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Optimal decay rate for the compressible Navier–Stokes–Poisson system in the critical L^p framework ☆

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Abstract

In this paper, we consider the large time behavior of global solutions to the initial value problem for the compressible Navier–Stokes–Poisson system *in the L^p critical framework and in any dimension $N \geq 3$*. We obtain the time decay rates, not only for Lebesgue spaces, but also for a family of Besov norms *with negative or nonnegative regularity exponents*, which improves the decay results in high Sobolev regularity. The proof is mainly based on the Littlewood–Paley theory and refined time weighted inequalities in Fourier space.

MSC: 35Q35; 35B40; 76N15

Keywords: Time-decay rates; Navier–Stokes–Poisson equations; Critical spaces; L^p framework

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1. Introduction

In this paper, we investigate the long-time behavior of global strong solutions for the following compressible Navier–Stokes–Poisson equations (NSP) in $\mathbb{R}^+ \times \mathbb{R}^N$ $(N \geq 3)$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \lambda)\nabla \operatorname{div}\mathbf{u} + \nabla P(\rho) = \rho \nabla \Phi, \\ \Delta \Phi = \rho - \bar{\rho}, \\ (\rho, \mathbf{u})(0) = (\rho_0, \mathbf{u}_0), \end{cases} \tag{1.1}$$

where ρ, $\mathbf{u}$ and Φ represent the electron density, the electron velocity and the electrostatic potential, respectively. The pressure P is a smooth function of ρ with $P'(\rho) > 0$ for $\rho > 0$, and the viscosity coefficients μ, λ are constants and satisfy $\mu > 0$ and $\nu := \lambda + 2\mu > 0$. Such a condition ensures ellipticity for the operator $\mu\Delta + (\lambda + \mu)\nabla \operatorname{div}$ and is satisfied in the physical cases. $\bar{\rho} > 0$ describes the background doping profile, and in this paper, for simplicity, we set $\bar{\rho} = 1$ and suppose that $P'(1) = 1$. The compressible NSP system could be used to model and simulate the transportation of charged particles in semiconductor devices [26].

The main purpose of this paper is to investigate the time decay rates of strong solutions to system (1.1) in the critical L^p framework. Here we observe that system (1.1) is invariant by the transformation

$$\tilde{\rho} = \rho(l^2 t, lx), \quad \tilde{\mathbf{u}} = l\mathbf{u}(l^2 t, lx)$$

up to a change of the pressure law $\tilde{P} = l^2 P$. A critical space is a space in which the norm is invariant under the scaling $(\tilde{e}, \tilde{\mathbf{f}})(x) = (e(lx), l\mathbf{f}(lx))$.

As regarding the existence of solutions to the compressible NSP equations, there are many important progresses. In terms of the local and global weak solutions, one can refer to [10–12,37] and references therein. In [32], Tan and Wang considered the global existence of weak solutions to the compressible magnetohydrodynamics with the Poisson term of Coulomb force in two dimensions. The global existence of small strong solutions to the compressible NSP equations in H^N Sobolev spaces was shown by Li, Matsumura and Zhang [23] in $\mathbb{R}^3$, while global existence of small solutions in the critical L^2 type hybrid Besov spaces in $\mathbb{R}^N$ $(N \geq 3)$ was obtained in [14]. Later on, Zheng [38] extended the result of [14] to the critical L^p framework (see Theorem 2.1 below).

One may wonder how the global solutions established in the critical L^p framework behave for large time. To make a clearer introduction to our result, we will recall some known convergence results for the compressible Navier–Stokes and NSP equations, respectively.

For the compressible Navier–Stokes equations, the first achievement is due to Matsumura and Nishida [27,28]. There, they proved the global existence of classical solutions for the initial perturbation small in $(L^1 \cap H^3)(\mathbb{R}^3)$ and established the following decay estimate:

$$\|(\rho - 1, \mathbf{u})(t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}. \tag{1.2}$$

This is the same as for the heat equation with data in $L^1(\mathbb{R}^3)$ and it turns out to be the optimal one for the corresponding linearized system. As a consequence, it is often referred to as the optimal time-decay rate.

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Later on, for the small initial perturbation belonging to $(H^m \cap W^{m,1})(\mathbb{R}^N)$ $(N = 2, 3)$ with $m \geq 4$, Ponce [30] obtained the optimal L^p decay rate

$$\|\nabla^k(\rho - 1, \mathbf{u})(t)\|_{L^p} \leq C(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k}{2}} \tag{1.3}$$

for $2 \leq p \leq \infty$ and $0 \leq k \leq 2$. These results were extended to some situations where the fluid domain is not $\mathbb{R}^N$: the exterior problem [21,22] or the half space problem [19,20]. In view of the analysis of Green function, the optimal L^p $(1 \leq p \leq \infty)$ decay rate was also derived for the small initial perturbation to $H^m \cap L^1$ with $m \geq 4$ (see e.g. [17,18,25]). Let us note that, by introducing the Sobolev space of negative order, Guo and Wang [13] developed a general energy method to prove the optimal time-decay rates.

Recently, the asymptotic behavior on the compressible NSP equations was studied in [23,24,33–35]. Li, Matsumura and Zhang [23] established the optimal decay estimates of global solutions in $\mathbb{R}^3$. Wang and Wu [33] obtained the pointwise estimates of the solutions by a detailed analysis of the Green's function to the corresponding linearized equations. Their results imply that the decay rate of the density ρ and the momentum m is $(1+t)^{-\frac{N}{4}}$ and $(1+t)^{-\frac{N-2}{4}}$, respectively. Wang [34] noted that the special structure of the NSP system and posed some stronger conditions on the initial value in low frequency region and then obtained the time decay rates of solutions to system (1.1) in $\mathbb{R}^3$ on the small initial data (see also [24]). Very recently, Wang and Wang [35] established the decay estimates of classical solutions in $\mathbb{R}^N$ $(N \geq 3)$ by an energy method in the Fourier space.

Let us emphasize that all the decay results mentioned above are concerned about solutions with high Sobolev regularity. Recently, Okati [29] performed low and high frequency decompositions and proved the time decay rate for strong solutions to the compressible Navier–Stokes equations in the L^2 critical framework and in dimension $N \geq 3$. In the survey paper [8], Danchin proposed another description of the time decay which allows to proceed with dimension $N \geq 2$ in the L^2 critical framework. Very recently, Danchin and Xu [9], developed the method of [8] to obtain optimal time decay rate in the general L^p type critical spaces and in any dimension $N \geq 2$.

Motivated by the papers of Danchin and Xu [9], Okati [29] and Haspot [15] on the Navier–Stokes equations, and by the papers of Wang [34] and Li and Zhang [24] regarding the compressible NSP equations, in the present paper, we will investigate the long-time behavior of solutions for (1.1) in the framework of critical spaces and in any dimension $N \geq 3$. Here scaling invariance will play an essential role. By the way, we will actually get an accurate description of the decay rates, not only for Lebesgue spaces, but also for a fully family of Besov norms with negative or nonnegative regularity indexes.

2. Main results

Before giving the main statement of this paper, we first introduce the homogeneous Littlewood–Paley decomposition, which relies upon a dyadic partition of unit. Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^N)$ supported in $\mathcal{C} = \{\xi \in \mathbb{R}^N, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1$ if $\xi \neq 0$. The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low frequency cut-off operators $\dot{S}_j$ are defined below:

$$\dot{\Delta}_j u \overset{\text{def}}{=} \varphi(2^{-j}D)u, \quad \dot{S}_j u \overset{\text{def}}{=} \sum_{k\leq j-1} \dot{\Delta}_k u \quad \text{for} \quad j \in \mathbb{Z}.$$

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Let us denote the space $\mathcal{Y}'(\mathbb{R}^N)$ by the quotient space of $\mathcal{S}'(\mathbb{R}^N)/\mathcal{P}$ with the polynomials space $\mathcal{P}$. The formal equality $u = \sum_{k\in\mathbb{Z}} \dot{\Delta}_k u$ holds true for $u \in \mathcal{Y}'(\mathbb{R}^N)$ and is called the homogeneous Littlewood–Paley decomposition.

For $s \in \mathbb{R}$, $1 \le p, r \le +\infty$, set the homogeneous Besov space

$$\dot{B}^s_{p,r} \stackrel{\text{def}}{=} \Big\{ f \in \mathcal{Y}'(\mathbb{R}^N) : \|f\|_{\dot{B}^s_{p,r}} < +\infty \Big\},$$

with

$$\|f\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \|2^{ks}\|\dot{\Delta}_k f\|_{L^p}\|_{\ell^r}.$$

Next, we introduce the following Besov–Chemin–Lerner space $\widetilde{L}^\rho_T(\dot{B}^s_{p,r})$ (see [4]):

$$\widetilde{L}^\rho_T(\dot{B}^s_{p,r}) = \Big\{ f \in (0,+\infty)\times\mathcal{Y}'(\mathbb{R}^N) : \|f\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})} < +\infty \Big\},$$

where

$$\|f\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \Big\| 2^{ks}\|\dot{\Delta}_k f(t)\|_{L^\rho(0,T;L^p)} \Big\|_{\ell^r}.$$

The index T will be omitted if $T = +\infty$. Using Minkowski's inequality implies that

$$L^\rho_T(\dot{B}^s_{p,r}) \hookrightarrow \widetilde{L}^\rho_T(\dot{B}^s_{p,r}) \text{ if } r \ge \rho, \quad \text{and} \quad \widetilde{L}^\rho_T(\dot{B}^s_{p,r}) \hookrightarrow L^\rho_T(\dot{B}^s_{p,r}) \text{ if } \rho \ge r.$$

In what follows, we define that for $f \in \mathcal{S}'(\mathbb{R}^N)$,

$$f^\ell \stackrel{\text{def}}{=} \sum_{2^j \le 2^{j_0}} \dot{\Delta}_j f \quad \text{and} \quad f^h \stackrel{\text{def}}{=} \sum_{2^j > 2^{j_0}} \dot{\Delta}_j f,$$

for some large enough nonnegative integer j_0. The corresponding semi-norms are defined by

$$\|f\|^\ell_{\dot{B}^\sigma_{p,r}} \stackrel{\text{def}}{=} \|f^\ell\|_{\dot{B}^\sigma_{p,r}} \quad \text{and} \quad \|f\|^h_{\dot{B}^\sigma_{p,r}} \stackrel{\text{def}}{=} \|f^h\|_{\dot{B}^\sigma_{p,r}}.$$

At last, we agree that throughout the paper, C stands for a generic constant, and $A \lesssim B$ means $A \le CB$. For a Banach space X, $p \in [1, +\infty]$ and $T > 0$, the notation $L^p(0, T; X)$ or $L^p_T(X)$ designates the set of measurable functions $f : [0, T] \to X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0, T)$, endowed with the norm

$$\|f\|_{L^p_T(X)} \stackrel{\text{def}}{=} \big\| \|f\|_X \big\|_{L^p(0,T)}.$$

The index T will be omitted if $T = +\infty$. The notation $\mathcal{C}([0, T]; X)$ denotes the set of continuous functions from $[0, T]$ to X, and $\widetilde{\mathcal{C}}_b([0, T]; \dot{B}^s_{p,r})$ represents the subset of functions of $\widetilde{L}^\infty_T(\dot{B}^s_{p,r})$ which are also continuous from $[0, T]$ to $\dot{B}^s_{p,r}$. It will be also understood that $\|(f, g)\|_X \stackrel{\text{def}}{=} \|f\|_X + \|g\|_X$ for all $f, g \in X$.

