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Stability of plane Couette flow for the compressible Navier–Stokes equations with Navier-slip boundary ☆

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Abstract

This paper is devoted to the stability analysis of the plane Couette flow for the 3D compressible Navier–Stokes equations with Navier-slip boundary condition at the bottom boundary. It is shown that the plane Couette flow is asymptotically stable for small perturbation provided that the slip length, Reynolds and Mach numbers satisfy $\frac{3(1+\widetilde{\nu})\alpha}{\gamma^2(\nu+\alpha)\gamma_0} \le 1$ and $\frac{2\alpha}{\nu(\nu+\alpha)} \le 1$ for some constant $\gamma_0 > 0$. In particular, the Reynolds number ν^{-1} can be large if the slip length α is suitably small. This means that the constraint required in [11] on the Reynolds number to guarantee the stability of the plane Couette flow can be relaxed and improved so long as the slip effect at the boundary is involved.

MSC: 76N15; 35Q30

Keywords: Compressible Navier–Stokes equation; Stability; Plane Couette flow; Navier-slip boundary

1. Introduction

This paper is concerned with the existence and asymptotic stability of the barotropic compressible Navier–Stokes equations

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$$\begin{cases} \partial_t \rho + div(\rho v) = 0, \\ \rho \partial_t v + \rho v \cdot \nabla v + \nabla P(\rho) = \nu \Delta v + (\nu + \bar{\nu}) \nabla div v, \end{cases} \tag{1.1}$$

in a three-dimensional infinite layer $\Omega = \mathbb{R}^2 \times (0, 1)$, where we denote the density and velocity by $\rho = \rho(x, t)$ and $v = (v_1(x, t), v_2(x, t), v_3(x, t))^\perp$ respectively with $\cdot^\perp$ standing for the transposition. Assume that the pressure $P(\rho)$ is a smooth function of ρ satisfying

$$P'(\rho_*) > 0,$$

for a given constant $\rho_* > 0$, ν and $\bar{\nu}$ are the viscosity coefficients satisfying

$$\nu > 0, \quad \frac{2}{3}\nu + \bar{\nu} \geq 0.$$

The corresponding Reynolds number Re, the second Reynolds number $\overline{Re}$ and the Mach number M_a are given by

$$Re = \nu^{-1}, \quad \overline{Re} = \bar{\nu}^{-1}, \quad M_a = \frac{1}{\sqrt{P'(1)}}.$$

We are interested in the stability of the plane Couette flow for compressible Navier–Stokes equations (1.1) with the Navier-slip boundary condition imposed on the bottom boundary, and expect that the boundary effect may play an important role in analyzing the global existence and asymptotical behaviors of solutions near the plane Couette flow. To this end, we assume for simplicity that the flow is driven by the top plate moving along x_1-direction with constant speed $\mathbf{v}_0 = (1, 0, 0)^\perp$ and that the boundary $\Sigma \cup \Sigma_b$ is not permeable, namely,

$$v \cdot \mathbf{n} = 0, \quad \text{on } \Sigma \cup \Sigma_b, \tag{1.2}$$

where $\mathbf{n}$ is the outward unit vector normal to the boundary, $\Sigma =: \{x_3 = 1\}$ denotes the top boundary of Ω, and $\Sigma_b =: \{x_3 = 0\}$ the bottom boundary. Moreover, we set the non-slip boundary condition at the top boundary

$$v = \mathbf{v}_0, \quad \text{on } \Sigma; \tag{1.3}$$

and the Navier-slip boundary condition at the bottom boundary

$$S\mathbf{n} \cdot \boldsymbol{\tau} + \alpha v \cdot \boldsymbol{\tau} = 0, \quad \text{on } \Sigma_b, \tag{1.4}$$

where $S = 2\nu\mathbb{D}(v) + (\bar{\nu} div v - P)I_3$ is the stress tensor, I_3 is an identity matrix of order 3, $\mathbb{D}(v)$ is the velocity deformation tensor with elements $D_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$, $\boldsymbol{\tau}$ is any tangent vector orthogonal to $\mathbf{n}$, and $\alpha > 0$ is a constant of slip length or friction coefficient. It should be mentioned that the conditions (1.2) and (1.4) are proposed by Navier [16] and imply that the component of the fluid velocity tangent to the surface is proportional to the rate of strain on the surface. Some recent experiments, generally with typical dimensions microns or smaller, have demonstrated that the phenomenon of slip actually occurs (refer to [4,6] and the references therein).

As in [5,7], we can show that the system (1.1)–(1.4) has a stationary solution $u_s = (\rho_s, v_s)^\perp$ satisfying

$$\rho_s = 1, \quad v_s = v_s^1 e_1 = (A_0 x_3 + B_0)e_1, \tag{1.5}$$

with $e_1 = (1,0,0)^\perp$, $A_0 = \frac{\alpha}{\nu+\alpha}$ and $B_0 = \frac{\nu}{\nu+\alpha}$, which is the so-called plane Couette flow. We are interested in the stability of the plane Couette flow solution (1.5) and justify the influence of the slip length, Reynolds and Mach numbers on the motion of the flow. As well known, the field of hydrodynamic stability is concerned with the stability of various flows as subjected to various disturbances. This is an important issue since a stationary unstable flow in general can not exist in reality. The type of stability that a flow may exhibit typically depends on the Reynolds number Re. Therefore, to analyze how the stability depends on the Reynolds number for different flows is of importance and an interesting issue in hydrodynamic stability. There are recently important progress on the stability analysis of plane Couette flow for viscous flow with non-slip boundary condition [1–3,8–11,15,17]. In particular, as for the incompressible Navier–Stokes equations, Romanov [17] first proved that the plane Couette flow nonlinear stable for any Reynolds number $Re > 0$ under sufficiently small perturbations. Inspired by the important work of Mouhot and Villani [15], Bedrossian and Masmoudi [3] have proven the nonlinear inviscid damping effect of the Couette flow in an infinite periodic channel for small Gevrey perturbation. Then, Bedrossian, Germain and Masmoudi [1,2] obtained the interesting threshold of stability of the periodic plane Couette flow in Gevrey-a class with $a \in (1,2)$ at high Reynolds number. On the other hand, Kagei [11] made the breakthrough on the stability of the plane Couette flow with respect to small Reynolds and Mach numbers for compressible Navier–Stokes equations under small perturbation, and showed that the solution behaved in large time as that of an $n-1$ dimensional linear heat equation in parallel with the motion of the plane Couette flow.

The main purpose of the present paper is to study the stability of the plane Couette flow for compressible Navier–Stokes equations with the Navier-slip boundary condition imposed at the bottom boundary. Our main results show that the plane Couette flow solution (1.5) is asymptotically stable for small perturbation provided that the slip length, Reynolds and Mach numbers satisfy $\frac{3(1+\tilde{\nu})\alpha}{\gamma^2(\nu+\alpha)\gamma_0} \le 1$ and $\frac{2\alpha}{\nu(\nu+\alpha)} \le 1$ for some constant $\gamma_0 > 0$. In particular, the Reynolds number ν^{-1} and the Mach number γ^{-1} can be large enough so long as the slip length α is suitably small. This implies that the constraint required in [11] on the Reynolds and Mach numbers to guarantee the stability of the plane Couette flow can be relaxed and improved in the present paper as the slip effect is involved.

The rest part of the paper is arranged as follows. In section 2, we present the reformulations of the system and the main results. In section 3, we derive the Stokes estimate with Navier-slip boundary which will be needed to obtain the global existence of the perturbed equation. Section 4 is devoted to the linearized problem. Finally, the main results are proved in section 5.

Notations. We introduce some notations that will be used throughout this paper. For a domain $\mathcal{D}$, we use the same notation for both scalar functions and vector fields in the L^p-space and Sobolev spaces $W^{k,p}(\mathcal{D})$ ($H^k(\mathcal{D})$ if $p=2$). For $u = (\phi, \omega)$ with $\phi \in W^{k,p}(\mathcal{D})$ and $\omega = (\omega_1, \omega_2, \omega_3)^\perp \in W^{l,q}(\mathcal{D})$, we define

$$\|u\|_{W^{k,p}(\mathcal{D})\times W^{l,q}(\mathcal{D})} = \|\phi\|_{W^{k,p}(\mathcal{D})} + \|\omega\|_{W^{l,q}(\mathcal{D})}.$$

We simply write $\|u\|_{W^{k,p}(\mathcal{D})\times W^{k,p}(\mathcal{D})} = \|u\|_{W^{k,p}(\mathcal{D})}$ for $k = l$ and $p = q$.

In the case $\mathcal{D} = \Omega$, we abbreviate $L^p(\Omega)$ as L^p (resp. $W^{k,p}$, H^k). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$. In the case $\mathcal{D} = (0,1)$, we denote the norm of $L^p(0,1)$ by $|\cdot|_p$, and the norms of $W^{k,p}(0,1)$ and $H^k(0,1)$ by $|\cdot|_{W^{k,p}}$ and $|\cdot|_{H^k}$ respectively. The inner product of $L^2(0,1)$ is denoted by

$$(f,g) = \int_0^1 f(x_3)\overline{g(x_3)}dx_3, \quad f,g \in L^2(0,1).$$

And $H_*^2(0,1)$ consists of all elements of $\omega \in H^2(0,1)$ that satisfy the following boundary conditions

$$\omega|_{x_3=1} = 0, \quad \omega_3|_{x_3=0} = 0, \quad \nu\partial_{x_3}\omega_j - \alpha w_j|_{x_3=0} = 0,\ j = 1,2.$$

It is easy to see that $H_*^2(0,1)$ is complete.

Furthermore, for $f \in L^1(0,1)$, we denote the mean value of f in $(0,1)$ by $\langle f\rangle$:

$$\langle f\rangle = (f,1) = \int_0^1 f(x_3)dx_3.$$

We often write $x \in \Omega$ as $x = (x', x_3)$ with $x' = (x_1, x_2) \in \mathbb{R}^2$. Partial derivatives of a function u in x, x', x_3 and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_3} u$ and $\partial_t u$, respectively.

We denote the $k \times k$ identity matrix by I_k. Q_0 and $\tilde{Q}$ denote the 4×4 diagonal matrices:

$$Q_0 = diag(1,0,0,0), \quad \tilde{Q} = diag(0,1,1,1).$$

For a function $f = f(x')$ $(x' \in \mathbb{R}^2)$, we denote its Fourier transform by $\hat{f}$ or $(\mathscr{F} f)(\xi)$:

$$\hat{f} = (\mathscr{F} f)(\xi) = \int_{\mathbb{R}^2} e^{-ix'\cdot\xi} f(x')dx',$$

with $\xi = (\xi_1, \xi_2)^\perp$. The inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$(\mathscr{F}^{-1} f)(x') = \int_{\mathbb{R}^2} e^{ix'\cdot\xi} f(\xi)d\xi.$$

We denote the resolvent set of a closed operator A by $\varrho(A)$ and the spectrum of A by $\sigma(A)$. For $\Lambda \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$, we denote the set $\{\lambda \in \mathbf{C}; |\arg(\lambda - \Lambda)| \le \theta\}$ by $\Sigma(\Lambda, \theta)$

$$\Sigma(\Lambda,\theta) = \{\lambda \in \mathbf{C}; |\arg(\lambda - \Lambda)| \le \theta\}.$$

2. Reformulation and main results

In this section, we reformulate the original problem (1.1)–(1.4) near the Couette flow solution (1.5) to obtain the corresponding one for the perturbations and state the main results. Define

$$u=(\phi,\omega)^{\perp}=(\gamma^2(\rho-\rho_s),v-v_s)^{\perp},\quad \gamma=M_a^{-1}. \tag{2.1}$$

Substituting (2.1) into (1.1), we obtain the following initial boundary value problem

$$\partial_t\phi+v_s^1\partial_{x_1}\phi+\gamma^2 div\omega=f_0, \tag{2.2}$$

$$\partial_t\omega-\nu\Delta\omega-\nu'\nabla div\omega+\nabla\phi+v_s^1\partial_{x_1}\omega+A_0\omega_3e_1=g, \tag{2.3}$$

$$\omega|_{\Sigma}=0,\quad \omega_3|_{\Sigma_b}=0, \tag{2.4}$$

$$\nu\partial_{x_3}\omega_j-\alpha w_j|_{\Sigma_b}=0,\ j=1,2, \tag{2.5}$$

$$(\phi,\omega)|_{t=0}=(\phi_0,\omega_0), \tag{2.6}$$

where $\nu'=\nu+\bar{\nu}$, f_0 and g denote the nonlinearities

$$f_0=-div(\phi\omega), \tag{2.7}$$

$$g=-\omega\cdot\nabla\omega-\frac{\phi}{\gamma^2+\phi}\{\nu\Delta\omega+\nu'\nabla div\omega+(P_2(\gamma,\phi)-1)\nabla\phi\}, \tag{2.8}$$

with

$$P_2(\gamma,\phi)=\frac{1}{\gamma^2}\int_0^1 P''(1+\gamma^2\phi\theta)d\theta.$$

Let us consider the linearized problem

$$\begin{cases}\partial_t u+Lu=0,\\ \omega|_{\Sigma}=0,\quad \omega_3|_{\Sigma_b}=0,\\ \nu\partial_{x_3}\omega_j-\alpha w_j|_{\Sigma_b}=0,\ j=1,2,\\ (\phi,\omega)|_{t=0}=(\phi_0,\omega_0),\end{cases} \tag{2.9}$$

where

$$L=\begin{pmatrix} v_s^1\partial_{x_1} & \gamma^2 div\\ \nabla & -\nu\Delta I_3-\nu'\nabla div\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & v_s^1\partial_{x_1}I_3+A_0e_1e_3^{\perp}\end{pmatrix},$$

and denote the solution operator of (2.9) by $S(t)$.

