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Classification of solutions of elliptic equations arising from a gravitational $O(3)$ gauge field model

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Abstract

In this paper, we study an elliptic equation arising from the self-dual Maxwell gauged $O(3)$ sigma model coupled with gravity. When the parameter τ equals 1 and there is only one singular source, we consider radially symmetric solutions. There appear three important constants: a positive parameter a representing a scaled gravitational constant, a nonnegative integer N_1 representing the total string number, and a nonnegative integer N_2 representing the total anti-string number. The values of the products $aN_1, aN_2 \in [0, \infty)$ play a crucial role in classifying radial solutions. By using the decay rates of solutions at infinity, we provide a complete classification of solutions for all possible values of aN_1 and aN_2. This improves previously known results.

MSC: 35J61; 35Q75; 81T13

Keywords: Topological solutions; Nontopological solutions; Gravitational Maxwell gauged $O(3)$ sigma model

1. Introduction

In this paper, we are interested in the following elliptic equation in $\mathbb{R}^2$:

$$\Delta v - \rho_\tau(x) f_\tau(v, a, \varepsilon) = 4\pi \sum_{j=1}^{d_1} n_{j,1}\delta_{p_{j,1}} - 4\pi \sum_{j=1}^{d_2} n_{j,2}\delta_{p_{j,2}}, \tag{1.1}$$

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where

$$\rho_\tau(x) = \left[\prod_{j=1}^{d_1} |x - p_{j,1}|^{(1-\tau)n_{j,1}} \prod_{j=1}^{d_2} |x - p_{j,2}|^{(1+\tau)n_{j,2}} \right]^{-a},$$

$$f_\tau(v, a, \varepsilon) = \frac{2e^{\frac{(1-\tau)}{2}av}\Big[(1+\tau)e^v - (1-\tau)\Big]}{\varepsilon^2(1+e^v)^{1+a}}.$$

Here, $p_{j,1}$'s and $p_{j,2}$'s are disjoint points and called the strings and the antistrings, respectively. The unknown is

$$v : \mathbb{R}^2 \backslash \{p_{j,k} : j = 1, \cdots, d_k,\ k = 1, 2\} \to \mathbb{R}.$$

Furthermore, a is a nonnegative real number, $\tau \in [-1, 1]$ is a real number, δ_p denotes the Dirac measure concentrated at the point p, and $n_{j,k}$'s are positive integers representing the multiplicity of the strings and the antistrings $p_{j,k}$. We define the total string and anti-string numbers as

$$N_1 = n_{1,1} + \cdots + n_{d_1,1}, \quad N_2 = n_{1,2} + \cdots + n_{d_2,2}.$$

It is not difficult to check that v is a solution of (1.1) for τ if and only if $-v$ is a solution of (1.1) for $-\tau$ with the change of roles of $\{(p_{j,1}, n_{j,1}) : 1 \le j \le d_1\}$ and $\{(p_{j,2}, n_{j,2}) : 1 \le j \le d_2\}$. So, hereafter we assume that $0 \le \tau \le 1$.

The equation (1.1) arises from a self-dual gauge field model coupled with Einstein Equations. By taking into account a gravity in classical self-dual gauge field models, we need to solve the self-dual equations of models together with Einstein Equations. Recently, mathematical studies on these equations have grown up as interesting problems in various gauge filed theories [1–3,5, 9,14–16,19,20,23,25]. In particular, the equation (1.1) describes the Maxwell gauged $O(3)$ sigma model in the Bogomol'nyi regime on a space–time manifold. This model was introduced as an extension of Schroers' $U(1)$ Maxwell gauged harmonic map model [17,18] to a general relativity frame. The constant a stands for a scaled gravitational constant and the classical Schroers' model corresponds to the equation (1.1) with $a = 0$. For the detail background and derivation of (1.1), one may refer to [12,23–25].

From the physical motivation, it is natural to find solutions which gives the finite integration of the nonlinear term, that is $\rho_\tau(x) f_\tau(v, a, \varepsilon) \in L^1(\mathbb{R}^2)$. Then, the integrability condition yields three kinds of boundary conditions:

$$\begin{cases} \text{topological condition: } v(x) \ \to \ \sigma \in \mathbb{R} \quad \text{as} \quad |x| \to \infty, \\ \text{nontopological condition of type I: } v(x) \ \to \ -\infty \quad \text{as} \quad |x| \to \infty, \\ \text{nontopological condition of type II: } v(x) \ \to \ \infty \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{1.2}$$

Solutions for each boundary condition are called topological solutions and nontopological solutions of type I and II, respectively. We often say a type I (resp. type II) solution simply for a nontopological solution of type I (resp. type II). The nature of solutions of (1.1) varies drastically according to the value $\tau \in [0, 1]$. In particular, we have different features of solutions according to $\tau = 1$ or $0 \leq \tau < 1$. The purpose of this paper is to classify all possible solutions of (1.1) for the case $\tau = 1$.

One way to classify of solutions of (1.1) is to investigate the decay rates of solutions at infinity. Here, we mean by the decay rate the leading term of solutions at infinity which determines the asymptotics of solutions at infinity. By an elementary potential analysis, one can derive that if v is a solution of (1.1), then

$$v(x) = (2N_1 - 2N_2 + \beta)\ln|x| + o(\ln|x|) \quad \text{as} \quad |x| \to \infty, \tag{1.3}$$

where $(2N_1 - 2N_2 + \beta)$ is the decay rate of v and

$$\beta = \frac{1}{2\pi}\int_{\mathbb{R}^2} \rho_\tau(x) f_\tau\big(v(x), a, \varepsilon\big)\, dx.$$

By examining $\rho_\tau(x) f_\tau\big(v(x), a, \varepsilon\big)$ at infinity, we obtain necessary conditions on the range of β. Indeed, if v is a solution of (1.1) satisfying (1.3), then it comes from the condition $\rho_\tau f_\tau \in L^1$ that $\beta \in J \subset \mathbb{R}$ where J is given by the following Tables 1.1 and 1.2 [12]:

Table 1.1
The set J for $a = 0$.

	Topological	Type I	Type II
$0 \leq \tau < 1$	$\{2N_2 - 2N_1\}$	$\emptyset$	$\emptyset$
$\tau = 1$	$\emptyset$	$(0, 2N_2 - 2N_1 - 2]$	$\emptyset$

Table 1.2
The set J for $a > 0$.

	Topological	Type I	Type II
$0 \leq \tau < 1$	$\{2N_2 - 2N_1\}$	$\big(-\infty, \frac{4aN_2-4}{(1-\tau)a}\big]$	$\big[\frac{4-4aN_1}{a(1+\tau)}, \infty\big)$
$\tau = 1$	$\{2N_2 - 2N_1\}$	$(0, 2aN_2 + 2N_2 - 2N_1 - 2]$	$\big[\frac{2-2aN_1}{a}, \infty\big)$

A natural question is whether the above necessary conditions are also sufficient or not. In other words, given $\beta \in J$, are there any topological, type I or type II solutions realizing (1.3)? If any, how many solutions exist? Can we identify J exactly for topological, type I or type II solutions if a and τ are fixed? Classification of solutions by their decay rate β at infinity is one of important tools in various self-dual gauge model equations [2–16]. Regarding our problems, there are several known results in this direction. We briefly introduce them and state the main result of this paper and see how it improves the previous results.

First, let us consider the case $a = 0$ which corresponds to the situation of no gravity in the physics literature. If $0 < \tau < 1$, then we have only topological solutions and it was shown in [22] that (1.1) possesses a unique topological solution. When $\tau = 1$, only type I solutions are in consideration and it was proved in [13,21] that for each $\beta \in (0, 2N_2 - 2N_1 - 2)$ (1.1) has a type I solution satisfying

$$v(x) = (2N_1 - 2N_2 + \beta)\ln|x| + O(1) \quad \text{as} \quad |x| \to \infty. \tag{1.4}$$

It is not yet completely known for the borderline case $\beta = 2N_2 - 2N_1 - 2$ in Table 1.1 whether there is a solution or not. In this case, $v(x) = -2\ln|x| + o(\ln|x|)$ as $|x| \to \infty$. If $N_1 = 0$ and $N_2 \geq 2$, then a partial answer was given in [13] such that (1.1) allows a type I solution satisfying that

$$v(x) = -2\ln|x| - 2\ln\ln|x| + O(1) \quad \text{as} \quad |x| \to \infty. \tag{1.5}$$

Next, we turn to the case $a > 0$. There have been several works on the existence and properties of solutions of (1.1) for the cases $\tau = 0$ and $\tau = 1$. First, we consider the case $\tau = 0$. In [15,25], the authors proved that if

$$0 < a(N_1 + N_2) < 1, \tag{1.6}$$

there exists a topological solution of $(1.1)_{\tau=0}$ with $\sigma = 0$ for any $\varepsilon > 0$. In [12], the authors constructed nontopological multi-string solutions. Up to now the existence of solutions of (1.1) is proved under the condition (1.6), and the other cases are not solved yet.

On the other hand, for more efficient approach on the existence of solutions without the restriction (1.6), one may consider the simplest case $d_2 = 0$ and $p_{1,1} = \cdots = p_{d_1,1} = 0$ and study radially symmetric solutions $v(r,s)$ of

$$\begin{cases} v'' + \dfrac{1}{r}v' = r^{-aN_1} f_0(v, a, \varepsilon), \quad r = |x| > 0, \\ v(r,s) = 2N_1 \ln r + s + o(1) \quad \text{near} \quad r = 0, \\ v(x) = (2N_1 + \beta)\ln r + O(1) \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{1.7}$$

Then, for any possible values of $aN_1 \geq 0$ which is not restricted to (1.6), the authors in [9, 14] completely classified the range of β for which (1.7) allows topological, type I and type II solutions. See [14] more details. When $\tau = 0$, another simple case is that $d_1 = 0$ and $p_{1,2} = \cdots = p_{d_2,2} = 0$ and we have the following radial equation:

$$\begin{cases} v'' + \dfrac{1}{r}v' = r^{-aN_2} f_0(v, a, \varepsilon), \quad r = |x| > 0, \\ v(r,s) = -2N_2 \ln r + s + o(1) \quad \text{near} \quad r = 0, \\ v(x) = (-2N_2 + \beta)\ln r + O(1) \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{1.8}$$

This equation can be studied by the symmetry property of f_0. Indeed, since $f_0(-v) = -f_0(v)$, if v is a solution of (1.7) with the replacement of N_1 by N_2, then $-v$ is a solution of (1.8). If $0 < \tau < 1$, such a symmetry is broken and we have to consider (1.7) and (1.8) separately. Furthermore, there might happen some difficulty in analyzing for the case $0 < \tau < 1$ which is distinguished from the case $\tau = 0$. We will report this aspect elsewhere later.

Next, we consider the case $\tau = 1$ which is the main interest of this article. In this case, the situation is quite different from that for $\tau = 0$. For the full equation (1.1), we have the following known result.

Theorem A *([6,19]). If $N_2 \geq N_1 + 2$, then for each*

$$\beta \in (0,\ 2N_2 - 2N_1 - 2), \tag{1.9}$$

there exists $a_0 \in (0, 1/N_2)$ such that for all $0 < a < a_0$, (1.1) possesses a type I solution such that

$$v(x) = (-2N_2 + \beta)\ln|x| + O(1) \quad as \quad |x| \to \infty \tag{1.10}$$

Theorem A provides the existence of only type I solutions under some restrictions on a: smallness of a and the condition

$$0 < aN_2 < 1. \tag{1.11}$$

Moreover, by comparing (1.9) and Table 1.2, we cannot be sure whether the range (1.9) of β is sufficient and necessary for the existence of type I solutions. To enhance such conditions and identify the better range of β, one may the simplest cases that there exists only one string or antistring point: either $d_1 = 0$ and $p_{1,2} = \cdots = p_{d_2,2} = 0$, or $d_2 = 0$ and $p_{1,1} = \cdots = p_{d_1,1} = 0$. The former case corresponds to

$$\begin{cases} v'' + \dfrac{1}{r}v' = r^{-2aN_2} f_1(v, a, \varepsilon), & r = |x| > 0, \\ v(r) = -2N_2 \ln r + s + o(1) \quad \text{near} & r = 0, \end{cases} \tag{1.12}$$

and the latter case is

$$\begin{cases} v'' + \dfrac{1}{r}v' = f_1(v, a, \varepsilon), & r > 0, \\ v(r) = 2N_1 \ln r + s + o(1) \quad \text{near} & r = 0, \end{cases} \tag{1.13}$$

where

$$f_1(v, a, \varepsilon) = \frac{4e^v}{\varepsilon^2(1 + e^v)^{1+a}}.$$

Regarding the radial equations, we have the following known result about type II solutions.

Theorem B *([23]). If $N_2 \geq 1$ and (1.11) is valid, there exists one parameter family of type II solutions v of (1.12) satisfying (1.10) for some $\beta > 2N_2$. If $0 < a < 1$, then β satisfies*

$$\beta > 4N_2 + 4(1 - aN_2). \tag{1.14}$$

The result of Theorem B also has the restriction (1.11) and the range (1.14) of β may not be completely identified in view of Table 1.2. In this point of view, even for the radial cases (1.12) and (1.13), only few results are known for the case $\tau = 1$. The value aN_2 is very important in the classification of solutions. In particular, Theorem A and Theorem B provides some ranges of β only for the case $0 < aN_2 < 1$. The aim of this article is to identify the sufficient and necessary conditions of β for the existence of radial type I and type II solutions of (1.12) and (1.13) without any restriction on aN_2 and thus classify all possible solutions.

To state the main results of this paper, we set up some notations. First, regarding the equation (1.12), if we set $u(r) = v(r) + 2N_2 \ln r$, then

$$\begin{cases} u'' + \dfrac{1}{r}u' = \dfrac{4r^{2aN_2}e^u}{\varepsilon^2(r^{2N_2} + e^u)^{1+a}} =: g_1(u, a, \varepsilon), \quad r = |x| > 0, \\ u(r) = s + o(1) \quad \text{near} \quad r = 0. \end{cases} \tag{1.15}$$

We note that if u is a solution of (1.15) if and only if

$$u(r) = s + \int_0^r \frac{1}{t} \int_0^t y g_1\big(u(y), a, \varepsilon\big) dy dt.$$

Then, by applying the standard Picard iteration, one can see that (1.15) possesses a unique global solution for any values of $a, N_2 > 0$ and $s \in \mathbb{R}$. We denote by $u(r, s, a, N_2)$ the unique global solution of (1.15) and write $v(r, s, a, N_2) = -2N_2 + u(r, s, a, N_2)$. As mentioned earlier, we are interested in the solution satisfying that

$$\beta(s) = \beta(s, a, N_2) = \int_0^\infty r^{1-2aN_2} f_1\big(v(r, s, a, N_2), a, \varepsilon\big) dr \tag{1.16}$$

is finite. Indeed, it is shown later that given $a > 0$ and $N_2 \geq 0$, $\beta(s, a, N_2) < \infty$ for any $s \in \mathbb{R}$. By a standard argument, one can see that

$$v(r, s, a, N_2) = \big[-2N_2 + \beta(s, a, N_2)\big] \ln r + o(\ln r) \quad \text{as} \quad r \to \infty. \tag{1.17}$$

In the following, we will write $\beta(s)$, $v(r, s)$ or $v(r)$ for simplicity as long as there is no confusion. The first main result of this paper is the following theorem providing a complete classification of solutions of (1.12) without any restrictions on the value aN_2 and identifying the exact range of β.

Theorem 1.1. *Let a be a positive real number, N_2 be a positive integer, and $v(r, s)$ be a solution of* (1.12).

