



# Spatially localized solutions of the Hammerstein equation with sigmoid type of nonlinearity

Anna Oleynik<sup>a,\*</sup>, Arcady Ponosov<sup>a</sup>, Vadim Kostrykin<sup>b</sup>,  
Alexander V. Sobolev<sup>c</sup>

<sup>a</sup> Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, Postboks 5003 NMBU, 1432 Ås, Norway

<sup>b</sup> FB 08 - Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudinger Weg 9, D-55099 Mainz, Germany

<sup>c</sup> Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

Received 19 November 2015; revised 31 May 2016

Available online 31 August 2016

## Abstract

We study the existence of fixed points to a parameterized Hammerstein operator  $\mathcal{H}_\beta$ ,  $\beta \in (0, \infty]$ , with sigmoid type of nonlinearity. The parameter  $\beta < \infty$  indicates the steepness of the slope of a nonlinear smooth sigmoid function and the limit case  $\beta = \infty$  corresponds to a discontinuous unit step function. We prove that spatially localized solutions to the fixed point problem for large  $\beta$  exist and can be approximated by the fixed points of  $\mathcal{H}_\infty$ . These results are of a high importance in biological applications where one often approximates the smooth sigmoid by discontinuous unit step function. Moreover, in order to achieve even better approximation than a solution of the limit problem, we employ the iterative method that has several advantages compared to other existing methods. For example, this method can be used to construct non-isolated homoclinic orbit of a Hamiltonian system of equations. We illustrate the results and advantages of the numerical method for stationary versions of the FitzHugh–Nagumo reaction–diffusion equation and a neural field model.

© 2016 Elsevier Inc. All rights reserved.

MSC: 47H30; 47N60; 47G10; 45L05; 45J05; 92B20

\* Corresponding author.

E-mail addresses: [anna.oleynik@nmbu.no](mailto:anna.oleynik@nmbu.no) (A. Oleynik), [arkadi@nmbu.no](mailto:arkadi@nmbu.no) (A. Ponosov), [kostrykin@mathematik.uni-mainz.de](mailto:kostrykin@mathematik.uni-mainz.de) (V. Kostrykin), [a.sobolev@ucl.ac.uk](mailto:a.sobolev@ucl.ac.uk) (A.V. Sobolev).

**Keywords:** Nonlinear integral equations; Sigmoid type nonlinearities; Neural field model; FitzHugh–Nagumo model; Bumps

## 1. Introduction

We study the existence of solutions to the fixed point problem

$$u = \mathcal{H}_\beta u, \quad (\mathcal{H}_\beta u)(x) := \int_{\mathbb{R}} \omega(x-y) f_\beta(u(y)) dy. \quad (1.1)$$

Here  $\mathcal{H}_\beta$  is the parameterized Hammerstein operator with  $\beta \in (0, \infty]$ ,  $\omega(x)$  is symmetric, and  $f_\beta(u) : \mathbb{R} \rightarrow [0, 1]$  is a smooth function of sigmoid shape that approaches (in some way which we specify later) the unit step function  $f_\infty = \chi_{[h, \infty)}$  for some  $h > 0$  as  $\beta \rightarrow \infty$ . Examples of this type of functions are

$$f_\beta(u) = S(\beta(u-h)), \quad S(u) := \frac{1}{1 + \exp(-|u|)}, \quad (1.2)$$

and

$$f_\beta(u) = S(\beta(u-h)), \quad S(u, p) := \frac{u^p}{u^p + 1} \chi_{[0, \infty)}(u), \quad p > 0, \quad (1.3)$$

see Fig. 1.

This problem arises in several biological applications, e.g., in studying the existence of steady state solutions to a neural field model and the FitzHugh–Nagumo equation. We give examples of these models in Section 2.

When the function  $f_\beta$  is such that  $f_\beta(u) = 0$  for all  $u < h$ , i.e.,  $\text{supp}(f_\beta) \subset [h, \infty)$ , all solutions to (1.1) can be divided into two categories: (i) localized solutions and (ii) non-localized solutions (e.g. periodic, quasi-periodic). Here we study the solutions of the first class which we define in detail in Section 4.2.

For the limit case  $\beta = \infty$ , one often can construct localized solutions analytically, see e.g. [1] and Chapter 3 in [2]. However, the case  $\beta = \infty$  is only a simplification of a more realistic model where  $f_\beta$ ,  $0 < \beta < \infty$  is a steep yet smooth function. Analytical tools do not work in the latter case, and the existence of solutions for the case of  $\beta < \infty$  and their continuous dependence on  $\beta$  as  $\beta \rightarrow 0$ , is often only conjectured from numerical simulations, see e.g. [3–5].

The main challenge of a proper justification of the transition between the cases  $\beta < \infty$  and  $\beta = \infty$ , is discontinuity of  $f_\infty$  which leads to discontinuity of the corresponding integral operator in any standard functional space. To avoid this difficulty we suggest to exploit a spatial structure of localized solutions for  $\beta = \infty$  and construct functional spaces such that the operator  $\mathcal{H}_\infty$  is not only continuous but Fréchet differentiable and the Implicit Function Theorem can be used. That is, we show that under the assumption that localized solution of (1.1) for  $\beta = \infty$  exists and satisfies some properties, solutions to (1.1) for large  $\beta < \infty$  exist, converge to a solution of

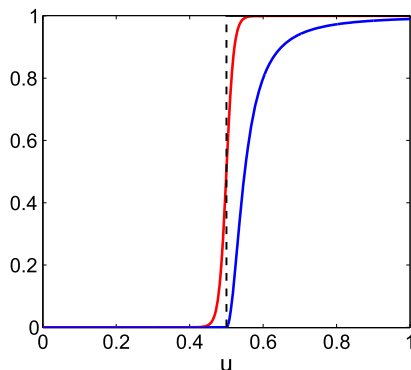


Fig. 1. Functions  $f_\beta$  given in (1.2) with  $\beta = 100$  (red curve) and in (1.3) with  $\beta = 20$  (blue curve), and  $f_\infty = \chi_{[h, \infty)}$  (black line), with  $h = 0.5$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the limit case as  $\beta \rightarrow \infty$ , and can be iteratively constructed. We emphasize that this result is quite important as it provides a motivation for a common in applied sciences approximation of a smooth sigmoid function by the discontinuous unit step function.

While the Implicit Function Theorem is well known to be a useful and powerful tool dealing with such problems, it has not been used due to the non-trivial choice of spaces for operator convergence. However there are similar results obtained in [6] and later extended by Burlakov et al. [7] for the case of a bounded spacial domain in  $\mathbb{R}^n$ ,  $n \geq 1$ , using the topological degree theory and properties of the Hammerstein operators for a smaller class of solutions and more strict assumptions on the integral kernel  $\omega$ .

The degree theory approach has some disadvantages compared to the method based on the Implicit Function Theorem proposed here. Firstly, it requires calculating the topological degree of  $\mathcal{H}_\infty$ . Though it seems like the necessary condition on the topological degree being non-zero coincides with an assumption we introduce on the localized solution of (1.1),  $\beta = \infty$ , its calculation could be involved. Secondly, the degree theory never gives uniqueness for  $\beta < \infty$  even if it is proven for  $\beta = \infty$ . Thirdly, as in infinite dimensional Banach spaces the degree is only defined for compact perturbations of the identity operator, see [8], the domain and the range space of  $\mathcal{H}_\beta$  must be the same. This restricts the choice of  $\omega$ . Finally, the degree theory approach does not give a method for constructing solutions numerically.

The paper is organized as follows. In Section 2 we give examples of relevant applications, that is, the FitzHugh–Nagumo equation (Section 2.1) and the neural field model (Section 2.2), that we use later to illustrate our results. We give a list of notations we use in Section 3. In Section 4 we introduce assumptions, describe the problem and state the results. In particular, in Section 4.1 we describe some properties of the operator  $\mathcal{H}_\beta$ , in Section 4.2 we give a definition of a bump and bump solution. The existence and assumptions on the solution to the problem (1.1) for the limiting case  $\beta = \infty$  is discussed in Section 4.3. We formulate our main results as Theorem 4.8 in Section 4.4. Section 5 is dedicated to the proof of Theorem 4.8. In Section 6 we apply our results to prove existence of stationary solutions to the FitzHugh–Nagumo equation (Section 6.1) and the neural field model (Section 6.2), and numerically construct them. We also use these examples to discuss the advantages of the method in comparison to other existing methods. Section 7 contains conclusions and outlook.

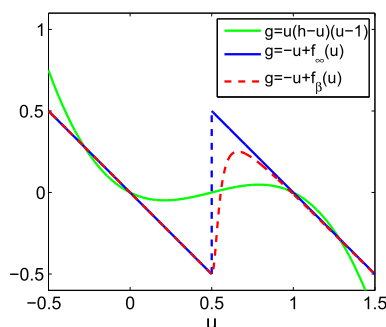


Fig. 2. Functions  $g = u(u - h)(1 - u)$  (green curve),  $g = -u + f_{\beta}(u)$  (red dashed curve) with  $f_{\beta}(u)$  given as in (1.3) with  $\beta = 20$  and  $g = -u + f_{\infty}(u)$  (blue line) given in (2.3), with  $h = 0.5$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

## 2. Examples

### 2.1. FitzHugh–Nagumo equation and its caricature

The famous FitzHugh–Nagumo equation introduced in [9–11] describes qualitative properties of nerve conduction. The equation can be written as

$$\begin{aligned} \frac{\partial u}{\partial t} &= g(u) - v + D \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial t} &= bu - \gamma v, \end{aligned} \quad (2.1)$$

where  $g$  is commonly assumed to be the cubic function

$$g(u) = u(u - h)(1 - u), \quad 0 < h < 1, \quad (2.2)$$

and  $b$ ,  $\gamma$ , and  $D$  are positive parameters.

McKean in [5] suggested replacing the cubic  $g$  with a broken line of the same general shape, that is,

$$g(u) = -u + f_{\infty}(u), \quad f_{\infty} = \chi_{[h, \infty)}, \quad 0 < h < 1, \quad (2.3)$$

see Fig. 2. The equation (2.1) with (2.3) is commonly referred to as a caricature of the FitzHugh–Nagumo equation. It is believed but not proven that the equation (2.1) with the cubic function (2.2) and with (2.3) have similar portraits in the large (and indeed they have similar phase portraits for the steady state solution equations for some parameter values). However, since discontinuous  $g = -u + f_{\infty}(u)$  does not provide a good approximation of the cubic function (2.2) but only resembles its shape, see Fig. 2, it is doubtful that one can draw any rigorous conclusion about one model from analysing another. And indeed, the analysis of the equation (2.1) with the function (2.2) and (2.3) are rather disconnected in the literature.

However, a smooth sigmoidal shape function  $g = -u + f_{\beta}(u)$  may provide a good approximation of  $g = -u + f_{\infty}(u)$  for large  $\beta$ , see Fig. 2. Thus, one would be able to draw a common

conclusion about existence and behavior of solutions obtained for either case. This leads us to study the reaction diffusion system

$$\begin{aligned}\frac{\partial u}{\partial t} &= f_\beta(u) - (u + v) + D \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial t} &= bu - \gamma v,\end{aligned}\tag{2.4}$$

with  $\beta \in (0, \infty]$ .

The steady states of (2.4) are solutions to

$$-u'' + k^2 u = f_\beta(u)\tag{2.5a}$$

$$v = \frac{b}{\gamma} u,\tag{2.5b}$$

where  $k = \sqrt{1 + b/\gamma}$ .

Any bounded solution of the ordinary differential equation in (2.5a) also satisfies the integral equation (1.1), i.e.,

$$u(x) = \int_{\mathbb{R}} \omega(x - y) f_\beta(u(y)),\tag{2.6}$$

with

$$\omega(x) = \frac{1}{2k} e^{-k|x|},\tag{2.7}$$

see Fig. 3.

Here (2.7) is the Green function to the linear part of the equation in (2.5a) which can be shown in a similar way as in [12].

## 2.2. Neural field model

The behavior of a single layer of neurons can be modeled by a nonlinear integro-differential equation of the Hammerstein type,

$$\partial_t u(x, t) = -u(x, t) + \int_{\mathbb{R}} \omega(x - y) f(u(y, t)) dy.\tag{2.8}$$

Here  $u(x, t)$  and  $f(u(x, t))$  represent the averaged local activity and the firing rate of neurons at the position  $x \in \mathbb{R}$  and time  $t > 0$ , respectively, and  $\omega(x - y)$  describes a coupling between neurons at positions  $x$  and  $y$ .

