

# Global existence and boundedness in a Keller–Segel–Stokes system involving a tensor-valued sensitivity with saturation

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## Abstract

This paper deals with a boundary-value problem in two-dimensional smoothly bounded domains for the coupled Keller–Segel–Stokes system

$$\begin{cases} n_t + \mathbf{u} \cdot \nabla n = \Delta n - \nabla \cdot (n \mathcal{S}(x, n, c) \cdot \nabla c), & (x, t) \in \Omega \times (0, T), \\ c_t + \mathbf{u} \cdot \nabla c = \Delta c - c + n, & (x, t) \in \Omega \times (0, T), \\ \mathbf{u}_t + \nabla P = \Delta \mathbf{u} + n \nabla \phi, & (x, t) \in \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} = 0, & (x, t) \in \Omega \times (0, T). \end{cases}$$

Here, one of the novelties is that the chemotactic sensitivity  $\mathcal{S}$  is not a scalar function but rather attains values in  $\mathbb{R}^{2 \times 2}$ , and satisfies  $|\mathcal{S}(x, n, c)| \leq C_S(1 + n)^{-\alpha}$  with some  $C_S > 0$  and  $\alpha > 0$ . We shall establish the existence of global bounded classical solutions for arbitrarily large initial data. In contrast to the corresponding case of scalar-valued sensitivities, this system does not possess any gradient-like structure due to the appearance of such matrix-valued  $\mathcal{S}$ . To overcome this difficulty, we will derive a series of *a priori* estimates involving a new interpolation inequality.

To the best of our knowledge, this is the first result on global existence and boundedness in a Keller–Segel–Stokes system with tensor-valued sensitivity, in which production of the chemical signal is involved.

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## 1. Introduction

**Keller–Segel model.** Chemotaxis is a biological process in which cells move toward a chemically more favorable environment. For example, bacteria often swim toward higher concentrations of a signaling substance to survive. To describe chemotaxis of cell populations, the classical Keller–Segel model, consisting of two equations of the form

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(x, n, c)\nabla c), \\ c_t = \Delta c - c + n, \end{cases} \quad (1.1)$$

has been widely used and extensively studied (see [12] for a survey), where  $n$  and  $c$  denote the cell density and chemosignal concentration, respectively. The function  $S$  measures the chemotactic sensitivity, which may depend on  $n$ ,  $c$  and also on the environmental variable  $x$ . When  $S \equiv 1$ , we arrive at the standard Keller–Segel model, which has rich and interesting properties including globally existing solutions, finite time blow-up and spatial pattern formation (see e.g. [12,13]). For instance, it is known that the Neumann problem associated with (1.1) in balls  $\Omega \subset \mathbb{R}^N$  possesses some solutions blowing up in finite time when either  $N \geq 3$ , or when  $N = 2$  and the total mass of cells  $\int_{\Omega} n_0$  is large [20,36], while all solutions remain bounded when either  $N = 1$ , or  $N = 2$  and the total mass  $\int_{\Omega} n_0$  is small [21,24]. More generally, when  $S = S(n)$  is a non-constant scalar function, then its asymptotic behavior decides whether or not the explosion phenomena may occur. For instance, Horstmann and Winkler [14] shows that if  $S(s) \leq C(1+s)^{-\alpha}$  for all  $s \geq 1$  and some  $\alpha > 1 - \frac{2}{N}$  then all solutions are global and uniformly bounded, while if  $\Omega$  is a ball in  $\mathbb{R}^N$  ( $N \geq 2$ ) and  $S(s) > cs^{-\alpha}$  for some  $\alpha < 1 - \frac{2}{N}$ , then under some technical assumptions the solution may blow up. Thus

$$\alpha_c = 1 - \frac{2}{N} \quad (1.2)$$

is the critical blow-up exponent, which is related to the presence of a so-called volume-filling effect. For the more related works in this direction, we mention that a corresponding quasilinear version has been deeply investigated by [4,5,27,33].

Many variants of the standard Keller–Segel system (1.1) have been proposed in the past few years to model chemotaxis mechanisms in various biological contexts. One particular class of models is concerned with situations when the signal is consumed, rather than produced, by the cells. A correspondingly modified chemotaxis system, in its simplest form, is then given by

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\nabla c) \\ c_t = \Delta c - nc. \end{cases} \quad (1.3)$$

It is known that the Neumann initial-boundary value problem of system (1.3) in bounded convex domains possesses global bounded classical solutions for fairly arbitrary initial data in the two-dimensional case [28]. The three-dimensional version is known to admit global weak solutions which after an initial waiting time eventually become smooth and bounded [28].

**Chemotaxis coupled with fluid.** In nature, cells often live in a viscous fluid so that cells and chemical substrates are also transported through the fluid, and the motion of the fluid is influenced by gravitational forcing generated by aggregation of cells. Thus, this kind of cell-fluid interaction becomes more complicated than fluid-free case as in (1.3) since it does not only account for of chemotaxis and diffusion, but also includes transportation and viscous fluid dynamics. Considering that the motion of the fluids is determined by the incompressible (Navier-)Stokes equations, Tuval et al. [31] proposed the following chemotaxis-(Navier-)Stokes system to describe such coupled biological phenomena in the context of signal consumption by cells:

$$\begin{cases} n_t + \mathbf{u} \cdot \nabla n = \Delta n - \nabla \cdot (n \mathcal{S} \nabla c), \\ c_t + \mathbf{u} \cdot \nabla c = \Delta c - n f(c), \\ \mathbf{u}_t + \kappa (\mathbf{u} \cdot \nabla) \mathbf{u} = \Delta \mathbf{u} + \nabla P + n \nabla \phi, \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \quad (1.4)$$

Here  $\mathbf{u}$  and  $P$  denote the fluid velocity field and the associated pressure, respectively. The coefficient  $\kappa$  is related to the strength of nonlinear fluid convection. The gravitational potential  $\phi$  and the oxygen consumption rate  $f(c)$  are supposed to be given functions. From a viewpoint of mathematical analysis, this system couples the known obstacles from the theory of fluid equations to the typical difficulties arising in the study of chemotaxis systems. Despite this challenge, numerous analytical approaches in the past several years have addressed issues of well-posedness for corresponding initial-value problems in either bounded or unbounded domains, with various assumptions on the scalar functions  $\mathcal{S}$ ,  $f$  and  $\phi$  (see e.g. [2,3,7,22,35,41,44]). A considerable literature also addresses related models with nonlinear diffusion (see e.g. [6,8,18,29,30,32]). Besides these works focused on the well-posedness theory, Winkler [37] further investigated the qualitative behavior of such solutions. He obtained a result on stabilization of global solutions toward some equilibrium for such a model. It is worth noticing that the results obtained so far indicate that in contrast to the standard Keller–Segel model, phenomena of finite-time blow-up, which represents maybe the most extreme facet of bacterial aggregation, do not occur for such system involving signal consumption even though the Stokes-fluid is included.

**Chemotaxis with rotational flux.** Recently, experimental findings and corresponding modeling approaches suggest that the environment for the bacterial cells may be more complicated and further external influences have to be considered. Accordingly, chemotactic migration need not necessarily be oriented along the gradient of the chemical substance, but may rather involve rotational flux components. This requires the sensitivity function  $\mathcal{S}$  in (1.1) to be a matrix possibly containing nontrivial off-diagonal entries (see [42] and [43] for a detailed model derivation) such as appearing e.g. in the prototype

$$\mathcal{S} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad a > 0, \quad b \in \mathbb{R}, \quad (1.5)$$

in the two-dimensional case. This generalization results in considerable mathematical difficulties due to the fact that chemotaxis systems with such rotational fluxes lose some energy structure, which has served as a key to the analysis for scalar-valued  $\mathcal{S}$  (cf. also the brief discussion below). Indeed, in contrast with large bodies of existing work on the scalar sensitivity case, only very few mathematical results seem to be available on models with such tensor-valued sensitivities. Recently, Li et al. [16] investigated the following chemotaxis model with rotational sensitivity  $\mathcal{S}$ :

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\mathcal{S}(x, n, c) \cdot \nabla c), \\ c_t = \Delta c - f(x, n, c). \end{cases}$$

Under the assumption that the rotational tensor  $\mathcal{S}$  and the source term  $f$  satisfy

$$|\mathcal{S}(x, n, c)| \leq \mathcal{S}_0(c), \quad f(x, n, 0) = 0 \quad \text{and} \quad 0 \leq f(x, n, c) \leq f_0(c)(n+1)$$

for some nondecreasing functions  $\mathcal{S}_0$  and  $f_0$ , they established the existence of globally defined bounded classical solutions for small initial signal concentrations (i.e.,  $\|c_0\|_{L^\infty(\Omega)}$  small enough) by deriving a series of *a priori* estimates involving a new interpolation inequality of Gagliardo–Nirenberg-type. The smallness assumption on the initial data was removed by Winkler in [38], where certain global weak solutions are constructed for arbitrarily large initial data. Apart from this, Cao and Ishida [1] studied the existence of globally bounded weak solutions to a variant involving nonlinear degenerate diffusion.

**Main results.** In his recent preprints [39,40], Winkler gave a complete analysis for the chemotaxis-Stokes system (1.4) with rotational sensitivity in the 2D and 3D case. The existence of global bounded solutions and the asymptotic of large-data solutions are established by presenting a novel *a priori* estimation method. Motivated by the above works, we will investigate the interaction of the standard Keller–Segel chemotaxis mechanism, the Stokes-fluid and the rotational sensitivity in this paper. Precisely, we shall consider the following initial-boundary problem

$$\begin{cases} n_t + \mathbf{u} \cdot \nabla n = \Delta n - \nabla \cdot (n\mathcal{S}(x, n, c) \cdot \nabla c), & x \in \Omega, t > 0, \\ c_t + \mathbf{u} \cdot \nabla c = \Delta c - c + n, & x \in \Omega, t > 0, \\ \mathbf{u}_t + \nabla P = \Delta \mathbf{u} + n \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot \mathbf{u} = 0, & x \in \Omega, t > 0, \\ (\nabla n - n\mathcal{S}(x, n, c) \nabla c) \cdot \nu = \nabla c \cdot \nu = 0, \quad \mathbf{u} = 0, & x \in \partial\Omega, t > 0, \\ n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x), \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x), & x \in \Omega, \end{cases} \quad (1.6)$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain with smooth boundary, and  $\nu$  denotes the outward normal vector field on  $\partial\Omega$ . Let the chemotactic sensitivity tensor  $\mathcal{S} = (\mathcal{S}_{ij})_{i,j \in \{1,2\}}$  satisfy

$$\mathcal{S}_{ij} \in C^2(\bar{\Omega} \times [0, \infty) \times [0, \infty)) \quad \text{for } i, j \in \{1, 2\}. \quad (1.7)$$

In the current setting, a fundamental mathematical question is whether or not system (1.6) can be solved globally in time for large initial data. Note that when  $\mathbf{u} = \mathbf{0}$  and  $\mathcal{S} = \mathbf{I}$  is the identity matrix, system (1.6) reduces to the classical Keller–Segel system, whose solution may blow up in finite time for large initial data as mentioned before. From this observation, we see that some suitable assumptions are needed to obtain the global existence of solutions to system (1.6) with large initial data. Here, we thus assume that furthermore

$$|\mathcal{S}(x, n, c)| \leq C_S(1+n)^{-\alpha} \quad \text{for some constants } \alpha > 0 \quad \text{and} \quad C_S > 0. \quad (1.8)$$

Indeed, this kind of assumption stems from the modeling approaches involving volume-filling effect: The chemotactic movement of cells is inhibited near points where the cells are densely packed (see [25]). The assumption  $\alpha > 0$  implies that when the cell density increases, the effect of chemotaxis is weakened. We shall show that this is enough to rule out blow-up phenomena.

In order to specify the framework for our analysis, let us assume throughout that the initial data satisfy

$$\begin{cases} n_0 \in C^0(\bar{\Omega}), & n_0 \geq 0 \text{ and } n_0 \not\equiv 0 \text{ in } \bar{\Omega} \\ c_0 \in W^{1,\infty}(\Omega), & c_0 \geq 0 \text{ and } c_0 \not\equiv 0 \text{ in } \bar{\Omega} \\ \mathbf{u}_0 \in D(\mathcal{A}_r^\beta) & \text{for all } r \in (1, \infty) \text{ and some } \beta \in \left(\frac{1}{2}, 1\right), \end{cases} \quad (1.9)$$

where  $\mathcal{A}_r$  denotes the Stokes operator with domain  $D(\mathcal{A}_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L_\sigma^r(\Omega)$  with  $L_\sigma^r(\Omega) := \{\mathbf{v} \in L^r(\Omega) \mid \nabla \cdot \mathbf{v} = 0\}$  for  $r \in (1, \infty)$ .