(i) *If $0 < aN_2 < 1$, then there exists a unique $s_* = s_*(a, N_2)$ such that the following hold.*
  (i-a) *If $s > s_*$, then $v(r, s)$ is a type II solution satisfying*

$$v(r, s) = [-2N_2 + \beta(s)] \ln r + I + O(r^{2-a\beta}) \tag{1.18}$$

for some constant $I = I(a, N_2, s)$ as $r \to \infty$. The function $\beta : (s_*, \infty) \to (4/a, \infty)$ is continuous, bijective, and strictly decreasing.

(i-b) *If $s \le s_*$, then $v(r,s)$ is a type I solution. The function $\beta : (-\infty, s_*] \to (0, 2aN_2 + 2N_2 - 2]$ is continuous, bijective, and strictly increasing. For $s < s_*$, we have*

$$v(r,s) = [-2N_2 + \beta(s)]\ln r + I + O(r^{2-2aN_2-2N_2+\beta}) \tag{1.19}$$

for some constant $I = I(a, N_2, s)$ as $r \to \infty$. If $s = s_*$, then

$$v(r,s) = [2aN_2 - 2]\ln r - 2\ln\ln r + O(1) \tag{1.20}$$

as $r \to \infty$.

(ii) *Suppose that $aN_2 = 1$.*

(ii-a) *If $0 < N_2 < 2\varepsilon^{-2}$, then for any $s \in \mathbb{R}$, $v(r,s)$ is a type II solution such that $\beta(s) = 4N_2$. Moreover,*

$$v(r,s) = 2N_2 \ln r + I + O(r^{-2})$$

for some constant $I = I(a, N_2, s)$ as $r \to \infty$.

(ii-b) *If $N_2 \ge 2\varepsilon^{-2}$, then for any $s \in \mathbb{R}$, $v(r,s)$ is a type I solution of (1.12) such that $\beta(s) = 2N_2 - 2\sqrt{N_2(N_2 - 2\varepsilon^{-2})}$. Moreover, if $N_2 > 2\varepsilon^{-2}$, then as $r \to \infty$,*

$$v(r,s) = -2\sqrt{N_2(N_2 - 2\varepsilon^{-2})}\ln r + I + O\left(r^{-2\sqrt{N_2(N_2-2\varepsilon^{-2})}}\right)$$

for some constant $I = I(a, N_2, s)$. If $N_2 = 2\varepsilon^{-2}$, then

$$v(r,s) = -2\ln\ln r + O(1) \quad \text{as} \quad r \to \infty.$$

(iii) *If $1 < aN_2 < 2$, then there exists a unique $s_* = s_*(a, N_2)$ such that the following hold.*

(iii-a) *For $s < s_*$, $v(r,s)$ is a type II solution of (1.12) and satisfies (1.18) The function $\beta : (\infty, s_*) \to (2N_2, 4/a)$ is continuous, bijective and strictly decreasing.*

(iii-b) *For $s > s_*$, $v(r,s)$ is a type I solution of (1.12) and satisfies (1.19) The function $\beta : (s_*, \infty) \to (0, 2N_2)$ is continuous, bijective and strictly decreasing.*

(iii-c) *For $s = s_*$, $v(r, s_*)$ is a topological solution of (1.12) and satisfies that*

$$v(r, s_*) = I_* + O(r^{2-2aN_2}) \quad \text{for some } I_* \text{ as } r \to \infty.$$

In particular, $\beta(s_*) = 2N_2$.

(iv) *If $aN_2 \ge 2$, then $v(r,s)$ is a type I solution of (1.12) for every $s \in \mathbb{R}$ and satisfies (1.19). The function $\beta : \mathbb{R} \to (0, 4/a)$ is continuous, bijective and strictly decreasing.*

Now, let us turn to the second problem (1.13). As for the equation (1.12), we have a global unique solution $v(r,s) = v(r,s,a,N_1)$ for any $s \in \mathbb{R}$. With the same notation

$$\beta(s) = \beta(s,a,N_1) = \int_0^\infty r f_1\big(v(r,s,a,N_1), a, \varepsilon\big) dr,$$

if $v(r,s)$ is a solution of (1.13), then

$$v(r,s) = \left[2N_1 + \beta(s)\right]\ln r + O(1) \quad \text{as} \quad r \to \infty. \tag{1.21}$$

The second main result of this paper is the following.

Theorem 1.2. *Let a be positive real number, N_1 be a nonnegative integer, and $v(r,s)$ be a solution of (1.13). Then, $v(r,s)$ is a type II solution for any $s \in \mathbb{R}$. The function*

$$\beta : (-\infty, \infty) \to \left(\max\left\{0, \frac{4(1-aN_1)}{a}\right\}, \infty\right)$$

is bijective and strictly decreasing. Furthermore,

$$v(r,s) = [2N_1 + \beta(s)]\ln r + I + O(r^{2+2aN_1-a\beta})$$

for some constant $I = I(a, N_1, s)$ as $r \to \infty$.

The proof of Theorem 1.1 is given in Section 2 and Section 3. The proof of Theorem 1.2 is given in Section 4.

We point out some important features of Theorem 1.1 and Theorem 1.2. First, Theorem 1.1 provides the possible range of antistring number N_2 in terms of a for the existence of type I and type II solutions. We give a complete description of radial solutions by classifying all the possible solutions. Second, Theorem 1.1 improves previous results on the range of β. The statement (i-a) says that if $0 < aN_2 < 1$, the optimal lower bound of β is $4/a$ for type II solutions, which improves (1.14) since $4/a > 4N_2 + 4(1 - aN_2)$ for $0 < a < 1$. Moreover, statement (i-a) tells us that if $0 < aN_2 < 1$, for each $\beta \in (0, 2aN_2 + 2N_2 - 2]$ we have type II solutions satisfying (1.17). This enhances the range of β in (1.9). Third, Theorem 1.1 also provides us the existence and nonexistence of type I and type II solutions for the case $aN \geq 1$. There have been known no results on the existence of solutions of (1.12) for the case $aN_2 \geq 1$ as far as we know. Thus, Theorem 1.1 exhibits a new result about the radial solutions in this case and may give an insight for the possible values of β for multi-string solutions of (1.1). In particular, we see that if $1 < aN_2 < 2$, then (1.12) possesses a topological solution which cannot be observed if $a = 0$. So, this gives us another difference between the case $a = 0$ and $a > 0$. In the physical literature, a represents Newton's gravitational constant and thus the existence of topological solutions manifests the effect of gravity in the underlying space–time manifold where the Maxwell gauged $O(3)$ sigma model is considered. Fourth, Theorem 1.2 gives a new result about the radial case when $N_1 \geq 0$ and $N_2 = 0$. Finally, it is an interesting open question to find type I and II multi-string solutions of (1.1) which satisfies the behaviors (1.4) and the decay rate β assumes the ranges in Theorem 1.1 and Theorem 1.2.

Here, we give outline of this paper and explain ideas for proof. In section 2, we consider the case $0 < aN_2 < 1$ and focus on the proof of Theorem 1.1 (i). We classify all possible solutions by using the decay rate β. Main tools for proof are standard shooting methods, Pohozaev identities, and Sturm–Liouville type comparison arguments. In particular, we deal with a generalized version of the equation (1.12) by allowing N_2 to be a positive real number. This setup will be very useful in constructing radial solutions of (1.12) when $1 < aN_2 < 2$. In section 3, we prove the other parts of Theorem 1.1, namely study (1.12) for the case $aN_2 \geq 1$. One of the main difficulty for this case is that r^{1-2aN_2} is not integrable near $r = 0$. So, in estimating an integral of the nonlinear term $r^{1-2aN_2} f_1(v, a)$ on a bounded interval including $r = 0$, we cannot extract the nonlinear term f_1 by bounding it by its sup norm. Moreover, in integrating by parts by multiplying

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(1.12) by rv', which is a very standard idea, it is inevitable to face the term $(2 - 2aN_2)$. Various arguments in Section 2 for the case $0 < aN_2 < 1$ depends on the positiveness of $(2 - 2aN_2)$ and thus are no longer valid for the case $aN_2 \geq 1$. To overcome such difficulties, we consider the transformation $\hat{v}(r, \hat{s}) = v(r^{-1}, s)$. Then, $v(r, s)$ is a solution of (1.12) with $1 < aN_2 < 2$ if and only if $\hat{v}(r, \hat{s})$ is a solution of

$$\hat{v}'' + \frac{1}{r}\hat{v}' = r^{-2\hat{a}\hat{N}_2} f_1(\hat{v}, a), \quad r > 0,$$
$$\hat{v}(r, \hat{s}) = \hat{s} + o(1), \quad \text{near} \quad r = 0,$$

with $0 < \hat{a}\hat{N}_2 < 1$. Here, $\hat{s}, \hat{a}, \hat{N}_2$ are suitably chosen. Then, from the result of Section 2, we can obtain the existence and properties of solutions $\hat{v}(r, \hat{s})$ of the transformed equation and then interpret them for $v(r, s)$. For $aN_2 > 2$, we use a similar argument by using a modified version of (1.13) as a transformed equation. Finally, in section 4, we prove Theorem 1.2. We get results for a generalized version the equation (1.13) which are used in Section 3 as transformed equations in several different places.

2. The case $0 < aN_2 < 1$ and $N_1 = 0$

In this section we prove Theorem 1.1 when a is a positive real number, N_2 is a positive integer, and $0 < aN_2 < 1$. In other words, we prove Theorem 1.1 (i). For later use, it deserves to introduce a generalized version of (1.12) as follows. Given *real* numbers a, b, N satisfying that

$$a, b > 0, \quad N > 0, \quad 0 < aN < 1, \tag{2.1}$$

we consider the following equation

$$\begin{cases} v'' + \dfrac{1}{r}v' = r^{-2aN} f_1(v, b, \varepsilon), & r > 0, \\ v(r) = -2N \ln r + s + o(1) & \text{near} \quad r = 0. \end{cases} \tag{2.2}$$

This generalized version will be used in the proof for the case $1 < aN_2 < 2$ in the next section. By Picard's iteration argument, it is not difficult to see that (2.2) allows a unique global solution for each $s \in \mathbb{R}$. We denote by $v(r, s, a, N, b)$ the unique global solution of (2.2). We also define

$$\beta(s, a, N, b) = \int_0^\infty r^{1-2aN} f_1\big(v(r, s, a, N, b), b, \varepsilon\big) dr \ > \ 0.$$

We want to find solutions such that $\beta(s, a, N, b) < \infty$. In fact, for every $s \in \mathbb{R}$, β is finite. See Lemma 2.2 below. Then, by a standard argument, it holds that

$$v(r, s, a, N, b) = \big[-2N + \beta(s, a, N, b)\big] \ln r + o(\ln r) \quad \text{as} \quad r \to \infty. \tag{2.3}$$

If there is no confusion, we will often write $v(r, s)$, $\beta(s)$ and $f(v, b)$ instead of $v(r, s, a, N, b)$, $\beta(s, a, N, b)$, $f(v, b, \varepsilon)$ and so on. One can easily check that v is a solution of (2.2) if and only if

$$v(r,s) = -2N\ln r + s + \int_0^r \frac{1}{t}\int_0^t y^{1-2aN} f_1\big(v(y,s),b\big)dydt. \tag{2.4}$$

We will often use this formula. In this section, we want to classify all solutions according to the shooting parameter s and the asymptotic decay rate $\beta(s)$ at infinity. Proposition 2.1 gives us the answer for this question.

Proposition 2.1. *Let a, b, N be positive real numbers such that $0 < aN < 1$.*

(i) *If $0 < aN \le 1 - N$, then $v(r,s)$ is a type II solution for all $s \in \mathbb{R}$. The function $\beta : (-\infty,\infty) \to (\bar{\beta}_{a,N,b},\infty)$ is continuous, onto, and strictly decreasing, where*

$$\bar{\beta}_{a,N,b} = \frac{4(1+bN-aN)}{b}.$$

In addition,

$$v(r,s) = [-2N + \beta(s)]\ln r + I + O(r^{2-2aN+2bN-b\beta}) \tag{2.5}$$

for some constant $I = I(a,N,b,s)$ as $r \to \infty$.

(ii) *If $1 - N < aN < 1$, then there exists a unique number $s_* = s_*(a,N)$ such that the following hold.*
  - (ii-a) *If $s > s_*$, then $v(r,s)$ is a type II solution satisfying (2.5). The function $\beta : (s_*,\infty) \to (\bar{\beta}_{a,N,b},\infty)$ is continuous, onto, and strictly decreasing.*
  - (ii-b) *If $s \le s_*$, then $v(r,s)$ is a type I solution. The function $\beta : (-\infty, s_*] \to (0, 2aN + 2N - 2]$ is continuous, onto, and strictly increasing. For $s < s_*$, we have*

$$v(r,s) = [-2N + \beta(s)]\ln r + I + O(r^{2-2aN-2N+\beta}) \tag{2.6}$$

for some constant $I = I(a,N,s)$ as $r \to \infty$. If $s = s_*$, then

$$v(r,s) = [2aN - 2]\ln r - 2\ln\ln r + O(1) \tag{2.7}$$

as $r \to \infty$.

We note that if N is a positive integer with $0 < aN < 1$, then we are led to the case (ii) in Proposition 2.1. Thus, letting $b = a$, we have the result of Theorem 1.1 (i). The basic strategy of the proof of Proposition 2.1 is the standard argument of shooting methods and Sturm–Liouville type comparison arguments as in [7,10,11,14]. The verification of each statement of Proposition 2.1 is carried out by a series of Lemmas below.

According to the shooting parameter $s \in \mathbb{R}$, we obtain two different kinds of solutions of (1.12) as follows. Let us define

$$\mathcal{A}^+ = \left\{ s \in \mathbb{R} \mid v'(r_0,s) = 0 \ \text{for some} \ r_0 > 0 \right\},$$
$$\mathcal{A}^- = \left\{ s \in \mathbb{R} \mid v'(r,s) < 0 \ \text{for all} \ r > 0 \right\}.$$

Since rv' is increases with respect to $r > 0$, for each $s \in \mathcal{A}^+$ there exists a unique point $z(s)$ such that $v'(z(s), s) = 0$. Since $v''(z(s), s) > 0$, it comes from the Implicit Function Theorem that $z(s)$ is C^1 and $\mathcal{A}^+$ is open. Thus, $\mathcal{A}^- = \mathbb{R} \setminus \mathcal{A}^+$ is closed. We also denote by $m(s)$ the minimum of $v(r, s)$ for $s \in \mathcal{A}^+$, namely,

$$m(s) = v(z(s), s) = \min_{r>0} v(r, s) \quad \text{for} \quad s \in \mathcal{A}^+.$$

We note that $v(r, s)$ is a type II (type I, resp.) solution of (1.12) for $s \in \mathcal{A}^+ (s \in \mathcal{A}^-$, resp.). Indeed, suppose that $s \in \mathcal{A}^+$. Since $v(r, s)$ is increasing for $r > z(s)$, there exists $\lim_{r\to\infty} v(r, s) = \lambda \in (-\infty, \infty]$. Suppose $\lambda < \infty$. Then, there exists $r_0 > 0$ such that $\lambda - 1 < v(r, s) < \lambda$ for all $r > r_0$. Thus, $(rv')' \geq c_0 r^{1-2aN}$ for some constant $c_0 > 0$. Integrating this inequality twice on $[r_0, r]$, we obtain

$$\begin{aligned} v(r, s) \geq v(r_0, s) + \left\{ r_0 v'(r_0, s) - \frac{c_0 r_0^{2-2aN}}{2-2aN} \right\} \ln\left(\frac{r}{r_0}\right) + \\ \frac{c_0}{(2-2aN)^2}(r^{2-2aN} - r_0^{2-2aN}). \end{aligned} \tag{2.8}$$

Letting $r \to \infty$, we see that $v(r, s) \to \infty$, a contradiction. Hence, $v(r, s) \to \infty$ as $r \to \infty$, which implies that $v(r, s)$ is a type II solution. Similarly, if $s \in \mathcal{A}^-$, then $v(r, s) \to \alpha \in [-\infty, \infty)$ as $r \to \infty$. If $\alpha > -\infty$, then $f_1(v, a)$ is bounded below by a positive constant at infinity such that (2.8) is still valid for a different value of c_0. Hence, $v(r, s) \to \infty$ as $r \to \infty$, a contradiction. So, for all $s \in \mathcal{A}^-$, $v(r, s) \to -\infty$ as $r \to \infty$.