Now, let us recall the global existence result of system (1.1) in the critical L^p framework (see [38]).

Theorem 2.1. *([38]) Let $N \geq 3$ and p satisfy*

$$2 \leq p < N \quad \text{and} \quad p \leq \min\{4, 2N/(N-2)\}. \tag{2.1}$$

There exists a constant $c = c(p, N, \lambda, \mu, P)$ such that if $b_0 \stackrel{\text{def}}{=} \rho_0 - 1 \in \dot{B}^{\frac{N}{p}}_{p,1}$, $\mathbf{u}_0 \in \dot{B}^{\frac{N}{p}-1}_{p,1}$ and if in addition $b_0^\ell \in \dot{B}^{\frac{N}{2}-2}_{2,1}$, $\mathbf{u}_0^\ell \in \dot{B}^{\frac{N}{2}-1}_{2,1}$ with

$$\mathcal{X}_{p,0} \stackrel{\text{def}}{=} \|b_0\|^\ell_{\dot{B}^{\frac{N}{2}-2}_{2,1}} + \|\mathbf{u}_0\|^\ell_{\dot{B}^{\frac{N}{2}-1}_{2,1}} + \|b_0\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}} + \|\mathbf{u}_0\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \leq c, \tag{2.2}$$

then (1.1) has a global solution $(\rho, \mathbf{u}, \Phi)$ with $\rho = b + 1$ and $(\rho, \mathbf{u}, \Phi)$ in the space X_p defined by

$$\begin{aligned}
&b^\ell \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{N}{2}-2}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{2}}_{2,1}), \quad &&\mathbf{u}^\ell \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{N}{2}-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{2}+1}_{2,1}),\\
&b^h \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{N}{p}}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{p}}_{p,1}), \quad &&\mathbf{u}^h \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{N}{p}-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{p}+1}_{p,1}),\\
&\Phi^\ell \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{N}{2}}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{2}+2}_{2,1}), \quad &&\Phi^h \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{N}{p}+2}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{p}+2}_{p,1}).
\end{aligned}$$

Moreover, we have for some constant $C = C(p, N, \lambda, \mu, P)$,

$$\mathcal{X}_p \leq C\mathcal{X}_{p,0}, \tag{2.3}$$

with

$$\begin{aligned}
\mathcal{X}_p \stackrel{\text{def}}{=} &\|b\|^\ell_{\widetilde{L}^\infty(\dot{B}^{\frac{N}{2}-2}_{2,1}) \cap L^1(\dot{B}^{\frac{N}{2}}_{2,1})} + \|\mathbf{u}\|^\ell_{\widetilde{L}^\infty(\dot{B}^{\frac{N}{2}-1}_{2,1}) \cap L^1(\dot{B}^{\frac{N}{2}+1}_{2,1})} + \|\Phi\|^\ell_{\widetilde{L}^\infty(\dot{B}^{\frac{N}{2}}_{2,1}) \cap L^1(\dot{B}^{\frac{N}{2}+2}_{2,1})}\\
&+ \|b\|^h_{\widetilde{L}^\infty(\dot{B}^{\frac{N}{p}}_{p,1}) \cap L^1(\dot{B}^{\frac{N}{p}}_{p,1})} + \|\mathbf{u}\|^h_{\widetilde{L}^\infty(\dot{B}^{\frac{N}{p}-1}_{p,1}) \cap L^1(\dot{B}^{\frac{N}{p}+1}_{p,1})} + \|\Phi\|^h_{\widetilde{L}^\infty(\dot{B}^{\frac{N}{p}+2}_{p,1}) \cap L^1(\dot{B}^{\frac{N}{p}+2}_{p,1})}.
\end{aligned} \tag{2.4}$$

The main results of this paper are stated as follows.

Theorem 2.2. *Let $N \geq 3$ and p satisfy condition (2.1). Let $(\rho_0, \mathbf{u}_0)$ fulfill the assumptions of Theorem 2.1, and denote by $(\rho, \mathbf{u}, \Phi)$ the corresponding global solution of system (1.1). Set $s_0 \stackrel{\text{def}}{=} N(\frac{2}{p} - \frac{1}{2})$ and $\alpha \stackrel{\text{def}}{=} \frac{N}{p} + \frac{1}{2} - \varepsilon$ for some arbitrarily small $\varepsilon > 0$. Then there exists a positive constant $c = c(p, N, \lambda, \mu, P)$ such that if in addition*

$$\mathcal{Y}_{p,0} \stackrel{\text{def}}{=} \|b_0\|^\ell_{\dot{B}^{-s_0-1}_{2,\infty}} + \|\mathbf{u}_0\|^\ell_{\dot{B}^{-s_0}_{2,\infty}} \leq c, \tag{2.5}$$

we have for all $t \geq 0$,

$$\mathcal{Y}_p(t) \lesssim \mathcal{Y}_{p,0} + \|(\nabla b_0, \mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}, \tag{2.6}$$

where the norm $\mathcal{Y}_p(t)$ is defined by

$$
\begin{aligned}
\mathcal{Y}_p(t) \stackrel{\text{def}}{=} & \sup_{s\in(-s_0,\, \frac{N}{2}+1]} \|\langle\tau\rangle^{\frac{s_0+s}{2}}(\Lambda^{-1}b, \mathbf{u}, \nabla\Phi)\|^{\ell}_{L^\infty_t(\dot{B}^s_{2,1})} \\
& + \|\langle\tau\rangle^{\alpha}(\nabla b, \mathbf{u}, \nabla\Delta\Phi)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} + \|\tau\mathbf{u}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}+1}_{p,1})},
\end{aligned}
\tag{2.7}
$$

here $\langle t\rangle \stackrel{\text{def}}{=} \sqrt{1+t^2}$.

Remark 2.1. The value chosen for s_0 is based on the fact that: in the standard case $p = 2$ and for classical solutions, it is generally assumed that the initial data are in L^1, namely, L^1–L^2 type decay rates. By the definition of s_0, it is obvious that $s_0 = \frac{N}{2}$ when $p = 2$. Then by the critical embedding, there holds $L^1 \hookrightarrow \dot{B}^{-\frac{N}{2}}_{2,\infty}$. Naturally, in our L^p framework, i.e., $L^{\frac{p}{2}}$–L^p type decay rates would bring us to replace L^1 by $L^{\frac{p}{2}}$. Choosing s_0 as above corresponds exactly to the critical embedding $L^{\frac{p}{2}} \hookrightarrow \dot{B}^{-s_0}_{2,\infty}$.

Remark 2.2. The first term of $\mathcal{Y}_p$, i.e., the decay rate for the low frequencies of the solution is optimal since it corresponds to that of the linearized system for (1.1) about $(1, \mathbf{0})$ for general datum $(\rho_0 - 1, \mathbf{u}_0)$ belonging to $\dot{B}^{-s_0-1}_{2,\infty} \times \dot{B}^{-s_0}_{2,\infty}$. On the other hand, as pointed out in [38], the compressible NSP system (1.1) and the Navier–Stokes system have the same behavior in the high frequency regime. Therefore, excluding the electrostatic potential Φ, the definitions of the second and third terms in $\mathcal{Y}_p$ involving $(b, \mathbf{u})$ in the high frequencies are same as those of the compressible Navier–Stokes system [9].

As a direct consequence, we have the following corollary.

Corollary 2.1. *The solution* $(\rho, \mathbf{u})$ *constructed in Theorem 2.2, satisfies that if* $-s_0 - 1 < s < \frac{N}{p}$, *then*

$$
\|\Lambda^s(\rho-1)\|_{L^p} \le C\Big(\mathcal{Y}_{p,0} + \|(\nabla b_0, \mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\Big)\langle t\rangle^{-\frac{N}{p}+\frac{N}{4}-\frac{s}{2}-\frac{1}{2}}. \tag{2.8}
$$

If $-s_0 < s \le \frac{N}{p} - 1$, *then*

$$
\|\Lambda^s\mathbf{u}\|_{L^p} \le C\Big(\mathcal{Y}_{p,0} + \|(\nabla b_0, \mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\Big)\langle t\rangle^{-\frac{N}{p}+\frac{N}{4}-\frac{s}{2}}. \tag{2.9}
$$

And if $-s_0 < s < \frac{N}{p} + 1$, *then*

$$
\|\Lambda^s\nabla\Phi\|_{L^p} \le C\Big(\mathcal{Y}_{p,0} + \|(\nabla b_0, \mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\Big)\langle t\rangle^{-\frac{N}{p}+\frac{N}{4}-\frac{s}{2}}. \tag{2.10}
$$

Remark 2.3. Let $p = 2$ and $s = 0$ in Corollary 2.1, then the standard optimal L^1–L^2 decay rate of $(b, \mathbf{u}, \nabla\Phi)$ is obtained. Notice however that our estimates also hold in the general L^p critical framework.

Remark 2.4. The regularity index s can take both negative and nonnegative values, rather than only nonnegative integers. Therefore, our results could be seen as complementary results of the works [23,24,33–35] which considered in high Sobolev regularity.

We can also obtain the following decay rates for the integrability index $2 \le r \le \infty$.

Corollary 2.2. *Let the assumptions of Theorem 2.2 be satisfied with $p = 2$. Then the corresponding solution $(\rho, \mathbf{u}, \nabla\Phi)$ satisfies*

$$\|\Lambda^m(\rho - 1)\|_{L^r} \le C\Big(\mathcal{Y}_{2,0} + \|(\nabla b_0, \mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{2}-1}_{2,1}}\Big)\langle t\rangle^{\frac{N}{2r}-\frac{N}{2}-\frac{m}{2}-\frac{1}{2}} \tag{2.11}$$

for all $2 \le r \le \infty$ and $m \in \mathbb{R}$ fulfilling $-\frac{N}{2} - 1 < m + N(\frac{1}{2} - \frac{1}{r}) < \frac{N}{2}$, and $(\mathbf{u}, \nabla\Phi)$ satisfies

$$\|\Lambda^k(\mathbf{u}, \nabla\Phi)\|_{L^r} \le C\Big(\mathcal{Y}_{2,0} + \|(\nabla b_0, \mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{2}-1}_{2,1}}\Big)\langle t\rangle^{\frac{N}{2r}-\frac{N}{2}-\frac{k}{2}} \tag{2.12}$$

for all $2 \le r \le \infty$ and $k \in \mathbb{R}$ fulfilling $-\frac{N}{2} < k + N(\frac{1}{2} - \frac{1}{r}) \le \frac{N}{2} - 1$.