The model above is often referred to as the Amari model and is a version of neural field models that constitute a special class of models where the neural tissue is treated as a continuous structure. The model (2.8) has been studied in numerous mathematical papers, for a review see,

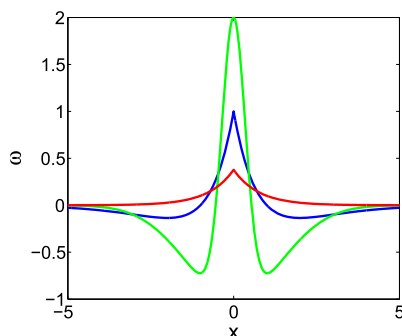


Fig. 3. Functions  $\omega$  given by (2.7) with  $k = 1.3229$  (red curve),  $\omega$  given by (2.9) with  $k = 1$  (blue curve) and  $\omega$  given by (2.10) with  $K = 3$ ,  $k = 2$ ,  $M = 1$ ,  $m = 0.5$  (green curve). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

e.g., [3,13] and [2]. In particular, the global existence and uniqueness of solutions to the initial value problem for (2.8) under rather mild assumptions on  $f$  and  $\omega$  has been proven in [14].

In 1977, Amari studied pattern formation in (2.8) for a model where  $f$  is the unit step function and  $\omega$  is assumed to be of the lateral-inhibitory type, i.e., continuous, integrable and even, with  $\omega(0) > 0$  and having exactly one positive zero. In particular, he showed the existence of stable and unstable time independent spatially localized solutions to (2.8) which he referred to as bumps. For more general  $f$  and  $\omega$  the existence of solutions of this kind has been shown by Kishimoto and Amari in [15] and later generalized in [16] and [17].

In what follows we use the Amari terminology, i.e., spatially localized solutions will be called bumps.

Since the work by Amari the lateral-inhibitory type of connectivity function is the common choice when one studies pattern formation in neural field models. Examples of this type of connectivity are

$$\omega(x) = (1 - |x|)e^{-k|x|}, \quad k > 0 \quad (2.9)$$

and

$$\omega(x) = Ke^{-(kx)^2} - Me^{-(mx)^2}, \quad K > M, \quad k > m, \quad (2.10)$$

see Fig. 3.

When the Fourier transform of the connectivity function is real and rational, e.g.,  $\omega$  in (2.9), the time independent version of (2.8) can be converted to a higher order nonlinear differential equation which in turn can be represented as a Hamiltonian system. Bumps correspond to homoclinic orbits of this system, [18,19].

Despite there are some analytical methods for studying existence of homoclinic orbits for higher order Hamiltonian systems, see e.g. [20,21], due to the specifics of the considered model these methods are not straightforwardly applicable here and the most results are numerical, for details see [18]. Alternatively, the existence of homoclinic orbits for such systems can be studied using methods of computer assisted proofs, see e.g. [22–26]. However, for the case when  $\omega$  does not admit a real rational Fourier transform, as for example in (2.10), the mentioned differential equations methods are not available and the other approaches must be used.

For  $f = f_\beta$ , where  $f_\beta \rightarrow f_\infty = \chi_{[h, \infty]}$  as  $\beta \rightarrow \infty$ , and  $\text{supp}(f_\beta) \subset [h, \infty)$ , the existence of bumps of a particular kind (so called 1-bumps) and their continuous dependence on the parameter  $\beta$  has been shown using the topological degree theory and collectively compactness and continuity of the Hammerstein operators in [6]. Later these results were extended to the case of  $n$ -dimensional,  $n \in \mathbb{N}$  bounded spacial domain in [7].

### 3. Notation

For readers convenience we give a list of functional spaces and specify other notations we use. Let  $\Omega$  be a bounded or unbounded subset in  $\mathbb{R}$ .

- $L^p(\Omega)$ ,  $1 \leq p \leq \infty$ , is the space of all functions such that the  $p$ th power of the absolute value is Lebesgue integrable and the norm is given as

$$\|v\|_{L^p(\Omega)} := \left( \int_{\Omega} |v(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|v\|_{L^\infty(\Omega)} := \text{ess sup}_{x \in \Omega} |v(x)|.$$

When  $\Omega = \mathbb{R}$  we use the common notation for the norm  $\|v\|_\infty := \|v\|_{L^\infty(\mathbb{R})}$ .

- $B(\Omega)$  is the linear space of all bounded functions.
- $C(\Omega)$  is the linear space of all continuous (but not necessarily bounded) functions on  $\Omega$ .
- $BC(\Omega)$  is the Banach space of all continuous bounded functions on  $\Omega$  with the norm

$$\|v\|_{C(\Omega)} := \sup_{x \in \Omega} |v(x)|.$$

When  $\Omega = \mathbb{R}$  we often use  $\|\cdot\|_\infty$  notation for the norm above.

- $C^n(\Omega)$ ,  $n \in \mathbb{N}$ , is the linear space of all continuous (but not necessarily bounded) functions with continuous  $k$ th derivatives,  $k = 1, \dots, n$ , on  $\Omega$ .
- $BC^n(\Omega)$ ,  $n \in \mathbb{N}$ , is the Banach space of all continuous bounded functions with continuous and bounded  $k$ th derivatives, for  $k = 1, \dots, n$ , on  $\Omega$  equipped with the norm

$$\|v\|_{C^n(\Omega)} := \sum_{k=0}^n \sup_{x \in \Omega} |v^{(k)}(x)|.$$

- $C^{0,1}(\Omega)$  is the space of all Lipschitz continuous functions on  $\Omega$ .
- $C^{0,\alpha}(\Omega)$  is the space of all Hölder continuous functions on  $\Omega$  with the exponent  $0 < \alpha < 1$ .

When  $\Omega$  is a compact set then we prefer the notation  $C^n(\Omega)$  over  $BC^n(\Omega)$ . Moreover, in this case we treat  $C^{0,\alpha}(\Omega)$ ,  $0 < \alpha \leq 1$  as the Banach spaces equipped with the norm

$$\|v\|_{C^{0,\alpha}(\Omega)} := \sup_{x \in \Omega} |v(x)| + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^\alpha}$$

When there is no confusion what  $\Omega$  is we use the notation  $\|\cdot\|_\alpha$  instead of  $\|\cdot\|_{C^{0,\alpha}(\Omega)}$ .

We denote  $C_{\text{even}}(\Omega)$  the space of all even continuous function on  $\Omega$ . The same notation applies for the other spaces, e.g.,  $L_{\text{even}}^p(\Omega)$ ,  $1 \leq p \leq \infty$  and etc.

If  $\Omega = \mathbb{R}$  we omit the set specification in the norm notation, i.e., we write  $\|\cdot\|_{C^n}$  instead of  $\|\cdot\|_{C^n(\mathbb{R})}$ .

We use boldface to denote vectors, e.g.,  $\mathbf{a}$ ,  $\mathbf{G}$ , and the `textsf` font for matrices, e.g.,  $\mathbf{S}$ ,  $\mathbf{I}$ .

#### 4. Framework and main results

We study existence of solutions to the fixed point problem (1.1) under the following assumptions on  $f_\beta$  and  $\omega$ .

**Assumption A.** Let  $h > 0$  be fixed, and let  $\{f_\beta\}$ ,  $\beta \in (0, \infty]$  define a family of functions with the following properties:

- (1)  $f_\beta : \mathbb{R} \rightarrow [0, 1]$  is non-decreasing for any  $\beta \in (0, \infty]$ ,
- (2)  $\text{supp}(f_\beta) \subset [h, \infty)$  for any  $\beta \in (0, \infty]$ ,
- (3)  $f_\infty = \chi_{(h, \infty)}$ , i.e., the characteristic function of the half-line set  $(h, \infty)$ ,
- (4)  $f_\beta(t)$  is continuous in  $\beta \in (0, \infty)$  uniformly in  $t$  on any bounded interval,
- (5)  $f_\beta \in C^{0,1}(\mathbb{R})$  for  $\beta < \infty$ , and for any  $\xi > 0$

$$C_\beta(\xi) := \text{ess sup}_{t \in (h+\xi, \infty)} |f'_\beta(t)| \rightarrow 0 \text{ as } \beta \rightarrow \infty.$$

The function in (1.3) with  $p > 1$  serves as an example of such a function.

We also notice that from Assumption A(5)  $f_\beta$  has the following convergence property

$$\sup_{t \in \mathbb{R} \setminus (h-\xi, h+\xi)} |f_\beta(t) - f_\infty(t)| \rightarrow 0 \text{ as } \beta \rightarrow \infty. \quad (4.1)$$

**Assumption B.** The function  $\omega$  in (1.1) satisfies the following conditions:

- (1)  $\omega$  is symmetric, i.e.,  $\omega(-x) = \omega(x)$ ,
- (2)  $\omega$  is a Lipschitz function, i.e.,  $\omega \in C^{0,1}(\mathbb{R})$ ,
- (3)  $\omega \in L^1(\mathbb{R})$ , and
- (4)  $\omega$  is bounded, i.e.,  $\omega \in L^\infty(\mathbb{R})$ .

The functions in (2.7), (2.9), and in (2.10) clearly satisfy the assumption above.

Since the function  $f_\beta$  is such that  $f_\beta(u) = 0$  for all  $u < h$ , see Assumption A(2), all the solutions to (1.1) can be divided into two categories: (i) localized solutions (so called bumps, see e.g. [1,6]) and (ii) non-localized solutions (e.g., periodic, quasi-periodic).

Here we study the existence of solutions of the first type. We introduce only a few properties of the operator  $\mathcal{H}_\beta$  that are needed here. For more general description of  $\mathcal{H}_\beta$  we refer to [6] and [7].



#### 4.1. Properties of $\mathcal{H}_\beta$

**Lemma 4.1.** *Let  $f_\beta$  and  $\omega$  satisfy [Assumption A](#) and [Assumption B](#). Then for any real valued measurable function  $u$  we have  $\mathcal{H}_\beta u \in BC(\mathbb{R})$  for any  $\beta \in (0, \infty]$ . Moreover, if  $u(x) \leq h$  for all  $x \in \mathbb{R}$  except a subset  $X \subset \mathbb{R}$  of a finite measure, then  $\mathcal{H}_\beta u \in L^1(\mathbb{R}) \cap BC^1(\mathbb{R})$  and  $(\mathcal{H}_\beta u)(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .*

**Proof.** For a general  $u$  we define a set  $X := \{x : u(x) \geq h\} \subseteq \mathbb{R}$ . We have the following estimate

$$|(\mathcal{H}_\beta u)(x)| = \left| \int_X \omega(x-y) f_\beta(u(y)) dy \right| \leq \int_X |\omega(x-y)| dy.$$

[Assumption B\(3\)](#) immediately yields  $\|\mathcal{H}_\beta u\|_\infty \leq \|\omega\|_{L^1} < \infty$ . To show continuity of  $\mathcal{H}_\beta$  let  $v = \mathcal{H}_\beta u$  and  $x \rightarrow x_0$ . Then we obtain

$$|v(x) - v(x_0)| \leq \int_X |\omega(x-y) - \omega(x_0-y)| dy \rightarrow 0 \text{ as } x \rightarrow x_0,$$

which follows from the continuity of translations in  $L^1(\mathbb{R})$ , see, e.g., Proposition 2.5 in [\[27\]](#).

Assume now that  $X$  has a finite measure  $\mu(X) < \infty$ . From [Assumption B\(2\)](#) and [Assumption B\(3\)](#) it follows that  $\|\mathcal{H}_\beta u\|_{L^1} \leq \|\omega\|_{L^1} \mu(X) < \infty$ . From [Assumption B\(4\)](#) the derivative of  $v = \mathcal{H}_\beta u$  with respect to  $x$  exists and is uniformly bounded, that is,

$$\|v'\|_\infty \leq \int_X |\omega'(x-y) f_\beta(u(y))| dy \leq \|\omega'\|_\infty \mu(X).$$

Next, we let  $x \rightarrow x_0$  and obtain the estimate

$$|v'(x) - v'(x_0)| \leq \int_X |\omega'(x-y) - \omega'(x_0-y)| dy.$$

[Assumption B\(2\)](#) implies that  $\omega' \in L^\infty(\mathbb{R})$  and thus  $\omega' \in L^1(X)$ . From the continuity of translations in  $L^1(X)$  we deduce that  $|v'(x) - v'(x_0)| \rightarrow 0$  as  $x \rightarrow x_0$ .

Hence we conclude that  $\mathcal{H}_\beta u \in BC^1(\mathbb{R})$  and the following estimate is valid

$$\|\mathcal{H}_\beta u\|_{C^1} \leq (\|\omega\|_\infty + \|\omega'\|_\infty) \mu(X) < \infty.$$

The property  $v(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  follows from [Assumption B\(2\)](#) and [Assumption B\(3\)](#).  $\square$

From the lemma above any solution to [\(1.1\)](#) is continuous and bounded.