As for the gravitational potential  $\phi$  in (1.6), we require that it is independent of time and satisfies

$$\phi \in C^2(\bar{\Omega}). \quad (1.10)$$

Under these assumptions, we can establish global existence of a classical solution to system (1.6) for general (large) data. Precisely, we have the following global existence result.

**Theorem 1.1.** *Suppose that (1.7)–(1.10) holds. Then system (1.6) possesses a unique global classical solution  $(n, c, \mathbf{u}, P)$  which enjoys the regularity properties*

$$\begin{aligned} n &\in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ c &\in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ \mathbf{u} &\in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ P &\in C^{1,0}(\bar{\Omega} \times (0, \infty)). \end{aligned}$$

Moreover, this solution is uniformly bounded in the sense that

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|\mathbf{u}(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all } t \in (0, \infty) \quad (1.11)$$

with some positive constant  $C$ .

**Remark 1.1.** Notice that our condition (1.8) is optimal and thus  $\alpha_c = 0$  is the critical blow-up exponent, which is consistent with (1.2) with  $N = 2$ . Indeed, if  $\alpha < 0$ , the corresponding fluid-free case admits finite-time blow-up solutions (see Corollary 1.4 in [5]).

Let us underline the challenge we encounter in the present setting. From the mathematical point of view, besides the complex fluid interaction being introduced to the Keller–Segel system (1.1), passing from system (1.1) to system (1.6) by allowing for more complex cross-diffusion mechanisms in (1.6) appears to bring about another considerable mathematical challenge resulting from the fact that thereby system (1.6) with rotational tensor apparently loses a favorable quasi-energy structure. Indeed, for system (1.1) with constant scalar sensitivity  $\mathcal{S}$ , the integral

$$\int_{\Omega} \left( n \log n - nc\mathcal{S} + \frac{\mathcal{S}}{2}(c^2 + |\nabla c|^2) \right)$$

plays the role of an energy functional. A corresponding gradient-like structure, along with all its consequences for the *a priori* knowledge on the regularity of solutions, apparently cannot be expected for tensor-valued  $\mathcal{S}$  in system (1.6). On the other hand, compared with the signal consumption case as in system (1.4), the quantity  $c$  of system (1.6) is no longer *a priori* bounded by its initial norm in  $L^\infty$ , which means that we have less regularity information on  $c$ .

**Main ideas and plan of the paper.** Our approach underlying the derivation of Theorem 1.1 will be based on an entropy-like estimate involving the functional

$$y(t) := \int_{\Omega} n^p + \int_{\Omega} |\nabla c|^{2q}$$

for solutions of certain regularized versions of (1.6) (see Section 2), where we eventually intend to choose  $p > 1$  and  $q > 1$  arbitrarily large. To this end, we will first perform some standard testing procedures to obtain some basic information on the time evolution of this sum (see Lemma 2.9). Then after estimating each ill-signed contribution appropriately, we can establish an ordinary differential inequality for  $y(t)$  containing an absorptive linear term, which thus implies an upper bound for  $y(t)$  (see Lemma 2.10). Its derivation will in an essential way rely on an interpolation inequality (Lemma 2.7) of Gagliardo–Nirenberg type, which differs from the inequality used in the analysis of the model with signal consumption (cf. Lemma 2.1 of [16]) in that our inequality no longer involves the  $L^\infty(\Omega)$  norm of  $c$ . We remark that the spatial dimension two plays an important role in the derivation of the interpolation inequality and the establishment of the entropy-like estimate for  $y(t)$ . The final step to gain our main result is following an approximating process (see Section 3), in which some estimates of the solution for regularized problems will be used, as well as some regularity properties of time derivatives.

To the best of our knowledge, this is the first result on global existence and boundedness in a 2D Keller–Segel–Stokes system with *tensor-valued sensitivities* of type (1.5), in which *production* of the signal is involved. It seems that the method used in this paper is not suitable for the corresponding 3D system. Indeed, we will use a different approach to establish the global boundedness for the 3D system in our forthcoming work. On the other hand, in our recent preprint [17], we have also investigated the global existence of system (1.6) with nonlinear diffusion.

The rest of this paper is organized as follows. In Section 2, we first establish the global existence of bounded solutions to system (1.6) under the assumption of  $\mathcal{S} = \mathbf{0}$  on  $\partial\Omega$ . Then we will deal with the general case by an approximation procedure in Section 3.

**Notations:** In some places of this paper, we will use  $C$  to denote generic constants, which may vary from line to line.

## 2. Global existence for $\mathcal{S} = \mathbf{0}$ on $\partial\Omega$

In this section, we shall first consider the case when besides (1.7) and (1.8), the sensitivity satisfies

$$\mathcal{S}(x, n, c) = \mathbf{0}, \quad (x, n, c) \in \partial\Omega \times [0, \infty) \times [0, \infty). \quad (2.1)$$

Under this assumption, the boundary condition for  $n$  in (1.6) actually reduces to the homogeneous Neumann condition

$$\frac{\partial n}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

This enables us to apply the well-known arguments to establish the local existence of solutions. Then, by establishing suitable *a priori* estimates, we can obtain the existence of global bounded classical solutions.

### 2.1. Local existence of classical solution

We first state the local solvability of system (1.6), which can be proved by a straightforward adaptation of the corresponding procedure in Lemma 2.1 of Winkler [35] to our current setting.

**Lemma 2.1.** *Suppose that (1.7)–(1.10) and (2.1) hold. Then there exist  $T_{\max} \in (0, \infty]$  and a classical solution  $(n, c, \mathbf{u}, P)$  of system (1.6) in  $\Omega \times (0, T_{\max})$  such that*

$$\begin{aligned} n &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ c &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ \mathbf{u} &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ P &\in C^{1,0}(\bar{\Omega} \times (0, T_{\max})). \end{aligned}$$

Moreover, we have  $n > 0$  and  $c > 0$  in  $\bar{\Omega} \times [0, T_{\max})$ , and

if  $T_{\max} < \infty$ , then  $\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|\mathcal{A}^\beta \mathbf{u}(\cdot, t)\|_{L^2(\Omega)} \rightarrow \infty$  as  $t \rightarrow T_{\max}$ ,

where  $\beta$  is taken from (1.9). This solution is unique, up to addition of constants to  $P$ .

The following lemma states the mass conservation property of  $n$ .

**Lemma 2.2.** *If (1.7)–(1.10) and (2.1) hold, then the solution of (1.6) satisfies*

$$\int_{\Omega} n(x, t) dx = \int_{\Omega} n_0(x) dx \quad \text{for all } t \in (0, T_{\max}). \quad (2.2)$$

**Proof.** The conclusion directly results from an integration of the first equation in (1.6) over  $\Omega$ .  $\square$

### 2.2. Regularity of $\mathbf{u}$

We next plan to derive some appropriate estimates for  $\mathbf{u}$  and, in particular, to obtain a statement on  $W^{1,r}$  regularity of  $\mathbf{u}$  implied by  $L^p$  regularity of  $n$ . To this end, let us first recall some facts concerning the Stokes operator.

For each  $r \in (1, \infty)$ , the Helmholtz projection acts as a bounded linear operator  $\mathcal{P}_r$  from  $L^r(\Omega)$  onto its subspace  $L_\sigma^r(\Omega) := \{\mathbf{v} \in L^r(\Omega) \mid \nabla \cdot \mathbf{v} = 0\}$  of all solenoidal vector fields. The realization  $\mathcal{A}_r$  of the Stokes operator  $\mathcal{A}$  in  $L_\sigma^r(\Omega)$  with domain  $D(\mathcal{A}_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L_\sigma^r(\Omega)$  is sectorial in  $L_\sigma^r(\Omega)$ . For any  $\beta \in \mathbb{R}$ , this operator possesses closed fractional powers

$\mathcal{A}_r^\beta$  with dense domains [9,11], and  $\mathcal{A}_r$  generates an analytic semigroup  $(e^{-t\mathcal{A}_r})_{t \geq 0}$  in  $L_\sigma^r(\Omega)$ . Since  $\mathcal{P}_r$ ,  $\mathcal{A}_r^\beta$  and  $(e^{-t\mathcal{A}_r})_{t \geq 0}$  are all actually independent of  $r \in (1, \infty)$  whenever applied to smooth functions, we will omit an explicit index  $r$  whenever there is no danger of confusion in the remaining part of this paper.

The following conclusion can be obtained by a modification of its three-dimensional version in Winkler [40] (see Lemma 3.3 there).

**Lemma 2.3.** *Suppose that  $1 \leq p < p_0 < \infty$  and  $\gamma \in (0, 1)$  are such that  $\gamma > \frac{1}{p} - \frac{1}{p_0}$ . Then there exists a positive constant  $C$  such that*

$$\|\mathcal{A}^{-\gamma} \mathcal{P}\varphi\|_{L^{p_0}(\Omega)} \leq C \|\varphi\|_{L^p(\Omega)}$$

for all  $\varphi \in C_0^\infty(\Omega)$ .

This lemma shows that, roughly speaking, up to projection to divergence-free vector fields, functions from  $L^p(\Omega)$  ( $p \geq 1$ ) can be viewed as elements of  $D(\mathcal{A}_{p_0}^{-\gamma})$  for  $p_0 > p$  and suitable  $\gamma > 0$ . Applying this estimate, we can establish some regularity estimates for  $\mathbf{u}$  and  $c$ . Indeed, as its first application, we have the following  $L^p$  estimate for  $\mathbf{u}$  (see also Lemma 3.1(i) in Winkler [39]).

**Lemma 2.4.** *Suppose that (1.7)–(1.10) and (2.1) hold. For any given  $p \in (1, \infty)$ , there exists a positive constant  $C = C(p, \mathbf{u}_0, n_0, \phi)$  such that*

$$\|\mathbf{u}(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}).$$

**Proof.** First of all, we have the variation-of-constants representation

$$\mathbf{u}(t) = e^{-t\mathcal{A}}\mathbf{u}_0 + \int_0^t e^{-(t-\tau)\mathcal{A}} \mathcal{P}(n(\cdot, \tau) \cdot \nabla \phi) d\tau \quad \text{for all } t \in (0, T_{\max}).$$

For any  $p > 1$ , we can fix  $\gamma$  such that  $\gamma \in (1 - \frac{1}{p}, 1)$ . It then follows that

$$\|\mathbf{u}(t)\|_{L^p(\Omega)} \leq \|e^{-t\mathcal{A}}\mathbf{u}_0\|_{L^p(\Omega)} + \int_0^t \|\mathcal{A}^\gamma e^{-(t-\tau)\mathcal{A}} \mathcal{A}^{-\gamma} \mathcal{P}(n(\cdot, \tau) \nabla \phi)\|_{L^p(\Omega)} d\tau \quad (2.3)$$

for all  $t \in (0, T_{\max})$ . Since  $\mathbf{u}_0 \in L_\sigma^p(\Omega)$  as a consequence of (1.9),

$$\|e^{-t\mathcal{A}}\mathbf{u}_0\|_{L^p(\Omega)} \leq C_1 \quad (2.4)$$

with some positive constant  $C_1$ . On the other hand, by the smoothing effect and decay estimates of the Stokes semigroup (see [11]), we can find a constant  $\lambda > 0$  such that for all  $\eta \geq 0$  it holds that

$$\|\mathcal{A}^\eta e^{-t\mathcal{A}}\varphi\|_{L^p(\Omega)} \leq C_2(\eta) t^{-\eta} e^{-\lambda t} \|\varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in L_\sigma^p(\Omega)$$



for some  $C_2(\eta) > 0$ , which implies that

$$\|\mathcal{A}^\gamma e^{-(t-\tau)\mathcal{A}} \mathcal{A}^{-\gamma} \mathcal{P}(n(\cdot, \tau) \nabla \phi)\|_{L^p(\Omega)} \leq C_2(\gamma) (t-\tau)^{-\gamma} e^{-\lambda(t-\tau)} \|\mathcal{A}^{-\gamma} \mathcal{P}(n(\cdot, \tau) \nabla \phi)\|_{L^p(\Omega)}. \quad (2.5)$$

The fact  $\gamma > 1 - \frac{1}{p}$  enables us to apply [Lemma 2.3](#) to obtain

$$\|\mathcal{A}^{-\gamma} \mathcal{P}(n(\cdot, \tau) \nabla \phi)\|_{L^p(\Omega)} \leq C_3 \|n \nabla \phi\|_{L^1(\Omega)} \leq C_4 \quad (2.6)$$

for some positive constants  $C_3$  and  $C_4$ . Here we also used the mass conservation [\(2.2\)](#) and the boundedness of  $\nabla \phi$ .

