\mathcal{S}_k(\psi) = \mu_0$. The solution operator $\mathcal{S}_k^{-1} : \dot{H}^{-1}(\Omega) \to \dot{H}^1(\Omega)$ is continuous. Moreover, if $\mu_0 \in \dot{H}^1(\Omega)$, then the solution satisfies $\psi \in \dot{H}^3(\Omega)$ and $\mathcal{S}_k^{-1} : \dot{H}^1(\Omega) \to \dot{H}^3(\Omega)$ is bounded and continuous.*

Proof. The unique solvability of equation $\mathcal{S}_k(\psi) = \mu_0$ for given source function μ_0 can be obtained by the theory of monotone operators.

We note that $\mathcal{S}_k$ is well defined for any given function $\varphi^k \in H^1(\Omega)$. Indeed, using the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ for $d = 2, 3$, we can see that for any $\psi \in \dot{H}^1(\Omega)$,

$$\big|\big\langle \mathcal{S}_k(\psi), \phi\big\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)}\big| \leq C(\epsilon)\big(\|\psi\|^3_{H^1(\Omega)} + \big|\overline{\varphi^k}\big|^3 + \|\psi\|_{H^1(\Omega)}\big)\|\phi\|_{H^1(\Omega)},$$

which implies the boundedness of $\mathcal{S}_k$ in $H^1(\Omega)$. Moreover, if a sequence $\psi_n \to \psi$ in $\dot{H}^1(\Omega)$ as $n \to \infty$, by Hölder's inequality and the Sobolev embedding, we deduce that for any $\phi \in \dot{H}^1(\Omega)$,

$$\begin{aligned}
&\left|\left\langle \mathcal{S}_k(\psi_n) - \mathcal{S}_k(\psi), \phi\right\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)}\right| \\
&\quad \le C(\epsilon)\left(\left\|\left(\psi_n + \overline{\varphi^k}\right)^3 - \left(\psi + \overline{\varphi^k}\right)^3\right\|_{L^{\frac{6}{5}}(\Omega)} + \left\|\nabla(\psi_n - \psi)\right\|_{\mathbf{L}^2(\Omega)}\right)\|\phi\|_{H^1(\Omega)} \\
&\quad \le C(\epsilon)\left(\left\|\psi_n^2 + \psi^2 + \left(\overline{\varphi^k}\right)^2\right\|_{L^2(\Omega)}\|\psi_n - \psi\|_{L^3(\Omega)} + \left\|\nabla(\psi_n - \psi)\right\|_{\mathbf{L}^2(\Omega)}\right)\|\phi\|_{H^1(\Omega)} \\
&\quad \to 0.
\end{aligned}$$

Hence, the nonlinear operator $\mathcal{S}_k : \dot{H}^1(\Omega) \to \dot{H}^{-1}(\Omega)$ is continuous. Concerning the coercivity of $\mathcal{S}_k$, using the Young inequality, we have for any $\psi \in \dot{H}^1(\Omega)$,

$$\begin{aligned}
&\left\langle \mathcal{S}_k(\psi), \psi\right\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)} \\
&\quad = \frac{1}{\epsilon}\int_\Omega \left(\psi + \overline{\varphi^k}\right)^3 \psi\,dx + \epsilon\int_\Omega |\nabla\psi|^2\,dx \\
&\quad \ge \frac{1}{\epsilon}\int_\Omega |\psi|^4\,dx - \frac{3|\overline{\varphi^k}|}{\epsilon}\int_\Omega |\psi|^3 dx - \frac{3|\overline{\varphi^k}|^2}{\epsilon}\int_\Omega |\psi|^2 dx - \frac{|\overline{\varphi^k}|^3}{\epsilon}\int_\Omega |\psi| dx \\
&\qquad + \epsilon\int_\Omega |\nabla\psi|^2\,dx \\
&\quad \ge C(\epsilon)\|\psi\|^2_{H^1(\Omega)} - C\left(\epsilon, |\Omega|, \left|\overline{\varphi^k}\right|\right),
\end{aligned} \tag{3.25}$$

which yields that

$$\frac{\langle \mathcal{S}_k(\psi), \psi\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)}}{\|\psi\|_{H^1(\Omega)}} \to +\infty, \quad \text{as } \|\psi\|_{H^1(\Omega)} \to \infty.$$

Finally, the strict monotonicity of $\mathcal{S}_k$ follows from the following identity

$$\begin{aligned}
&\left\langle \mathcal{S}_k(\psi_1) - \mathcal{S}_k(\psi_2), \psi_1 - \psi_2\right\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)} \\
&\quad = \frac{1}{\epsilon}\int_\Omega (\psi_1 - \psi_2)^2\left[\left(\psi_1 + \overline{\varphi^k}\right)^2 + \left(\psi_2 + \overline{\varphi^k}\right)^2 + \left(\psi_1 + \overline{\varphi^k}\right)\left(\psi_2 + \overline{\varphi^k}\right)\right] dx \\
&\qquad + \epsilon\int_\Omega \left|\nabla(\psi_1 - \psi_2)\right|^2 dx \\
&\quad \ge 0, \quad \forall \psi_1, \psi_2 \in \dot{H}^1(\Omega)
\end{aligned} \tag{3.26}$$

and the equal sign holds if and only if $\psi_1 = \psi_2$.

Based on the above observations, we can apply the Browder–Minty theorem (cf. e.g., [45, p. 39, Theorem 2.2]) to conclude the existence of a unique solution $\psi \in \dot{H}^1(\Omega)$ to the nonlinear equation $\mathcal{S}_k(\psi) = \mu_0$ for a given source function $\mu_0 \in \dot{H}^{-1}(\Omega)$. The coercive estimate (3.25) also implies that

$$\|\psi\|^2_{H^1(\Omega)} \le C(\epsilon)\left(\|\mu_0\|^2_{\dot{H}^{-1}(\Omega)} + \left|\overline{\varphi^k}\right|^4 + 1\right). \tag{3.27}$$

For the continuous dependence of the solution ψ on μ_0, i.e., if a sequence $\mu_{0n} \to \mu_0$ strongly in $\dot{H}^{-1}(\Omega)$ and $\mathcal{S}_k(\psi_n) = \mu_{0n}$, $\mathcal{S}_k(\psi) = \mu_0$, then $\psi_n, \psi \in \dot{H}^1(\Omega)$ and as $n \to +\infty$, it holds

$$\left\langle \mathcal{S}_k(\psi_n) - \mathcal{S}_k(\psi), \psi_n - \psi\right\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)} = \langle \mu_{0n} - \mu_0, \psi_n - \psi\rangle_{\dot{H}^{-1}(\Omega),\dot{H}^1(\Omega)} \to 0. \tag{3.28}$$

Then a similar estimate like (3.26) yields that $\psi_n \to \psi$ strongly in $\dot{H}^1(\Omega)$. As a consequence, the solution operator $\mathcal{S}_k^{-1} : \dot{H}^{-1}(\Omega) \to \dot{H}^1(\Omega)$ is continuous.

If we further assume that $\mu_0 \in \dot{H}^1(\Omega)$, the weak solution ψ indeed has higher regularity. To this end, we rewrite the weak form of the equation $\mathcal{S}_k(\psi) = \mu_0$ as

$$\epsilon(\nabla\psi, \nabla\phi) = \left(\mu_0 - G\left(\psi, \varphi^k\right), \phi\right), \quad \forall \phi \in \dot{H}^1(\Omega),$$

where $G(\psi, \varphi^k) = \epsilon^{-1}(\psi + \overline{\varphi^k})^3 \in L^2(\Omega)$. Then ψ is a weak solution to the following elliptic equation with homogeneous Neumann boundary condition:

$$\begin{cases} -\epsilon\Delta\psi = \mu_0 - G_0, & \text{in } \Omega, \\ \dfrac{\partial\psi}{\partial\mathbf{n}} = 0, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \psi\, dx = 0, \end{cases} \tag{3.29}$$

with $G_0 = G(\psi, \varphi^k) - \overline{G}(\psi, \varphi^k)$. Since the source function $\mu_0 - G_0 \in \dot{L}^2(\Omega)$, one deduces from the classical elliptic regularity theory (cf. [39]) that $\psi \in H^2(\Omega)$ if Ω is $C^{1,1}$ or a convex bounded domain. In particular, one can derive from (3.29) that

$$\|\psi\|_{H^2(\Omega)} \le C(\epsilon)\left(\|\mu_0\|_{L^2(\Omega)} + \|\psi\|^3_{H^1(\Omega)} + \left|\overline{\varphi^k}\right|^3 + \|\psi\|_{L^2(\Omega)}\right). \tag{3.30}$$

Since $H^2(\Omega)$ is an algebra with respect to point-wise multiplication in $\mathbb{R}^d$ ($d \le 3$), one has $\mu_0 - G_0 \in \dot{H}^1(\Omega)$. Then it follows from (3.29), (3.30) that

$$\begin{aligned} \|\psi\|_{H^3} &\le C(\epsilon)\left(\|\mu_0\|_{H^1(\Omega)} + \left|\overline{\varphi^k}\right|^3 + \left\|\psi^3\right\|_{H^1(\Omega)} + \|\psi\|_{L^2(\Omega)}\right) \\ &\le C(\epsilon)\left(\|\mu_0\|_{H^1(\Omega)} + \left|\overline{\varphi^k}\right|^3 + \|\psi\|_{L^2(\Omega)}\right) \\ &\quad + C(\epsilon)\left(\|\psi\|^2_{L^\infty(\Omega)}\|\nabla\psi\|_{\mathbf{L}^2(\Omega)} + \|\psi\|^3_{L^6(\Omega)}\right) \\ &\le C\left(\epsilon, \Omega, \|\mu_0\|_{H^1(\Omega)}, \left|\overline{\varphi^k}\right|\right), \end{aligned} \tag{3.31}$$

which yields that the solution operator $\psi = \mathcal{S}_k^{-1}(\mu_0)$ is bounded from $\dot{H}^1(\Omega)$ to $\dot{H}^3(\Omega)$. Consider the difference problem

$$\begin{cases} -\epsilon\Delta(\psi_n-\psi)=(\mu_{0n}-\mu_0)-(G_{0n}-G_0), \\ \dfrac{\partial(\psi_n-\psi)}{\partial\mathbf{n}}=0, \quad \text{on } \partial\Omega, \end{cases} \tag{3.32}$$

with $G_{0n}=G(\psi_n,\varphi^k)-\overline{G}(\psi_n,\varphi^k)$ and $G_0=G(\psi,\varphi^k)-\overline{G}(\psi,\varphi^k)$. Assuming that $\mu_{0n}\to\mu_0$ strongly in $\dot{H}^1(\Omega)$, similar to (3.30), we can first derive the H^2 estimates for ψ_n, ψ, and then use the elliptic estimates as in (3.31) to get

$$\begin{aligned} \|\psi_n-\psi\|_{H^3(\Omega)} &\le C\big(\|\mu_{0n}-\mu_0\|_{H^1(\Omega)}+\|G_{0n}-G_0\|_{H^1(\Omega)}+\|\psi_n-\psi\|_{L^2(\Omega)}\big) \\ &\le C\big(\|\psi_n\|_{L^\infty(\Omega)},\|\psi\|_{L^\infty(\Omega)},\|\nabla\psi_n\|_{\mathbf{L}^3(\Omega)},\|\nabla\psi\|_{\mathbf{L}^3(\Omega)}\big)\|\psi_n-\psi\|_{H^1(\Omega)} \\ &\quad +C\|\mu_{0n}-\mu_0\|_{H^1(\Omega)}. \end{aligned}$$

We have already shown that $\mathcal{S}_k^{-1}:\dot{H}^{-1}(\Omega)\to\dot{H}^1(\Omega)$ is continuous, which combining the above estimate further yields that $\mathcal{S}_k^{-1}:\dot{H}^1(\Omega)\to\dot{H}^3(\Omega)$ is also (strongly) continuous. The proof is complete. □

Step 2. *Definition of operators $\mathcal{T}_k$, $\mathcal{G}_k$ and their properties.*

We introduce the following product spaces

$$\begin{cases} \mathbf{X}=\mathbf{V}\times\dot{H}^1(\Omega)\times\dot{H}^{3-\sigma}(\Omega)\times\mathbb{R}, \\ \mathbf{Y}=\mathbf{V}'\times\dot{H}^{-1}(\Omega)\times\dot{L}^2(\Omega)\times\mathbb{R}, \end{cases} \tag{3.33}$$

where $\sigma\in(0,\frac{1}{2})$ is a constant.

Owing to the mass-conservation property (3.7) of the approximate scheme and for the convenience of the norm of $\dot{H}^1(\Omega)$, we will project the unknowns φ and μ into $\dot{L}^2(\Omega)$ such that

$$\varphi=\psi+\overline{\varphi^k}, \qquad \mu=\mu_0+\overline{S}_k,$$

where $\overline{\varphi^k}$ and $\overline{S}_k$ are the averages of φ_k and $\tilde{f}(\varphi,\varphi^k)$ on Ω, respectively.