We remark that the above argument may not hold for the case $aN \geq 1$ since the right-hand side of (2.8) might not blow up to ∞. In other words, for $s \in \mathcal{A}^+$ (resp. $s \in \mathcal{A}^-$), it might happen that $v(r, s) \nearrow \sigma$ (resp. $v(r, s) \searrow \sigma$) for some number $\sigma \in \mathbb{R}$. In this case, $v(r, s)$ corresponds to a topological solution. This indeed happens for the case $1 < aN < 2$. See the next section.

Let $\varphi(r, s, a, N, b) = \partial v(r, s, a, N, b)/\partial s$ be the unique solution of

$$\begin{cases} \varphi'' + \dfrac{1}{r}\varphi' = r^{-2aN} f_1'\big(v(r, s), b\big)\varphi, \\ \varphi(0, s) = 1, \qquad \varphi'(0, s) = 0, \end{cases} \tag{2.9}$$

where

$$f_1'(v, b) = \frac{4e^v(1 - be^v)}{\varepsilon^2(1 + e^v)^{2+b}}.$$

We denote $\varphi(r, s) = \varphi(r, s, a, N, b)$ for simplicity if there is no risk of confusion. We also define $w_c(r, s) = rv'(r, s) + c$ for $c \in \mathbb{R}$. Then, w_c satisfies

$$\begin{cases} w_c'' + \dfrac{1}{r}w_c' = r^{-2aN} f_1'\big(v(r, s), b\big)w_c + r^{-2aN}\Phi_c(r, s), \\ w_c(0, s) = -2N + c, \qquad w_c'(0, s) = 0, \end{cases} \tag{2.10}$$

where

$$\begin{aligned}\Phi_c(r,s) &= (2-2aN)f_1\big(v(r,s),b\big) - cf_1'\big(v(r,s),b\big) \\ &= \frac{4e^{v(r,s)}\big\{\big[(2-2aN)+cb\big]e^{v(r,s)} + \big[(2-2aN)-c\big]\big\}}{\varepsilon^2(1+e^{v(r,s)})^{2+b}}.\end{aligned} \tag{2.11}$$

We note that

$$\text{if } v'(r,s) \le 0, \text{ then } \varphi(r,s) > 0. \tag{2.12}$$

Otherwise, let r_0 be the first zero of φ such that $v'(r_0,s) \le 0$. Then, comparing φ and w_0, we obtain

$$0 \ge -r_0 w_0(r_0,s)\varphi'(r_0,s) = \int_0^{r_0} (2-2aN)r^{1-2aN} f_1(v,b)\varphi > 0,$$

which leads to a contradiction. A simple consequence of (2.12) is the following:

$$\text{if } s \in \mathcal{A}^+, \text{ then } m(s) \text{ is increasing.} \tag{2.13}$$

Indeed, we have

$$m'(s) = v'\big(z(s),s\big)z'(s) + \varphi\big(z(s),s\big) = \varphi\big(z(s),s\big) \ge 0.$$

Lemma 2.2. *For all $s \in \mathbb{R}$, $\beta(s)$ is finite and continuous on $\mathcal{A}^+$ and $\mathcal{A}^-$. Moreover,*

$$\begin{cases} \beta(s) \ge [2+2(b-a)N]/b & \text{for} \quad s \in \mathcal{A}^+, \\ 0 < \beta(s) \le 2aN + 2N - 2 & \text{for} \quad s \in \mathcal{A}^-. \end{cases} \tag{2.14}$$

In particular, if $0 < aN \le 1 - N$, then $\mathcal{A}^+ = \mathbb{R}$.

Proof. Since $rv'(r,s)$ is monotone, there exists

$$-\infty < c_s := \lim_{r\to\infty} rv'(r,s) = -2N + \int_0^{\infty} r^{1-2aN} f_1(v(r,s),b)dr = -2N + \beta(s)$$

for all $s \in \mathbb{R}$. If $c_s = \infty$, then $r^{1-2aN} f_1(v(r,s),b)$ is integrable on $\mathbb{R}$. So $\beta(s)$ is finite which implies that $c_s < \infty$, a contradiction. Hence, $|c_s| < \infty$. Moreover, by the integrability condition, $1 - 2aN + 2bN - b\beta \le -1$ for any $s \in \mathcal{A}^+$. Thus we have $\beta(s) \ge [2 + 2(b-a)N]/b$ for $s \in \mathcal{A}^+$. Similarly, if $s \in \mathcal{A}^-$, then $0 < \beta(s) \le 2aN + 2N - 2$ which implies that $aN > 1 - N$. In particular, if $0 < aN \le 1 - N$, then $\mathcal{A}^- = \emptyset$. The continuity of β is a simple consequence of the Lebesgue Convergence Theorem. □

Remark 2.3. In the proof of Lemma 2.2, to show the finiteness of $\beta(s)$, we only used the integrability condition in the contradiction argument. So, the range (2.14) hold true for any values of $a, b, N > 0$, and $s \in \mathbb{R}$.

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Lemma 2.4. *Suppose that $1-N<aN<1$. Then, there exists a number $s_*=s_*(a,N,b)\in\mathbb{R}$ such that $\mathcal{A}^-=(-\infty,s_*]$ and $\mathcal{A}^+=(s_*,\infty)$.*

Proof. First, we claim that $s\in\mathcal{A}^+$ for $s\gg 1$. Otherwise, there exists a sequence $s_n\to\infty$ such that $s_n\in\mathcal{A}^-$. Let r_n, R_n be unique numbers such that $v(r_n,s_n)=1$ and $v(R_n,s_n)=0$. Obviously, $r_n<R_n$. Since $1\ge -2N\ln r_n+s_n$ by (2.4), $r_n\to\infty$ as $s_n\to\infty$. Since

$$rv'(r,s)\ge -2N \quad \text{for all} \quad r>0, \tag{2.15}$$

integrating this inequality on $[r_n,R_n]$, we obtain that $(r_n/R_n)\le\exp(-1/2N)$. Then it follows that

$$\begin{aligned}2N\ge R_nv'(R_n,s_n)-r_nv'(r_n,s_n)&=\int_{r_n}^{R_n} r^{1-2aN}f_1\big(v(r,s_n),b\big)dr\\&\ge\frac{R_n^{2-2aN}}{2-2aN}\cdot\left[1-\left(\frac{r_n}{R_n}\right)^{2-2aN}\right]\cdot\Big(\inf_{0\le v\le 1}f_1(v,b)\Big)\\&\ge\frac{R_n^{2-2aN}}{2-2aN}\cdot\Big(1-e^{-\frac{2-2aN}{2N}}\Big)\cdot\Big(\inf_{0\le v\le 1}f_1(v,b)\Big)\quad\to\quad\infty\end{aligned}$$

as $s_n\to\infty$, which yields a contradiction.

Next, we claim that $s\in\mathcal{A}^-$ for $s\ll -1$. Assume the contrary. Then, there exists a sequence $s_n\to-\infty$ such that $s_n\in\mathcal{A}^+$. Since

$$v(1,s_n)=s_n+\int_0^1\frac{1}{r}\int_0^r y^{1-2aN}f_1\big(v(y,s_n),b\big)\,dydr=s_n+O(1),$$

we have that $v(1,s_n)\to-\infty$ as $s_n\to-\infty$. Let $t_n<1$ be the first point such that $v(t_n,s_n)=v(1,s_n)/2$. Then, integrating (2.15) on $[t_n,1]$, we see that $v(1,s_n)/2\ge 2N\ln t_n$, which implies that $t_n\to 0$. Moreover, it holds that

$$\begin{aligned}v'(1,s_n)+2N&=\int_0^1 r^{1-2aN}f_1(v,a)dr=\int_0^{t_n}+\int_{t_n}^1\\&\le\frac{t_n^{2-2aN}\|f_1\|_\infty}{2-2aN}+\frac{4e^{v(1,s_n)/2}(1-t_n^{2-2aN})}{\varepsilon^2(2-2aN)}\quad\to\quad 0,\end{aligned}$$

which tells us that $v'(1,s_n)\to-2N$ as $s_n\to-\infty$. In particular, $z(s_n)>1$ for all large n. Since $1-N<aN<1$, we can choose $\delta\in(2-2aN-N,N)$ and $T_n\in(1,z(s_n))$ such that $T_nv'(T_n,s_n)=-(N+\delta)$. Since $rv'(r,s_n)<-(N+\delta)$ for $r\in(1,T_n)$, we have $e^{v(r,s_n)}<e^{v(1,s_n)}r^{-(N+\delta)}$ for $r\in(1,T_n)$. Then it follows that

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$$
\begin{aligned}
0 < N - \delta + o(1) = T_n v'(T_n, s_n) - v'(1, s_n) &= \int_1^{T_n} r^{1-2aN} f_1(v, a) dr \\
&\le \frac{4}{\varepsilon^2} \int_1^{T_n} r^{1-2aN} e^{v(r,s_n)} dr \le \frac{4e^{v(1,s_n)}}{\varepsilon^2} \int_1^{\infty} r^{1-(2a+1)N-\delta} dr \quad \to \quad 0
\end{aligned}
$$

as $s_n \to -\infty$, a contradiction.

Finally, we show that both $\mathcal{A}^+$ and $\mathcal{A}^-$ are infinite intervals. It suffices to show that if $(s_1, s_2) \subset \mathcal{A}^+$ is a finite interval, then $s_2 \in \mathcal{A}^+$. Fix a number $s_0 \in (s_1, s_2)$. Then, by (2.13), $m(s) \ge m(s_0)$ for all $s \ge s_0$. So, $v(r, s) \ge m(s_0)$ for all $r > 0$ and $s \ge s_0$. Since $v(r, s) \to v(r, s_2)$ locally uniformly on $\mathbb{R}$, we conclude that $\inf_{r>0} v(r, s_2) \ge m(s_0)$ and thus $s_2 \in \mathcal{A}^+$. Now we finish the proof by letting $s_* = \inf\{s : s \in \mathcal{A}^+\}$. □

Lemma 2.5. *We have*

$$
\begin{cases}
\lim\limits_{s\to\infty} m(s) = \infty, & \textit{for} \quad 0 < aN < 1, \\
\lim\limits_{s\searrow s_*} m(s) = -\infty, & \textit{for} \quad 1 - N < aN < 1, \\
\lim\limits_{s\to-\infty} m(s) = -\infty, & \textit{for} \quad 0 < aN \le 1 - N.
\end{cases}
$$

Proof. For the first limit, by (2.13) it suffices to show that $\sup_{s\in\mathcal{A}^+} m(s) = \infty$. Assume to the contrary such that $\xi = \sup_{s\in\mathcal{A}^+} m(s) < \infty$. Choose an increasing sequence $s_n \to \infty$ and let $z_n = z(s_n)$. By (2.4),

$$
\xi \ge m(s_n) = v(z_n, s_n) \ge -2N \ln z_n + s_n.
$$

Hence, $z_n \to \infty$. Let $r_n \in (0, z_n)$ be the unique number such that $v(r_n, s_n) = m(s_n) + 1$. As in the proof of Lemma 2.4, we have $(r_n/z_n) \le \exp(-1/2N)$. Since $\xi - 1 \le m(s_n) \le v(r, s_n) \le \xi + 1$ for $r_n < r < z_n$ and large n, it follows that

$$
\begin{aligned}
2N &\ge \int_{r_n}^{z_n} r^{1-2aN} f_1\big(v(r, s_n), b\big) dr \\
&\ge \frac{z_n^{2-2aN}}{2 - 2aN} \cdot \big(1 - e^{-\frac{2-2aN}{2N}}\big) \cdot \Big(\inf_{\xi-1\le v\le\xi+1} f_1(v, b)\Big) \quad \to \quad \infty
\end{aligned}
$$

as $n \to \infty$. This gives us a contradiction.

The second limit comes from the continuous dependence of solutions on the shooting parameter s. Indeed, by a standard argument one can show that $v(r, s) \to v(r, s_*)$ locally uniformly on $(0, \infty)$ as $s \searrow s_*$. Since $v(r, s_*) \to -\infty$ as $r \to \infty$, the result follows. The third limit comes from the observation that by (2.4), $v(1, s) = s + O(1) \to -\infty$ as $s \to -\infty$. □

We introduce two functions:

$$\begin{cases} G(v,b) = -\dfrac{4}{b\varepsilon^2(1+e^v)^b}, \\ H(v,b) = G(v,b) + \dfrac{4}{b\varepsilon^2} = \dfrac{4}{b\varepsilon^2}\left\{1 - \dfrac{1}{(1+e^v)^b}\right\}. \end{cases} \tag{2.16}$$

Then, $\partial G/\partial v = \partial H/\partial v = f_1(v,b)$, and

$$G(v,b) = -\frac{1}{b} f_1(v,b) - \frac{4}{b\varepsilon^2(1+e^v)^{1+b}},$$
$$H(v,b) = -\frac{1}{b} f_1(v,b) + \frac{4}{b\varepsilon^2}\left\{1 - \frac{1}{(1+e^v)^{1+b}}\right\}.$$

Multiplying (2.2) by $rv'(r,s)$, we obtain the following identities: for $s \in \mathcal{A}^+$,

$$\begin{aligned} E(r,s) &:= \frac{1}{2}\left(rv'\right)^2 - 2N^2 - r^{2-2aN} G(v(r,s),b) \\ &= -(2-2aN)\int_0^r t^{1-2aN} G\big(v(t,s),b\big)dt, \end{aligned} \tag{2.17}$$

and for $s \in \mathcal{A}^-$,

$$\begin{aligned} F(r,s) &:= \frac{1}{2}\left(rv'\right)^2 - 2N^2 - r^{2-2aN} H(v(r,s),b) \\ &= -(2-2aN)\int_0^r t^{1-2aN} H\big(v(t,s),b\big)dt. \end{aligned} \tag{2.18}$$

Using (2.17) and (2.18), we get the following Pohozaev type identities.

Lemma 2.6. *If $s \in \mathcal{A}^+$ and $0 < aN < 1$, then*

$$\beta(\beta - 4N) = -(4-4aN)\int_0^\infty r^{1-2aN} G\big(v(r,s),b\big)\,dr, \tag{2.19}$$

$$\beta\big(\beta - \bar\beta_{a,N,b}\big) = \frac{16-16aN}{b\varepsilon^2}\int_0^\infty \frac{r^{1-2aN}}{(1+e^v)^{1+b}}\,dr, \tag{2.20}$$

where $\bar\beta_{a,N,b} = 4(1+bN-aN)/b$. If $s \in \mathcal{A}^-$ and $1-N < aN < 1$, then

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$$\beta(\beta-4N)=-(4-4aN)\int_0^\infty r^{1-2aN}H\big(v(r,s),b\big)\,dr, \tag{2.21}$$

$$\beta\big(\beta-\bar\beta_{a,N,b}\big)=-\frac{16-16aN}{b\varepsilon^2}\int_0^\infty\Big[1-\frac{1}{(1+e^v)^{1+b}}\Big]r^{1-2aN}dr. \tag{2.22}$$

In particular,

$$\begin{cases} \text{if } s\in\mathcal{A}^+ \text{ and } 0<aN<1, \text{ then } \beta(s)>\bar\beta_{a,N,b}>4N,\\ \text{if } s\in\mathcal{A}^- \text{ and } 1-N<aN<1, \text{ then } 0<\beta(s)\le 2aN+2N-2<2N. \end{cases} \tag{2.23}$$

Proof. First, suppose that $s\in\mathcal{A}^+$. Since by Lemma 2.2

$$\lim_{r\to\infty} r^{2-2aN}G\big(v(r,s),b\big)=0,$$

we deduce from (2.17) that

$$\begin{aligned}\frac12[-2N+\beta(s)]^2-2N^2&=-(2-2aN)\int_0^\infty r^{1-2aN}G\big(v(r,s),b\big)dr\\ &=\frac{(2-2aN)}{b}\left\{\beta(s)+\frac{4}{\varepsilon^2}\int_0^\infty\frac{r^{1-2aN}}{(1+e^v)^{1+b}}dr\right\},\end{aligned}$$

which implies (2.19) and (2.20).