Thereupon, substituting [\(2.4\)](#), [\(2.5\)](#) and [\(2.6\)](#) into [\(2.3\)](#), we conclude that

$$\|\mathbf{u}(t)\|_{L^p(\Omega)} \leq C_1 + C_2 C_4 \int_0^t (t-\tau)^{-\gamma} e^{-\lambda(t-\tau)} d\tau \leq C$$

for all  $t \in (0, T_{\max})$  with some  $C > 0$  depending on  $p, u_0, n_0, \phi$ .  $\square$

Another application of [Lemma 2.3](#) is the following  $W^{1,r}$  estimate of  $\mathbf{u}$ , whose proof is similar to that of its three-dimensional version (see Corollary 3.4 in Winkler [\[40\]](#)). Considering that this result plays an important role in the subsequence of this paper, we give a sketch for the sake of completeness.

**Lemma 2.5.** *Suppose that [\(1.7\)–\(1.10\)](#) and [\(2.1\)](#) hold. Let  $p \in [1, \infty)$  and  $r \in [1, \infty]$  be such that*

$$\begin{cases} r < \frac{2p}{2-p}, & p \leq 2, \\ r \leq \infty, & p > 2. \end{cases} \quad (2.7)$$

Then for all  $K > 0$  there exists  $C = C(p, r, K, \mathbf{u}_0, \phi)$  such that if for some  $T > 0$  we have

$$\|n(\cdot, t)\|_{L^p(\Omega)} \leq K \quad \text{for all } t \in (0, T), \quad (2.8)$$

then

$$\|D\mathbf{u}(\cdot, t)\|_{L^r(\Omega)} \leq C \quad \text{for all } t \in (0, T).$$

**Proof.** The case  $r \leq p$  can be proved by taking the procedure as the proof of [Lemma 2.4](#). Thus we only need to prove the case  $r > p$ . For all  $\beta > \frac{1}{2}$  given by [\(1.9\)](#), we fix  $r_0$  such that  $r_0 \in (p, r)$  and

$$\beta > \frac{1}{2} + \frac{1}{r_0} - \frac{1}{r}.$$

Noticing that

$$\left(\frac{1}{2} + \frac{1}{r_0} - \frac{1}{r}\right) - \left(1 - \frac{1}{p} + \frac{1}{r_0}\right) = -\frac{1}{2} + \frac{1}{p} - \frac{1}{r} < 0$$

by (2.7), we can choose  $\beta_0 \in (\frac{1}{2}, \beta)$  fulfilling

$$\frac{1}{2} + \frac{1}{r_0} - \frac{1}{r} < \beta_0 < 1 - \frac{1}{p} + \frac{1}{r_0}.$$

Then we can pick  $\delta \in (0, 1)$  small enough such that

$$\beta_0 + \delta < 1 - \frac{1}{p} + \frac{1}{r_0}, \quad (2.9)$$

and finally fix some  $p_0 > p$  sufficiently close to  $p$  to satisfy

$$\delta > \frac{1}{p} - \frac{1}{p_0}. \quad (2.10)$$

The fact  $\beta_0 > \frac{1}{2} + \frac{1}{r_0} - \frac{1}{r}$  ensures that the embedding  $D(\mathcal{A}_{r_0}^{\beta_0}) \hookrightarrow W^{1,r}(\Omega)$  holds. Thus we can apply  $\mathcal{A}^{\beta_0}$  to both sides of the variation-of-constants representation of  $\mathbf{u}$  to obtain

$$\begin{aligned} \|D\mathbf{u}(\cdot, t)\|_{L^r(\Omega)} &\leq C_1 \|\mathcal{A}^{\beta_0} \mathbf{u}(\cdot, t)\|_{L^{r_0}(\Omega)} \\ &\leq C_1 \|\mathcal{A}^{\beta_0} e^{-t\mathcal{A}} \mathbf{u}_0\|_{L^{r_0}(\Omega)} + C_1 \int_0^t \|\mathcal{A}^{\beta_0+\delta} e^{-(t-\tau)\mathcal{A}} \mathcal{A}^{-\delta} \mathcal{P}(n(\cdot, \tau) \nabla \phi)\|_{L^{r_0}(\Omega)} d\tau \end{aligned}$$

for all  $t \in (0, T)$  and for some  $C_1 > 0$ . It follows from  $\beta_0 < \beta$  and  $\mathbf{u}_0 \in D(\mathcal{A}_r^\beta)$  that

$$\|\mathcal{A}^{\beta_0} e^{-t\mathcal{A}} \mathbf{u}_0\|_{L^{r_0}(\Omega)} = \|e^{-t\mathcal{A}} \mathcal{A}^{\beta_0} \mathbf{u}_0\|_{L^{r_0}(\Omega)} \leq C_2$$

with some  $C_2 > 0$ . On the other hand, thanks to Lemma 2.3 with  $\gamma = \delta$  satisfying (2.10) and the boundedness of  $\nabla \phi$ , we can achieve from (2.8) that

$$\|\mathcal{A}^{-\delta} \mathcal{P}(n(\cdot, \tau) \nabla \phi)\|_{L^{p_0}(\Omega)} \leq C_3 \|n \nabla \phi\|_{L^p(\Omega)} \leq C_4 K \quad \text{for all } \tau \in (0, T)$$

and thus

$$\begin{aligned} &\|\mathcal{A}^{\beta_0+\delta} e^{-(t-\tau)\mathcal{A}} \mathcal{A}^{-\delta} \mathcal{P}(n(\cdot, \tau) \nabla \phi)\|_{L^{r_0}(\Omega)} \\ &\leq C_5 (t - \tau)^{-\beta_0 - \delta - (\frac{1}{p_0} - \frac{1}{r_0})} e^{-\lambda(t-\tau)} \|\mathcal{A}^{-\delta} \mathcal{P}(n(\cdot, \tau) \nabla \phi)\|_{L^{p_0}(\Omega)} \\ &\leq C_4 C_5 K (t - \tau)^{-\beta_0 - \delta - (\frac{1}{p_0} - \frac{1}{r_0})} e^{-\lambda(t-\tau)} \end{aligned}$$

for some positive constant  $C_3$ ,  $C_4$  and  $C_5$  from the property of Stokes semigroup. Noticing that (2.9) implies that  $\int_0^\infty \tau^{-\beta_0 - \delta - (\frac{1}{p_0} - \frac{1}{r_0})} e^{-\lambda\tau} d\tau$  is finite, we can deduce the desired result.  $\square$

### 2.3. An estimate for $c$

We next give an  $L^p$  estimate for  $c$  by using the regularity information on  $n$  and  $\mathbf{u}$  obtained so far.

**Lemma 2.6.** *Suppose that (1.7)–(1.10) and (2.1) hold. For all  $p \in [1, \infty)$ , there exists a positive constant  $C$  depending on  $p$ ,  $n_0$ ,  $c_0$  and  $\mathbf{u}_0$  such that*

$$\|c(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}).$$

**Proof.** An integration of the second equation in (1.6) shows that  $\|c(\cdot, t)\|_{L^1(\Omega)}$  is bounded by some positive constant  $C_1 = \max\{\int_{\Omega} c_0, \int_{\Omega} n_0\}$  for all  $t \in (0, T_{\max})$ .

Next we consider the case of  $p > 1$ . The second equation in (1.6) shows that

$$\begin{aligned} c(\cdot, t) &= e^{t(\Delta-1)}c_0 + \int_0^t e^{(t-\tau)(\Delta-1)}n(\cdot, \tau)d\tau - \int_0^t e^{(t-\tau)(\Delta-1)}(\mathbf{u} \cdot \nabla c)(\cdot, \tau)d\tau \\ &:= c_1(\cdot, t) + c_2(\cdot, t) + c_3(\cdot, t) \end{aligned}$$

for  $t \in (0, T_{\max})$ , where  $(e^{t\Delta})_{t \geq 0}$  denotes the Neumann heat semigroup in  $\Omega$ .

According to standard smoothing estimates (see Lemma 1.3 in Winkler [34]), we can find a positive constant  $C_2$  such that

$$\|c_1(\cdot, t)\|_{L^p(\Omega)} = \|e^{t(\Delta-1)}c_0\|_{L^p(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \leq C_2 \quad (2.11)$$

for all  $t \in (0, T_{\max})$ , and

$$\begin{aligned} \|c_2(\cdot, t)\|_{L^p(\Omega)} &\leq \int_0^t \|e^{(t-\tau)(\Delta-1)}n(\cdot, \tau)\|_{L^p(\Omega)}d\tau \\ &= \int_0^t e^{-(t-\tau)} \|e^{(t-\tau)\Delta}(n(\cdot, \tau) - \bar{n}(\cdot, \tau))\|_{L^p(\Omega)}d\tau \\ &\quad + \int_0^t e^{-(t-\tau)} \|e^{(t-\tau)\Delta}\bar{n}(\cdot, \tau)\|_{L^p(\Omega)}d\tau \\ &\leq C_3 \int_0^t e^{-(t-\tau)} \left(1 + (t-\tau)^{-(1-\frac{1}{p})}\right) \|n(\cdot, \tau)\|_{L^1(\Omega)}d\tau \\ &\leq C_4 \end{aligned} \quad (2.12)$$

for all  $t \in (0, T_{\max})$  with some positive constants  $C_3$  and  $C_4$ , where  $\bar{n} = \frac{1}{|\Omega|} \int_{\Omega} n dx$ . Here we used the mass conservation (2.2) in the last inequality.

We fix a  $\sigma$  such that  $\max\{\frac{2p}{p+2}, 1\} < \sigma < p$ . Then there exists a positive constant  $C_5$  such that

$$\begin{aligned} \|c_3(\cdot, t)\|_{L^p(\Omega)} &\leq \int_0^t \|e^{(t-\tau)(\Delta-1)} \nabla \cdot (\mathbf{u}c)(\cdot, \tau)\|_{L^p(\Omega)} d\tau \\ &\leq C_5 \int_0^t \left(1 + (t-\tau)^{-\frac{1}{2} - (\frac{1}{\sigma} - \frac{1}{p})}\right) e^{-(\lambda_1+1)(t-\tau)} \|(\mathbf{u}c)(\cdot, \tau)\|_{L^\sigma(\Omega)} d\tau, \end{aligned} \quad (2.13)$$

where  $\lambda_1 > 0$  denotes the first nonzero eigenvalue of  $-\Delta$  in  $\Omega$  under the homogeneous Neumann boundary conditions. Choosing  $s > 1$  such that  $1 < s\sigma < p$ , we can see from Hölder's inequality and Lemma 2.4 that

$$\|\mathbf{u}(\cdot, t)c(\cdot, t)\|_{L^\sigma(\Omega)} \leq \|\mathbf{u}(\cdot, t)\|_{L^{\frac{s\sigma}{s-1}}(\Omega)} \|c(\cdot, t)\|_{L^{s\sigma}(\Omega)} \leq C_6 \|c(\cdot, t)\|_{L^{s\sigma}(\Omega)} \quad (2.14)$$

for all  $t \in (0, T_{\max})$  with some positive constant  $C_6$  depending on  $s, \sigma, \mathbf{u}_0, n_0, \phi$ .

Combining (2.11), (2.12), (2.13) and (2.14), one conclude that

$$\|c(\cdot, t)\|_{L^p(\Omega)} \leq C_2 + C_4 + C_5 C_6 \int_0^t \left(1 + (t-\tau)^{-\frac{1}{2} - (\frac{1}{\sigma} - \frac{1}{p})}\right) e^{-(\lambda_1+1)(t-\tau)} \|c(\cdot, \tau)\|_{L^{s\sigma}(\Omega)} d\tau \quad (2.15)$$

for all  $t \in (0, T_{\max})$ . By  $1 < s\sigma < p$ , we have

$$\|c(\cdot, t)\|_{L^{s\sigma}(\Omega)} \leq \|c(\cdot, t)\|_{L^p(\Omega)}^a \|c(\cdot, t)\|_{L^1(\Omega)}^{1-a},$$

where  $a \in (0, 1)$  satisfies that  $\frac{1}{s\sigma} = \frac{a}{p} + (1-a)$ . Thus we have from (2.15) that

$$\begin{aligned} \|c(\cdot, t)\|_{L^p(\Omega)} &\leq C_2 + C_4 \\ &\quad + C_1^{1-a} C_5 C_6 \int_0^t \left(1 + (t-\tau)^{-\frac{1}{2} - (\frac{1}{\sigma} - \frac{1}{p})}\right) e^{-(\lambda_1+1)(t-\tau)} \|c(\cdot, \tau)\|_{L^p(\Omega)}^a d\tau \end{aligned}$$

for all  $t \in (0, T_{\max})$ .

By setting  $M(T) := \sup_{t \in (0, T)} \|c(\cdot, t)\|_{L^p(\Omega)}$  for all  $T \in (0, T_{\max})$ , we conclude that

$$M(T) \leq C_7 + C_7 M^a(T) \quad (2.16)$$

for some positive constant  $C_7$ , which is independent on  $T$ . Here we used the fact  $\sigma > \frac{2p}{p+2}$ . An application of Young's inequality to (2.16) immediately yields the existence of  $T$ -independent positive constant  $C$  satisfying

$$\sup_{t \in (0, T_{\max})} \|c(\cdot, t)\|_{L^p(\Omega)} \leq M(T) \leq C,$$

which implies the desired conclusion.  $\square$

#### 2.4. A coupled entropy estimate

We next plan to give a new interpolation inequality, which may be independent interesting and will help us to make efficient use of the boundedness of  $c$  stated in [Lemma 2.6](#) to the later estimates.