We have the following results on the global existence and large time behavior of linearized problem (2.9).

Theorem 2.1. *Let $\widetilde{\nu} = \nu + \nu'$. Suppose that $u_0 = (\phi_0, \omega_0)^\perp \in H^1 \times L^2$ and $\partial_{x'}\omega_0 \in L^2$. Then the initial value problem (2.9) has a unique solution $u(t) = S(t)u_0$ and satisfies the estimates*

$$\|\partial_x^l S(t)u_0\|_2 \le C\Big\{t^{-\frac{l}{2}}\|u_0\|_{H^1\times L^2} + \|\partial_{x'}\omega_0\|_2\Big\}, \quad l = 0, 1, \tag{2.10}$$

for $0 < t \le 1$.

Moreover, there exists a positive constant γ_0, such that if $\frac{3(1+\widetilde{\nu})\alpha}{\gamma^2(\nu+\alpha)\gamma_0} \le 1$ and $\frac{2\alpha}{\nu(\nu+\alpha)} \le 1$, then the following estimates hold uniformly in $t \ge 1$, $u_0 = (\phi_0, \omega_0)^\perp \in H^1 \times L^2$ with $\partial_{x'}\omega_0 \in L^2$ and $u_0 \in L^1$,

$$\|\partial_x^l S(t)u_0\|_2 \le C\Big\{t^{-\frac{1}{2}-\frac{l}{2}}\|u_0\|_1 + e^{-\delta t}\|u_0\|_{H^1}\Big\}, \quad l = 0, 1, \tag{2.11}$$

$$\|S(t)u_0 - G_{t*x'}\Pi^{(0)}u_0\|_2 \le C\Big\{t^{-1}\|u_0\|_1 + e^{-\delta t}\|u_0\|_{H^1}\Big\}, \tag{2.12}$$

with some constant $\delta > 0$, where

$$G_{t*x'}\Pi^{(0)}u_0 = \mathscr{F}^{-1}\Big[e^{-((\frac{A_0}{2}+B_0)i\xi_1+\kappa_1\xi_1^2+\kappa_2\xi_2^2)t}\,\widehat{\Pi}_0\hat{u}_0\Big]$$

with $\widehat{\Pi}_0\hat{u}_0 = \langle Q_0\hat{u}_0\rangle$, where κ_1 and κ_2 are some positive constants that will be given in Section 4.

Remark 2.1. It is not difficult to show that

$$\|G_{t*x'}\Pi_0u_0\|_2 \le t^{-\frac{1}{2}}\|u_0\|_1.$$

And $G_{t*x'}\Pi_0u_0$ can be written in the form $G_{t*x'}\Pi_0u_0 = (\phi^{(0)}(x', t), 0)$ with $\phi^{(0)}(x', t)$ satisfying

$$\partial_t\phi^{(0)} - \kappa_1\partial_{x_1}^2\phi^{(0)} - \kappa_2\partial_{x_2}^2\phi^{(0)} + (\frac{A_0}{2} + B_0)\partial_{x_1}\phi^{(0)} = 0,$$

$$\phi^{(0)}|_{t=0} = \int_0^1 \phi^{(0)}(x', x_3)dx_3.$$

With the help of Theorem 2.1, we can establish the following result on the stability of the plane Couette flow of compressible Navier–Stokes equations.

Theorem 2.2.

(i) *Let s be an integer satisfying $s \ge 2$. There exist constants $\varepsilon_0 > 0$ and $\gamma_0 > 0$ such that if $\frac{3(1+\widetilde{\nu})\alpha}{\gamma^2(\nu+\alpha)\gamma_0} \le 1$ and $\frac{2\alpha}{\nu(\nu+\alpha)} \le 1$, then for any $u_0 = (\phi_0, \omega_0)^\perp \in H^s$ satisfying the appropriate compatibility condition with $\|u_0\|_{H^s} \le \varepsilon_0$, there exists a unique global solution $u(t) = (\phi(t), \omega(t))^\perp \in C([0, \infty); H^s)$ of (2.2)–(2.6), which satisfies*

$$\|(\phi,\omega)(t)\|_{H^s}^2+\int_0^t \|\partial_x\phi\|_{H^{s-1}}^2+\|\partial_x\omega\|_{H^s}^2 d\tau \le C\|u_0\|_{H^s}^2, \tag{2.13}$$

and

$$\|(\phi,\omega)(t)\|_\infty \to 0, \quad as \quad t\to\infty.$$

(ii) *Furthermore, let $s\ge 3$ and assume that $\frac{3(1+\widetilde{\nu})\alpha}{\gamma^2(\nu+\alpha)\gamma_0}\le 1$ and $\frac{2\alpha}{\nu(\nu+\alpha)}\le 1$. Assume also that $u_0=(\phi_0,\omega_0)^\perp\in H^s\cap L^1$ and satisfies the appropriate compatibility condition. There exists a constant $\varepsilon_1\in(0,\varepsilon_0]$ such that if $\|u_0\|_{H^s\cap L^1}\le\varepsilon_1$, then the solution $u(t)=(\phi(t),\omega(t))^\perp$ satisfies*

$$\|\partial_x^l(\phi(t),\omega(t))\|_2=\mathcal{O}(t^{-\frac{1}{2}-\frac{l}{2}}), \quad as \quad t\to\infty, \tag{2.14}$$

for $l=0,1,2$ and

$$\|u(t)-G_{t*x'}\Pi_0u_0\|_2=\mathcal{O}(t^{-1}), \quad as \quad t\to\infty. \tag{2.15}$$

Remark 2.2. As in the paper [11], the disturbance behaves the same time convergence rates in L^2 norm as the solution of a two dimensional linear heat equation with a convective term.

3. Stokes estimate with Navier-slip boundary

To prove Theorem 2.1–2.2, we need to deal with the Stokes problem with Navier-slip boundary condition and establish the estimates for the linearized problem. Consider the following Stokes problem with Navier-slip boundary condition at the bottom boundary and Dirichlet boundary condition at the top boundary,

$$\begin{cases} -\nu\Delta\omega+\nabla q=\vec{F}, \quad \nabla\cdot\omega=F_0, \\ \nu\frac{\partial\omega_j}{\partial x_3}-\alpha\omega_j\big|_{\Sigma_b}=0, \quad j=1,2, \\ \omega_3\big|_{\Sigma_b}=0, \quad \omega\big|_{\Sigma}=0. \end{cases} \tag{3.1}$$

We have the existence and uniqueness of solution to (3.1) below.

Theorem 3.1. *Let $\Omega=\mathbb{R}^2\times(0,1)$. Suppose*

$$\vec{F}\in H^{n-2}(\Omega),\, F_0\in H^{n-1}(\Omega),\, \int_\Omega F_0dx=0,\ n\ge 2,$$

then the problem (3.1) has a unique solution (w,q) such that

$$\|\omega\|_{H^n(\Omega)}^2+\|\nabla q\|_{H^{n-2}(\Omega)}^2\le C(\|\vec{F}\|_{H^{n-2}(\Omega)}^2+\|F_0\|_{H^{n-1}(\Omega)}^2), \tag{3.2}$$

where the constant C is proportional to $\nu^2+\dfrac{1}{\nu^2}$.

To prove Theorem 3.1, we reformulate Stokes problem (3.1) as follows

$$\begin{cases} -\nu\Delta\omega + \nabla q = 0, \quad \nabla\cdot\omega = 0, \\ \nu\frac{\partial\omega_j}{\partial x_3} - \alpha\omega_j\big|_{\Sigma_b} = b_j, \quad j = 1, 2, \\ \omega_3\big|_{\Sigma_b\cup\Sigma} = 0, \quad \omega_j\big|_{\Sigma} = b_{j+2}, \quad j = 1, 2. \end{cases} \tag{3.3}$$

Applying the Fourier transform to (3.3) with respect to $x' = (x_1, x_2)$, we have the following system of ordinary differential equations

$$\begin{cases} \nu(|\xi|^2 - \frac{d^2}{dx_3^2})\hat{\omega}_j + i\xi_j\hat{q} = 0, \quad (j = 1, 2), \\ \nu(|\xi|^2 - \frac{d^2}{dx_3^2})\hat{\omega}_3 + \frac{d\hat{q}}{dx_3} = 0, \\ i\xi_1\hat{\omega}_1 + i\xi_2\hat{\omega}_2 + \frac{d\hat{\omega}_3}{dx_3} = 0, \end{cases} \tag{3.4}$$

with the boundary conditions

$$\begin{cases} \nu\frac{\partial\hat{\omega}_j}{\partial x_3} - \alpha\hat{\omega}_j\big|_{\Sigma_b} = \hat{b}_j, \quad j = 1, 2, \\ \hat{\omega}_3\big|_{\Sigma_b\cap\Sigma} = 0, \quad \hat{\omega}_j\big|_{\Sigma} = \hat{b}_{j+2}, \quad j = 1, 2. \end{cases} \tag{3.5}$$

The solution to (3.4)–(3.5) has the form

$$\begin{cases} \hat{\omega}_1 = C_5 e^{|\xi|x_3} + C_6 e^{-|\xi|x_3} + i\xi_1 C_1 x_3 e^{|\xi|x_3} - i\xi_1 C_2 x_3 e^{-|\xi|x_3}, \\ \hat{\omega}_2 = C_7 e^{|\xi|x_3} + C_8 e^{-|\xi|x_3} + i\xi_2 C_1 x_3 e^{|\xi|x_3} - i\xi_2 C_2 x_3 e^{-|\xi|x_3}, \\ \hat{\omega}_3 = C_3 e^{|\xi|x_3} + C_4 e^{-|\xi|x_3} + C_1|\xi|x_3 e^{|\xi|x_3} + C_2|\xi|x_3 e^{-|\xi|x_3}, \\ \hat{q} = 2\nu C_1|\xi|e^{|\xi|x_3} + 2\nu C_2|\xi|e^{-|\xi|x_3}, \end{cases} \tag{3.6}$$

with the coefficient C_j, $j = 1, \cdots, 8$ to be determined below. Substituting (3.6) into (3.5) and $(3.4)_3$, we obtain the following equations

$$C_5 e^{|\xi|} + C_6 e^{-|\xi|} + i\xi_1 C_1 e^{|\xi|} - i\xi_1 C_2 e^{-|\xi|} = \hat{b}_3, \tag{3.7}$$

$$\nu(C_5|\xi| - C_6|\xi| + i\xi_1 C_1 - i\xi_1 C_2) - \alpha'(C_5 + C_6) = \hat{b}_1, \tag{3.8}$$

$$C_7 e^{|\xi|} + C_8 e^{-|\xi|} + i\xi_2 C_1 e^{|\xi|} - i\xi_2 C_2 e^{-|\xi|} = \hat{b}_4, \tag{3.9}$$

$$\nu(C_7|\xi| - C_8|\xi| + i\xi_2 C_1 - i\xi_2 C_2) - \alpha'(C_7 + C_8) = \hat{b}_2, \tag{3.10}$$

$$C_3 e^{|\xi|} + C_4 e^{-|\xi|} + C_1|\xi|e^{|\xi|} + C_2|\xi|e^{-|\xi|} = 0, \tag{3.11}$$

$$i\xi_1 C_5 + i\xi_2 C_7 + C_3|\xi| + C_1|\xi| = 0, \tag{3.12}$$

$$i\xi_1 C_6 + i\xi_2 C_8 - C_4|\xi| + C_2|\xi| = 0, \tag{3.13}$$

$$C_3 + C_4 = 0. \tag{3.14}$$

By (3.7)–(3.10), we can obtain after a tedious computation

$$\begin{cases} C_5 = \frac{1}{A}\Big\{i\xi_1(\nu|\xi|+\alpha)(C_2e^{-|\xi|} - C_1e^{|\xi|}) + (\nu|\xi|+\alpha)\hat{b}_3 + (\hat{b}_1 + i\xi_1\nu(C_2-C_1))e^{-|\xi|}\Big\}, \\ C_6 = \frac{1}{A}\Big\{i\xi_1(\nu|\xi|-\alpha)(C_2e^{-|\xi|} - C_1e^{|\xi|}) + (\nu|\xi|-\alpha)\hat{b}_3 - (\hat{b}_1 + i\xi_1\nu(C_2-C_1))e^{|\xi|}\Big\}, \\ C_7 = \frac{1}{A}\Big\{i\xi_2(\nu|\xi|+\alpha)(C_2e^{-|\xi|} - C_1e^{|\xi|}) + (\nu|\xi|+\alpha)\hat{b}_4 + (\hat{b}_2 + i\xi_2\nu(C_2-C_1))e^{-|\xi|}\Big\}, \\ C_8 = \frac{1}{A}\Big\{i\xi_2(\nu|\xi|-\alpha)(C_2e^{-|\xi|} - C_1e^{|\xi|}) + (\nu|\xi|-\alpha)\hat{b}_4 - (\hat{b}_2 + i\xi_2\nu(C_2-C_1))e^{|\xi|}\Big\}, \end{cases} \tag{3.15}$$

with

$$A = (\nu|\xi|+\alpha)e^{|\xi|} + (\nu|\xi|-\alpha)e^{-|\xi|} > 0, \quad (|\xi|>0).$$