Next, we assume that $s\in\mathcal{A}^-$. By Lemma 2.2,

$$\lim_{r\to\infty} r^{2-2aN}H\big(v(r,s),b\big)=0.$$

So, (2.18) implies that

$$\begin{aligned}&\frac12[-2N+\beta(s)]^2-2N^2\\ &=-(2-2aN)\int_0^\infty r^{1-2aN}H\big(v(r,s),b\big)dr\\ &=-\frac{(2-2aN)}{b}\left\{-\beta(s)+\frac{4}{\varepsilon^2}\int_0^\infty\Big[1-\frac{1}{(1+e^v)^{1+b}}\Big]r^{1-2aN}dr\right\}.\end{aligned}$$

Thus, (2.21) and (2.22) follows.

Finally, in view of Lemma 2.2 and (2.19)–(2.20), we deduce that

$$\begin{cases} s \in \mathcal{A}^+ \Rightarrow \beta(s) > \max\{4N, \bar{\beta}_{a,N,b}\} = \bar{\beta}_{a,N,b}, \\ s \in \mathcal{A}^- \Rightarrow 0 < \beta(s) \le \min\{2aN + 2N - 2, \bar{\beta}_{a,N,b}\} = 2aN + 2N - 2, \end{cases} \tag{2.24}$$

which proves (2.23). □

In the following four lemmas, we identify the exact ranges of $\beta(s)$ for $s \in \mathcal{A}^+$ or $s \in \mathcal{A}^-$, and prove that β is monotone on both $\mathcal{A}^+$ and $\mathcal{A}^-$. This means that we can classify solutions of (1.12) by their decay rates. We begin with the set $\mathcal{A}^+$.

Lemma 2.7. *If $0 < aN \le 1 - N$, then*

$$\lim_{s\to\infty} \beta(s) = \bar{\beta}_{a,N,b}, \qquad \lim_{s\to-\infty} \beta(s) = \infty. \tag{2.25}$$

Hence, $\beta : (-\infty, \infty) \to (\bar{\beta}_{a,N,b}, \infty)$ is onto.

Proof. We recall that $s \in \mathcal{A}^+$ for all $s \in \mathbb{R}$. If follows from (2.20) that

$$\beta\big(\beta - \bar{\beta}_{a,N,b}\big) = \frac{4 - 4aN}{b} \int_0^\infty r^{1-2aN} e^{-v} f_1(v, a) dr \le \frac{4 - 4aN}{b} e^{-m(s)} \beta.$$

Letting $s \to \infty$, we obtain the first limit of (2.25) by Lemma 2.5.

For the second part of (2.25), we claim that $z(s) \to \infty$ as $s \to -\infty$. To see this, we recall from Lemma 2.5 that $m(s) \to -\infty$ as $s \to -\infty$. Since $f_1(v, b)$ is uniformly bounded for all $v \in \mathbb{R}$, we can choose small $\delta > 0$, which is independent of s, such that

$$\delta v'(\delta, s) = -2N + \int_0^\delta r^{1-2aN} f_1(v(r, s), b)\, dr < -N.$$

Given any $R > \delta$, it follows from (2.4) that $v(r, s) = s + O(1) \to -\infty$ uniformly for all $r \in [\delta, R]$ as $s \to -\infty$. Here, $O(1)$ is a quantity independent of $r \in [\delta, R]$. Now, for $s \ll -1$,

$$Rv'(R, s) = \delta v'(\delta, s) + \int_\delta^R r^{1-2aN} f_1(v(r, s), b) dr < -N + o(1) < 0.$$

Thus, $R < z(s)$ for all $s \ll -1$ and the claim follows.

Suppose that the second part of (2.25) is not true. Then, there exists a sequence $s_n \to -\infty$ such that $\beta(s_n) \le c$ for some constant $c > 0$. Then

$$0 < v'(r, s_n) \le (c - 2N)/r \quad \text{for all} \quad r > z_n = z(s_n). \tag{2.26}$$

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Since $m(s_n) \to -\infty$, we can choose $r_n, R_n \in (z_n, \infty)$ such that $v(r_n, s_n) = 0$ and $v(R_n, s_n) = 1$. We note that $r_n < R_n$ and $R_n \to \infty$ as $s_n \to -\infty$. By (2.26),

$$1 = \int_{r_n}^{R_n} v'(r, s_n) dr \le (c - 2N) \ln\left(\frac{R_n}{r_n}\right).$$

Then, we deduce that

$$\begin{aligned} c \ge \beta(s_n) &\ge \left[\inf_{0\le v\le 1} f_1(v, b)\right] \int_{r_n}^{R_n} r^{1-2aN} dr \\ &\ge \left[\inf_{0\le v\le 1} f_1(v, b)\right] \cdot \frac{R_n^{2-2aN}}{(2-2aN)} \cdot \left(1 - e^{-\frac{2-2aN}{c-2N}}\right) \quad \to \quad \infty, \end{aligned}$$

a contradiction. □

Lemma 2.8. *If $1 - N < aN < 1$ and $s < s_*$, then $\beta(s)$ is strictly increasing.*

Proof. For $s < s_*$, let

$$\begin{aligned} \lambda(s) &= (4 - 4aN) \int_0^\infty r^{1-2aN} H\big(v(r, s), b\big)\, dr \\ &= \frac{16 - 16aN}{b\varepsilon^2} \int_0^\infty r^{1-2aN} \left\{1 - \frac{1}{(1 + e^v(r, s))^b}\right\} dr. \end{aligned}$$

We note from (2.12) that $\lambda(s)$ is a strictly increasing function. Let us rewrite (2.21) as

$$\beta(s)\big[\beta(s) - 4N\big] = -\lambda(s).$$

So, we have

$$\beta(s) = 2N \pm \sqrt{4N^2 - \lambda(s)} =: \beta^{\pm}(s).$$

If $\beta = \beta^+$, then $2N < \beta \le 2aN + 2N - 2$ by (2.23) and thus $aN > 1$, a contradiction. Therefore, $\beta = \beta^-$ and β^- is an increasing function of s. This gives us the desired result. □

Lemma 2.9. *If $1 - N < aN < 1$, then*

$$\lim_{s\to\infty} \beta(s) = \bar{\beta}_{a,N,b}, \qquad \lim_{s\searrow s_*} \beta(s) = \infty, \tag{2.27}$$

and

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$$\lim_{s\to-\infty}\beta(s)=0, \qquad \beta(s_*)=2aN+2N-2. \tag{2.28}$$

In particular, $\beta:(s_*,\infty)\to(\bar{\beta}_{a,N,b},\infty)$ is onto and $\beta:(-\infty,s_*]\to(0,2aN+2N-2]$ is onto.

Proof. The first part of (2.27) is the same as the proof of the first part of (2.25). Since $v(r,s)\to v(r,s_*)$ locally uniformly as $s\searrow s_*$, it holds that $z(s)\to\infty$ and $m(s)\to-\infty$ as $s\searrow s_*$. Then, the second part of (2.27) follows from the same argument for the proof of the second part of (2.25).

Now we turn to the proof of (2.28). By (2.23) and Lemma 2.8,

$$\limsup_{s\to-\infty}\beta(s)<2aN+2N-2. \tag{2.29}$$

We note from (2.4) that $v(1,s)=s+O(1)\to-\infty$ as $s\to-\infty$. Hence, there exists a unique $r_s<1$ such that $v(r_s,s)=s/2$ as $s\to-\infty$. Moreover, $r_s\to0$ as $s\to-\infty$. Since $v(r,s)\ll-1$ for $r>r_s$, it comes from the Taylor expansion that as $s\to-\infty$,

$$\begin{aligned}&\int_{r_s}^{\infty}\Big[1-\frac{1}{(1+e^v)^{1+b}}\Big]r^{1-2aN}dr\\ &\le\int_{r_s}^{\infty}(1+b)e^v r^{1-2aN}dr\\ &\le(1+b)e^{v(r_s,s)}\int_{r_s}^{1}r^{1-2aN}dr+(1+b)e^{v(1,s)}\int_{1}^{\infty}r^{1-2aN-2N+\beta}dr\;=\;o(1).\end{aligned}$$

Here, we used the fact that $rv'<-2N+\beta$ for all $r>0$ such that $v(r,s)<v(1,s)+(-2N+\beta)\ln r$ for $r>1$. We also utilized (2.29) to see the last integral is finite. Then, by (2.22), it holds that

$$\begin{aligned}\beta\big(\beta-\bar{\beta}_{a,N,b}\big)&=-\frac{16-16aN}{b\varepsilon^2}\int_0^{\infty}\Big[1-\frac{1}{(1+e^v)^{1+b}}\Big]r^{1-2aN}dr\\ &=-\frac{16-16aN}{b\varepsilon^2}\Big(\int_0^{r_s}+\int_{r_s}^{\infty}\Big)=o(1)\end{aligned}$$

as $s\to-\infty$. Now, keeping (2.23) in mind, we conclude that $\beta(s)\to0$ as $s\to-\infty$.

It remains to show the second part of (2.28). We recall that $0<\beta(s)\le2aN+2N-2$ for any $s\in(-\infty,s_*]$. Suppose that $\beta(s_*)<2aN+2N-2$. Choose a constant $\xi\in(0,\frac{2-2aN}{4})$ such that $\beta(s_*)<2aN+2N-2-4\xi$. Let $r_0>0$ be the unique zero of $v(r,s_*)$ such that $v(r,s_*)>0$ for $r<r_0$ and $v(r,s_*)<0$ for $r>r_0$. For $k=1,2$, let $h_k:(0,\infty)\to\mathbb{R}$ be a smooth function such that $h_k(r)=-1+(2aN-2-k\xi)\ln(r/r_0)$ for $r\ge r_0$ and $h_k(r)<0$ for $r<r_0$. Thus, $h_k(r)<v(r,s_*)$ for $r<r_0$. By comparing the decay rates of $h_k(r)$ and $v(r,s_*)$ at ∞, we notice

that $h_k(r)$ and $v(r,s_*)$ must intersect. If r_k is the first intersection point of $h_k(r)$ and $v(r,s_*)$, then there are no more intersection points. In fact, if $\bar{r}_k$ is the second intersection point of $h_k(r)$ and $v(r,s_*)$, then

$$-2N+\beta(s_*) > \bar{r}v'(\bar{r},s_*) \geq \bar{r}h_k'(\bar{r}) = 2aN+2-k\xi,$$

which contradict to the choice of ξ. Thus, $h_1(r)$ and $v(r,s_*)$ intersect exactly once at r_1.

Now, let $s_n \in \mathcal{A}^+$ be a sequence such that $s_n \searrow s_*$. Then, for all s_n sufficiently close to s_*, the graphs of $h_k(r)$ and $v_n(r) = v(r,s_n)$ meet at least twice. Let $t_{n,k}$ and $T_{n,k}$ be the first two intersection points of h_k and v_n with $r_k < t_{n,k} < T_{n,k}$. Since rv_n' is increasing, if $r > T_{n,k}$, then

$$rv_n'(r) > T_{n,k}v_n'(T_{n,k}) > T_{n,k}h_k'(T_{n,k}) = rh_k'(r).$$

This implies that $v_n(r) > h_k(r)$ for $r > T_{n,k}$, namely, the graphs of h_k and v_n meet exactly twice at $t_{n,k}$ and $T_{n,k}$. We also note that $t_{n,1} < t_{n,2} < T_{n,2} < T_{n,1}$.

Since $h_k(r)$ and v_n coincide at $t_{n,k}$ and $T_{n,k}$, by Rolles' Theorem there exists $\eta_{n,k} \in (t_{n,k}, T_{n,k})$ such that

$$v_n'(\eta_{n,k}) = h_k'(\eta_{n,k}) = \frac{2aN-2-k\xi}{\eta_{n,k}} < 0. \tag{2.30}$$

Obviously, $t_{n,k} < \eta_{n,k} < z_n = z(s_n)$. Moreover, from the continuous dependence of solutions on s, it follows that $\eta_{n,k} \to \infty$ and $v_n(\eta_{n,k}) \to -\infty$ as $s_n \searrow s_*$. Since the graph of h_k lies above the graph of v_n on $(\eta_{n,k}, T_{n,k})$, we deduce from the Taylor expansion of H that for $r \in (\eta_{n,k}, T_{n,k})$,

$$r^{1-2aN}H\big(v_n(r)\big) \leq \frac{4}{\varepsilon^2}r^{1-2aN}e^{v_n(r)} \leq Cr^{1-2aN}e^{h_k(r)} \leq Cr^{-1-k\xi}. \tag{2.31}$$

We have two choices: either $t_{n,1} < z_n < T_{n,1}$ or $T_{n,1} \leq z_n$. If $t_{n,1} < z_n < T_{n,1}$, then it comes from (2.17) and (2.31) that

$$E(z_n,s_n) - E(\eta_{n,1},s_n) = -(2-2aN)\int_{\eta_{n,1}}^{z_n} r^{1-2aN}H\big(v_n(r),b\big)dr = o(1)$$

as $s_n \searrow s_*$. However, a direct computation yields from (2.17) and (2.30) that

$$\begin{aligned} &E(z_n,s_n) - E(\eta_{n,1},s_n) \\ &= -z_n^{2-2aN}H\big(v(z_n),b\big) + \eta_{n,1}^{2-2aN}H\big(v(\eta_{n,1}),b\big) - \frac{1}{2}(2aN-2-\xi)^2 \\ &\to -\frac{1}{2}(2aN-2-\xi)^2 \end{aligned}$$

as $s_n \searrow s_*$, a contradiction. Similarly, if $T_{n,1} \leq z_n$, then we also have a contradiction:

$$\begin{aligned}
o(1) &= -(2-2aN)\int_{\eta_{n,2}}^{T_{n,1}} r^{1-2aN} H\big(v(r,s_n),b\big)dr \\
&= E(T_{n,1},s_n) - E(\eta_{n,2},s_n) \\
&= \frac{1}{2}\Big[T_{n,1}v_n'(T_{n,1})\Big]^2 - \frac{1}{2}\Big[\eta_{n,2}v_n'(\eta_{n,2})\Big]^2 + o(1) \\
&< \frac{1}{2}\Big[T_{n,1}h_1'(T_{n,1})\Big]^2 - \frac{1}{2}\Big[\eta_{n,2}v_n'(\eta_{n,2})\Big]^2 + o(1) \\
&= \frac{1}{2}(2aN-2-\xi)^2 - \frac{1}{2}(2aN-2-2\xi)^2 + o(1) \\
&\to \frac{1}{2}\xi(4aN-4-3\xi) \ < \ 0.
\end{aligned}$$

This completes the proof of Lemma 2.9. □

Lemma 2.10. *For $0 < aN < 1$ and $s \in \mathcal{A}^+$, φ has exactly one zero and $\beta(s)$ is strictly decreasing.*

Proof. For $s \in \mathcal{A}^+$, by the Lebesgue Dominated Convergence Theorem, one can check that the limit

$$\beta'(s) = \int_0^\infty r^{1-2aN} f_1'\big(v(r,s),b\big)\varphi'(r,s)\,dr = \lim_{r\to\infty} r\varphi'(r,s) \tag{2.32}$$

is finite. We will show that $\beta'(s)$ never vanishes for $s \neq s_*$. Then, due to the limit (2.27), it follows that $\beta'(s) < 0$ and the proof is complete.

