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Global existence and boundedness in a Keller–Segel–Stokes system involving a tensor-valued sensitivity with saturation: The 3D case

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Abstract

In this paper we continue to deal with the initial–boundary value problem for the coupled Keller–Segel–Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\mathcal{S}(x,n,c)\cdot\nabla c), & (x,t)\in\Omega\times(0,T),\\ c_t + u\cdot\nabla c = \Delta c - c + n, & (x,t)\in\Omega\times(0,T),\\ u_t + \nabla P = \Delta u + n\nabla\phi, & (x,t)\in\Omega\times(0,T),\\ \nabla\cdot u = 0, & (x,t)\in\Omega\times(0,T), \end{cases}$$

where $\Omega\subset\mathbb{R}^d$ is a bounded domain with smooth boundary and the chemotactic sensitivity $\mathcal{S}$ is not a scalar function but rather attains values in $\mathbb{R}^{d\times d}$, and satisfies $|\mathcal{S}(x,n,c)|\le C_{\mathcal{S}}(1+n)^{-\alpha}$ with some $C_{\mathcal{S}}>0$ and $\alpha>0$. When $d=2$, our previous work (J. Differential Equations, 2015) has established the existence of global bounded classical solutions under the subcritical assumption $\alpha>0$, which is consistent with the corresponding results of the fluid-free system, but the method seems to be invalid in the three-dimensional setting.

In this paper, for the case $d=3$, we develop a new method to establish the existence and boundedness of global classical solutions for arbitrarily large initial data under the assumption $\alpha>\frac{1}{2}$, which is slightly stronger than the corresponding subcritical assumption $\alpha>\frac{1}{3}$ on the fluid-free system, where such an

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assumption is essentially necessary and sufficient for the existence of global bounded solutions. The key idea here is to establish the general L^p regularity of u from a rather low L^p regularity of n, which will be obtained by a new combinational functional.

MSC: 35K55; 35Q92; 35Q35; 92C17

Keywords: Keller–Segel–Stokes system; Tensor-valued sensitivity; Global existence; Boundedness

1. Introduction

In this paper, we shall consider the following initial–boundary value problem

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot\big(n\mathcal{S}(x,n,c)\cdot\nabla c\big), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega,\ t>0,\\ u_t + \nabla P = \Delta u + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\ \big(\nabla n - n\mathcal{S}(x,n,c)\nabla c\big)\cdot\nu = \nabla c\cdot\nu = 0,\ \ u=0, & x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x),\ c(x,0)=c_0(x),\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

for the unknown bacterial density n, the nutrient concentration c, the fluid velocity u and the associated pressure P in a bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary $\partial\Omega$, where ν denotes the unit outward normal vector field on $\partial\Omega$, the gravitational potential function ϕ and the tensor-valued sensitivity function $\mathcal{S}:\Omega\times[0,\infty)^2\to\mathbb{R}^{3\times3}$ are supposed to be given parameter functions. Systems of this type arise in the modeling of bacterial populations, like *Escherichia coli*, in which the cells live in a viscous fluid so that cells and chemical substrates are transported with fluid, and that the motion of the fluid is under the influence of gravitational forcing generated by aggregation of cells [26,35]. This kind of model can also be used to the studies of coral broadcast spawning [12,18].

Before going into our mathematical analysis, we recall some important progresses on system (1.1) and its variants. In biological contexts, many simple life-forms can exhibit a complex collective behavior. Chemotaxis is one particular mechanism responsible for some instances of such demeanor, where the organism, like bacteria, adapts its movement according to the concentrations of a chemical signal. Considering that in nature some bacteria, like *Bacillus subtilis* and *Escherichia coli*, often live in a viscous fluid, Tuval et al. [35] conducted a detailed experiment to describe the chemotaxis–fluid interaction. More precisely, they observed the large-scale convection patterns in a water drop sitting on a glass surface containing *Bacillus subtilis*, oxygen diffusing into the drop through the fluid-air interface and proposed the following mathematical model to describe the dynamics of the cell concentration, oxygen concentration, and fluid velocity:

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(nS(c)\nabla c),\\ c_t + u\cdot\nabla c = \Delta c - nf(c),\\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u + n\nabla\phi,\\ \nabla\cdot u = 0 \end{cases} \tag{1.2}$$

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with $\kappa = 1$. The chemotactic sensitivity function $S(c)$ and the per-capita oxygen consumption rate $f(c)$ are given scalar functions. The coefficient κ is related to the strength of nonlinear fluid convection. In particular, when the fluid flow is slow, we can use the Stokes equation instead of the Navier–Stokes one, i.e., $\kappa = 0$ (see [9,25]). As to the mathematical analysis of system (1.2), numerous results in the past several years have concentrated on the natural first question of local and global solvability of corresponding initial(–boundary) value problems in either bounded or unbounded domains Ω under various assumptions of the scalar functions S and f. For instance, in the two-dimensional setting, the diffusive mechanism turns out to be strong enough so as to allow for the construction of global bounded weak/classical solutions [4,10,24,42,54], while in the three-dimensional setting, some more restrictive assumptions on S and f are needed to ensure the existence of global bounded weak solutions for $\kappa = 0$ [42] or of global classical solutions for general $\kappa \in \mathbb{R}$ under some smallness assumptions on the initial data [4,5,10,30,51], and until recently, the existence of global bounded weak solutions for general $\kappa \in \mathbb{R}$ and large initial data has been demonstrated by Winkler [47]. Very recently, the large time behavior of classical solutions to (1.2) in the two-dimensional case (see [44,52]) and the eventual smoothness of weak solutions in the three-dimensional case (see [48]) have also been investigated. Quite a few results on global existence and boundedness properties have also been obtained for the variant of (1.2) obtained on replacing linear diffusion Δn by the porous medium-type nonlinear diffusion Δn^m for several ranges of $m > 1$ (see e.g. [9,11,24,32,33,36,46,53]). On the other hand, when the chemotactic sensitivity $S(c)$ in system (1.2) is replaced by the matrix $\mathcal{S}(x, n, c)$, which is justified in the recent experimental findings and suggests that chemotactic migration need not necessarily be oriented along the gradient of the chemical substance, but may rather involve rotational flux components [49,50], some new approaches have been developed to construct the global bounded weak/classical solutions [3,17,37,46,45].

The results obtained so far indicate that finite-time blow-up phenomena of solutions, which related to the extreme facet of bacterial aggregation, do not occur for system (1.2) even though the Stokes fluid is included. However, this is not necessarily true for system (1.1), of which a typical feature is the production of chemoattractant by the bacteria, in contrast to (1.2) with consumption of nutrient. Indeed, the signal production mechanism may bring about the spontaneous formation of aggregates even in the classical Keller–Segel system without fluid interaction, as given by letting $S \equiv 1$ in

$$\begin{cases} n_t = \Delta n - \nabla \cdot \big(n\mathcal{S}(n)\nabla c\big), \\ c_t = \Delta c - c + n \end{cases} \tag{1.3}$$

in $\Omega \subset \mathbb{R}^d$. For instance, it is well-known that for large classes of initial data, solutions of system (1.3) blow up when either $d \geq 3$, or $d = 2$ and the total mass of cells is large, while global bounded solutions can be constructed under appropriate smallness conditions on the initial data [27,43]. On the other hand, such explosion phenomena can be ruled out when $S(n)$ is related to the prototypical assumption of volume-filling effect. Precisely, it has been shown that for the Keller–Segel system with n-dependent sensitivities $S(n)$, all solutions are global and uniformly bounded provided that

$$S(n) \leq \overline{C}(1+n)^{-\alpha} \quad \text{with} \quad \alpha > 1 - \frac{2}{d} \tag{1.4}$$

(see [15,19]), while under some technical assumptions the solution may blow up if $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) is a ball and

$$S(n) > \underline{C} n^{-\alpha} \quad \text{with} \quad \alpha < 1 - \frac{2}{d}$$

(see [15,40]). For the more related works in this direction, we mention that a corresponding quasilinear version has been deeply investigated by [6–8,31] (see also the recent survey [1]). From the view of biology, the chemotactic movement of cells is inhibited near points where the cells are densely packed (see [28]) under such hypothesis. For the Keller–Segel-fluid system of the form (1.1), the mechanism for preventing blow-up is more subtle. As far as we know, there are only few results which deal with chemotaxis–fluid interaction in the presence of a signal production mechanism. In particular, for $\mathcal{S} = \mathbf{I}$, the identity matrix, the literature so far concentrates on either the construction of small-data solutions [20], on systems involving logistic growth restrictions as an additional dissipative mechanism [12,34] or on the sublinear signal production [2]. For general matrix $\mathcal{S}$, there are only two results (see [38,39]) which deal with the two-dimensional Keller–Segel–(Navier–)Stokes system of the form (1.1) and demonstrate the existence of global bounded classical solutions with large initial data under the assumption of

$$|\mathcal{S}(x,n,c)| \leq C_{\mathcal{S}}(1+n)^{-\alpha} \tag{1.5}$$

with $\alpha > 0$, which is consistent with the corresponding result (1.4) for the fluid-free system (1.3).

Main results. In this paper, we investigate the initial–boundary value problem for the coupled Keller–Segel–Stokes system (1.1) with the matrix-valued sensitivity function $\mathcal{S}$ in the three-dimensional setting. In contrast to the previous studies on system (1.2), such a problem is much more delicate, for which the regularization signal absorption mechanism from (1.2) is no longer available, and for which thus a much more colorful dynamics must be expected as already indicated by known facts for the fluid-free Keller–Segel system (1.3).