Substituting (3.15) into (3.11)–(3.14), we have by a complicated computation

$$\begin{cases} C_1 = -e^{-2|\xi|}C_2 - |\xi|^{-1}(1-e^{-2|\xi|})C_3, \\ C_2 = \frac{1}{B}\big\{(\nu|\xi|+\alpha)e^{2|\xi|} - 2(\nu|\xi|^2+\alpha|\xi|+\alpha') - (\nu|\xi|-\alpha)(1+2|\xi|)e^{-2|\xi|}\big\}(i\xi_1\hat{b}_1 + i\xi_2\hat{b}_2) \\ \quad + \frac{1}{B}\{-(\nu|\xi|+\alpha)(2\nu|\xi|+2\alpha|\xi|-\alpha)e^{|\xi|} + (-2\alpha\nu|\xi|^2+4\alpha\nu|\xi|+2\alpha^2|\xi|-2\alpha^2)e^{-|\xi|} \\ \quad + (2\nu|\xi|-\alpha)(\nu|\xi|-\alpha)e^{-3|\xi|}\}(i\xi_1\hat{b}_3 + i\xi_2\hat{b}_4), \\ C_3 = \frac{-2|\xi|^2}{B}\big\{\nu|\xi|+\alpha+(\nu|\xi|-\alpha)e^{-2|\xi|}\big\}(i\xi_1\hat{b}_1+i\xi_2\hat{b}_2) + \frac{1}{B}\{|\xi|(\nu|\xi|+\alpha)(2\nu|\xi|+\alpha)e^{|\xi|} \\ \quad + (4\nu^2|\xi|^3 - 2\alpha\nu|\xi|)e^{-|\xi|} + |\xi|(2\nu|\xi|-\alpha)(\nu|\xi|-\alpha)e^{-3|\xi|}\}(i\xi_1\hat{b}_3+i\xi_2\hat{b}_4), \\ C_4 = -C_3, \end{cases} \tag{3.16}$$

with

$$\begin{aligned} B = {} & \alpha^2|\xi|(1-e^{-2|\xi|})(e^{2|\xi|}+e^{-2|\xi|}-2-4|\xi|^2) + \alpha\nu|\xi|^2\{2(1-e^{-2|\xi|})(e^{2|\xi|}-e^{-2|\xi|}-4|\xi|) \\ & + e^{2|\xi|} + e^{-4|\xi|} - e^{-2|\xi|} - 1 - 4|\xi|^2 - 4|\xi|^2e^{-2|\xi|}\} + 2\nu^2|\xi|^3(1+e^{-2|\xi|})(e^{2|\xi|}-e^{-2|\xi|}-4|\xi|) \\ & > 0, \quad \text{for} \quad |\xi|>0. \end{aligned}$$

It is not difficult to conclude that

$$\begin{cases} B > \frac{16\nu^2}{3}|\xi|^6, \ \text{ as } |\xi|>0; \ \ B \sim (\nu|\xi|+\alpha)(2\nu|\xi|+\alpha)|\xi|e^{2|\xi|} \ \text{ as } |\xi|>1, \\ B \sim \frac{8}{3}(4\nu+\alpha)(\nu+\alpha)|\xi|^6, \ \text{ as } |\xi|\to 0, \\ C_1 \sim \frac{4}{B}(\nu+\alpha)|\xi|^3(i\xi_1\hat{b}_1+i\xi_2\hat{b}_2) - \frac{4}{B}(\nu+\alpha)(2\nu+\alpha)|\xi|^3(i\xi_1\hat{b}_3+i\xi_2\hat{b}_4), \ \text{ as } |\xi|\to 0, \\ C_2 \sim \frac{4}{B}(\nu+\alpha)|\xi|^3(i\xi_1\hat{b}_1+i\xi_2\hat{b}_2) - \frac{4}{B}(\nu+\alpha)(2\nu+\alpha)|\xi|^3(i\xi_1\hat{b}_3+i\xi_2\hat{b}_4), \ \text{ as } |\xi|\to 0, \\ C_3 \sim -\frac{4}{B}(\nu+\alpha)|\xi|^3(i\xi_1\hat{b}_1+i\xi_2\hat{b}_2) + \frac{4}{B}(\nu+\alpha)(2\nu+\alpha)|\xi|^3(i\xi_1\hat{b}_3+i\xi_2\hat{b}_4), \ \text{ as } |\xi|\to 0, \\ C_2 - C_1 \sim -\frac{16}{3B}(\nu+\alpha)|\xi|^4(i\xi_1\hat{b}_1+i\xi_2\hat{b}_2) + \frac{8}{3B}\alpha(\nu+\alpha)|\xi|^4(i\xi_1\hat{b}_3+i\xi_2\hat{b}_4), \ \text{ as } |\xi|\to 0, \\ C_2e^{-|\xi|} - C_1e^{|\xi|} \sim \frac{8}{3B}(\nu+\alpha)|\xi|^4(i\xi_1\hat{b}_1+i\xi_2\hat{b}_2) - \frac{16}{3B}\alpha(\nu+\alpha)|\xi|^4(i\xi_1\hat{b}_3+i\xi_2\hat{b}_4), \ \text{ as } |\xi|\to 0. \end{cases} \tag{3.17}$$

As for the functions $e^{|\xi|x_3}$, $e^{-|\xi|x_3}$, $x_3e^{|\xi|x_3}$ and $x_3e^{-|\xi|x_3}$, we have the following estimates.

Lemma 3.1. *Let $k = 0, 1, 2$, then we have*

$$\int_0^1 \left| \left(\frac{d}{dx_3}\right)^k e^{|\xi| x_3} \right|^2 dx_3 \leq C|\xi|^{2k-1}(e^{2|\xi|} - 1),$$

$$\int_0^1 \left| \left(\frac{d}{dx_3}\right)^k e^{-|\xi| x_3} \right|^2 dx_3 \leq C|\xi|^{2k-1}(1 - e^{-2|\xi|}),$$

$$\int_0^1 \left| \left(\frac{d}{dx_3}\right)^k (x_3 e^{|\xi| x_3}) \right|^2 dx_3 \leq C|\xi|^{(k-1)_+} e^{2|\xi|}(|\xi| + 2)^k,$$

$$\int_0^1 \left| \left(\frac{d}{dx_3}\right)^k (x_3 e^{-|\xi| x_3}) \right|^2 dx_3 \leq C|\xi|^{(k-1)_+},$$

where $(k-1)_+ = max\{0, k-1\}$.

For any nonnegative integer $l \geq 1$, we introduce the norm

$$\|\omega\|_{l,\Omega}^2 = \sum_{k=1}^{l} \int_{\mathbb{R}^2} \left\| \left(\frac{d}{dx_3}\right)^k \hat{\omega}(\xi, \cdot) \right\|_{L^2(0,1)}^2 |\xi|^{2(l-k)} d\xi,$$

$$\|\omega\|_{\dot{H}^l(\Omega)} = \sum_{k=1}^{l} \|\partial_x^k \omega\|_{L^2}^2$$

It is easy to verify that the norm $\sum_{k=1}^{l} \|\omega\|_{k,\Omega}$ is equivalent to $\|\omega\|_{\dot{H}^l(\Omega)}$ due to Parseval's equality.

By (3.6), (3.15)–(3.17) and Lemma 3.1, we can obtain the following lemma.

Lemma 3.2. *Suppose $\vec{b} = (b_1, b_2) \in H^{n-\frac{3}{2}}(\Sigma_b)$, $\vec{d} = (b_3, b_4) \in H^{n-\frac{1}{2}}(\Sigma)$, $n \geq 2$, then the solution (3.6) to the problem (3.3) satisfies*

$$\|\omega\|_{\dot{H}^n(\Omega)}^2 + \|\nabla q\|_{H^{n-2}(\Omega)}^2 \leq C\left(\|\vec{b}\|_{H^{n-\frac{3}{2}}(\Sigma_b)}^2 + \|\vec{d}\|_{H^{n-\frac{1}{2}}(\Sigma)}^2 \right).$$

Furthermore, by Poincaré inequality, we have

$$\|\omega\|_{H^n(\Omega)}^2 + \|\nabla q\|_{H^{n-2}(\Omega)}^2 \leq C\left(\|\vec{b}\|_{H^{n-\frac{3}{2}}(\Sigma_b)}^2 + \|\vec{d}\|_{H^{n-\frac{1}{2}}(\Sigma)}^2 \right).$$

Proof of Theorem 3.1. We construct the solution to (3.1) in the form

$$(\omega, q) = (\omega^{(1)} + \omega^{(2)} + \omega^{(3)},\ \nu F_0' + q^{(3)}), \tag{3.18}$$

where $\omega^{(1)}$ is a solution of the Dirichlet problem

$$-\nu\Delta\omega^{(1)} = \vec{F} \quad \text{in} \quad \Omega, \quad \omega^{(1)} = 0 \quad \text{on} \quad \Sigma_b \cup \Sigma, \tag{3.19}$$

$\omega^{(2)} = \nabla\varphi$ with φ being a solution of the Neumann problem

$$\Delta\varphi = F_0 - \nabla\cdot\omega^{(1)} = F_0' \quad \text{in} \quad \Omega, \quad \frac{\partial\varphi}{\partial x_3}\Big|_{\Sigma_b\cup\Sigma} = 0, \tag{3.20}$$

and $(\omega^{(3)}, q^{(3)})$ is a solution of problem (3.3) with $\vec{b} = (b_1, b_2)$ and $\vec{d} = (b_3, b_4)$ defined by

$$b_j = -\nu\frac{\partial}{\partial x_3}\left(\omega_j^{(1)} + \omega_j^{(2)}\right) + \alpha\left(\omega_j^{(1)} + \omega_j^{(2)}\right)\Big|_{\Sigma_b}, \quad b_{j+1} = -\omega_j^{(1)} - \omega_j^{(2)}\Big|_{\Sigma}, \quad j = 1, 2.$$

By the standard elliptic estimates of equations (3.19) and (3.20) together with Lemma 3.2, we can obtain (3.2). □

4. The linearized problem

In this section, we consider the linearized problem (2.9) and prove Theorem 2.1.