Scheme of the proof and structure of the paper. Firstly, defining

$$\mathcal{A} \stackrel{\text{def}}{=} \mu\Delta + (\mu + \lambda)\nabla\text{div}, \quad k(b) \stackrel{\text{def}}{=} 1 - \frac{P'(1+b)}{1+b}, \quad I(b) \stackrel{\text{def}}{=} \frac{b}{1+b}, \tag{2.13}$$

we consider the following linearized system corresponding to system (1.1) at point $(1, \mathbf{0})$, i.e.,

$$\begin{cases} \partial_t b + \text{div}\mathbf{u} = f, \\ \partial_t \mathbf{u} - \mathcal{A}\mathbf{u} + \nabla b - \nabla\Delta^{-1} b = \mathbf{g}, \end{cases} \tag{2.14}$$

with

$$f \stackrel{\text{def}}{=} -\text{div}(b\mathbf{u}), \quad \mathbf{g} \stackrel{\text{def}}{=} -\mathbf{u}\cdot\nabla\mathbf{u} + k(b)\nabla b - I(b)\mathcal{A}\mathbf{u}. \tag{2.15}$$

Then, given the global existence result of Theorem 2.1, similar to [9], according to the definition of $\mathcal{Y}_p$, we proceed in three steps. Roughly speaking, in the first step, we combine the low frequency decay properties of the semi-group defined by the left hand side of (2.14), and Duhamel principle to handle the terms in the right hand side of (2.14). In the second step, in order to exhibit the decay of the high frequencies part of the solution, we resort to the effective velocity $\mathbf{w} \stackrel{\text{def}}{=} \nabla(-\Delta)^{-1}(b - \text{div}\mathbf{u})$ introduced by [15,16]. In the last step, we establish gain of regularity and decay altogether for the high frequencies of $\mathbf{u}$.

However, compared with [9], here thanks to the appearance of the Poisson equation, we have some new observations. More concretely, in the low frequencies, the authors in [9] considered the following homogeneous linearized compressible Navier–Stokes equations about $(1, \mathbf{0})$,

$$\begin{cases} \partial_t b + \text{div}\mathbf{u} = 0, \\ \partial_t \mathbf{u} - \mathcal{A}\mathbf{u} + \nabla b = \mathbf{0}. \end{cases} \tag{2.16}$$

Therefore, after spectral localization and from an explicit computation of the action of the semi-group associated to (2.16) (see e.g. [3]), there exists a constant $c > 0$ such that for all $j \in \mathbb{Z}$ and $j \leq j_0$,

$$\|b_j\|_{L^2} + \|\mathbf{u}_j\|_{L^2} \lesssim e^{-c2^{2j}t}(\|b_{0j}\|_{L^2} + \|\mathbf{u}_{0j}\|_{L^2}), \tag{2.17}$$

here and in what follows, we set $z_j \stackrel{\text{def}}{=} \dot{\Delta}_j z$ for any tempered distribution z and $j \in \mathbb{Z}$. In this paper, we will consider the following homogeneous linear system corresponding to (2.14), namely,

$$\begin{cases} \partial_t b + \operatorname{div}\mathbf{u} = 0, \\ \partial_t \mathbf{u} - \mathcal{A}\mathbf{u} + \nabla b - \nabla\Delta^{-1} b = \mathbf{0}. \end{cases} \tag{2.18}$$

It should be noted that in the low frequencies, if we treat system (1.1) in the same way as that in [9], the term $\nabla\Delta^{-1}b$ would not be controlled. So here term $\nabla\Delta^{-1}b$ is placed to the left hand side. From the explicit expression of the Green matrix of (2.18) (see [38]), we know that it has smoothing effect in the low frequency regime. Concretely, assuming that $(\tilde{b}, \tilde{\mathbf{u}})$ is a solution of system (2.18), one has after spectral localization that for any $j_0 \in \mathbb{Z}$ and $j \leq j_0$ (see [38]),

$$2^{-j}\|\tilde{b}_j\|_{L^2} + \|\tilde{\mathbf{u}}_j\|_{L^2} \lesssim e^{-c2^{2j}t}(2^{-j}\|b_{0j}\|_{L^2} + \|\mathbf{u}_{0j}\|_{L^2}). \tag{2.19}$$

Here compared with (2.17), the appearance of the coefficient 2^{-j} of $\|\tilde{b}_j\|_{L^2}$ is due to the term $\nabla\Delta^{-1}b$. Just because of this, different from the definition of $\mathcal{Y}_p$ in [9], here we replace b with $\Lambda^{-1}b$.

This paper is organized as follows. In Section 3, we present some material concerning paradifferential calculus, product and commutator estimates. Section 4 is devoted to the proofs of Theorem 2.2 and Corollaries 2.1 and 2.2, respectively.

3. Littlewood–Paley decomposition and Besov spaces

The following proposition (referred to as Bernstein's inequalities) describes the way derivatives act on spectrally localized functions.

Proposition 3.1. *(See [5].) Let $\mathcal{C}$ be an annulus and $\mathcal{B}$ a ball, $1 \leq p \leq q \leq +\infty$. Assume that $f \in L^p(\mathbb{R}^N)$, then for any nonnegative integer k, there exists constant C independent of f, k such that*

$$\operatorname{supp}\hat{f} \subset \lambda\mathcal{B} \Rightarrow \|D^k f\|_{L^q(\mathbb{R}^N)} \stackrel{\text{def}}{=} \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^q(\mathbb{R}^N)} \leq C^{k+1}\lambda^{k+N(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^N)},$$

$$\operatorname{supp}\hat{f} \subset \lambda\mathcal{C} \Rightarrow C^{-k-1}\lambda^k\|f\|_{L^p(\mathbb{R}^N)} \leq \|D^k f\|_{L^p(\mathbb{R}^N)} \leq C^{k+1}\lambda^k\|f\|_{L^p(\mathbb{R}^N)}.$$

Let us now recall some classical properties for the Besov spaces.

Proposition 3.2. *The following properties hold true:*

1) *Derivation: There exists a universal constant C such that*

$$C^{-1}\|f\|_{\dot{B}^s_{p,r}} \leq \|\nabla f\|_{\dot{B}^{s-1}_{p,r}} \leq C\|f\|_{\dot{B}^s_{p,r}}.$$

2) *Sobolev embedding: If $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-\frac{N}{p_1}+\frac{N}{p_2}}_{p_2,r_2}$.*
3) *Real interpolation: $\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|f\|^{\theta}_{\dot{B}^{s_1}_{p,r}}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}$.*
4) *Algebraic properties: for $s>0$, $\dot{B}^s_{p,1} \cap L^\infty$ is an algebra.*

Next we state a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Firstly introduced by Bony in [2], the paraproduct between f and g is defined by

$$T_f g = \sum_{q\in\mathbb{Z}} \dot{S}_{q-1} f \dot{\Delta}_q g,$$

and the remainder is given by

$$R(f,g) = \sum_{q\in\mathbb{Z}} \dot{\Delta}_q f \tilde{\dot{\Delta}}_q g$$

with

$$\tilde{\dot{\Delta}}_q g \stackrel{\text{def}}{=} (\dot{\Delta}_{q-1} + \dot{\Delta}_q + \dot{\Delta}_{q+1})g.$$

We have the following so-called Bony's decomposition:

$$fg = T_g f + T_f g + R(f,g). \tag{3.1}$$

The paraproduct T and the remainder R operators satisfy the following continuous properties (see e.g. [1]).

Proposition 3.3. *Suppose that $s\in\mathbb{R}$, $\sigma>0$, and $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$. Then we have:*

(1) *The paraproduct T is a bilinear, continuous operator from $L^\infty \times \dot{B}^s_{p,r}$ to $\dot{B}^s_{p,r}$, and from $\dot{B}^{-\sigma}_{\infty,r_1} \times \dot{B}^s_{p,r_2}$ to $\dot{B}^{s-\sigma}_{p,r}$ with $\frac{1}{r} \stackrel{\text{def}}{=} \min\{1, \frac{1}{r_1}+\frac{1}{r_2}\}$.*
(2) *The remainder R is bilinear continuous from $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ to $\dot{B}^{s_1+s_2}_{p,r}$ with $s_1+s_2>0$, $\frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1}+\frac{1}{p_2} \leq 1$, and $\frac{1}{r} \stackrel{\text{def}}{=} \frac{1}{r_1}+\frac{1}{r_2} \leq 1$.*
(3) *The remainder R is bilinear continuous from $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ to $\dot{B}^0_{p,\infty}$ with $s_1+s_2=0$, $\frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1}+\frac{1}{p_2} \leq 1$, and $\frac{1}{r_1}+\frac{1}{r_2} \geq 1$.*

From (3.1) and Proposition 3.3, we have the following more accurate product estimates:

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Corollary 3.1. *([7]) If $u \in \dot{B}^{s_1}_{p_1,1}$ and $v \in \dot{B}^{s_2}_{p_2,1}$ with $1 \le p_1 \le p_2 \le \infty$, $s_1 \le \frac{N}{p_1}$, $s_2 \le \frac{N}{p_2}$ and $s_1 + s_2 > 0$, then $uv \in \dot{B}^{s_1+s_2-\frac{N}{p_1}}_{p_2,1}$ and there exists a constant C, depending only on N, s_1, s_2, p_1 and p_2, such that*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{N}{p_1}}_{p_2,1}} \le C\|u\|_{\dot{B}^{s_1}_{p_1,1}}\|v\|_{\dot{B}^{s_2}_{p_2,1}}. \tag{3.2}$$

Corollary 3.2. *([9]) For exponents $\sigma > 0$ and $1 \le p_1, p_2, q \le \infty$ satisfying*

$$\frac{N}{p_1} + \frac{N}{p_2} - N \le \sigma \le \min\left(\frac{N}{p_1}, \frac{N}{p_2}\right) \text{ and } \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma}{N},$$

we have

$$\|fg\|_{\dot{B}^{-\sigma}_{q,\infty}} \lesssim \|f\|_{\dot{B}^{\sigma}_{p_1,1}}\|g\|_{\dot{B}^{-\sigma}_{p_2,\infty}}.$$

We also need the following composition result (see [6,31]).