Our main purpose is to examine how far the volume-filling effect of the form in (1.5) continues to determine the existence of global bounded classical solutions to system (1.1) in the three-dimensional setting. To formulate this more precisely, let us suppose that $\mathcal{S} = (S_{ij})_{3\times 3}$ satisfies

$$S_{ij} \in C^2\big(\bar{\Omega} \times [0,\infty) \times [0,\infty)\big) \quad \text{for } i,j \in \{1,2,3\} \tag{1.6}$$

and

$$|\mathcal{S}(x,n,c)| \leq C_{\mathcal{S}}(1+n)^{-\alpha} \quad \text{for some constants} \quad \alpha > 0 \quad \text{and} \quad C_{\mathcal{S}} > 0. \tag{1.7}$$

The assumption $\alpha > 0$ implies that when the cell density increases, the effect of chemotaxis is weakened. In view of the analysis for the fluid-free Keller–Segel system (1.3), in the three dimensional case, some further algebraic saturation assumptions are needed to obtain the existence of global bounded solutions to system (1.1) with large initial data. Indeed, we will show that $\alpha > \frac{1}{2}$ is enough to rule out the blow-up in finite or infinite time, which is slightly stronger than the corresponding assumption $\alpha > \frac{1}{3}$ on the fluid-free system. Such a result is new even in the case of scalar chemotactic sensitivity S, but this matrix-valued generalization results in considerable mathematical difficulties due to the fact that chemotaxis systems with such rotational fluxes lose some energy structure, which has served as a key to the analysis for scalar-valued S.

We shall assume throughout this paper that the initial data satisfy

$$\begin{cases} n_0 \in C^0(\overline{\Omega}), \quad n_0 \geq 0 \text{ and } n_0 \not\equiv 0 \text{ in } \bar{\Omega} \\ c_0 \in W^{1,\infty}(\Omega), \quad c_0 \geq 0 \text{ and } c_0 \not\equiv 0 \text{ in } \bar{\Omega} \\ u_0 \in D(\mathcal{A}_r^{\beta}) \qquad \text{for all } r \in (1,\infty) \text{ and some } \beta \in \left(\frac{3}{4}, 1\right), \end{cases} \tag{1.8}$$

where $\mathcal{A}_r$ denotes the Stokes operator with domain $D(\mathcal{A}_r) := W^{2,r}(\Omega) \bigcap W_0^{1,r}(\Omega) \bigcap L_\sigma^r(\Omega)$ with $L_\sigma^r(\Omega) := \left\{\varphi \in L^r(\Omega) \mid \nabla \cdot \varphi = 0\right\}$ for $r \in (1,\infty)$.

As for the gravitational potential ϕ in (1.1), we require that it is independent of time and satisfies

$$\phi \in W^{2,\infty}(\Omega). \tag{1.9}$$

Under these assumptions, we can establish the existence of global bounded classical solutions to system (1.1) for general (large) data. Precisely, we have the following global existence result.

Theorem 1.1. *Suppose that (1.6)–(1.9) hold with $\alpha > \frac{1}{2}$. Then system (1.1) admits a global classical solution (n, c, u, P), which is uniformly bounded in the sense that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \qquad \textit{for all} \quad t \in (0,\infty) \tag{1.10}$$

with some positive constant C. This solution is unique, up to addition of constants to P.

Remark 1.1. One way to relax the restriction on α is to replace the linear diffusion Δn by the porous medium diffusion Δn^m with m suitably large. This problem has been investigated by the recent preprint [29] (see also [23] for the 2D case).

Main idea and plan of this paper. Due to the loss of a favorable quasi-energy structure and of the regularization signal absorption mechanism, compared with the constant scalar sensitivity S and with the signal consumption case, respectively, our approach underlying the derivation of Theorem 1.1 will be based on an entropy-like estimate involving the functional of the form

$$\|n(\cdot,t)\|_{L^p(\Omega)} + \|\nabla c(\cdot,t)\|_{L^{2q}(\Omega)}$$

for solutions of system (1.1), where we eventually intend to choose p and q arbitrarily large. This idea is similar to its two-dimensional version [38]. However, in contrast to the two-dimensional case, a major technical difficulty here lies in that the evident mass conservation property $\int_\Omega n \equiv \int_\Omega n_0$ is not sufficient to derive some useful regularity information of u and c. Indeed, in [38], we can use some regular properties of Stokes operator to directly obtain the key L^p estimates of u and c for any $p > 1$ from the basic spatial L^1 bound of n. In this paper, for the three-dimensional case, we will develop a new approach to do this, which consists of some bootstrap arguments on solutions of the boundary regularized system (2.1). Precisely, we shall first track the time evolution of the combinational functional of the form $\int_\Omega n_\varepsilon^{2\alpha} + \int_\Omega c_\varepsilon^2$ (Lemma 2.3). Taking this as a starting point of a series of arguments, we establish the L^{p_1} boundedness of u_ε

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for some $p_1 > 3$ (Corollary 2.1). Then it is possible for us to obtain a coupled estimate of $\int_\Omega n_\varepsilon^p$ and $\int_\Omega |\nabla c_\varepsilon|^2$ for some $p < \frac{8}{3}\alpha + \frac{1}{3}$ (Lemma 2.7), which further enforces the regularity of u_ε in arbitrary L^p space. Next we can derive the desired entropy estimate of the form $\|n_\varepsilon\|_{L^p(\Omega)} + \|\nabla c_\varepsilon\|_{L^{2q}(\Omega)}$, which eventually shows the boundedness of n_ε and ∇c_ε in arbitrary L^p space (Corollary 2.3). These boundedness together with the extension criterion of local solution ensures the existence of global bounded solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ for the regularized system (2.1). Finally, Theorem 1.1 is proved through a limit procedure in the regularized system in Section 3.

2. Approximation by homogeneous Neumann boundary problems

To overcome the difficulties brought by the nonlinear boundary condition, we shall first deal with some boundary regularized approximate problems in this section. We follow an idea from [22] and introduce an appropriate regularization in which $\mathcal{S}_\varepsilon$ defined below vanishes near the lateral boundary.

Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subset C_0^\infty(\Omega)$ be a family of standard cut-off functions satisfying $0 \le \rho_\varepsilon \le 1$ in Ω for all $\varepsilon \in (0,1)$ and $\rho_\varepsilon \to 1$ in Ω pointwisely as $\varepsilon \to 0$. Then we define

$$\mathcal{S}_\varepsilon(x,n,c) = \rho_\varepsilon(x)\mathcal{S}(x,n,c), \qquad (x,n,c) \in \overline{\Omega} \times [0,\infty) \times [0,\infty)$$

for $\varepsilon \in (0,1)$. It is clear that for each fixed $\varepsilon \in (0,1)$, $\mathcal{S}_\varepsilon$ vanishes on $\partial\Omega$ and still satisfies (1.7) with the values of $C_\mathcal{S}$ and α unchanged.

We now consider the following approximate problem

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot \big(n_\varepsilon \mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon\big), & x \in \Omega,\ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, & x \in \Omega,\ t > 0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \Delta u_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\ \dfrac{\partial n_\varepsilon}{\partial\nu} = \dfrac{\partial c_\varepsilon}{\partial\nu} = 0,\ u_\varepsilon = 0, & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_0(x),\ c_\varepsilon(x,0) = c_0(x),\ u_\varepsilon(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

By an adaptation of well-established fixed point arguments, we can establish the following local existence result for regularized problem (2.1).

2.1. *Local existence of classical solutions to the regularized system*

We first state the local solvability of system (2.1), which can by proved by a straightforward adaptation of the corresponding procedure in Lemma 2.1 of Winkler [42] to our current setting.

Lemma 2.1. *Suppose that (1.6)–(1.9) hold. Then for each $\varepsilon \in (0,1)$, there exist $T_{\max,\varepsilon} \in (0,\infty]$ and a classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ to system (2.1) in $\Omega \times (0, T_{\max,\varepsilon})$ such that*

$$\begin{aligned} &n_\varepsilon \in C^0\big(\bar\Omega \times [0, T_{\max,\varepsilon})\big) \cap C^{2,1}\big(\bar\Omega \times (0, T_{\max,\varepsilon})\big), \\ &c_\varepsilon \in C^0\big(\bar\Omega \times [0, T_{\max,\varepsilon})\big) \cap C^{2,1}\big(\bar\Omega \times (0, T_{\max,\varepsilon})\big), \\ &u_\varepsilon \in C^0\big(\bar\Omega \times [0, T_{\max,\varepsilon})\big) \cap C^{2,1}\big(\bar\Omega \times (0, T_{\max,\varepsilon})\big), \\ &P_\varepsilon \in C^{1,0}\big(\bar\Omega \times (0, T_{\max,\varepsilon})\big). \end{aligned}$$

Moreover, we have $n_\varepsilon > 0$ *and* $c_\varepsilon > 0$ *in* $\bar{\Omega} \times (0, T_{\max,\varepsilon})$*, and*

$$\begin{aligned} &\text{if } T_{\max,\varepsilon} < \infty, \text{ then } \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\mathcal{A}^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty \\ &\quad \text{as } t \to T_{\max,\varepsilon}, \end{aligned} \tag{2.2}$$

where β *is taken from* (1.8)*. This solution is unique, up to addition of constants to* P_ε.

The following lemma is very basic but important and will be frequently used in the sequel.

Lemma 2.2. *For each* $\varepsilon \in (0, 1)$*, the solution of* (2.1) *satisfies*

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \quad \text{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{2.3}$$

and

$$\int_\Omega c_\varepsilon(\cdot,t) \le \max\left\{\int_\Omega c_0, \int_\Omega n_0\right\} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.4}$$

Proof. The first conclusion directly results from an integration of the first equation in (2.1) over Ω. Then from the second equation in (2.1) we have that

$$\frac{d}{dt}\int_\Omega c_\varepsilon + \int_\Omega c_\varepsilon = \int_\Omega n_\varepsilon = \int_\Omega n_0 \qquad \text{for all } t \in (0, T_{\max,\varepsilon}),$$

which implies (2.4) through an comparison argument. □

2.2. Some low regularity estimates for n_ε *and* c_ε

In this section, we shall propose some low regularity estimates for n_ε and c_ε, by tracking the time evolution of a certain combinational functional of them, which will be a starting point of a series of arguments. Such a functional is motivated by a similar energy-like structure used in [39] (see Section 5 there).

Lemma 2.3. *Suppose that* (1.6)–(1.9) *hold with* $\alpha > \frac{1}{2}$*. There exists a positive constant* C *depending on* α*,* n_0*,* c_0 *and* u_0 *such that for all* $\varepsilon \in (0, 1)$ *we have*

$$\int_\Omega n_\varepsilon^{2\alpha}(\cdot,t) \le C \tag{2.5}$$

and

$$\int_\Omega c_\varepsilon^2(\cdot,t) \le C \tag{2.6}$$

on $(0, T_{\max,\varepsilon})$.