**Lemma 2.7.** Suppose that  $\Omega \subset \mathbb{R}^2$  is a bounded domain with smooth boundary. Let  $q > 1$ ,  $\gamma > 1$ ,  $\varrho \geq \frac{2\gamma}{2\gamma-1}(q+1)$ . Then there exists a constant  $C = C(q, \gamma, \varrho) > 0$  such that for any  $c \in C^2(\bar{\Omega})$  satisfying  $c \frac{\partial c}{\partial \nu} = 0$  on  $\partial\Omega$ , the inequality

$$\|\nabla c\|_{L^{\varrho}(\Omega)}^{\varrho} \leq C \left\| |\nabla c|^{q-1} D^2 c \right\|_{L^2(\Omega)}^{\frac{(2\gamma-1)\varrho-2\gamma}{q(2\gamma-1)}} \left( \|c\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}^{\frac{2\gamma}{2\gamma-1}} + \|c\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}^{\frac{\varrho}{q+1}} \right) + C \|c\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}^{\frac{\varrho}{q+1}} \quad (2.17)$$

holds, where  $D^2 c$  denotes the Hessian of  $c$ .

**Proof.** Since  $c \frac{\partial c}{\partial \nu} = 0$  on  $\partial\Omega$ , an integration by parts yields

$$\int_{\Omega} |\nabla c|^{\varrho} = -\frac{\varrho-2}{q} \int_{\Omega} c |\nabla c|^{\varrho-q-2} \nabla c \cdot \nabla |\nabla c|^q - \int_{\Omega} c |\nabla c|^{\varrho-2} \Delta c. \quad (2.18)$$

By using Hölder's inequality twice, we have

$$\begin{aligned} \left| \int_{\Omega} c |\nabla c|^{\varrho-q-2} \nabla c \cdot \nabla |\nabla c|^q \right| &\leq \left( \int_{\Omega} |\nabla |\nabla c|^q|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} c^2 |\nabla c|^{2(\varrho-q-1)} \right)^{\frac{1}{2}} \\ &\leq \|\nabla |\nabla c|^q\|_{L^2(\Omega)} \|c\|_{L^{2\gamma'}(\Omega)} \|\nabla c|^{\varrho-q-1}\|_{L^{2\gamma}(\Omega)} \end{aligned} \quad (2.19)$$

for any  $\gamma > 1$  and  $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$ . Similarly, we have

$$\begin{aligned} \left| \int_{\Omega} c |\nabla c|^{\varrho-2} \Delta c \right| &\leq \int_{\Omega} |\nabla c|^{q-1} \Delta c \cdot |c| \cdot |\nabla c|^{\varrho-q-1} \\ &\leq \|\nabla c|^{q-1} \Delta c\|_{L^2(\Omega)} \|c\|_{L^{2\gamma'}(\Omega)} \|\nabla c|^{\varrho-q-1}\|_{L^{2\gamma}(\Omega)}. \end{aligned} \quad (2.20)$$

Substituting (2.19) and (2.20) into (2.18), we obtain

$$\begin{aligned} \|\nabla c\|_{L^{\varrho}(\Omega)}^{\varrho} &\leq C_1 \|c\|_{L^{2\gamma'}(\Omega)} \|\nabla c|^{\varrho-q-1}\|_{L^{2\gamma}(\Omega)} \left( \|\nabla |\nabla c|^q\|_{L^2(\Omega)} + \|\nabla c|^{q-1} \Delta c\|_{L^2(\Omega)} \right) \\ &=: C_1 \|c\|_{L^{2\gamma'}(\Omega)} \|\nabla c|^{\varrho-q-1}\|_{L^{2\gamma}(\Omega)} I \end{aligned} \quad (2.21)$$

with some positive constant  $C_1$ .

If  $\varrho > \frac{2\gamma}{2\gamma-1}(q+1)$ , we can invoke the Gagliardo–Nirenberg inequality to find a positive constant  $C_2$  such that

$$\begin{aligned} \|\nabla c\|^{\varrho-q-1}_{L^{2\gamma}(\Omega)} &= \|\nabla c\|^q \left\| \frac{\varrho-q-1}{L^{\frac{2\gamma(\varrho-q-1)}{q}}(\Omega)} \right\| \\ &\leq C_2 \left( \|\nabla|\nabla c|^q\|_{L^2(\Omega)}^b \cdot \|\nabla c\|^q \left\| \frac{1-b}{L^{\frac{\varrho}{q}}(\Omega)} \right\| + \|\nabla c\|^q \left\| \frac{\varrho}{L^{\frac{\varrho}{q}}(\Omega)} \right\| \right)^{\frac{\varrho-q-1}{q}}, \end{aligned}$$

where

$$b = \frac{(2\gamma-1)\varrho-2\gamma(q+1)}{2\gamma(\varrho-q-1)} \in (0, 1).$$

It then follows from Young's inequality that

$$\begin{aligned} \|\nabla c\|^{\varrho-q-1}_{L^{2\gamma}(\Omega)} &\leq C_3 \left( \|\nabla|\nabla c|^q\|_{L^2(\Omega)}^{\frac{(2\gamma-1)\varrho-2\gamma(q+1)}{2\gamma q}} \|\nabla c\|^q \left\| \frac{\varrho}{L^{\frac{\varrho}{q}}(\Omega)} \right\|^{\frac{2\gamma q}{q}} + \|\nabla c\|^q \left\| \frac{\varrho-q-1}{L^{\frac{\varrho}{q}}(\Omega)} \right\|^{\frac{2\gamma q}{q}} \right) \\ &\leq C_3 \left( I^{\frac{(2\gamma-1)\varrho-2\gamma(q+1)}{2\gamma q}} \|\nabla c\|_{L^{\varrho}(\Omega)}^{\frac{\varrho}{2\gamma}} + \|\nabla c\|_{L^{\varrho}(\Omega)}^{\varrho-q-1} \right) \end{aligned} \quad (2.22)$$

for some  $C_3 > 0$ . Substituting (2.22) into (2.21), we can achieve that

$$\|\nabla c\|_{L^{\varrho}(\Omega)}^{\varrho} \leq C_4 \|c\|_{L^{2\gamma'}(\Omega)} I^{\frac{(2\gamma-1)\varrho-2\gamma}{2\gamma q}} \|\nabla c\|_{L^{\varrho}(\Omega)}^{\frac{\varrho}{2\gamma}} + C_4 \|c\|_{L^{2\gamma'}(\Omega)} I \|\nabla c\|_{L^{\varrho}(\Omega)}^{\varrho-q-1}$$

with  $C_4 = C_1 C_3$ . Upon applying Young's inequality to the two terms on the right hand, respectively, we find the existence of a positive constant  $C_5$  fulfilling

$$\|\nabla c\|_{L^{\varrho}(\Omega)}^{\varrho} \leq C_5 \|c\|_{L^{2\gamma'}(\Omega)}^{\frac{2\gamma}{2\gamma-1}} I^{\frac{(2\gamma-1)\varrho-2\gamma}{(2\gamma-1)q}} + C_5 \|c\|_{L^{2\gamma'}(\Omega)}^{\frac{\varrho}{q+1}} I^{\frac{\varrho}{q+1}}. \quad (2.23)$$

Since

$$|\nabla|\nabla c|^q| = q|\nabla c|^{q-2}|D^2 c| |\nabla c| \leq q|\nabla c|^{q-1}|D^2 c|$$

and

$$|\Delta c| \leq \sqrt{2}|D^2 c|,$$

there exists  $C_6 > 0$  such that

$$I \leq C_6 \|\nabla c\|^{q-1} |D^2 c|_{L^2(\Omega)}. \quad (2.24)$$

Combining this with (2.23), we arrive at

$$\|\nabla c\|_{L^{\varrho}(\Omega)}^{\varrho} \leq C_7 \|c\|_{L^{2\gamma'}(\Omega)}^{\frac{2\gamma}{2\gamma-1}} \|\nabla c\|^{q-1} |D^2 c|_{L^2(\Omega)}^{\frac{(2\gamma-1)\varrho-2\gamma}{(2\gamma-1)q}} + C_7 \|c\|_{L^{2\gamma'}(\Omega)}^{\frac{\varrho}{q+1}} \|\nabla c\|^{q-1} |D^2 c|_{L^2(\Omega)}^{\frac{\varrho}{q+1}}$$

for some positive constant  $C_7$ . The fact  $\varrho > \frac{2\gamma}{2\gamma-1}(q+1)$  implies  $\frac{(2\gamma-1)\varrho-2\gamma}{(2\gamma-1)q} > \frac{\varrho}{q+1}$ , which enables us to use Young's inequality to the rightmost term to obtain (2.17).

If  $\varrho = \frac{2\gamma}{2\gamma-1}(q+1)$ , we can easily see that

$$\| |\nabla c|^{\varrho-q-1} \|_{L^{2\gamma}(\Omega)} = \| \nabla c \|_{L^{\varrho}(\Omega)}^{\varrho-q-1}.$$

Thus by (2.21) and (2.24), we can assert the existence of some positive constant  $C_8$  such that

$$\| \nabla c \|_{L^{\varrho}(\Omega)}^{\varrho} \leq C_8 \| c \|_{L^{2\gamma'}(\Omega)}^{\frac{\varrho}{q+1}} I^{\frac{\varrho}{q+1}} \leq C_8 C_6^{\frac{\varrho}{q+1}} \| c \|_{L^{2\gamma'}(\Omega)}^{\frac{\varrho}{q+1}} \| |\nabla c|^{q-1} D^2 c \|_{L^2(\Omega)}^{\frac{\varrho}{q+1}},$$

which implies the desired (2.17) by  $\varrho = \frac{2\gamma}{2\gamma-1}(q+1)$ .  $\square$

To give an estimate for arbitrarily high Lebesgue norm of  $u$  and  $\nabla c$ , we first need to make sure that we can choose certain parameters appropriately to be used later.

**Lemma 2.8.** *For any  $\alpha > 0$ ,  $\bar{p} \geq 1$  and  $\bar{q} \geq 2$ , there exist numbers  $p \geq \bar{p}$ ,  $q \geq \bar{q}$ ,  $\zeta > 1$ ,  $\theta > 1$  and  $\mu > 1$  such that*

$$p - 2\alpha > 1 - \frac{1}{\theta}, \quad (2.25)$$

$$\theta \geq \frac{\zeta}{2\zeta-1}(q+1), \quad (2.26)$$

$$(q-1)\mu \geq \frac{\zeta}{2\zeta-1}(q+1), \quad (2.27)$$

$$1 - \frac{2\alpha}{p} - \frac{1}{p} \left(1 - \frac{1}{\theta}\right) + \frac{(2\zeta-1)\theta - \zeta}{q(2\zeta-1)\theta} < 1, \quad (2.28)$$

$$\frac{2}{p} - \frac{1}{p} \left(1 - \frac{1}{\mu}\right) + \frac{(2\zeta-1)(q-1)\mu - \zeta}{q(2\zeta-1)\mu} < 1. \quad (2.29)$$

**Proof.** Let  $\alpha > 0$ . It is sufficient to prove that there exist numbers  $p \geq \bar{p}$ ,  $q \geq \bar{q}$ ,  $\frac{1}{2} < s < 1$ ,  $\theta > 1$  and  $\mu > 1$  such that

$$p - 2\alpha > 1 - \frac{1}{\theta}, \quad (2.30)$$

$$\theta \geq s(q+1), \quad (2.31)$$

$$(q-1)\mu \geq s(q+1), \quad (2.32)$$

$$\frac{1}{p} \left( \frac{1}{\theta} - 2\alpha - 1 \right) + \frac{1}{q} - \frac{s}{q\theta} < 0, \quad (2.33)$$

$$\frac{1}{p} \left( 1 + \frac{1}{\mu} \right) - \frac{1}{q} - \frac{s}{q\mu} < 0. \quad (2.34)$$

To this end, we first fix  $q > \max\{\bar{q}, \frac{1}{6\alpha} - 1, \frac{\bar{p} + \frac{4}{3}}{2\alpha+1} - 1\}$  and

$$\mu > \max \left\{ \frac{1}{4\alpha+1 - \frac{8}{3(q+1)}}, \frac{3(q+1)^2}{4(q-1)}, \frac{(q+1)(\frac{1}{4} - \frac{3}{2}\alpha)}{(2\alpha+1)(q+1) - \frac{4}{3} - q} \right\}.$$