Theorem 4.1. *There exists a constant γ_0 such that if $\frac{3(1+\tilde{\nu})\alpha}{\gamma^2(\nu+\alpha)\gamma_0} \le 1$ and $\frac{2\alpha}{\nu(\nu+\alpha)} \le 1$, then for any $u_0 = (\phi_0, \omega_0)^\perp \in (H^1 \times L^2) \cap L^1$ with $\partial_{x'}\omega_0 \in L^2$, the solution $u(t) = S(t)u_0$ of the problem (2.9) can be decomposed as*

$$S(t)u_0 = S^{(0)}(t)u_0 + S^{(\infty)}(t)u_0.$$

Moreover, it holds that

(i) *The function $S^{(0)}(t)u_0$ satisfies the following estimates uniformly for $t \ge 1$,*

$$\|\partial_x^l S^{(0)}(t)u_0\|_2 \le Ct^{-\frac{1}{2}-\frac{l}{2}}\|u_0\|_1, \quad l = 0, 1, \tag{4.1}$$

$$\|S^{(0)}(t)u_0 - G_{t*x'}\Pi^{(0)}u_0\|_2 \le Ct^{-1}\|u_0\|_1, \tag{4.2}$$

and

$$\|\partial_x^l S^{(0)}(t)[\tilde{Q}u_0]\|_2 \le Ct^{-1-\frac{l}{2}}\|\tilde{Q}u_0\|_1, \quad l = 0, 1. \tag{4.3}$$

(ii) *There exists a constant $\delta > 0$ such that $S^{(\infty)}(t)u_0$ satisfies*

$$\|\partial_x^l S^{(\infty)}(t)u_0\|_2 \le Ce^{-\delta t}\|u_0\|_{H^1}, \quad l = 0, 1, \tag{4.4}$$

for all $t \ge 1$.

To prove Theorem 4.1, we decompose $S(t)u_0$ as follows. Let $R_0 > 0$, define $\chi^{(0)}(\xi)$ and $\chi^{(\infty)}(\xi)$ by

$$\chi^{(0)}(\xi) = 1, \text{if } |\xi| \le R_0; \quad \chi^{(0)}(\xi) = 0, \text{if } |\xi| > R_0; \quad \text{and } \chi^{(\infty)}(\xi) = 1 - \chi^{(0)}(\xi).$$

We decompose $S(t)u_0$ as

$$S(t)u_0 = U_0(t)u_0 + U_\infty(t)u_0,$$

where

$$U_j(t)u_0 = \mathscr{F}^{-1}[\chi^{(j)}(\xi)e^{-t\widehat{L}_\xi}\hat{u}_0], \quad j = 1, \infty. \tag{4.5}$$

Here $\widehat{L}_\xi$ is the operator and has the form

$$\widehat{L}_\xi = \begin{pmatrix} i v_s^1 \xi_1 & i\gamma^2 \xi^\perp & \gamma^2 \partial_{x_3} \\ i\xi & \{\nu(|\xi|^2 - \partial_{x_3}^2) + i v_s^1 \xi_1\} I_2 + \nu' \xi \xi^\perp & -i\nu' \xi \partial_{x_3} + A_0 e_1' \\ \partial_{x_3} & -i\nu' \xi^\perp \partial_{x_3} & \nu(|\xi|^2 - \partial_{x_3}^2) - \nu' \partial_{x_3}^2 + i v_s^1 \xi_1 \end{pmatrix},$$

which is a closed operator on $H^1(0,1) \times L^2(0,1)$ with the domain of definition $D(\widehat{L}_\xi) = H^1(0,1) \times H_*^2(0,1)$.

Proposition 4.1. *There is a constant $r_0 > 0$ such that if $R_0 \le r_0$, then $U_0(t)u_0$ defined by (4.5) can be written as*

$$U_0(t)u_0 = S^{(0)}(t)u_0 + R^{(0)}(t)u_0,$$

where $S^{(0)}(t)u_0$ has the properties (i) of Theorem 4.1 and $R^{(0)}(t)u_0$ satisfies the estimate (ii) of Theorem 4.1 with $S^{(\infty)}(t)u_0$ replaced by $R^{(0)}(t)u_0$.

Proposition 4.2. *There exists a positive constant γ_0 depending only on R_0 such that if $\frac{3(1+\widetilde{\nu})\alpha}{\gamma^2(\nu+\alpha)\gamma_0} \le 1$ and $\frac{2\alpha}{\nu(\nu+\alpha)} \le 1$, then the following estimate holds for any fixed $R_0 > 0$ uniformly in $t \ge 1$,*

$$\|U_\infty(t)u_0\|_{H^1} \le Ce^{-\delta t}\|u_0\|_{H^1}, \tag{4.6}$$

where $\delta = \delta(R_0) > 0$.

Proof of Theorem 2.1. The proof of (2.11)–(2.12) in Theorem 2.1 follows from Proposition 4.1 and 4.2 by setting $R_0 = r_0$ and $S^{(\infty)}u_0 = R^{(0)}(t)u_0 + U_\infty(t)u_0$. And the proof of (2.10) in Theorem 2.1 follows from Proposition 4.4. □

4.1. Proof of Proposition 4.1

In this subsection, we prove Proposition 4.1. To this end, we take the Fourier transform of (2.9) with respect to $x' \in \mathbb{R}^2$ and obtain the following initial boundary value problem for the functions $\hat{u} = (\hat{\phi}(x_3,t), \hat{\omega}(x_3,t)^{\perp}, x_3 \in (0,1), t \geq 0$:

$$\frac{d}{dt}\hat{u} + \widehat{L}_\xi \hat{u} = 0, \quad \hat{u}|_{t=0} = \hat{u}_0 =: (\hat{\phi}_0(x_3), \hat{\omega}_0(x_3))^{\perp}. \tag{4.7}$$

We decompose $\hat{L}_\xi$ in the following form:

$$\widehat{L}_\xi = \widehat{L}_0 + \sum_{j=1}^{2} \xi_j \widehat{L}_j^{(1)} + \sum_{j,k=1}^{2} \xi_j \xi_k \widehat{L}_{jk}^{(2)},$$

where

$$\widehat{L}_0 = \begin{pmatrix} 0 & 0 & \gamma^2 \partial_{x_3} \\ 0 & -\nu \partial_{x_3}^2 I_2 & A_0 e_1' \\ \partial_{x_3} & 0 & -\widetilde{\nu}\partial_{x_3}^2 \end{pmatrix}, \quad \widetilde{\nu} = \nu + \nu',$$

$$\widehat{L}_j^{(1)} = \begin{pmatrix} i v_s^1 \delta_{1j} & i\gamma^2 e_j'^{\perp} & 0 \\ i e_j' & i v_s^1 \delta_{1j} I_2 & -i\nu' e_j' \partial_{x_3} \\ 0 & -i\nu' e_j'^{\perp} \partial_{x_3} & i v_s^1 \delta_{1j} \end{pmatrix},$$

$$\widehat{L}_{jk}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu \delta_{jk} I_2 + \nu' e_j' e_k'^{\perp} & 0 \\ 0 & 0 & \nu \delta_{jk} \end{pmatrix},$$

with e_j' denoting the unit vector in ξ_j-direction of $\mathbb{R}^2$.

We consider

$$\lambda \hat{u} + \widehat{L}_\xi \hat{u} = f, \tag{4.8}$$

where $\lambda \in \mathbf{C}$ is the resolvent parameter, $\hat{u} = (\hat{\phi}(x_3), \hat{\omega}', \hat{\omega}_3)^{\perp}$ and $f = (f_0, f', f_3)^{\perp}$. We treat the operator $\widehat{L}_\xi$ as a perturbation of $\widehat{L}_0$ and then we start with the resolvent problem (4.8) for $\xi = 0$ as

$$\lambda \hat{u} + \widehat{L}_0 \hat{u} = f. \tag{4.9}$$

For $k = 1, 2, \cdots$, we define $\lambda_{\pm,k}$ and $\lambda_{1,k}$ by

$$\lambda_{\pm,k} = -\frac{\widetilde{\nu}}{2}(k\pi)^2 \pm \frac{1}{2}\sqrt{\widetilde{\nu}^2 (k\pi)^4 - 4\gamma^2 (k\pi)^2},$$

and

$$\lambda_{1,k} = -\nu a_k,$$

where $\frac{(2k-1)\pi}{2} < a_k < k\pi$ is the solution of the function

$$\alpha \tan x + \nu x = 0, \quad x > 0.$$

It is easy to see that $\lambda_{\pm,k}$ are two roots of the equations $\lambda^2 + \widetilde{\nu}(k\pi)^2\lambda + \gamma^2(k\pi)^2 = 0$ satisfying $\lambda_{-,k} = \overline{\lambda_{+,k}}$ for $|k\pi| < \frac{2\gamma}{\widetilde{\nu}}$ and $\lambda_{\pm,k} \in \mathbb{R}$ for $k\pi \geq \frac{2\gamma}{\widetilde{\nu}}$, and it holds

$$\lambda_{+,k} = -\frac{\gamma^2}{\widetilde{\nu}} + O(k^{-2}), \quad \lambda_{-,k} = -\widetilde{\nu}(k\pi)^2 + O(1), \quad \text{as} \quad k \to \infty.$$

We have the following lemma on the resolvent estimate of the operator $-\widehat{L}_0$.

Lemma 4.1.

(i) *It holds that*

$$\sigma(-\widehat{L}_0) = \{0\} \cup \{\lambda_{1,k}\}_{k=1}^{\infty} \cup \{\lambda_{\pm,k}\}_{k=1}^{\infty} \cup \{-\frac{\gamma^2}{\widetilde{\nu}}\}.$$

In particular, $\lambda = 0$ *is a simple eigenvalue of* $-\widehat{L}_0$ *with eigenprojection*

$$\widehat{\Pi}_0 \hat{u} = \langle \hat{\phi} \rangle u^{(0)}, \quad \text{for} \quad \hat{u} = (\hat{\phi}, \hat{\omega})^{\perp},$$

where $u^{(0)} = (1, 0, 0, 0)^{\perp}$.

(ii) *There exist positive numbers* η_0 *and* θ_0 *with* $\theta_0 \in (\frac{\pi}{2}, \pi)$ *such that the following estimates hold uniformly for* $\lambda \in \varrho(-\widehat{L}_0) \cap \Sigma(-\eta_0, \theta_0)$*:*

$$\begin{aligned}
&\left|(\lambda + \widehat{L}_0)^{-1} f\right|_{H^l \times L^2} \leq \frac{C}{|\lambda|} |f|_{H^l \times L^2}, \quad l = 0, 1, \\
&\left|\partial_{x_3}^l \tilde{Q}(\lambda + \widehat{L}_0)^{-1} f\right|_2 \leq \frac{C}{(|\lambda| + 1)^{1-\frac{l}{2}}} |f|_{H^{l-1} \times L^2}, \quad l = 1, 2, \\
&\left|\partial_{x_3}^2 Q_0(\lambda + \widehat{L}_0)^{-1} f\right|_2 \leq \frac{C}{(|\lambda| + 1)^{\frac{1}{2}}} |f|_{H^2 \times H^1}.
\end{aligned}$$

Proof. The proof of Lemma 4.1 can be made by similar arguments as used in [8]. Here we only show the difference due to the different boundary condition. Let us consider the eigenvalue problem

$$\begin{cases}
\lambda \hat{\phi} + \gamma^2 \partial_{x_3} \hat{\omega}_3 = f_0, \\
\partial_{x_3} \hat{\phi} + \lambda \hat{\omega}_3 - \widetilde{\nu} \partial_{x_3}^2 \hat{\omega}_3 = f_3, \\
\hat{\omega}_3 \big|_{\Sigma \cup \Sigma_b} = 0,
\end{cases} \tag{4.10}$$

and

$$\begin{cases} \lambda\hat{\omega}' - \nu\partial_{x_3}^2\hat{\omega}' = f' - A_0\hat{\omega}_3 e_1', \\ \nu\partial_{x_3}\hat{\omega}' - \alpha\hat{\omega}'\big|_{\Sigma_b} = 0, \quad \hat{\omega}'\big|_{\Sigma} = 0. \end{cases} \tag{4.11}$$

We first consider the eigenvalue problem for (4.10). The spectrum analysis and the resolvent estimates of (4.10) can be made by applying the same way with some modification as in [8], so we omit the details. Indeed, we can show that there exist positive numbers η_0 and $\theta_0 \in (\frac{\pi}{2}, \pi)$ so that it holds for λ with $|\arg(\lambda + \eta_0)| \le \theta_0$ that

$$\begin{cases} \{0\} \cup \{\lambda_{\pm,k}\}_{k=1}^{\infty} \cup \{-\frac{\gamma^2}{\tilde{\nu}}\} \subset \sigma(-\hat{L}_0), \\ |\hat{\phi}|_2 \le \dfrac{C|f|_2}{|\lambda|}, \\ |\partial_{x_3}^{l+1}\hat{\phi}|_2 \le \dfrac{C|f|_{H^{l+1}\times H^l}}{(|\lambda|+1)^{1-\frac{l}{2}}}, \quad \text{for} \quad l = 0, 1, \\ |\partial_{x_3}^{l}\hat{\omega}_3|_2 \le \dfrac{C|f|_{H^{(l-1)_+}\times L^2}}{(|\lambda|+1)^{1-\frac{l}{2}}}, \quad \text{for} \quad l = 0, 1, 2. \end{cases} \tag{4.12}$$

Then we turn to investigate eigenvalue the problem (4.11) for $\lambda \neq 0$, $\lambda \neq \lambda_{\pm,k}$ and $\lambda \neq -\frac{\gamma^2}{\nu_1}$. Since there exists a unique solution $(\hat{\phi}, \hat{\omega}_3)$ to the problem (4.10) for any given f_0 and f_3, by using Fourier series expansion, it is easy to conclude that (4.11) has a unique solution $\hat{\omega}' \in H_*^2(0,1)$ for any given $f' - A_0\hat{\omega}_3 e_1' \in L^2(0,1)$ if and only if $\lambda \neq \lambda_{1,k}$ for any $k = 1, 2, \cdots$. Furthermore, one can establish the estimates

$$|\partial_{x_3}^l\hat{\omega}'|_2 \le \frac{C}{(|\lambda|+1)^{1-\frac{l}{2}}}|f|_2, \quad l = 0, 1, 2. \tag{4.13}$$

The fact that $\lambda = 0$ is an simple eigenvalue of $-\widehat{L}_0$ with eigenprojection $\widehat{\Pi}_0\hat{u} = \langle\hat{\phi}\rangle u^{(0)}$ can be proved in the same way as in [8], we omit the detail here. This completes the proof. □

We next give some estimates for $(\lambda + \widehat{L}_\xi)^{-1}$. Based on Lemma 4.1 and then by a similar way as Theorem 3.2 in [8] and Theorem 5.2 in [11], we have the following lemma for $|\xi| \le \widetilde{r}_0$.