Proposition 3.4. *Let $s > 0$, $p \in [1,\infty]$ and $u \in \dot{B}^s_{p,1} \cap L^\infty$. Let $F \in W^{[s]+2,\infty}_{\text{loc}}(\mathbb{R}^N)$ such that $F(0) = 0$. Then $F(u) \in \dot{B}^s_{p,1}$ and there exists a constant $C = C(s, p, N, F, \|u\|_{L^\infty})$ such that*

$$\|F(u)\|_{\dot{B}^s_{p,1}} \le C\|u\|_{\dot{B}^s_{p,1}}.$$

Next, we list the following commutator estimate which was proved in [9].

Proposition 3.5. *Let $1 \le p, p_1 \le \infty$ and*

$$-\min\left\{\frac{N}{p_1}, \frac{N}{p'}\right\} < \sigma \le 1 + \min\left\{\frac{N}{p}, \frac{N}{p_1}\right\}. \tag{3.3}$$

There exists a constant $C > 0$ depending only on σ such that for all $j \in \mathbb{Z}$ and $i \in \{1, \cdots, N\}$, we have

$$\|[v \cdot \nabla, \partial_i \dot{\Delta}_j]a\|_{L^p} \le C c_j 2^{-j(\sigma-1)}\|\nabla v\|_{\dot{B}^{\frac{N}{p_1}}_{p_1,1}}\|\nabla a\|_{\dot{B}^{\sigma-1}_{p,1}}, \tag{3.4}$$

where the commutator $[\cdot,\cdot]$ is defined by $[f, g] = fg - gf$, and $(c_j)_{j\in\mathbb{Z}}$ denotes a sequence such that $\|(c_j)\|_{\ell^1} \le 1$ and $\frac{1}{p'} + \frac{1}{p} = 1$.

Finally, we give the optimal regularity estimates for the heat equation (see e.g. [1]).

Proposition 3.6. *Let $\sigma \in \mathbb{R}$, $(p, r) \in [1,\infty]^2$ and $1 \le \rho_2 \le \rho_1 \le \infty$. Let u satisfy*

$$\begin{cases} \partial_t u - \mu\Delta u = f, \\ u|_{t=0} = u_0. \end{cases} \tag{3.5}$$

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Then for all $T > 0$, the following a priori estimate is satisfied:

$$\mu^{\frac{1}{\rho_1}}\|u\|_{\widetilde{L}_T^{\rho_1}(\dot{B}_{p,r}^{\sigma+\frac{2}{\rho_1}})} \lesssim \|u_0\|_{\dot{B}_{p,r}^{\sigma}} + \mu^{\frac{1}{\rho_2}-1}\|f\|_{\widetilde{L}_T^{\rho_2}(\dot{B}_{p,r}^{\sigma-2+\frac{2}{\rho_2}})}. \tag{3.6}$$

Remark 3.1. The solutions to the following Lamé system

$$\begin{cases} \partial_t u - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u = f, \\ u|_{t=0} = u_0, \end{cases} \tag{3.7}$$

where λ and μ are constant coefficients such that $\mu > 0$ and $\lambda + 2\mu > 0$, also satisfy (3.6).

4. The proof of main results

In this section, with the global existence result in Theorem 2.1 at hand, we prove Theorem 2.2 and Corollaries 2.1 and 2.2. In what follows, we will apply repeatedly that for $0 < \gamma_1 \le \gamma_2$ with $\gamma_2 > 1$, it holds that

$$\int_0^t \langle t-\tau\rangle^{-\gamma_1}\langle\tau\rangle^{-\gamma_2}d\tau \lesssim \langle t\rangle^{-\gamma_1}. \tag{4.1}$$

Step 1: Bounds for the low frequencies. This step is devoted to the estimates of the first term of $\mathcal{Y}_p(t)$.

Proposition 4.1. *Under the assumptions of Theorem 2.2, we have*

$$\langle t\rangle^{\frac{s_0+s}{2}}\|(\Lambda^{-1}b, \mathbf{u}, \nabla\Phi)(t)\|^{\ell}_{\dot{B}_{2,1}^{s}} \lesssim \mathcal{Y}_{p,0} + \mathcal{X}_p^2(t) + \mathcal{Y}_p^2(t)$$

for all $t \ge 0$, provided that $-s_0 < s \le \frac{N}{2}+1$.

Proof. We first deal with the homogeneous linear system (2.18). As mentioned at the end of Section 2, it is clear that multiplying by $t^{\frac{s_0+s}{2}}2^{js}$ in two sides of (2.19) and summing up on $j \le j_0$, we write

$$\begin{aligned} & t^{\frac{s_0+s}{2}}\sum_{j\le j_0}\Big(2^{j(s-1)}\|\tilde{b}_j\|_{L^2} + 2^{js}\|\tilde{\mathbf{u}}_j\|_{L^2}\Big) \\ & \lesssim \Big(\|b_0\|^{\ell}_{\dot{B}_{2,\infty}^{-1-s_0}} + \|\mathbf{u}_0\|^{\ell}_{\dot{B}_{2,\infty}^{-s_0}}\Big)\sum_{j\le j_0}(\sqrt{t}2^j)^{s_0+s}e^{-c(\sqrt{t}2^j)^2}. \end{aligned} \tag{4.2}$$

Thanks to the following fact (see e.g. [1]): for any $\delta > 0$, there exists a constant C_δ so that

$$\sup_{t\ge 0}\sum_{j\in\mathbb{Z}} t^{\frac{\delta}{2}}2^{\delta j}e^{-c_0 2^{2j}t} \le C_\delta,$$

then (4.2) implies that for $s_0 + s > 0$,

$$\sup_{t\geq 0} t^{\frac{s_0+s}{2}} \|(\Lambda^{-1}\tilde{b}, \tilde{\mathbf{u}})(t)\|^{\ell}_{\dot{B}^{s}_{2,1}} \lesssim \|b_0\|^{\ell}_{\dot{B}^{-1-s_0}_{2,\infty}} + \|\mathbf{u}_0\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}.$$

Moreover, it is obvious that for $s + s_0 > 0$,

$$\|(\Lambda^{-1}\tilde{b}, \tilde{\mathbf{u}})(t)\|^{\ell}_{\dot{B}^{s}_{2,1}} \lesssim \left(\|b_0\|^{\ell}_{\dot{B}^{-1-s_0}_{2,\infty}} + \|\mathbf{u}_0\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}\right) \sum_{j\leq j_0} 2^{j(s_0+s)} \lesssim \|b_0\|^{\ell}_{\dot{B}^{-1-s_0}_{2,\infty}} + \|\mathbf{u}_0\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}.$$

As a consequence, we obtain

$$\sup_{t\geq 0} \langle t\rangle^{\frac{s_0+s}{2}} \|(\Lambda^{-1}\tilde{b}, \tilde{\mathbf{u}})(t)\|^{\ell}_{\dot{B}^{s}_{2,1}} \lesssim \|b_0\|^{\ell}_{\dot{B}^{-1-s_0}_{2,\infty}} + \|\mathbf{u}_0\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}. \tag{4.3}$$

Given the above result, we now return to system (2.14). Let $(b, \mathbf{u})$ be a solution of (2.14). Then by Duhamel's formula, we need to bound the following term:

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \left(\|f(\tau)\|^{\ell}_{\dot{B}^{-1-s_0}_{2,\infty}} + \|\mathbf{g}(\tau)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}\right) d\tau. \tag{4.4}$$

In the following we will prove that if p satisfies (2.1), then we have for all $t \geq 0$,

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \left(\|f(\tau)\|^{\ell}_{\dot{B}^{-1-s_0}_{2,\infty}} + \|\mathbf{g}(\tau)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}\right) d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}} (\mathcal{Y}_p^2(t) + \mathcal{X}_p^2(t)), \tag{4.5}$$

where $\mathcal{X}_p$ and $\mathcal{Y}_p$ have been defined in (2.4) and (2.7), respectively.

Let us begin with the estimate of the term with f. Thanks to the embedding $L^{p/2} \hookrightarrow \dot{B}^{-s_0}_{2,\infty}$ and the definition of f in (2.15), it follows that

$$\begin{aligned} \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|f(\tau)\|^{\ell}_{\dot{B}^{-1-s_0}_{2,\infty}} d\tau &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(b\mathbf{u})(\tau)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} d\tau \\ &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(b\mathbf{u})(\tau)\|^{\ell}_{L^{\frac{p}{2}}} d\tau \lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|b(\tau)\|_{L^p} \|\mathbf{u}(\tau)\|_{L^p} d\tau. \end{aligned} \tag{4.6}$$

We claim that

$$\|\mathbf{u}(\tau)\|_{L^p} \lesssim \langle \tau\rangle^{-\frac{s_0}{2}+\frac{N}{2p}-\frac{N}{4}} \mathcal{Y}_p(\tau) \tag{4.7}$$

and

$$\|b(\tau)\|_{L^p} \lesssim \langle \tau\rangle^{-\frac{s_0}{2}+\frac{N}{2p}-\frac{N}{4}-\frac{1}{2}} \mathcal{Y}_p(\tau). \tag{4.8}$$

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Indeed, by virtue of Minkowski's inequality and embedding, we get for $2 \le p < N$,

$$\|\mathbf{u}\|_{L^p} \le \|\mathbf{u}^\ell\|_{L^p} + \|\mathbf{u}^h\|_{L^p} \lesssim \|\mathbf{u}\|^\ell_{\dot{B}^{\frac{N}{2}-\frac{N}{p}}_{2,1}} + \|\mathbf{u}\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}},$$

which together with the definitions of α and of $\mathcal{Y}_p(t)$ gives (4.7). Similarly, one obtains

$$\begin{aligned}
\|b(\tau)\|_{L^p} &\le \|b^\ell(\tau)\|_{L^p} + \|b^h(\tau)\|_{L^p} \lesssim \|b(\tau)\|^\ell_{\dot{B}^{\frac{N}{2}-\frac{N}{p}}_{2,1}} + \|b(\tau)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \\
&\lesssim \langle\tau\rangle^{-\frac{s_0+\frac{N}{2}-\frac{N}{p}+1}{2}} \left(\langle\tau\rangle^{\frac{s_0+\frac{N}{2}-\frac{N}{p}+1}{2}} \|\Lambda^{-1} b(\tau)\|^\ell_{\dot{B}^{\frac{N}{2}-\frac{N}{p}+1}_{2,1}}\right) + \langle\tau\rangle^{-\alpha}\left(\langle\tau\rangle^\alpha \|b(\tau)\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}}\right) \\
&\lesssim \langle\tau\rangle^{-\frac{s_0}{2}+\frac{N}{2p}-\frac{N}{4}-\frac{1}{2}} \mathcal{Y}_p(\tau),
\end{aligned} \tag{4.9}$$

which yields (4.8). Thus (4.6) could be written as

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|f(\tau)\|^\ell_{\dot{B}^{-1-s_0}_{2,\infty}} d\tau \lesssim \mathcal{Y}_p^2(t) \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \langle\tau\rangle^{-s_0+\frac{N}{p}-\frac{N}{2}-\frac{1}{2}} d\tau. \tag{4.10}$$

Due to $2 \le p < N$ and $s_0 = \frac{2N}{p} - \frac{N}{2}$, we get that for all $-s_0 < s \le \frac{N}{2}+1$,

$$\frac{s_0}{2} + \frac{s}{2} \le \frac{N}{p} + \frac{1}{2}.$$

It is obvious that $\frac{N}{p} + \frac{1}{2} > 1$, then inequality (4.1) guarantees

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \langle\tau\rangle^{-s_0+\frac{N}{p}-\frac{N}{2}-\frac{1}{2}} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}}.$$

Consequently, it follows from (4.10) that

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|f(\tau)\|^\ell_{\dot{B}^{-1-s_0}_{2,\infty}} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}} \mathcal{Y}_p^2(t). \tag{4.11}$$

Next, we turn to estimate the term with $\mathbf{g}$ in (4.5). By the definition of $\mathbf{g}$ in (2.15), we start with $\mathbf{u}\cdot\nabla\mathbf{u}$ as follows.