According to the formulation of the system (3.16)–(3.18), we now introduce the nonlinear operators $\mathcal{T}_k$, $\mathcal{G}_k:\mathbf{X}\to\mathbf{Y}$. For any given functions $\varphi^k\in H^1(\Omega)$, $\mathbf{u}_c^k\in\mathbf{L}^2(\Omega_c)$ and for $\mathbf{w}=(\mathbf{u}_c,P_m,\mu_0,\psi,\overline{S}_k)\in\mathbf{X}$, we define

$$\mathcal{T}_k(\mathbf{w})=\begin{pmatrix} \mathcal{L}_k(\mathbf{u}_c,P_m) \\ \mathcal{D}_k(\mu_0) \\ \mathcal{S}_k(\psi) \\ \overline{S}_k \end{pmatrix}, \tag{3.34}$$

and

$$\mathcal{G}_k(\mathbf{w})=\begin{pmatrix} \mathcal{J}_k(\mathbf{w}) \\ \mathrm{P}_0(\delta^{-1}(\psi+\overline{\varphi^k}-\varphi^k)+\mathbf{u}\cdot\nabla\psi) \\ \mu_0+\epsilon^{-1}(\varphi^k-\overline{\varphi^k}) \\ |\Omega|^{-1}\epsilon^{-1}\int_\Omega((\psi+\overline{\varphi^k})^3-\varphi^k)dx \end{pmatrix}. \tag{3.35}$$

The operators $\mathcal{L}_k$, $\mathcal{D}_k$, $\mathcal{S}_k$ in (3.34) are defined in (3.20), (3.23) and (3.24) (associated with the given function φ^k), respectively. In (3.35), the operator $\mathcal{J}_k : \mathbf{X} \to \mathbf{V}'$ is given by

$$\begin{aligned}
&\langle \mathcal{J}_k(\mathbf{w}), (\mathbf{v}_c, q_m) \rangle_{\mathbf{V}',\mathbf{V}} \\
&= \left(-\frac{\varpi}{\delta}(\mathbf{u}_c - \mathbf{u}_c^k) + (\mu_{0c} + \bar{S}_k)\nabla\psi_c, \mathbf{v}_c \right)_c + (\mathbf{u}_c, \mathbf{v}_c)_c \\
&\quad + \left(\frac{\Pi}{\nu(\varphi_m^k)}(\mu_{0m} + \bar{S}_k)\nabla\psi_m, \nabla q_m \right)_m, \quad \forall (\mathbf{v}_c, q_m) \in \mathbf{V}.
\end{aligned} \tag{3.36}$$

Here, one recalls that P_0 is the projection operator from $L^2(\Omega)$ into $\dot{L}^2(\Omega)$ and the facts $\mu_{0c} = \mu_0|_{\Omega_c}$, $\mu_{0m} = \mu_0|_{\Omega_m}$. The velocity $\mathbf{u}$ in (3.35) fulfills $\mathbf{u}|_{\Omega_c} = \mathbf{u}_c$, $\mathbf{u}|_{\Omega_m} = \mathbf{u}_m$ and $\mathbf{u}_m$ is given by (3.4).

From the definition of $\mathcal{T}_k$ and Lemmas 3.3–3.5 obtained in the previous step, one can conclude that

Lemma 3.6. $\mathcal{T}_k : \mathbf{X} \to \mathbf{Y}$ *is an invertible mapping and its inverse* $\mathcal{T}_k^{-1} : \mathbf{Y} \to \mathbf{X}$ *is continuous. In particular,* $\mathcal{T}_k^{-1}(\mathbf{0}) = \mathbf{0}$.

Then concerning the operator $\mathcal{G}_k$, one has

Lemma 3.7. $\mathcal{G}_k : \mathbf{X} \to \mathbf{Y}$ *is a continuous and bounded mapping. Moreover, it is compact.*

Proof. For all $\mathbf{w} = (\mathbf{u}_c, P_m, \mu_0, \psi, \bar{S}_k) \in \mathbf{X}$, using the Sobolev embedding theorems ($d \le 3$) such that $H^1 \hookrightarrow L^6$ and $H^{1-\sigma} \hookrightarrow L^3$, $H^{2-\sigma} \hookrightarrow L^\infty$ for $\sigma \in (0, \frac{1}{2})$, it is straightforward to show that

$$\mathcal{G}_k(\mathbf{w}) \in \left(\mathbf{L}^2(\Omega_c) \times \left(H^{1-\sigma}(\Omega_m)\right)'\right) \times \dot{L}^2(\Omega) \times \dot{H}^1(\Omega) \times K \hookrightarrow\hookrightarrow \mathbf{Y},$$

where K is a bounded set in $\mathbb{R}$. Our conclusion easily follows. $\square$

We now interpret the relation between the abstract equation $\mathcal{T}_k(\mathbf{w}) = \mathcal{G}_k(\mathbf{w})$ for $\mathbf{w} \in \mathbf{X}$ and the elliptic system (3.1)–(3.3). The following equivalence result can be easily seen from the definitions (3.20)–(3.24) and (3.34)–(3.36):

Proposition 3.1. $\{\mathbf{u}_c, P_m, \varphi, \mu\} \in \mathbf{H}_{c,\mathrm{div}} \times X_m \times H^3(\Omega) \times H^1(\Omega)$ *is a solution of the system* (3.1)–(3.3) *if and only if* $\mathbf{w} = (\mathbf{u}_c, P_m, \mu_0, \psi, \bar{S}_k) \in \mathbf{X}$ *satisfies* $\mathcal{T}_k(\mathbf{w}) = \mathcal{G}_k(\mathbf{w})$ *with* $\varphi = \psi + \overline{\varphi^k}$, $\mu = \mu_0 + \bar{S}_k$.

Step 3. *Solvability of the nonlinear system (3.1)–(3.4).*

We proceed to show that there exists a $\mathbf{w} \in \mathbf{X}$ such that $\mathcal{T}_k(\mathbf{w}) = \mathcal{G}_k(\mathbf{w})$. Since $\mathcal{T}_k$ is invertible, this abstract equation can be rewritten equivalently as $\mathbf{w} = \mathcal{T}_k^{-1}(\mathcal{G}_k(\mathbf{w}))$, namely,

$$(\mathcal{I} - \mathcal{N}_k)(\mathbf{w}) = \mathbf{0}, \tag{3.37}$$

where $\mathcal{I}$ is the identity operator on $\mathbf{X}$ and the nonlinear operator $\mathcal{N}_k$ is defined by

$$\mathcal{N}_k(\mathbf{w}) := \mathcal{T}_k^{-1}\big(\mathcal{G}_k(\mathbf{w})\big) : \mathbf{X} \to \mathbf{X}, \quad \forall \mathbf{w} \in \mathbf{X} \tag{3.38}$$

and it is a compact operator on $\mathbf{X}$ due to Lemmas 3.6 and 3.7. Thus we only have to prove that there exists a vector $\mathbf{w} = (\mathbf{u}_c, P_m, \mu_0, \psi, \bar{S}_k) \in \mathbf{X}$ that satisfies Eq. (3.37). This can be done by a homotopy argument based on the Leray–Schauder degree (cf. [25,26]).

Lemma 3.8. *Assume that assumptions (A1)–(A3) are satisfied. For any $\mathbf{u}_c^k \in \mathbf{L}^2(\Omega_c)$ and $\varphi^k \in H^1(\Omega)$, the equation $\mathcal{T}_k(\mathbf{w}) = \mathcal{G}_k(\mathbf{w})$ admits a solution $\mathbf{w} = (\mathbf{u}_c, P_m, \mu_0, \psi, \bar{S}_k) \in \mathbf{X}$.*

Proof. For $s \in [0, 1]$, we define

$$\mathbf{u}_c^k(s) = (1-s)\mathbf{u}_c^k, \qquad \varphi^k(s) = (1-s)\varphi^k.$$

Replace $\mathbf{u}_c^k$, φ^k in the system (3.16)–(3.18) by $\mathbf{u}_c^k(s)$, $\varphi^k(s)$, respectively. Then we denote by $\mathcal{T}_k^{(s)}$, $\mathcal{G}_k^{(s)}$ the corresponding operators under the above transformation. In particular, $\mathcal{T}_k^{(0)} = \mathcal{T}_k$, $\mathcal{G}_k^{(0)} = \mathcal{G}_k$. It is easy to see that $\mathcal{T}_k^{(s)}$, $\mathcal{G}_k^{(s)}$ have the same properties as in Lemmas 3.6–3.7. Then we denote $\mathcal{N}_k^{(s)} = (\mathcal{T}_k^{(s)})^{-1}\mathcal{G}_k^{(s)}$, which is a compact operator. Moreover, $\mathcal{N}_k^{(0)} = \mathcal{N}_k$.

In analogy to (3.9), we can derive the following discrete energy law with respect to the parameter s:

$$\begin{aligned}
&\mathcal{E}(\mathbf{u}_c, \varphi) + \delta\big(\nu\big(\varphi_m^k(s)\big)\Pi^{-1}\mathbf{u}_m, \mathbf{u}_m\big)_m + 2\delta\big(\nu\big(\varphi_c^k(s)\big)\mathbb{D}(\mathbf{u}_c), \mathbb{D}(\mathbf{u}_c)\big)_c \\
&\quad + \delta\int_\Omega \mathrm{M}\big(\varphi^k(s)\big)|\nabla\mu|^2 dx + \frac{\delta\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}}\sum_{i=1}^{d-1}\int_{\Gamma_{cm}} \nu\big(\varphi_m^k(s)\big)|\mathbf{u}_c\cdot\boldsymbol{\tau}_i|^2 dS \\
&\quad + \frac{\varpi}{2}\big(\mathbf{u}_c - \mathbf{u}_c^k(s), \mathbf{u}_c - \mathbf{u}_c^k(s)\big)_c + \frac{\epsilon}{2}\big\|\nabla\big(\varphi - \varphi^k(s)\big)\big\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad + \frac{1}{2\epsilon}\big\|\varphi - \varphi^k(s)\big\|^2_{L^2(\Omega)} \\
&\le \mathcal{E}\big(\mathbf{u}_c^k(s), \varphi^k(s)\big).
\end{aligned} \tag{3.39}$$

For any given $\mathbf{u}_c^k \in \mathbf{L}^2(\Omega_c)$ and $\varphi^k \in H^1(\Omega)$, there exists a constant $R > 0$ depending only on $\|\mathbf{u}_c^k\|_{\mathbf{L}^2(\Omega_c)}$, $\|\varphi^k\|_{H^1(\Omega)}$, ϖ, ϵ and Ω such that $\mathcal{E}(\mathbf{u}_c^k(s), \varphi^k(s)) \le R$ for all $s \in [0, 1]$. By the energy estimate (3.39), there exists $C_0 > 0$ depending on R and coefficients of the system but independent of s such that the solution $\mathbf{w} = \mathbf{w}^{(s)}$ to the equation $\mathcal{T}_k^{(s)}(\mathbf{w}) = \mathcal{G}_k^{(s)}(\mathbf{w})$, if it exists, will satisfy

$$\big\|\mathbf{w}^{(s)}\big\|_{\mathbf{X}} \le C_0, \quad \forall s \in [0, 1].$$

Taking the ball in $\mathbf{X}$ centered at $\mathbf{0}$ with radius $2C_0$:

$$\mathbf{B} = \big\{\mathbf{w} \in \mathbf{X}:\ \|\mathbf{w}\|_{\mathbf{X}} \le 2C_0\big\},$$

we infer from the above *a priori* estimate that for all $s \in [0, 1]$, $(\mathcal{I} - \mathcal{N}_k^{(s)})(\mathbf{w}) \ne \mathbf{0}$ for any $\mathbf{w} \in \partial\mathbf{B}$. Therefore, the Leray–Schauder degree of the operator $\mathcal{I} - \mathcal{N}_k^{(s)}$ at $\mathbf{0}$ in the ball $\mathbf{B}$, denoted by $\deg(\mathcal{I} - \mathcal{N}_k^{(s)}, \mathbf{B}, \mathbf{0})$, is well-defined for $s \in [0, 1]$.