Then we take  $\theta := \frac{3}{4}(q+1)$ . Notice that  $\mu > \frac{1}{4\alpha+1-\frac{8}{3(q+1)}}$  is equivalent to

$$\frac{1}{2}\left(\frac{1}{\mu} + 1\right) < 2\alpha + 1 - \frac{1}{\theta}, \quad (2.35)$$

while  $\mu > \frac{(q+1)(\frac{1}{4}-\frac{3}{2}\alpha)}{(2\alpha+1)(q+1)-\frac{4}{3}-q}$  is equivalent to

$$\frac{q(\mu+1)}{\frac{3}{4}+\mu} < (2\alpha+1)(q+1) - \frac{4}{3}. \quad (2.36)$$

Thus, inequalities (2.35), (2.36) and the choice of  $q$  enable us to find  $p$  satisfying

$$\max\left\{\frac{1}{2}\left(\frac{1}{\mu} + 1\right)(q+1), 2\alpha + 1 - \frac{4}{3(q+1)}, \frac{q(\mu+1)}{\frac{3}{4}+\mu}, \bar{p}\right\} < p < \left(2\alpha + 1 - \frac{1}{\theta}\right)(q+1), \quad (2.37)$$

which gives (2.30). Moreover, for such  $p$ , we have  $\frac{1}{2}(\frac{1}{\mu} + 1) < \frac{p}{q+1} < 2\alpha + 1 - \frac{1}{\theta}$ , which implies that

$$\left[\frac{q}{p}\left(\frac{1}{\theta} - 2\alpha - 1\right) + 1\right]\theta < \frac{1}{q+1}\theta \quad (2.38)$$

and

$$\left[\frac{q}{p}\left(1 + \frac{1}{\mu}\right) - 1\right]\mu < \frac{q-1}{q+1}\mu. \quad (2.39)$$

From (2.37), we also have

$$\left[\frac{q}{p}\left(1 + \frac{1}{\mu}\right) - 1\right]\mu < \frac{3}{4} = \frac{1}{q+1}\theta \quad (2.40)$$

by the choice of  $\theta$ , and

$$\left[\frac{q}{p}\left(\frac{1}{\theta} - 2\alpha - 1\right) + 1\right]\theta < \frac{q-1}{q+1}\mu \quad (2.41)$$

because of  $\mu > \frac{3(q+1)^2}{4(q-1)}$ . It follows from the fact  $\mu > \frac{3(q+1)^2}{4(q-1)} > \frac{q+1}{2(q-1)}$  that

$$\frac{q-1}{q+1}\mu > \frac{1}{2}. \quad (2.42)$$

Thus, the estimates (2.38)–(2.42) make it possible to choose  $s \in (\frac{1}{2}, 1)$  such that

$$\left[\frac{q}{p}\left(\frac{1}{\theta} - 2\alpha - 1\right) + 1\right]\theta < s \leq \frac{1}{q+1}\theta \quad (2.43)$$



and

$$\left[ \frac{q}{p} \left( 1 + \frac{1}{\mu} \right) - 1 \right] \mu < s \leq \frac{q-1}{q+1} \mu. \quad (2.44)$$

Then (2.43) gives (2.31) and (2.33), while (2.44) gives (2.32) and (2.34). This completes the proof of Lemma 2.8.  $\square$

Next we shall devote our attention to establish the time evolution for the sum of  $\int_{\Omega} n^p$  and  $\int_{\Omega} |\nabla c|^{2q}$ .

**Lemma 2.9.** Suppose that (1.7)–(1.10) and (2.1) hold. Then for any  $p \in (1, \infty)$  and  $q \in (1, \infty)$ , it holds that

$$\begin{aligned} & \frac{d}{dt} \left( \int_{\Omega} n^p + \int_{\Omega} |\nabla c|^{2q} \right) + \frac{3(p-1)}{p} \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 + \frac{q}{2} \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \\ & \leq p(p-1) C_S^2 \int_{\Omega} n^{p-2\alpha} |\nabla c|^2 + 2q^2 \int_{\Omega} n^2 |\nabla c|^{2q-2} + 2q \int_{\Omega} |\nabla c|^{2q} |D\mathbf{u}| + C \quad \text{on } (0, T_{\max}) \end{aligned}$$

with some positive constant  $C = C(q)$ , where  $C_S$  is defined by (1.8).

**Proof.** Multiplying the first equation in (1.6) by  $n^{p-1}$  and integrating by parts over  $\Omega$ , we obtain that

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} n^p + (p-1) \int_{\Omega} n^{p-2} |\nabla n|^2 \\ & = (p-1) \int_{\Omega} n^{p-1} \nabla n \cdot (\mathcal{S}(x, n, c) \nabla c) \\ & \leq \frac{(p-1)}{4} \int_{\Omega} n^{p-2} |\nabla n|^2 + (p-1) C_S^2 \int_{\Omega} n^{p-2\alpha} |\nabla c|^2 \end{aligned} \quad (2.45)$$

for all  $t \in (0, T_{\max})$ , which implies that

$$\frac{d}{dt} \int_{\Omega} n^p + \frac{3(p-1)}{p} \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 \leq p(p-1) C_S^2 \int_{\Omega} n^{p-2\alpha} |\nabla c|^2 \quad (2.46)$$

for all  $t \in (0, T_{\max})$ .

We next differentiate the second equation in (1.6) to achieve the point-wise identity

$$(|\nabla c|^2)_t = 2\nabla c \cdot \nabla \Delta c - 2|\nabla c|^2 + 2\nabla c \cdot \nabla n - 2\nabla c \cdot \nabla(\mathbf{u} \cdot \nabla c),$$

which together with the identity  $\Delta |\nabla c|^2 = 2\nabla c \cdot \nabla \Delta c + 2|D^2 c|^2$  gives that

$$(|\nabla c|^2)_t = \Delta |\nabla c|^2 - 2|D^2 c|^2 - 2|\nabla c|^2 + 2\nabla c \cdot \nabla n - 2\nabla c \cdot \nabla(\mathbf{u} \cdot \nabla c)$$

for all  $x \in \Omega$  and  $t \in (0, T_{\max})$ . Testing this against  $|\nabla c|^{2q-2}$  yields

$$\begin{aligned} & \frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla c|^{2q} \\ &= \int_{\partial\Omega} \frac{\partial |\nabla c|^2}{\partial \nu} |\nabla c|^{2q-2} - (q-1) \int_{\Omega} |\nabla c|^{2q-4} |\nabla |\nabla c|^2|^2 - 2 \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \\ & \quad - 2 \int_{\Omega} |\nabla c|^{2q} + 2 \int_{\Omega} |\nabla c|^{2q-2} \nabla c \cdot \nabla n - 2 \int_{\Omega} |\nabla c|^{2q-2} \nabla c \cdot \nabla(\mathbf{u} \cdot \nabla c) \end{aligned} \quad (2.47)$$

for all  $t \in (0, T_{\max})$ . It follows from the integration by parts and Young's inequality that

$$\begin{aligned} & 2 \int_{\Omega} |\nabla c|^{2q-2} \nabla c \cdot \nabla n \\ &= -2(q-1) \int_{\Omega} n |\nabla c|^{2q-4} \nabla |\nabla c|^2 \cdot \nabla c - 2 \int_{\Omega} n |\nabla c|^{2q-2} \Delta c \\ &\leq \frac{q-1}{2} \int_{\Omega} |\nabla c|^{2q-4} |\nabla |\nabla c|^2|^2 + 2(q-1) \int_{\Omega} n^2 |\nabla c|^{2q-2} \\ & \quad + \frac{1}{2} \int_{\Omega} |\nabla c|^{2q-2} |\Delta c|^2 + 2 \int_{\Omega} n^2 |\nabla c|^{2q-2} \\ &\leq \frac{q-1}{2} \int_{\Omega} |\nabla c|^{2q-4} |\nabla |\nabla c|^2|^2 + 2q \int_{\Omega} n^2 |\nabla c|^{2q-2} + \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2. \end{aligned} \quad (2.48)$$

As for the last integral in (2.47), a direct calculation yields that

$$- \int_{\Omega} |\nabla c|^{2q-2} \nabla c \cdot \nabla(\mathbf{u} \cdot \nabla c) = - \int_{\Omega} |\nabla c|^{2q-2} \nabla c \cdot (\nabla c \cdot D\mathbf{u}) - \int_{\Omega} |\nabla c|^{2q-2} \nabla c \cdot (D^2 c \cdot \mathbf{u})$$

for all  $t \in (0, T_{\max})$ . Since

$$- \int_{\Omega} |\nabla c|^{2q-2} \nabla c \cdot (D^2 c \cdot \mathbf{u}) = - \frac{1}{2q} \int_{\Omega} \nabla |\nabla c|^{2q} \cdot \mathbf{u} = \frac{1}{2q} \int_{\Omega} (\nabla \cdot \mathbf{u}) |\nabla c|^{2q} = 0$$

by  $|\nabla c|^{2q-2} \nabla c \cdot (D^2 c \cdot \mathbf{u}) = \frac{1}{2q} \nabla |\nabla c|^{2q} \cdot \mathbf{u}$ , we have

$$- \int_{\Omega} |\nabla c|^{2q-2} \nabla c \cdot \nabla(\mathbf{u} \cdot \nabla c) = - \int_{\Omega} |\nabla c|^{2q-2} \nabla c \cdot (\nabla c \cdot D\mathbf{u}) \leq \int_{\Omega} |\nabla c|^{2q} \cdot |D\mathbf{u}| \quad (2.49)$$

for all  $t \in (0, T_{\max})$ .

Substituting (2.48) and (2.49) into (2.47), we can infer that

$$\begin{aligned} & \frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla c|^{2q} + \frac{q-1}{2} \int_{\Omega} |\nabla c|^{2q-4} |\nabla |\nabla c|^2|^2 + \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \\ & \leq \int_{\Omega} \frac{\partial |\nabla c|^2}{\partial \nu} |\nabla c|^{2q-2} + 2q \int_{\Omega} n^2 |\nabla c|^{2q-2} + 2 \int_{\Omega} |\nabla c|^{2q} |D\mathbf{u}|. \end{aligned} \quad (2.50)$$

On the other hand, we know from (3.7) and (3.8) of [15] that there exists  $C_1 > 0$  which depends only on  $\Omega$  such that

$$\int_{\partial\Omega} \frac{\partial |\nabla c|^2}{\partial \nu} |\nabla c|^{2q-2} \leq C_1 \left\| |\nabla c|^q \right\|_{W^{r+\frac{1}{2},2}(\Omega)}^2$$

for any  $r \in (0, \frac{1}{2})$ . The Gagliardo–Nirenberg inequality implies that

$$\left\| |\nabla c|^q \right\|_{W^{r+\frac{1}{2},2}(\Omega)} \leq C_2 \left\| \nabla |\nabla c|^q \right\|_{L^2}^a \cdot \left\| |\nabla c|^q \right\|_{L^{\frac{3(q+1)}{2q}}(\Omega)}^{1-a} + C_2 \left\| |\nabla c|^q \right\|_{L^{\frac{3(q+1)}{2q}}(\Omega)} \quad (2.51)$$

for some  $C_2 > 0$  and  $a \in (0, 1)$ . We notice that by Lemma 2.6 and Lemma 2.7, there exists a positive constant  $C_3$  such that

$$\left\| \nabla c \right\|_{L^{\varrho}(\Omega)}^{\varrho} \leq C_3 \left\| |\nabla c|^{q-1} D^2 c \right\|_{L^2(\Omega)}^{\frac{(2\gamma-1)\varrho-2\gamma}{q(2\gamma-1)}} + C_3 \quad (2.52)$$

for any  $\gamma > 1$  and  $\varrho \geq \frac{2\gamma}{2\gamma-1}(q+1)$ . In particular, we can take  $\gamma = \frac{3}{2}$ ,  $\varrho = \frac{2\gamma}{2\gamma-1}(q+1) = \frac{3(q+1)}{2}$  in (2.52) and then substitute the corresponding inequality into (2.51) to obtain that

$$\begin{aligned} \left\| |\nabla c|^q \right\|_{W^{r+\frac{1}{2},2}(\Omega)} & \leq C_4 \left\| \nabla |\nabla c|^q \right\|_{L^2}^a \cdot \left\| |\nabla c|^{q-1} D^2 c \right\|_{L^2(\Omega)}^{\frac{q}{q+1}(1-a)} \\ & \quad + C_4 \left\| |\nabla c|^{q-1} D^2 c \right\|_{L^2(\Omega)}^{\frac{q}{q+1}} + C_4 \left\| \nabla |\nabla c|^q \right\|_{L^2}^a + C_4, \end{aligned}$$

for some  $C_4 > 0$ . Since  $a + \frac{q}{q+1}(1-a) < 1$ , we can apply Young's inequality twice to conclude that for some positive constant  $C_5$ ,

$$\int_{\partial\Omega} \frac{\partial |\nabla c|^2}{\partial \nu} |\nabla c|^{2q-2} \leq \frac{q-1}{q^2} \int_{\partial\Omega} |\nabla |\nabla c|^q|^2 + \frac{1}{2} \int_{\partial\Omega} |\nabla c|^{2q-2} \cdot |D^2 c|^2 + C_5.$$

Recalling the identity

$$|\nabla c|^{2q-4} |\nabla |\nabla c|^2|^2 = \frac{4}{q^2} |\nabla |\nabla c|^q|^2,$$

we immediately conclude from (2.50) that there exists some positive constant  $C$  depending on  $q$  fulfilling

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla c|^{2q} + \frac{q-1}{q} \int_{\Omega} |\nabla |\nabla c|^q|^2 + \frac{q}{2} \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \\ & \leq 2q^2 \int_{\Omega} n^2 |\nabla c|^{2q-2} + 2q \int_{\Omega} |\nabla c|^{2q} |D\mathbf{u}| + C. \end{aligned} \quad (2.53)$$

Whereupon, our conclusion comes from (2.46) and (2.53).  $\square$

We can now establish the desired combined estimate of  $\|n(t)\|_{L^p(\Omega)}$  and  $\|\nabla c(t)\|_{L^{2q}(\Omega)}$ .