For simplicity, let us write $v(r)$, $\varphi(r)$, $f(v)$ instead of $v(r,s)$, $\varphi(r,s)$, $f(v,b)$ and so on. We note that for all large r

$$rw_0'(r) = r^{2-2aN} f_1(v(r)) = O(r^{2-2aN+2bN-b\beta}) = O(r^{-b\bar{\beta}/2}).$$

Suppose that φ does not have a zero. Then, it is necessary that $\varphi > 0$ for all r. Moreover, since $f_1'(v(r)) < 0$ for all large r, it follows that $(r\varphi')' < 0$ for all large r and $\lim_{r\to\infty} r\varphi'(r) \geq 0$. In particular, $|\varphi(r)| \leq C\ln r$ for some constant $C > 0$ and for all large r. So, we have

$$0 > -(2-2aN)\int_0^\infty f_1\big(v(r)\big)r^{1-2aN}\varphi dr = \lim_{r\to\infty}[r\varphi'(r)w_0(r) - rw_0'(r)\varphi(r)] \geq 0,$$

a contradiction.

Let us denote by $r_1 = r_1(s)$ the first zero of φ. By (2.12), $r_1 > z = z(s)$. Suppose that φ has the second zero at r_2. Set R be the minimum point of φ/w_0 on (r_1,r_2). Then,

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$$0 = \Big(\frac{\varphi}{w_0}\Big)'(R) = \Big(\frac{r\varphi' w_0 - r w_0'\varphi}{r w_0^2}\Big)(R) = \frac{-(2-2aN)}{R w_0^2(R)} \int_0^R r^{1-2aN} f_1\big(v(r)\big)\varphi dr. \tag{2.33}$$

In particular, since $w_0(R) > 0$ for $R > z$, we deduce that $\varphi'(R) < 0$. We note that

$$\Big(\frac{f_1'}{f_1}\Big)'(v) = -\frac{(1+b)e^v}{(1+e^v)^2} < 0. \tag{2.34}$$

Let $\mu = f_1'\big(v(r_1)\big)/f_1\big((r_1)\big)$. Then, $f_1'\big(v(r)\big) - \mu f_1\big(v(r)\big) > 0$ on $(0, r_1)$ and $f_1'\big(v(r)\big) - \mu f_1\big(v(r)\big) < 0$ on (r_1, ∞). So, $[f_1'\big(v(r)\big) - \mu f_1\big(v(r)\big)]\varphi(r) > 0$ on $(0, R)$. By (2.9) and (2.33), it follows that

$$0 < \int_0^R r^{1-2aN}[f_1'(v) - \mu f_1(v)]\varphi dr = \int_0^R r^{1-2aN} f_1'(v)\varphi dr = R\varphi'(R) < 0,$$

which is a contradiction. In the sequel, φ has exactly one zero at r_1.

We note from (2.9) that $(r\varphi'(r))' > 0$ for all large r. Hence, there exists $\lim_{r\to\infty} r\varphi'(r) = \delta$. If $\delta \neq 0$, then $\beta'(s) \neq 0$ by (2.32). This implies by Lemma 2.9 that β is strictly decreasing and the proof is complete. Now, let us assume to the contrary that $\delta = 0$. If $c_0 = -(2-2aN)/b < 0$, then $\Phi_{c_0}(r) > 0$ for all $r > 0$. Hence,

$$0 > -\int_0^{r_1} r^{1-2aN}\Phi_{c_0}(r)\varphi(r) dr = r_1\varphi'(r_1) w_{c_0}(r_1).$$

So, $w_{c_0}(r_1) > 0$ such that $c_1 = -r_1 v'(r_1) < c_0$. Then $\Phi_{c_1}(r)$ has a zero before r_1. Otherwise, we have $\Phi_{c_1}(r) < 0$ for all $r < r_1$ such that

$$0 < -\int_0^{r_1} r^{1-2aN}\Phi_{c_1}\varphi dr = \big(r\varphi' w_{c_1} - r w_{c_1}'\varphi\big)(r_1) = 0,$$

a contradiction. We note that $\Phi_{c_1}(r) \nearrow 0$ as $r \to 0$ or $r \to \infty$. Hence $\Phi_{c_1}(r)$ have exactly two zeros y_1 and y_2 with $y_1 < z < y_2$ such that $\Phi_{c_1} > 0$ on (y_1, y_2) and $\Phi_{c_1} < 0$ on $(0, y_1) \cup (y_2, \infty)$. We have two possibilities: either $y_2 \le r_1$ or $y_1 < r_1 < y_2$. First, if $y_2 \le r_1$, then

$$0 < \int_{r_1}^{\infty} r^{1-2aN}\Phi_{c_1}(r)\varphi(r) dr = \lim_{r\to\infty}\big(r w_{c_1}'\varphi - r\varphi' w_{c_1}\big) = 0,$$

a contradiction.

Next, suppose that $y_1 < r_1 < y_2$. One may refer to the Fig. 1 for this situation. Since

$$\lim_{r\to 0}\big(r\varphi' w_0 - r w_0'\varphi\big) = \lim_{r\to\infty}\big(r\varphi' w_0 - r w_0'\varphi\big) = 0$$

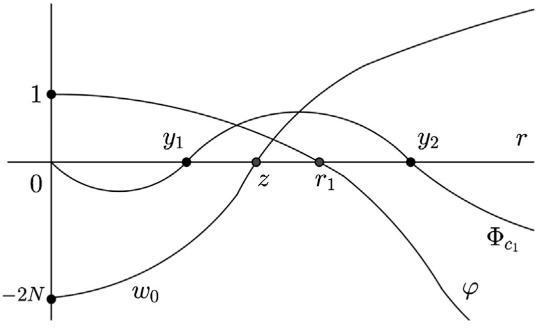

Fig. 1. Graph of w_0, φ, and Φ_{c_1} in the proof of Lemma 2.10.

and

$$[r\varphi' w_0 - r w_0' \varphi]' = -(2-2aN)r^{1-2aN} f_1(v(r))\varphi(r) = \begin{cases} <0 & \text{on} \quad (0, r_1), \\ >0 & \text{on} \quad (r_1, \infty), \end{cases}$$

it holds that $r\varphi' w_0 - r w_0' \varphi < 0$ for all r. This implies

$$\left(\frac{w_0}{\varphi}\right)'(r) = \frac{-(r\varphi' w_0 - r w_0' \varphi)}{r\varphi^2} > 0, \quad \text{for all} \quad r \in (0, r_1) \cup (r_1, \infty).$$

Thus, if we set $\lambda_1 = (w_0/\varphi)(y_1) < 0$, then $w_0 - \lambda_1 \varphi < 0$ for $0 < r < y_1$ and $w_0 - \lambda_1 \varphi > 0$ for $y_1 < r < z$. Moreover, $(w_0 - \lambda_1 \varphi)\Phi_{c_1} > 0$ and $\Phi_{c_1}(r)(r^{2-2aN} - y_1^{2-2aN}) > 0$ on $(0, z)$. As a consequence,

$$\begin{aligned} 0 < \lambda_1 \int_0^z r^{1-2aN} \Phi_{c_1} \varphi \, dr &< \int_0^z r^{2-2aN} \Phi_{c_1} v' \, dr \\ &< y_1^{2-2aN} \int_0^z \Phi_{c_1} v' \, dr = -y_1^{2-2aN} Q_{c_1}\big(v(z)\big). \end{aligned}$$

Here, we define

$$Q_{c_1}(v) = \int_v^\infty K_{c_1}(t) dt,$$

where $\Phi_c(r) = K_c(v(r))$ and K_c is given by

$$K_c(t) = \frac{4e^t \big\{[(2-2aN) + cb]e^t + [(2-2aN) - c]\big\}}{(1+e^t)^{2+b}}.$$

On the other hand, if we set $\lambda_2 = (w_0/\varphi)(y_2) < 0$, then $w_0 - \lambda_2\varphi > 0$ for $r_1 < r < y_2$ and $w_0 - \lambda_2\varphi < 0$ for $r > y_2$. Hence, $(w_0 - \lambda_2\varphi)\Phi_{c_1} > 0$ and $\Phi_{c_1}(r)(r^{2-2aN} - y_2^{2-2aN}) < 0$ on (r_1, ∞). Since $\delta = 0$, we obtain

$$0 = \lambda_2 \int_{r_1}^{\infty} r^{1-2aN}\Phi_{c_1}\varphi\, dr < \int_{r_1}^{\infty} r^{2-2aN}\Phi_{c_1} v'\, dr$$

$$< y_2^{2-2aN} \int_{r_1}^{\infty} \Phi_{c_1} v'\, dr = y_2^{2-2aN} Q_{c_1}\big(v(r_1)\big).$$

In the sequel, we get $Q_{c_1}\big(v(z)\big) < 0 < Q_{c_1}\big(v(r_1)\big)$. However, since $\big[Q_{c_1}\big(v(r)\big)\big]' = -\Phi_{c_1}(r)v'(r) < 0$ on (z, y_2), $Q_{c_1}\big(v(r)\big)$ is decreasing on (z, y_2) and this inequality yields a contradiction. □

Lemma 2.11. *Suppose that* $0 < aN < 1$.

(i) *If* $s \in \mathcal{A}^+$, *then* $v(r, s) = [-2N + \beta(s)]\ln r + I + O(r^{2-2aN+2bN-b\beta})$ *for some constant* $I = I(a, N, b, s)$ *as* $r \to \infty$.
(ii) *If* $1 - N < aN < 1$ *and* $s < s_*$, *then* $v(r, s) = [-2N + \beta(s)]\ln r + I + O(r^{2-2aN-2N+\beta})$ *for some constant* $I = I(a, N, b, s)$ *as* $r \to \infty$.
(iii) *If* $1 - N < aN < 1$ *and* $s = s_*$, *then* $v(r, s_*) = [2aN - 2]\ln r - 2\ln\ln r + O(1)$ *as* $r \to \infty$.

Proof. Let $u(r, s) = v(r, s) - (-2N + \beta)\ln r$ such that $u(r, s) = o(\ln r)$ and $ru'(r, s) = o(1)$ as $r \to \infty$. We will estimate u to prove each statement of this lemma.

(i) Suppose that $s \in \mathcal{A}^+$. Since $\beta > \bar\beta = \bar\beta_{a,N,b}$, given any small $0 < \eta < \beta - \bar\beta/2$, we can choose a large number $r_0 > 0$ such that for all $r \ge r_0$, $|u(r, s)| \le \eta \ln r$. So,

$$r^{1-2aN} f_1(v(r, s), b) = O(r^{1-2aN}e^{-bv}) = O(r^{1-2aN+2bN-b\beta+b\eta})$$
$$= O\Big(r^{-1+b(\frac{\bar\beta}{2}-\beta+\eta)}\Big).$$

Integrating $(ru')' = r^{1-2aN} f_1(v)$ on (r, ∞), we obtain that for $r > r_0$

$$ru'(r) = -\int_r^{\infty} t^{1-2aN} f_1(v(t, s), b)dt,$$

and thus

$$u(r, s) = u(r_0, s) - \int_{r_0}^{r} \frac{1}{t} \int_t^{\infty} y^{1-2aN} f_1(v(y, s), b)dydt.$$

Since

$$\int_{r_0}^{\infty} \frac{1}{t} \int_t^{\infty} y^{1-2aN} f_1(v(y,s),b)dydt \le C{r_0}^{b(\frac{\bar{\beta}}{2}-\beta+\eta)},$$

we conclude that $u(r,s) = O(1)$ as $r \to \infty$. In the sequel,

$$u(r,s) = I + \int_r^{\infty} \frac{1}{t} \int_t^{\infty} y^{1-2aN} f_1(v(y,s),b)dydt,$$

where

$$I = I(a,N,b,s) = u(r_0,s) - \int_{r_0}^{\infty} \frac{1}{t} \int_t^{\infty} y^{1-2aN} f_1(v(y,s),b)dydt.$$

Since $u(r,s) = O(1)$ as $r \to \infty$, we deduce that

$$\begin{aligned}\int_r^{\infty} \frac{1}{t} \int_t^{\infty} y^{1-2aN} f_1(v(y,s),b)dydt &\le C \int_r^{\infty} \frac{1}{t} \int_t^{\infty} y^{1-2aN+2bN-b\beta} dydt \\ &\le Cr^{2-2aN+2bN-b\beta},\end{aligned}$$

which yields the desired estimate.

(ii) Assume that $1 - N < aN < 1$ and $s < s_*$. We fix a number η such that $0 < \eta < 2aN + 2N - 2 - \beta$ and select a large number $r_0 > 0$ such that for all $r \ge r_0$, $|u(r,s)| \le \eta \ln r$. We note that for $r > r_0$

$$r^{1-2aN} f_1(v(r,s),b) = O\big(r^{1-2aN}e^v\big) = O\big(r^{-1+(2-2aN-2N+\beta+\eta)}\big).$$

Now, proceeding as in (i), we can obtain the desired result. We omit the detail.

(iii) Assume that $1 - N < aN < 1$ and $s = s_*$. We note that

$$-2N + \beta(s_*) = -2N + (2aN + 2N - 2) = 2aN - 2,$$

and $u(r, s_*)$ satisfies

$$(ru')' = \frac{4r^{-1}e^u}{\varepsilon^2\big(1 + r^{2aN-2}e^u\big)^{1+b}}. \tag{2.35}$$

We also notice that

$$2aN + 2N - 2 = \beta(s_*) = \int_0^{\infty} \frac{4r^{-1}e^u}{\varepsilon^2\big(1 + r^{2aN-2}e^u\big)^{1+b}}\, dr. \tag{2.36}$$

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Since $(ru')' > 0$, it holds that $ru' < 0$ for all $r > 0$. Indeed, if $u'(r_0, s_*) > 0$ for some $r_0 > 0$, then it holds that

$$u(r, s_*) \geq u(r_0, s_*) + r_0 u'(r_0, s_*) \ln \frac{r}{r_0} \quad \forall\, r > r_0,$$

a contradiction. Moreover, the finiteness of $\beta(s_*)$ implies by (2.36) that $u(r, s_*) \to -\infty$ as $r \to \infty$.

Let r_1 be the unique number such that $v(r_1, s_*) = 0$. Then, it comes from (2.35) that for all $r \geq r_1$

$$2^{1-b} r^{-1} e^{u(r,s_*)} \leq \varepsilon^2 \big(ru'(r, s_*)\big)' \leq 4r^{-1} e^{u(r,s_*)}$$

Multiplying this inequality by ru' and integrating on $[r, \infty]$ with $r > r_1$, we have

$$2^{2-b} e^{u(r,s_*)} \leq \varepsilon^2 \big[ru'(r, s_*)\big]^2 \leq 8e^{u(r,s_*)},$$

namely,

$$-\frac{\sqrt{2}}{r} \leq \frac{\varepsilon u' e^{-\frac{u}{2}}}{2} \leq -\frac{2^{-b/2}}{r}.$$

Integrating this inequality on $[r_1, r]$, we see that for all $r > r_1$

$$-2 \ln \left(\frac{\sqrt{2}}{\varepsilon} \ln \frac{r}{r_1} + e^{-\frac{u(r_1,s_*)}{2}} \right) \leq u(r, s_*) \leq -2 \ln \left(\frac{2^{-b/2}}{\varepsilon} \ln \frac{r}{r_1} + e^{-\frac{u(r_1,s_*)}{2}} \right).$$

Consequently, $u(r, s_*) = -2 \ln \ln r + O(1)$ as $r \to \infty$. $\square$

Now, by gathering the above Lemmas, we establish Proposition 2.1.