On the other hand, since $\mathcal{N}_k^{(0)} = \mathcal{N}_k$, then by the homotopy invariance of the Leray–Schauder degree, we have

$$\deg(\mathcal{I} - \mathcal{N}_k, \mathbf{B}, \mathbf{0}) = \deg\bigl(\mathcal{I} - \mathcal{N}_k^{(0)}, \mathbf{B}, \mathbf{0}\bigr) = \deg\bigl(\mathcal{I} - \mathcal{N}_k^{(1)}, \mathbf{B}, \mathbf{0}\bigr). \tag{3.40}$$

Next, we shall prove that $\deg(\mathcal{I} - \mathcal{N}_k^{(1)}, \mathbf{B}, \mathbf{0}) = 1$. For this purpose, we define a further homotopy for $s \in [1, 2]$ such that

$$\mathcal{N}_k^{(s)}(\mathbf{w}) = \bigl(\mathcal{T}_k^{(1)}\bigr)^{-1}\bigl[(2-s)\mathcal{G}_k^{(1)}(\mathbf{w})\bigr], \quad \forall \mathbf{w} \in \mathbf{X}. \tag{3.41}$$

For $s \in [1, 2)$, $(\mathcal{I} - \mathcal{N}_k^{(s)})(\mathbf{w}) = \mathbf{0}$ if and only if for $\mathbf{w} = (\mathbf{u}_c, P_m, \mu_0, \psi, \bar{S}_k) \in \mathbf{X}$, the vector $(\mathbf{u}_c, P_m, \varphi, \mu)$ with $\varphi = \psi$, $\mu = \mu_0 + \bar{S}_k(2-s)^{-2}$ is a solution of the following system

$$\begin{aligned}
&\frac{\varpi(2-s)}{\delta}(\mathbf{u}_c, \mathbf{v}_c)_c + 2\bigl(\nu(0)\mathbb{D}(\mathbf{u}_c), \mathbb{D}(\mathbf{v}_c)\bigr)_c \\
&\quad + (s-1)(\mathbf{u}_c, \mathbf{v}_c)_c + \left(\frac{\Pi}{\nu(0)}\nabla P_m, \nabla q_m\right)_m \\
&\quad + \sum_{i=1}^{d-1} \frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}} \int_{\Gamma_{cm}} \nu(0)(\mathbf{u}_c \cdot \boldsymbol{\tau}_i)(\mathbf{v}_c \cdot \boldsymbol{\tau}_i)\,dS \\
&\quad + \int_{\Gamma_{cm}} P_m(\mathbf{v}_c \cdot \mathbf{n}_{cm})\,dS - \int_{\Gamma_{cm}} (\mathbf{u}_c \cdot \mathbf{n}_{cm})q_m\,dS \\
&= (2-s)(\mu_c\nabla\varphi_c, \mathbf{v}_c)_c + (2-s)\left(\frac{\Pi}{\nu(0)}\mu_m\nabla\varphi_m, \nabla q_m\right)_m,
\end{aligned} \tag{3.42}$$

$$\frac{2-s}{\delta}(\varphi, \phi) + (2-s)(\mathbf{u}\cdot\nabla\varphi, \phi) = -\bigl(\mathrm{M}(0)\nabla\mu, \nabla\phi\bigr), \tag{3.43}$$

$$(2-s)(\mu, \phi) = \frac{1}{\epsilon}\bigl(\varphi^3, \phi\bigr) + \epsilon(\nabla\varphi, \nabla\phi), \tag{3.44}$$

for any $q_m \in X_m$, $\mathbf{v}_c \in \mathbf{H}_{c,\mathrm{div}}$, $\phi \in H^1(\Omega)$, and $\mathbf{u}_m$ is given by

$$\mathbf{u}_m = -\frac{\Pi}{\nu(0)}\bigl[\nabla P_m - \mu(\varphi_m)\nabla\varphi_m\bigr]. \tag{3.45}$$

Taking the testing functions $\mathbf{v}_c = \mathbf{u}_c$, $q_m = P_m$ in (3.42), $\phi = \mu$ in (3.43) and $\phi = \varphi$ in (3.44), summing up, we obtain that

$$\begin{aligned}
&\frac{\varpi(2-s)}{\delta}(\mathbf{u}_c, \mathbf{u}_c)_c + \frac{\epsilon}{\delta}(\nabla\varphi, \nabla\varphi) + \frac{1}{\delta\epsilon}\int_{\Omega} \varphi^4\,dx \\
&\quad + 2\bigl(\nu(0)\mathbb{D}(\mathbf{u}_c), \mathbb{D}(\mathbf{u}_c)\bigr)_c + (s-1)(\mathbf{u}_c, \mathbf{u}_c)_c
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\Pi}{\nu(0)}\nabla P_m, \nabla P_m\right)_m + \sum_{i=1}^{d-1}\frac{\alpha_{BJSJ}}{\sqrt{\mathrm{trace}(\Pi)}}\int_{\Gamma_{cm}} \nu(0)|\mathbf{u}_c\cdot\boldsymbol{\tau}_i|^2 dS \\
&+\big(\mathrm{M}(0)\nabla\mu, \nabla\mu\big) \\
=\,& 0.
\end{aligned}
\tag{3.46}
$$

The above estimate implies that for $s \in (1,2)$, $(\mathcal{I}-\mathcal{N}_k^{(s)})(\mathbf{w}) = \mathbf{0}$ if and only if $\mathbf{w} = \mathbf{0}$. Moreover, it is straightforward to check that $\mathcal{I}-\mathcal{N}_k^{(2)} = \mathcal{I}$ (cf. Lemmas 3.6, 3.7, in particular, $(\mathcal{T}_k^{(1)})^{-1}(\mathbf{0}) = \mathbf{0}$) and thus $(\mathcal{I}-\mathcal{N}_k^{(2)})(\mathbf{w}) = \mathbf{0}$ if and only if $\mathbf{w} = \mathbf{0}$. Thus, for $s \in [1,2]$, we have $(\mathcal{I}-\mathcal{N}_k^{(s)})(\mathbf{w}) \neq \mathbf{0}$ for any $\mathbf{w} \in \partial\mathbf{B}$. As a consequence, the homotopy invariance of the Leray–Schauder degree yields that

$$
\deg\big(\mathcal{I}-\mathcal{N}_k^{(1)}, \mathbf{B}, \mathbf{0}\big) = \deg(\mathcal{I}, \mathbf{B}, \mathbf{0}) = 1. \tag{3.47}
$$

In summary, we can conclude from (3.40) and (3.47) that $\deg(\mathcal{I}-\mathcal{N}_k, \mathbf{B}, \mathbf{0}) = 1$, which implies that the abstract equation (3.37) admits a solution $\mathbf{w} = (\mathbf{u}_c, P_m, \mu_0, \psi, \bar{S}_k) \in \mathbf{X}$ that solves $\mathcal{T}_k(\mathbf{w}) = \mathcal{G}_k(\mathbf{w})$.

The proof of Lemma 3.8 is complete. □

Finally, we can conclude the existence of weak solutions to the system (3.1)–(3.3) from Lemmas 3.1, 3.2, 3.5, 3.8 and Proposition 3.1,

Theorem 3.1 *(Existence of solutions to the discrete problem). For every $\mathbf{u}_c^k \in \mathbf{L}^2(\Omega_c)$ and $\varphi^k \in H^1(\Omega)$, there exists a weak solution $\{\mathbf{u}_c, \mathbf{u}_m, P_m, \varphi, \mu\}$ to the nonlinear discrete problem (3.1)–(3.4) such that*

$$
\mathbf{u}_c \in \mathbf{H}_{c,\mathrm{div}}, \qquad \mathbf{u}_m \in \mathbf{H}_{m,\mathrm{div}}, \qquad P_m \in X_m, \qquad \varphi \in H^3(\Omega), \qquad \mu \in H^1(\Omega).
$$

Moreover, the solution satisfies the mass-conservation property (3.7) and the energy-dissipation inequality (3.9).

3.3. Construction of approximate solution and passage to limit

The existence of weak solutions to the time-discrete system (3.1)–(3.4) enables us to construct approximate solutions to the time-continuous system (2.15)–(2.18). Recall that $\delta = \frac{T}{N}$, where $T > 0$ and N is a positive integer. We set

$$
t_k = k\delta, \quad k = 0, 1, \cdots, N.
$$

Let $\{\mathbf{u}_c^{k+1}, P_m^{k+1}, \varphi^{k+1}, \mu^{k+1}\}$ $(k = 0, 1, \cdots, N-1)$ be chosen successively as a solution of the discretized problem (3.1)–(3.4) with $(\mathbf{u}_c^k, \varphi^k)$ being the “initial value” (cf. Theorem 3.1). In particular, $(\mathbf{u}_c^0, \varphi^0) = (\mathbf{u}_0, \varphi_0)$. Then for $k = 0, 1, \cdots, N-1$, we define the approximate solutions as follows

$$\varphi^\delta := \frac{t_{k+1}-t}{\delta}\varphi^k + \frac{t-t_k}{\delta}\varphi^{k+1}, \quad \text{for } t \in [t_k, t_{k+1}],$$

$$\mathbf{u}_c^\delta := \frac{t_{k+1}-t}{\delta}\mathbf{u}_c^k + \frac{t-t_k}{\delta}\mathbf{u}_c^{k+1}, \quad \text{for } t \in [t_k, t_{k+1}],$$

$$\hat{P}_m^\delta := P_m^{k+1}, \quad \text{for } t \in (t_k, t_{k+1}],$$

$$\hat{\mathbf{u}}_m^\delta := -\frac{\Pi}{\nu(\varphi_m^k)}\big(\nabla P_m^{k+1} - \mu^{k+1}\nabla\varphi_m^{k+1}\big), \quad \text{for } t \in (t_k, t_{k+1}],$$

$$\hat{\mathbf{u}}_c^\delta := \mathbf{u}_c^{k+1}, \quad \text{for } t \in (t_k, t_{k+1}],$$

$$\hat{\mathbf{u}}^\delta\big|_{\Omega_c} = \hat{\mathbf{u}}_c^\delta, \qquad \hat{\mathbf{u}}^\delta\big|_{\Omega_m} = \hat{\mathbf{u}}_m^\delta, \quad \text{for } t \in (t_k, t_{k+1}],$$

$$\hat{\varphi}^\delta := \varphi^{k+1}, \quad \text{for } t \in (t_k, t_{k+1}],$$

$$\tilde{\varphi}^\delta := \varphi^k, \quad \text{for } t \in [t_k, t_{k+1}),$$

$$\hat{\mu}^\delta := \mu^{k+1}, \quad \text{for } t \in (t_k, t_{k+1}].$$

Remark 3.3. It follows from the above definitions that φ^δ, $\mathbf{u}_c^\delta$ are continuous piecewise linear functions in time, while $\hat{\mathbf{u}}_c^\delta$, $\hat{P}_m^\delta$, $\hat{\varphi}^\delta$, $\hat{\mu}^\delta$ are piecewise constant (in time) functions being right continuous at the nodes $\{t_{k+1}\}$ and $\tilde{\varphi}^\delta$ is left continuous at the nodes $\{t_k\}$.

Using the above definition of approximate solutions, one can derive from the discrete problem (3.1)–(3.4) that the following identities hold:

$$\begin{aligned}
&\varpi\int_0^T \big(\partial_t\mathbf{u}_c^\delta, \mathbf{v}_c\big)_c dt + 2\int_0^T \big(\nu\big(\tilde{\varphi}_c^\delta\big)\mathbb{D}\big(\hat{\mathbf{u}}_c^\delta\big), \mathbb{D}(\mathbf{v}_c)\big)_c dt \\
&\quad + \int_0^T \left(\frac{\Pi}{\nu(\tilde{\varphi}_m^\delta)}\big(\nabla\hat{P}_m^\delta - \hat{\mu}_m^\delta\nabla\hat{\varphi}_m^\delta\big), \nabla q_m\right)_m dt \\
&\quad + \sum_{i=1}^{d-1}\frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}}\int_0^T\int_{\Gamma_{cm}} \nu\big(\tilde{\varphi}_m^\delta\big)\big(\hat{\mathbf{u}}_c^\delta\cdot\boldsymbol{\tau}_i\big)(\mathbf{v}_c\cdot\boldsymbol{\tau}_i)\,dS\,dt \\
&\quad + \int_0^T\int_{\Gamma_{cm}} \hat{P}_m^\delta(\mathbf{v}_c\cdot\mathbf{n}_{cm})\,dS\,dt - \int_0^T\int_{\Gamma_{cm}} \big(\hat{\mathbf{u}}_c^\delta\cdot\mathbf{n}_{cm}\big)q_m\,dS\,dt \\
&= \int_0^T \big(\hat{\mu}_c^\delta\nabla\hat{\varphi}_c^\delta, \mathbf{v}_c\big)_c dt,
\end{aligned} \tag{3.48}$$

$$\int_0^T \big(\partial_t\varphi^\delta, \phi\big)dt - \int_0^T \big(\hat{\mathbf{u}}^\delta\hat{\varphi}^\delta, \nabla\phi\big)dt = -\int_0^T \big(\mathrm{M}\big(\tilde{\varphi}^\delta\big)\nabla\hat{\mu}^\delta, \nabla\phi\big)dt, \tag{3.49}$$

$$\int_0^T (\hat{\mu}^\delta, \phi)dt = \frac{1}{\epsilon}\int_0^T (\tilde{f}(\hat{\varphi}^\delta, \tilde{\varphi}^\delta), \phi)dt + \epsilon \int_0^T (\nabla\hat{\varphi}^\delta, \nabla\phi)dt, \tag{3.50}$$

$$\int_0^T (\hat{\mathbf{u}}_m^\delta, \mathbf{v}_m)_m dt = -\int_0^T \left(\frac{\Pi}{\nu(\tilde{\varphi}_m^\delta)}(\nabla \hat{P}_m^\delta - \hat{\mu}_m^\delta \nabla\hat{\varphi}_m^\delta), \mathbf{v}_m\right)_m dt \tag{3.51}$$

for any $\mathbf{v}_c \in C_0^\infty([0,T]; \mathbf{H}_{c,\text{div}})$, $q_m \in C^\infty([0,T]; X_m)$, $\phi \in C_0^\infty([0,T]; H^1(\Omega))$ and $\mathbf{v}_m \in C^\infty([0,T]; \mathbf{L}^2(\Omega_m))$.