Lemma 4.2. *Let η_0 and θ_0 be the numbers given in Lemma 4.1. Then, there exists a positive number $\widetilde{r}_0 = \widetilde{r}_0(\eta_0, \theta_0)$ such that the set $\Sigma(-\eta_0, \theta_0) \cap \{\lambda; |\lambda| \ge \frac{\eta_0}{2}\} \subset \varrho(-\widehat{L}_\xi)$ for $|\xi| \le \widetilde{r}_0$. Furthermore, the following estimates hold for any multi-index β uniformly in $\Sigma(-\eta_0, \theta_0) \cap \{\lambda; |\lambda| \ge \frac{\eta_0}{2}\}$ and ξ with $|\xi| \le \widetilde{r}_0$*

$$\left|\partial_\xi^\beta(\lambda + \widehat{L}_\xi)^{-1} f\right|_{H^l\times L^2} \le \frac{C_\beta}{|\lambda|}|f|_{H^l\times L^2}, \quad l = 0, 1,$$

$$\left|\partial_\xi^\beta\partial_{x_3}^l\widetilde{Q}(\lambda + \widehat{L}_\xi)^{-1} f\right|_2 \le \frac{C_\beta}{(|\lambda|+1)^{1-\frac{l}{2}}}|f|_{H^{l-1}\times L^2}, \quad l = 1, 2,$$

$$\left|\partial_\xi^\beta\partial_{x_3}^2 Q_0(\lambda + \widehat{L}_\xi)^{-1} f\right|_2 \le \frac{C_\beta}{(|\lambda|+1)^{\frac{1}{2}}}|f|_{H^2\times H^1}.$$

In case $|\xi| \geq \frac{\widetilde{r}_0}{2}$, we have the following result.

Lemma 4.3. *Let $\widetilde{r}_0$ be the given number in Lemma 4.2.*

(i) *There exist positive numbers $\widetilde{\eta}$ and $\widetilde{\theta}$ with $\widetilde{\theta} \in (\frac{\pi}{2}, \pi)$ such that the set $\Sigma(-\widetilde{\eta}, \widetilde{\theta}) \subset \varrho(-\widehat{L}_\xi)$ for $|\xi| \geq \frac{\widetilde{r}_0}{2}$.*
(ii) *The following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})$ for $|\xi| \geq \frac{\widetilde{r}_0}{2}$.*

$$\left|(\lambda + \widehat{L}_\xi)^{-1} f\right|_{H^l \times L^2} \leq \frac{C}{|\lambda|} |f|_{H^l \times L^2}, \quad l = 0, 1.$$

Lemma 4.3 can be proved in a similar manner to the proof of Theorem 2.5 in [9]. So we omit the proof.

As for the spectrum of $-\widehat{L}_\xi$ near $\lambda = 0$ in the case that $|\xi|$ is sufficiently small, we have the following result.

Lemma 4.4. *Let η_0 and $\widetilde{r}_0$ be the numbers given in Lemma 4.2. Then, there exists a positive number r_0 with $r_0 \leq \widetilde{r}_0$ such that for each ξ with $|\xi| \leq r_0$ it holds*

$$\sigma(-\widehat{L}_\xi) \cap \{\lambda; |\lambda| \leq \eta_0\} = \{\lambda_0(\xi)\},$$

where $\lambda_0(\xi)$ is a simple eigenvalue of $-\widehat{L}_\xi$ that has the form

$$\lambda_0(\xi) = -(\frac{A_0}{2} + B_0) i \xi_1 - \kappa_1 |\xi_1|^2 - \kappa_2 |\xi_2|^2 + O(|\xi|^3)$$

as $|\xi| \to 0$, and κ_1 and κ_2 are positive numbers given by

$$\kappa_1 = \frac{A_0^2 \widetilde{\nu}}{12\gamma^2} + \frac{\gamma^2}{\nu}\left(\frac{1}{3} - \frac{1}{4}A_0 + \frac{7}{240}\frac{A_0^2}{\gamma^2} - \frac{1}{48}\frac{A_0^3}{\gamma^2}\right), \quad \kappa_2 = \frac{\gamma^2}{\nu}(\frac{1}{3} - \frac{A_0}{4}).$$

Remark 4.1. In the case $\alpha \to \infty$, i.e. $A_0 \to 1$, $B_0 \to 0$, then we have

$$\lambda_0(\xi) = -\frac{i}{2}\xi_1 - \kappa_1 |\xi_1|^2 - \kappa_2 |\xi_2|^2 + O(|\xi|^3)$$

with

$$\kappa_1 = \frac{1}{12}\left\{(\frac{\widetilde{\nu}}{\gamma^2} + \frac{1}{10\nu}) + \frac{\gamma^2}{\nu}\right\}, \quad \kappa_2 = \frac{\gamma^2}{12\nu},$$

which is the same as Theorem 5.3 in [11].

Proof. We first observe that

$$\left|\widehat{L}_j^{(1)} \hat{u}\right|_{H^l \times H^{(l-1)_+}} \leq C\{|Q_0 \hat{u}|_{H^l} + |\widetilde{Q}\hat{u}|_{H^{(l-1)_+ + 1}}\} \tag{4.14}$$

and

$$\left|\widehat{L}^{(2)}_{jk}\hat{u}\right|_{H^l\times H^{(l-1)_+}} \le C|\widetilde{Q}\hat{u}|_{H^{(l-1)_+}}, \tag{4.15}$$

where $l=0,1,2$ and $\lambda\in\Sigma(-\eta_0,\theta_0)\cap\{\lambda;|\lambda|\ge\frac{\eta_0}{2}\}$. By Lemma 4.1, (4.14) and (4.15), we conclude that if $|\lambda|=\eta_0$, then $\lambda\in\varrho(-\widehat{L}_\xi)$ for $|\xi|\le\widetilde{r}_0$. In particular,

$$\widehat{\Pi}(\xi)\hat{u}=\frac{1}{2\pi i}\int\limits_{|\lambda|=\eta_0}(\lambda+\widehat{L}_\xi)^{-1}\hat{u}d\lambda$$

is the eigenprojection for the eigenvalues of $-\widehat{L}_\xi$ lying inside the circle $|\lambda|=\eta_0$. The continuity of $(\lambda+\widehat{L}_\xi)^{-1}$ in (λ,ξ) implies that $\dim\widehat{\Pi}(\xi)=\dim\widehat{\Pi}_0=1$ (see Chap. 1, Lemma 4.10 and Chap. 4, Theorem 3.16 in [12]). Based on Lemma 4.1 and (4.14), (4.15), we can apply the analytic perturbation theory (see Chap. 2, Sect. 2.2 and Chap. 7, Remark 2.10 in [12]) to obtain that if ξ is sufficiently small, then

$$\sigma(-\widehat{L}_\xi)\cap\{\lambda;|\lambda|\le\eta_0\}=\lambda_0(\xi),$$

where $\lambda_0(\xi)$ is a simple eigenvalue. Furthermore, we can decompose $\lambda_0(\xi)$ and $\widehat{\Pi}(\xi)$ as

$$\lambda_0(\xi)=\lambda_0+\sum_{j=1}^{2}\xi_j\lambda_j^{(1)}+\sum_{j,k=1}^{2}\xi_j\xi_k\lambda_{jk}^{(2)}+O(|\xi|^3),$$
$$\widehat{\Pi}(\xi)=\widehat{\Pi}_0+\sum_{j=1}^{2}\xi_j\widehat{\Pi}_j^{(1)}+O(|\xi|^2),$$

with $\lambda_0=0$, $\lambda_{jk}^{(2)}=\lambda_{kj}^{(2)}$ and

$$\begin{aligned}
\lambda_j^{(1)}&=-(\widehat{L}_j^{(1)}u^{(0)},u^{(0)}),\\
\lambda_{jk}^{(2)}&=-\left(\frac{1}{2}(\widehat{L}_{jk}^{(2)}+\widehat{L}_{kj}^{(2)})u^{(0)},u^{(0)}\right)+\left(\frac{1}{2}(\widehat{L}_j^{(1)}\widehat{S}\widehat{L}_k^{(1)}+\widehat{L}_k^{(1)}\widehat{S}\widehat{L}_j^{(1)})u^{(0)},u^{(0)}\right),\\
\widehat{\Pi}_j^{(1)}&=-\widehat{\Pi}_0\widehat{L}_j^{(1)}\widehat{S}-\widehat{S}\widehat{L}_j^{(1)}\widehat{\Pi}_0,
\end{aligned}$$

where $\widehat{S}=\left\{(I-\widehat{\Pi}_0)\widehat{L}_0(I-\widehat{\Pi}_0)\right\}^{-1}$.

We have

$$\widehat{L}_j^{(1)}u^{(0)}=i\begin{pmatrix}v_s^1\delta_{1j}\\ e_j'\\ 0\end{pmatrix},$$

from which it follows

$$\lambda_j^{(1)} = -(\frac{A_0}{2} + B_0) i\delta_{1j}.$$

It is easy to see that $\widehat{L}_{jk}^{(2)} u^{(0)} = 0$. Let us compute $\widehat{L}_j^{(1)} \widehat{S}\widehat{L}_1^{(1)} u^{(0)}$ and $\widehat{L}_j^{(1)} \widehat{S}\widehat{L}_2^{(1)} u^{(0)}$ one by one. Set $f = (I - \widehat{\Pi}_0))\widehat{L}_1^{(1)} u^{(0)}$. Then $\widehat{S}\widehat{L}_1^{(1)} u^{(0)}$ is a unique solution $\hat{u} = (\hat{\phi}, \hat{\omega})$ to the following problem

$$\begin{aligned}
&\widehat{L}_0 \hat{u} = f, \quad \langle \hat{\phi} \rangle = 0, \\
&\hat{\omega}\big|_{\Sigma} = 0, \quad \hat{\omega}_3\big|_{\Sigma_b} = 0, \\
&\nu \partial_{x_3} \hat{\omega}_j - \alpha \hat{\omega}_j\big|_{\Sigma_b} = 0, \ j = 1, 2
\end{aligned}$$

By a direct computation, we have

$$\widehat{S}\widehat{L}_1^{(1)} u^{(0)} = \begin{pmatrix} \frac{iA_0 \widetilde{\nu}}{\gamma^2}(x_3 - \frac{1}{2}) \\ \frac{i}{\nu}\{\frac{1-x_3^2}{2} + \frac{A_0^2}{2\gamma^2}(\frac{1-x_3^3}{6} - \frac{1-x_3^4}{12}) - (\frac{A_0}{2} + \frac{A_0^3}{24\gamma^2})(1 - x_3)\} e_1' \\ \frac{iA_0}{2\gamma^2}(x_3^2 - x_3) \end{pmatrix}.$$

It follows that

$$\lambda_{11}^{(2)} = -\frac{A_0^2 \widetilde{\nu}}{12\gamma^2} - \frac{\gamma^2}{\nu}\left(\frac{1}{3} - \frac{1}{4}A_0 + \frac{7}{240}\frac{A_0^2}{\gamma^2} - \frac{1}{48}\frac{A_0^3}{\gamma^2}\right).$$

Similarly, we have

$$\widehat{S}\widehat{L}_2^{(1)} u^{(0)} = \begin{pmatrix} 0 \\ \frac{i}{\nu}\{\frac{1-x_3^2}{2} - \frac{A_0}{2})(1 - x_3)\} e_2' \\ 0 \end{pmatrix},$$

and

$$\lambda_{22}^{(2)} = -\frac{\gamma^2}{\nu}(\frac{1}{3} - \frac{A_0}{4}), \quad \lambda_{12}^{(2)} = 0.$$

This completes the proof. □

As for the eigenprojection $\widehat{\Pi}(\xi)$ associated with $\lambda_0(\xi)$, we have the following result by a similar argument as Theorem 3.3 in [8].