**Lemma 2.10.** Suppose that (1.7)–(1.10) and (2.1) hold with  $\alpha > 0$ . Then for any  $p \in [1, \infty)$  and  $q \in [1, \infty)$ , there exists a positive constant  $C = C(p, q, C_s, \alpha, n_0, c_0)$  such that

$$\|n(t)\|_{L^p(\Omega)} + \|\nabla c(t)\|_{L^{2q}(\Omega)} \leq C \quad (2.54)$$

for all  $t \in (0, T_{\max})$ .

**Proof.** It is sufficient to prove that for any  $p_0 > 1$  and  $q_0 > 2$ , we can find some  $p \geq p_0$  and  $q \geq q_0$  such that (2.54) holds. To achieve this, given such  $p_0$  and  $q_0$ , we first fix  $p \geq p_0$ ,  $q \geq q_0$ ,  $\zeta > 1$ ,  $\theta > 1$  and  $\mu > 1$  as provided by Lemma 2.8. Then from Lemma 2.9, we know that there exists  $C(q) > 0$  such that

$$\begin{aligned} & \frac{d}{dt} \left( \int_{\Omega} n^p + \int_{\Omega} |\nabla c|^{2q} \right) + \frac{3(p-1)}{p} \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 + \frac{q}{2} \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \\ & \leq p(p-1)C_S^2 \int_{\Omega} n^{p-2\alpha} |\nabla c|^2 + 2q^2 \int_{\Omega} n^2 |\nabla c|^{2q-2} + 2q \int_{\Omega} |\nabla c|^{2q} |D\mathbf{u}| + C(q). \end{aligned} \quad (2.55)$$

We shall estimate the three integrals on the right hand of (2.55) one by one. We begin with estimating the first one. By using Hölder's inequality, we have

$$\int_{\Omega} n^{p-2\alpha} |\nabla c|^2 \leq \left( \int_{\Omega} n^{(p-2\alpha)\theta'} \right)^{\frac{1}{\theta'}} \left( \int_{\Omega} |\nabla c|^{2\theta} \right)^{\frac{1}{\theta}} \quad (2.56)$$

with  $\theta' = \frac{\theta}{\theta-1}$ . Notice that (2.25) yields  $(p-2\alpha)\theta' > 1$ . To estimate the first factor on the right hand side, we use the Gagliardo–Nirenberg inequality to find a positive constant  $C_1$  such that

$$\left( \int_{\Omega} n^{(p-2\alpha)\theta'} \right)^{\frac{1}{\theta'}} = \|n^{\frac{p}{2}}\|_{L^{\frac{2(p-2\alpha)\theta'}{p}}(\Omega)}^{\frac{2(p-2\alpha)}{p}} \leq C_1 \left( \|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(p-2\alpha)}{p} b_1} \|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p-2\alpha)}{p} (1-b_1)} + \|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p-2\alpha)}{p}} \right)$$

with  $b_1 = 1 - \frac{1}{(p-2\alpha)\theta'} \in (0, 1)$ , which together with the mass conservation (2.2) gives that

$$\left( \int_{\Omega} n^{(p-2\alpha)\theta'} \right)^{\frac{1}{\theta'}} \leq C_2 \left( \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 \right)^{\frac{p-2\alpha}{p} - \frac{1}{p\theta'}} + C_2 \quad (2.57)$$

for some  $C_2 > 0$ . To estimate the second factor on the right hand of (2.56), we notice that by Lemma 2.6 and Lemma 2.7, there exists a positive constant  $C_3$  such that

$$\|\nabla c\|_{L^{\varrho}(\Omega)}^{\varrho} \leq C_3 \left\| |\nabla c|^{q-1} D^2 c \right\|_{L^2(\Omega)}^{\frac{(2\gamma-1)\varrho-2\gamma}{q(2\gamma-1)}} + C_3 \quad (2.58)$$

for any  $\gamma > 1$  and  $\varrho \geq \frac{2\gamma}{2\gamma-1}(q+1)$ . In particular, we can take  $\varrho = 2\theta$  and  $\gamma = \zeta$  by (2.26), and then deduce that

$$\begin{aligned} \left( \int_{\Omega} |\nabla c|^{2\theta} \right)^{\frac{1}{\theta}} &\leq C_4 \left\| |\nabla c|^{q-1} D^2 c \right\|_{L^2(\Omega)}^{\frac{(2\zeta-1)2\theta-2\zeta}{q\theta(2\zeta-1)}} + C_4 \\ &= C_4 \left( \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \right)^{\frac{(2\zeta-1)\theta-\zeta}{q\theta(2\zeta-1)}} + C_4 \end{aligned} \quad (2.59)$$

with some positive constant  $C_4$ . Combining (2.56), (2.57) and (2.59), we have

$$\begin{aligned} \int_{\Omega} n^{p-2\alpha} |\nabla c|^2 &\leq C_2 C_4 \left( \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 \right)^{\frac{p-2\alpha}{p} - \frac{1}{p\theta'}} \left( \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \right)^{\frac{(2\zeta-1)\theta-\zeta}{q\theta(2\zeta-1)}} \\ &\quad + C_2 C_4 \left( \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 \right)^{\frac{p-2\alpha}{p} - \frac{1}{p\theta'}} \\ &\quad + C_2 C_4 \left( \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \right)^{\frac{(2\zeta-1)\theta-\zeta}{q\theta(2\zeta-1)}} + C_2 C_4. \end{aligned} \quad (2.60)$$

Noticing that

$$\frac{p-2\alpha}{p} - \frac{1}{p\theta'} + \frac{(2\zeta-1)\theta-\zeta}{q\theta(2\zeta-1)} = 1 - \frac{2\alpha}{p} - \frac{1}{p} \left(1 - \frac{1}{\theta}\right) + \frac{(2\zeta-1)\theta-\zeta}{q\theta(2\zeta-1)} < 1$$

by (2.28), we can apply the Young inequality to (2.60) and obtain that for any  $\eta > 0$  there exists some  $C_5(\eta) > 0$  such that

$$\int_{\Omega} n^{p-2\alpha} |\nabla c|^2 \leq \eta \left( \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 + \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \right) + C_5(\eta). \quad (2.61)$$

Next we consider the second integral on the right hand side of (2.55). By Hölder's inequality, we have the following estimate

$$\int_{\Omega} n^2 |\nabla c|^{2q-2} \leq \left( \int_{\Omega} n^{2\mu'} \right)^{\frac{1}{\mu'}} \left( \int_{\Omega} |\nabla c|^{2(q-1)\mu} \right)^{\frac{1}{\mu}} \quad (2.62)$$

with  $\mu' = \frac{\mu}{\mu-1}$ . We first use the Gagliardo–Nirenberg inequality to find some positive constant  $C_6$  fulfilling

$$\left(\int_{\Omega} n^{2\mu'}\right)^{\frac{1}{\mu'}} = \|n^{\frac{p}{2}}\|_{L^{\frac{4\mu'}{p}}(\Omega)}^{\frac{4}{p}} \leq C_6 \left( \|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^{2(\frac{2}{p} - \frac{1}{p\mu'})} \|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p\mu'}} + \|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{4}{p}} \right),$$

which ensures that

$$\left(\int_{\Omega} n^{2\mu'}\right)^{\frac{1}{\mu'}} \leq C_7 \left(\int_{\Omega} |\nabla n^{\frac{p}{2}}|^2\right)^{\frac{2}{p} - \frac{1}{p\mu'}} + C_7 \quad (2.63)$$

with some  $C_7 > 0$  from the mass conservation (2.2). On the other hand, we have  $2(q-1)\mu \geq \frac{2\zeta}{2\zeta-1}(q+1)$  by (2.27). Thus we can use (2.58) with  $\varrho = 2(q-1)\mu$  and  $\gamma = \zeta$  to obtain

$$\left(\int_{\Omega} |\nabla c|^{2(q-1)\mu}\right)^{\frac{1}{\mu}} \leq C_8 \left(\int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2\right)^{\frac{(2\zeta-1)(q-1)\mu-\zeta}{q\mu(2\zeta-1)}} + C_8 \quad (2.64)$$

for some positive constant  $C_8$ . Combining (2.62), (2.63) and (2.64), we arrive at

$$\begin{aligned} \int_{\Omega} n^2 |\nabla c|^{2q-2} &\leq C_7 C_8 \left(\int_{\Omega} |\nabla n^{\frac{p}{2}}|^2\right)^{\frac{2}{p} - \frac{1}{p\mu'}} \left(\int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2\right)^{\frac{(2\zeta-1)(q-1)\mu-\zeta}{q\mu(2\zeta-1)}} \\ &\quad + C_7 C_8 \left(\int_{\Omega} |\nabla n^{\frac{p}{2}}|^2\right)^{\frac{2}{p} - \frac{1}{p\mu'}} \\ &\quad + C_7 C_8 \left(\int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2\right)^{\frac{(2\zeta-1)(q-1)\mu-\zeta}{q\mu(2\zeta-1)}} + C_7 C_8. \end{aligned}$$

Since

$$\frac{2}{p} - \frac{1}{p\mu'} + \frac{(2\zeta-1)(q-1)\mu-\zeta}{q\mu(2\zeta-1)} < 1$$

by (2.29), we can use Young's inequality to obtain that for any  $\epsilon > 0$ , there exists  $C_9(\epsilon) > 0$  such that

$$\int_{\Omega} n^2 |\nabla c|^{2q-2} \leq \epsilon \left(\int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 + \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2\right) + C_9(\epsilon). \quad (2.65)$$

We now turn to the third integral on the right hand side of (2.55). Fix a constant  $\vartheta$  such that  $\vartheta \in (1, 2)$ . By using the Hölder inequality, we have

$$\int_{\Omega} |\nabla c|^{2q} |D\mathbf{u}| \leq \|D\mathbf{u}\|_{L^{\vartheta}(\Omega)} \|\nabla c\|_{L^{\vartheta'}(\Omega)}^{2q} \quad (2.66)$$

with  $\vartheta' = \frac{\vartheta}{\vartheta-1}$ . Since  $\|n(\cdot, t)\|_{L^1(\Omega)} \leq \|n_0\|_{L^1(\Omega)}$ , we can use [Lemma 2.5](#) to obtain

$$\|D\mathbf{u}\|_{L^{\vartheta}(\Omega)} \leq C_{10} \quad (2.67)$$

for some positive constant  $C_{10}$ . On the other hand, the fact  $\vartheta' > 2$  and  $q > 2$  shows that  $2q\vartheta' \geq \frac{2\zeta}{2\zeta-1}(q+1)$ . Thus we can apply inequality [\(2.58\)](#) with  $\varrho = 2q\vartheta'$  and  $\gamma = \zeta$  to find a positive constant  $C_{11}$  such that

$$\| |\nabla c|^{2q} \|_{L^{\vartheta'}(\Omega)} = \| \nabla c \|_{L^{2q\vartheta'}(\Omega)}^{2q} \leq C_{11} \left( \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \right)^{\frac{(2\zeta-1)q\vartheta'-\zeta}{q\vartheta'(2\zeta-1)}} + C_{11},$$

which together with [\(2.66\)](#) and [\(2.67\)](#) yields that

$$\int_{\Omega} |\nabla c|^{2q} |D\mathbf{u}| \leq C_{10} C_{11} \left( \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \right)^{\frac{(2\zeta-1)q\vartheta'-\zeta}{q\vartheta'(2\zeta-1)}} + C_{10} C_{11}$$

By using the Young inequality, we see that for any  $\delta > 0$ , there exists  $C_{12}(\delta) > 0$  such that

$$\int_{\Omega} |\nabla c|^{2q} |D\mathbf{u}| \leq \delta \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 + C_{12}(\delta). \quad (2.68)$$