Besides, let $\mathcal{E}^\delta(t)$ be the piecewise linear interpolation of the discrete energy $\mathcal{E}(\mathbf{u}_c^k, \varphi^k)$ at t_k such that

$$\mathcal{E}^\delta(t) = \frac{t_{k+1}-t}{\delta}\mathcal{E}(\mathbf{u}_c^k, \varphi^k) + \frac{t-t_k}{\delta}\mathcal{E}(\mathbf{u}_c^{k+1}, \varphi^{k+1}), \quad \text{for } t \in [t_k, t_{k+1}], \tag{3.52}$$

and $\mathcal{D}^\delta(t)$ be the interpolated approximate dissipation

$$\begin{aligned}
\mathcal{D}^\delta(t) &= 2\big(\nu(\varphi_c^k)\mathbb{D}(\mathbf{u}_c^{k+1}), \mathbb{D}(\mathbf{u}_c^{k+1})\big)_c + \big(\nu(\varphi_m^k)\Pi^{-1}\mathbf{u}_m^{k+1}, \mathbf{u}_m^{k+1}\big)_m \\
&\quad + \int_\Omega \mathrm{M}(\varphi^k)\big|\nabla\mu^{k+1}\big|^2 dx \\
&\quad + \frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}}\sum_{i=1}^{d-1}\int_{\Gamma_{cm}} \nu(\varphi_m^k)\big|\mathbf{u}_c^{k+1}\cdot\boldsymbol{\tau}_i\big|^2 dS, \quad \text{for } t \in (t_k, t_{k+1}).
\end{aligned}$$

Then it follows from the discrete energy estimate (3.9) that for $k = 0, 1, \cdots, N-1$

$$\frac{d}{dt}\mathcal{E}^\delta(t) = \frac{1}{\delta}\big(\mathcal{E}(\mathbf{u}_c^{k+1}, \varphi^{k+1}) - \mathcal{E}(\mathbf{u}_c^k, \varphi^k)\big) \le -\mathcal{D}^\delta(t), \quad \text{for } t \in (t_k, t_{k+1}). \tag{3.53}$$

In particular, we have for $t \in [0,T]$,

$$\mathcal{E}\big(\hat{\mathbf{u}}_c^\delta(t), \hat{\varphi}^\delta(t)\big) + \int_0^t \mathcal{D}^\delta(t)dt \le \mathcal{E}(\mathbf{u}_0, \varphi_0), \quad \forall t \in [0,T]. \tag{3.54}$$

3.4. *Proof of Theorem 2.1*

We now proceed to prove our main result Theorem 2.1 on the existence of finite energy weak solutions to system (2.15)–(2.18). To this end, we shall distinguish the two cases such that $\varpi > 0$ and $\varpi = 0$.

3.4.1. Case $\varpi > 0$

In order to pass to the limit as $\delta \to 0$, we first derive some *a priori* estimates on the approximate solutions that are uniform in δ. First, recall the mass-conservation from Lemma 3.1

$$\int_\Omega (\varphi^{k+1} - \varphi^k) dx = 0, \quad \text{for } k = 0, ..., N-1,$$

which immediately yields

$$\int_\Omega \varphi^\delta dx = \int_\Omega \hat{\varphi}^\delta dx = \int_\Omega \tilde{\varphi}^\delta dx = \int_\Omega \varphi_0 dx.$$

Besides, it follows from the energy estimate (3.54) that

$$\varpi \|\hat{\mathbf{u}}_c^\delta\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_c))} + \|\hat{\varphi}^\delta\|_{L^\infty(0,T;H^1(\Omega))} \le C, \tag{3.55}$$

$$\|\mathbb{D}(\hat{\mathbf{u}}_c^\delta)\|_{L^2(0,T;\mathbf{L}^2(\Omega_c))} + \sum_{i=1}^{d-1} \|\hat{\mathbf{u}}_c^\delta \cdot \boldsymbol{\tau}_i\|_{L^2(0,T;L^2(\Gamma_{cm}))} \le C, \tag{3.56}$$

$$\|\hat{\mathbf{u}}_m^\delta\|_{L^2(0,T;\mathbf{L}^2(\Omega_m))} \le C, \tag{3.57}$$

$$\|\nabla \hat{\mu}^\delta\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \le C, \tag{3.58}$$

where the constant C depends on $\mathcal{E}(\mathbf{u}_0, \varphi_0)$ and Ω but is independent of δ. Taking $\phi = 1$ in (3.3), we have for $k = 0, 1, ..., N-1$

$$\left| \int_\Omega \mu^{k+1} dx \right| \le \epsilon^{-1} \int_\Omega \left(|\varphi^{k+1}|^3 + |\varphi^k| \right) dx \le C,$$

which combined with the Poincaré inequality and (3.58) implies that

$$\|\hat{\mu}^\delta\|_{L^2(0,T;H^1(\Omega))} \le C_T,$$

where the constant C_T depends on $\mathcal{E}(\mathbf{u}_0, \varphi_0)$, Ω and T. Then similar to the Neumann problem (3.29), we can apply the elliptic estimate (similar to (3.31)) to get

$$\|\hat{\varphi}^\delta\|_{L^2(0,T;H^3(\Omega))} \le C_T. \tag{3.59}$$

Using (3.4), the above estimates, the Hölder inequality and the Gagliardo–Nirenberg inequality, we can obtain the following estimates for $\hat{P}_m$ such that when $d = 3$

$$
\begin{aligned}
&\int_0^T \|\nabla \hat{P}_m^\delta\|_{\mathbf{L}^2(\Omega_m)}^{\frac{8}{5}} dt \\
&\quad \le C \int_0^T \left(\|\hat{\mathbf{u}}_m^\delta\|_{\mathbf{L}^2(\Omega_m)}^{\frac{8}{5}} + \|\nabla \hat{\varphi}_m^\delta\|_{\mathbf{L}^3(\Omega_m)}^{\frac{8}{5}} \|\hat{\mu}_m^\delta\|_{L^6(\Omega_m)}^{\frac{8}{5}}\right) dt \\
&\quad \le C \int_0^T \left(\|\hat{\mathbf{u}}_m^\delta\|_{\mathbf{L}^2(\Omega_m)}^2 + 1\right) dt + C \sup_{0\le t\le T} \|\hat{\varphi}_m^\delta\|_{H^1(\Omega_m)}^{\frac{6}{5}} \int_0^T \|\hat{\varphi}_m^\delta\|_{H^3(\Omega_m)}^{\frac{2}{5}} \|\hat{\mu}_m^\delta\|_{H^1(\Omega_m)}^{\frac{8}{5}} dt \\
&\quad \le C \int_0^T \left(\|\hat{\mathbf{u}}_m^\delta\|_{\mathbf{L}^2(\Omega_m)}^2 + 1\right) dt \\
&\qquad + C \sup_{0\le t\le T} \|\hat{\varphi}_m^\delta\|_{H^1(\Omega_m)}^{\frac{6}{5}} \left(\int_0^T \|\hat{\varphi}_m^\delta\|_{H^3(\Omega_m)}^2 dt\right)^{\frac{1}{5}} \left(\int_0^T \|\hat{\mu}_m^\delta\|_{H^1(\Omega_m)}^2 dt\right)^{\frac{4}{5}} \\
&\quad \le C_T,
\end{aligned}
\tag{3.60}
$$

and when $d = 2$

$$
\begin{aligned}
&\int_0^T \|\nabla \hat{P}_m^\delta\|_{\mathbf{L}^2(\Omega_m)}^{\frac{2r}{1+r}} dt \\
&\quad \le C \int_0^T \left(\|\hat{\mathbf{u}}_m^\delta\|_{\mathbf{L}^2(\Omega_m)}^{\frac{2r}{1+r}} + \|\nabla \hat{\varphi}_m^\delta\|_{\mathbf{L}^{\frac{2r}{r-2}}(\Omega_m)}^{\frac{2r}{1+r}} \|\hat{\mu}_m^\delta\|_{L^r(\Omega_m)}^{\frac{2r}{1+r}}\right) dt \\
&\quad \le C \int_0^T \left(\|\hat{\mathbf{u}}_m^\delta\|_{\mathbf{L}^2(\Omega_m)}^2 + 1\right) dt + C \sup_{0\le t\le T} \|\hat{\varphi}_m^\delta\|_{H^1(\Omega_m)}^{\frac{2(r-1)}{1+r}} \int_0^T \|\hat{\varphi}_m^\delta\|_{H^3(\Omega_m)}^{\frac{2}{1+r}} \|\hat{\mu}_m^\delta\|_{H^1(\Omega_m)}^{\frac{2r}{1+r}} dt \\
&\quad \le C \int_0^T \left(\|\hat{\mathbf{u}}_m^\delta\|_{\mathbf{L}^2(\Omega_m)}^2 + 1\right) dt \\
&\qquad + C \sup_{0\le t\le T} \|\hat{\varphi}_m^\delta\|_{H^1(\Omega_m)}^{\frac{2(r-1)}{1+r}} \left(\int_0^T \|\hat{\varphi}_m^\delta\|_{H^3(\Omega_m)}^2 dt\right)^{\frac{1}{1+r}} \left(\int_0^T \|\hat{\mu}_m^\delta\|_{H^1(\Omega_m)}^2 dt\right)^{\frac{r}{1+r}} \\
&\quad \le C_T, \quad \text{for any } r \in (2, +\infty).
\end{aligned}
\tag{3.61}
$$

Based on the above estimates (3.55)–(3.61) which are independent of δ, we can see that there exists a subsequence $\{(\hat{\mathbf{u}}_c^\delta, \hat{P}_m^\delta, \hat{\varphi}^\delta, \hat{\mu}^\delta)\}$ (still denoted by the same symbols for simplicity) as $\delta \to 0$ (or equivalently $N \to +\infty$) such that

$$\begin{cases} \hat{\mathbf{u}}_c^\delta \to \mathbf{u}_c & \text{weakly star in } L^\infty\big(0,T;\mathbf{L}^2(\Omega_c)\big), \\ & \text{weakly in } L^2\big(0,T;\mathbf{H}^1(\Omega_c)\big), \\ \hat{P}_m \to P_m & \text{weakly in } L^\alpha(0,T;X_m), \\ \hat{\mathbf{u}}_m^\delta \to \mathbf{u}_m & \text{weakly in } L^2\big(0,T;\mathbf{L}^2(\Omega_m)\big), \\ \hat{\varphi}^\delta \to \varphi & \text{weakly star in } L^\infty\big(0,T;H^1(\Omega)\big), \\ & \text{weakly in } L^2\big(0,T;H^3(\Omega)\big), \\ \hat{\mu}^\delta \to \mu & \text{weakly in } L^2\big(0,T;H^1(\Omega)\big), \end{cases} \tag{3.62}$$

for certain functions $(\mathbf{u}_c, P_m, \mathbf{u}_m, \varphi, \mu)$ satisfying

$$\begin{aligned} \mathbf{u}_c &\in L^\infty\big(0,T;\mathbf{L}^2(\Omega_c)\big) \cap L^2\big(0,T;\mathbf{H}^1(\Omega_c)\big), \\ P_m &\in L^\alpha(0,T;X_m), \\ \mathbf{u}_m &\in L^2\big(0,T;\mathbf{L}^2(\Omega_m)\big), \\ \varphi &\in L^\infty\big(0,T;H^1(\Omega)\big) \cap L^2\big(0,T;H^3(\Omega)\big), \\ \mu &\in L^2\big(0,T;H^1(\Omega)\big), \end{aligned}$$

where $\alpha = \frac{8}{5}$ when $d = 3$ and $\alpha \in (\frac{4}{3}, 2)$ that can be arbitrary close to 2 when $d = 2$.