3. The case $aN_2 \geq 1$ and $N_1 = 0$

This section deals with (1.12) under the condition $aN_2 \geq 1$ and Theorem 1.1 (ii)–(iv) are proved by Proposition 3.1, Proposition 3.3, and Proposition 3.6.

If $aN_2 \geq 1$, then we need more careful approach. For instance, in the previous section, there appears the term $1 - aN_2$ in many places and its sign was very important in deriving appropriate conclusion. Since we have reverse sign for the case $aN_2 \geq 1$, it is not possible to draw some desired results in estimates or contradiction argument through the same manner. Moreover, if $aN_2 < 1$, then r^{1-2aN_2} is integrable near $r = 0$ such that

$$\int_0^r r^{1-2aN_2} f_1(v(r, s), a)dt \leq \|f_1\|_\infty \int_0^r r^{1-2aN_2} dr = O(1)$$

for any fixed $r > 0$, where $O(1)$ denote a quantity independent of $s \in \mathbb{R}$. Thus, if r is fixed, then we get

$$v(r,s) = -2N_2 \ln r + s + O(1) \quad \text{for any} \quad s \in \mathbb{R}. \tag{3.1}$$

We used this type estimates in many places in the previous section. However, if $aN_2 \geq 1$, then r^{1-2aN_2} is no more integrable near $r = 0$ and we cannot obtain simple estimate like (3.1). To analyze the case $aN_2 \geq 1$, we take the approach of [14] by using the result for the generalized version (2.2). We employ the same notations $\mathcal{A}^{\pm}_{a,N,b}$ as before. Then, by the Implicit Function Theorem, $\mathcal{A}^{+}_{a,N_2,a}$ is open and thus $\mathcal{A}^{-}_{a,N_2,a}$ is closed. In this section, we will often write $\mathcal{A}^{\pm}_{a,N_2,a}$ as $\mathcal{A}^{\pm}$ if there is no risk of confusion. As in the previous section, $z(s)$ and $m(s)$ will stand for the minimum point and minimum value of $v(r,s)$ for each $s \in \mathcal{A}^+$. We also recall the solution formula: $v(r,s)$ is a solution of (1.12) if and only if

$$v(r,s) = -2N_2 \ln r + s + \int_0^r \frac{1}{t} \int_0^t y^{1-2aN_2} f_1\big(v(y,s), a\big)\,dy\,dt. \tag{3.2}$$

Here, we start with the case $aN_2 = 1$.

Proposition 3.1. *Suppose that N_2 is a positive integer and $aN_2 = 1$.*

(i) *If $0 < N_2 < 2\varepsilon^{-2}$, then for any $s \in \mathbb{R}$, $v(r,s)$ is a type II solution of (1.12) such that $\beta(s) = 4N_2$. Moreover, as $r \to \infty$,*

$$v(r,s) = 2N_2 \ln r + I + O(r^{-2})$$

for some constant $I = I(a, N_2, s)$.

(ii) *If $N_2 \geq 2\varepsilon^{-2}$, then for any $s \in \mathbb{R}$, $v(r,s)$ is a type I solution of (1.12) such that $\beta(s) = 2N_2 - 2\sqrt{N_2(N_2 - 2\varepsilon^{-2})}$. Moreover, if $N_2 > 2\varepsilon^{-2}$, then as $r \to \infty$,*

$$v(r,s) = -2\sqrt{N_2(N_2 - 2\varepsilon^{-2})}\ln r + I + O\Big(r^{-2\sqrt{N_2(N_2-2\varepsilon^{-2})}}\Big)$$

for some constant $I = I(a, N_2, s)$. If $N_2 = 2\varepsilon^{-2}$, then

$$v(r,s) = -2\ln\ln r + O(1) \quad \text{as} \quad r \to \infty.$$

Proof. Let us rewrite (1.12) as

$$(rv')' = \frac{4r^{-1}e^v}{\varepsilon^2(1+e^v)^{1+a}}. \tag{3.3}$$

First, we claim that for any $s \in \mathbb{R}$, either $v(r,s) \to \infty$ or $v(r,s) \to -\infty$, that is, $v(r,s)$ is a type I solution for $s \in \mathcal{A}^-$ and a type II solution for $\mathcal{A}^+$. To see this, let us assume that $v(r,s) \to \sigma$ for some $\sigma \in \mathbb{R}$. Then, there exists $c_0, r_0 > 0$ such that $(rv')' \geq c_0 r^{-1}$ for all $r > r_0$. Integrating this inequality twice on $[r_0, r]$, we obtain

$$v(r,s) \geq v(r_0,s) + \big[r_0 v'(r_0,s) - c_0 \ln r_0\big] \ln\Big(\frac{r}{r_0}\Big) + \frac{c_0}{2}\big[(\ln r)^2 - (\ln r_0)^2\big] \;\to\; \infty$$

as $r \to \infty$, a contradiction. Multiplying (3.3) by rv' and integrating it on $(0, r)$, we obtain that

$$\frac{1}{2}\left(rv'(r,s)\right)^2 - 2N_2^2 = -\frac{4}{\varepsilon^2 a(1+e^{v(r,s)})^a}. \tag{3.4}$$

If $s \in \mathcal{A}^+$, then by letting $r \to \infty$, we obtain $(\beta - 2N_2)^2 = 4N_2^2$, that is, $\beta(s) = 4N_2$. Moreover, if $s \in \mathcal{A}^+$, then by letting $r = z(s)$ in (3.4), we get

$$N_2 = \frac{2}{\varepsilon^2(1+e^{v(z,s)})^a} < \frac{2}{\varepsilon^2}. \tag{3.5}$$

Hence, if $N_2 \geq 2\varepsilon^{-2}$, then $\mathcal{A}^+ = \emptyset$ and $\mathcal{A}^- = \mathbb{R}$.

On the other hand, if $s \in \mathcal{A}^-$, by letting $r \to \infty$ in (3.4), we deduce that

$$0 \leq (\beta - 2N_2)^2 = 4N_2^2 - \frac{8}{\varepsilon^2 a} = 4N_2\left(N_2 - \frac{2}{\varepsilon^2}\right). \tag{3.6}$$

Hence, $\beta(s) = 2N_2 - 2\sqrt{N_2(N_2 - 2\varepsilon^{-2})}$. Moreover, if $N_2 < 2\varepsilon^{-2}$, then $\mathcal{A}^- = \emptyset$ and $\mathcal{A}^+ = \mathbb{R}$.

Now, we turn to the asymptotic behavior of solutions. If $N_2 < 2\varepsilon^{-2}$, then $s \in \mathcal{A}^+$ for all $s \in \mathbb{R}$ such that as $r \to \infty$,

$$r^{1-2aN_2} f_1(v(r,s), a) = O(r^{1-2aN_2}e^{-av}) = O(r^{1-4aN_2}) = O(r^{-3}).$$

Then, following the argument of the proof of Lemma 2.11, we deduce that

$$v(r,s) = 2N_2 \ln r + I + O(r^{-2}) \quad \text{for some constant } I = I(a, N_2, s) \text{ as } r \to \infty.$$

Similarly, if $N_2 > 2\varepsilon^{-2}$, then $s \in \mathcal{A}^-$ for all $s \in \mathbb{R}$ such that as $r \to \infty$,

$$r^{1-2aN_2} f_1(v(r,s), a) = O(r^{1-2aN_2}e^{v}) = O(r^{-1-2\sqrt{N_2(N_2-2\varepsilon^{-2})}}),$$

and hence

$$v(r,s) = -2\sqrt{N_2(N_2 - 2\varepsilon^{-2})}\ln r + I + O\left(r^{-2\sqrt{N_2(N_2-2\varepsilon^{-2})}}\right) \quad \text{as} \quad r \to \infty.$$

Finally, suppose that $N_2 = 2\varepsilon^{-2}$. Then, for any $s \in \mathbb{R}$, $v(r,s) = o(\ln r)$ and $rv'(r,s) = o(1)$ as $r \to \infty$. Then, employing the same argument to (3.3) as in the proof of Lemma 2.11 (iii), we conclude that $v(r,s) = -2\ln\ln r + O(1)$ as $r \to \infty$. This completes the proof. □

We now proceed with the case $aN_2 > 1$.

Lemma 3.2. *Suppose that N_2 is a positive integer such that $aN_2 > 1$.*

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(i) *If* $1 < aN_2 < 2$, *then*

$$\begin{cases} 2N_2 < \beta(s) < \dfrac{4}{a} & \textit{for} \quad s \in \mathcal{A}^+_{a,N_2,a}, \\ 0 < \beta(s) \le 2N_2 & \textit{for} \quad s \in \mathcal{A}^-_{a,N_2,a}. \end{cases}$$

(ii) *If* $aN_2 \ge 2$, *then* $\mathcal{A}^+_{a,N_2,a} = \emptyset$ *and* $\mathcal{A}^-_{a,N_2,a} = \mathbb{R}$ *such that*

$$0 < \beta(s) < \frac{4}{a} \quad \textit{for} \quad s \in \mathbb{R}.$$

Proof. Let us assume that $aN_2 > 1$. By proceeding as in the proof of Lemma 2.2, one can check that $\beta(s)$ is finite for any values of a and N_2 with $aN_2 > 1$. We note that the function $G(v, a)$, defined by (2.16), is bounded in v. So, if $aN_2 > 1$, then for any $s \in \mathbb{R}$, we have

$$\lim_{r\to\infty} r^{2-2aN_2} G\big(v(r,s), a\big) = 0.$$

Therefore, the Pohozaev identities (2.19) and (2.20) are still valid such that

$$\beta(\beta - 4N_2) < 0, \quad \beta\big(\beta - \bar{\beta}_{a,N,a}\big) < 0, \quad \forall\, s \in \mathbb{R}.$$

However, it is worthwhile to mention that the identities (2.21) and (2.22) are no more true since $r^{1-2aN_2} H\big(v(r,s), a\big)$ is not integrable near $r = 0$. Since $\bar{\beta}_{a,N_2,a} = 4/a$, we obtain that

$$0 < \beta(s) < \min\left\{4N_2, \frac{4}{a}\right\} = \frac{4}{a}, \quad \forall\, s \in \mathbb{R}.$$

Suppose that $s \in \mathcal{A}^+$. Then, we recall from Remark 2.3 that (2.14) is true and so $\beta(s) \ge 2/a$. Since $-2N_2 + \beta(s) \ge 0$ for $s \in \mathcal{A}^+$, we conclude that

$$2N_2 = \max\left\{2N_2, \frac{2}{a}\right\} \le \beta < \frac{4}{a}.$$

In particular, $\mathcal{A}^+ = \emptyset$ if $aN_2 \ge 2$. We also note that $2N_2 \notin \text{Range}\,\beta$. Indeed, if $2N_2 = \beta(s_1)$ for some $s_1 \in \mathcal{A}^+$, then $rv'(r, s_1) \to 0$ as $r \to \infty$. Hence,

$$0 = \int_{z(s_1)}^{\infty} (rv')'(r, s_1)dr = \int_{z(s_1)}^{\infty} r^{1-2aN_2} f_1\big(v(r,s_1), a\big)dr > 0,$$

a contradiction.

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On the other hand, if $s \in \mathcal{A}^-$, then $-2N_2 + \beta(s) < 0$ such that

$$aN_2 < 2 \quad \Rightarrow \quad 0 < \beta \le \min\left\{2N_2, \frac{4}{a}\right\} = 2N_2,$$
$$aN_2 \ge 2 \quad \Rightarrow \quad 0 < \beta < \min\left\{2N_2, \frac{4}{a}\right\} = \frac{4}{a}.$$

This gives us the desired result. □

As the second subject of this section, we deal with the case $1 < aN_2 < 2$ in the following proposition.

Proposition 3.3. *Suppose that N_2 is a positive integer such that $1 < aN_2 < 2$. Then, there exists a number $s_* \in \mathbb{R}$ such that we have the following.*

(i) *For $s < s_*$, $v(r,s)$ is a type II solution of (1.12) and satisfies that*

$$v(r,s) = [-2N_2 + \beta(s)]\ln r + I + O(r^{2-a\beta}) \quad as \quad r \to \infty, \tag{3.7}$$

where $I = I(a, N_2, s)$ is a constant. The function $\beta : (-\infty, s_*) \to (2N_2, 4/a)$ is bijective and strictly decreasing.

(ii) *For $s > s_*$, $v(r,s)$ is a type I solution of (1.12) and satisfies that*

$$v(r,s) = [-2N_2 + \beta(s)]\ln r + I + O(r^{2-2aN_2-2N_2+\beta}) \quad as \quad r \to \infty, \tag{3.8}$$

where $I = I(a, N_2, s)$ is a constant. The function $\beta : (s_*, \infty) \to (0, 2N_2)$ is bijective and strictly decreasing.

(iii) *For $s = s_*$, $v(r, s_*)$ is a topological solution of (1.12) and satisfies that*

$$v(r, s_*) = I_* + O(r^{2-2aN_2}) \quad for\ some\ I_* \in \mathbb{R}\ as\ r \to \infty. \tag{3.9}$$

Moreover, $\beta(s_*) = 2N_2$.

The proof of Proposition 3.3 follows from Lemma 3.4 and Lemma 3.5 below.

Lemma 3.4. *If N_2 is a positive integer and $1 < aN_2 < 2$, then one of the following alternatives holds:*

(i) *$\mathcal{A}^+_{a,N_2,a} = \mathbb{R}$. $\beta : \mathcal{A}^+_{a,N_2,a} \to (2N_2, 4/a)$ is continuous and bijective.*
(ii) *$\mathcal{A}^+_{a,N_2,a} = (-\infty, s_*)$ for some $s_* \in \mathbb{R}$. $\beta : \mathcal{A}^+_{a,N_2,a} \to (2N_2, 4/a)$ is continuous, bijective, and strictly decreasing.*
(iii) *$\mathcal{A}^+_{a,N_2,a} = (s_*, \infty)$ for some $s_* \in \mathbb{R}$. $\beta : \mathcal{A}^+_{a,N_2,a} \to (2N_2, 4/a)$ is continuous, bijective, and strictly increasing.*

Moreover, the asymptotic behavior (3.7) holds true for $s \in \mathcal{A}^+_{a,N_2,a}$.

Proof. We divide the proof into four steps.

Step 1. $\mathcal{A}^+_{a,N_2,a} \neq \emptyset$.