Proof. Testing the first equation in (2.1) by $n_\varepsilon^{2\alpha-1}$ and using the solenoidality of u_ε, the subcriticality assumption (1.7) and Young's inequality, we see that

$$\begin{aligned}\frac{1}{2\alpha}\frac{d}{dt}\int_\Omega n_\varepsilon^{2\alpha} + (2\alpha-1)\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 &= (2\alpha-1)\int_\Omega n_\varepsilon^{2\alpha-1}\nabla n_\varepsilon\cdot\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\\ &\le (2\alpha-1)C_s\int_\Omega n_\varepsilon^{\alpha-1}|\nabla n_\varepsilon|\,|\nabla c_\varepsilon|\\ &\le \frac{2\alpha-1}{2}\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 + \frac{2\alpha-1}{2}C_s^2\int_\Omega |\nabla c_\varepsilon|^2\end{aligned}$$

for all $t\in(0,T_{\max,\varepsilon})$, which implies that

$$\frac{d}{dt}\int_\Omega n_\varepsilon^{2\alpha} + \frac{2\alpha-1}{\alpha}\int_\Omega |\nabla n_\varepsilon^\alpha|^2 \le \alpha(2\alpha-1)C_s^2\int_\Omega |\nabla c_\varepsilon|^2 := C_1\int_\Omega |\nabla c_\varepsilon|^2 \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{2.7}$$

In order to absorb the rightmost integral in (2.7) appropriately, we multiply the second equation in (2.1) by c_ε to obtain from Hölder's inequality that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^2 = \int_\Omega n_\varepsilon c_\varepsilon \le \|c_\varepsilon\|_{L^6(\Omega)}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \tag{2.8}$$

for all $t\in(0,T_{\max,\varepsilon})$. Due to the Sobolev embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ in the three-dimensional setting, there exists $C_2>0$ such that

$$\|c_\varepsilon\|_{L^6(\Omega)}^2 \le C_2\|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + C_2\|c_\varepsilon\|_{L^1(\Omega)}^2$$

for all $t\in(0,T_{\max})$ which together with (2.4) gives that

$$\|c_\varepsilon\|_{L^6(\Omega)}^2 \le C_2\|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + C_3$$

for all $t\in(0,T_{\max})$ with some constant $C_3>0$. Thus by means of Young's inequality, in (2.8) we proceed to estimate

$$\begin{aligned}\int_\Omega n_\varepsilon c_\varepsilon &\le \frac{1}{2C_2}\|c_\varepsilon\|_{L^6(\Omega)}^2 + \frac{C_2}{2}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}^2\\ &\le \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + \frac{C_3}{2C_2} + \frac{C_2}{2}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}^2 \qquad \text{for all } t\in(0,T_{\max,\varepsilon}).\end{aligned} \tag{2.9}$$

We then invoke the Gagliardo–Nirenberg inequality to find $C_4>0$ such that

$$\begin{aligned}\frac{C_2}{2}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} &= \frac{C_2}{2}\|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{6}{5\alpha}}(\Omega)}\\ &\le C_4\|\nabla n_\varepsilon^\alpha\|^{\frac{2}{\alpha}\cdot a}_{L^2(\Omega)}\|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}\cdot(1-a)}_{L^{\frac{1}{\alpha}}(\Omega)} + C_4\|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)}\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $a = \frac{\alpha}{6\alpha-1} \in (0, 1)$. Recalling the mass conservation (2.3) and noticing that $a < \alpha$ due to $\alpha > \frac{1}{3}$, we assert from Young's inequality that

$$\frac{C_2}{2}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \le C_5\|\nabla n_\varepsilon^\alpha\|^{\frac{2}{\alpha}\cdot a}_{L^2(\Omega)} + C_5 \le \frac{2\alpha-1}{8C_1\alpha}\int_\Omega |\nabla n_\varepsilon^\alpha|^2 + C_6$$

for all $t \in (0, T_{\max})$ with some positive constants C_5 and C_6. This together with (2.9) and (2.8) leads to

$$\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \int_\Omega |\nabla c_\varepsilon|^2 + 2\int_\Omega c_\varepsilon^2 \le \frac{2\alpha-1}{4C_1\alpha}\int_\Omega |\nabla n_\varepsilon^\alpha|^2 + 2C_6 + \frac{C_3}{C_2} \tag{2.10}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Then we can infer from an appropriate linear combination of (2.10) and (2.7) that

$$\begin{aligned}\frac{d}{dt}\left\{\int_\Omega n_\varepsilon^{2\alpha} + 2C_1\int_\Omega c_\varepsilon^2\right\} + \frac{2\alpha-1}{2\alpha}\int_\Omega |\nabla n_\varepsilon^\alpha|^2 + 2C_1\int_\Omega |\nabla c_\varepsilon|^2 + 4C_1\int_\Omega c_\varepsilon^2\\ \le 4C_1C_6 + \frac{2C_1C_3}{C_2} \qquad \text{for all } t \in (0, T_{\max,\varepsilon}).\end{aligned} \tag{2.11}$$

If we integrate (2.11) directly, we can only assert that the functional $\int_\Omega n_\varepsilon^{2\alpha}(\cdot,t) + 2C_1\int_\Omega c_\varepsilon^2(\cdot,t)$ grows in time at most linearly. To establish its uniform bound further, we employ the Gagliardo–Nirenberg inequality again to estimate

$$\int_\Omega n_\varepsilon^{2\alpha} = \|n_\varepsilon^\alpha\|^2_{L^2(\Omega)} \le C_7\left(\|\nabla n_\varepsilon^\alpha\|^{\frac{2(2\alpha-1)}{2\alpha-\frac{1}{3}}}_{L^2(\Omega)}\|n_\varepsilon^\alpha\|^{\frac{4}{6\alpha-1}}_{L^{\frac{1}{\alpha}}(\Omega)} + \|n_\varepsilon^\alpha\|^2_{L^{\frac{1}{\alpha}}(\Omega)}\right)$$

for all $t \in (0, T_{\max})$ with some positive constant C_7. Then Young's inequality and the mass conservation (2.3) entail the existence of positive constants C_8 and C_9 such that

$$\int_\Omega n_\varepsilon^{2\alpha} \le C_8\left(\|\nabla n_\varepsilon^\alpha\|^{\frac{2(2\alpha-1)}{2\alpha-\frac{1}{3}}}_{L^2(\Omega)} + 1\right) \le \frac{(2\alpha-1)}{2\alpha}\int_\Omega |\nabla n_\varepsilon^\alpha|^2 + C_9 \quad \text{for all } t \in (0, T_{\max}). \tag{2.12}$$

A similar argument together with (2.4) shows that

$$2C_1\int_\Omega c_\varepsilon^2 \le 2C_1\int_\Omega |\nabla c_\varepsilon|^2 + C_{10} \tag{2.13}$$

with some $C_{10} > 0$. Setting $y(t) := \int_\Omega n_\varepsilon^{2\alpha}(\cdot,t) + 2C_1 \int_\Omega c_\varepsilon^2(\cdot,t)$ for $t \in (0, T_{\max,\varepsilon})$ and then using (2.11), (2.12) and (2.13), we have

$$y'(t) + y(t) \le 4C_1C_6 + \frac{2C_1C_3}{C_2} + C_9 + C_{10},$$

which immediately implies that

$$y(t) \le C_{11} := \max\left\{\int_\Omega n_0^{2\alpha} + 2C_1\int_\Omega c_0^2,\ 4C_1C_6 + \frac{2C_1C_3}{C_2} + C_9 + C_{10}\right\}$$

for all $t \in (0, T_{\max,\varepsilon})$. This uniform bound gives the desired low regularity estimates (2.5) and (2.6). □

2.3. *Regularity of u_ε in arbitrary L^p spaces*

In this section, we will establish the L^p regularity of u_ε for any $p > 1$. From the third equation of (1.1), our argument will be based on the L^p regularity of n_ε, which is rather low as we see from Lemma 2.3. To overcome this difficulty, we will use a bootstrap argument. Precisely, we first use the third equation of (1.1) and the L^p regularity of n_ε obtained in Lemma 2.3 to derive some low L^p regularity for u_ε. Then by tracking the evolution of a new functional, we obtain the higher regularity of n_ε from this low regularity of u_ε, which in turn gives the L^p regularity of u_ε for arbitrarily large p.

We begin with recalling several basic facts related to the Stokes operator [13,14]. For each $r \in (1,\infty)$, the Helmholtz projection acts as a bounded linear operator $\mathcal{P}_r$ from $L^r(\Omega)$ onto its solenoidal subspace $L^r_\sigma(\Omega) := \{\varphi \in L^r(\Omega) \mid \nabla\cdot\varphi = 0\}$. The realization $\mathcal{A}_r$ of the Stokes operator $\mathcal{A}$ in $L^r_\sigma(\Omega)$ with domain $D(\mathcal{A}_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L^r_\sigma(\Omega)$ is sectorial and generates an analytic semigroup $(e^{-t\mathcal{A}_r})_{t\ge 0}$ in $L^r_\sigma(\Omega)$. Moreover, $\mathcal{A}_r$ possesses closed fractional powers $\mathcal{A}_r^\beta$ with dense domains for any $\beta \in \mathbb{R}$. We shall omit an explicit index r whenever there is no danger of confusion in the remaining part of this paper, for that $\mathcal{P}_r$, $\mathcal{A}_r^\beta$ and $(e^{-t\mathcal{A}_r})_{t\ge 0}$ are all actually independent of $r \in (1,\infty)$ whenever applied to smooth functions.

We then introduce the following lemma, which shows that, roughly speaking, up to projection to divergence-free vector fields, functions from L^p $(p \ge 1)$ can be viewed as elements of $D(\mathcal{A}_{p_0}^{-\gamma})$ for any $p_0 > p$ and suitable $\gamma > 0$.

Lemma 2.4 *(see [46], Lemma 3.3). Suppose that $1 \le p < p_0 < \infty$, and that $\gamma \in (0,1)$ is such that $2\gamma - \frac{3}{p} > -\frac{3}{p_0}$. Then there exists $C > 0$ such that*

$$\|\mathcal{A}^{-\gamma}\mathcal{P}\varphi\|_{L^{p_0}(\Omega)} \le C\|\varphi\|_{L^p(\Omega)}$$

for all $\varphi \in C_0^\infty(\Omega)$.