In order to pass to the limit in nonlinear terms, we need to show the strong convergence of $\hat{\varphi}^\delta$ (up to a subsequence). It follows from Eq. (3.49), the Gagliardo–Nirenberg inequality and the Sobolev embedding theorem that

$$\begin{aligned} &\|\partial_t \varphi^\delta\|^{\frac{8}{5}}_{L^{\frac{8}{5}}(0,T;(H^1(\Omega))')} \\ &\le C \int_0^T \big(\|\nabla\hat{\mu}^\delta\|^{\frac{8}{5}}_{\mathbf{L}^2(\Omega)} + \|\hat{\varphi}^\delta\|^{\frac{8}{5}}_{L^\infty(\Omega)}\|\hat{\mathbf{u}}^\delta\|^{\frac{8}{5}}_{\mathbf{L}^2(\Omega)}\big)dt \\ &\le C \int_0^T \|\nabla\hat{\mu}^\delta\|^{\frac{8}{5}}_{\mathbf{L}^2(\Omega)} dt + C \sup_{0\le t\le T} \|\hat{\varphi}^\delta\|^{\frac{6}{5}}_{L^6(\Omega)} \int_0^T \|\hat{\varphi}^\delta\|^{\frac{2}{5}}_{H^3(\Omega)} \|\hat{\mathbf{u}}^\delta\|^{\frac{8}{5}}_{\mathbf{L}^2(\Omega)} dt \\ &\le C \int_0^T \big(\|\nabla\hat{\mu}^\delta\|^2_{\mathbf{L}^2(\Omega)} + 1\big)dt + C \sup_{0\le t\le T} \|\hat{\varphi}^\delta\|^{\frac{6}{5}}_{H^1(\Omega)} \int_0^T \big(\|\hat{\varphi}^\delta\|^2_{H^3(\Omega)} + \|\hat{\mathbf{u}}^\delta\|^2_{\mathbf{L}^2(\Omega)}\big)dt \\ &\le C_T, \quad \text{when } d = 3. \end{aligned} \tag{3.63}$$

For $d = 2$, we use the Brézis–Gallouet interpolation inequality (cf. [46])

$$\|g\|_{L^\infty(\Omega)} \le C\|g\|_{H^1(\Omega)}\sqrt{\ln\big(1 + \|g\|_{H^2(\Omega)}\big)} + C\big(1 + \|g\|_{H^1(\Omega)}\big), \quad \forall g \in H^2(\Omega)$$

to obtain that for any $\alpha \in (1, 2)$, it holds

$$
\begin{aligned}
&\left\|\partial_t\varphi^\delta\right\|^\alpha_{L^\alpha(0,T;(H^1(\Omega))')}\\
&\quad\le C\int_0^T\left(\left\|\nabla\hat\mu^\delta\right\|^\alpha_{\mathbf{L}^2(\Omega)}+\left\|\hat\varphi^\delta\right\|^\alpha_{L^\infty(\Omega)}\left\|\hat{\mathbf{u}}^\delta\right\|^\alpha_{\mathbf{L}^2(\Omega)}\right)dt\\
&\quad\le C\int_0^T\left\|\nabla\hat\mu^\delta\right\|^\alpha_{\mathbf{L}^2(\Omega)}dt\\
&\qquad+C\Big(1+\sup_{0\le t\le T}\left\|\hat\varphi^\delta\right\|^\alpha_{H^1(\Omega)}\Big)\int_0^T\Big(1+\sqrt{\ln\big(1+\|\varphi\|_{H^2(\Omega)}\big)}\Big)^\alpha\left\|\hat{\mathbf{u}}^\delta\right\|^\alpha_{\mathbf{L}^2(\Omega)}dt\\
&\quad\le C\int_0^T\left(\left\|\nabla\hat\mu^\delta\right\|^2_{\mathbf{L}^2(\Omega)}+1\right)dt\\
&\qquad+C\int_0^T\Big[\Big(1+\sqrt{\ln\big(1+\|\varphi\|_{H^2(\Omega)}\big)}\Big)^{\frac{2\alpha}{2-\alpha}}+\left\|\hat{\mathbf{u}}^\delta\right\|^2_{\mathbf{L}^2(\Omega)}\Big]dt\\
&\quad\le C_T,\quad \text{when } d=2.
\end{aligned}
\tag{3.64}
$$

As a result, it follows that

$$\partial_t\varphi^\delta\to\partial_t\varphi\quad\text{weakly in } L^\alpha\big(0,T;\big(H^1(\Omega)\big)'\big),$$

where $\alpha=\frac{8}{5}$ when $d=3$ and $\alpha\in(1,2)$ that can be arbitrary close to 2 when $d=2$.

Since

$$\left\|\hat\varphi^\delta-\varphi^\delta\right\|_{(H^1)'}=\left\|(t_{k+1}-t)\frac{(\varphi^{k+1}-\varphi^k)}{\delta}\right\|_{(H^1)'}\le\delta\left\|\partial_t\varphi^\delta\right\|_{(H^1)'},\quad t\in(t_k,t_{k+1}],$$

for $k=0,1,\dots,N-1$, we have

$$\int_0^T\left\|\hat\varphi^\delta-\varphi^\delta\right\|^\alpha_{(H^1)'}dt\le\delta^\alpha\int_0^T\left\|\partial_t\varphi^\delta\right\|^\alpha_{(H^1)'}dt\to0,\quad\text{as }\delta\to0,\tag{3.65}$$

which implies

$$\hat\varphi^\delta-\varphi^\delta\to0,\quad\text{strongly in } L^\alpha\big(0,T;\big(H^1\big)'\big),\ \text{as }\delta\to0.$$

Similarly, one can show $\|\tilde\varphi^\delta-\hat\varphi^\delta\|_{L^\alpha(0,T;(H^1)')}\to0$, as $\delta\to0$. Thus, the sequences $\{\varphi^\delta\}$, $\{\hat\varphi^\delta\}$ and $\{\tilde\phi^\delta\}$, if convergent, should converge to the same limit. On the other hand, by the definition of φ^δ, it satisfies the estimates similar to (3.55), (3.59) for $\hat\varphi^\delta$. Hence, applying Simon's compactness lemma (cf. e.g., [47]), we deduce that there exists $\varphi^* \in L^2(0,T;H^{3-\beta}(\Omega)) \cap C([0,T];H^{1-\beta}(\Omega))$, for a suitable subsequence,

$$\varphi^\delta \to \varphi^*, \quad \text{strongly in } L^2\big(0,T;H^\beta(\Omega)\big), \text{ as } \delta \to 0,$$

for certain $0 < \beta \le 1$. In particular, we have $\varphi^* = \varphi$ and up to a subsequence,

$$\hat{\varphi}^\delta, \tilde{\varphi}^\delta \to \varphi \quad \text{strongly in } L^2\big(0,T;H^{3-\beta}(\Omega)\big) \cap C\big([0,T];H^{1-\beta}(\Omega)\big), \text{ as } \delta \to 0. \tag{3.66}$$

Concerning the initial data, since by definition $\varphi^\delta|_{t=0} = \varphi_0$, we infer from (3.66) that

$$\varphi|_{t=0} = \varphi_0.$$

Indeed, by (3.55), (3.63) and [26, Lemma 4.1], we also have $\varphi \in C_w([0,T];H^1(\Omega))$.

Using similar arguments for (3.60) and (3.61), we can deduce from (3.48) and (3.60) that (taking $q_m = 0$)

$$\begin{aligned}
&\|\partial_t \mathbf{u}_c^\delta\|^{\frac{8}{5}}_{L^{\frac{8}{5}}(0,T;(\mathbf{H}^1(\Omega))')} \\
&\quad \le C\int_0^T \Big(\|\hat{\mathbf{u}}_c^\delta\|^{\frac{8}{5}}_{\mathbf{H}^1(\Omega_c)} + \|\hat{P}_m^\delta\|^{\frac{8}{5}}_{H^1(\Omega_m)} + \|\hat{\mu}_c^\delta\|^{\frac{8}{5}}_{L^6(\Omega_c)}\|\nabla\hat{\varphi}_c^\delta\|^{\frac{8}{5}}_{\mathbf{L}^3(\Omega_c)}\Big)dt \\
&\quad \le C_T, \quad \text{when } d = 3
\end{aligned} \tag{3.67}$$

and

$$\begin{aligned}
&\|\partial_t \mathbf{u}_c^\delta\|^{\frac{2r}{1+r}}_{L^{\frac{2r}{1+r}}(0,T;(\mathbf{H}^1(\Omega))')} \\
&\quad \le C\int_0^T \Big(\|\hat{\mathbf{u}}_c^\delta\|^{\frac{2r}{1+r}}_{\mathbf{H}^1(\Omega_c)} + \|\hat{P}_m^\delta\|^{\frac{2r}{1+r}}_{H^1(\Omega_m)} + \|\hat{\mu}_c^\delta\|^{\frac{2r}{1+r}}_{L^r(\Omega_c)}\|\nabla\hat{\varphi}_c^\delta\|^{\frac{2r}{1+r}}_{\mathbf{L}^{\frac{2r}{r-2}}(\Omega_c)}\Big)dt \\
&\quad \le C_T, \quad \forall r \in (2+\infty), \text{ when } d = 2.
\end{aligned} \tag{3.68}$$

Parallel to the arguments for φ^δ, $\hat{\varphi}^\delta$, the above estimates yield that as $\delta \to 0$,

$$\hat{\mathbf{u}}_c^\delta - \mathbf{u}_c^\delta \to 0, \quad \text{strongly in } L^\alpha\big(0,T;\big(\mathbf{H}^1(\Omega_c)\big)'\big), \tag{3.69}$$

$$\hat{\mathbf{u}}_c^\delta, \mathbf{u}_c^\delta \to \mathbf{u}_c, \quad \text{strongly in } L^2\big(0,T;\mathbf{H}^\beta(\Omega_c)\big) \cap C\big([0,T];\mathbf{H}^{-\beta}(\Omega_c)\big), \tag{3.70}$$

for some $\beta \in (0,1)$, $\alpha = \frac{8}{5}$ when $d = 3$ and $\alpha \in (\frac{4}{3}, 2)$ that can be arbitrary close to 2 when $d = 2$. Moreover, we have $\mathbf{u}_c|_{t=0} = \mathbf{u}_0$ and $\mathbf{u}_c \in C_w([0,T];\mathbf{L}^2(\Omega_c))$.

Based on the strong convergence (3.66) and the Sobolev embedding theorem, we can derive that

$$\tilde{f}\big(\hat{\varphi}^\delta, \tilde{\varphi}^\delta\big) \to f(\varphi), \quad \text{strongly in } L^2\big(0,T;L^2(\Omega)\big). \tag{3.71}$$

By the assumptions (A1)–(A2), we get

$$\nu(\tilde{\varphi}^\delta) \to \nu(\varphi), \quad \text{strongly in } C([0,T]; H^{1-\beta}(\Omega)),$$
$$\mathrm{M}(\tilde{\varphi}^\delta) \to \mathrm{M}(\varphi), \quad \text{strongly in } C([0,T]; H^{1-\beta}(\Omega)).$$

Similar to the argument in (3.60), we have $\hat{\mu}^\delta \nabla \hat{\varphi}^\delta \in L^\alpha(0,T;\mathbf{L}^2(\Omega))$ with α being the parameter specified above. Moreover, we infer from the strong convergence of $\hat{\varphi}^\delta$ (see (3.66)) and the weak convergence of $\hat{\mu}^\delta$ (see (3.62)) that

$$\hat{\mu}^\delta \nabla \hat{\varphi}^\delta \to \mu \nabla \varphi$$

in the distribution sense. At last, we note that in (3.48)–(3.49), after integration by parts, we get

$$\int_0^T (\partial_t \mathbf{u}_c^\delta, \mathbf{v}_c)_c dt = -\int_0^T (\mathbf{u}_c^\delta, \partial_t \mathbf{v}_c)_c dt,$$
$$\int_0^T (\partial_t \varphi^\delta, \phi) dt = -\int_0^T (\varphi^\delta, \partial_t \phi) dt.$$

Using the above convergence results, we are able to pass to the limit in Eqs. (3.48)–(3.51) to see that the limit functions $(\mathbf{u}_c, P_m, \mathbf{u}_m, \varphi, \mu)$ satisfy the weak formulation (2.15)–(2.18) (see Definition 2.1).

Finally, we show that $(\mathbf{u}_c, \mathbf{u}_m, \varphi, \mu)$ also fulfills the energy inequality (2.19). The energy estimate (3.53) yields that

$$\mathcal{E}(\mathbf{u}_0, \varphi_0)h(0) + \int_0^T \mathcal{E}^\delta(t)h'(t)dt \geq \int_0^T \mathcal{D}^\delta(t)h(t)dt, \tag{3.72}$$

for all $h(t) \in W^{1,1}(0,T)$ with $h \geq 0$ and $h(T) = 0$. On the other hand, it follows from the strong convergence results (3.66) and (3.70) that as $\delta \to 0$, for almost every $t \in (0,T)$, we have (up to a subsequence),

$$\hat{\mathbf{u}}_c^\delta(t) \to \mathbf{u}_c(t), \quad \text{strongly in } \mathbf{L}^2(\Omega_c),$$
$$\hat{\varphi}^\delta(t) \to \varphi(t), \quad \text{strongly in } H^1(\Omega),$$

which imply that

$$\mathcal{E}^\delta(t) \to \mathcal{E}(\mathbf{u}_c(t), \varphi(t)), \quad \text{for almost every } t \in (0,T).$$

By the lower semi-continuity of norms and the almost everywhere convergence of $\nu(\tilde{\varphi}^\delta)$, $\mathrm{M}(\tilde{\varphi}^\delta)$, we have

$$\liminf_{\delta\to 0}\int_s^t \mathcal{D}^\delta(\tau)h(\tau)d\tau \ge \int_s^t \mathcal{D}(\tau)h(\tau)d\tau, \quad \text{for } 0\le s<t\le T,$$

where $\mathcal{D}(t)$ is defined as in (2.3). Passing to the limit in (3.72), we get

$$\mathcal{E}(\mathbf{u}_0,\varphi_0)h(0)+\int_0^T \mathcal{E}\big(\mathbf{u}_c(t),\varphi(t)\big)h'(t)dt \ge \int_0^T \mathcal{D}(t)h(t)dt.$$

Then we can conclude from [26, Lemma 4.3] that the energy inequality (2.19) holds for all $s\le t<T$ and almost all $0\le s<T$ including $s=0$.