Put $N_0 = (2 - aN_2)/a$. Given $\hat{N}_2 \in (0, N_0)$, set $\hat{a} = (2 - aN_2)/\hat{N}_2$ and consider the following initial value problem:

$$\begin{cases} \hat{v}'' + \frac{1}{r}\hat{v}' = r^{-2\hat{a}\hat{N}_2} f_1(\hat{v}, a), & r > 0, \\ \hat{v}(r, \hat{s}) = -2\hat{N}_2 \ln r + \hat{s} + o(1), & \text{near} \quad r = 0. \end{cases} \tag{3.10}$$

We notice that $0 < \hat{a}\hat{N}_2 < 1$. Hence, by Proposition 2.1, there exists a number $\hat{s}_* \in [-\infty, \infty)$ such that $\mathcal{A}^-_{\hat{a},\hat{N}_2,a} = (-\infty, \hat{s}^*]$ and $\mathcal{A}^+_{\hat{a},\hat{N}_2,a} = (\hat{s}^*, \infty)$ associated with (3.10). We also note that $\hat{s}^* > -\infty$ if $1 - \hat{N}_2 < \hat{a}\hat{N}_2 < 1$, whereas $\hat{s}^* = -\infty$ and hence $\mathcal{A}^-_{\hat{a},\hat{N}_2,a} = \emptyset$ if $0 < \hat{a}\hat{N}_2 \leq 1 - \hat{N}_2$. Moreover, for each $\hat{s} > \hat{s}^*$,

$$\hat{v}(r, \hat{s}) = \left[-2\hat{N}_2 + \hat{\beta}(\hat{s})\right] \ln r + \hat{I} + o(1) \quad \text{as} \quad r \to \infty, \tag{3.11}$$

where $\hat{I}$ is a constant and

$$\hat{\beta}(\hat{s}) = \int_0^\infty r^{1-2\hat{a}\hat{N}_2} f_1(\hat{v}(r, \hat{s}), a) dr > \bar{\beta}_{\hat{a},\hat{N}_2,a} = \frac{4}{a}(1 + a\hat{N}_2 - \hat{a}\hat{N}_2). \tag{3.12}$$

It follows from the choice of $\hat{N}_2$ that

$$2N_2 + 2\hat{N}_2 - \bar{\beta}_{\hat{a},\hat{N}_2,a} = \frac{2}{a}(2 - aN_2 - a\hat{N}_2) > \frac{2}{a}(2 - aN_2 - aN_0) = 0. \tag{3.13}$$

Since $\hat{\beta} : (\hat{s}_*, \infty) \to (\bar{\beta}_{\hat{a},\hat{N}_2,a}, \infty)$ is bijective by Proposition 2.1, there exists $\hat{s}_0 = \hat{s}_0(\hat{N}_2) \in \mathcal{A}^+_{\hat{a},\hat{N}_2,a}$ such that

$$\hat{\beta}(\hat{s}_0) = 2N_2 + 2\hat{N}_2 \tag{3.14}$$

and thus by (3.11)

$$\hat{v}(r, \hat{s}_0) = 2N_2 \ln r + \hat{I}_0 + o(1) \quad \text{for some} \quad \hat{I}_0 = \hat{I}_0(\hat{N}_2) \quad \text{as} \quad r \to \infty. \tag{3.15}$$

If we set $v(r, \hat{I}_0) = \hat{v}(r^{-1}, \hat{s}_0)$, then $v(r, \hat{I}_0)$ satisfies

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$$\begin{cases} v'' + \dfrac{1}{r}v' = r^{-2aN_2} f_1(v,a), & r > 0, \\ v(r,\hat{I}_0) = -2N_2 \ln r + \hat{I}_0 + o(1), & \text{near } r = 0, \\ v(r,\hat{I}_0) = 2\hat{N}_2 \ln r + \hat{s}_0 + o(1), & \text{as } r \to \infty, \end{cases} \tag{3.16}$$

which implies that $\hat{I}_0 \in \mathcal{A}^+_{a,N_2,a}$. In other words, $v(r,\hat{I}_0)$ is a type II solution of (1.12).

We write the third line of (3.16) as

$$v(r,\hat{I}_0) = [-2N_2 + \beta(\hat{I}_0)] \ln r + \hat{s}_0 + o(1) \quad \text{as } r \to \infty,$$

where $\beta(\hat{I}_0) = \hat{\beta}(\hat{s}_0) = 2N_2 + 2\hat{N}_2$. It follows from Lemma 3.2 that $\beta(\hat{I}_0) \in (2N_2, 4/a)$. Moreover, β is continuous by the Lebesgue Dominated Convergence Theorem.

Step 2. *Let $\hat{m}(\hat{N}_2) = \min_{r>0} \hat{v}(r,\hat{s}_0) = \min_{r>0} v(r,\hat{I}_0) = m(\hat{I}_0)$. Then,*

$$\lim_{\hat{N}_2 \nearrow N_0} \hat{m}(\hat{N}_2) = \infty. \tag{3.17}$$

Let $\hat{z} = \hat{z}(\hat{N}_2)$ be the unique minimum point of $\hat{v}(r,\hat{s}_0)$ and let $r_1 = r_1(\hat{N}_2) < \hat{z}$ be the unique point such that $\hat{v}(r_1,\hat{s}_0) = \hat{m}(\hat{N}_2) + 1$. Since $r\hat{v}'(r,\hat{s}_0) \geq -2\hat{N}_2$ for all $r > 0$, we obtain that $r_1/\hat{z} \leq e^{-1/2\hat{N}_2}$. Suppose that $\sup_{\hat{N}_2 \nearrow N_0} \hat{m}(\hat{N}_2) \leq \hat{\xi}$ for some $\hat{\xi} > 0$. Then, the integral representation (2.4) implies that $\infty > \hat{\xi} \geq \hat{m}(\hat{N}_2) \geq -2\hat{N}_2 \ln \hat{z} + \hat{s}_0$ and hence $\hat{z} \geq e^{(\hat{s}_0 - \hat{\xi})/2N_0}$. We note from (3.13) that $\hat{\beta}(\hat{s}_0) - \bar{\beta}_{\hat{a},\hat{N}_2,a} \to 0$ as $\hat{N}_2 \nearrow N_0$. Thus, by the Pohozaev identity (2.20), as $\hat{N}_2 \nearrow N_0$,

$$\begin{aligned} o(1) = \hat{\beta}(\hat{s}_0)\left(\hat{\beta}(\hat{s}_0) - \bar{\beta}_{\hat{a},\hat{N}_2,a}\right) &> \frac{16(1-\hat{a}\hat{N}_2)}{a\varepsilon^2} \int_{r_1}^{\hat{z}} \frac{r^{1-2\hat{a}\hat{N}_2}}{(1+e^{\hat{v}(r,\hat{s}_0)})^{1+a}} dr \\ &\geq \frac{8e^{-(1+a)(\hat{\xi}+1)}}{a\varepsilon^2} \cdot \hat{z}^{2-2\hat{a}\hat{N}_2} \left[1 - e^{-\frac{(1-\hat{a}\hat{N}_2)}{\hat{N}_2}}\right] \\ &\geq \frac{8e^{-(1+a)(\hat{\xi}+1)}}{a\varepsilon^2} \cdot e^{\frac{(\hat{s}_0-\hat{\xi})(aN_2-1)}{N_0}} \left[1 - e^{-\frac{(aN_2-1)}{N_0}}\right] > 0, \end{aligned}$$

which is a contradiction.

Step 3. *Proof of alternatives.*

By (3.14),

$$\lim_{\hat{N}_2 \searrow 0} \beta(\hat{I}_0) = 2N_2 \qquad \text{and} \qquad \lim_{\hat{N}_2 \nearrow N_0} \beta(\hat{I}_0) = 2N_2 + 2N_0 = \frac{4}{a}. \tag{3.18}$$

Hence, $\beta : \mathcal{A}^+ \to (2N_2, a/4)$ is onto. To see the injectivity of β, let us suppose that $\beta_0 = \beta(s_1) = \beta(s_2) \in (2N_2, 4/a)$ for some $s_1, s_2 \in \mathcal{A}^+$. Then, for each s_j, we have

$$v(r, s_j) = -2N_2 \ln r + s_j + o(1) \quad \text{as} \quad r \to 0,$$
$$v(r, s_j) = (-2N_2 + \beta_0) \ln r + I_j + o(1) \quad \text{as} \quad r \to \infty.$$

Let $\hat{N}_2 = (\beta_0 - 2N_2)/2 \in (0, N_0)$ and $\hat{a} = (2 - aN_2)/\hat{N}_2$. Then, $\hat{v}(r, \hat{s}_j)$ satisfies (3.10) and (3.11) with $\hat{s}_j = I_j$, $\hat{I}_j = s_j$ and $\hat{\beta}(\hat{s}_j) = \beta_0$. Since $\hat{\beta}$ is a bijective map on $\hat{\mathcal{A}}^+_{\hat{a},\hat{N}_2,a}$ by Proposition 2.1, it follows that $\hat{s}_1 = \hat{s}_2$. Hence, $\hat{v}(r, \hat{s}_1) = \hat{v}(r, \hat{s}_2)$ such that $s_1 = s_2$. As a consequence, $\beta : \mathcal{A}^+_{a,N_2,a} \to (2N_2, 4/a)$ is a homeomorphism. There are four alternatives: (a) $\mathcal{A}^+ = \mathbb{R}$, (b) $\mathcal{A}^+ = (\mu_1, \mu_2)$ for some $\mu_i \in \mathbb{R}$, (c) $\mathcal{A}^+ = (-\infty, s_*)$ for some $s_* \in \mathbb{R}$, or (d) $\mathcal{A}^+ = (s_*, \infty)$ for some $s_* \in \mathbb{R}$.

Let $\hat{M}_k \nearrow N_0$ be any sequence and set $s_k = \hat{I}_0(\hat{M}_k)$. Then, by Step 2, $m(s_k) \to \infty$ as $\hat{M}_k \nearrow N_0$. If s_k is bounded such that $s_k \to \bar{s} \in \mathcal{A}^+$ up to a subsequence, then

$$\infty = \lim_{\hat{M}_k \nearrow N_0} \hat{m}(\hat{M}_k) = \lim_{s_k \to \bar{s}} m(s_k) = m(\bar{s}) < \infty,$$

a contradiction. If $\bar{s} = \mu_i$ for the alternative (b) or $\bar{s} = s_*$ for (c) and (d), then $\mu_i, s_* \in \mathcal{A}^-$ such that

$$\infty = \lim_{\hat{M}_k \nearrow N_0} \hat{m}(\hat{M}_k) = \lim_{s_k \to \bar{s}} m(s_k) = -\infty,$$

a contradiction. In the sequel, the alternative (b) is excluded. Moreover, if $\hat{M}_k \nearrow N_0$, then $s_k = \hat{I}_0(\hat{M}_k) \to -\infty$ for (c) and $s_k = \hat{I}_0(\hat{M}_k) \to \infty$ for (d). Since $\beta(s_k) \to 4/a$ by (3.18), β is strictly decreasing for (c) and strictly increasing for (d).

Step 4. *Proof of* (3.7).

Let $u(r, s) = v(r, s) - (-2N_2 + \beta) \ln r$ such that $u(r, s) = o(\ln r)$ and $ru'(r, s) = o(1)$ as $r \to \infty$. Since $1 < aN_2 < 2$ and $2N_2 < \beta < 4/a$, we have $2 < a\beta < 4$. Given a positive number $\eta < (a\beta - 2)/a$, there exists a number r_0 such that $|u(r, s)| \le \eta \ln r$ for all $r \ge r_0$. Hence, $\delta = a\beta - a\eta - 2 > 0$ and

$$r^{1-2aN_2} f_1\big(v(r, s), a\big) = O\big(r^{1-2aN_2} e^{-av}\big) = O\big(r^{1-a\beta+a\eta}\big) = O\big(r^{-1-\delta}\big). \tag{3.19}$$

Integrating $(ru')' = r^{1-2aN_2} f_1(v, a)$ on (r, ∞), we obtain by (3.19) that for $r > r_0$

$$u(r, s) = u(r_0, s) - \int_{r_0}^{r} \frac{1}{t} \int_{t}^{\infty} y^{1-2aN_2} f_1(v(y, s), a) dy dt = O(1)$$

as $r \to \infty$. Hence, we can rewrite (3.19) as $r^{1-2aN_2} f_1\big(v(r, s), a\big) = O\big(r^{1-a\beta}\big)$. Consequently,

$$u(r,s) = I + \int_r^\infty \frac{1}{t} \int_t^\infty y^{1-2aN_2} f_1(v(y,s),a)dydt = I + O\big(r^{2-a\beta}\big),$$

where

$$I = u(r_0,s) - \int_{r_0}^\infty \frac{1}{t} \int_t^\infty y^{1-2aN_2} f_1(v(y,s),a)dydt.$$

This completes the proof. □

Lemma 3.5. *If N_2 is a positive integer and $1 < aN_2 < 2$, then there exists a number $s_* \in \mathbb{R}$ such that $\mathcal{A}^+ = (-\infty, s_*)$ and $\mathcal{A}^- = [s_*, \infty)$. Consequently, the dichotomy (ii) holds in Lemma 3.4. The function $\beta : \mathcal{A}^- \to (0, 2N_2]$ is continuous, bijective and strictly decreasing. Moreover, we have (3.8) for $s > s_*$ and (3.9) for $s = s_*$.*

Proof. The proof is split into five parts.

Step 1. *Either $\mathcal{A}^- = [s_*, \infty)$ or $\mathcal{A}^- = (-\infty, s_*]$ for some $s_* \in \mathbb{R}$.*

Given a number $\hat{N}_2 \in (0, N_2)$, set $\hat{a} = (2 - aN_2)/\hat{N}_2$ such that $0 < \hat{a}\hat{N}_2 < 1$. Let us consider the following problem:

$$\begin{cases} \hat{v}'' + \dfrac{1}{r}\hat{v}' = r^{-2\hat{a}\hat{N}_2} f_1(\hat{v}, a), & r > 0, \\ \hat{v}(r, \hat{s}) = 2\hat{N}_2 \ln r + \hat{s} + o(1), & \text{near} \quad r = 0. \end{cases} \tag{3.20}$$

This equation is a kind of the equation (4.1) with the hypothesis (A1) which will be studied in detail in the next section. Then, by Proposition 4.1 in the next section, it follows that

$$\hat{v}(r,\hat{s}) = \big[2\hat{N}_2 + \hat{\beta}(\hat{s})\big] \ln r + \hat{I} + o(1), \quad \text{as } r \to \infty \tag{3.21}$$

where

$$\hat{\beta}(\hat{s}) = \int_0^\infty r^{1-2\hat{a}\hat{N}_2} f_1\big(\hat{v}(r,\hat{s}), a\big)dr.$$

In addition, $\hat{\beta} : (-\infty, \infty) \to \big(\max\big\{\tilde{\beta}_{\hat{a},\hat{N}_2,a,\hat{N}_2}, 0\big\}, \infty\big)$ is bijective, where

$$\tilde{\beta}_{\hat{a},\hat{N}_2,a,\hat{N}_2} = \frac{4 - 4(\hat{a}\hat{N}_2 + a\hat{N}_2)}{a}.$$

We note that $2N_2 - 2\hat{N}_2 > 0$ and

$$2N_2 - 2\hat{N}_2 - \frac{4 - 4(\hat{a}\hat{N}_2 + a\hat{N}_2)}{a} = \frac{4 - 2aN_2 + 2a\hat{N}_2}{a} > 0.$$

Hence, there exists $\hat{s}_0 = \hat{s}_0(\hat{N}_2) \in \mathbb{R}$ such that

$$\begin{cases} \hat{\beta}(\hat{s}_0) = 2N_2 - 2\hat{N}_2, \\ \hat{v}(r, \hat{s}_0) = 2N_2 \ln r + \hat{I}_0 + o(1) \;\text{ for some } \hat{I}_0 = \hat{I}_0(\hat{N}_2) \text{ as } r \to \infty. \end{cases} \tag{3.22}$$

If we set $v(r, \hat{I}_0) = \hat{v}(r^{-1}, \hat{s}_0)$, then $v(r, \hat{I}_0)$ satisfies

$$\begin{cases} v'' + \dfrac{1}{r} v' = r^{-2aN_2} f_1(v, a), \quad r > 0, \\ v(r, \hat{I}_0) = -2N_2 \ln r + \hat{I}_0 + o(1), \quad \text{near } r = 0, \\ v(r, \hat{I}_0) = -2\hat{N}_2 \ln r + \hat{s}_0 + o(1), \quad \text{as } r \to \infty, \end{cases} \tag{3.23}$$

which implies that $\hat{I}_0 \in \mathcal{A}^-_{a,N_2,a}$. So, $v(r, \hat{I}_0)$ is a type I solution of (1.12). Furthermore, Lemma 3.4 tells us that either $\mathcal{A}^-_{a,N_2,a} = [s_*, \infty)$ or $\mathcal{A}^-_{a,N_2,a} = (-\infty, s_*]$ for some $s_* \in \mathbb{R}$.