Lemma 4.5. *Let $\widehat{\Pi}(\xi)$ be the eigenprojection associated with $\lambda_0(\xi)$. Then there exists a positive number r_0 such that for any ξ with $|\xi| \le r_0$ the projection $\widehat{\Pi}(\xi)$ can be written in the form*

$$\widehat{\Pi}(\xi)\hat{u} = \int_0^1 \widehat{\Pi}(\xi, x_3, y_3)\hat{u}(y_3) dy_3$$

with

$$\widehat{\Pi}(\xi, x_3, y_3) = \widehat{\Pi}_0 + \sum_{j=1}^{2} \xi_j \widehat{\Pi}_j^{(1)}(x_3, y_3) + \widehat{\Pi}^{(2)}(\xi, x_3, y_3),$$

where $\widehat{\Pi}_0 = Q_0$, $\widehat{\Pi}_j^{(1)}(x_3, y_3)$ *and* $\widehat{\Pi}^{(2)}(\xi, x_3, y_3)$ *satisfy*

$$\left|\partial_{x_3}^k \partial_{y_3}^l \widehat{\Pi}_j^{(1)}(\cdot,\cdot)\right|_{L^\infty((0,1)\times(0,1))} \le C,$$
$$\left|\partial_{x_3}^k \partial_{y_3}^l \partial_\xi^\beta \widehat{\Pi}^{(2)}(\xi,\cdot,\cdot)\right|_{L^\infty((0,1)\times(0,1))} \le C_\beta |\xi|^{2-|\beta|},$$

uniformly in ξ *with* $|\xi| \le r_0$ *for* $0 \le k, l \le 1$ *and multi-index* β.

Proof of Proposition 4.1. By Lemma 4.2 and 4.3, we can obtain that the operator $-\widehat{L}_\xi$ generates an analytic semigroup $e^{-t\widehat{L}_\xi}$ on $H^1(0,1) \times L^2(0,1)$ for each fixed $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Then, by Lemma 4.2, $U_0(t)u_0$ can be expressed as

$$U_0(t)u_0 = \mathscr{F}^{-1}\left[\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\chi^{(0)}(\xi)(\lambda + \widehat{L}_\xi)^{-1}\hat{u}_0 d\lambda\right],$$

where $\Gamma = \{\lambda = \eta + se^{\pm i\theta}\}$ with some $\eta > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$.

By Lemma 4.2 and 4.4, we can deform the contour Γ into $\Gamma_0 \cup \widetilde{\Gamma}$ and a suitable circle around origin point 0 with Γ_0 and $\widetilde{\Gamma}$ defined by

$$\Gamma_0 = \{\lambda = -\eta_0 + is; |s| \le s_0\}, \quad \widetilde{\Gamma} = \{\lambda = \eta + se^{\pm i\theta}; |s| \ge \widetilde{s}_0\},$$

where the positive numbers s_0 and $\widetilde{s}_0$ are chosen such that Γ_0 connects with Γ at the end points of Γ_0. It then follows from Lemma 4.4–4.5 and the residue theorem that $U_0(t)u_0$ can be decomposed as

$$U_0(t)u_0 = S^{(0)}(t)u_0 + R^{(0)}(t)u_0,$$

with

$$S^{(0)}(t)u_0 = \mathscr{F}^{-1}\left[\chi^{(0)}(\xi)e^{\lambda_0(\xi)t}\widehat{\Pi}(\xi)\hat{u}_0\right]$$

and

$$R^{(0)}(t)u_0 = \mathscr{F}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_0\cup\widetilde{\Gamma}} e^{\lambda t}\chi^{(0)}(\xi)(\lambda + \widehat{L}_\xi)^{-1}\hat{u}_0 d\lambda\right].$$

By Lemma 4.2, one can see that $R^{(0)}(t)u_0$ has the desired estimate (4.6) in Proposition 4.2.

Let us consider $S^{(0)}(t)u_0$. We rewrite it as

$$S^{(0)}(t)u_0 = G_{t*x'}\widehat{\Pi}_0 u_0 + S_1^{(0)}(t)u_0 + S_2^{(0)}(t)u_0 + S_3^{(0)}(t)u_0 + S_4^{(0)}(t)u_0,$$

where

$$G_{t*x'}\widehat{\Pi}_0 u_0 = \mathscr{F}^{-1}\left[e^{-((\frac{A_0}{2}+B_0)i\xi_1+\kappa_1\xi_1^2+\kappa_2\xi_2^2)t}\widehat{\Pi}_0\hat{u}_0\right],$$

$$S_1^{(0)}(t)u_0 = \mathscr{F}^{-1}\left[(\chi^{(0)}(\xi)-1)e^{-((\frac{A_0}{2}+B_0)i\xi_1+\kappa_1\xi_1^2+\kappa_2\xi_2^2)t}\widehat{\Pi}_0\hat{u}_0\right],$$

$$S_2^{(0)}(t)u_0 = \mathscr{F}^{-1}\left[\chi^{(0)}(\xi)e^{-((\frac{A_0}{2}+B_0)i\xi_1+\kappa_1\xi_1^2+\kappa_2\xi_2^2)t}\widehat{\Pi}^{(1)}(\xi)\hat{u}_0\right],$$

$$S_3^{(0)}(t)u_0 = \mathscr{F}^{-1}\left[\chi^{(0)}(\xi)e^{-((\frac{A_0}{2}+B_0)i\xi_1+\kappa_1\xi_1^2+\kappa_2\xi_2^2)t}\widehat{\Pi}^{(2)}(\xi)\hat{u}_0\right],$$

$$S_4^{(0)}(t)u_0 = \mathscr{F}^{-1}\left[\chi^{(0)}(\xi)(e^{\lambda_0(\xi)t} - e^{-((\frac{A_0}{2}+B_0)i\xi_1+\kappa_1\xi_1^2+\kappa_2\xi_2^2)t})\widehat{\Pi}(\xi)\hat{u}_0\right],$$

with

$$\widehat{\Pi}^{(1)}(\xi) = \sum_{j=1}^{2}\xi_j\widehat{\Pi}_j^{(1)}.$$

Then, the expected estimates for $S^{(0)}(t)u_0$ follow from Lemma 4.4 and 4.5. □

4.2. *Proof of Proposition 4.2*

As mentioned before, for each fixed $\xi \in \mathbb{R}^2$, the operator $-\widehat{L}_\xi$ generates an analytic semigroup $e^{-t\widehat{L}_\xi}$ on $H^1(0,1)\times L^2(0,1)$. This implies that $\hat{u}(t) = e^{-t\widehat{L}_\xi}\hat{u}_0 = (\hat{\phi}(t), \hat{\omega}(t))^\perp$ is a unique solution of (4.7), and we have

$$\frac{1}{\gamma^2}\partial_t\hat{\phi} + \frac{1}{\gamma^2}i\xi_1 v_s^1\hat{\phi} + i\xi\cdot\hat{\omega}' + \partial_{x_3}\hat{\omega}_3 = 0, \tag{4.16}$$

$$\partial_t\hat{\omega}' + \nu(|\xi|^2 - \partial_{x_3}^2)\hat{\omega}' - i\nu'\xi(i\xi\cdot\hat{\omega}' + \partial_{x_3}\hat{\omega}_3) + i\xi\hat{\phi} + i\xi_1 v_s^1\hat{\omega}' + A_0\hat{\omega}_3 e_1' = 0, \tag{4.17}$$

$$\partial_t\hat{\omega}_3 + \nu(|\xi|^2 - \partial_{x_3}^2)\hat{\omega}_3 - \nu'\partial_{x_3}(i\xi\cdot\hat{\omega}' + \partial_{x_3}\hat{\omega}_3) + \partial_{x_3}\hat{\phi} + i\xi_1 v_s^1\hat{\omega}_3 = 0, \tag{4.18}$$

$$\hat{\omega}\Big|_{\Sigma} = 0,\ \hat{\omega}_3\Big|_{\Sigma_b} = 0,\ \nu\partial_{x_3}\hat{\omega}' - \alpha\hat{\omega}'\Big|_{\Sigma_b} = 0, \tag{4.19}$$

$$\hat{u}|_{t=0} = \hat{u}_0 = (\hat{\phi}_0, \hat{\omega}_0)^\perp, \tag{4.20}$$

for $t > 0$.

We denote the material derivative $\partial_t\hat{\phi} + i\xi_1 v_s^1\hat{\phi}$ by $\dot{\phi}$, i.e.,

$$\dot{\phi} = \partial_t\hat{\phi} + i\xi_1 v_s^1\hat{\phi}.$$

In what follows $\hat{u}(t) = (\hat{\phi}(t), \hat{\omega}(t))^\perp$ will denote the unique solution of problem (4.16)–(4.20).

Proposition 4.3. *Let $\hat{u}(t) = (\hat{\phi}(t), \hat{\omega}(t))^{\perp}$ denote the unique solution of problem (4.16)–(4.20). There exist constants γ_0 depending only on R_0 such that if $\frac{3(1+\widetilde{\nu})\alpha}{\gamma^2(\nu+\alpha)\gamma_0} \le 1$ and $\frac{2\alpha}{\nu(\nu+\alpha)} \le 1$, then for any $R_0 > 0$ there exists a constant $\delta = \delta(R_0) > 0$ such that the estimate*

$$|\hat{u}|_2^2 + |\widehat{\nabla u}|_2^2 \le Ce^{-\delta(t-1)}(|\hat{u}|_2^2 + |\widehat{\nabla u}|_2^2)(1) \tag{4.21}$$

holds uniformly for $t \ge 1$, provided that $|\xi| \ge R_0$.

Proposition 4.4. *Let $\hat{u}(t) = (\hat{\phi}(t), \hat{\omega}(t))^{\perp}$ denote the unique solution of problem (4.16)–(4.20). For $0 < t \le 1$ and $\xi \in \mathbb{R}^2$, we have the following estimate*

$$|\hat{u}|_2^2 + |\widehat{\nabla u}|_2^2 \le C\{(1+|\xi|^2)|\hat{u}_0|_2^2 + |\partial_{x_3}\hat{\phi}_0|_2^2 + t^{-1}|\hat{\omega}_0|_2^2\}. \tag{4.22}$$

Remark 4.2. The proof of Proposition 4.4 can be proved in a similar way as in [11] (Proposition 6.11), we omit the details here.

The proof of Proposition 4.3 consists of following Lemmas 4.6–4.9.