3.4.2. Case $\varpi=0$

If $\varpi=0$, one does not have a direct estimate on $\|\hat{\mathbf{u}}_c^\delta\|_{\mathbf{L}^2(\Omega_c)}$ (compare to (3.55)). Recall also that in our domain setting, the boundary $\Gamma_c=\emptyset$ is allowed, i.e., Ω_c can be enclosed completely by Ω_m. As a consequence, the classical Korn's inequality (3.22) does not apply. To overcome this difficulty, we shall derive an equivalent norm on the following space

$$\mathbf{Z}=\{\mathbf{u}\mid \mathbf{u}_c=\mathbf{u}|_{\Omega_c}\in\mathbf{H}_{c,\mathrm{div}},\ \mathbf{u}_m=\mathbf{u}|_{\Omega_m}\in\mathbf{H}_{m,\mathrm{div}},\ \mathbf{u}_c\cdot\mathbf{n}_{cm}=\mathbf{u}_m\cdot\mathbf{n}_{cm} \text{ on } \Gamma_{cm}\}.$$

Lemma 3.9. *The norm given by*

$$\|\mathbf{u}\|_{\mathbf{Z}}^2 := \big\|\mathbb{D}(\mathbf{u}_c)\big\|_{\mathbf{L}^2(\Omega_c)}^2+\sum_{i=1}^{d-1}\|\mathbf{u}_c\cdot\boldsymbol{\tau}_i\|_{L^2(\Gamma_{cm})}^2+\|\mathbf{u}_m\|_{\mathbf{L}^2(\Omega_m)}^2, \tag{3.73}$$

is an equivalent norm on $\mathbf{Z}$.

Proof. The case that Γ_c has positive measure is trivial in view of Korn's inequality (3.22). Below we focus on the situation where Ω_m encloses completely Ω_c. It is clear from Korn's inequality (3.21) and the trace theorem that the following quantity defines an equivalent norm on $\mathbf{Z}$

$$|||\mathbf{u}|||^2 := \big\|\mathbb{D}(\mathbf{u}_c)\big\|_{\mathbf{L}^2(\Omega_c)}^2+\|\mathbf{u}_c\|_{\mathbf{L}^2(\Omega_c)}^2+\sum_{i=1}^{d-1}\|\mathbf{u}_c\cdot\boldsymbol{\tau}_i\|_{L^2(\Gamma_{cm})}^2+\|\mathbf{u}_m\|_{\mathbf{L}^2(\Omega_m)}^2. \tag{3.74}$$

One only needs to prove that there exists a constant C independent of the function $\mathbf{u}$ such that

$$|||\mathbf{u}|||\le C\|\mathbf{u}\|_{\mathbf{Z}}, \quad \forall \mathbf{u}\in\mathbf{Z}.$$

Suppose by contradiction that for a sequence $\{\mathbf{u}_n\}$ in $\mathbf{Z}$ it holds

$$|||\mathbf{u}_n|||\ge n\|\mathbf{u}_n\|_{\mathbf{Z}}. \tag{3.75}$$

By homogeneity, we may normalize $|||\mathbf{u}_n||| = 1$. Then $\{\mathbf{u}_n\}$ is a bounded sequence in $\mathbf{Z}$. There exists a subsequence, still denoted by $\{\mathbf{u}_n\}$, such that $\mathbf{u}_n$ converges to $\mathbf{u}_\infty$ weakly in $\mathbf{Z}$. In particular, one has by Sobolev compact embedding $\mathbf{u}_{c_n} := \mathbf{u}_n|_{\Omega_c}$ converges to $\mathbf{u}_{c_\infty}$ strongly in $\mathbf{L}^2(\Omega_c)$. On the other hand, due to (3.75),

$$\|\mathbf{u}_n\|_{\mathbf{Z}} \to 0. \tag{3.76}$$

It follows from the definitions (3.73) and (3.74) that $\|\mathbf{u}_{c_n}\|_{\mathbf{L}^2(\Omega_c)}$ converges to 1, which implies

$$\|\mathbf{u}_{c_\infty}\|_{\mathbf{L}^2(\Omega_c)} = 1. \tag{3.77}$$

Using the facts that $\mathbf{u}_{m_n} := \mathbf{u}_n|_{\Omega_m} \in \mathbf{H}_{m,\mathrm{div}}$, (3.76) and the trace theorem, we see that

$$\mathbf{u}_{m_n} \cdot \mathbf{n}_{cm}|_{\Gamma_{cm}} \to 0, \quad \text{in } H^{-\frac{1}{2}}(\Gamma_{cm}).$$

Since $\mathbf{u}_n \in \mathbf{Z}$, by the continuity condition on the interface Γ_{cm}, one concludes

$$\mathbf{u}_{c_n} \cdot \mathbf{n}_{cm}|_{\Gamma_{cm}} = \mathbf{u}_{m_n} \cdot \mathbf{n}_{cm}|_{\Gamma_{cm}} \to 0, \quad \text{in } H^{\frac{1}{2}}(\Gamma_{cm}).$$

On the other hand, (3.76) implies $\|\mathbf{u}_{c_n} \cdot \boldsymbol{\tau}_i\|_{L^2(\Gamma_{cm})} \to 0$ $(i = 1, \ldots, d-1)$. As a consequence of the above estimates and the fact that $\mathbf{u}_{c_\infty}$ is the weak limit of $\mathbf{u}_{c_n}$ in $\mathbf{H}^1(\Omega_c)$, we obtain

$$\mathbf{u}_{c_\infty}|_{\Gamma_{cm}} = \mathbf{0}. \tag{3.78}$$

Finally, by the weak lower semi-continuity of norm, one has

$$\left\|\mathbb{D}(\mathbf{u}_{c_\infty})\right\|_{\mathbf{L}^2(\Omega_c)} \le \liminf_{n\to\infty}\left\|\mathbb{D}(\mathbf{u}_{c_n})\right\|_{\mathbf{L}^2(\Omega_c)} = 0. \tag{3.79}$$

By virtue of (3.78) and (3.79), we infer from the Korn inequality (3.22) that

$$\|\mathbf{u}_{c_\infty}\|_{\mathbf{L}^2(\Omega_c)} = 0.$$

This leads to a contradiction with (3.77). The proof is complete. $\square$

Now we return to the proof of Theorem 2.1. It follows easily from Lemma 3.1 and the definition of $\hat{\mathbf{u}}_c^\delta, \hat{\mathbf{u}}_m^\delta$ that

$$\hat{\mathbf{u}}_m^\delta \in \mathbf{H}_{m,\mathrm{div}}, \qquad \hat{\mathbf{u}}_m^\delta \cdot \mathbf{n}_{cm} = \hat{\mathbf{u}}_c^\delta \cdot \mathbf{n}_{cm} \quad \text{in } H^{\frac{1}{2}}(\Gamma_{cm}).$$

Thus, the equivalent norm (3.73) in Lemma 3.9 is applicable, and one can derive estimate on $\|\hat{\mathbf{u}}_c^\delta\|_{L^2(0,T;\mathbf{H}^1(\Omega_c))}$ from the energy estimate (3.54). Then one can conclude the proof as in the case of $\varpi > 0$.

The proof of Theorem 2.1 is complete.

4. Weak–strong uniqueness

In this section, we prove the uniqueness result Theorem 2.2. Below we just give the proof for $d = 3$, while the proof for $d = 2$ can be obtained with minor modifications under certain weaker regularity assumptions.

First, we recall that the finite energy weak solutions $(\mathbf{u}_c, P_m, \mathbf{u}_m, \varphi, \mu)$ to CHSD system (1.1)–(1.22) satisfy the energy inequality (2.19), i.e.,

$$\begin{aligned}
&\int_{\Omega_c} \frac{\varpi}{2}\big|\mathbf{u}_c(t)\big|^2 dx + \int_{\Omega}\left[\frac{\epsilon}{2}|\nabla\varphi|^2 + \frac{1}{\epsilon}F(\varphi)\right]dx \\
&\quad + \int_0^t\int_{\Omega_m} \nu(\varphi_m)\Pi^{-1}|\mathbf{u}_m|^2 dx d\tau + \int_0^t\int_{\Omega_c} 2\nu(\varphi_c)\big|\mathbb{D}(\mathbf{u}_c)\big|^2 dx d\tau \\
&\quad + \int_0^t\int_{\Omega} \mathrm{M}(\varphi)\big|\nabla\mu(\varphi)\big|^2 dx d\tau \\
&\quad + \frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}}\sum_{i=1}^{d-1}\int_0^t\int_{\Gamma_{cm}} \nu(\varphi_m)|\mathbf{u}_c\cdot\boldsymbol{\tau}_i|^2 dS d\tau \\
&\le \int_{\Omega_c}\frac{\varpi}{2}|\mathbf{u}_0|^2 dx + \int_{\Omega}\left[\frac{\epsilon}{2}|\nabla\varphi_0|^2 + \frac{1}{\epsilon}F(\varphi_0)\right]dx.
\end{aligned} \tag{4.1}$$

On the other hand, the regular solutions $(\tilde{\mathbf{u}}_c, \tilde{P}_m, \tilde{\mathbf{u}}_m, \tilde{\varphi}, \tilde{\mu})$ are allowed to be used as the test functions for the CHSD system and the following energy equality holds

$$\begin{aligned}
&\int_{\Omega_c} \frac{\varpi}{2}\big|\tilde{\mathbf{u}}_c(t)\big|^2 dx + \int_{\Omega}\left[\frac{\epsilon}{2}|\nabla\tilde{\varphi}|^2 + \frac{1}{\epsilon}F(\tilde{\varphi})\right]dx \\
&\quad + \int_0^t\int_{\Omega_m} \nu(\tilde{\varphi}_m)\Pi^{-1}|\tilde{\mathbf{u}}_m|^2 dx d\tau + \int_0^t\int_{\Omega_c} 2\nu(\tilde{\varphi}_c)\big|\mathbb{D}(\tilde{\mathbf{u}}_c)\big|^2 dx d\tau \\
&\quad + \int_0^t\int_{\Omega} \mathrm{M}(\tilde{\varphi})\big|\nabla\tilde{\mu}(\tilde{\varphi})\big|^2 dx d\tau \\
&\quad + \frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}}\sum_{i=1}^{d-1}\int_0^t\int_{\Gamma_{cm}} \nu(\tilde{\varphi}_m)|\tilde{\mathbf{u}}_c\cdot\boldsymbol{\tau}_i|^2 dS d\tau \\
&= \int_{\Omega_c}\frac{\varpi}{2}|\mathbf{u}_0|^2 dx + \int_{\Omega}\left[\frac{\epsilon}{2}|\nabla\varphi_0|^2 + \frac{1}{\epsilon}F(\varphi_0)\right]dx.
\end{aligned} \tag{4.2}$$

Next, taking $\tilde{\mathbf{u}}$ and $-\epsilon\Delta\tilde{\varphi}$ as test functions in the weak formulation for the finite energy weak solution $(\mathbf{u}_c, P_m, \mathbf{u}_m, \varphi, \mu)$ and using the equations for $\tilde{\mathbf{u}}_c$, $\tilde{\varphi}$, we obtain that

$$
\begin{aligned}
&\varpi\big(\mathbf{u}_c(t), \tilde{\mathbf{u}}_c(t)\big)_c - \varpi\int_{\Omega_c}|\mathbf{u}_0|^2dx \\
&\quad= \varpi\int_0^t(\mathbf{u}_c, \partial_t\tilde{\mathbf{u}}_c)_c d\tau - \frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}}\sum_{i=1}^{d-1}\int_0^t\int_{\Gamma_{cm}}\nu(\varphi_m)(\mathbf{u}_c\cdot\boldsymbol{\tau}_i)(\tilde{\mathbf{u}}_c\cdot\boldsymbol{\tau}_i)dSd\tau \\
&\qquad-\int_0^t\int_{\Gamma_{cm}}P_m(\tilde{\mathbf{u}}_c\cdot\mathbf{n}_{cm})dSd\tau - \int_0^t\int_{\Omega_c}2\nu(\varphi_c)\mathbb{D}(\mathbf{u}_c):\mathbb{D}(\tilde{\mathbf{u}}_c)dxd\tau \\
&\qquad+\int_0^t(\mu_c\nabla\varphi_c, \tilde{\mathbf{u}}_c)_c d\tau \\
&\quad= -\frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}}\sum_{i=1}^{d-1}\int_0^t\int_{\Gamma_{cm}}\nu(\varphi_m)(\mathbf{u}_c\cdot\boldsymbol{\tau}_i)(\tilde{\mathbf{u}}_c\cdot\boldsymbol{\tau}_i)dSd\tau \\
&\qquad-\frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}}\sum_{i=1}^{d-1}\int_0^t\int_{\Gamma_{cm}}\nu(\tilde{\varphi}_m)(\mathbf{u}_c\cdot\boldsymbol{\tau}_i)(\tilde{\mathbf{u}}_c\cdot\boldsymbol{\tau}_i)dSd\tau \\
&\qquad-\int_0^t\int_{\Gamma_{cm}}P_m(\tilde{\mathbf{u}}_c\cdot\mathbf{n}_{cm})dSd\tau - \int_0^t\int_{\Gamma_{cm}}\tilde{P}_m(\mathbf{u}_c\cdot\mathbf{n}_{cm})dSd\tau \\
&\qquad-\int_0^t\int_{\Omega_c}2\nu(\varphi_c)\mathbb{D}(\mathbf{u}_c):\mathbb{D}(\tilde{\mathbf{u}}_c)dxd\tau - \int_0^t\int_{\Omega_c}2\nu(\tilde{\varphi}_c)\mathbb{D}(\mathbf{u}_c):\mathbb{D}(\tilde{\mathbf{u}}_c)dxd\tau \\
&\qquad+\int_0^t(\mu_c\nabla\varphi_c, \tilde{\mathbf{u}}_c)_c d\tau + \int_0^t(\tilde{\mu}_c\nabla\tilde{\varphi}_c, \mathbf{u}_c)_c d\tau,
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&\epsilon\int_{\Omega}\nabla\varphi(t)\cdot\nabla\tilde{\varphi}(t)dx - \epsilon\int_{\Omega}|\nabla\varphi_0|^2dx \\
&\quad= \epsilon\int_0^t\int_{\Omega}M(\varphi)\nabla\mu\cdot\nabla\Delta\tilde{\varphi}dxd\tau + \epsilon\int_0^t\int_{\Omega}M(\tilde{\varphi})\nabla\tilde{\mu}\cdot\nabla\Delta\varphi dxd\tau \\
&\qquad+\epsilon\int_0^t\int_{\Omega}(\mathbf{u}\cdot\nabla\varphi)\Delta\tilde{\varphi}dxd\tau + \epsilon\int_0^t\int_{\Omega}(\tilde{\mathbf{u}}\cdot\nabla\tilde{\varphi})\Delta\varphi dxd\tau,
\end{aligned}
\tag{4.4}
$$