The following lemma is the foundation of our bootstrap argument, which shows the gain of regularity of u_ε from the *a priori* regularity of n_ε.

Lemma 2.5. *Let $p \in [1, +\infty)$ and $r \in [1, +\infty]$ be such that*

$$\begin{cases} r < \dfrac{3p}{3-2p} & \text{if } p \le \frac{3}{2}, \\ r \le \infty & \text{if } p > \frac{3}{2}. \end{cases} \tag{2.14}$$

Then for all $K > 0$ there exists $C = C(p, r, K, u_0, \phi)$ such that if for some $\varepsilon \in (0, 1)$ we have

$$\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \le K \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{2.15}$$

then

$$\|u_\varepsilon(\cdot, t)\|_{L^r(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.16}$$

Proof. For any p and r satisfying (2.14), we can fix r_0 such that $r_0 \in \big(\frac{3}{2}(\frac{1}{p} - \frac{1}{r}), 1\big)$ for $r \in [1, \infty)$ and $r_0 \in \big(\frac{3}{2p}, 1\big)$ for $r = \infty$. It then follows from the variation-of-constants representation

$$u_\varepsilon(t) = e^{-t\mathcal{A}} u_0 + \int_0^t e^{-(t-\tau)\mathcal{A}} \mathcal{P}(n_\varepsilon(\cdot, \tau) \cdot \nabla\phi) d\tau \qquad \text{for all } \ t \in (0, T_{\max,\varepsilon})$$

and Young's inequality that

$$\|u_\varepsilon(t)\|_{L^r(\Omega)} \le \|e^{-t\mathcal{A}} u_0\|_{L^r(\Omega)} + \int_0^t \big\| \mathcal{A}^{r_0} e^{-(t-\tau)\mathcal{A}} \mathcal{A}^{-r_0} \mathcal{P}\big(n_\varepsilon(\cdot, \tau)\nabla\phi\big)\big\|_{L^r(\Omega)} d\tau \tag{2.17}$$

for all $t \in (0, T_{\max,\varepsilon})$. Since $u_0 \in L^r_\sigma(\Omega)$ as a consequence of (1.8), we have

$$\|e^{-t\mathcal{A}} u_0\|_{L^r(\Omega)} \le C_1 \tag{2.18}$$

for some positive constant C_1. On the other hand, when $r \in [1, \infty)$, by the smoothing effect and decay estimates of the Stokes semigroup (see [14]), we can find a constant $\lambda > 0$ such that

$$\big\| \mathcal{A}^{r_0} e^{-(t-\tau)\mathcal{A}} \mathcal{A}^{-r_0} \mathcal{P}\big(n_\varepsilon(\cdot, \tau)\nabla\phi\big)\big\|_{L^r(\Omega)} \le C_2 (t-\tau)^{-r_0} e^{-\lambda(t-\tau)} \big\| \mathcal{A}^{-r_0} \mathcal{P}\big(n_\varepsilon(\cdot, \tau)\nabla\phi\big)\big\|_{L^r(\Omega)}$$

with some $C_2 > 0$. The fact $r_0 > \frac{3}{2}(\frac{1}{p} - \frac{1}{r})$ enables us to apply Lemma 2.4 to obtain

$$\|\mathcal{A}^{-r_0} \mathcal{P}\big(n_\varepsilon(\cdot, \tau)\nabla\phi\big)\|_{L^r(\Omega)} \le C_3 \|n_\varepsilon(\cdot, \tau)\nabla\phi\|_{L^p(\Omega)}$$

for some $C_3 > 0$, which together with the regularity (2.15) of n_ε and the boundedness of $\nabla\phi$ gives that

$$\|\mathcal{A}^{-r_0} \mathcal{P}\big(n_\varepsilon(\cdot, \tau)\nabla\phi\big)\|_{L^r(\Omega)} \le C_4$$

for some positive constant C_4 and thus

$$\left\|\mathcal{A}^{r_0}e^{-(t-\tau)\mathcal{A}}\mathcal{A}^{-r_0}\mathcal{P}\big(n_\varepsilon(\cdot,\tau)\nabla\phi\big)\right\|_{L^r(\Omega)} \le C_2C_4(t-\tau)^{-r_0}e^{-\lambda(t-\tau)}. \tag{2.19}$$

When $r=\infty$, we first fix $\delta\in(0,1-r_0)$ and then choose $r_1>\frac{3}{\delta}$ such that $W^{\delta,r_1}(\Omega)\hookrightarrow L^\infty(\Omega)$. Thus the Sobolev embedding together with the same way as $r\in[1,\infty)$ yields that

$$\begin{aligned}
&\left\|\mathcal{A}^{r_0}e^{-(t-\tau)\mathcal{A}}\mathcal{A}^{-r_0}\mathcal{P}\big(n_\varepsilon(\cdot,\tau)\nabla\phi\big)\right\|_{L^\infty(\Omega)}\\
&\le \left\|\mathcal{A}^{r_0+\delta}e^{-(t-\tau)\mathcal{A}}\mathcal{A}^{-r_0}\mathcal{P}\big(n_\varepsilon(\cdot,\tau)\nabla\phi\big)\right\|_{L^{r_1}(\Omega)}\\
&\le C_5(t-\tau)^{-r_0-\delta}e^{-\lambda(t-\tau)}\left\|\mathcal{A}^{-r_0}\mathcal{P}\big(n_\varepsilon(\cdot,\tau)\nabla\phi\big)\right\|_{L^{r_1}(\Omega)}
\end{aligned}$$

with some $C_5>0$. Due to $r_0>\frac{3}{2p}>\frac{3}{2}(\frac{1}{p}-\frac{1}{r_1})$, Lemma 2.4 gives that

$$\|\mathcal{A}^{-r_0}\mathcal{P}\big(n_\varepsilon(\cdot,\tau)\nabla\phi\big)\|_{L^{r_1}(\Omega)} \le C_6\|n_\varepsilon(\cdot,\tau)\nabla\phi\|_{L^p(\Omega)}$$

for some $C_6>0$, which together with the regularity (2.15) of n_ε and the boundedness of $\nabla\phi$ gives that

$$\|\mathcal{A}^{-r_0}\mathcal{P}\big(n_\varepsilon(\cdot,\tau)\nabla\phi\big)\|_{L^{r_1}(\Omega)} \le C_7$$

for some positive constant C_7 and thus

$$\left\|\mathcal{A}^{r_0}e^{-(t-\tau)\mathcal{A}}\mathcal{A}^{-r_0}\mathcal{P}\big(n_\varepsilon(\cdot,\tau)\nabla\phi\big)\right\|_{L^\infty(\Omega)} \le C_5C_7(t-\tau)^{-r_0-\delta}e^{-\lambda(t-\tau)}. \tag{2.20}$$

Thereupon, substituting (2.18), (2.19) or (2.20) into (2.17), we conclude from $0<r_0<r_0+\delta<1$ and $\lambda>0$ that

$$\begin{aligned}
\|u_\varepsilon(t)\|_{L^r(\Omega)} &\le C_1+\max\left\{C_2C_4\int_0^t(t-\tau)^{-r_0}e^{-\lambda(t-\tau)}d\tau,\ C_5C_7\int_0^t(t-\tau)^{-r_0-\delta}e^{-\lambda(t-\tau)}d\tau\right\}\\
&\le C_8 \qquad \text{for all } t\in(0,T_{\max,\varepsilon})
\end{aligned}$$

with some $C_8>0$ depending on p, r, K, u_0 and ϕ. □

As the first application of Lemma 2.5, we can deduce some L^p bound for u_ε with p suitably large from the low integrability of n_ε.

Corollary 2.1. *Suppose that (1.6)–(1.9) hold with $\alpha>\frac{1}{2}$. Then there exist $p_1>3$ and a positive constant $C=C(p_1,\alpha,u_0,\phi)$ such that for all $\varepsilon\in(0,1)$ and $t\in(0,T_{\max,\varepsilon})$, we have*

$$\|u_\varepsilon(\cdot,t)\|_{L^{p_1}(\Omega)} \le C.$$

Proof. Setting $p=2\alpha$, we see that $\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}$ is bounded by Lemma 2.3. Notice that $\frac{3p}{3-2p}>3$ if $1<p\le\frac{3}{2}$. Thus the desired conclusion can be directly derived from Lemma 2.5. □

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To deduce the eventual L^p boundedness of u_ε for any $p > 1$, we need to derive further regularity properties of the solutions to (2.1), and in particular of n_ε by Lemma 2.5. Thus we now investigate a rudimentary time evolution outcome of $\int_\Omega n_\varepsilon^p$ for all $p > 1$, which will also be used in next subsection.

Lemma 2.6. *Let $p > 1$. Then for all $\varepsilon \in (0, 1)$,*

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{3(p-1)}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 \le p(p-1)C_{\mathcal{S}}^2 \int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{2.21}$$

with $C_{\mathcal{S}}$ taken from (1.7).

Proof. Multiplying the first equation in (2.1) by n_ε^{p-1}, integrating by parts over Ω and using Young's inequality and the upper estimate (1.7) for $\mathcal{S}$, we obtain that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + (p-1)\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 &= (p-1)\int_\Omega n_\varepsilon^{p-1}\nabla n_\varepsilon \cdot \big(\mathcal{S}(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon\big) \\ &\le \frac{(p-1)}{4}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + (p-1)C_{\mathcal{S}}^2\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, which immediately yields

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{3(p-1)}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 \le p(p-1)C_{\mathcal{S}}^2 \int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \tag{2.22}$$

for all $t \in (0, T_{\max,\varepsilon})$. □

From Lemma 2.6, it seems to be necessary to investigate the time evolution of $\int_\Omega |\nabla c_\varepsilon|^2$, which motivates our consideration on the combinational functional of $\int_\Omega n_\varepsilon^p$ and $\int_\Omega |\nabla c_\varepsilon|^2$.