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(\mathbf{u}\cdot\nabla\mathbf{u})(\tau)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(\mathbf{u}\cdot\nabla\mathbf{u})(\tau)\|^{\ell}_{L^{\frac{p}{2}}} d\tau \\ &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|\mathbf{u}(\tau)\|_{L^p}\|\nabla\mathbf{u}(\tau)\|_{L^p} d\tau.\end{aligned} \tag{4.12}$$

Following the procedure leading to (4.8), we also can obtain

$$\|\nabla\mathbf{u}(\tau)\|_{L^p} \lesssim \langle\tau\rangle^{-\frac{s_0}{2}+\frac{N}{2p}-\frac{N}{4}-\frac{1}{2}}\mathcal{Y}_p(\tau), \tag{4.13}$$

this together with (4.7) and inequality (4.1) gives

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(\mathbf{u}\cdot\nabla\mathbf{u})(\tau)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{Y}_p^2(t). \tag{4.14}$$

In what follows, let us proceed with the second term of $\mathbf{g}$, i.e., $k(b)\nabla b$. Similar to (4.6), one writes

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(k(b)\nabla b)(\tau)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(k(b)\nabla b)(\tau)\|^{\ell}_{L^{\frac{p}{2}}} d\tau \\ &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|b(\tau)\|_{L^p}\|\nabla b(\tau)\|_{L^p} d\tau.\end{aligned} \tag{4.15}$$

Now, we observe that

$$\begin{aligned}\|\nabla b(\tau)\|_{L^p} &\le \|\nabla b^{\ell}(\tau)\|_{L^p} + \|\nabla b^{h}(\tau)\|_{L^p} \lesssim \|\nabla b(\tau)\|^{\ell}_{\dot{B}^{\frac{N}{2}-\frac{N}{p}}_{2,1}} + \|\nabla b(\tau)\|^{h}_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \\ &\lesssim \|\Lambda^{-1} b(\tau)\|^{\ell}_{\dot{B}^{\frac{N}{2}-\frac{N}{p}+2}_{2,1}} + \|\nabla b(\tau)\|^{h}_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \\ &\lesssim \langle\tau\rangle^{-\frac{s_0+\frac{N}{2}-\frac{N}{p}+2}{2}}\left(\langle\tau\rangle^{\frac{s_0+\frac{N}{2}-\frac{N}{p}+2}{2}}\|\Lambda^{-1}b(\tau)\|^{\ell}_{\dot{B}^{\frac{N}{2}-\frac{N}{p}+2}_{2,1}}\right) + \langle\tau\rangle^{-\alpha}\left(\langle\tau\rangle^{\alpha}\|b(\tau)\|^{h}_{\dot{B}^{\frac{N}{p}}_{p,1}}\right) \\ &\lesssim \langle\tau\rangle^{-\frac{s_0}{2}-\frac{N}{4}+\frac{N}{2p}-1}\mathcal{Y}_p(\tau).\end{aligned} \tag{4.16}$$

Combining this with (4.8), we infer that (4.15) becomes

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(k(b)\nabla b)(\tau)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} d\tau \lesssim \mathcal{Y}_p^2(t)\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-s_0-\frac{N}{2}+\frac{N}{p}-\frac{3}{2}} d\tau. \tag{4.17}$$

Thus inequality (4.1) ensures that

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(k(b)\nabla b)(\tau)\|^{\ell}_{\dot B^{-s_0}_{2,\infty}} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}} \mathcal{Y}_p^2(t). \tag{4.18}$$

On the last term $I(b)\mathcal{A}\mathbf{u}$ of $\mathbf{g}$, we split it into two terms $I(b)\mathcal{A}\mathbf{u}^{\ell}$ and $I(b)\mathcal{A}\mathbf{u}^{h}$. Firstly, one has that

$$\begin{aligned} \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(I(b)\mathcal{A}\mathbf{u}^{\ell})(\tau)\|_{\dot B^{-s_0}_{2,\infty}} d\tau &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(I(b)\mathcal{A}\mathbf{u}^{\ell})(\tau)\|_{L^{\frac{p}{2}}} d\tau \\ &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|b\|_{L^p} \|\nabla^2\mathbf{u}^{\ell}\|_{L^p} d\tau. \end{aligned} \tag{4.19}$$

It is clear that

$$\|\nabla^2\mathbf{u}^{\ell}(\tau)\|_{L^p} \lesssim \|\mathbf{u}(\tau)\|^{\ell}_{\dot B^{\frac{N}{2}-\frac{N}{p}+2}_{2,1}} \lesssim \langle\tau\rangle^{-\frac{s_0+\frac{N}{2}-\frac{N}{p}+2}{2}} \mathcal{Y}_p(\tau).$$

Along the same line to the derivation of (4.18), we write

$$\begin{aligned} &\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(I(b)\mathcal{A}\mathbf{u}^{\ell})(\tau)\|_{\dot B^{-s_0}_{2,\infty}} d\tau \\ &\lesssim \mathcal{Y}_p^2(t) \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \langle\tau\rangle^{-s_0-\frac{N}{2}+\frac{N}{p}-\frac{3}{2}} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}} \mathcal{Y}_p^2(t). \end{aligned} \tag{4.20}$$

To bound the term involving $I(b)\mathcal{A}\mathbf{u}^{h}$, we distinguish two cases $t \geq 2$ and $t \leq 2$. If $t \geq 2$, we have

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(I(b)\mathcal{A}\mathbf{u}^{h})(\tau)\|_{\dot B^{-s_0}_{2,\infty}} d\tau = \left(\int_0^1 + \int_1^t\right)(\cdots)d\tau \stackrel{\text{def}}{=} I_1 + I_2.$$

Keeping the definitions of $\mathcal{X}_p(t)$ and $\mathcal{Y}_p(t)$ in mind, we derive that

$$
\begin{aligned}
I_1 &= \int_0^1 \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(I(b)\mathcal{A}\mathbf{u}^h)(\tau)\|_{\dot B^{-s_0}_{2,\infty}} d\tau \\
&\lesssim \int_0^1 \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|b(\tau)\|_{L^p} \|\nabla^2 \mathbf{u}^h(\tau)\|_{L^p} d\tau \\
&\lesssim \langle t\rangle^{-\frac{s_0+s}{2}} \Big(\sup_{0\le\tau\le1} \|b(\tau)\|_{L^p}\Big) \int_0^1 \|\nabla^2\mathbf{u}^h(\tau)\|_{L^p} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}} \mathcal{Y}_p(1)\mathcal{X}_p(1).
\end{aligned}
$$

For I_2, we notice that $\langle\tau\rangle \approx \tau$ when $\tau \ge 1$. Applying Corollary 3.2, Proposition 3.4 and the embedding $\dot B^{1-\frac{N}{p}}_{\frac{pN}{p+N},\infty} \hookrightarrow \dot B^{-s_0}_{2,\infty}$, we get that if $2 \le p < N$,

$$
\|I(b)\mathcal{A}\mathbf{u}^h\|_{\dot B^{-s_0}_{2,\infty}} \lesssim \|I(b)\mathcal{A}\mathbf{u}^h\|_{\dot B^{1-\frac{N}{p}}_{\frac{pN}{p+N},\infty}} \lesssim \|b\|_{\dot B^{\frac{N}{p}-1}_{p,1}} \|\nabla^2\mathbf{u}^h\|_{\dot B^{1-\frac{N}{p}}_{p,1}}.
$$

Note that

$$
\|b(\tau)\|_{\dot B^{\frac{N}{p}-1}_{p,1}} \lesssim \|b^\ell(\tau)\|_{\dot B^{\frac{N}{2}-1}_{2,1}} + \|b^h(\tau)\|_{\dot B^{\frac{N}{p}-1}_{p,1}} \lesssim \langle\tau\rangle^{-(\frac{N}{p}-\frac12)}\mathcal{Y}_p(\tau),
$$

and that if $\frac{N}{2} < p < N$, then setting $\theta \overset{\text{def}}{=} 2-\frac{N}{p}$, we have by interpolation that

$$
\|\nabla^2\mathbf{u}^h(\tau)\|_{\dot B^{1-\frac{N}{p}}_{p,1}} \lesssim \|\mathbf{u}^h(\tau)\|^{\theta}_{\dot B^{\frac{N}{p}+1}_{p,1}} \|\mathbf{u}^h(\tau)\|^{1-\theta}_{\dot B^{\frac{N}{p}-1}_{p,1}} \lesssim \langle\tau\rangle^{-\theta}\langle\tau\rangle^{-\alpha(1-\theta)}\mathcal{Y}_p(\tau),
$$

where the last step comes from the definition of $\mathcal{Y}_p(t)$. Then thanks to (4.1), one gets if $\frac{N}{2} < p < N$,

$$
\begin{aligned}
I_2 &= \int_1^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(I(b)\mathcal{A}\mathbf{u}^h)(\tau)\|_{\dot B^{-s_0}_{2,\infty}} d\tau \\
&\lesssim \mathcal{Y}_p^2(t) \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \langle\tau\rangle^{-\frac{N}{p}+\frac12-\theta-\alpha+\alpha\theta} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}} \mathcal{Y}_p^2(t).
\end{aligned} \tag{4.21}
$$