Summarily, substituting [\(2.61\)](#), [\(2.65\)](#) and [\(2.68\)](#) into [\(2.55\)](#), we conclude that

$$\begin{aligned} & \frac{d}{dt} \left( \int_{\Omega} n^p + \int_{\Omega} |\nabla c|^{2q} \right) + \frac{3(p-1)}{p} \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 + \frac{q}{2} \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \\ & \leq \left( p(p-1)C_S^2\eta + 2q(q\epsilon + \delta) \right) \left( \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 + \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \right) + C_{13}(\eta, \epsilon, \delta) \end{aligned}$$

with some positive constant  $C_{13}(\eta, \epsilon, \delta)$ . Then we choose  $\eta, \epsilon$  and  $\delta$  in [\(2.61\)](#), [\(2.65\)](#) and [\(2.68\)](#) such that

$$p(p-1)C_S^2\eta + 2q(q\epsilon + \delta) = \min \left\{ \frac{2(p-1)}{p}, \frac{q}{4} \right\}$$

and thus

$$\frac{d}{dt} \left( \int_{\Omega} n^p + \int_{\Omega} |\nabla c|^{2q} \right) + \frac{(p-1)}{p} \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 + \frac{q}{4} \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 \leq C_{13}(\eta, \epsilon, \delta) \quad (2.69)$$

with some positive constant  $C_{13}(\eta, \epsilon, \delta)$ . It follows from the Gagliardo–Nirenberg inequality, the mass conservation [\(2.2\)](#) and Young's inequality that

$$\begin{aligned}
\int_{\Omega} n^p &= \|n^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \leq C_{14} \left( \|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^{2(1-\frac{1}{p})} \|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} + \|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)} \right) \\
&\leq C_{15} \|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^{2(1-\frac{1}{p})} + C_{15} \\
&\leq \frac{p-1}{p} \int_{\Omega} |\nabla n^{\frac{p}{2}}|^2 + C_{16}(p),
\end{aligned} \tag{2.70}$$

for some positive constants  $C_{14}$ ,  $C_{15}$  and  $C_{16}(p)$ . On the other hand, the fact  $q > 2$  implies  $2q \geq \frac{4}{3}(q+1)$  and thus the inequality (2.58) with  $\varrho = 2q$  and  $\gamma = 2$  holds. That is, there exists  $C_{17} > 0$  such that

$$\int_{\Omega} |\nabla c|^{2q} \leq C_{17} \left( \int_{\Omega} |\nabla c|^{2q-2} |D^2 c| \right)^{\frac{3q-2}{3q}} + C_{17},$$

which together with Young's inequality gives that

$$\int_{\Omega} |\nabla c|^{2q} \leq \frac{q}{4} \int_{\Omega} |\nabla c|^{2q-2} |D^2 c|^2 + C_{18}(q) \tag{2.71}$$

with some positive constant  $C_{18}(q)$ .

Substituting (2.70) and (2.71) into (2.69), we thus achieve that

$$\frac{d}{dt} \left( \int_{\Omega} n^p + \int_{\Omega} |\nabla c|^{2q} \right) + \left( \int_{\Omega} n^p + \int_{\Omega} |\nabla c|^{2q} \right) \leq C_{19}$$

for some positive constant  $C_{19} = C_{13}(\eta, \epsilon, \delta) + C_{16}(p) + C_{18}(q)$ . Then an ODE comparison argument shows that

$$\int_{\Omega} n^p + \int_{\Omega} |\nabla c|^{2q} \leq \max \left\{ \left( \int_{\Omega} n_0^p + \int_{\Omega} |\nabla c_0|^{2q} \right), C_{19} \right\}$$

for all  $t \in (0, T_{\max})$ . This completes the proof of Lemma 2.10.  $\square$

## 2.5. Global existence for $\mathcal{S} = 0$ on $\partial\Omega$

With all of above estimates at hand, we can now establish the global existence result in the case  $\mathcal{S} = 0$  on the boundary  $\partial\Omega$ .

**Theorem 2.1.** Suppose that (1.7)–(1.10) and (2.1) hold. Then system (1.6) admits a unique global classical solution satisfying (1.11).

**Proof.** To obtain the existence of global bounded solution, by the extension criterion in Lemma 2.1, we only need to show that



$$\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|\mathcal{A}^\beta \mathbf{u}(\cdot, t)\|_{L^2(\Omega)} \leq C \quad (2.72)$$

for any  $t \in (0, T_{\max})$  for some positive constant  $C$  independent of  $T_{\max}$ . To this end, we first use [Lemma 2.5](#) and [Lemma 2.10](#) to obtain

$$\|D\mathbf{u}(\cdot, t)\|_{L^r(\Omega)} \leq C_1 \quad \text{for all } 1 \leq r \leq \infty \text{ and some constant } C_1 > 0,$$

which together with [Lemma 2.4](#) and the interpolation inequality yields the existence of  $C_2 > 0$  satisfying

$$\|\mathbf{u}(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_2 \quad (2.73)$$

for any  $t \in (0, T_{\max})$ .

We now establish the boundedness of  $n$ . Fix a constant  $q$  such that  $q > 2$ . According to the known smoothing estimate for the Neumann heat semigroup in  $\Omega$  (see [\[34\]](#), for instance), we can invoke the variation-of-constants formula for  $n$  to find that some positive constant  $C_3$  such that

$$\begin{aligned} & \|n(\cdot, t)\|_{L^\infty(\Omega)} \\ & \leq \|e^{t\Delta} n_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-\tau)\Delta} \nabla \cdot (n\mathcal{S}(x, n, c)\nabla c + n\mathbf{u})(\cdot, \tau)\|_{L^\infty(\Omega)} d\tau \\ & \leq \|n_0\|_{L^\infty(\Omega)} + C_3 \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{q}} e^{-\lambda_1(t-\tau)} \|n\mathcal{S}(x, n, c)\nabla c + n\mathbf{u}\|_{L^q(\Omega)} d\tau. \end{aligned} \quad (2.74)$$

We then take  $r > q$ . By Hölder's inequality, [Lemma 2.10](#) and [\(2.73\)](#), we obtain

$$\|n\mathcal{S}(x, n, c)\nabla c\|_{L^q(\Omega)} \leq C_S \|n\nabla c\|_{L^q(\Omega)} \leq C_S \|n(\cdot, s)\|_{L^r} \|\nabla c(\cdot, s)\|_{L^{\frac{rq}{r-q}}} \leq C_4$$

and

$$\|n\mathbf{u}\|_{L^q(\Omega)} \leq \|\mathbf{u}\|_{L^\infty(\Omega)} \|n\|_{L^q(\Omega)} \leq C_5$$

with some positive constants  $C_4$  and  $C_5$ . Noticing that  $\int_0^\infty (t-\tau)^{-\frac{1}{2}-\frac{1}{q}} e^{-\lambda_1(t-\tau)} d\tau$  is finite, we can infer from [\(2.74\)](#) that

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} \leq C_6 \quad (2.75)$$

for some positive constant  $C_6$ .

Similarly, applying the variation-of-constants formula for  $c$

$$c(t) = e^{t(\Delta-1)} c_0 + \int_0^t e^{(t-\tau)(\Delta-1)} (n(\tau) - \mathbf{u}(\tau) \cdot \nabla c(\tau)) d\tau$$

and using [Lemma 2.10](#), we can assert that there exists  $C_7 > 0$  such that

$$\|c(\cdot, t)\|_{w^{1,\infty}(\Omega)} \leq C_7. \quad (2.76)$$

We now turn to the estimate for  $\mathbf{u}$ . Let  $\beta \in (\frac{1}{2}, 1)$  be the constant given by [\(1.9\)](#). Applying the fractional power  $\mathcal{A}^\beta$  to the variation-of-constants formula

$$\mathbf{u}(\cdot, t) = e^{-t\mathcal{A}}\mathbf{u}_0 + \int_0^t e^{-(t-\tau)\mathcal{A}}\mathcal{P}(n(\cdot, \tau)\nabla\phi)d\tau, \quad t \in (0, T_{\max}),$$

we can obtain

$$\|\mathcal{A}^\beta \mathbf{u}(\cdot, t)\|_{L^2(\Omega)} \leq \|\mathcal{A}^\beta e^{-t\mathcal{A}}\mathbf{u}_0\|_{L^2(\Omega)} + \int_0^t \|\mathcal{A}^\beta e^{-(t-\tau)\mathcal{A}}\mathcal{P}(n(\cdot, \tau)\nabla\phi)\|_{L^2(\Omega)}d\tau. \quad (2.77)$$

Since  $\mathbf{u}_0 \in D(\mathcal{A}^\beta)$ , there exists positive constant  $C_8$  satisfying

$$\|\mathcal{A}^\beta e^{-t\mathcal{A}}\mathbf{u}_0\|_{L^2(\Omega)} = \|e^{-t\mathcal{A}}\mathcal{A}^\beta \mathbf{u}_0\|_{L^2(\Omega)} \leq C_8 \quad \text{for all } t \in (0, T_{\max}). \quad (2.78)$$

On the other hand, we notice that the estimate of Stokes operator (see e.g. [\[9\]](#))

$$\|\mathcal{A}^\beta e^{-t\mathcal{A}}\varphi\|_{L^2(\Omega)} \leq C_9 t^{-\beta} e^{-\lambda t} \|\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in L^2_\sigma(\Omega),$$

valid for all  $t > 0$  and some positive constants  $C_9$  and  $\lambda > 0$ . Since  $\mathcal{P}$  is a bounded operator from  $L^2(\Omega)$  to  $L^2_\sigma(\Omega)$ , we thereupon obtain

$$\begin{aligned} & \int_0^t \|\mathcal{A}^\beta e^{-(t-s)\mathcal{A}}\mathcal{P}(n(\cdot, s)\nabla\phi)\|_{L^2(\Omega)}ds \\ & \leq C_9 \|n\|_{L^\infty(\Omega)} \int_0^\infty (t-s)^{-\beta} e^{-\lambda(t-s)} ds \leq C_{10} \end{aligned} \quad (2.79)$$

with some  $C_{10} > 0$ .

Substituting [\(2.78\)](#) and [\(2.79\)](#) into [\(2.77\)](#), we conclude that there exists a positive constant  $C_{11}$  such that

$$\|\mathcal{A}^\beta \mathbf{u}(\cdot, t)\|_{L^2(\Omega)} \leq C_{11}. \quad (2.80)$$

Then combining [\(2.75\)](#), [\(2.76\)](#) and [\(2.80\)](#), we infer that [\(2.72\)](#) holds. We therefore conclude that  $T_{\max} = \infty$  and that  $(n, c, \mathbf{u}, P)$  is a global in time. The boundedness estimate [\(1.11\)](#) is a direct consequence of [\(2.73\)](#), [\(2.75\)](#) and [\(2.76\)](#). This completes the proof of [Theorem 2.1](#).  $\square$

### 3. Global classical solutions for general $\mathcal{S}$

In this section, we shall construct global solution to system (1.6) with general tensor-valued sensitivity  $\mathcal{S}$ , which satisfies (1.7) and (1.8), but no longer necessarily vanishes on  $\partial\Omega$ . For this purpose, we shall perform adequate approximation procedure.

Let  $\{\rho_\varepsilon\}_{\varepsilon \in (0,1)} \subset C_0^\infty(\Omega)$  be a family of standard cut-off functions satisfying  $0 \leq \rho_\varepsilon \leq 1$  in  $\Omega$  for all  $\varepsilon \in (0, 1)$  and  $\rho_\varepsilon \rightarrow 1$  in  $\Omega$  pointwisely as  $\varepsilon \rightarrow 0$ . Then we define

$$\mathcal{S}_\varepsilon(x, n, c) = \rho_\varepsilon(x) \mathcal{S}(x, n, c), \quad (x, n, c) \in \bar{\Omega} \times [0, \infty) \times [0, \infty)$$

for  $\varepsilon \in (0, 1)$ . It is clear that  $\mathcal{S}_\varepsilon$  vanishes on  $\partial\Omega$  for each fixed  $\varepsilon \in (0, 1)$ . Thus Theorem 2.1 implies that each of the approximating problem

$$\begin{cases} n_{\varepsilon t} + \mathbf{u}_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon \mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon), & x \in \Omega, t > 0, \\ c_{\varepsilon t} + \mathbf{u}_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, & x \in \Omega, t > 0, \\ \mathbf{u}_{\varepsilon t} + \nabla P_\varepsilon = \Delta \mathbf{u}_\varepsilon + n_\varepsilon \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot \mathbf{u}_\varepsilon = 0, & x \in \Omega, t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = 0, \quad \mathbf{u}_\varepsilon = 0, & x \in \partial\Omega, t > 0, \\ n_\varepsilon(x, 0) = n_0(x), \quad c_\varepsilon(x, 0) = c_0(x), \quad \mathbf{u}_\varepsilon(x, 0) = \mathbf{u}_0(x), & x \in \Omega \end{cases} \quad (3.1)$$

possesses a global classical solution  $(n_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon, P_\varepsilon)$ . Moreover, there exists some positive constant  $M_1$  such that

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|\mathbf{u}_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq M_1 \quad \text{for all } t \in (0, \infty). \quad (3.2)$$

Thus we can integrate (2.46) with  $p = 2$  to conclude that there exists a  $M_2(T) > 0$  such that

$$\int_0^T \int_\Omega |\nabla n_\varepsilon|^2 \leq M_2(T) \quad \text{for all } T > 0. \quad (3.3)$$

Note that since the initial data of system (3.1) is as same as that of (1.6), we know from former lemmata that  $M_1$  and  $M_2(T)$  are both independent of  $\varepsilon$ .