**Lemma 4.2.** *Let  $f_\beta$  and  $\omega$  satisfy [Assumption A](#) and [Assumption B](#), respectively, and let the operator  $\mathcal{H}_\beta : BC(\mathbb{R}) \rightarrow BC(\mathbb{R})$  be defined as in [\(1.1\)](#). Then the following statements are true.*  
*(i) Any solution of [\(1.1\)](#) is translation invariant, i.e., if  $u(x)$  is a solution so is  $u(x-c)$  for*

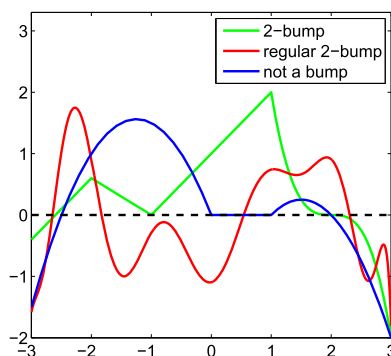


Fig. 4. Examples of a bump (gree curve), regular bump (red curve) and not a bump (blue curve), when  $h = 0$  and assuming that the functions are smooth enough and do not increase outside the interval  $[-3, 3]$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

any  $c \in \mathbb{R}$ . (ii) The operator  $\mathcal{H}_\beta$  preserves the symmetry, i.e., for any  $u(x) = u(-x)$  we have  $(\mathcal{H}_\beta u)(-x) = (\mathcal{H}_\beta u)(x)$ . (iii) If for a fixed point  $u(x)$  the corresponding  $\text{supp}(f_\beta(u(\cdot)))$  is a symmetric set, then  $u(x)$  is even.

The proof is straightforward.

**Remark 4.3.** For Lemma 4.1 and Lemma 4.2 we did not use all the conditions of Assumption A. It suffices to assume that  $u \mapsto f_\beta(u) \in [0, 1]$  is measurable and Assumption A(2)–(3) are satisfied.

As mentioned before we intend to investigate the existence of localized solutions to (1.1). In the next section we describe the class of functions we are interested in.

## 4.2. Bumps and regular bumps

**Definition 4.4.** Let  $h \in \mathbb{R}$  and  $\{b_i\}_{i=1}^{2N} \subset \mathbb{R}$  be an increasing sequence of  $2N$  points. The function  $u \in C(\mathbb{R})$  is called a  $(h; b_1, b_2, \dots, b_{2N})$ -bump if the following conditions are satisfied:

- (i)  $\{b_i\}_{i=1}^{2N}$  are the only roots to  $u(x) = h$ ,
- (ii) there exists  $\gamma > 0$  and  $A > 0$  such that  $u(x) < h - \gamma$  for all  $|x| > A$ .

We call the  $(h; b_1, b_2, \dots, b_{2N})$ -bump regular if in addition  $u \in C^1(\mathbb{R})$  and  $u'(b_i) \neq 0$  for all  $i = 1, \dots, 2N$ .

When  $h$  is assumed to be given and there is no need to specify the roots  $\{b_i\}_{i=1}^{2N}$ , we often refer to  $u$  as a (regular)  $N$ -bump or, simply a (regular) bump.

We illustrate the definition with Fig. 4.

Regular bumps are stable under small perturbations in  $C^1(\mathbb{R})$ . Indeed, let  $u$  be a regular  $(h; b_1, b_2, \dots, b_{2N})$ -bump, and for some  $l < b_1$  and  $L > b_{2N}$  define the set

$$K_\varepsilon(u; l, L) := \{v \in C^1(\mathbb{R}) : \sup_{x \in \mathbb{R}} |u(x) - v(x)| + \max_{x \in [l, L]} |u'(x) - v'(x)| < \varepsilon\}. \quad (4.2)$$

We formulate the following lemma.

**Lemma 4.5.** *Let  $u$  and  $K_\varepsilon(u; l, L)$  be given as above. Then there exists  $\varepsilon \in (0, 1/2 \min_i |u'(b_i)|)$  such that for any  $v \in K_\varepsilon(u; l, L)$ ,  $v$  is a regular  $(h; b_1(\varepsilon), b_2(\varepsilon), \dots, b_{2N}(\varepsilon))$ -bump where  $b_i(\varepsilon) \rightarrow b_i$  as  $\varepsilon \rightarrow 0$ .*

The proof is rather straightforward and can be found in [6].

**Corollary 4.6.** *Observe that there are values  $b_i^\pm(\varepsilon) : b_i^-(\varepsilon) < b_i < b_i^+(\varepsilon)$  where  $\bigcap_{i=1}^{2N} (b_i^-(\varepsilon), b_i^+(\varepsilon)) = \emptyset$  associated with  $K_\varepsilon(u; l, L)$  such that  $b_i^\pm(\varepsilon) \rightarrow b_i$  as  $\varepsilon \rightarrow 0$ , and for any  $v \in K_\varepsilon(u; l, L)$  we have the following estimates*

$$\begin{cases} |v(x) - h| > \varepsilon, & x \in \mathbb{R} \setminus \bigcup_{i=1}^{2N} (b_i^-(\varepsilon), b_i^+(\varepsilon)) \\ |v'(x)| > \varepsilon, & x \in \bigcup_{i=1}^{2N} (b_i^-(\varepsilon), b_i^+(\varepsilon)) \end{cases}$$

**Definition 4.7.** A (regular) bump that is a solution to (1.1) we call a (regular) bump solution.

Lemma 4.1 implies that any bump solution  $u(x)$  of (1.1) is in  $BC^1(\mathbb{R})$  and  $|u(x)| \rightarrow 0$  as  $|x| \rightarrow \infty$ . Thus, the threshold  $h$  cannot be chosen negative.

#### 4.3. Bump solutions to (1.1) with $\beta = \infty$

In [1] Amari studied the equation (1.1) under the assumption that  $f_\beta = f_\infty = \chi_{[h, \infty)}$  with some  $h > 0$ . In this case one can find analytic expressions for the bump solutions. Let  $\omega$  satisfy Assumption B and suppose that (1.1) with  $\beta = \infty$  has a  $(h; b_1, \dots, b_{2N})$ -bump solution, say  $u_\infty$ .

Then it immediately follows from (1.1) that

$$u_\infty(x) = \sum_{k=1}^N \int_{b_{2k-1}}^{b_{2k}} \omega(x-y) dy$$

or, rewriting the equation above in terms of the anti-derivative

$$\begin{aligned} W(x) &= \int_0^x \omega(y) dy, \\ u_\infty(x) &= \sum_{k=1}^N \left( W(x - b_{2k-1}) - W(x - b_{2k}) \right). \end{aligned} \quad (4.3)$$

Next, we assume that  $u_\infty$  is symmetric and, thus,  $b_i = -a_{N-i+1}$ , and  $b_{N+i} = a_i$ , for  $i = 1, \dots, N$ . In this new notation,  $u_\infty$  is a  $(h; -a_N, \dots, -a_1, a_1, \dots, a_N)$ -bump and we can rewrite (4.3) as

$$u_{\infty}(x) = \sum_{k=1}^N (-1)^{N-k+1} \left( W(x - a_k) - W(x + a_k) \right). \quad (4.4)$$

The vector  $\mathbf{a} = (a_1, \dots, a_N)^T$  with  $a_i < a_{i+1}$ ,  $i = 1, \dots, N-1$ , must be a solution of the system of  $N$  nonlinear equations

$$u_{\infty}(a_i) = \sum_{k=1}^N (-1)^{N-k+1} \left( W(a_i - a_k) - W(a_i + a_k) \right) = h. \quad (4.5)$$

Once  $\mathbf{a}$  is found one can construct  $u_{\infty}$  using the formula (4.4) and then verify that the obtained function is indeed a bump, and thus, a bump solution to (1.1). By Lemma 4.2(iii) the function  $u$  is automatically even.

Alternatively, when  $\omega$  has a rational real Fourier transform one can obtain an analytical expression for bump solutions by solving the corresponding piecewise linear ordinary differential equation, see e.g. [18,19,12]. We do not focus on this problem here but refer the reader to [1,18,28] for more details.

Further on we assume that the bump solution exists, it is symmetric, regular, and impose one extra assumption on the intersection points  $a_i$  whose role will be more clear later.

**Assumption C.** Let  $h > 0$  be fixed and  $u_{\infty}$  given by (4.4) be a  $(h; -a_N, \dots, -a_1, a_1, \dots, a_N)$ -bump solution to (1.1) with  $f_{\beta} = f_{\infty} = \chi_{[h, \infty)}$ .

(1) The solution  $u_{\infty}$  is a regular bump, that is,

$$|u'_{\infty}(a_i)| = \sum_{k=1}^N (-1)^{k+i} (\omega(a_i - a_k) - \omega(a_i + a_k)) > 0, \quad i = 1, \dots, N. \quad (4.6)$$

(2) From (4.5)  $\mathbf{a}$  is a solution to  $\mathbf{G}(\mathbf{a}) = \mathbf{0}$  where  $\mathbf{G} = (g_1, \dots, g_N)^T$  with  $g_i = (-1)^{i+1} \times (u_{\infty}(a_i) - h)$ . We assume that the Jacobian matrix of  $\mathbf{G}$  derived at  $\mathbf{a}$ ,  $J(\mathbf{a}) = \left( \partial g_i / \partial a_j \right)_{ij}(\mathbf{a})$  with

$$\begin{aligned} \frac{\partial g_i}{\partial a_i}(\mathbf{a}) &= |u'_{\infty}(a_i)| - \omega(0) - \omega(2a_i) \\ \frac{\partial g_i}{\partial a_j}(\mathbf{a}) &= (-1)^{i+j+1} (\omega(a_i - a_j) + \omega(a_i + a_j)), \end{aligned} \quad \begin{aligned} i, j &= 1, \dots, N, \\ i &\neq j, \end{aligned}$$

has an inverse, that is,  $\det(J(\mathbf{a})) \neq 0$ .

In particular, Assumption C(ii) guarantees that  $\mathbf{a}$  is the isolated solution of  $\mathbf{G}(\mathbf{a}) = \mathbf{0}$ .

#### 4.4. Existence and approximation of bump solutions for large $\beta < \infty$

We formulate our main results.

**Theorem 4.8.** Let  $h > 0$  be fixed and  $f_\beta$  and  $\omega$  satisfy [Assumptions A and B](#). Moreover, we assume that  $\omega$  is such that for  $\beta = \infty$  there exists a symmetric  $(h; -a_N, \dots, -a_1, a_1, \dots, a_N)$ -bump solution  $u_\infty$  of (1.1) and [Assumption C](#) is satisfied. Then we have the following result.

- There is an  $\varepsilon > 0$  such that for sufficiently large  $\beta > 0$  the operator  $\mathcal{H}_\beta : K_\varepsilon(u_\infty; -d, d) \rightarrow BC^1(\mathbb{R})$  for any  $d > a_N$  has a symmetric fixed point  $u_\beta$  which is a regular bump. Moreover  $\|u_\beta - u_\infty\|_{C^1} \rightarrow 0$  as  $\beta \rightarrow \infty$ .
- The bump solution  $u_\beta \in K_\varepsilon(u_\infty; -d, d)$  can be iteratively constructed and the sequence of successive approximations  $\{u_n\}_{n \in \mathbb{N} \cup \{0\}}$  converges to the solution  $u_\beta$  in  $BC^1(\mathbb{R})$ -norm. The sequence  $u_n$ ,  $n = 0, 1, \dots$ , is defined by

$$u_{n+1} = \int_{-d}^d \omega(x-y) f_\beta(U_{n+1}(y)) dy, \quad x \in \mathbb{R} \quad (4.7a)$$

$$U_{n+1}(x) = (H_\beta(U_n))(x) - \mathbf{s}^T(x)(\mathbf{S} - \mathbf{I})^{-1} \mathbf{p}_n, \quad x \in [-d, d] \quad (4.7b)$$

with  $U_0$  being the restriction of  $u_\infty$  on  $[-d, d]$ ,

$$(H_\beta U)(x) := \int_{-d}^d \omega(x-y) f_\beta(U(y)) dy \in C^1([-d, d]),$$

$$\mathbf{s}(x) = \left( \frac{\omega(x-a_1) + \omega(x+a_1)}{|u'_\infty(a_1)|}, \dots, \frac{\omega(x-a_N) + \omega(x+a_N)}{|u'_\infty(a_N)|} \right)^T,$$

and  $\mathbf{S}$  is an  $N \times N$  matrix with the elements

$$S_{ij} = \frac{\omega(a_i - a_j) + \omega(a_i + a_j)}{|u'_\infty(a_j)|}, \quad i, j = 1, \dots, N,$$

and  $\mathbf{I}$  is the  $N \times N$  identity matrix. The vector  $\mathbf{p}_n$  is defined as

$$\mathbf{p}_n := (p_n(a_1), \dots, p_n(a_N))^T, \quad p_n(a_i) = (H_\beta U_n - U_n)(a_i), \quad i = 1, \dots, N.$$

## 5. Proof of [Theorem 4.8](#)

Let  $h > 0$  be fixed and let  $f_\beta$  and  $\omega$  satisfy [Assumption A](#) and [Assumptions B](#), respectively. Moreover, we assume that  $\omega$  is such that for  $\beta = \infty$  there exists a symmetric regular  $(h; -a_N, \dots, -a_1, a_1, \dots, a_N)$ -bump solution  $u_\infty$  and [Assumption C](#) is satisfied.