While in the case of $2 \le p \le \frac{N}{2}$, we get just by embedding that

$$
\|\nabla^2\mathbf{u}^h(\tau)\|_{\dot B^{1-\frac{N}{p}}_{p,1}} \lesssim \|\mathbf{u}(\tau)\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}} \le \langle\tau\rangle^{-\alpha}\mathcal{Y}_p(\tau),
$$

which yields if $2 \le p \le \frac{N}{2}$,

$$
\begin{aligned}
I_2 &= \int_1^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(I(b)\mathcal{A}\mathbf{u}^h)(\tau)\|_{\dot{B}^{-s_0}_{2,\infty}} d\tau \\
&\lesssim \mathcal{Y}_p^2(t) \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \langle\tau\rangle^{-\frac{N}{p}+\frac{1}{2}-\alpha} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}} \mathcal{Y}_p^2(t).
\end{aligned} \tag{4.22}
$$

Therefore, for $t \geq 2$, we conclude that

$$
\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(I(b)\mathcal{A}\mathbf{u}^h)(\tau)\|_{\dot{B}^{-s_0}_{2,\infty}} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}} (\mathcal{X}_p(t)\mathcal{Y}_p(t) + \mathcal{Y}_p^2(t)). \tag{4.23}
$$

The case $t \leq 2$ is easy as $\langle t\rangle \approx 1$ and $\langle t-\tau\rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$, and

$$
\begin{aligned}
\int_0^t \|(I(b)\mathcal{A}\mathbf{u}^h)(\tau)\|_{\dot{B}^{-s_0}_{2,\infty}} d\tau &\lesssim \int_0^t \|(I(b)\mathcal{A}\mathbf{u}^h)(\tau)\|_{L^{\frac{p}{2}}} d\tau \\
&\lesssim \|b\|_{L^\infty_t(L^p)} \|\nabla^2 \mathbf{u}^h\|_{L^1_t(L^p)} \lesssim \mathcal{Y}_p(t)\mathcal{X}_p(t).
\end{aligned} \tag{4.24}
$$

Collecting the estimates (4.14), (4.18), (4.20), (4.23) and (4.24) together, we have

$$
\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|\mathbf{g}(\tau)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}} (\mathcal{Y}_p^2(t) + \mathcal{X}_p^2(t)). \tag{4.25}
$$

From which and (4.11), we complete the proof of (4.5).

Thus (4.5) together with (4.3) for the bound of the term pertaining to the data, we conclude that

$$
\langle t\rangle^{\frac{s_0+s}{2}} \|(\Lambda^{-1}b, \mathbf{u})(t)\|^{\ell}_{\dot{B}^s_{2,1}} \lesssim \mathcal{Y}_{p,0} + \mathcal{X}_p^2(t) + \mathcal{Y}_p^2(t)
$$

for all $t \geq 0$, provided that $-s_0 < s \leq \frac{N}{2} + 1$.

On the other hand, it follows from the Poisson equation in (1.1) that

$$
\langle t\rangle^{\frac{s_0+s}{2}} \|\nabla\Phi(t)\|^{\ell}_{\dot{B}^s_{2,1}} \lesssim \langle t\rangle^{\frac{s_0+s}{2}} \|\Lambda^{-1}b\|^{\ell}_{\dot{B}^s_{2,1}} \lesssim \mathcal{Y}_{p,0} + \mathcal{X}_p^2(t) + \mathcal{Y}_p^2(t)
$$

for all $t \geq 0$, provided that $-s_0 < s \leq \frac{N}{2} + 1$. This completes the proof of Proposition 4.1. $\square$

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Step 2: Bounds for the high frequencies. In this step we focus on the decay for the high frequencies of $(\nabla b, \mathbf{u}, \nabla\Delta\Phi)$. In this case, we can treat the term $\nabla\Delta^{-1}b$ in (2.14)$_2$ as source term because of its smallness of suitable norm in the high frequency regime. Thus, system (2.14) could be written as

$$\begin{cases} \partial_t b + \mathrm{div}\mathbf{u} = f, \\ \partial_t \mathbf{u} - \mathcal{A}\mathbf{u} + \nabla b = \mathbf{g} + \nabla\Delta^{-1}b \stackrel{\text{def}}{=} \mathbf{g_1}. \end{cases} \tag{4.26}$$

Proposition 4.2. *Under the assumptions of Theorem 2.2, we have for all $t \geq 0$,*

$$\|\langle\tau\rangle^{\alpha}(\nabla b, \mathbf{u}, \nabla\Delta\Phi)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|(\nabla b_0, \mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \mathcal{X}_p^2(t) + \mathcal{Y}_p^2(t) + 2^{-2j_0}\mathcal{Y}_p(t).$$

Proof. Denote $R^i_j \stackrel{\text{def}}{=} [\mathbf{u}\cdot\nabla, \partial_i\dot{\Delta}_j]b$ $(i = 1, \cdots, N)$, where the commutator $[\cdot,\cdot]$ is defined by $[f, g] = fg - gf$. According to [9, the estimate (3.47)], it follows that

$$\begin{aligned} \|\langle\tau\rangle^{\alpha}(\nabla b, \mathbf{u})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} &\lesssim \|(\nabla b_0, \mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \\ &\quad + \sum_{j\geq j_0}\sup_{0\leq t\leq T}\Big(\langle t\rangle^{\alpha}\int_0^t e^{-c_0(t-\tau)}2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau\Big), \end{aligned} \tag{4.27}$$

with $S_j \stackrel{\text{def}}{=} S^1_j + \cdots + S^5_j$ and

$$\begin{aligned} &S^1_j \stackrel{\text{def}}{=} \|\dot{\Delta}_j(b\mathbf{u})\|_{L^p}, \quad S^2_j \stackrel{\text{def}}{=} \|\mathbf{g_1}{}_j\|_{L^p}, \\ &S^3_j \stackrel{\text{def}}{=} \|\nabla\dot{\Delta}_j(b\,\mathrm{div}\mathbf{u})\|_{L^p}, \quad S^4_j \stackrel{\text{def}}{=} \|R_j\|_{L^p}, \quad S^5_j \stackrel{\text{def}}{=} \|\mathrm{div}\mathbf{u}\|_{L^\infty}\|\nabla b_j\|_{L^p}. \end{aligned} \tag{4.28}$$

To proceed with the sum, we first consider the case for $0 \leq t \leq 2$.

$$\sum_{j\geq j_0}\sup_{0\leq t\leq 2}\Big(\langle t\rangle^{\alpha}\int_0^t e^{-c_0(t-\tau)}2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau\Big) \lesssim \int_0^2\sum_{j\geq j_0}2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau. \tag{4.29}$$

It follows from Corollary 3.1 and Proposition 3.5 that

$$\int_0^2\sum_{j\geq j_0}2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau \lesssim \int_0^2\Big(\|b\mathbf{u}\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \|\mathbf{g_1}\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \|\nabla\mathbf{u}\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|b\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\Big)d\tau. \tag{4.30}$$

For the last term, we write

$$\int_0^2\|\nabla\mathbf{u}\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|b\|_{\dot{B}^{\frac{N}{p}}_{p,1}}d\tau \leq \|\mathbf{u}\|_{L^1_t(\dot{B}^{\frac{N}{p}+1}_{p,1})}\|b\|_{L^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})} \lesssim \mathcal{X}_p^2(2).$$

Due to inequality (3.2), one arrives at

$$\|b\mathbf{u}\|^h_{L^1_t(\dot B^{\frac{N}{p}-1}_{p,1})} \lesssim \|b\|_{L^1_t(\dot B^{\frac{N}{p}}_{p,1})}\|\mathbf{u}\|_{L^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{X}^2_p(2).$$

Next, combining Corollary 3.1 and Proposition 3.4, recalling the definition of $\mathbf{g}$ in (2.15), we derive

$$\|\mathbf{g_1}\|_{L^1_t(\dot B^{\frac{N}{p}-1}_{p,1})} \lesssim \|\mathbf{u}\|_{L^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})}\|\mathbf{u}\|_{L^1_t(\dot B^{\frac{N}{p}+1}_{p,1})} + \|b\|_{L^\infty_t(\dot B^{\frac{N}{p}}_{p,1})}\|\mathbf{u}\|_{L^1_t(\dot B^{\frac{N}{p}+1}_{p,1})}$$
$$+ \|b\|_{L^\infty_t(\dot B^{\frac{N}{p}}_{p,1})}\|b\|_{L^1_t(\dot B^{\frac{N}{p}}_{p,1})} + 2^{-2j_0}\|b\|_{L^1_t(\dot B^{\frac{N}{p}}_{p,1})}.$$

Then we conclude that

$$\sum_{j\ge j_0}\sup_{0\le t\le 2}\Big(\langle t\rangle^\alpha\int_0^t e^{-c_0(t-\tau)}2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau\Big) \lesssim \mathcal{X}^2_p(2) + 2^{-2j_0}\mathcal{X}_p(2). \tag{4.31}$$

Now, assuming that $T \ge 2$, we turn to bound the supremum for $2 \le t \le T$ in the r.h.s. of (4.27). To this end, we split the integral on $[0, t]$ into integrals on $[0, 1]$ and $[1, t]$. The integral on $[0, 1]$ is not difficult to handle. Indeed, as $e^{-c_0(t-\tau)} \le e^{-c_0t/2}$ for $t \in [2, T]$ and $\tau \in [0, 1]$, one can write that

$$\sum_{j\ge j_0}\sup_{2\le t\le T}\Big(\langle t\rangle^\alpha\int_0^1 e^{-c_0(t-\tau)}2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau\Big)$$
$$\le \sum_{j\ge j_0}\sup_{2\le t\le T}\Big(\langle t\rangle^\alpha e^{-\frac{c_0}{2}t}\int_0^1 2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau\Big) \lesssim \int_0^1\sum_{j\ge j_0}2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau.$$

Along the same line of the estimate for (4.29), we conclude that

$$\sum_{j\ge j_0}\sup_{2\le t\le T}\Big(\langle t\rangle^\alpha\int_0^1 e^{-c_0(t-\tau)}2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau\Big) \lesssim \mathcal{X}^2_p(1) + 2^{-2j_0}\mathcal{X}_p(1). \tag{4.32}$$

In order to bound the $[1, t]$ part of the integral for $2 \le t \le T$, we observe that

$$\sum_{j\ge j_0}\sup_{2\le t\le T}\Big(\langle t\rangle^\alpha\int_1^t e^{-c_0(t-\tau)}2^{j(\frac{N}{p}-1)}S_j(\tau)d\tau\Big) \lesssim \sum_{j\ge j_0}2^{j(\frac{N}{p}-1)}\sup_{1\le t\le T}t^\alpha S_j(t). \tag{4.33}$$

Next, we are going to bound the terms in the r.h.s. of (4.33) one by one. Firstly, we claim that

$$\|\tau\nabla\mathbf{u}\|_{\widetilde L^\infty_t(\dot B^{\frac{N}{p}}_{p,1})} \lesssim \mathcal{Y}_p(t). \tag{4.34}$$