$$\int_0^t \int_{\Omega_m} \nu(\varphi_m)\Pi^{-1}\mathbf{u}_m \cdot \tilde{\mathbf{u}}_m dx d\tau + \int_0^t \int_{\Omega_m} \nu(\tilde{\varphi}_m)\Pi^{-1}\mathbf{u}_m \cdot \tilde{\mathbf{u}}_m dx d\tau$$

$$= -\int_0^t \big((\nabla P_m - \mu_m \nabla\varphi_m), \tilde{\mathbf{u}}_m\big)_m d\tau - \int_0^t \big((\nabla \tilde{P}_m - \tilde{\mu}_m \nabla\tilde{\varphi}_m), \mathbf{u}_m\big)_m d\tau$$

$$= \int_0^t \int_{\Gamma_{cm}} P_m(\tilde{\mathbf{u}}_c \cdot \mathbf{n}_{cm}) dS d\tau + \int_0^t \int_{\Gamma_{cm}} \tilde{P}_m(\mathbf{u}_c \cdot \mathbf{n}_{cm}) dS d\tau$$

$$+ \int_0^t (\mu_m \nabla\varphi_m, \tilde{\mathbf{u}}_m)_m d\tau + \int_0^t (\tilde{\mu}_m \nabla\tilde{\varphi}_m, \mathbf{u}_m)_m d\tau. \tag{4.5}$$

Adding (4.1) with (4.2) and subtracting the sum of (4.3)–(4.5) from the resultant, by a direct computation we obtain that

$$\frac{\varpi}{2}\int_{\Omega_c} \big|\mathbf{u}_c(t) - \tilde{\mathbf{u}}_c(t)\big|^2 dx + \frac{\epsilon}{2}\int_{\Omega} \big|\nabla\varphi(t) - \nabla\tilde{\varphi}(t)\big|^2 dx$$

$$+ \int_0^t \int_{\Omega_c} 2\nu(\varphi_c)\big|\mathbb{D}(\mathbf{u}_c) - \mathbb{D}(\tilde{\mathbf{u}}_c)\big|^2 dx d\tau$$

$$+ \int_0^t \int_{\Omega_m} \nu(\varphi_m)\Pi^{-1}|\mathbf{u}_m - \tilde{\mathbf{u}}_m|^2 dx d\tau$$

$$+ \epsilon^2 \int_0^t \int_{\Omega} \mathrm{M}(\varphi)|\nabla\Delta\varphi - \nabla\Delta\tilde{\varphi}|^2 dx d\tau$$

$$+ \frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}} \sum_{i=1}^{d-1} \int_0^t \int_{\Gamma_{cm}} \nu(\varphi)\big|(\mathbf{u}_c - \tilde{\mathbf{u}}_c)\cdot \boldsymbol{\tau}_i\big|^2 dS d\tau$$

$$\le -\int_0^t \int_{\Omega_c} 2\big(\nu(\tilde{\varphi}_c) - \nu(\varphi_c)\big)\mathbb{D}(\tilde{\mathbf{u}}_c) : \big(\mathbb{D}(\tilde{\mathbf{u}}_c) - \mathbb{D}(\mathbf{u}_c)\big) dx d\tau$$

$$- \int_0^t \int_{\Omega_m} \big(\nu(\tilde{\varphi}_m) - \nu(\varphi_m)\big)\Pi^{-1}\tilde{\mathbf{u}}_m(\tilde{\mathbf{u}}_m - \mathbf{u}_m) dx d\tau$$

$$- \epsilon^2 \int_0^t \int_{\Omega} \big(\mathrm{M}(\tilde{\varphi}) - \mathrm{M}(\varphi)\big)\nabla\Delta\tilde{\varphi} \cdot (\nabla\Delta\tilde{\varphi} - \nabla\Delta\varphi) dx d\tau$$

$$
\begin{aligned}
&- \frac{\alpha_{BJSJ}}{\sqrt{\operatorname{trace}(\Pi)}} \sum_{i=1}^{d-1} \int_0^t \int_{\Gamma_{cm}} \big(\nu(\tilde{\varphi}_m) - \nu(\varphi_m)\big)(\tilde{\mathbf{u}}_c \cdot \boldsymbol{\tau}_i)\big((\tilde{\mathbf{u}}_c - \mathbf{u}_c) \cdot \boldsymbol{\tau}_i\big) dS d\tau \\
&+ 2 \int_0^t \int_\Omega \big(\mathrm{M}(\varphi)\nabla\Delta\varphi \cdot \nabla f(\varphi) + \mathrm{M}(\tilde{\varphi})\nabla\Delta\tilde{\varphi} \cdot \nabla f(\tilde{\varphi})\big) dx d\tau \\
&- \int_0^t \int_\Omega \big(\mathrm{M}(\varphi)\nabla f(\varphi) \cdot \nabla\Delta\tilde{\varphi} + \mathrm{M}(\tilde{\varphi})\nabla f(\tilde{\varphi}) \cdot \nabla\Delta\varphi\big) dx d\tau \\
&- \frac{1}{\epsilon^2} \int_0^t \int_\Omega \big(\mathrm{M}(\varphi)\big|\nabla f(\varphi)\big|^2 + \mathrm{M}(\tilde{\varphi})\big|\nabla f(\tilde{\varphi})\big|^2\big) dx d\tau \\
&+ \frac{1}{\epsilon} \int_\Omega \big(2F(\varphi_0) - F(\varphi) - F(\tilde{\varphi})\big) dx \\
&+ \epsilon \int_0^t \int_\Omega (\Delta\varphi\nabla\varphi \cdot \tilde{\mathbf{u}} + \Delta\tilde{\varphi}\nabla\tilde{\varphi} \cdot \mathbf{u} - \mathbf{u} \cdot \nabla\varphi\Delta\tilde{\varphi} - \tilde{\mathbf{u}} \cdot \nabla\tilde{\varphi}\Delta\varphi) dx d\tau \\
:= &\sum_{j=1}^{9} I_j,
\end{aligned}
\tag{4.6}
$$

where we have used the incompressibility condition and the fact

$$
\int_\Omega (\mathbf{u} \cdot \nabla\varphi) f(\varphi) dx = \int_\Omega \mathbf{u} \cdot \nabla F(\varphi) dx = 0.
$$

Using the mass conservation property $\int_\Omega (\tilde{\varphi} - \varphi) dx = 0$ (due to the choice of initial data), the Poincaré inequality, the Sobolev embedding theorem and the Gagliardo–Nirenberg inequality, we have the following estimates for $\phi = \tilde{\varphi} - \varphi$

$$
\begin{aligned}
\|\phi\|_{L^\infty(\Omega)} &\le C\big(\|\nabla\Delta\phi\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{4}} \|\phi\|_{L^6(\Omega)}^{\frac{3}{4}} + \|\phi\|_{L^6(\Omega)}\big) \\
&\le C\big(\|\nabla\Delta\phi\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{4}} \|\nabla\phi\|_{\mathbf{L}^2(\Omega)}^{\frac{3}{4}} + \|\nabla\phi\|_{\mathbf{L}^2(\Omega)}\big), \\
\|\Delta\phi\|_{L^3(\Omega)} &\le C\big(\|\nabla\Delta\phi\|_{\mathbf{L}^2(\Omega)}^{\frac{3}{4}} \|\nabla\phi\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{4}} + \|\nabla\phi\|_{\mathbf{L}^2(\Omega)}\big), \\
\|\nabla\phi\|_{\mathbf{L}^6(\Omega)} &\le C\big(\|\nabla\Delta\phi\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}} \|\nabla\phi\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}} + \|\nabla\phi\|_{\mathbf{L}^2(\Omega)}\big).
\end{aligned}
$$

Combining the above estimates with the Young inequality, we get

$$\begin{aligned}
I_1 &\le C\int_0^t \big\|\nu(\tilde\varphi_c)-\nu(\varphi_c)\big\|_{L^\infty(\Omega_c)}\big\|\mathbb{D}(\tilde{\mathbf{u}}_c)\big\|_{\mathbf{L}^2(\Omega_c)}\big\|\mathbb{D}(\tilde{\mathbf{u}}_c)-\mathbb{D}(\mathbf{u}_c)\big\|_{\mathbf{L}^2(\Omega_c)}d\tau\\
&\le C\int_0^t \|\tilde\varphi-\varphi\|_{L^\infty(\Omega)}\big\|\mathbb{D}(\tilde{\mathbf{u}}_c)\big\|_{\mathbf{L}^2(\Omega_c)}\big\|\mathbb{D}(\tilde{\mathbf{u}}_c)-\mathbb{D}(\mathbf{u}_c)\big\|_{\mathbf{L}^2(\Omega_c)}d\tau\\
&\le C\int_0^t \big(\big\|\nabla\Delta(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^{\frac14}\big\|\nabla(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^{\frac34}+\big\|\nabla(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}\big)\\
&\quad\times\big\|\mathbb{D}(\tilde{\mathbf{u}}_c)\big\|_{\mathbf{L}^2(\Omega_c)}\big\|\mathbb{D}(\tilde{\mathbf{u}}_c)-\mathbb{D}(\mathbf{u}_c)\big\|_{\mathbf{L}^2(\Omega_c)}d\tau\\
&\le \zeta\int_0^t \big\|\nabla\Delta(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2 d\tau+\zeta\int_0^t\big\|\mathbb{D}(\tilde{\mathbf{u}}_c)-\mathbb{D}(\mathbf{u}_c)\big\|_{\mathbf{L}^2(\Omega_c)}^2d\tau\\
&\quad+C\int_0^t\big(\big\|\mathbb{D}(\tilde{\mathbf{u}}_c)\big\|_{\mathbf{L}^2(\Omega_c)}^{\frac83}+1\big)\big\|\nabla(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2d\tau,
\end{aligned}$$

where $\zeta>0$ is a small constant to be chosen later. In a similar manner, we have the following estimates for I_2, I_3 and I_4:

$$\begin{aligned}
I_2 &\le \zeta\int_0^t \big\|\nabla\Delta(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2 d\tau+\zeta\int_0^t \|\tilde{\mathbf{u}}_m-\mathbf{u}_m\|_{\mathbf{L}^2(\Omega_m)}^2d\tau\\
&\quad+C\int_0^t\big(\|\tilde{\mathbf{u}}_m\|_{\mathbf{L}^2(\Omega_m)}^{\frac83}+1\big)\big\|\nabla(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2d\tau,\\
I_3 &\le \zeta\int_0^t \big\|\nabla\Delta(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2 d\tau\\
&\quad+C\int_0^t\big(\|\nabla\Delta\tilde\varphi\|_{\mathbf{L}^2(\Omega)}^{\frac83}+1\big)\big\|\nabla(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2d\tau,\\
I_4 &\le \zeta\int_0^t \big\|\nabla\Delta(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2 d\tau+\zeta\sum_{i=1}^{d-1}\int_0^t\int_{\Gamma_{cm}}\big|(\tilde{\mathbf{u}}_c-\mathbf{u}_c)\cdot\boldsymbol{\tau}_i\big|^2dSd\tau\\
&\quad+C\int_0^t\big(\|\tilde{\mathbf{u}}_c\|_{\mathbf{L}^2(\Omega_c)}^{\frac83}+1\big)\big\|\nabla(\tilde\varphi-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2d\tau.
\end{aligned}$$