Let  $d = a_N + \delta$  for some  $\delta > 0$  and define a set of even functions

$$K_\varepsilon(u_\infty) := K_\varepsilon(u_\infty; -d, d) \cap C_{\text{even}}^1(\mathbb{R})$$

where  $K_\varepsilon(u_\infty; -d, d)$  is given as in (4.2). We assume that  $\varepsilon > 0$  is small enough that  $K_\varepsilon(u_\infty)$  contains only regular symmetric  $N$ -bump solutions, see [Lemma 4.5](#).

If  $u \in K_\varepsilon(u_\infty)$  is a solution to (1.1) then due to [Corollary 4.6](#) and the choice of  $d$ , it is given as

$$u(x) = \int_{-d}^d \omega(x-y) f_{\beta}(u(y)) dy.$$

Now let  $T_1 : BC^1(\mathbb{R}) \rightarrow B([-d, d])$  be the restriction operator given as  $T_1 u = u|_{[-d, d]}$ , and  $T_2$  be the reconstruction operator

$$(T_2 U)(x) = \int_{-d}^d \omega(x-y) f_{\beta}(U(y) - h) dy, \quad (5.1)$$

where by [Lemma 4.1](#) we have  $T_2 : B([-d, d]) \rightarrow BC^1(\mathbb{R})$ .

From now on we use capital letters to denote the restriction of  $u \in K_{\varepsilon}(u_{\infty})$ , that is,  $U = T_1 u$  and, in particular,  $U_{\infty} = T_1 u_{\infty}$ . Notice that  $T_1 K_{\varepsilon}(u_{\infty}) = B_{\varepsilon}(U_{\infty})$  is the  $\varepsilon$ -ball in  $C_{\text{even}}^1([-d, d])$ , that is,

$$B_{\varepsilon}(U_{\infty}) := \{U : \|U - U_{\infty}\|_{C^1([-d, d])} < \varepsilon\} \cap C_{\text{even}}^1(\mathbb{R}).$$

It is obvious that [Lemma 4.5](#) and [Corollary 4.6](#) can be directly reformulated for  $B_{\varepsilon}(U_{\infty})$  as  $a_i \in [0, d]$ ,  $i = 1, \dots, N$ . We formulate this as a remark.

**Remark 5.1.** From [Corollary 4.6](#) there are  $a_i^{\pm}(\varepsilon)$ ,  $i = 1, \dots, N$ , such that  $a_i^{\pm}(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and for any  $U \in B_{\varepsilon}(U_{\infty})$

$$|U(x) - h| > \varepsilon, \quad x \in \mathbb{R} \setminus \bigcup_{i=1}^N (\pm a_i^{\mp}(\varepsilon), \pm a_i^{\pm}(\varepsilon)), \quad (5.2a)$$

$$|U'(x)| > \varepsilon, \quad x \in \bigcup_{i=1}^N (\pm a_i^{\mp}(\varepsilon), \pm a_i^{\pm}(\varepsilon)). \quad (5.2b)$$

Hence, if  $u \in K_{\varepsilon}(u_{\infty})$  is the solution to (1.1) then  $u = T_2(T_1 u)$  and  $U = T_1 u \in B_{\varepsilon}(U_{\infty})$  is a solution to the fixed point problem

$$U = H_{\beta} U, \quad (H_{\beta} U)(x) := \int_{-d}^d \omega(x-y) f_{\beta}(U(y)) dy. \quad (5.3)$$

On the other hand, if  $U \in B_{\varepsilon}(U_{\infty})$  is the solution to (5.3) there is no guarantee that any  $T_1$ -preimage of  $U$  is a fixed point of  $\mathcal{H}_{\beta}$  in (1.1). However, if it is, then it must be given as  $u = T_2 U$ .

With the next proposition we claim there exist sufficiently small  $\varepsilon > 0$  and sufficiently large  $\beta > 0$  such that for any solution  $U \in B_{\varepsilon}(U_{\infty})$  of (5.3) the corresponding  $u = T_2 U$  is a  $N$ -bump solution to (1.1). We need an auxiliary lemma.

**Lemma 5.2.** *The Nemytskii operator  $N(\beta; U) = f_{\beta}(U) : (0, \infty) \times L^{\infty}([-d, d]) \rightarrow L^p([-d, d])$ ,  $1 \leq p \leq \infty$  is jointly continuous in  $(\beta_0, U_0)$  for any  $\beta_0 \in (0, \infty)$  and  $U_0 \in L^{\infty}([-d, d])$ .*

Moreover,  $N(\beta; U)$  is jointly continuous in  $(\infty, U_0)$  for any  $U_0 \in B_\varepsilon(U_\infty)$  as a map from  $(0, \infty] \times B_\varepsilon(U_\infty)$  to  $L^p_{\text{even}}([-d, d])$  for  $1 \leq p < \infty$ .

**Proof.** Note that  $f_\beta, \beta \in (0, \infty]$  is uniformly bounded and thus,  $N(\beta, U)$  is in  $L^p([-d, d])$  and, if  $U$  is even, in  $L^p_{\text{even}}([-d, d])$ ,  $1 \leq p \leq \infty$ .

Let  $\beta, \beta_0 \in (0, \infty]$  and  $U, U_0 \in L^\infty([-d, d])$  then we have the estimate

$$\|N(\beta, U) - N(\beta_0, U_0)\|_{L^p([-d, d])}^p = \int_{-d}^d |f_\beta(U(x)) - f_\infty(U_0(x))|^p dx \leq \Sigma_1(U) + \Sigma_2(\beta_0)$$

where

$$\Sigma_1(U) := \int_{-d}^d |f_\beta(U(x)) - f_{\beta_0}(U(x))|^p dx,$$

and

$$\Sigma_2(\beta_0) := \int_{-d}^d |f_{\beta_0}(U(x)) - f_{\beta_0}(U_0(x))|^p dx.$$

For  $\beta \rightarrow \beta_0$ ,  $\beta, \beta_0 \in (0, \infty)$ ,  $\Sigma_1(U) \rightarrow 0$  uniformly in  $U \in L^\infty([-d, d])$  by [Assumption A\(4\)](#). From [Assumption A\(5\)\[Lip\]](#) we obtain

$$\Sigma_2(\beta_0) \leq 2d \|f'_{\beta_0}\|_\infty \|U - U_0\|_{L^\infty([-d, d])} \rightarrow 0 \text{ as } \|U - U_0\|_{L^\infty([-d, d])} \rightarrow 0.$$

Thus,  $(0, \infty) \times L^\infty([-d, d]) \mapsto N(\beta, U)$  is jointly continuous in  $(\beta_0, U_0)$ .

Now we let  $\beta \rightarrow \infty$  and show that  $\Sigma_1(U)$  converges to zero uniformly for all  $U \in B_\varepsilon(U_\infty)$ . We introduce  $D := \bigcup_{i=1}^N (a_i^-(\varepsilon), a_i^+(\varepsilon))$  with  $a_i^\pm(\varepsilon)$  given as in [\(5.2\)](#). Then we have

$$\begin{aligned} \Sigma_1(U) &= \int_{[-d, d] \setminus D} |f_\beta(U(x)) - f_{\beta_0}(U(x))|^p dx + \\ &\quad \int_D |f_\beta(U(x)) - f_{\beta_0}(U(x))|^p dx. \end{aligned} \tag{5.4}$$

[Assumption A](#), or more precisely [\(4.1\)](#), yields the uniform in  $U$  convergence of the first integral in [\(5.4\)](#) as  $\beta \rightarrow \infty$ .

To estimate the second integral in [\(5.4\)](#) note that  $|U'(x)| > \varepsilon$  on  $(a_i^-(\varepsilon), a_i^+(\varepsilon))$ ,  $i = 1, \dots, N$ , see [Remark 5.1](#), [\(5.2b\)](#). Thus, there exists an inverse of  $U(x)$ ,  $k_i(t) : (U(a_i^-(\varepsilon)), U(a_i^+(\varepsilon))) \rightarrow (a_i^-(\varepsilon), a_i^+(\varepsilon))$  with  $|k'_i(t)| = 1/|U'(x)| \leq 1/\varepsilon$ .

Define  $M := \max_{x \in [-d, d]} U_\infty(x) + \varepsilon$  and  $m := \min_{x \in [-d, d]} U_\infty(x) - \varepsilon$ . We have

$$\begin{aligned} \int_{a_i^-(\varepsilon)}^{a_i^+(\varepsilon)} |f_\beta(U(x)) - f_\infty(U(x))|^p dx &= \int_{U(a_i^-(\varepsilon))}^{U(a_i^+(\varepsilon))} |f_\beta(t) - f_\infty(t)| |k'_i(t)| dt \leq \\ &\varepsilon^{-p} \int_m^M |f_\beta(t) - f_\infty(t)|^p dt. \end{aligned}$$

Hence,

$$\int_D |f_\beta(U_n(x)) - f_{\beta_0}(U_0(x))|^p dx \leq N\varepsilon^{-p} \int_m^M |f_\beta(t) - f_\infty(t)|^p dt \rightarrow 0, \quad \beta \rightarrow \infty$$

by the Lebesgue dominant convergence theorem. Thus  $\Sigma_1(U) \rightarrow 0$  as  $\beta \rightarrow \infty$ .

Next, we consider  $\Sigma_2(\infty)$ . Let  $U = U_n$ ,  $U_0 \in B_\varepsilon(U_\infty)$  and  $\|U_n - U_0\|_{C^1([-d,d])} \rightarrow 0$  as  $n \rightarrow \infty$ . In order to avoid introducing new notation we assume  $U_0 = U_\infty$ . For  $U_0 \neq U_\infty$  the same analysis applies. Note that as  $n \rightarrow \infty$ ,  $U_n \in B_{1/n}(U_\infty)$ . Define  $D_n := \bigcup_{i=1}^N (a_i^-(1/n), a_i^+(1/n))$  with  $a_i^\pm(\varepsilon)$ ,  $\varepsilon = 1/n$ , see Remark 5.1. Observe that as  $n \rightarrow \infty$  the sequences  $a_i^\pm(1/n) \rightarrow a_i$  and  $\mu(D_n) \rightarrow 0$ .

From (5.2a)  $f_\infty(U_n(x)) = f_\infty(U_\infty(x))$  for all  $x \in [-d, d] \setminus D_n$  and therefore

$$\Sigma_2(\infty) = \int_{D_n} |f_\infty(U_n(x)) - f_\infty(U_0(x))|^p dx \leq 2^{p+1} \mu(D_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Convergence properties of  $\Sigma_1(U)$  and  $\Sigma_2(\infty)$  result in the joint continuity of  $(0, \infty] \times B_\varepsilon(U_\infty) \mapsto N(\beta, U)$  at  $(\infty, U_0)$ .  $\square$

**Proposition 5.3.** Let  $f_\beta$  and  $\omega$  satisfy Assumptions A and Assumptions B, respectively. Moreover, assume there exists an  $(h; -a_N, \dots, -a_1, a_1, \dots, a_N)$ -bump solution  $u_\infty$  of (1.1) for  $\beta = \infty$  that satisfies Assumption C, and  $U_\infty = T_1 u_\infty$  being its restriction on  $[-d, d]$ . Then there exists  $\varepsilon > 0$  such that starting from sufficiently large  $\beta$ , if  $U_\beta \in B_\varepsilon(U_\infty)$  is a solution to (5.3) then  $u_\beta := T_2 U_\beta$  is the solution to (1.1).

**Proof.** It is sufficient to show that  $u_\beta \in K_\rho(u_\infty)$  where  $\rho > 0$  is small enough so that  $u_\beta$  is a regular  $N$ -bump, see Lemma 4.5. We derive the estimate

$$\begin{aligned} \|u_\beta - u_\infty\|_\infty &\leq \int_{-d}^d |\omega(x-y)(f_\beta(u_\beta(y)) - f_\infty(u_\infty(y)))| dy \leq \\ &\|\omega\|_\infty \left( \int_{-d}^d |f_\beta(u_\beta(y)) - f_\beta(u_\infty(y))| dy + \int_{-d}^d |f_\beta(u_\infty(y)) - f_\infty(u_\infty(y))| dy \right) \leq \\ &\|\omega\|_\infty \left( 2d \|f'_\beta\|_\infty \|U_\beta - U_\infty\|_\infty + \|N(\beta, U_\infty) - N(\infty, U_\infty)\|_{L_1([-d,d])} \right). \end{aligned}$$



From Lemma 4.2  $N(\beta, U_\infty)$  is continuous in  $\beta$  and, thus,  $\|N(\beta, U_\infty) - N(\infty, U_\infty)\|_{L_1([-d, d])} \leq \varepsilon$  for sufficiently large  $\beta > 0$ .