Lemma 2.7. *Suppose that (1.6)–(1.9) hold with $\alpha > \frac{1}{2}$. If $\max\left\{\frac{4}{3}, 2\alpha + \frac{1}{3}\right\} < p < \frac{8}{3}\alpha + \frac{1}{3}$, then there exists a positive constant C such that for all $\varepsilon \in (0, 1)$, we have*

$$\int_\Omega n_\varepsilon^p(\cdot, t) + \int_\Omega |\nabla c_\varepsilon(\cdot, t)|^2 \le C \qquad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

Proof. We shall derive our conclusion by establishing an ODI for the combinational functional

$$\int_\Omega n_\varepsilon^p(\cdot, t) + \int_\Omega |\nabla c_\varepsilon(\cdot, t)|^2.$$

To this end, we first pay our attention to estimate the rightmost integral in (2.22). Using Hölder's inequality we see

$$\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \le \left(\int_\Omega n_\varepsilon^{3(p-2\alpha)}\right)^{\frac{1}{3}} \left(\int_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{2.23}$$

For the first factor on the right hand side of (2.23), we invoke the Gagliardo–Nirenberg inequality to find $C_1>0$ such that

$$\left(\int_\Omega n_\varepsilon^{3(p-2\alpha)}\right)^{\frac{1}{3}} = \|n_\varepsilon^{\frac{p}{2}}\|^{\frac{2(p-2\alpha)}{p}}_{L^{\frac{6(p-2\alpha)}{p}}(\Omega)} \le C_1\left(\|\nabla n_\varepsilon^{\frac{p}{2}}\|^{b_1}_{L^2(\Omega)}\|n_\varepsilon^{\frac{p}{2}}\|^{1-b_1}_{L^{\frac{2}{p}}(\Omega)} + \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}\right)^{\frac{2(p-2\alpha)}{p}}$$

for all $t\in(0,T_{\max,\varepsilon})$, where

$$b_1=\frac{3p-\frac{p}{p-2\alpha}}{3p-1}\in(0,1)$$

due to $p>2\alpha+\frac{1}{3}$. Recalling the mass conservation (2.3), we can obtain that there exists $C_2>0$ fulfilling

$$\left(\int_\Omega n_\varepsilon^{3(p-2\alpha)}\right)^{\frac{1}{3}} \le C_2\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right)^{\frac{3(p-2\alpha)-1}{3p-1}} + C_2 \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{2.24}$$

On the other hand, for the second factor on the right hand side of (2.23), we apply the Gagliardo–Nirenberg inequality again to find $C_3>0$ such that

$$\left(\int_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} = \|\nabla c_\varepsilon\|^2_{L^3(\Omega)} \le C_3\left(\|\Delta c_\varepsilon\|^{\frac{3}{4}}_{L^2(\Omega)}\|c_\varepsilon\|^{\frac{1}{4}}_{L^2(\Omega)} + \|c_\varepsilon\|_{L^2(\Omega)}\right)^2$$

for all $t\in(0,T_{\max,\varepsilon})$, which together with (2.6) gives that

$$\left(\int_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} \le C_4\left(\int_\Omega |\Delta c_\varepsilon|^2\right)^{\frac{3}{4}} + C_4 \quad \text{for all } t\in(0,T_{\max,\varepsilon}) \tag{2.25}$$

with some constant $C_4>0$. Substituting (2.24) and (2.25) into (2.23), we obtain

$$\begin{aligned}\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \le{}& C_2C_4\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right)^{\frac{3(p-2\alpha)-1}{3p-1}}\left(\int_\Omega |\Delta c_\varepsilon|^2\right)^{\frac{3}{4}} + C_2C_4\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right)^{\frac{3(p-2\alpha)-1}{3p-1}}\\ &+ C_2C_4\left(\int_\Omega |\Delta c_\varepsilon|^2\right)^{\frac{3}{4}} + C_2C_4\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$. Then by using Young's inequality, we see that there exists $C_5 > 0$ such that

$$\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \le \frac{1}{p^2 C_{\mathcal{S}}^2}\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right) + C_5\left(\int_\Omega |\Delta c_\varepsilon|^2\right)^{\frac{3}{4}\cdot\frac{3p-1}{6\alpha}} + C_2C_4\left(\int_\Omega |\Delta c_\varepsilon|^2\right)^{\frac{3}{4}} + C_2C_4(C_5+1)$$

for all $t \in (0, T_{\max,\varepsilon})$. Since $p < \frac{8}{3}\alpha + \frac{1}{3}$ implies that

$$\frac{3}{4}\cdot\frac{p-\frac{1}{3}}{2\alpha} < 1,$$

Young's inequality once more entails that there exists $C_6 > 0$ such that

$$\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \le \frac{1}{p^2 C_{\mathcal{S}}^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{1}{4p(p-1)C_{\mathcal{S}}^2}\int_\Omega |\Delta c_\varepsilon|^2 + C_6 \tag{2.26}$$

for all $t \in (0, T_{\max,\varepsilon})$. We can thereof infer from (2.22) and (2.26) that

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{2(p-1)}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 \le \frac{1}{4}\int_\Omega |\Delta c_\varepsilon|^2 + C_7 \qquad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{2.27}$$

with $C_7 = C_6 p(p-1)C_{\mathcal{S}}^2$.

To absorb the rightmost integral in (2.27) appropriately, we multiply the second equation by $-\Delta c_\varepsilon$ and integrate on Ω to obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2 = -\int_\Omega \Delta c_\varepsilon \cdot n_\varepsilon - \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\Delta c_\varepsilon \tag{2.28}$$

for all $t \in (0, T_{\max,\varepsilon})$. For the second integral on the right-hand side of (2.28), we first use Hölder's inequality to conclude that

$$\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\Delta c_\varepsilon \le \|\Delta c_\varepsilon\|_{L^2(\Omega)}\cdot\|u_\varepsilon\|_{L^{p_1}(\Omega)}\cdot\|\nabla c_\varepsilon\|_{L^{\frac{2p_1}{p_1-2}}(\Omega)} \tag{2.29}$$

for all $t \in (0, T_{\max,\varepsilon})$, where p_1 is taken from Corollary 2.1. By the Gagliardo–Nirenberg inequality, we have

$$\|\nabla c_\varepsilon\|_{L^{\frac{2p_1}{p_1-2}}(\Omega)} \le C_8\|\Delta c_\varepsilon\|_{L^2(\Omega)}^a\cdot\|c_\varepsilon\|_{L^2(\Omega)}^{1-a} + C_8\|c_\varepsilon\|_{L^2(\Omega)} \qquad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{2.30}$$

where $a = \frac{3}{2}\left(\frac{5}{6} - \frac{p_1-2}{2p_1}\right)$. Notice that the fact $p_1 > 3$ implies $\frac{p_1-2}{2p_1} > \frac{1}{6}$, which means $a \in (0,1)$.

Then (2.29) and (2.30) yield that

$$\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon \le C_8 \|\Delta c_\varepsilon\|_{L^2(\Omega)}^{1+a} \cdot \|u_\varepsilon\|_{L^{p_1}(\Omega)} \cdot \|c_\varepsilon\|_{L^2(\Omega)}^{1-a} + C_8 \|\Delta c_\varepsilon\|_{L^2(\Omega)} \cdot \|u_\varepsilon\|_{L^{p_1}(\Omega)} \cdot \|c_\varepsilon\|_{L^2(\Omega)}$$

for all $t \in (0, T_{\max,\varepsilon})$. Thus the boundedness of $\|u_\varepsilon\|_{L^{p_1}(\Omega)}$ and $\|c_\varepsilon\|_{L^2}$ obtained in Corollary 2.1 and Lemma 2.3 together with Young's inequality implies that

$$\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon \le \frac{1}{4}\int_\Omega |\Delta c_\varepsilon|^2 + C_9 \qquad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{2.31}$$

with some $C_9 > 0$. On the other hand, for the first integral on the right of (2.28), we have

$$\int_\Omega \Delta c_\varepsilon \cdot n_\varepsilon \le \frac{1}{4}|\Delta c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 \qquad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.32}$$

Substituting (2.31) and (2.32) into (2.28), we obtain

$$\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 + 2\int_\Omega |\nabla c_\varepsilon|^2 \le 2\int_\Omega n_\varepsilon^2 + 2C_9 \qquad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.33}$$

By using Gagliardo–Nirenberg inequality again, we have

$$\int_\Omega n_\varepsilon^2 = \|n^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p}} \le C_{10}\left(\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{b_2} \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{1-b_2} + \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}\right)^{\frac{4}{p}} \tag{2.34}$$

for all $t \in (0, T_{\max,\varepsilon})$ with some $C_{10} > 0$, where

$$b_2 = \frac{\frac{p}{4}}{\frac{p}{2} - \frac{1}{6}} \in (0, 1).$$

Due to $p > \frac{4}{3}$, we have $b_2 \cdot \frac{4}{p} < 2$. Then similar to above reasoning, we deduce from (2.34) that there exists $C_{11} > 0$ fulfilling

$$\int_\Omega n_\varepsilon^2 \le \frac{p-1}{2p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + C_{11} \qquad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.35}$$

Thereupon we can infer from (2.35) and (2.33) that

$$\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 \le \frac{p-1}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + C_{12} \tag{2.36}$$

for all $t \in (0, T_{\max,\varepsilon})$ with some $C_{12} > 0$.

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Summing up (2.27) and (2.36), we have

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c|^2\right\} + \frac{p-1}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{3}{4}\int_\Omega |\Delta c_\varepsilon|^2 \le C_{13} \tag{2.37}$$

holds for all $t \in (0, T_{\max,\varepsilon})$ with $C_{13} = C_7 + C_{12}$. To obtain the uniform bound of the functional, we notice that

$$\int_\Omega n_\varepsilon^p \le \frac{p-1}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + C_{14} \qquad \text{for all } \ t \in (0, T_{\max,\varepsilon})$$

with some $C_{14} > 0$, which follows from a similar procedure as (2.12), and that

$$\int_\Omega |\nabla c_\varepsilon|^2 \le \frac{3}{4}\int_\Omega |\Delta c|^2 + C_{15} \qquad \text{for all } \ t \in (0, T_{\max,\varepsilon})$$

for some positive constant C_{15}. Inserting these two inequalities into (2.37), we conclude that there exists a positive constant C_{16} such that

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^2\right\} + \left\{\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^2\right\} \le C_{16}$$

for all $t \in (0, T_{\max,\varepsilon})$, which immediately leads to

$$\int_\Omega n_\varepsilon^p(\cdot,t) + \int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2 \le C$$

with some positive constant C for all $t \in (0, T_{\max,\varepsilon})$. $\square$

It is clear that $\alpha > \frac{1}{2}$ is equivalent to $\frac{8}{3}\alpha + \frac{1}{3} > \frac{5}{3}$. Thus Lemma 2.7 implies that $\int_\Omega n_\varepsilon^{\frac{5}{3}}(\cdot,t)$ is bounded for all $t \in (0, T_{\max})$. Thereupon, we can finally achieve the regularity for u_ε in arbitrary L^p space from Lemma 2.5 due to $\frac{5}{3} > \frac{3}{2}$.