Lemma 4.6. *Let $\hat{u}(t) = (\hat{\phi}(t), \hat{\omega}(t))^{\perp}$ denote the unique solution of problem (4.16)–(4.20). If $\frac{2\alpha}{\nu(\nu+\alpha)} \le 1$, then the following estimate holds*

$$\frac{d}{dt}(\frac{1}{\gamma^2}|\hat{\phi}|_2^2 + |\hat{\omega}|_2^2) + \nu|\widehat{\nabla\omega}|_2^2 + \nu'|\widehat{div\omega}|_2^2 + \frac{\widetilde{\nu}}{12\gamma^4}|\dot{\phi}|_2^2 \le 0. \tag{4.23}$$

Proof. Taking the inner product of (4.16), (4.17) and (4.18) with $\hat{\phi}$, $\hat{\omega}'$ and $\hat{\omega}_3$, respectively, and integrating the resulted by parts, we have

$$\frac{1}{\gamma^2}(\partial_t\hat{\phi}, \hat{\phi}) + \frac{1}{\gamma^2}i\xi_1(v_s^1\hat{\phi}, \hat{\phi}) + (i\xi\cdot\hat{\omega}' + \partial_{x_3}\hat{\omega}_3, \hat{\phi}) = 0, \tag{4.24}$$

$$\begin{aligned}&(\partial_t\hat{\omega}', \hat{\omega}') + \nu D_0[\hat{\omega}'] - i\nu'\xi(i\xi\cdot\hat{\omega}' + \partial_{x_3}\hat{\omega}_3, \hat{\omega}')\\ &+(i\xi\hat{\phi}, \hat{\omega}') + i\xi_1 v_s^1\hat{\omega}', \hat{\omega}') + A_0(\hat{\omega}_3, \hat{\omega}_1) + \alpha\nu|\hat{\omega}'|^2\big|_{\Sigma_b} = 0,\end{aligned} \tag{4.25}$$

$$\begin{aligned}&(\partial_t\hat{\omega}_3, \hat{\omega}_3) + \nu D_0[\hat{\omega}_3] - \nu'(\partial_{x_3}(i\xi\cdot\hat{\omega}' + \partial_{x_3}\hat{\omega}_3), \hat{\omega}_3)\\ &+(\partial_{x_3}\hat{\phi}, \hat{\omega}_3) + i\xi_1(v_s^1\hat{\omega}_3, \hat{\omega}_3) + \alpha\nu|\hat{\omega}_3|^2\big|_{\Sigma_b} = 0.\end{aligned} \tag{4.26}$$

Summing (4.24)–(4.26) together and taking the real part of the resulting identity, one obtains

$$\frac{1}{2}\frac{d}{dt}(\frac{1}{\gamma^2}|\hat{\phi}|_2^2 + |\hat{\omega}|_2^2) + \nu|\widehat{\nabla\omega}|_2^2 + \nu'|\widehat{div\omega}|_2^2 + A_0 Re(\hat{\omega}_3, \hat{\omega}_1) \le 0, \quad A_0 = \frac{\alpha}{\nu+\alpha}. \tag{4.27}$$

By the Poincaré inequality, we have

$$|(\hat{\omega}_3, \hat{\omega}_1)| \le \frac{1}{2}|\partial_{x_3}\hat{\omega}|_2^2.$$

It then follows from (4.27) that

$$\frac{1}{2}\frac{d}{dt}(\frac{1}{\gamma^2}|\hat{\phi}|_2^2+|\hat{\omega}|_2^2)+\frac{3\nu}{4}|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2\le 0, \tag{4.28}$$

provided that

$$\frac{2\alpha}{\nu(\nu+\alpha)}\le 1.$$

Furthermore, by (4.16), we have

$$\widetilde{\nu}|\dot{\phi}|_2^2\le 3\gamma^4(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2).$$

This together with (4.28) gives rise to the desired estimate (4.23). This completes the proof of Lemma 4.6. □

Lemma 4.7. *Let $\hat{u}(t)=(\hat{\phi}(t),\hat{\omega}(t))^{\perp}$ denote the unique solution of problem (4.16)–(4.20). If $\frac{2\alpha}{\nu(\nu+\alpha)}\le 1$, then the following estimate holds for any $\eta>0$*

$$\begin{aligned}&\frac{d}{dt}E_1(t)+(1+|\xi|^2)(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2)+\frac{\widetilde{\nu}}{6\gamma^4}(1+|\xi|^2)|\dot{\phi}|_2^2+|\partial_t\hat{\omega}|_2^2\\&\le\frac{\eta}{\widetilde{\nu}}|\xi|^2|\hat{\phi}|_2^2+\frac{3}{\eta}(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2),\end{aligned}\tag{4.29}$$

where

$$\begin{aligned}E_1(t)=&\ 2(1+\frac{2+6\gamma^2}{\nu})(1+|\xi|^2)(\frac{1}{\gamma^2}|\hat{\phi}|_2^2+|\hat{\omega}|_2^2)+\alpha\nu|\hat{\omega}|^2\big|_{\Sigma_b}\\&+(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2)-2Re(\hat{\phi},i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3).\end{aligned}$$

Proof. Taking the inner product of (4.17) and (4.18) with $\partial_t\hat{\omega}'$ and $\partial_t\hat{\omega}_3$, respectively, one has after taking the real part,

$$\begin{aligned}&|\partial_t\hat{\omega}|_2^2+\frac{1}{2}\frac{d}{dt}(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2+\alpha\nu|\hat{\omega}|^2\big|_{\Sigma_b})\\&=Re\{(\hat{\phi},\partial_t(i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3))+i\xi_1(v_s^1\hat{\omega},\partial_t\hat{\omega})-A_0(\hat{\omega}_3,\partial_t\hat{\omega}_1)\}.\end{aligned}\tag{4.30}$$

The first term on the right-hand side of (4.30) can be estimated as follows

$$\begin{aligned}&(\hat{\phi},\partial_t(i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3))=\frac{d}{dt}(\hat{\phi},i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3)-(\partial_t\hat{\phi},i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3)\\&=\frac{d}{dt}(\hat{\phi},i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3)+i\xi_1\Big(v_s^1\hat{\phi},i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3\Big)+\gamma^2|i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3|_2^2.\end{aligned}\tag{4.31}$$

By (4.30)–(4.31), we have

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2+\alpha\nu|\hat{\omega}|^2\big|_{\Sigma_b})-Re(\hat{\phi},i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3)\right\}+|\partial_t\hat{\omega}|_2^2\\
&\le\frac{1}{2}|\partial_t\hat{\omega}|_2^2+|\widehat{\nabla\omega}|_2^2+(|\xi||\hat{\phi}|_2+\gamma^2|i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3|_2)|i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3|_2.
\end{aligned}$$

Since $|i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3|_2^2\le\frac{3}{\widetilde{\nu}}(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2+\alpha\nu|\hat{\omega}|^2\big|_{\Sigma_b})-2Re(\hat{\phi},i\xi\cdot\hat{\omega}'+\partial_{x_3}\hat{\omega}_3)\right\}+|\partial_t\hat{\omega}|_2^2\\
&\le 2|\widehat{\nabla\omega}|_2^2+\frac{\eta}{\widetilde{\nu}}|\xi|^2|\hat{\phi}|_2^2+3(\frac{2\gamma^2}{\widetilde{\nu}}+\frac{1}{\eta})(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2).
\end{aligned}\tag{4.32}$$

Summing $2(1+\frac{2+6\gamma^2}{\nu})(1+|\xi|^2)\times$ (4.23) and (4.32) together, we obtain (4.29). □

Lemma 4.8. *Let $\hat{u}(t)=(\hat{\phi}(t),\hat{\omega}(t))^{\perp}$ denote the unique solution of problem (4.16)–(4.20). If $\frac{2\alpha}{\nu(\nu+\alpha)}\le 1$, then the following estimate holds for any $\eta>0$,*

$$\begin{aligned}
&\frac{d}{dt}E_2(t)+\frac{1}{\widetilde{\nu}}|\partial_{x_3}\hat{\phi}|_2^2+(1+\frac{1}{\widetilde{\nu}\nu})(1+|\xi|^2)(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2)\\
&+(1+\frac{1}{\widetilde{\nu}\nu})|\partial_t\hat{\omega}|_2^2+\frac{2\widetilde{\nu}}{3\gamma^4}(1+|\xi|^2)|\dot{\phi}|_2^2+\frac{\widetilde{\nu}}{4\gamma^4}|\partial_{x_3}[\hat{\phi}]_t|_2^2\\
&\le\left(\frac{2\widetilde{\nu}A_0^2}{\gamma^4}+\frac{4\eta}{\widetilde{\nu}}(1+\frac{1}{\widetilde{\nu}\nu})\right)|\xi|^2|\hat{\phi}|_2^2+\frac{12}{\eta}(1+\frac{1}{\widetilde{\nu}\nu})(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2),
\end{aligned}\tag{4.33}$$

where

$$E_2(t)=4(1+\frac{1}{\widetilde{\nu}\nu})E_1[u]+\frac{1}{\gamma^2}|\partial_{x_3}\hat{\phi}|_2^2.$$

Proof. Differentiating (4.16) in x_3, we have

$$\frac{1}{\gamma^2}\partial_t\partial_{x_3}\hat{\phi}+\frac{1}{\gamma^2}i\xi_1v_s^1\partial_{x_3}\hat{\phi}+\frac{A_0}{\gamma^2}i\xi_1\hat{\phi}+i\xi\cdot\partial_{x_3}\hat{\omega}'+\partial_{x_3}^2\hat{\omega}_3=0.\tag{4.34}$$

We rewrite (4.18) as

$$\partial_{x_3}\hat{\phi}-\widetilde{\nu}\partial_{x_3}^2\hat{\omega}_3=-\{\partial_t\hat{\omega}_3+\nu|\xi|^2\hat{\omega}_3-i\nu'\xi\cdot\partial_{x_3}\hat{\omega}'+i\xi_1v_s^1\hat{\omega}_3\}\tag{4.35}$$

It then follows from (4.34) and $\frac{1}{\widetilde{\nu}}\times$ (4.35) that

$$\begin{aligned}
&\frac{1}{\gamma^2}\partial_t\partial_{x_3}\hat{\phi}+\frac{1}{\gamma^2}i\xi_1v_s^1\partial_{x_3}\hat{\phi}+\frac{1}{\widetilde{\nu}}\partial_{x_3}\hat{\phi}+\frac{A_0}{\gamma^2}i\xi_1\hat{\phi}\\
&=-\frac{1}{\widetilde{\nu}}\{\partial_t\hat{\omega}_3+\nu|\xi|^2\hat{\omega}_3-i\nu\xi\cdot\partial_{x_3}\hat{\omega}'+i\xi_1v_s^1\hat{\omega}_3\}.
\end{aligned}\tag{4.36}$$

Taking the inner product of (4.36) with $\partial_{x_3}\hat{\phi}$ and then taking the real part of the resulted equation, we have

$$\begin{aligned}&\frac{1}{2\gamma^2}\frac{d}{dt}|\partial_{x_3}\hat{\phi}|_2^2+\frac{1}{\widetilde{\nu}}|\partial_{x_3}\hat{\phi}|_2^2\\ &\le \frac{1}{4\widetilde{\nu}}|\partial_{x_3}\hat{\phi}|_2^2+\frac{\widetilde{\nu}A_0^2}{\gamma^4}|\xi|^2|\hat{\phi}|_2^2+\frac{1}{\widetilde{\nu}}|\partial_t\hat{\omega}|_2^2+\frac{1+\nu^2}{\widetilde{\nu}}(1+|\xi|^2)|\widehat{\nabla\omega}|_2^2. \end{aligned} \tag{4.37}$$

We thus obtain

$$\begin{aligned}&\frac{1}{\gamma^2}\frac{d}{dt}|\partial_{x_3}\hat{\phi}|_2^2+\frac{3}{2\widetilde{\nu}}|\partial_{x_3}\hat{\phi}|_2^2\\ &\le \frac{2\widetilde{\nu}A_0^2}{\gamma^4}|\xi|^2|\hat{\phi}|_2^2+\frac{2}{\widetilde{\nu}}|\partial_t\hat{\omega}|_2^2+2\frac{1+\nu^2}{\widetilde{\nu}\nu}(1+|\xi|^2)(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2). \end{aligned} \tag{4.38}$$

The combination of $4(1+\frac{1}{\widetilde{\nu}\nu})\times$ (4.29) and (4.38) shows that

$$\begin{aligned}&\frac{d}{dt}E_2(t)+\frac{3}{2\widetilde{\nu}}|\partial_{x_3}\hat{\phi}|_2^2+2(1+\frac{1}{\widetilde{\nu}\nu})(1+|\xi|^2)(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2)\\ &+2(1+\frac{1}{\widetilde{\nu}\nu})|\partial_t\hat{\omega}|_2^2+\frac{2\widetilde{\nu}}{3\gamma^4}(1+|\xi|^2)|\dot{\phi}|_2^2\\ &\le \left(\frac{2\widetilde{\nu}A_0^2}{\gamma^4}+\frac{4\eta}{\widetilde{\nu}}(1+\frac{1}{\widetilde{\nu}\nu})\right)|\xi|^2|\hat{\phi}|_2^2+\frac{12}{\eta}(1+\frac{1}{\widetilde{\nu}\nu})(\nu|\widehat{\nabla\omega}|_2^2+\nu'|\widehat{div\omega}|_2^2). \end{aligned} \tag{4.39}$$

We next rewrite (4.36) as

$$\partial_{x_3}\dot{\phi}+\frac{\gamma^2}{\widetilde{\nu}}\partial_{x_3}\hat{\phi}=-\frac{\gamma^2}{\widetilde{\nu}}\{\partial_t\hat{\omega}_3+\nu|\xi|^2\hat{\omega}_3-i\nu\xi\cdot\partial_{x_3}\hat{\omega}'+i\xi_1v_s^1\hat{\omega}_3\}.$$