Thus, we have

$$\|u_\beta - u_\infty\|_\infty + \|u'_\beta - u'_\infty\|_{C([-d, d])} \leq \varepsilon \left(1 + \|\omega\|_\infty + 2d\|f'_\beta\|_\infty\|\omega\|_\infty\right).$$

Assigning  $\varepsilon < \rho / \left(1 + \|\omega\|_\infty + 2d\|f'_\beta\|_\infty\|\omega\|_\infty\right)$  we secure that  $u_\beta \in K_\rho(u_\infty)$ . This completes the proof.  $\square$

In the view of the proposition above we can study existence of solutions to (1.1) with  $\mathcal{H}_\beta : K_\rho(u_\infty) \rightarrow C^1_{\text{even}}(\mathbb{R})$  by studying existence of the solutions to (5.3) on  $B_\varepsilon(U_\infty)$ .

We make use the following classical result.

**Theorem 5.4** (Implicit Function Theorem, e.g., Section 4.7 in [8]). Let  $\mathcal{V}$ ,  $\mathcal{U}$ , and  $\mathcal{W}$  be Banach spaces,  $(v_0, u_0) \in \mathcal{V} \times \mathcal{U}$ , and  $\Omega \subset \mathcal{V} \times \mathcal{U}$  be a neighbourhood of  $(v_0, u_0)$ . Let the operator  $P : \Omega \rightarrow \mathcal{W}$  satisfy the following properties

- (i)  $P(v_0, u_0) = 0$ ,
- (ii)  $P$  is continuous at  $(v_0, u_0)$ ,
- (iii) there exist  $\Omega \mapsto P'_u[v, u]$  such that it is continuous in  $(v_0, u_0)$ , i.e.,

$$\lim_{(v, u) \rightarrow (v_0, u_0)} \|P'[v, u] - P'[v_0, u_0]\|_{\mathcal{W}} = 0$$

- (iv) the operator  $P'_u[v_0, u_0] : \mathcal{U} \rightarrow \mathcal{W}$  is a bounded linear operator with the bounded inverse

$$\Gamma = (P'_u[v_0, u_0])^{-1} : \mathcal{W} \rightarrow \mathcal{U}.$$

Then the following are true:

- There exist an operator  $F : O \rightarrow \mathcal{U}$ , where  $O \subset \mathcal{V}$  is some neighbourhood of  $v_0$ , with the following properties
  - (a)  $P(v, Fv) = 0$  for all  $v \in O$
  - (b)  $Fv_0 = u_0$
  - (c)  $F$  is continuous in  $v_0$ .
 Moreover, the operator  $F$  is uniquely defined, i.e., if there exists  $F_1$  that satisfies (a)–(c) then there is  $\varepsilon > 0$  such that  $F_1v = Fv$  for all  $\|v - v_0\|_{\mathcal{V}} < \varepsilon$ .
- The sequence of successive approximations  $\{F_n\}$  defined by  $F_0v \equiv u_0$  and

$$F_{n+1}v = F_nv - (P'_u[v_0, u_0])^{-1} \circ P(v, F_nv) \quad (5.5)$$

converges to the solution  $Fv$  as  $n \rightarrow \infty$  for all  $v \in O$ .

Define the operator  $P(\beta, U) : (0, \infty] \rightarrow C^{0, \alpha}_{\text{even}}([-d, d])$  for some  $0 < \alpha < 1$  as

$$P(\beta, U) = -U + H_\beta(U). \quad (5.6)$$

Using the notations in [Theorem 5.4](#) we have  $\mathcal{V} = \mathbb{R}$ ,  $\mathcal{U} = C^1_{\text{even}}([-d, d])$ ,  $\mathcal{W} = C^{0,\alpha}_{\text{even}}([-d, d])$  and  $\Omega = (0, \infty] \times B_\varepsilon(U_\infty)$ . Though it follows from [Lemma 4.1](#) and [Lemma 4.2](#) that  $(0, \infty] \times B_\varepsilon(U_\infty) \mapsto P(\beta, U) \in C^1_{\text{even}}([-d, d])$  we were not able to prove the condition (iii) of [Theorem 5.4](#) for  $C^1_{\text{even}}([-d, d])$  but  $C^{0,\alpha}_{\text{even}}([-d, d])$ . We will comment on it later.

We outline the idea of the proof of [Theorem 4.8](#).

- Step 1. We verify the conditions (i)–(iv) of [Theorem 5.4](#) for the operator  $P$  in (5.6).  
 Step 2. We prove the first part of [Theorem 4.8](#). First, we apply the first part of [Theorem 5.4](#) to the operator  $P$  to obtain existence and uniqueness of the fixed point  $U_\beta \in B_\varepsilon(U_\infty)$  of the operator  $H_\beta$  for large  $\beta > 0$  and  $\|U_\beta - U_\infty\|_{C^1([-d, d])} \rightarrow 0$  as  $\beta \rightarrow \infty$ . Next, we define  $u_\beta = T_2 U_\beta$  with  $T_2$  given in (5.1) and use [Proposition 5.3](#) to obtain the results for  $u_\beta$ .  
 Step 3. We prove the second part of [Theorem 4.8](#). Firstly, we validate (4.7b) using [Theorem 5.4](#). Secondly, we show that  $u_n = T_2 U_n$  converges to  $u_\beta$ .

**Step 1.** The condition (i) of [Theorem 5.4](#) follows directly for  $\beta = \infty$  and  $U = U_\infty$ , that is,  $P(\infty, U_\infty) = 0$ . The condition (ii) follows from [Lemma 5.2](#). Indeed,

$$\|H_\beta(U) - H_\infty(U_\infty)\|_{C^1([-d, d])} \leq (\|\omega\|_\infty + \|\omega'\|_\infty) \|N(\beta; U) - N(\infty; U_\infty)\|_{L^1([-d, d])} \rightarrow 0$$

as  $\|U - U_\infty\|_{C^1([-d, d])} \rightarrow 0$  and  $\beta \rightarrow \infty$  due to [Lemma 5.2](#) which implies the continuity of the operator  $P(\beta; U)$  at  $(\infty, U_\infty)$ .

Next we show Fréchet differentiability of the operator  $H_\beta$  for  $\beta < \infty$ .

**Lemma 5.5.** *The operator  $H_\beta : B_\varepsilon(U_\infty) \subset C^1_{\text{even}}([-d, d]) \rightarrow C^1_{\text{even}}([-d, d])$ ,  $\beta < \infty$ , given in (5.3) is Fréchet differentiable at  $U \in B_\varepsilon(U_\infty)$  with the derivative*

$$H'_\beta(U) = S[\beta, U], \quad S[\beta, U]v = \int_{-d}^d \omega(x-y) f'_\beta(U(y))v(y)dy. \quad (5.7)$$

**Proof.** Computing the Gâteaux derivative of  $H_\beta$  at  $U \in B_\varepsilon(U_\infty)$  we obtain  $dH_\beta(U; v) = S[\beta, U]v$ , with  $S[\beta, U]$  given in (5.7). The operator  $H_\beta$  is Fréchet differentiable if

$$\lim_{t \rightarrow 0} \left\| \frac{H_\beta(U + tv) - H_\beta(U)}{t} - S[\beta, U]v \right\|_{C^1([-d, d])} = 0$$

uniformly for all  $\|v\|_{C^1([-d, d])} < 1$ , see [Proposition 4.8 (b), [\[8\]](#)].

We obtain the estimate

$$\begin{aligned} & \left\| \frac{H_\beta(U + tv) - H_\beta(U)}{t} - S[\beta, U]v \right\|_{C^1([-d, d])} \leq \\ & \left( \|\omega\|_\infty + \|\omega'\|_\infty \right) \int_{-d}^d \left| \frac{f_\beta(U(y) + tv(y)) - f_\beta(U(y))}{t} - f'_\beta(U(y))v(y) \right| dy. \end{aligned} \quad (5.8)$$

From  $f_\beta \in C^{0,1}(\mathbb{R})$ , see [Assumption A\(5\)](#),

$$\frac{f_\beta(U(y) + tv(y)) - f_\beta(U(y))}{t} \rightarrow f'_\beta(U(y))v(y)$$

almost everywhere on  $[-d, d]$  uniformly in  $v$  on  $\{v : \|v\|_{C^1([-d, d])} < 1\}$ , and

$$|f_\beta(U(y) + tv(y)) - f_\beta(U(y))|/|t| \leq \|f'_\beta\|_\infty \|v\|_{C^1([-d, d])}.$$

Thus, by the Lebesgue Dominated Convergence theorem the integral in (5.8) converges to zero uniformly.  $\square$

To show the Fréchet differentiability of the operator  $H_\infty$  we must proceed in a different way due to the discontinuity of  $f_\infty$ .

**Lemma 5.6.** *The operator  $H_\infty : B_\varepsilon(U_\infty) \subset C^1_{\text{even}}([-d, d]) \rightarrow C^{0,1}_{\text{even}}([-d, d])$  given in (5.3) is Fréchet differentiable at  $U \in B_\varepsilon(U_\infty)$  with the derivative*

$$H'_\infty(U) = S[\infty, U], \quad S[\infty, U]v = \sum_{i=1}^N \frac{\omega(x - b_i) + \omega(x + b_i)}{|U(b_i)|} v(b_i), \quad (5.9)$$

where  $b_i \in (a_i^-(\varepsilon), a_i^+(\varepsilon))$ ,  $i = 1, \dots, N$ , are the positive solutions to  $U(x) = h$ .

**Proof.** We start by calculating the Gâteaux derivative of  $H_\infty$  at  $U \in B_\varepsilon(U_\infty)$ . Consider

$$H_\infty(U + tv) - H_\infty(U) = \int_{\mathcal{O}^+(t)} \omega(x - y)dy - \int_{\mathcal{O}^-(t)} \omega(x - y)dy \quad (5.10)$$

where

$$\mathcal{O}^+(t) = \{y : U(y) \leq h \text{ and } U(y) + tv(y) \geq h\}$$

and

$$\mathcal{O}^-(t) = \{y : U(y) \geq h \text{ and } U(y) + tv(y) \leq h\}.$$

Since  $U \in B_\varepsilon(U_\infty)$  is a restriction of a regular  $(h; -b_N, \dots, -b_1, b_1, \dots, b_N)$  bump on  $[-d, d]$  it is clear that  $\pm b_i$  for all  $i = 1, \dots, N$ , belongs either to  $\mathcal{O}^+(t)$  or  $\mathcal{O}^-(t)$ . As  $U$  and  $v$  are even function, the sets  $\mathcal{O}^\pm(t)$  are symmetric. Thus, without loss of generality we further consider  $y \in \mathcal{O}^\pm(t)$  such that  $y \geq 0$ .

Let  $b_i$ ,  $i \in \{1, \dots, N\}$  belong to either  $\mathcal{O}^-(t)$  or  $\mathcal{O}^+(t)$  and  $y \neq b_i$  be a limiting point of this set, and hence  $U(y) + tv(y) = h$ . By the mean value theorem

$$y = b_i - \frac{tv(b_i)}{U'(\xi) + tv(\xi)}$$

where  $\xi$  lies in between of  $y$  and  $b_i$ . Then we get

$$\lim_{t \rightarrow 0} \frac{|y - b_i|}{t} = \lim_{t \rightarrow 0} \frac{|t|}{t} \frac{|v(b_i)|}{|U'(b_i)|}.$$

If  $b_i \in \mathcal{O}^+(t)$  then either  $v(b_i) < 0, t > 0$  or  $v(b_i) > 0, t < 0$  and therefore

$$\lim_{t \rightarrow 0} \frac{|y - b_i|}{t} = \frac{v(b_i)}{|U'(b_i)|}, \quad b_i \in \mathcal{O}^+(t).$$

Similarly, for  $b_i \in \mathcal{O}^-(t)$  we conclude that

$$\lim_{t \rightarrow 0} \frac{|y - b_i|}{t} = -\frac{v(b_i)}{|U'(b_i)|}, \quad b_i \in \mathcal{O}^-(t).$$

Making use of (5.10) and limits above we obtain the Gâteaux derivative of  $H_\infty$  at  $U \in B_\varepsilon(U_\infty)$ ,

$$\begin{aligned} dH_\infty(U; v) &= \lim_{t \rightarrow 0} \frac{H_\infty(U_\infty + tv) - H_\infty(U)}{t} = \\ &= \sum_{\pm b_i \in \mathcal{O}^+(t) \cup \mathcal{O}^-(t)} \frac{\omega(\pm b_i - x)}{|U'(b_i)|} v(b_i) = \\ &= \sum_{i=1}^N \frac{\omega(b_i - x) + \omega(b_i + x)}{|U'(b_i)|} v(b_i) = S[\infty, U]v. \end{aligned}$$

As  $dH_\infty(U; v)$  is continuous at  $U$  for any  $U \in B_\varepsilon(U_\infty)$  we conclude that  $H_\infty$  is the Fréchet differentiable at  $U$  with  $H'_\infty(U)v = dH_\infty(U; v)$ , see Proposition 4.8(c) in [8].  $\square$

**Remark 5.7.** When  $U = U_\infty$  the derivative  $H'_\infty(U_\infty)$  is given as

$$H'_\infty(U_\infty) = S[\infty, U_\infty], \quad S[\infty, U_\infty]v = \sum_{i=1}^N \frac{\omega(x - a_i) + \omega(x + a_i)}{|u'_\infty(a_i)|} v(a_i). \quad (5.11)$$

From Lemma 5.5 and Lemma 5.6 the Fréchet derivative with respect to the second variable at  $(\beta, U)$  exists and is given as  $P'_U[\beta, U] = I - S[\beta, U]$  with  $S[\beta, U]$  given in (5.7) for  $\beta < \infty$  and (5.9) for  $\beta = \infty$ .