Corollary 2.2. *Suppose that (1.6)–(1.9) hold with $\alpha > \frac{1}{2}$. Then for any $p > 1$, there exists a positive constant $C = C(p, u_0, n_0, \phi)$ such that for all $\varepsilon \in (0, 1)$, we have*

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \qquad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}).$$

2.4. Regularity of n_ε and ∇c_ε in arbitrary L^p spaces

By making use of the regularity information obtained so far, we now devote our attention to establish a coupled entropy estimate for n_ε and ∇c_ε, from which we can eventually deduce the integrability of n_ε and ∇c_ε in arbitrary L^p spaces.

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Lemma 2.8. *Suppose that (1.6)–(1.9) hold with $\alpha > \frac{1}{2}$. Then for any $q > 1$, it holds for all $\varepsilon \in (0,1)$ that*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2q} + \frac{q-1}{2q}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2 \\
&\leq (2q^2+q)\int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} + (4q^2+2q)\int_\Omega |\nabla c_\varepsilon|^{2q}|u_\varepsilon|^2 + C
\end{aligned} \tag{2.38}$$

on $(0, T_{\max,\varepsilon})$ with some positive constant $C = C(q)$.

Proof. Applying ∇ to equation $(2.1)_2$ and then multiplying the resulting equation by $2q|\nabla c_\varepsilon|^{2(q-1)}\nabla c_\varepsilon$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2q} - 2q\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}\nabla c_\varepsilon \cdot \Delta\nabla c_\varepsilon + 2q\int_\Omega |\nabla c_\varepsilon|^{2q} \\
&= 2q\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}\nabla c_\varepsilon \cdot \nabla n_\varepsilon - 2q\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}\nabla c_\varepsilon \cdot \nabla\big(u \cdot \nabla c_\varepsilon\big)
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$. Noticing the pointwise identity $2\nabla c_\varepsilon \cdot \nabla\Delta c_\varepsilon = \Delta|\nabla c_\varepsilon|^2 - 2|D^2 c_\varepsilon|^2$ and using the integration by parts, we deduce

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2q} + q(q-1)\int_\Omega |\nabla c_\varepsilon|^{2(q-2)}\left|\nabla|\nabla c_\varepsilon|^2\right|^2 \\
&\quad + 2q\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}|D^2 c_\varepsilon|^2 dx + 2q\int_\Omega |\nabla c_\varepsilon|^{2q} \\
&= 2q\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}\nabla n_\varepsilon \cdot \nabla c_\varepsilon dx + 2q(q-1)\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)|\nabla c_\varepsilon|^{2(q-2)}\nabla c_\varepsilon \cdot \nabla|\nabla c_\varepsilon|^2 \\
&\quad + 2q\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)|\nabla c_\varepsilon|^{2(q-1)}\Delta c_\varepsilon + q\int_{\partial\Omega} |\nabla c_\varepsilon|^{2(q-1)}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}
\end{aligned} \tag{2.39}$$

for all $t \in (0, T_{\max,\varepsilon})$.

We now estimate the right hand side of (2.39) one by one. For the first term, due to $|\Delta c_\varepsilon|^2 \leq 3|D^2 c_\varepsilon|^2$, we can use the integration by parts and Young's inequality to obtain

$$\begin{aligned}
&2q\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}\nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
&\quad = -2q\int_\Omega |\nabla c_\varepsilon|^{2(q-1)} n_\varepsilon \Delta c_\varepsilon - 2q(q-1)\int_\Omega |\nabla c_\varepsilon|^{2(q-2)} n_\varepsilon \nabla c_\varepsilon \cdot \nabla|\nabla c_\varepsilon|^2
\end{aligned}$$

$$\leq 2\sqrt{3}q\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}n_\varepsilon|D^2c_\varepsilon| + 2q(q-1)\int_\Omega |\nabla c_\varepsilon|^{2q-3}n_\varepsilon\left|\nabla|\nabla c_\varepsilon|^2\right|$$

$$\leq q\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}|D^2c_\varepsilon|^2 + \frac{q(q-1)}{2}\int_\Omega |\nabla c_\varepsilon|^{2(q-2)}\left|\nabla|\nabla c_\varepsilon|^2\right|^2$$

$$+ (2q^2+q)\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2(q-1)} \tag{2.40}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Similarly, for the second and third terms on the right hand side of (2.39), we have

$$\begin{aligned} &2q(q-1)\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2(q-2)}\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 \\ &\leq \frac{q(q-1)}{4}\int_\Omega |\nabla c_\varepsilon|^{2(q-2)}\left|\nabla|\nabla c_\varepsilon|^2\right|^2 + 4q(q-1)\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^{2q} \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} 2q\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2(q-1)}\Delta c_\varepsilon &\leq 2\sqrt{3}q\int_\Omega |u_\varepsilon||\nabla c_\varepsilon|^{2q-1}|D^2c_\varepsilon| \\ &\leq \frac{q}{2}\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}|D^2c_\varepsilon|^2 + 6q\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^{2q} \end{aligned} \tag{2.42}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Finally, for the last term, we know from (3.7) and (3.8) of [16] that there exists $C_1 > 0$ such that

$$\int_{\partial\Omega} \frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}|\nabla c_\varepsilon|^{2q-2} \leq C_1\left\||\nabla c_\varepsilon|^q\right\|^2_{W^{\frac{3}{4},2}(\Omega)}.$$

On the other hand, the Gagliardo–Nirenberg inequality implies that

$$\left\||\nabla c_\varepsilon|^q\right\|_{W^{\frac{3}{4},2}(\Omega)} \leq C_2\left\|\nabla|\nabla c_\varepsilon|^q\right\|^a_{L^2(\Omega)}\cdot\left\||\nabla c_\varepsilon|^q\right\|^{1-a}_{L^{\frac{2}{q}}(\Omega)} + C_2\left\||\nabla c_\varepsilon|^q\right\|_{L^{\frac{2}{q}}(\Omega)}$$
$$\text{for all } t \in (0, T_{\max,\varepsilon})$$

with some $C_2 > 0$ and $a = \frac{\frac{q}{2}-\frac{1}{4}}{\frac{q}{2}-\frac{1}{6}} \in (0,1)$. Thereupon, the boundedness of $\|\nabla c_\varepsilon\|_{L^2(\Omega)}$ obtained in Lemma 2.7 and Young's inequality ensure the existence of a positive constant C_3 satisfying

$$q\int_{\partial\Omega} |\nabla c_\varepsilon|^{2(q-1)}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} \leq \frac{q-1}{2q}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2 + C_3 \qquad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{2.43}$$

Substituting (2.40)–(2.43) into (2.39) and noticing the identity $\frac{1}{4}|\nabla c_\varepsilon|^{2(q-2)}\left|\nabla|\nabla c_\varepsilon|^2\right|^2 = \frac{1}{q^2}\left|\nabla|\nabla c_\varepsilon|^q\right|^2$, we obtain

$$\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2q} + \frac{q-1}{2q}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2 + \frac{q}{2}\int_\Omega |\nabla c_\varepsilon|^{2(q-1)}|D^2 c_\varepsilon|^2 + 2q\int_\Omega |\nabla c_\varepsilon|^{2q}$$
$$\le (2q^2+q)\int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2(q-1)}dx + (4q^2+2q)\int_\Omega |\nabla c_\varepsilon|^{2q}|u_\varepsilon|^2 + C_3$$

for all $t \in (0, T_{\max,\varepsilon})$, which immediately leads to our conclusion. □

We can now establish the desired combined estimate of $\|n_\varepsilon(t)\|_{L^p(\Omega)}$ and $\|\nabla c_\varepsilon(t)\|_{L^{2q}(\Omega)}$.

Lemma 2.9. *Suppose that (1.6)–(1.9) hold with $\alpha > \frac{1}{2}$. Let $q \ge 2$ and $p > 1$ is such that*

$$\max\left\{\frac{4}{3}, \frac{2\alpha}{q-\frac{1}{3}}\right\} < \frac{p-\frac{1}{3}}{q-\frac{1}{3}} < 6\alpha, \tag{2.44}$$

then there exists a positive constant $C = C(p, q, C_s, \alpha, n_0, c_0)$ such that for all $\varepsilon \in (0, 1)$

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} + \|\nabla c_\varepsilon(\cdot,t)\|_{L^{2q}(\Omega)} \le C \tag{2.45}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Proof. Combining Lemma 2.6 and Lemma 2.8, we see that for all $\varepsilon \in (0, 1)$

$$\frac{d}{dt}\left(\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^{2q}\right) + \frac{3(p-1)}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{q-1}{2q}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2$$
$$\le p(p-1)C_{\mathcal{S}}^2\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 + (2q^2+q)\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2q-2} + (4q^2+2q)\int_\Omega |\nabla c_\varepsilon|^{2q}|u_\varepsilon|^2 + C_1 \tag{2.46}$$

on $(0, T_{\max,\varepsilon})$ with some constant $C_1 > 0$. We shall show that each term of the three integrals on the right hand of (2.46) can be controlled by the two integrals on the left hand of (2.46) provided (2.44) is fulfilled.