Our goal is to show the solutions of system (3.1) approach a classical solution of system (1.6) as  $\varepsilon \rightarrow 0$ . To this end, we need now to proceed to establish some further regularity estimates.

**Lemma 3.1.** *Suppose that (1.7)–(1.10) hold. Then there exists  $C > 0$  such that*

$$\|n_{\varepsilon t}\|_{(W_0^{2,2}(\Omega))^*} \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1), \quad (3.4)$$

and also

$$\|n_\varepsilon(\cdot, t) - n_\varepsilon(\cdot, s)\|_{(W_0^{2,2}(\Omega))^*} \leq C|t - s| \quad \text{for all } t > 0, \quad s \geq 0 \quad \text{and} \quad \varepsilon \in (0, 1). \quad (3.5)$$

**Proof.** Choosing  $\psi \in C_0^\infty(\Omega)$  as a test function of the first equation in (3.1), we can obtain

$$\begin{aligned} \int_{\Omega} n_{\varepsilon t} \psi &= - \int_{\Omega} \nabla n_{\varepsilon} \cdot \nabla \psi + \int_{\Omega} n_{\varepsilon} (\mathcal{S}_{\varepsilon} \cdot \nabla c_{\varepsilon}) \cdot \nabla \psi - \int_{\Omega} (\mathbf{u}_{\varepsilon} \cdot \nabla n_{\varepsilon}) \psi \\ &= \int_{\Omega} n_{\varepsilon} \Delta \psi + \int_{\Omega} n_{\varepsilon} (\mathcal{S}_{\varepsilon} \cdot \nabla c_{\varepsilon}) \cdot \nabla \psi + \int_{\Omega} (\mathbf{u}_{\varepsilon} n_{\varepsilon}) \cdot \nabla \psi. \end{aligned}$$

Using (3.2), one can easily conclude

$$\left| \int_{\Omega} n_{\varepsilon t} \psi \right| \leq M_1 \int_{\Omega} |\Delta \psi| + M_1^2 (C_S + 1) \int_{\Omega} |\nabla \psi|$$

for all  $t > 0$  and  $\varepsilon \in (0, 1)$ . This helps us to establish (3.4) and moreover (3.5).  $\square$

Thanks to the boundedness of  $n_{\varepsilon}$ ,  $c_{\varepsilon}$  and  $u_{\varepsilon}$  as in (3.2), we can obtain the following uniform Hölder continuity of  $c_{\varepsilon}$ ,  $\nabla c_{\varepsilon}$  and  $u_{\varepsilon}$  from the standard parabolic regularity theory (see e.g. Lemma 3.18 and Lemma 3.19 in Winkler [40]).

**Lemma 3.2.** Suppose that (1.7)–(1.10) hold. Then there exist  $\sigma \in (0, 1)$  and  $C > 0$  such that

$$\|c_{\varepsilon}\|_{C^{\sigma, \frac{\sigma}{2}}(\bar{\Omega} \times [t, t+1])} \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1)$$

and

$$\|\mathbf{u}_{\varepsilon}\|_{C^{\sigma, \frac{\sigma}{2}}(\bar{\Omega} \times [t, t+1])} \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1),$$

and that for each  $t_0 > 0$  we can find  $C(t_0) > 0$  such that

$$\|\nabla c_{\varepsilon}\|_{C^{\sigma, \frac{\sigma}{2}}(\bar{\Omega} \times [t, t+1])} \leq C(t_0) \quad \text{for all } t > t_0 \quad \text{and} \quad \varepsilon \in (0, 1).$$

With all these preparation, we can prove our main results.

**Proof of Theorem 1.1.** We first give a series of convergence results. By Lemma 3.2 and the Arzelà–Ascoli theorem along with a standard extraction procedure, we can find a sequence  $\{\varepsilon_j\}_{j \in \mathbb{N}} \subset (0, 1)$  with  $\varepsilon_j \searrow 0$  as  $j \rightarrow \infty$  such that

$$c_{\varepsilon} \rightarrow c \quad \text{in } C_{loc}^0(\bar{\Omega} \times [0, \infty)), \quad (3.6)$$

$$\nabla c_{\varepsilon} \rightarrow \nabla c \quad \text{in } C_{loc}^0(\bar{\Omega} \times [0, \infty)), \quad (3.7)$$

and

$$\mathbf{u}_{\varepsilon} \rightarrow \mathbf{u} \quad \text{in } C_{loc}^0(\bar{\Omega} \times [0, \infty)) \quad (3.8)$$

for some limited functions  $c$  and  $\mathbf{u}$  as  $\varepsilon = \varepsilon_j \searrow 0$ . On the other hand, (3.2) ensures the existence of a subsequence such that

$$n_\varepsilon \rightarrow n \quad \text{weak* in } L^\infty(\Omega \times (0, \infty)), \quad (3.9)$$

$$\nabla c_\varepsilon \rightarrow \nabla c \quad \text{weak* in } L^\infty(\Omega \times (0, \infty)), \quad (3.10)$$

and

$$D\mathbf{u}_\varepsilon \rightarrow D\mathbf{u} \quad \text{weak* in } L^\infty(\Omega \times (0, \infty)) \quad (3.11)$$

for some  $n \in L^\infty(\Omega \times (0, \infty))$ , and (3.3) shows that

$$\nabla n_\varepsilon \rightharpoonup \nabla n \quad \text{in } L^2_{loc}(\bar{\Omega} \times [0, \infty)). \quad (3.12)$$

Estimate (3.3) also shows that  $\{n_\varepsilon\}_{\varepsilon \in (0,1)}$  is bounded in  $L^2((0, T); W^{1,2}(\Omega))$ . In view of Lemma 3.1, the Aubin–Lions lemma (see Chapter IV, [19]) yields the strong precompactness of  $n_\varepsilon$  in  $L^2_{loc}(\bar{\Omega} \times [0, \infty))$ , which in conjunction with the Egorov’s theorem yield

$$n_\varepsilon \rightarrow n \quad \text{a.e. in } \Omega \times (0, \infty). \quad (3.13)$$

Finally, noticing that  $L^\infty(\Omega) \hookrightarrow (W^{2,2}_0(\Omega))^*$  is compact, we can use Arzelà–Ascoli theorem again to assert

$$n_\varepsilon \rightarrow n \quad \text{in } C^0_{loc}([0, \infty); (W^{2,2}_0(\Omega))^*). \quad (3.14)$$

Next we shall prove  $(n, c, u)$  is a weak solution of problem (1.6). Testing the first equation in (3.1) by  $\varphi \in C^\infty_0(\bar{\Omega} \times [0, \infty))$ , we obtain

$$\begin{aligned} - \int_0^\infty \int_\Omega n_\varepsilon \varphi_t &= \int_\Omega n_{0\varepsilon} \varphi(\cdot, 0) - \int_0^\infty \int_\Omega \nabla n_\varepsilon \cdot \nabla \varphi \\ &\quad + \int_0^\infty \int_\Omega n_\varepsilon \mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon \cdot \nabla \varphi - \int_0^\infty \int_\Omega n_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \varphi \end{aligned}$$

for all  $\varepsilon \in (0, 1)$ . Then (3.7)–(3.8), (3.12), (3.14) and the dominated convergence theorem enables us to conclude that

$$- \int_0^\infty \int_\Omega n \varphi_t = \int_\Omega n_0 \varphi(\cdot, 0) - \int_0^\infty \int_\Omega \nabla n \cdot \nabla \varphi + \int_0^\infty \int_\Omega n \mathcal{S}(x, n, c) \nabla c \cdot \nabla \varphi - \int_0^\infty \int_\Omega n \mathbf{u} \cdot \nabla \varphi$$

by a limit procedure. Along with a similar procedure applied to the second equation in (3.1), we can achieve

$$-\int_0^\infty \int_\Omega c \varphi_t = \int_\Omega c_0 \varphi(\cdot, 0) - \int_0^\infty \int_\Omega \nabla c \cdot \nabla \varphi - \int_0^\infty \int_\Omega (c - n) \varphi - \int_0^\infty \int_\Omega c \mathbf{u} \cdot \nabla \varphi.$$

Then testing the third equation in (3.1) by  $\mathbf{w} \in (C_0^\infty(\bar{\Omega} \times [0, \infty)))^3$  and using a limit procedure, we obtain

$$-\int_0^\infty \int_\Omega \mathbf{u} \mathbf{w}_t = \int_\Omega \mathbf{u}_0 \mathbf{w}(\cdot, 0) - \int_0^\infty \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{w} + \int_0^\infty \int_\Omega n \nabla \phi \cdot \mathbf{w}.$$

This means that  $(n, c, \mathbf{u})$  is a weak solution of system (1.6).

Now we shall show that this weak solution actually is a classical solution. Our argument is similar to that of [16]. In fact, we can rewrite the first equation of (1.6) as

$$n_t = \nabla \cdot \mathbf{a}(x, t, n, \nabla n) + b(x, t, \nabla n)$$

with boundary data  $\mathbf{a}(x, t, n, \nabla n) \cdot \nu = 0$  on  $\partial\Omega$ , where  $\mathbf{a}(x, t, n, \nabla n) := \nabla n - n\mathcal{S}(x, n, c) \cdot \nabla c$  and  $b(x, t, \nabla n) = -\mathbf{u} \cdot \nabla n$ . Let  $\mathbf{p} := \nabla n$ . From Young's inequality and the boundedness of  $n$  and  $\mathbf{u}$ , we can easily conclude that, there exists a positive constant  $C$  such that

$$\begin{aligned} \mathbf{a}(x, t, n, \mathbf{p}) \cdot \mathbf{p} &\geq \frac{1}{2} |\mathbf{p}|^2 - C |\nabla c|^2, \\ |\mathbf{a}(x, t, n, \mathbf{p})| &\leq |\mathbf{p}| + C |\nabla c|, \end{aligned}$$

and

$$|b(x, t, \mathbf{p})| \leq C + \frac{1}{2} |\mathbf{p}|^2.$$

Since  $\nabla c$  belongs to  $L_{loc}^\infty((0, \infty); L^2(\Omega))$  by (2.76), we can deduce that  $n$  is continuous in  $\bar{\Omega} \times [0, \infty)$  and Hölder continuous in  $\bar{\Omega} \times (0, \infty)$  (see Theorem 1.3 and Remark 1.4, [26]). Likewise, we can claim that  $c(x, t)$  is locally Hölder continuous in  $\bar{\Omega} \times (0, \infty)$ . By using the regularity theories for Stokes semigroup (see [10]), we can obtain the continuity of  $\mathbf{u}$  from the regularity of  $n$  and  $\phi$ .

Finally, the higher order regularity can be obtained by a straightforward bootstrap argument. Indeed, applying Theorem IV.5.3 of [23] to the equation

$$c_t - \Delta c + \mathbf{u} \cdot \nabla c + c = n$$

with homogeneous Neumann boundary condition, we can obtain that  $c$  belongs to  $C^{2+l, 1+\frac{l}{2}}(\bar{\Omega} \times (0, \infty))$  for some  $l \in (0, 1)$ . Here we also used the Hölder continuity of  $n$  and  $\mathbf{u}$ . Note that  $n$  actually satisfies the equation

$$n_t - \Delta n + (\mathbf{u} + (\mathcal{S} \cdot \nabla c)) \cdot \nabla n - (\nabla \cdot (\mathcal{S} \cdot \nabla c))n = 0,$$

with boundary condition  $(\nabla n - \mathcal{S}(x, n, c)n\nabla c) \cdot \nu = 0$ . Thus, with the aforementioned regularity properties of  $c$  and  $\mathbf{u}$  at hand, we can obtain the higher order Hölder continuity of  $n$  from Theorem IV.5.3 of [23] again. At last, the regularity of  $\mathbf{u}$  results from the standard bootstrap arguments involving the regularity theories for the Stokes semigroup (see Lemma 2.1 of [35]).

The stated boundedness of the classical solution comes from (3.2) and the aforementioned convergence results (3.6)–(3.11) directly. This completes the proof of Theorem 1.1.  $\square$

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