Now we turn to the proof of the norm convergence of  $P'_U[\beta, U]$ , see (iv) in Theorem 5.4. As

$$\|P'_U[\beta, U] - P'_U[\infty, U_\infty]\|_{C^{0,\alpha}([-d,d])} \leq \|U - U_\infty\|_{C^{0,\alpha}} + \|S[\beta, U] - S[\infty, U_\infty]\|_{C^{0,\alpha}([-d,d])},$$

it suffices to show that  $\|S[\beta, U] - S[\infty, U_\infty]\|_{C^{0,\alpha}([-d,d])} \rightarrow 0$  as  $(\beta, U) \rightarrow (\infty, U_\infty)$ . Before we prove this operator convergence we need the following lemma.

**Lemma 5.8.** Let  $O(a_i) \subset (a_i^-(\varepsilon), a_i^+(\varepsilon))$ ,  $i = 1, \dots, N$ , be an open neighbourhood of  $a_i(\varepsilon)$  and  $\mathcal{O}$  non-empty subset of  $[0, d]/\bigcup_{i=1}^N O(a_i)$ . Then for any  $\varphi \in C([-d, d])$  we have

$$\int_{\mathcal{O}} f'_{\beta_n}(U_n(x))\varphi(x)dx \rightarrow 0 \quad (5.12)$$

and

$$\int_{O(a_i)} f'_{\beta_n}(U_n(x))\varphi(x)dx \rightarrow \frac{\varphi(a_i)}{|u'_\infty(a_i)|} \quad (5.13)$$

for  $\beta_n \rightarrow \infty$  and  $U_n \rightarrow U_\infty$  in the  $C^1([-d, d])$  norm as  $n \rightarrow \infty$ .

**Proof.** We first prove (5.12). Let  $\eta < \varepsilon$  be such that  $[a_i^-(\eta), a_i^+(\eta)] \subset O(a_i)$ ,  $i = 1, \dots, N$ . As  $n \rightarrow \infty$  we have  $U_n \in B_\eta(U_\infty)$  and thus  $|U_n(x) - h| > \eta$  on the set  $D_\eta := [0, d]/\bigcup_{i=1}^N [a_i^-(\eta), a_i^+(\eta)]$ , see Remark 5.1. From Assumption A(5) and 4.1 we have  $|f'_{\beta_n}(U_n(x))| \leq C_\eta(\beta_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, we conclude that

$$\left| \int_{\mathcal{O}} f'_{\beta_n}(U_n(x))\varphi(x)dx \right| \leq \|\varphi\|_{L^\infty([-d, d])} \int_{D_\eta} |f'_{\beta_n}(U_n(x))|dx \leq d \|\varphi\|_{L^\infty([-d, d])} C_\eta(\beta_n) \rightarrow 0.$$

Now we turn our attention to proving (5.13). Due to (5.12) we can without loss of generality assume that  $O(a_i) = (a_i^-(\varepsilon), a_i^+(\varepsilon))$ ,  $i = 1, \dots, N$ . Let us fix some  $i = 1, \dots, N$ . Due to [29],  $U_\infty$  and  $U_n$  have the inverse functions  $k_\infty(t)$  and  $k_n(t)$ , respectively, defined on  $[h - \varepsilon, h + \varepsilon]$ . Moreover, by the Implicit Function Theorem  $\|k_n - k_\infty\|_{C^1([h - \varepsilon, h + \varepsilon])} \rightarrow 0$  as  $n \rightarrow \infty$ . Using a change of variables and (5.12) we obtain

$$\int_{O(a_i)} f'_{\beta_n}(U_n(x))\varphi(x)dx - \int_{h-\varepsilon}^{h+\varepsilon} f'_{\beta_n}(t)|k'_n(t)|\varphi(k_n(t))dt \rightarrow 0, \quad n \rightarrow \infty. \quad (5.14)$$

We note that for any positive  $\eta < \varepsilon$

$$\int_{h-\eta}^{h+\eta} f'_{\beta_n}(t)|k'_\infty(h)|\varphi(k_\infty(h))dt = \frac{\varphi(a_i)}{|u'_\infty(a_i)|} \quad (5.15)$$

and

$$I(\eta, n) = \int_{h-\eta}^{h+\eta} f'_{\beta_n}(t) \left( |k'_n(t)|\varphi(k_n(t)) - k'_\infty(h)\varphi(k_\infty(h)) \right) dt. \quad (5.16)$$

Let  $I(\eta, n) = I_1(\eta, n) + I_2(\eta, n)$  where

$$I_1(\eta, n) = \int_{h-\varepsilon}^{h+\varepsilon} f'_{\beta_n}(t) \left( |k'_n(t)|\varphi(k_n(t)) - |k'_\infty(t)|\varphi(k_\infty(t)) \right) dt$$

and

$$I_2(\eta, n) = \int_{h-\varepsilon}^{h+\varepsilon} f'_{\beta_n}(t) \left( |k'_\infty(t)|\varphi(k_\infty(t)) - |k'_\infty(h)|\varphi(k_\infty(h)) \right) dt.$$

We observe that  $|I_1(\eta, n)| \rightarrow 0$  and  $|I_2(\eta, n)| \rightarrow o(\eta)$  as  $n \rightarrow \infty$  since

$$|I_1(\eta, n)| \leq \max_{t \in [h-\eta, h+\eta]} \left( \varphi(k_n(t))|k'_n(t)| - \varphi(k_\infty(t))|k'_\infty(t)| \right) \rightarrow 0$$

due to  $\|k_n - k_\infty\|_{C^1([-d, d])} \rightarrow 0$  and

$$|I_2(\eta, n)| \leq \max_{t \in [h-\eta, h+\eta]} \left( \varphi(k_\infty(t))|k'_\infty(t)| - \varphi(k_\infty(h))|k'_\infty(h)| \right) \rightarrow o(\eta).$$

Thus, we obtain

$$\lim_{n \rightarrow 0} |I(\eta, n)| \leq \lim_{n \rightarrow 0} |I_1(\eta, n)| + \lim_{n \rightarrow 0} |I_2(\eta, n)| = o(\eta)$$

that combined with (5.16) and (5.15) results in

$$\lim_{n \rightarrow 0} \int_{h-\eta}^{h+\eta} f'_{\beta_n}(t) |k'_n(t)| \varphi(k_n(t)) dt = \frac{\varphi(a_i)}{|u'_\infty(a_i)|} + o(\eta). \quad (5.17)$$

From (5.12) we get

$$\int_{a_i^-(\varepsilon)}^{a_i^+(\varepsilon)} f'_{\beta_n}(U_n(x)) \varphi(x) dx - \int_{a_i^-(\eta)}^{a_i^+(\eta)} f'_{\beta_n}(U_n(x)) \varphi(x) dx \rightarrow 0, \quad n \rightarrow \infty,$$

independently on  $\eta$ . This observation and (5.14) yields  $o(\eta) \equiv 0$  in (5.17) which completes the proof.  $\square$

**Proposition 5.9.** Let *Assumption A*, *Assumption B* and *Assumption C* be satisfied and  $U_\infty = T_1 u_\infty$  be a restriction of  $u_\infty$  on  $[-d, d]$ . Then the map  $(0, \infty] \times B_\varepsilon(U_\infty) \mapsto S[\beta, U]$  is continuous at  $(\infty, U_\infty)$ , that is,

$$\|S[\beta, U] - S[\infty, U_\infty]\|_{C^{0,\alpha}([-d, d])} \rightarrow 0$$

as  $\beta \rightarrow \infty$  and  $\|U - U_\infty\|_{C^1([-d, d])} \rightarrow 0$ .

**Proof.** Let  $\beta < \infty$ ,  $\beta_n \rightarrow \infty$  and  $\|U - U_\infty\|_{C^1([-d, d])} \rightarrow 0$  as  $n \rightarrow \infty$ . We introduce the partition of  $[0, d]$ ,  $0 = y_0 < y_1 < \dots < y_N = d$  where  $y_i = (a_i + a_{i-1})/2$ ,  $i = 1, \dots, N$  and represent  $S[\beta_n, U_n]$  as

$$S[\beta_n, U_n] = \sum_{i=1}^N (S_i^+(n) + S_i^-(n))$$

with

$$S_i^\pm(n)v = \pm \int_{\pm y_i}^{\pm y_{i+1}} \omega(x - y) f'_{\beta_n}(U_n(y)) v(y) dy.$$

For  $\beta = \infty$  we have

$$S[\infty, U_n] = \sum_{i=1}^N (S_i^+ + S_i^-), \quad S_i^\pm v = \frac{\omega(x \pm a_i)}{|u'_\infty(a_i)|} v(a_i).$$

Next we prove that  $\|S_i^+(n) - S_i^+\|_{C^{0,\alpha}([-d, d])} \rightarrow 0$ ,  $n \rightarrow \infty$ , for any  $i = 1, \dots, N$ .

For an arbitrary fixed  $i \in \{1, \dots, N\}$

$$(S_i^+(n)v)(x) - (S_i^+v)(x) = g_1(x, n) + g_2(x, n)$$

where

$$g_1(x, n) = \int_{y_i}^{y_{i+1}} (\omega(x - y) - \omega(x - a_i)) f'_{\beta_n}(U_n(y)) v(y) dy$$

and

$$g_2(x, n) = \omega(x - a_i) \int_{y_i}^{y_{i+1}} f'_{\beta_n}(U_n(y)) \left( v(y) - \frac{v(a_i)}{u'_\infty(a_i)} \right) dy.$$

Next we show that  $\|g_k(\cdot, n)\|_\alpha = o(1/n)\|v\|_\alpha$ ,  $k = 1, 2$ , as  $n \rightarrow \infty$ . We start with  $g_1(x, n)$ . Let  $x \in [y_i, y_{i+1}]$  then

$$|g_1(x, n)| \leq \int_{y_i}^{y_{i+1}} |\omega(x - y) - \omega(x - a_i)| f'_{\beta_n}(U_n(y)) |v(y)| dy \leq$$

$$L \|v\|_\alpha \int_{y_i}^{y_{i+1}} |y - a_i| f'_{\beta_n}(U_n(y)) dy$$

with  $L \leq \|\omega'\|_\infty$  being the Lipschitz constant of  $\omega(x)$  on the interval  $[-d - y_{i+1}, d + y_{i+1}]$ .

Using (5.12) with  $\varphi(y) = |y - a_i|$  the integral on the right hand side tends to zero as  $n \rightarrow \infty$ . In order to bound the Hölder constant of  $g_1(x, n)$  we use two different estimates

$$|\omega(x - y) - \omega(x - a_i) - \omega(z - y) + \omega(z - a_i)| \leq 2L|y - a_i|$$

and

$$|\omega(x - y) - \omega(x - a_i) - \omega(z - y) + \omega(z - a_i)| \leq 2L|x - z|$$

so that

$$|\omega(x - y) - \omega(x - a_i) - \omega(z - y) + \omega(z - a_i)| \leq 2L|y - a_i|^{1-\alpha}|x - z|^\alpha.$$

Thus for  $x \neq z$  we have

$$\frac{|g_1(x, n) - g_1(y, n)|}{|x - z|^\alpha} \leq 2L\|v\|_\alpha \int_{y_i}^{y_{i+1}} |y - a_i|^{1-\alpha} f'_{\beta_n}(U_n(y)) dy.$$

Again by (5.12) the integral on the right hand side tends to zero as  $n \rightarrow 0$ .

To handle the term  $g_2(x, n)$  we represent it as a sum  $g_2(x, n) = \omega(x - a_i)(I_1(n) - I_2(n))$  with

$$I_1(n) = \int_{y_i}^{y_{i+1}} f'_{\beta_n}(U_n(y))(v(y) - v(a_i)) dy$$

and

$$I_2(n) = v(a_i) \left( \int_{y_i}^{y_{i+1}} f'_{\beta_n}(U_n(y)) dy - \frac{1}{|u'_\infty(a_i)|} \right).$$

We have the estimates

$$|I_1(n)| \leq \|v\|_\alpha \int_{y_i}^{y_{i+1}} f'_{\beta_n}(U_n(y)) |y - a_i| dy$$

and

$$|I_2(n)| \leq \|v\|_\alpha \left( \int_{y_i}^{y_{i+1}} f'_{\beta_n}(U_n(y)) dy - \frac{1}{|u'_\infty(a_i)|} \right),$$

where from Lemma 5.8 both integrals on the right hand sides tend to zero as  $n \rightarrow 0$ .