For the first one, we first use Hölder's inequality to obtain

$$\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \le \left(\int_\Omega n_\varepsilon^{3(p-2\alpha)}\right)^{\frac{1}{3}} \cdot \left(\int_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.47}$$

Noticing that $p > 2\alpha + \frac{1}{3}$, we know from (2.24) that

$$\left(\int_\Omega n_\varepsilon^{3(p-2\alpha)}\right)^{\frac{1}{3}} = \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{6(p-2\alpha)}{p}}(\Omega)}^{\frac{2(p-2\alpha)}{p}} \le C_2\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right)^{\frac{p-2\alpha-\frac{1}{3}}{p-\frac{1}{3}}} + C_2 \quad \text{for all } t\in(0,T_{\max,\varepsilon}), \tag{2.48}$$

with some constant $C_2 > 0$. On the other hand, the Gagliardo–Nirenberg inequality shows that

$$\begin{aligned}\left(\int_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} &= \left\||\nabla c_\varepsilon|^q\right\|_{L^{\frac{3}{q}}(\Omega)}^{\frac{2}{q}} \\ &\le C_3\left(\left\|\nabla|\nabla c_\varepsilon|^q\right\|_{L^2(\Omega)}^{\frac{q}{3q-1}}\cdot\left\|\nabla|c_\varepsilon|^q\right\|_{L^{\frac{2}{q}}(\Omega)}^{\frac{2q-1}{3q-1}} + \left\|\nabla|c_\varepsilon|^q\right\|_{L^{\frac{2}{q}}(\Omega)}\right)^{\frac{2}{q}} \\ &\le C_4\left(\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2\right)^{\frac{1}{3q-1}} + C_4 \end{aligned}\tag{2.49}$$

for all $t\in(0,T_{\max,\varepsilon})$ with some positive constants C_3 and C_4, where in last step we used the boundedness of $\|\nabla c_\varepsilon\|_{L^2(\Omega)}$. Submitting (2.48) and (2.49) into (2.47), we arrive at

$$\begin{aligned}\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \le{}& C_2C_4\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right)^{\frac{p-2\alpha-\frac{1}{3}}{p-\frac{1}{3}}}\cdot\left(\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2\right)^{\frac{1}{3q-1}} \\ &+ C_2C_4\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right)^{\frac{p-2\alpha-\frac{1}{3}}{p-\frac{1}{3}}} + C_2C_4\left(\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2\right)^{\frac{1}{3q-1}} \\ &+ C_2C_4 \end{aligned}\tag{2.50}$$

for all $t\in(0,T_{\max,\varepsilon})$.

Since

$$\frac{p-2\alpha-\frac{1}{3}}{p-\frac{1}{3}} + \frac{1}{3q-1} < 1$$

due to $\frac{p-\frac{1}{3}}{q-\frac{1}{3}} < 6\alpha$, Young's inequality entails that for any $\zeta > 0$,

$$\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \le \zeta\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right) + \zeta\left(\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2\right) + C_5 \tag{2.51}$$

for all $t\in(0,T_{\max,\varepsilon})$ with some positive constant $C_5 = C(\zeta) > 0$.

Next, in quite a similar manner, we can estimate the second summands on the right of (2.46). Indeed, Hölder's inequality shows that

$$\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2(q-1)} \le \left(\int_\Omega n^3\right)^{\frac{2}{3}} \cdot \left(\int_\Omega |\nabla c_\varepsilon|^{6(q-1)}\right)^{\frac{1}{3}} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.52}$$

Then we can make use of Young's inequality to estimate $\left(\int_\Omega n_\varepsilon^3\right)^{\frac{2}{3}}$ and $\left(\int_\Omega |\nabla c_\varepsilon|^{6(q-1)}\right)^{\frac{1}{3}}$ as (2.48) and (2.49), respectively. This eventually leads to

$$\begin{aligned}\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2(q-1)} \le {}& C_5\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right)^{\frac{\frac{4}{3}}{p-\frac{1}{3}}} \cdot \left(\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2\right)^{\frac{q-\frac{4}{3}}{q-\frac{1}{3}}} + C_5\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right)^{\frac{\frac{4}{3}}{p-\frac{1}{3}}} \\ &+C_5\left(\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2\right)^{\frac{q-\frac{4}{3}}{q-\frac{1}{3}}} + C_5\end{aligned} \tag{2.53}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Since $\frac{p-\frac{1}{3}}{q-\frac{1}{3}} > \frac{4}{3}$ from (2.44), we have

$$\frac{\frac{4}{3}}{p-\frac{1}{3}} + \frac{q-\frac{4}{3}}{q-\frac{1}{3}} < 1 \qquad \text{and} \qquad \frac{\frac{4}{3}}{p-\frac{1}{3}} < 1.$$

Whereupon, Young's inequality once more shows that for any $\gamma > 0$, there exists positive constant $C_6 = C(\gamma) > 0$ such that

$$\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2(q-1)} \le \gamma\left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right) + \gamma\left(\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2\right) + C_6 \tag{2.54}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Finally, for the last term on the right of (2.46), we first take $1 < \rho < 3$ and use Hölder's inequality to obtain

$$\int_\Omega |\nabla c_\varepsilon|^{2q}|u_\varepsilon|^2 \le \|u_\varepsilon^2\|_{L^{\rho'}(\Omega)} \cdot \left\||\nabla c_\varepsilon|^{2q}\right\|_{L^\rho(\Omega)} \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{2.55}$$

where $\rho' = \frac{\rho-1}{\rho}$. We then apply the interpolation inequality and Young's inequality to obtain that for all $t \in (0, T_{\max,\varepsilon})$,

$$\big\| |\nabla c_\varepsilon|^{2q} \big\|_{L^\rho(\Omega)} = \big\| |\nabla c_\varepsilon|^q \big\|_{L^{2\rho}(\Omega)}^2 \le C_7 \big\| \nabla |\nabla c_\varepsilon|^q \big\|_{L^2(\Omega)}^{2\cdot\frac{q-\frac{1}{\rho}}{q-\frac{1}{3}}} + C_7 \le \varsigma \int_\Omega \big|\nabla |\nabla c_\varepsilon|^q\big|^2 + C_8 \tag{2.56}$$

for any $\varsigma > 0$ and some positive constants C_7 and $C_8 = C(\varsigma)$, where we used Lemma 2.7 in the first inequality. For any $\sigma > 0$, we use Corollary 2.2 to find a constant $C_9 = C(\sigma) > 0$ such that

$$\int_\Omega |\nabla c_\varepsilon|^{2q} |u_\varepsilon|^2 \le \sigma \int_\Omega \big|\nabla |\nabla c_\varepsilon|^q\big|^2 + C_9 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{2.57}$$

by (2.55) and (2.56).

Now choosing suitable ζ in (2.51), γ in (2.54), and σ in (2.57) satisfying

$$p(p-1)C_{\mathcal{S}}^2 \zeta + (2q^2+q)\gamma + (4q^2+2q)\sigma \le \min\left\{\frac{2(p-1)}{p}, \frac{q-1}{2q}\right\},$$

and substituting these inequalities into (2.46), we can achieve

$$\frac{d}{dt}\left(\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^{2q}\right) + \frac{(p-1)}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{q-1}{4q}\int_\Omega \big|\nabla |\nabla c_\varepsilon|^q\big|^2 \le C_{10} \tag{2.58}$$

on $(0, T_{\max,\varepsilon})$ with some constant $C_{10} > 0$. To establish the uniform estimates of the functional, we once more employ the Gagliardo–Nirenberg inequality to estimate

$$\int_\Omega n_\varepsilon^p = \|n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{4}{p}} \le C_{11}\left(\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(p-1)}{p-\frac{1}{3}}} \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{4}{3p-1}} + \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^2 \right)$$

for all $t \in (0, T_{\max,\varepsilon})$ with some constant $C_{11} > 0$. Then Young's inequality and the mass conservation (2.3) entail the existence of positive constant C_{12} such that

$$\int_\Omega n_\varepsilon^p \le \frac{(p-1)}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + C_{12} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.59}$$

Similarly, by applying the Gagliardo–Nirenberg inequality again, Young's inequality and Lemma 2.7, we can find $C_{13} > 0$ such that

$$\int_\Omega |\nabla c_\varepsilon|^{2q} \le \frac{q-1}{4q}\int_\Omega \big|\nabla |\nabla c_\varepsilon|^q\big|^2 + C_{13} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.60}$$

Thereupon, we infer from (2.58), (2.59) and (2.60) that

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$$\frac{d}{dt}\left(\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^{2q}\right) + \left(\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^{2q}\right) \le C_{14}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $C_{14} = C_{10} + C_{12} + C_{13}$. Then an ODE comparison argument shows that

$$\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^{2q} \le C_{15} := \max\left\{\left(\int_\Omega n_0^p + \int_\Omega |\nabla c_0|^{2q}\right), C_{14}\right\}$$

for all $t \in (0, T_{\max,\varepsilon})$ and thus conclude (2.45). □

We now derive the regularity of n_ε and ∇c_ε in arbitrary L^p spaces.

Corollary 2.3. *For any $p > 1$, $q > 1$, there exists some positive constant C such that for any $\varepsilon \in (0, 1)$,*

$$\int_\Omega n_\varepsilon^p(\cdot, t) \le C \qquad \textit{and} \qquad \int_\Omega |\nabla c_\varepsilon(\cdot, t)|^q \le C \quad \textit{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.61}$$

Proof. For any fixed $q \ge 2$, we can choose p satisfies (2.44), which entails

$$\int_\Omega |\nabla c_\varepsilon|^q \le C_1$$

for all $t \in (0, T_{\max,\varepsilon})$ with some positive constant C_1 from Lemma 2.9. Similarly, for any fixed $p > 2\alpha + \frac{1}{3}$, we can take q such that $\frac{1}{6\alpha} < \frac{q-\frac{1}{3}}{p-\frac{1}{3}} < \frac{3}{4}$, which implies (2.44) is valid. Thereupon, Lemma 2.9 asserts that $\int_\Omega n_\varepsilon^p$ is bounded for such p for all $t \in (0, T_{\max,\varepsilon})$. □

2.5. Existence of global bounded classical solutions to the regularized system

With the above regularization properties of each component n_ε, c_ε, u_ε at hand, we are now in the position to make sure that all approximate problems (2.1) are in fact globally solvable. To this end, we need the following regularity features of Du_ε implied by the boundedness property of n_ε.

Lemma 2.10. *Let $p \in [1, \infty)$ and $r \in [1, \infty]$ be such that*

$$\begin{cases} r < \frac{3p}{3-p} & \textit{if} \quad p \le 3, \\ r \le \infty & \textit{if} \quad p > 3. \end{cases} \tag{2.62}$$

Then for all $K > 0$ there exists $C = C(p, r, K, u_0, \phi)$ such that if for some $\varepsilon \in (0, 1)$ and $T > 0$ we have

$$\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \le K \qquad \textit{for all} \quad t \in (0, T), \tag{2.63}$$

then

$$\|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le C \qquad \text{for all} \quad t \in (0,T).$$

Proof. This lemma can be exactly proved as Corollary 3.4 in [46]. □

Thus by Corollary 2.3 and Lemma 2.10, we immediately obtain the boundedness of Du_ε.