Since $\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega))$, by the Gagliardo–Nirenberg inequality we deduce that

$$\begin{aligned}\int_0^T \|f(\varphi)\|_{H^1(\Omega)}^4 dt &\le C\int_0^T \big(\|\varphi\|_{H^1(\Omega)}^4 + \|\varphi\|_{L^6(\Omega)}^{12} + 81\big\|\varphi^2\nabla\varphi\big\|_{\mathbf{L}^2(\Omega)}^4\big)dt \\ &\le C_T + C\int_0^t \|\varphi\|_{L^\infty(\Omega)}^8 \|\nabla\varphi\|_{\mathbf{L}^2(\Omega)}^4 dt \\ &\le C_T + C\int_0^T \big(\|\nabla\Delta\varphi\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{4}} \|\varphi\|_{L^6(\Omega)}^{\frac{3}{4}} + \|\varphi\|_{L^6(\Omega)}\big)^8 dt \\ &\le C_T + C\int_0^T \|\nabla\Delta\varphi\|_{\mathbf{L}^2(\Omega)}^2 dt \\ &\le C_T,\end{aligned}$$

which implies $f(\varphi) \in L^4(0,T;H^1(\Omega)) \subset L^{\frac{8}{3}}(0,T;H^1(\Omega))$. Thus we can take $f(\varphi) = \varphi^3 - \varphi$ as a test function in the Cahn–Hilliard equation for φ. Since the nonlinear part φ^3 is monotone increasing, similar to [48, Proposition 4.2], we see that the dual product satisfies $\langle \varphi_t, f(\varphi)\rangle_{(H^1)',H^1} = \frac{d}{dt}\int_\Omega F(\varphi)dx$ for a.e. $t \in (0,T)$. Then integrating with respect to t we deduce that

$$\int_\Omega F(\varphi)dx - \int_\Omega F(\varphi_0)dx = \epsilon \int_0^t \int_\Omega \mathrm{M}(\varphi)\nabla\Delta\varphi \cdot \nabla f(\varphi)dxd\tau - \frac{1}{\epsilon}\int_0^t \int_\Omega \mathrm{M}(\varphi)\big|\nabla f(\varphi)\big|^2 dxd\tau.$$

In a similar way, we have the same identity for the regular solution $\tilde{\varphi}$

$$\int_\Omega F(\tilde{\varphi})dx - \int_\Omega F(\varphi_0)dx = \epsilon \int_0^t \int_\Omega \mathrm{M}(\tilde{\varphi})\nabla\Delta\tilde{\varphi} \cdot \nabla f(\tilde{\varphi})dxd\tau - \frac{1}{\epsilon}\int_0^t \int_\Omega \mathrm{M}(\tilde{\varphi})\big|\nabla f(\tilde{\varphi})\big|^2 dxd\tau.$$

As a consequence, we obtain that

$$\begin{aligned}&I_5 + I_6 + I_7 + I_8 \\ &\quad = \int_0^t \int_\Omega \big[-\mathrm{M}(\varphi)\nabla\Delta(\tilde{\varphi} - \varphi) \cdot \nabla f(\varphi) + \mathrm{M}(\tilde{\varphi})\nabla\Delta(\tilde{\varphi} - \varphi) \cdot \nabla f(\tilde{\varphi})\big]dxd\tau \\ &\quad = \int_0^t \int_\Omega \big[\mathrm{M}(\tilde{\varphi}) - \mathrm{M}(\varphi)\big]\nabla\Delta(\tilde{\varphi} - \varphi) \cdot \nabla f(\tilde{\varphi})dxd\tau\end{aligned}$$

$$+\int_0^t\int_\Omega \mathrm{M}(\varphi)\nabla\Delta(\tilde{\varphi}-\varphi)\cdot\big(\nabla f(\tilde{\varphi})-\nabla f(\varphi)\big)dx\,d\tau$$
$$:= J_1 + J_2. \tag{4.7}$$

The term J_1 can be estimated like I_1 such that

$$\begin{aligned}
J_1 &\le C\int_0^t \big\|\mathrm{M}(\tilde{\varphi})-\mathrm{M}(\varphi)\big\|_{L^\infty(\Omega)}\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}\big\|\nabla f(\tilde{\varphi})\big\|_{\mathbf{L}^2(\Omega)}d\tau\\
&\le C\int_0^t \|\tilde{\varphi}-\varphi\|_{L^\infty(\Omega)}\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}\big\|\nabla f(\tilde{\varphi})\big\|_{\mathbf{L}^2(\Omega)}d\tau\\
&\le C\int_0^t \big(\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^{\frac14}\big\|\nabla(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^{\frac34}+\big\|\nabla(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}\big)\\
&\quad\times\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}\big(\|\tilde{\varphi}\|_{L^\infty}^2+1\big)\|\nabla\tilde{\varphi}\|_{\mathbf{L}^2(\Omega)}d\tau\\
&\le \zeta\int_0^t \big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2 d\tau\\
&\quad+C\int_0^t \big(\|\tilde{\varphi}\|_{L^\infty(\Omega)}^{\frac{16}{3}}+1\big)\|\nabla\tilde{\varphi}\|_{\mathbf{L}^2(\Omega)}^{\frac83}\big\|\nabla(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2 d\tau\\
&\quad+C\int_0^t \big(\|\tilde{\varphi}\|_{L^\infty(\Omega)}^4+1\big)\|\nabla\tilde{\varphi}\|_{\mathbf{L}^2(\Omega)}^2\big\|\nabla(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2 d\tau\\
&\le \zeta\int_0^t \big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2 d\tau + C\int_0^t \big(\|\nabla\Delta\tilde{\varphi}\|_{L^2(\Omega)}^{\frac43}+1\big)\big\|\nabla(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}^2 d\tau.
\end{aligned}$$

For J_2, it holds

$$\begin{aligned}
J_2 &\le \int_0^t \big\|\mathrm{M}(\varphi)\big\|_{L^\infty(\Omega)}\big\|\nabla\big(f(\tilde{\varphi})-f(\varphi)\big)\big\|_{\mathbf{L}^2(\Omega)}\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}d\tau\\
&\le C\int_0^t \big\|\nabla(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}d\tau\\
&\quad+C\int_0^t \big\|\tilde{\varphi}^2\big\|_{L^\infty(\Omega)}\big\|\nabla(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}d\tau
\end{aligned}$$

$$+ C\int_0^t \|\tilde{\varphi}+\varphi\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{\mathbf{L}^2(\Omega)}\|\tilde{\varphi}-\varphi\|_{L^\infty(\Omega)}\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}d\tau$$

$$\leq C\int_0^t \big(1+\|\tilde{\varphi}\|^2_{L^\infty(\Omega)}\big)\big\|\nabla(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}d\tau$$

$$+ C\int_0^t \big(\|\tilde{\varphi}\|_{L^\infty(\Omega)}+\|\varphi\|_{L^\infty(\Omega)}\big)\|\nabla\varphi\|_{\mathbf{L}^2(\Omega)}\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}$$

$$\times\big(\big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|^{\frac14}_{\mathbf{L}^2(\Omega)}\big\|\nabla(\tilde{\varphi}-\varphi)\big\|^{\frac34}_{\mathbf{L}^2(\Omega)}+\big\|\nabla(\tilde{\varphi}-\varphi)\big\|_{\mathbf{L}^2(\Omega)}\big)d\tau$$

$$\leq \zeta\int_0^t \big\|\nabla\Delta(\tilde{\varphi}-\varphi)\big\|^2_{\mathbf{L}^2(\Omega)}d\tau$$

$$+ C\int_0^t \big(\|\nabla\Delta\tilde{\varphi}\|^2_{\mathbf{L}^2(\Omega)}+\|\nabla\Delta\varphi\|^2_{\mathbf{L}^2(\Omega)}+1\big)\big\|\nabla(\tilde{\varphi}-\varphi)\big\|^2_{\mathbf{L}^2(\Omega)}d\tau.$$

Now we estimate the last term I_9,

$$I_9=\epsilon\int_0^t\int_\Omega \tilde{\mathbf{u}}\cdot\nabla(\varphi-\tilde{\varphi})\Delta(\varphi-\tilde{\varphi})dx\,d\tau+\epsilon\int_0^t\int_\Omega \Delta\tilde{\varphi}(\tilde{\mathbf{u}}-\mathbf{u})\cdot\nabla(\varphi-\tilde{\varphi})dx\,d\tau$$

$$\leq C\int_0^t \|\tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\big\|\nabla(\varphi-\tilde{\varphi})\big\|_{\mathbf{L}^6(\Omega)}\big\|\Delta(\varphi-\tilde{\varphi})\big\|_{L^3(\Omega)}d\tau$$

$$+ C\int_0^t \|\Delta\tilde{\varphi}\|_{L^6(\Omega)}\|\tilde{\mathbf{u}}-\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\big\|\nabla(\varphi-\tilde{\varphi})\big\|_{\mathbf{L}^3(\Omega)}d\tau$$

$$\leq C\int_0^t \|\tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\big(\big\|\nabla\Delta(\varphi-\tilde{\varphi})\big\|^{\frac12}_{\mathbf{L}^2(\Omega)}\big\|\nabla(\varphi-\tilde{\varphi})\big\|^{\frac12}_{\mathbf{L}^2(\Omega)}+\big\|\nabla(\varphi-\tilde{\varphi})\big\|_{\mathbf{L}^2(\Omega)}\big)$$

$$\times\big(\big\|\nabla\Delta(\varphi-\tilde{\varphi})\big\|^{\frac34}_{\mathbf{L}^2(\Omega)}\big\|\nabla(\varphi-\tilde{\varphi})\big\|^{\frac14}_{\mathbf{L}^2(\Omega)}+\big\|\nabla(\varphi-\tilde{\varphi})\big\|_{\mathbf{L}^2(\Omega)}\big)d\tau$$

$$+ C\int_0^t \big(\|\nabla\Delta\tilde{\varphi}\|_{L^2(\Omega)}+\|\nabla\tilde{\varphi}\|_{L^2(\Omega)}\big)\|\tilde{\mathbf{u}}-\mathbf{u}\|_{\mathbf{L}^2(\Omega)}$$

$$\times\big(\big\|\nabla\Delta(\varphi-\tilde{\varphi})\big\|^{\frac14}_{\mathbf{L}^2(\Omega)}\big\|\nabla(\varphi-\tilde{\varphi})\big\|^{\frac34}_{\mathbf{L}^2(\Omega)}+\big\|\nabla(\varphi-\tilde{\varphi})\big\|_{\mathbf{L}^2(\Omega)}\big)d\tau$$

$$\leq \zeta \int_0^t \|\nabla\Delta(\tilde{\varphi}-\varphi)\|^2_{\mathbf{L}^2(\Omega)} d\tau + \zeta \int_0^t \|\tilde{\mathbf{u}}-\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} d\tau$$

$$+ C \int_0^t \big(\|\tilde{\mathbf{u}}\|^{\frac{8}{3}}_{\mathbf{L}^2(\Omega)} + \|\nabla\Delta\tilde{\varphi}\|^{\frac{8}{3}}_{L^2(\Omega)} + 1\big)\|\nabla(\tilde{\varphi}-\varphi)\|^2_{\mathbf{L}^2(\Omega)} d\tau. \tag{4.8}$$

Combining the above estimates, using the equivalent norm $\|\mathbf{u}\|_{\mathbf{Z}}$ given by (3.73) in Lemma 3.9 and the assumptions (A1)–(A3), by taking $\zeta > 0$ sufficiently small, we deduce that

$$\begin{aligned}
&\varpi \|(\tilde{\mathbf{u}}_c - \mathbf{u}_c)(t)\|^2_{\mathbf{L}^2(\Omega_c)} + \epsilon \|\nabla(\tilde{\varphi}-\varphi)(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad + \gamma_1 \int_0^t \big(\|(\tilde{\mathbf{u}}-\mathbf{u})(\tau)\|^2_{\mathbf{Z}} + \|\nabla\Delta(\tilde{\varphi}-\varphi)(\tau)\|^2_{\mathbf{L}^2(\Omega)}\big) d\tau \\
&\leq \gamma_2 \int_0^t h(\tau)\|\nabla(\tilde{\varphi}-\varphi)(\tau)\|^2_{\mathbf{L}^2(\Omega)} d\tau,
\end{aligned} \tag{4.9}$$

where

$$h(t) = \|\tilde{\mathbf{u}}(t)\|^{\frac{8}{3}}_{\mathbf{Z}} + \|\nabla\Delta\tilde{\varphi}(t)\|^{\frac{8}{3}}_{\mathbf{L}^2(\Omega)} + \|\nabla\Delta\varphi(t)\|^2_{\mathbf{L}^2(\Omega)} + 1,$$

and the constants $\gamma_1, \gamma_2 > 0$ may depend on the initial energy $\mathcal{E}(0)$ as well as the coefficients of the CHSD system.

Since by our assumption $(\mathbf{u}_c, \varphi)|_{t=0} = (\mathbf{0}, 0)$ and $h(t) \in L^1(0, T)$, then it follows from (4.9) and the Gronwall inequality that for $t \in [0, T]$,

$$\varpi \|(\tilde{\mathbf{u}}_c - \mathbf{u}_c)(t)\|^2_{\mathbf{L}^2(\Omega_c)} + \epsilon \|\nabla(\tilde{\varphi}-\varphi)(t)\|^2_{\mathbf{L}^2(\Omega)} = 0 \tag{4.10}$$

and then

$$\int_0^T \|(\tilde{\mathbf{u}}-\mathbf{u})(t)\|^2_{\mathbf{Z}} dt = 0. \tag{4.11}$$

Recalling the fact $\int_\Omega (\tilde{\varphi}-\varphi) dx = 0$ for $t \in [0, T]$, by the Poincaré inequality and the definition of the norm $\|\cdot\|_{\mathbf{Z}}$ (see (3.73)), we infer that

$$(\mathbf{u}_c, \mathbf{u}_m, \varphi) = (\tilde{\mathbf{u}}_c, \tilde{\mathbf{u}}_m, \tilde{\varphi}). \tag{4.12}$$

Finally, we remark that for the case of $\varpi = 0$, one can proceed as above and conclude (4.10), (4.11) with $\varpi = 0$ in (4.10), which again yield the uniqueness result (4.12).

The proof of Theorem 2.2 is complete.

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