It follows that for any  $x \in (y_i, y_{i+1})$ ,

$$|g_2(x, n)| \leq \|\omega\|_\infty (|I_1(n)| + |I_2(n)|) = o(1/n) \|v\|_\alpha, \text{ as } n \rightarrow 0,$$

and

$$\frac{|g_2(x, n) - g_2(z, n)|}{|x - z|^\alpha} \leq \sup_{\substack{x, z \in [y_i, y_{i+1}] \\ x \neq z}} \frac{|\omega(x - a_i) - \omega(z - a_i)|}{|x - z|^\alpha} \|v\|_\alpha (|I_1(n)| + |I_2(n)|) \leq$$

$$L \sup_{x, z} |x - z|^{1-\alpha} (|I_1(n)| + |I_2(n)|) < (d)^{1-\alpha} L \|v\|_\alpha o(1/n), n \rightarrow \infty.$$

Collecting the estimates for  $\|g_1(x, n)\|_\alpha$  and  $\|g_2(x, n)\|_\alpha$  we conclude that  $\|S_i^+(n) - S_i^+\|_\alpha = o(1/n)$ ,  $n \rightarrow \infty$ . Due to the symmetry  $\|S_i^-(n) - S_i^-\|_\alpha = o(1/n)$  as well.  $\square$

**Remark 5.10.** Notice that the need for  $C^{0,\alpha}([-d, d])$  space with  $0 < \alpha < 1$  comes when estimating the convergence of  $|g_i(x, n) - g_i(z, n)|/|x - z|^\alpha$ ,  $n \rightarrow 0$ ,  $i = 1, 2$ , as similar arguments would fail for  $\alpha = 1$ .

With the next proposition we prove that the condition (iv) of [Theorem 5.4](#) is satisfied.

**Proposition 5.11.** Let  $u_\infty$  be a symmetric bump solution to (1.1),  $U_\infty = T_1 u_\infty$  be its restriction on  $[-d, d]$ , and assume that [Assumptions B](#) and [Assumption C](#) are satisfied. Then the Fréchet derivative of  $P(\infty, U)$  at  $U_\infty$ ,  $P'[\infty, U_\infty] = I - S[\infty, U_\infty] : C_{\text{even}}^1([-d, d]) \rightarrow C^{0,\alpha}([-d, d])$  with  $S[\infty, U_\infty]$  defined in (5.9) is invertible.

**Proof.** We will show that [Assumption C\(2\)](#) implies  $P'[\infty, U_\infty]$  has no zero eigenvalue. First we notice that the operator  $S[\infty, U_\infty]$  has the same eigenvalues as the  $N \times N$  matrix  $S = (S_{ij})$

$$S_{ij} = \frac{\omega(a_i - a_j) + \omega(a_i + a_j)}{|u'_\infty(a_j)|}, \quad i, j = 1, \dots, N.$$

Introduce the real diagonal matrix  $C := \text{diag}(|u'_\infty(a_1)|, \dots, |u'_\infty(a_N)|)$ . The matrix  $S$  can be made symmetric as  $C^{1/2} S C^{-1/2}$  and thus has only real eigenvalues. The operator  $P'[\infty, U_\infty]$  has in turn the same eigenvalues as the matrix  $P := I - S$  with  $I$  being the identity matrix. We notice that the elements of  $P = (P_{ij})$ ,  $i, j = 1, \dots, N$  are

$$P_{ii} = \frac{|u'_\infty(a_i)| - \omega(0) - \omega(2a_i)}{|u'_\infty(a_i)|} = \frac{1}{|u'(a_i)|} \frac{\partial g_i}{\partial a_i}(\mathbf{a}),$$

$$P_{ij} = -\frac{\omega(a_i - a_j) + \omega(a_i + a_j)}{|u'_\infty(a_j)|} = \frac{(-1)^{i+j}}{|u'(a_i)|} \frac{\partial g_i}{\partial a_j}(\mathbf{a}), \quad i \neq j,$$
(5.18)

with  $\partial g_i / \partial a_j(\mathbf{a})$  defined in [Assumption C\(2\)](#).

Let  $J$  be the Jacobian matrix defined in [Assumption C\(2\)](#),  $D := \text{diag}((-1)^1, \dots, (-1)^N)$ , and  $C$  given as above. Then we have  $P \sim PC = DJD \sim J$ . As  $\det(J) \neq 0$  we conclude that the matrix  $P$  has no zero eigenvalue as well as the operator  $P'[\infty, U_\infty]$ .  $\square$

Thus, we are ready to apply [Theorem 5.4](#) for the operator  $P$  in [\(5.6\)](#).

**Step 2.** [Theorem 5.4](#) (a)–(c) yields the existence and uniqueness of the fixed point  $U_\beta \in B_\varepsilon(U_\infty)$  of the operator  $H_\beta$  for large  $\beta > 0$  and the convergence  $\|U_\beta - U_\infty\|_{C^1([-d,d])} \rightarrow 0$  as  $\beta \rightarrow \infty$ .

Next, we apply the reconstruction operator  $T_2$  given in [\(5.1\)](#) to  $U_\beta$ , that is,  $u_\beta = T_2 U_\beta$ . From [Proposition 5.3](#)  $u_\beta$  is then a N-bump solution to [\(1.1\)](#). Let  $\beta \rightarrow \infty$ , from the estimate

$$\|u_\beta - u_\infty\|_{C^1} \leq (\|\omega\|_\infty + \|\omega'\|_\infty) \|N(\beta, U_\beta) - N(\infty, U_\infty)\|_{L^1([-d,d])},$$

[Lemma 5.2](#), and  $\|U_\beta - U_\infty\|_{C^1([-d,d])} \rightarrow 0$  the convergence  $\|u_\beta - u_\infty\|_{C^1} \rightarrow 0$  follows.

**Step 3.** From the second part of [Theorem 5.4](#) the fixed point  $U_\beta \in B_\varepsilon(U_\infty)$  of  $H_\beta$  for sufficiently large  $\beta$  can be obtain by the sequence of successive approximations  $\{U_n\}$ ,  $\|U_n - U_\beta\|_{C^1([-d,d])} \rightarrow 0$ ,  $n \rightarrow \infty$ , as

$$U_{n+1} = U_n - (P'_U[\infty, U_\infty])^{-1} \circ P(\beta, U_n), \quad U_0 = U_\infty,$$

or, using [\(5.6\)](#) and [\(5.11\)](#)

$$U_{n+1} = U_n - (I - S[\infty, U_\infty])^{-1} \circ (I - H_\beta(U_n)), \quad U_0 = U_\infty. \quad (5.19)$$

First we obtain the expression for  $(I - S[\infty, U_\infty])^{-1}$ . Let  $(I - S[\infty, U_\infty])v = w$  and introduce  $\mathbf{v} = (v(a_1), \dots, v(a_N))^T$  and  $\mathbf{w} = (w(a_1), \dots, w(a_N))^T$ . Then  $\mathbf{v} = (I - \mathbf{S})^{-1}\mathbf{w}$  and  $v(x) = w(x) + S[\infty, U_\infty]v(x) = w(x) + \mathbf{s}^T \mathbf{v}$  where  $\mathbf{l}$ ,  $\mathbf{s}$  and  $\mathbf{S}$  are as in [Theorem 4.8](#).

From the last two formulae we derive

$$v(x) = w(x) + \mathbf{s}^T (I - \mathbf{S})^{-1} \mathbf{w}. \quad (5.20)$$

Combining [5.20](#) and [\(5.19\)](#) we arrive at [\(4.7\)](#).

On each iteration step we can obtain the corresponding  $u_n = T_2 U_n$  with  $T_2$  given in [\(5.1\)](#). It remains to notice that

$$\|u_n - u_\beta\|_{C^1} = \|T_2 U_n - T_2 U_\beta\|_{C^1} \leq (\|\omega\|_\infty + \|\omega'\|_\infty) \|N(\beta, U_n) - N(\beta, U_\beta)\|_{L^1([-d,d])} \rightarrow 0$$

as  $n \rightarrow \infty$  due to [Lemma 4.5](#).

## 6. Advantages for numerical construction

In this section we apply [Theorem 4.8](#) to demonstrate the existence of 1-bump solutions of the FitzHugh–Nagumo equation [\(2.4\)](#) and 2-bump solutions to the Amari model [\(2.8\)](#) with  $\omega$  given in [\(2.10\)](#) and  $f_\beta$  as in [\(1.3\)](#). We also compute the approximations of the bump solutions using [\(4.7\)](#) and discuss the advantages of this numerical approximation compared to other approaches.

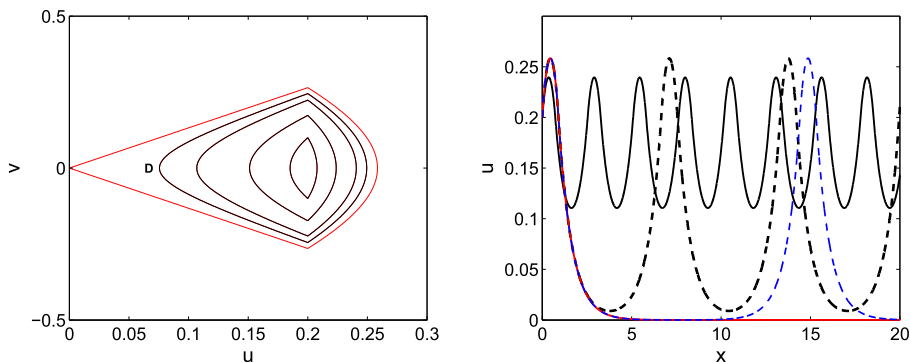


Fig. 5. For  $\beta = \infty$ ,  $k = 1.339$ , and  $h = 0.2$  we plot (a) periodic orbits (black) and the homoclinic orbit (red) of (6.1) and (b) periodic solutions (black and blue) and the bump solution (red) to (2.5a). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

### 6.1. 1-Bump solution of FitzHugh–Nagumo equation

Let us investigate the equation (2.5a) with  $f_\beta$  given by (1.3) with  $p = 2$  and its corresponding integral equation (2.6), for the existence and numerical construction of bump solutions. A bump solution corresponds to the homoclinic orbit in the phase plane of the equation

$$\begin{aligned} u' &= v \\ v' &= k^2 u - f_\beta(u) \end{aligned} \quad (6.1)$$

which exists when  $k < 1/\sqrt{2h}$ . Any bounded solution to (6.1) is confined in the closure of the bounded open region  $D$  of the phase plane with the homoclinic orbit being its boundary  $\partial D$ , see e.g. Fig. 5a.

Notice that the system (6.1) is reversible and conservative. From the reversibility it follows that any solution of (6.1) with the initial conditions in  $D$  is a periodic orbit and thus periodic orbits are dense in  $D$ , see Fig. 5a. This fact causes some difficulties to obtain the homoclinic orbit numerically. We have plotted in Fig. 5b the bump solution obtained analytically (red line) and by solving (6.1) numerically with  $(u(0), v(0)) = (h, kh)$  (blue dashed line). This illustrates that in order to obtain a good approximation of the bump solution on a large interval using the shooting method one must increase the precision of the method accordingly. Moreover, the shooting method would not be straightforwardly applicable if  $f_\beta$  does not satisfy Assumption A(2). This supports our argument for replacing the cubic  $g(u)$  in (2.1) with  $-u + f_\beta$ , see Section 2.1.

The proof of existence and rigorous construction of homoclinic orbits for (6.1) and the equations of a higher order with even more general types of nonlinearity can be studied using computer assisted proofs, see e.g. [22–26]. We however do not consider this approach here.