This gives

$$|\partial_{x_3}\dot{\phi}|_2^2\le\frac{\gamma^4}{\widetilde{\nu}^2}\left\{|\partial_{x_3}\hat{\phi}|_2^2+|\partial_t\hat{\omega}_3|_2^2+(1+\nu^2)(1+|\xi|^2)|\widehat{\nabla\omega}|_2^2\right\}. \tag{4.40}$$

The combination of (4.39) and $\frac{\widetilde{\nu}}{4\gamma^4}\times$ (4.40) gives the desire estimates (4.33). □

To control $|\xi|^2|\hat{\phi}|_2^2$, we make use of the estimates for the Stokes system under the Fourier transform. We rewrite (4.16)–(4.19) as

$$\begin{cases} i\xi_1\hat{\omega}_1+i\xi_2\hat{\omega}_2+\partial_{x_3}\hat{\omega}_3=\hat{F}_0,\\ \nu(|\xi|^2-\partial_{x_3}^2)\hat{\omega}'+i\xi_j\hat{\phi}=\hat{F}',\\ \nu(|\xi|^2-\partial_{x_3}^2)\hat{\omega}_3+\partial_{x_3}\hat{\phi}=\hat{F}_3, \end{cases} \tag{4.41}$$

$$\begin{cases} \nu\partial_{x_3}\hat{\omega}_j - \alpha\hat{\omega}_j\big|_{\Sigma_b} = 0, & j = 1, 2, \\ \hat{\omega}_3\big|_{\Sigma_b} = 0, \quad \hat{\omega}\big|_{\Sigma} = 0, \end{cases} \tag{4.42}$$

with

$$\hat{F}_0 = -\frac{1}{\gamma^2}\dot{\phi},$$

$$\hat{F}' = -\partial_t\hat{\omega}' - v_s^1 i\xi_1\hat{\omega}' - A_0\hat{\omega}_3 e_1' - \frac{\nu'}{\gamma^2} i\xi\dot{\phi},$$

$$\hat{F}_3 = -\partial_t\hat{\omega}_3 - i\xi_1 v_s^1\hat{\omega}_3 - \frac{\nu'}{\gamma^2}\partial_{x_3}\dot{\phi}.$$

Lemma 4.9. *Let $\hat{u}(t) = (\hat{\phi}(t), \hat{\omega}(t))^{\perp}$ denote the unique solution of problem (4.16)–(4.20). For any constant $R_0 > 0$, if $|\xi| \geq R_0$, then the following estimate holds for the solution $(\hat{\phi}, \hat{\omega})$ of (4.41)–(4.42),*

$$|\hat{\phi}|_2^2 + |\widehat{\nabla\phi}|_2^2 \leq C_0\left\{|\partial_t\hat{\omega}|_2^2 + |\widehat{\nabla\omega}|_2^2 + \frac{1+\widetilde{\nu}^2}{\gamma^4}(|\xi|^2|\dot{\phi}|_2^2 + |\partial_{x_3}\dot{\phi}|_2^2) + \frac{1}{\gamma^4}|\dot{\phi}|_2^2\right\}, \tag{4.43}$$

where the constant C_0 depends only on R_0 and $\nu^2 + \dfrac{1}{\nu^2}$.

Proof. By (3.18), one can see that

$$(\hat{\omega}, \hat{\phi}) = (\hat{\omega}^{(1)} + \hat{\omega}^{(2)} + \hat{\omega}^{(3)},\ \nu\hat{F}_0' + \hat{q}^{(3)}),$$

is a unique solution to the system (4.41)–(4.42). By (3.6), (3.15)–(3.17), we have

$$|\hat{q}^{(3)}|_2^2 + |\xi|^2|\hat{q}^{(3)}|_2^2 + |\partial_{x_3}\hat{q}^{(3)}|_2^2 \leq C(|\hat{\vec{b}}|^2 + |\hat{\vec{d}}|^2), \tag{4.44}$$

where $\hat{\vec{b}} = (\hat{b}_1, \hat{b}_2)$ and $\hat{\vec{d}} = (\hat{b}_3, \hat{b}_4)$ defined by

$$\hat{b}_j = -\nu\frac{\partial}{\partial x_3}\left(\hat{\omega}_j^{(1)} + \hat{\omega}_j^{(2)}\right) + \alpha\left(\hat{\omega}_j^{(1)} + \hat{\omega}_j^{(2)}\right)\Big|_{\Sigma_b}, \quad b_{j+1} = -\hat{\omega}_j^{(1)} - \hat{\omega}_j^{(2)}\Big|_{\Sigma}, \quad j = 1, 2.$$

Then we apply the Fourier transform to (3.19), and obtain

$$\nu(|\xi|^2 - \partial_{x_3}^2)\hat{\omega}^{(1)} = \hat{\vec{F}} \quad \text{in} \quad \mathbb{R}_\xi^2 \times (0,1), \quad \hat{\omega}^{(1)} = 0 \quad \text{on} \quad \mathbb{R}_\xi^2 \times \{0,1\}. \tag{4.45}$$

Taking the inner product of (4.45) with $\hat{\omega}^{(1)}$, one can obtain

$$|\xi|^2|\hat{\omega}^{(1)}|_2^2 + |\partial_{x_3}\hat{\omega}^{(1)}|_2^2 \leq \frac{1}{\nu}\int_0^1 \hat{\vec{F}}\,\overline{\hat{\omega}^{(1)}}\,dx_3 \leq \frac{1}{2}|\xi|^2|\hat{\omega}^{(1)}|_2^2 + \frac{1}{2\nu^2}|\xi|^{-2}|\hat{\vec{F}}|_2^2,$$

so we have

$$|\xi|^2|\hat{\omega}^{(1)}|_2^2 + |\partial_{x_3}\hat{\omega}^{(1)}|_2^2 \le \frac{1}{\nu^2|\xi|^2}|\hat{\vec{F}}|_2^2. \tag{4.46}$$

And by (4.45), it holds

$$|\partial_{x_3}^2\hat{\omega}^{(1)}|_2^2 \le |\xi|^4|\hat{\omega}^{(1)}|_2^2 + \frac{1}{\nu^2}|\hat{\vec{F}}|_2^2 \le \frac{2}{\nu^2}|\hat{\vec{F}}|_2^2. \tag{4.47}$$

Similar result is also valid for $\hat{\omega}^{(2)}$,

$$\begin{cases} |\xi|^2|\hat{\omega}^{(2)}|_2^2 + |\partial_{x_3}\hat{\omega}^{(2)}|_2^2 \le |\hat{F}_0|_2^2 + \frac{1}{\nu^2|\xi|^2}|\hat{\vec{F}}|_2^2, \\ |\partial_{x_3}^2\hat{\omega}^{(2)}|_2^2 \le |\xi|^2|\hat{F}_0|_2^2 + |\partial_{x_3}\hat{F}_0|_2^2 + \frac{1}{\nu^2}|\hat{\vec{F}}|_2^2. \end{cases} \tag{4.48}$$

Combining the inequality (4.46)–(4.48) with (4.44) and using the trace theorem, the desired estimate (4.43) holds by a direct computation. □

Proof of Proposition 4.2. Set $C_1 = \dfrac{1}{8C_0}$ and $\eta = \dfrac{C_1\widetilde{\nu}^3\nu}{16(1+\widetilde{\nu}\nu)(1+\widetilde{\nu}^2)}$. The combination of $\dfrac{12}{\eta}(1+\dfrac{1}{\widetilde{\nu}\nu}) \times$ (4.23), (4.33) and $\dfrac{C_1\widetilde{\nu}}{1+\widetilde{\nu}^2} \times$ (4.43) shows that

$$\begin{aligned} &\frac{d}{dt}E_3(t) + \frac{1}{\widetilde{\nu}}|\partial_{x_3}\hat{\phi}|_2^2 + \frac{1}{2}(1+\frac{1}{\widetilde{\nu}\nu})(1+|\xi|^2)(\nu|\widehat{\nabla\omega}|_2^2 + \nu'|\widehat{div\omega}|_2^2) + \frac{1}{2}(1+\frac{1}{\widetilde{\nu}\nu})|\partial_t\hat{\omega}|_2^2 \\ &+ \frac{\widetilde{\nu}}{3\gamma^4}(1+|\xi|^2)|\dot{\phi}|_2^2 + \frac{\widetilde{\nu}}{8\gamma^4}|\partial_{x_3}\dot{\phi}|_2^2 + \frac{C_1\widetilde{\nu}}{2(1+\widetilde{\nu}^2)}(|\hat{\phi}|_2^2 + |\widehat{\nabla\phi}|_2^2) \\ \le\ &\frac{2\widetilde{\nu}A_0^2}{\gamma^4}|\xi|^2|\hat{\phi}|_2^2, \end{aligned} \tag{4.49}$$

where

$$E_3(t) = E_2(t) + \frac{12}{\eta}(1+\frac{1}{\widetilde{\nu}\nu})(\frac{1}{\gamma^2}|\hat{\phi}|_2^2 + |\hat{\omega}|_2^2).$$

Therefore, if $\frac{3(1+\widetilde{\nu})\alpha}{\gamma^2(\nu+\alpha)} \le \sqrt{C_1}$, we have

$$\begin{aligned} &\frac{d}{dt}E_3(t) + \frac{1}{\widetilde{\nu}}|\partial_{x_3}\hat{\phi}|_2^2 + \frac{1}{2}(1+\frac{1}{\widetilde{\nu}})(1+|\xi|^2)\widetilde{D}_0[\hat{\omega}] + \frac{1}{2}(1+\frac{1}{\widetilde{\nu}})|\partial_t\hat{\omega}|_2^2 \\ &+ \frac{\widetilde{\nu}}{3\gamma^4}(1+|\xi|^2)|\dot{\phi}|_2^2 + \frac{\widetilde{\nu}}{8\gamma^4}|\partial_{x_3}\dot{\phi}|_2^2 + \frac{C_1\widetilde{\nu}}{4(1+\widetilde{\nu}^2)}(|\hat{\phi}|_2^2 + |\widehat{\nabla\phi}|_2^2) \le 0, \end{aligned}$$

which leads to the desired estimate (4.21) together with $\gamma_0(\nu) = \sqrt{C_1}$ and the fact that $E_3(t)$ is equivalent to $|\hat{u}|_2^2 + |\widehat{\nabla u}|_2^2$. □

5. The nonlinear problem

In this section, we prove Theorem 2.2. Inspired by [11], the proof of our stability result is based on the energy method and the spectral analysis of the linearized operator. The argument is similar to those in [10,11,13,14], so we only give an outline of proof here.

With the help of the Stokes estimates (under Navier-slip boundary condition) in Theorem 3.1, the global existence result (i) of Theorem 2.2 can be proved by the energy method by Matsumura and Nishida [14], which also implies the H^s-energy estimates.

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x \phi\|_{H^{s-1}}^2 + \|\partial_x \omega\|_{H^s}^2 d\tau \leq C\|u_0\|_{H^s}^2, \tag{5.1}$$

uniformly for $t \geq 0$ with $s \geq 2$. We omit the details here.

The proof of (ii) of Theorem 2.2 is based on the H^s-energy estimates (5.1) and the decay estimates for the solutions of the linearized problem Theorem 4.1. Indeed, by Duhamel principle, the solution of inhomogeneous equation (2.2)–(2.6) has the following form

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau) f(\tau) d\tau, \tag{5.2}$$

where $f = (\gamma^2 f_0, g)$ with f_0 and g defined in (2.7) and (2.8), namely,

$$\begin{aligned} f_0 &= -div(\phi\omega), \\ g &= -\omega \cdot \nabla\omega - \frac{\phi}{\gamma^2+\phi}\{\nu\Delta\omega + \nu'\nabla div\omega + (P_2(\gamma,\phi)-1)\nabla\phi\}. \end{aligned}$$

There exists a constant $\varepsilon_1 > 0$ such that if $\|u_0\|_{H^s \cap L^1} \leq \varepsilon_1$, one can obtain by Theorem 2.1, Theorem 4.1 and (5.1) the following decay estimate after a tedious computation

$$\|u(t) - S(t)u_0\|_{H^1} \leq C(1+t)^{-1}, \quad t \geq 0. \tag{5.3}$$

The decay estimate (5.3) together with Theorem 2.1 yields the decay estimates (2.14) and (2.14) of Theorem 2.2. The proof is completed. □

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