Indeed, as regards the high-frequencies $\|\tau\nabla\mathbf{u}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}$, it holds true from the definition of $\mathcal{Y}_p(t)$. And for the low-frequencies, applying Bernstein's inequalities yields

$$\begin{aligned}\|\tau\nabla\mathbf{u}\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})} &\lesssim \|\tau\mathbf{u}\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{2}+1}_{2,1})} \lesssim \|\tau\mathbf{u}\|^\ell_{L^\infty_t(\dot{B}^{\frac{N}{2}+1-2\varepsilon}_{2,1})}\\ &\lesssim \|\langle\tau\rangle^{\frac{s_0}{2}+\frac{N}{4}+\frac{1}{2}-\varepsilon}\mathbf{u}\|^\ell_{L^\infty_t(\dot{B}^{\frac{N}{2}+1-2\varepsilon}_{2,1})} \le \mathcal{Y}_p(t).\end{aligned}$$

To bound the terms with S^1_j and S^2_j in (4.33), we use the fact that

$$\sum_{j\ge j_0} 2^{j(\frac{N}{p}-1)} \sup_{1\le t\le T} t^\alpha(S^1_j(t)+S^2_j(t)) \lesssim \|t^\alpha(b\mathbf{u},\mathbf{g_1})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}.$$

For the term with $t^\alpha b\mathbf{u}$, we use that $t^\alpha b\mathbf{u} = t^\alpha b\mathbf{u}^h + t^\alpha b^h\mathbf{u}^\ell + t^\alpha b^\ell\mathbf{u}^\ell$. By the product laws adapted to tilde spaces (see (3.2)), it follows that

$$\begin{aligned}\|t^\alpha b\mathbf{u}^h\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} &\lesssim \|b\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|t^\alpha\mathbf{u}^h\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{X}_p(T)\mathcal{Y}_p(T),\\ \|t^\alpha b^h\mathbf{u}^\ell\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} &\lesssim \|t^\alpha b^h\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|\mathbf{u}^\ell\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{Y}_p(T)\mathcal{X}_p(T).\end{aligned}$$

Using Bernstein's inequality combined with embedding and product law (3.2) yields

$$\|t^\alpha b^\ell\mathbf{u}^\ell\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|t^\alpha b^\ell\mathbf{u}^\ell\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{2}-1}_{2,1})} \lesssim \|t^{\frac{\alpha}{2}}\Lambda^{-1}b^\ell\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{2}}_{2,1})}\|t^{\frac{\alpha}{2}}\mathbf{u}^\ell\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{2}}_{2,1})}.$$

From the definitions of tilde norms and $\mathcal{Y}_p$, we deduce that

$$\|t^{\frac{\alpha}{2}}(\Lambda^{-1}b^\ell,\mathbf{u}^\ell)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{2}}_{2,1})} \lesssim \|t^{\frac{\alpha}{2}}(\Lambda^{-1}b^\ell,\mathbf{u}^\ell)\|_{L^\infty_T(\dot{B}^{\frac{N}{2}-\varepsilon}_{2,1})} \le \mathcal{Y}_p(T). \tag{4.35}$$

Thus, we conclude that

$$\|t^\alpha(b\mathbf{u})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{Y}_p(T)(\mathcal{Y}_p(T)+\mathcal{X}_p(T)).$$

To bound the term with $\mathbf{u}\cdot\nabla\mathbf{u}$ in $\mathbf{g_1}$, we note that

$$\|t^\alpha(\mathbf{u}\cdot\nabla\mathbf{u})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|t^{\alpha-1}\mathbf{u}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}\|t\nabla\mathbf{u}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}.$$

On one hand, it is clear that $\|t^{\alpha-1}\mathbf{u}\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \le \mathcal{Y}_p(T)$. On the other hand, we have for small enough ε:

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$$\|t^{\alpha-1}\mathbf{u}\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|t^{\alpha-1}\mathbf{u}\|^{\ell}_{L^{\infty}_T(\dot{B}^{\frac{N}{2}-1-2\varepsilon}_{2,1})} \le \mathcal{Y}_p(T)$$

as $\alpha-1=\frac{N}{p}-\frac{1}{2}-\varepsilon=\frac{s_0}{2}+\frac{N}{4}-\frac{1}{2}-\varepsilon$. As a consequence, it follows from (4.34) that

$$\|t^{\alpha}(\mathbf{u}\cdot\nabla\mathbf{u})\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{Y}_p^2(T).$$

To bound the term with $k(b)\nabla b$, using inequality (3.2) again along with Proposition 3.4, we deduce

$$\|t^{\alpha}(k(b)\nabla b^{h})\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|b\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|t^{\alpha}b\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}}_{p,1})} \le \mathcal{X}_p(T)\mathcal{Y}_p(T), \tag{4.36}$$

$$\|t^{\alpha}(k(b)\nabla b^{\ell})\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|t^{\frac{\alpha}{2}}b\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|t^{\frac{\alpha}{2}}b\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{2}}_{2,1})} \le \mathcal{Y}_p^2(T), \tag{4.37}$$

where we have used the fact that according to (4.35), there hold

$$\|t^{\frac{\alpha}{2}}b\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{2}}_{2,1})} \lesssim \|t^{\frac{\alpha}{2}}\Lambda^{-1}b\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{2}}_{2,1})} \le \mathcal{Y}_p(T),$$

and then

$$\|t^{\frac{\alpha}{2}}b\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}}_{p,1})} \lesssim \|t^{\frac{\alpha}{2}}b\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{2}}_{2,1})} + \|t^{\frac{\alpha}{2}}b\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}}_{p,1})} \lesssim \mathcal{Y}_p(T). \tag{4.38}$$

To bound the term with $I(b)\mathcal{A}\mathbf{u}$, we have that

$$\|t^{\alpha}I(b)\mathcal{A}\mathbf{u}\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|t\nabla^2\mathbf{u}\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}\left(\|t^{\alpha-1}b\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{2}}_{2,1})} + \|t^{\alpha-1}b\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}}_{p,1})}\right).$$

The first term on the right-hand side has been estimated in (4.34), and it is obvious that the last term can also be bounded by $\mathcal{Y}_p(T)$. To proceed with the second one, we have for small enough ε:

$$\|t^{\alpha-1}b\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{2}}_{2,1})} \lesssim \|t^{\alpha-1}\Lambda^{-1}b\|^{\ell}_{L^{\infty}_T(\dot{B}^{\frac{N}{2}+1-2\varepsilon}_{2,1})} \le \mathcal{Y}_p(T) \tag{4.39}$$

as $\alpha-1=\frac{N}{p}-\frac{1}{2}-\varepsilon\le\frac{s_0}{2}+\frac{N}{4}+\frac{1}{2}-\varepsilon$. Thus, we arrive at

$$\|t^{\alpha}I(b)\mathcal{A}\mathbf{u}\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{Y}_p^2(T).$$

For the last term $\nabla\Delta^{-1}b$ of $\mathbf{g_1}$, we have

$$\|t^{\alpha}\nabla\Delta^{-1}b\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim 2^{-2j_0}\|t^{\alpha}b\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p}}_{p,1})} \le 2^{-2j_0}\mathcal{Y}_p(T).$$

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As regarding the term S_j^3, we have

$$\|t^\alpha \nabla(b\,\mathrm{div}\mathbf{u})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|t^\alpha(b\,\mathrm{div}\mathbf{u})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} \lesssim \|t^{\alpha-1}b\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|t\nabla\mathbf{u}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}.$$

Consequently, applying (4.34) along with (4.39) yields that

$$\|t^\alpha \nabla(b\,\mathrm{div}\mathbf{u})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{Y}_p^2(T).$$

As for the term S_j^4, one notes that a small modification of Proposition 3.5 (i.e., just treat the time variables as a parameter) gives rise to

$$\sum_{j\in\mathbb{Z}} 2^{j(\frac{N}{p}-1)} \sup_{1\le t\le T} t^\alpha \|R_j(t)\|_{L^p} \lesssim \|t\nabla\mathbf{u}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|t^{\alpha-1}\nabla b\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}. \tag{4.40}$$

Then from (4.34) and (4.39) again, we deduce that

$$\sum_{j\in\mathbb{Z}} 2^{j(\frac{N}{p}-1)} \sup_{1\le t\le T} t^\alpha \|R_j(t)\|_{L^p} \lesssim \mathcal{Y}_p^2(T).$$

The term with S_j^5 is obviously bounded by the r.h.s. of (4.40). Collecting all the above estimates together, we conclude that

$$\sum_{j\ge j_0} 2^{j(\frac{N}{p}-1)} \sup_{1\le t\le T} t^\alpha S_j(t) \lesssim \mathcal{X}_p^2(T) + \mathcal{Y}_p^2(T) + 2^{-2j_0}\mathcal{Y}_p(T). \tag{4.41}$$

Substituting (4.41) into (4.33), and recalling (4.27), (4.31) and (4.32), we derive that for all $t \ge 0$,

$$\|\langle\tau\rangle^\alpha(\nabla b,\mathbf{u})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \mathcal{X}_p^2(t) + \mathcal{Y}_p^2(t) + 2^{-2j_0}\mathcal{Y}_p(t).$$

At last, it follows from the Poisson equation that for all $t \ge 0$,

$$\begin{aligned}\|\langle\tau\rangle^\alpha\nabla\Delta\Phi\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} &\lesssim \|\langle\tau\rangle^\alpha\nabla b\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\ &\lesssim \|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \mathcal{X}_p^2(t) + \mathcal{Y}_p^2(t) + 2^{-2j_0}\mathcal{Y}_p(t),\end{aligned}$$

which completes the proof of Proposition 4.2. □

Step 3: Decay estimates with gain of regularity for the high frequencies of **u**. The following result gives the decay estimates with gain of regularity for the high frequencies of **u**.