Now we analyse the equation (2.6) where Assumption A and Assumption B are satisfied. For  $\beta = \infty$  one can obtain an explicit formula for the  $(h; a)$ -bump solution  $u_\infty$  to (2.6) using the Amari technique [1] or by solving the ordinary differential equation (2.5a). We calculate  $a = -\ln(1 - 2k^2h)/2k$  and  $u'_\infty(a) = kh > 0$ , that is the bump is regular and Assumption C(1) is satisfied. The condition (2) of Assumption C is reduced to  $\omega(0) \neq 0$  which is also fulfilled. Thus, by Theorem 4.8 there exists a regular 1-bump solution to (2.6) that converges in  $C^1(\mathbb{R})$ -norm

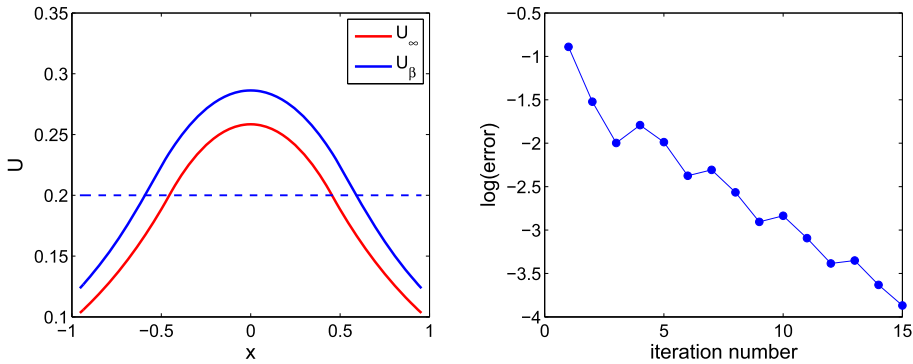


Fig. 6. (a) Approximation of the bump solution  $U_\beta$  (blue curve) to (2.6) on  $[-d, d]$  with  $d = a + 0.5$  and  $\beta = 100$  after the 15th iteration, and  $U_\infty$  (red curve). (b) The logarithm of the relative error (6.2) of the iteration process. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

to  $u_\infty$  and can be constructed using the iteration scheme (4.7). In Fig. 6a we have plotted the approximation of the restriction of bump solution, that is,  $U_\beta$ . In Fig. 6b we have plotted the base 10 logarithm of the relative error

$$\text{error}(n+1) = \frac{\|U_{n+1} - U_n\|_{C^1([-d,d])}}{\|U_n\|_{C^1([-d,d])}}, \quad n = 0, 1, \dots, \quad (6.2)$$

where  $U_n$  is given in (4.7b).

Notice, that despite the bump solution  $u_\infty$  is non-isolated fixed point of (1.1) with  $\beta = \infty$  on  $C^1(\mathbb{R})$ , it is isolated in  $K_\varepsilon(u_\infty)$ . Consequently,  $U_\beta = T_1 u_\beta$  is an isolated fixed point of the operator  $H_\beta$  already in the whole space  $C^1([-d, d])$ . Thus, the proposed method allows us to isolate solutions which can be extremely useful when dealing with higher order ordinary differential equations.

## 6.2. 2-Bump solutions of the neural field model

In this section we illustrate the obtained result for the Amari model with the firing rate function  $f_\beta$ ,  $\beta \in (0, \infty]$  as in (1.3),  $p = 2$  and the connectivity function  $\omega$  given as in (2.10) with  $K = 3$ ,  $k = 2$ ,  $M = 1$ , and  $m = 0.5$ , see Fig. 3. Hence, Assumption A and Assumption B are satisfied. When  $\beta = \infty$  one can employ the Amari technique [1,28] to show that there exist 2-bump solutions. In particular, for  $h = 0.3$  there is a pair of 2-bump solutions  $u_\infty^{(1)}$  and  $u_\infty^{(2)}$ : the first one is the  $(h; -a_2, -a_1, a_1, a_2)$ -bump with  $a_1 = 0.2948$ ,  $a_2 = 0.8506$  and the second one is with  $a_1 = 0.3786$ ,  $a_2 = 0.6782$ . We verified numerically that Assumption C is fulfilled for both bump solutions, thus Theorem 4.8 can be applied.

In Fig. 7a we have plotted the restriction of  $u_\infty^{(i)}$ ,  $i = 1, 2$ , on  $[-1, 1]$  and the restriction of the approximation of the bump solutions of (1.1) with  $\beta = 100$ .

For both cases we have computed the relative error using (6.2) with  $U_n = U_n^{(i)}$  and have plotted the base 10 logarithm of the error in Fig. 7b.

As the Fourier transform of  $\omega(x)$  is not a real rational function, the ordinary differential methods cannot be used here. Yet, when  $f_\beta$  is such that  $\chi_{[h+1/\beta, \infty)} \leq f_\beta < \chi_{[h, \infty)}$  it might be possible to iteratively construct  $u_\beta$  using the theory of monotone operators in Banach spaces

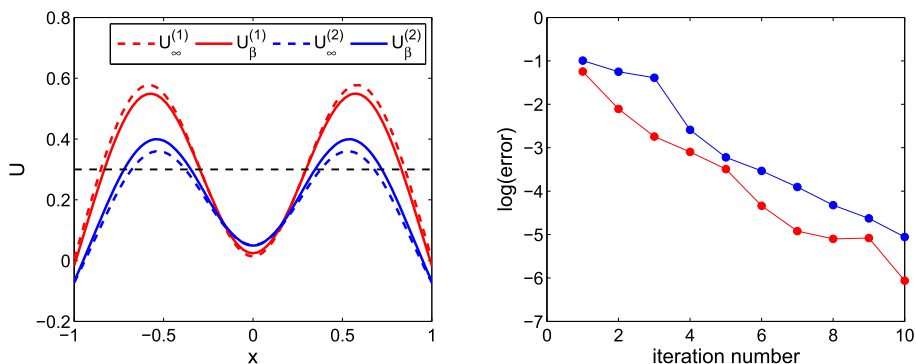


Fig. 7. (a) Approximation of the 2-bump solutions  $U_{\beta}^{(1)}$  (red solid curve) and  $U_{\beta}^{(2)}$  (blue solid curve) to (2.6) on  $[-d, d]$  with  $d = a_2 + 0.5$  and  $\beta = 100$  after the 10th iteration, and the corresponding  $U_{\infty}^{(1)}$  (red dashed curve) and  $U_{\infty}^{(2)}$  (blue dashed curve). (b) The logarithm of the relative error (6.2) of the iteration process for obtaining  $U_{\beta}^{(1)}$  (red curve) and  $U_{\beta}^{(2)}$  (blue curve). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

similarly as it has been done for 1-bump solutions in [16]. However, this method can be tricky (or not even possible) to use for  $N$ -bumps solutions when  $N$  is large. Moreover, some of the bump solutions cannot be obtained by this method as e.g. already for the case of 1-bumps, the limiting solution  $u_{\infty}$  is required to be linearly stable. As one of the 2-bump solutions is unstable, see [18], it is doubtful the iterative method as in [16] can be successful.

## 7. Conclusions and outlook

To summarize, we would like to emphasize few important points: (i) With Theorem 4.8 we justify the approximation of a smooth sigmoid function by the discontinuous unit step function for (1.1) on the class of bump solutions. In particular, one can be assured that a bump solution  $u_{\beta}$  for large  $\beta$  exists and can be approximated by  $u_{\infty}$ . (ii) Theorem 4.8 does not require  $\omega$  to be smooth nor to have a real rational Fourier transform. (iii) In order to obtain a better approximation of  $u_{\beta}$  than  $u_{\infty}$  one may utilize the iteration scheme in Theorem 4.8. Compared to the ordinary differential methods it allows us to isolate the solution and thus does not require a high numerical precision to secure that the found solution is indeed localized. However it might not be as efficient as e.g. the shooting method.

The technique presented in this paper could be fruitful in more general situations, for instance, to study existence and uniqueness of spatially localized solutions in two and three dimensional neural field models. Moreover, this method can be turned into a computer assisted proof which could be an interesting topic of future study. However, the most serious obstacle to develop such a theory is the absence of a general scheme for studying bumps in the limit (discontinuous) case, the only exception being the theory of radially symmetric bumps [30,31].

Within the framework of the neural field model (2.8), the next step would be to study Lyapunov stability of found bumps (the work under preparation). This can be done using the properties of spectral asymptotic that follows from the norm convergence of the Fréchet derivatives, see Proposition 5.9.

## Acknowledgments

The authors would like to thank the anonymous reviewer for the suggestions that helped to improve and clarify this manuscript.

## References

- [1] S. Amari, Dynamics of pattern formation in lateral-inhibition type neural fields, *Biol. Cybernet.* 27 (2) (1977) 77–87.
- [2] S. Coombes, P. beim Graben, R. Potthast, J. Wright (Eds.), *Neural Fields: Theory and Applications*, 1 edn, Springer-Verlag, Berlin–Heidelberg, 2014.
- [3] S. Coombes, Waves, bumps, and patterns in neural field theories, *Biol. Cybernet.* 93 (2) (2005) 91–108.
- [4] D. Avitabile, H. Schmidt, Snakes and ladders in an inhomogeneous neural field model, arXiv:1403.1037v2, 2014.
- [5] H.P. McKean, Nagumo's equation, *Adv. Math.* 4 (1970) 209–223.
- [6] A. Oleynik, A. Ponosov, J.A. Wyller, On the properties of nonlinear nonlocal operators arising in neural field models, *J. Math. Anal. Appl.* 398 (1) (2013) 335–351.
- [7] E. Burlakov, A. Ponosov, J. Wyller, Stationary solutions of continuous and discontinuous neural field equations, *J. Math. Anal. Appl.* 444 (1) (2016) 47–68.
- [8] Eberhard Zeidler, Peter R. Wadsack, *Nonlinear Functional Analysis and Its Applications. I. Fixed-Point Theorems*, Springer, New York, Berlin, 1986.
- [9] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, *Biophys. J.* 1 (6) (1961) 445–466.
- [10] R. FitzHugh, Mathematical models of excitation and propagation in nerve, in: H.P. Schwan (Ed.), *Biological Engineering*, McGraw–Hill Book Co., N.Y., 1969, pp. 1–85, Chapter 1.
- [11] J. Nagumo, S. Arimoto, S. Yoshizawa, An active pulse transmission line simulating nerve axon, *Proc. IRE* 50 (1962) 2061–2070.
- [12] E.P. Krisner, The link between integral equations and higher order ODEs, *J. Math. Anal. Appl.* 291 (1) (2004) 165–179.
- [13] B. Ermentrout, Neural networks as spatio-temporal pattern-forming systems, *Rep. Progr. Phys.* 61 (4) (1998) 353.
- [14] R. Potthast, P. Beim Graben, Existence and properties of solutions for neural field equations, *Math. Methods Appl. Sci.* 33 (8) (2010) 935–949.
- [15] K. Kishimoto, S. Amari, Existence and stability of local excitations in homogeneous neural fields, *J. Math. Biol.* 7 (4) (1979) 303–318.
- [16] A. Oleynik, A. Ponosov, J.A. Wyller, Iterative schemes for bump solutions in neural field model, *Differ. Equ. Dyn. Syst.* 23 (1) (2015) 79–98.
- [17] V. Kostykin, A. Oleynik, On the existence of unstable bumps in neural networks, *Integral Equations Operator Theory* 75 (2013) 445–458.
- [18] C.R. Laing, W.C. Troy, B. Gutkin, B. Ermentrout, Multiple bumps in a neuronal model of working memory, *SIAM J. Appl. Math.* 63 (2002) 62–97.
- [19] Carlo R. Laing, William C. Troy, PDE methods for nonlocal models, *SIAM J. Appl. Dyn. Syst.* 2 (3) (2003) 487–516.
- [20] A.R. Champneys, Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics, *Phys. D* 112 (1–2) (1998) 158–186.
- [21] A.R. Champneys, Homoclinic orbits in reversible systems II: Multibumps and saddle-centers, *Quart. - Cent. Wiskd. Inform.* 12 (1999) 185–212.
- [22] Daniel Wilczak, The existence of Shilnikov homoclinic orbits in the Michelson system: a computer assisted proof, *Found. Comput. Math.* 6 (4) (2006) 495–535.
- [23] Brian A. Coomes, Hüseyin Koçak, Kenneth J. Palmer, Transversal connecting orbits from shadowing, *Numer. Math.* 106 (3) (2007) 427–469.
- [24] Daniel Wilczak, Piotr Zgliczyński, Computer assisted proof of the existence of homoclinic tangency for the Hénon map and for the forced damped pendulum, *SIAM J. Appl. Dyn. Syst.* 8 (4) (2009) 1632–1663.
- [25] Jean-Philippe Lessard, Jason D. Mireles James, Christian Reinhardt, Computer assisted proof of transverse saddle-to-saddle connecting orbits for first order vector fields, *J. Dynam. Differential Equations* 26 (2) (2014) 267–313.
- [26] Blake Barker, Numerical proof of stability of roll waves in the small-amplitude limit for inclined thin film flow, *J. Differential Equations* 257 (8) (2014) 2950–2983.

- [27] Elias M. Stein, Rami Shakarchi, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton Lectures in Analysis, Princeton University Press, Princeton (N.J.), Oxford, 2005.
- [28] J. Angela Murdock, Fernanda Botelho, James E. Jamison, Persistence of spatial patterns produced by neural field equations, *Phys. D* 215 (2) (2006) 106–116.
- [29] Erich Barvínek, Ivan Daler, Jan Francu, Convergence of sequences of inverse functions, *Arch. Math. (Brno)* 027 (3–4) (1991) 201–204.
- [30] M.R. Owen, C.R. Laing, S. Coombes, Bumps and rings in a two-dimensional neural field: splitting and rotational instabilities, *New J. Phys.* 9 (10) (2007) 378.
- [31] Evgenii Burlakov, John Wyller, Arcady Ponomov, Two-dimensional Amari neural field model with periodic microstructure: rotationally symmetric bump solutions, *Commun. Nonlinear Sci. Numer. Simul.* 32 (2016) 81–88.