Corollary 2.4. *Suppose that (1.6)–(1.9) hold with $\alpha > \frac{1}{2}$. Then there exists a positive constant C such that for all $\varepsilon \in (0,1)$*

$$\|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le C \qquad \text{for all } 1 \le r \le \infty. \tag{2.64}$$

With all above preparations, we can now establish the existence of global bounded classical solutions to the regularized system (2.1).

Theorem 2.1. *Suppose that (1.6)–(1.9) hold with $\alpha > \frac{1}{2}$. Then system (2.1) admits a global classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$, which is uniformly bounded in the sense that for all $\varepsilon \in (0,1)$,*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \qquad \text{for all } t \in (0,\infty) \tag{2.65}$$

with some positive constant C. This solution is unique, up to addition of constants to P_ε.

Proof. To obtain the existence of global classical solution, by the extension criterion in Lemma 2.1, we only need to show that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\mathcal{A}^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_1 \tag{2.66}$$

for all $t \in (0, T_{\max,\varepsilon})$ with some positive constant C_1 independent of $T_{\max,\varepsilon}$. To this end, we first notice that Corollary 2.4 together with the interpolation inequality yields the existence of $C_2 > 0$ satisfying

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C_2 \tag{2.67}$$

for all $t \in (0, T_{\max,\varepsilon})$. Then by using of the $L^p - L^q$ estimate for the Neumann heat semigroup and Stokes semigroup, the desired estimate (2.66) can be obtained in the same way as the proof of its two-dimensional version (see Theorem 2.1 in [38]). Here we give a sketch for completeness.

We first establish the boundedness of n_ε as follows. Fix two constants r and q such that $r > q > 3$. By the smoothing estimate for the Neumann heat semigroup in Ω (see e.g. [41]), we can invoke the variation-of-constants formula for n_ε to find that some positive constant C_3 such that

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$$
\begin{aligned}
\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le \|e^{t\triangle}n_0\|_{L^\infty(\Omega)} + \int_0^t \left\| e^{(t-\tau)\triangle}\nabla\cdot\big(n_\varepsilon \mathcal{S}(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon + n_\varepsilon u_\varepsilon\big)(\cdot,\tau)\right\|_{L^\infty(\Omega)} d\tau \\
&\le \|n_0\|_{L^\infty(\Omega)} + C_3\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2q}} e^{-\lambda_1(t-\tau)} \left\|\big(n_\varepsilon \mathcal{S}(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon + n_\varepsilon u_\varepsilon\big)(\cdot,\tau)\right\|_{L^q(\Omega)} d\tau
\end{aligned}
\tag{2.68}
$$

for all $t \in (0, T_{\max,\varepsilon})$. By Hölder's inequality, (1.7), (2.67), Corollary 2.3 and Corollary 2.4, we see for any $r > 1$

$$
\begin{aligned}
\left\|\big(n_\varepsilon \mathcal{S}(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\big)(\cdot,\tau)\right\|_{L^q(\Omega)} &\le C_{\mathcal{S}}\|(n_\varepsilon\nabla c_\varepsilon)(\cdot,\tau)\|_{L^q(\Omega)} \\
&\le C_{\mathcal{S}}\|n_\varepsilon(\cdot,\tau)\|_{L^r}\|\nabla c_\varepsilon(\cdot,\tau)\|_{L^{\frac{rq}{r-q}}} \le C_4
\end{aligned}
$$

and

$$
\|(n_\varepsilon u_\varepsilon)(\cdot,\tau)\|_{L^q(\Omega)} \le \|u_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)}\|n_\varepsilon(\cdot,\tau)\|_{L^q(\Omega)} \le C_5
$$

for all $t \in (0, T_{\max,\varepsilon})$ with some positive constants C_4 and C_5. Thus we can infer from (2.68) that

$$
\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|n_0\|_{L^\infty(\Omega)} + C_3(C_4+C_5)\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2q}} e^{-\lambda_1(t-\tau)} d\tau \le C_6 \tag{2.69}
$$

for all $t \in (0, T_{\max,\varepsilon})$ with some positive constant C_6 since the integral $\int_0^\infty \sigma^{-\frac{1}{2}-\frac{3}{2q}} e^{-\lambda_1\sigma} d\sigma$ is finite by $q > 3$ and $\lambda_1 > 0$.

Similarly, for c_ε, we can apply Corollary 2.3 to the variation-of-constants formula

$$
c_\varepsilon(\cdot,t) = e^{t(\Delta-1)}c_0 + \int_0^t e^{(t-\tau)(\Delta-1)}\big(n_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\big)(\cdot,\tau)d\tau
$$

to assert that

$$
\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C_7 \tag{2.70}
$$

for all $t \in (0, T_{\max,\varepsilon})$ with some $C_7 > 0$.

Finally, we turn to the estimate of u_ε. Let $\beta \in (\frac{3}{4}, 1)$ be the constant given by (1.8). Applying the fractional power $\mathcal{A}^\beta$ to the variation-of-constants formula

$$
u_\varepsilon(\cdot,t) = e^{-t\mathcal{A}}u_0 + \int_0^t e^{-(t-\tau)\mathcal{A}}\mathcal{P}\big(n_\varepsilon\nabla\phi\big)(\cdot,\tau)d\tau, \quad t \in (0, T_{\max,\varepsilon}),
$$

we can obtain

$$\|\mathcal{A}^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le \|\mathcal{A}^\beta e^{-t\mathcal{A}} u_0\|_{L^2(\Omega)} + \int_0^t \big\|\mathcal{A}^\beta e^{-(t-\tau)\mathcal{A}} \mathcal{P}\big(n_\varepsilon \nabla\phi\big)(\cdot,\tau)\big\|_{L^2(\Omega)} d\tau$$

for all $t \in (0, T_{\max,\varepsilon})$. Then by the decay estimate of Stokes semigroup (see e.g. [13]), the boundedness of $\|n(\cdot,\tau)\|_{L^\infty(\Omega)}$ and of $\|\nabla\phi\|_{L^2(\Omega)}$, and the initial condition (1.8), we can find three positive constants λ, C_8 and C_9 such that

$$\begin{aligned}\|\mathcal{A}^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &\le \|\mathcal{A}^\beta e^{-t\mathcal{A}} u_0\|_{L^2(\Omega)} \\ &\quad + C_8 \|n_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)} \|\nabla\phi\|_{L^2(\Omega)} \int_0^t \|(t-\tau)^{-\beta} e^{-\lambda(t-\tau)} d\tau \le C_9 \end{aligned} \tag{2.71}$$

for all $t \in (0, T_{\max,\varepsilon})$. Thus combining (2.69), (2.70) and (2.71), we establish the desired estimate (2.66). We therefore conclude that $T_{\max} = \infty$ and that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is global in time.

The boundedness estimate (2.65) is a direct consequence of (2.67), (2.69) and (2.70). This completes the proof of Theorem 2.1. □

3. Passing to the limit. Proof of Theorem 1.1

In this section, we shall use an approximate procedure to construct the global bounded classical solution to system (1.1) with general tensor-valued sensitivity $\mathcal{S}$, under the assumption of (1.6) and (1.7). For this purpose, we first recall that Theorem 2.1 shows that the regularized system (2.1) possesses a global classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ for any $\varepsilon \in (0,1)$, which satisfies the uniform bound (2.65). Moreover, integrating (2.37) from 0 to T with $p = 2$, we can also conclude that there exists a positive constant $M(T)$ such that

$$\int_0^T \int_\Omega |\nabla n_\varepsilon|^2 \le M(T) \quad \text{for all } T > 0. \tag{3.1}$$

From lemmata in Section 2, we know that $M(T)$ and the uniform bound in (2.65) are both independent of ε.

Then our goal is to show the solutions of the regularized system (2.1) will approach to a classical solution of system (1.1) as $\varepsilon \to 0$.

Proof of Theorem 1.1. The proof follows from a similar argument as its two-dimensional version in [38]. Here we just mention the key steps for completeness.

First of all, it follows from the uniform bound (2.65) that

$$\|n_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,2}(\Omega))^*} \le C_1 \qquad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0,1),$$

and

$$\|n_\varepsilon(\cdot,t) - n_\varepsilon(\cdot,s)\|_{(W_0^{2,2}(\Omega))^*} \le C_1 |t-s| \qquad \text{for all } t > 0,\ s \ge 0 \quad \text{and} \quad \varepsilon \in (0,1)$$

with some $C_1 > 0$.

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Similar to Lemma 3.18 and Lemma 3.19 in Winkler [46], we can also use the uniform bound (2.65) and the standard parabolic regularity theory to obtain the following uniform Hölder continuity for c_ε, ∇c_ε and u_ε: There exist $\sigma \in (0,1)$ and $C_2 > 0$ such that

$$\|c_\varepsilon\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le C_2 \qquad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0,1)$$

and

$$\|u_\varepsilon\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le C_2 \qquad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0,1),$$

and that for each $t_0 > 0$ we can find $C_3(t_0) > 0$ such that

$$\|\nabla c_\varepsilon\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le C_3(t_0) \qquad \text{for all } t > t_0 \quad \text{and} \quad \varepsilon \in (0,1).$$

Then these estimates (i.e., the uniform bound (2.65), the spatial-time integrability (3.1) and the above Hölder continuity) together with some Arzelà–Ascoli theorem and the dominated convergence theorem ensure that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ will converge to a weak solution (n, c, u, P) of system (1.1) as $\varepsilon \to 0$.

Next the higher regularity of solution (n, c, u, P) can be established by the standard parabolic regularity theory (see Chapter IV in [21]) and regularity theories for Stokes operator (see Lemma 2.1 in [42]).

Finally, the stated boundedness of the classical solution comes from the uniform bound (2.65) and the aforementioned convergence results of the solutions of the regularized system (2.1) to the weak solution of system (1.1). This completes the proof of Theorem 1.1. □

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