Proposition 4.3. *Under the assumptions of Theorem 2.2, we have for all $t \geq 0$,*

$$\|\tau \mathbf{u}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}+1}_{p,1})} \lesssim \mathcal{X}_{p,0} + \mathcal{X}^2_p(t) + \mathcal{Y}^2_p(t) + 2^{-2j_0}\mathcal{Y}_p(t).$$

Proof. Note that

$$\partial_t \mathbf{u} - \mathcal{A}\mathbf{u} = \mathbf{G} \overset{\text{def}}{=} -\mathbf{u}\cdot\nabla\mathbf{u} + (k(b)-1)\nabla b - I(b)\mathcal{A}\mathbf{u} + \nabla\Delta^{-1}b.$$

This implies that

$$\partial_t(t\mathcal{A}\mathbf{u}) - \mathcal{A}(t\mathcal{A}\mathbf{u}) = \mathcal{A}\mathbf{u} + t\mathcal{A}\mathbf{G}.$$

It thus follows from Proposition 3.6 and Remark 3.1 that

$$\|\tau\nabla^2\mathbf{u}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|\mathcal{A}\mathbf{u}\|^h_{L^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} + \|\tau\mathcal{A}\mathbf{G}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-3}_{p,1})}.$$

Therefore, using the bounds given by Theorem 2.1, one has

$$\|\tau\mathbf{u}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}+1}_{p,1})} \lesssim \|\mathbf{u}\|^h_{L^1_t(\dot{B}^{\frac{N}{p}+1}_{p,1})} + \|\tau\mathbf{G}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{X}_{p,0} + \|\tau\mathbf{G}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}. \tag{4.42}$$

Next, we focus on the estimate of the last term. Firstly, product and composition estimates adapted to tilde spaces give

$$\|\tau k(b)\nabla b\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|\tau^{\frac{1}{2}}b\|^2_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}. \tag{4.43}$$

In view of the fact that $\|\tau^{\frac{1}{2}}b\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})} \leq \mathcal{Y}_p(t)$ and that for small enough ε,

$$\|\tau^{\frac{1}{2}}b\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})} \lesssim \|\tau^{\frac{1}{2}}b\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{2}}_{2,1})} \lesssim \|\tau^{\frac{1}{2}}\Lambda^{-1}b\|^\ell_{L^\infty_t(\dot{B}^{\frac{N}{2}+1-2\varepsilon}_{2,1})} \leq \mathcal{Y}_p(t),$$

then (4.43) becomes

$$\|\tau k(b)\nabla b\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{Y}^2_p(t).$$

Similarly,

$$\|\tau\mathbf{u}\cdot\nabla\mathbf{u}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim \|\mathbf{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\|\tau\nabla\mathbf{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})} \lesssim \mathcal{X}_p(t)\mathcal{Y}_p(t)$$

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and

$$\|\tau I(b)\mathcal{A}\mathbf{u}\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})} \lesssim \|b\|_{\widetilde{L}^\infty_t(\dot B^{\frac{N}{p}}_{p,1})}\|\tau\nabla^2\mathbf{u}\|_{\widetilde{L}^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})} \lesssim \mathcal{X}_p(t)\mathcal{Y}_p(t),$$

where in the last inequality, we have used the estimate (4.34). By Proposition 4.2 and the fact that $\alpha \geq 1$, we have

$$\|\tau\nabla b\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})} \lesssim \|\langle\tau\rangle^\alpha\nabla b\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})} \lesssim \|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}} + \mathcal{X}_p^2(t)+\mathcal{Y}_p^2(t)+2^{-2j_0}\mathcal{Y}_p(t).$$

Finally, for the term $\nabla\Delta^{-1}b$, it follows that

$$\|\tau\nabla\Delta^{-1}b\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})} \lesssim 2^{-2j_0}\|\langle\tau\rangle^\alpha\nabla b\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})} \leq 2^{-2j_0}\mathcal{Y}_p(t).$$

Thus, reverting to (4.42), we complete the proof of Proposition 4.3. □

Proof of Theorem 2.2. Propositions 4.1, 4.2 and 4.3 together imply that

$$\mathcal{Y}_p(T) \lesssim \mathcal{X}_{p,0}+\mathcal{Y}_{p,0}+\mathcal{X}_p^2(T)+\mathcal{Y}_p^2(T)+2^{-2j_0}\mathcal{Y}_p(T).$$

Since the fact that

$$\|b_0\|^\ell_{\dot B^{\frac{N}{2}-2}_{2,1}} \lesssim \|b_0\|^\ell_{\dot B^{-s_0-1}_{2,\infty}}, \qquad \|\mathbf{u}_0\|^\ell_{\dot B^{\frac{N}{2}-1}_{2,1}} \lesssim \|\mathbf{u}_0\|^\ell_{\dot B^{-s_0}_{2,\infty}},$$

and that Theorem 2.1 guarantees $\mathcal{X}_p \leq \mathcal{X}_{p,0} \ll 1$, we conclude that if $\mathcal{Y}_{p,0}$ and $\|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}}$ are small enough, and j_0 is large enough, then (2.6) is fulfilled for all time. This ends the proof of Theorem 2.2. □

Finally, we are devoted to the proof of Corollary 2.1 and Corollary 2.2, respectively.

Proof of Corollary 2.1. From the embedding, there holds

$$\sup_{t\in[0,T]}\langle t\rangle^{\frac{s_0+s+1}{2}}\|\Lambda^s b\|_{\dot B^0_{p,1}} \lesssim \|\langle t\rangle^{\frac{s_0+s+1}{2}}\Lambda^{-1}b\|^\ell_{L^\infty_T(\dot B^{s+1}_{2,1})} + \|\langle t\rangle^{\frac{s_0+s+1}{2}} b\|^h_{L^\infty_T(\dot B^s_{p,1})}.$$

Thanks to (2.6) and the definition of $\mathcal{Y}_p$, we see that

$$\|\langle t\rangle^{\frac{s_0+s+1}{2}}\Lambda^{-1}b\|^\ell_{L^\infty_T(\dot B^{s+1}_{2,1})} \lesssim \mathcal{Y}_{p,0}+\|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}} \text{ if } -s_0-1<s\leq\frac{N}{2},$$

and that, because $\alpha \geq \frac{s_0+s+1}{2}$ for all $s<\frac{N}{p}$,

$$\|\langle t\rangle^{\frac{s_0+s+1}{2}} b\|^h_{L^\infty_T(\dot B^s_{p,1})} \lesssim \mathcal{Y}_{p,0}+\|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}} \text{ if } s<\frac{N}{p},$$

which implies (2.8).

To bound the velocity $\mathbf{u}$, similar to the above, one infers

$$\sup_{t\in[0,T]} \langle t\rangle^{\frac{s_0+s}{2}}\|\Lambda^s\mathbf{u}\|_{\dot B^0_{p,1}} \lesssim \|\langle t\rangle^{\frac{s_0+s}{2}}\mathbf{u}\|^{\ell}_{L^\infty_T(\dot B^s_{2,1})} + \|\langle t\rangle^{\frac{s_0+s}{2}}\mathbf{u}\|^{h}_{L^\infty_T(\dot B^s_{p,1})}.$$

Here,

$$\|\langle t\rangle^{\frac{s_0+s}{2}}\mathbf{u}\|^{\ell}_{L^\infty_T(\dot B^s_{2,1})} \lesssim \mathcal{Y}_{p,0} + \|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}} \quad \text{if } -s_0 < s \le \frac{N}{2}+1,$$

and that, because $\alpha \ge \frac{s_0+s}{2}$ for all $s \le \frac{N}{p}-1$,

$$\|\langle t\rangle^{\frac{s_0+s}{2}}\mathbf{u}\|^{h}_{L^\infty_T(\dot B^s_{p,1})} \lesssim \mathcal{Y}_{p,0} + \|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}} \quad \text{if } s \le \frac{N}{p}-1,$$

which yield (2.9).

For the estimate of $\nabla\Phi$, we also have

$$\sup_{t\in[0,T]} \langle t\rangle^{\frac{s_0+s}{2}}\|\Lambda^s\nabla\Phi\|_{\dot B^0_{p,1}} \lesssim \|\langle t\rangle^{\frac{s_0+s}{2}}\nabla\Phi\|^{\ell}_{L^\infty_T(\dot B^s_{2,1})} + \|\langle t\rangle^{\frac{s_0+s}{2}}\nabla\Phi\|^{h}_{L^\infty_T(\dot B^s_{p,1})}.$$

As above, one has

$$\|\langle t\rangle^{\frac{s_0+s}{2}}\nabla\Phi\|^{\ell}_{L^\infty_T(\dot B^s_{2,1})} \lesssim \mathcal{Y}_{p,0} + \|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}} \quad \text{if } -s_0 < s \le \frac{N}{2}+1,$$

and that as $\alpha \ge \frac{s_0+s}{2}$ for all $s < \frac{N}{p}+1$, it follows that

$$\|\langle t\rangle^{\frac{s_0+s}{2}}\nabla\Phi\|^{h}_{L^\infty_T(\dot B^s_{p,1})} \lesssim \mathcal{Y}_{p,0} + \|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}} \quad \text{if } s < \frac{N}{p}+1.$$

This yields (2.10) and so far we complete the proof of Corollary 2.1. □

Proof of Corollary 2.2. We first recall the following Gagliardo–Nirenberg type inequality (see e.g., [1] and [36]):

$$\|\Lambda^\alpha f\|_{L^r} \lesssim \|\Lambda^\beta f\|^{1-\theta}_{L^q}\|\Lambda^\gamma\|^{\theta}_{L^q},$$

where $0\le\theta\le 1$, $1\le q\le r\le\infty$ and

$$\alpha + N\left(\frac{1}{q}-\frac{1}{r}\right) = \beta(1-\theta)+\gamma\theta.$$

We take $p=2$ in Corollary 2.1. Then, it follows from the above Gagliardo–Nirenberg inequality with $q=2$ and $\alpha=m$ that

$$\begin{aligned}\|\Lambda^m b\|_{L^r} &\lesssim \|\Lambda^\beta b\|_{L^2}^{1-\theta}\|\Lambda^\gamma b\|_{L^2}^{\theta} \lesssim \left(\mathcal{Y}_{2,0}+\|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac N2-1}_{2,1}}\right)\\ &\quad\times\left(\langle t\rangle^{-\frac N4-\frac\beta2-\frac12}\right)^{1-\theta}\left(\langle t\rangle^{-\frac N4-\frac\gamma2-\frac12}\right)^{\theta}\\ &=\left(\mathcal{Y}_{2,0}+\|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac N2-1}_{2,1}}\right)\langle t\rangle^{-\frac N4-\frac12-\frac\beta2(1-\theta)-\frac\gamma2\theta}\\ &=\left(\mathcal{Y}_{2,0}+\|(\nabla b_0,\mathbf{u}_0)\|^h_{\dot B^{\frac N2-1}_{2,1}}\right)\langle t\rangle^{\frac{N}{2r}-\frac N2-\frac m2-\frac12},\end{aligned}$$

here we used that

$$m+N\left(\frac12-\frac1r\right)=\beta(1-\theta)+\gamma\theta. \tag{4.44}$$

By Corollary 2.1 again, β and γ should satisfy $-\frac N2-1<\beta,\gamma<\frac N2$. This combined with $0\le\theta\le1$ and (4.44) yields that

$$-\frac N2-1<m+N\left(\frac12-\frac1r\right)<\frac N2$$

which gives (2.11). Similarly, we could prove (2.12) and the proof of Corollary 2.2 is finished. □

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