We write the third line of (3.23) as

$$v(r, \hat{I}_0) = [-2N_2 + \beta(\hat{I}_0)] \ln r + \hat{s}_0 + o(1), \quad \text{as } r \to \infty,$$

where $\beta(\hat{I}_0) = \hat{\beta}(\hat{s}_0) = 2N_2 - 2\hat{N}_2$.

Step 2. $\beta : \mathring{\mathcal{A}}^-_{a,N_2,a} \to (0, 2N_2)$ *is continuous and bijective where* $\mathring{\mathcal{A}}^-_{a,N_2,a}$ *is the interior of* $\mathcal{A}^-_{a,N_2,a}$.

It follows from Lemma 3.2 that $\beta : \mathcal{A}^-_{a,N_2,a} \to (0, 2N_2]$. If $\beta(s_1) = \beta(s_2) \in (0, 2N_2]$ for some $s_1, s_2 \in \mathcal{A}^-_{a,N_2,a}$, then by proceeding as in the proof of Lemma 3.4, we deduce that $s_1 = s_2$. This implies that β cannot attain the value $2N_2$ on $\mathring{\mathcal{A}}^-_{a,N_2,a}$ and it is injective on $\mathring{\mathcal{A}}^-_{a,N_2,a}$. The continuity on $\mathring{\mathcal{A}}^-_{a,N_2,a}$ is a consequence of the Lebesgue Convergence Theorem. Moreover, by (3.22),

$$\lim_{\hat{N}_2 \searrow 0} \beta(\hat{I}_0) = 2N_2 \quad \text{and} \quad \lim_{\hat{N}_2 \nearrow N_2} \beta(\hat{I}_0) = 2N_2 - 2\hat{N}_2 = 0. \tag{3.24}$$

Hence, β is onto on $\mathring{\mathcal{A}}^-_{a,N_2,a}$. So, $\beta : \mathring{\mathcal{A}}^-_{a,N_2,a} \to (0, 2N_2)$ is a homeomorphism.

Step 3. $\hat{I}_0(\hat{N}_2) \to \infty$ *as* $\hat{N}_2 \nearrow N_2$. *So,* $\mathcal{A}^- = [s_*, \infty)$ *and* $\beta : \mathring{\mathcal{A}}^-_{a,N_2,a} \to (0, 2N_2)$ *is strictly decreasing. Moreover, the dichotomy (ii) holds in Lemma 3.4.*

For a solution $\hat{v}(r, \hat{s}_0)$ of (3.20) satisfying (3.22), let $\hat{u}(r, \hat{s}_0) = \hat{v}(r, \hat{s}_0) - (2\hat{N}_2 + \hat{\beta}) \ln r$ such that $r\hat{u}'(r, \hat{s}_0) = o(1)$ as $r \to \infty$. Since

$$(r\hat{u}')' = r^{1-2\hat{a}\hat{N}_2} f_1(\hat{v}, a) > 0, \tag{3.25}$$

$(r\hat{u}')(\cdot, \hat{s}_0)$ is increasing. Moreover, since $r\hat{u}' = r\hat{v}' - (2\hat{N}_2 + \hat{\beta}) < 0$, $\hat{u}(\cdot, \hat{s}_0)$ is decreasing.

Claim 1. If $\hat{R}_1 = \hat{R}_1(\hat{N}_2)$ is a unique number such that $\hat{v}(\hat{R}_1, \hat{s}_0) = 0$, then $\hat{R}_1 \to 0$ as $\hat{N}_2 \nearrow N_2$.

To see the Claim, let $\hat{R}_2 = \hat{R}_2(\hat{N}_2)$ be a unique number such that $\hat{v}(\hat{R}_2, \hat{s}_0) = 1$. Since $r\hat{v}' < 2\hat{N}_2 + \hat{\beta} = 2N_2$, we have $\hat{R}_1/\hat{R}_2 < e^{-1/(2N_2)}$. Then, by (3.24), as $\hat{N}_2 \nearrow N_2$,

$$0 \;\leftarrow\; \hat{\beta} > \int_{\hat{R}_1}^{\hat{R}_2} r^{1-2\hat{a}\hat{N}_2} f_1\big(\hat{v}(r, \hat{s}_0), a\big) dr$$

$$\geq \Big(\inf_{0\leq v\leq 1} f_1(v, a)\Big) \cdot \frac{\hat{R}_2^{2-2\hat{a}\hat{N}_2}\big(1 - e^{-\frac{1-\hat{a}\hat{N}_2}{N_2}}\big)}{2 - 2\hat{a}\hat{N}_2} \;>\; 0.$$

Hence, $\hat{R}_2 \to 0$.

Claim 2. Given $\hat{t} > 0$, $\hat{u}(\hat{t}, \hat{s}_0) \to \infty$ as $\hat{N}_2 \nearrow N_2$.

Otherwise, suppose that $\hat{u}(\hat{t}, \hat{s}_0) \leq c_0$ as $\hat{N}_2 \nearrow N_2$. By Claim 1, $\hat{v}(\hat{t}, \hat{s}_0) > 0$ as $\hat{N}_2 \nearrow N_2$. Since $r\hat{v}' < 2N_2$ for all $r > 0$, we have by (3.22)

$$\hat{v}(2\hat{t}, \hat{s}_0) < \hat{u}(\hat{t}, \hat{s}_0) + (2\hat{N}_2 + \hat{\beta}) \ln \hat{t} + 2N_2 \ln 2 \leq c_0 + 2N_2(\ln \hat{t} + \ln 2) =: c_1.$$

Hence, as $\hat{N}_2 \nearrow N_2$,

$$0 \;\leftarrow\; \hat{\beta} > \Big(\inf_{0\leq v\leq c_1} f_1(v, a)\Big) \cdot \frac{(2\hat{t})^{2-2\hat{a}\hat{N}_2} - \hat{t}^{2-2\hat{a}\hat{N}_2}}{2 - 2\hat{a}\hat{N}_2} > 0,$$

a contradiction.

Given a number $\delta \in \big(0, 1/(3a)\big)$, let $h_k(r) \in C^\infty(0, \infty)$ such that $h_k(r) = -k\delta \ln r$ for $r > 1$ and $h_k(r)$ is bounded for $0 < r < 1$.

Claim 3. $\hat{u}(r, \hat{s}_0) > h_2(r)$ for all $r > 0$ if $\hat{N}_2$ is sufficiently close to N_2.

Otherwise, since $r\hat{u}'(r, \hat{s}_0) = o(1)$ as $r \to \infty$, there exist the first and the second numbers $\hat{r}_1, \hat{r}_2 \in (1, \infty)$ such that $\hat{r}_1 < \hat{r}_2$ and $\hat{u}(\hat{r}_i, \hat{s}_0) = h_1(\hat{r}_i)$, $i = 1, 2$. By Claim 2, $\hat{r}_1 = \hat{r}_1(\hat{N}_2) \to \infty$ as $\hat{N}_2 \nearrow N_2$. Similarly, there exist the first and the second numbers $\hat{r}_3 < \hat{r}_4$ such that $\hat{u}(\hat{r}_i, \hat{s}_0) = h_2(\hat{r}_i)$, $i = 3, 4$. Obviously, $\hat{r}_1 < \hat{r}_3 < \hat{r}_4 < \hat{r}_2$ and $\hat{u}'(\hat{r}_1, \hat{s}_0) < h_1'(\hat{r}_1)$, $\hat{u}'(\hat{r}_3, \hat{s}_0) < h_2'(\hat{r}_3)$, $\hat{u}'(\hat{r}_4, \hat{s}_0) > h_2'(\hat{r}_4)$, and $\hat{u}'(\hat{r}_2, \hat{s}_0) > h_1'(\hat{r}_2)$. Hence, we can find two numbers $\hat{t}_1 \in (\hat{r}_3, \hat{r}_4)$ and $\hat{t}_2 \in (\hat{r}_1, \hat{r}_2)$ with $\hat{t}_1 < \hat{t}_2$ such that $\hat{t}_1\hat{u}'(\hat{t}_1, \hat{s}_0) = -2\delta$ and $\hat{t}_2\hat{u}'(\hat{t}_2, \hat{s}_0) = -\delta$. Since $r\hat{u}' = r\hat{v}' - (2N_2 + \hat{\beta}) > -\hat{\beta}$, we have by Claim 2

$$\hat{u}(r, \hat{s}_0) > u(1, \hat{s}_0) - \hat{\beta} \ln r > -\hat{\beta} \ln r \;\text{ for all } r > 1 \text{ as } \hat{N}_2 \nearrow N_2.$$

Hence, as $\hat{N}_2 \nearrow N_2$, for $r > 1$,

$$\begin{aligned} r^{2-2\hat{a}\hat{N}_2} G\big(\hat{v}(r, \hat{s}_0), a\big) &= O\big(r^{-2+2aN_2} e^{-a\hat{u} - a(2\hat{N}_2+\hat{\beta}) \ln r}\big) \\ &= O\big(r^{-2+2a(N_2-\hat{N}_2)}\big) = O\big(r^{-2+o(1)}\big), \end{aligned} \tag{3.26}$$

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where G is defined by (2.16). Now, multiplying (3.20) by $r^2\hat{v}'$ and reminding that $\hat{t}_1 \to \infty$, we infer from (3.26) that as $\hat{N}_2 \nearrow N_2$,

$$\frac{1}{2}\big[\hat{t}_2\hat{v}'(\hat{t}_2,\hat{s}_0)\big]^2 - \frac{1}{2}\big[\hat{t}_1\hat{v}'(\hat{t}_1,\hat{s}_0)\big]^2$$
$$= \Big[r^{2-2\hat{a}\hat{N}_2}G\big(\hat{v}(r,\hat{s}_0),a\big)\Big]_{r=\hat{t}_1}^{\hat{t}_2} - (2-2\hat{a}\hat{N}_2)\int_{\hat{t}_1}^{\hat{t}_2} r^{1-2\hat{a}\hat{N}_2}G\big(\hat{v}(r,\hat{s}_0),a\big)dr \ = o(1).$$

However, as $\hat{N}_2 \nearrow N_2$, we also get

$$\frac{1}{2}\big[\hat{t}_2\hat{v}'(\hat{t}_2,\hat{s}_0)\big]^2 - \frac{1}{2}\big[\hat{t}_1\hat{v}'(\hat{t}_1,\hat{s}_0)\big]^2 = \frac{1}{2}\big[-\delta+(2\hat{N}_2+\hat{\beta})\big]^2 - \frac{1}{2}\big[-2\delta+(2\hat{N}_2+\hat{\beta})\big]^2$$
$$= \frac{\delta}{2}(2N_2-3\delta)+o(1),$$

a contradiction and the claim follows.

Now, integrating (3.25) on $(1,r)$, we obtain that for $r>1$

$$\hat{u}(r,\hat{s}_0) = \hat{u}(1,\hat{s}_0) - \int_1^r \frac{1}{t}\int_t^\infty y^{1-2\hat{a}\hat{N}_2} f_1(\hat{v}(y,\hat{s}_0),a)dydt.$$

We note by Claim 3 that for $r>1$

$$r^{1-2\hat{a}\hat{N}_2} f_1(\hat{v}(r,\hat{s}_0),a) \le Cr^{1-2\hat{a}\hat{N}_2-a(2\hat{N}_2+\hat{\beta})}e^{-a\hat{u}} \le Cr^{-3+2a(N_2-\hat{N}_2)-a\hat{\beta}+2a\delta}$$
$$\le C(r^{-3+3a\delta})$$

as $\hat{N}_2 \nearrow N_2$. Here, C is independent of $\hat{N}_2$. Hence, $\hat{u}(r,\hat{s}_0) = \hat{u}(1,\hat{s}_0) + O(1)$ for $r>1$ as $\hat{N}_2 \nearrow N_2$. This implies by Claim 2 that $\hat{u}(r,\hat{s}_0) \to \infty$. Comparing this with (3.21), we deduce that $\hat{I}_0 \to \infty$ and $\beta(\hat{I}_0) = \hat{\beta}(\hat{s}_0) \to 0$ as $\hat{N}_2 \nearrow N_2$. Eventually, we conclude that $\mathcal{A}^- = [s_*,\infty)$ and $\beta : \mathring{\mathcal{A}}^-_{a,N_2,a} \to (0,2N_2)$ is strictly decreasing by (3.24). As a consequence, $\mathcal{A}^+ = (-\infty, s_*)$ and the dichotomy (ii) holds in Lemma 3.4.

Step 4. *$v(r,s_*)$ is a topological solution and $\beta(s_*) = 2N_2$.*

Given a number $\hat{N}_2 \in (0,N_2)$, set $\hat{a} = (2-aN_2)/\hat{N}_2$ and consider the following problem:

$$\begin{cases} \hat{v}'' + \frac{1}{r}\hat{v}' = r^{-2\hat{a}\hat{N}_2} f_1(\hat{v},a), & r>0, \\ \hat{v}(r,\hat{s}) = \hat{s} + o(1), & \text{near} \quad r=0. \end{cases} \tag{3.27}$$

This equation is a kind of the equation (4.1) with the hypothesis (A1) in the next section. We note that the equation (3.27) depends only on the product $\hat{a}\hat{N}_2 = 2 - aN_2$. By Proposition 4.1, it follows that

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$$\hat{v}(r,\hat{s}) = \hat{\beta}(\hat{s})\ln r + \hat{I} + o(1), \quad \text{as } r \to \infty \tag{3.28}$$

where

$$\hat{\beta}(\hat{s}) = \int_0^\infty r^{1-2\hat{a}\hat{N}_2} f_1\big(\hat{v}(r,\hat{s}),a\big)dr.$$

Moreover, $\hat{\beta} : (-\infty,\infty) \to (\tilde{\beta}_{\hat{a},0,a,\hat{N}_2},\infty)$ is bijective, where

$$\tilde{\beta}_{\hat{a},0,a,\hat{N}_2} = \frac{4-4\hat{a}\hat{N}_2}{a} = \frac{4aN_2-4}{a} > 0$$

is defined by (4.6) in the next section. Since $2N_2 > \tilde{\beta}_{\hat{a},0,a,\hat{N}_2}$, by Proposition 4.1 there exists $\hat{s}_* \in \mathbb{R}$ such that $\hat{\beta}(\hat{s}_*) = 2N_2$ and

$$\hat{v}(r,\hat{s}_*) = 2N_2\ln r + \hat{I}_* + o(1) \quad \text{for some} \quad \hat{I}_* \quad \text{as} \quad r \to \infty.$$

If we set $v(r,\hat{I}_*) = \hat{v}(r^{-1},\hat{s}_*)$, then $v(r,\hat{I}_*)$ satisfies

$$\begin{cases} v'' + \frac{1}{r}v' = r^{-2aN_2}f_1(v,a), & r>0, \\ v(r,\hat{I}_*) = -2N_2\ln r + \hat{I}_* + o(1), & \text{near } r=0, \\ v(r,\hat{I}_*) = \hat{s}_* + o(1), & \text{as } r \to \infty. \end{cases} \tag{3.29}$$

Since $\hat{v}(\cdot,\hat{s}_*)$ is strictly increasing, $v(\cdot,\hat{I}_*)$ is strictly decreasing, which implies that $\hat{I}_* \in \mathcal{A}^-_{a,N_2,a}$ and $\beta(\hat{I}_*) = 2N_2$. Since $\beta : \mathring{\mathcal{A}}^-_{a,N_2,a} = (s_*,\infty) \to (0,2N_2)$ is continuous, bijective, and strictly decreasing, we deduce that $\hat{I}_* = s_*$. In the sequel, we conclude that $\beta : [s_*,\infty) \to (0,2N_2]$ is continuous, bijective and strictly decreasing. Moreover, (3.29) tells us that $v(r,s_*)$ is a topological solution.

Step 5. *Proof of* (3.8) *and* (3.9).

For $s > s_*$, let $u(r,s) = v(r,s) - (-2N_2+\beta)\ln r$ such that $u(r,s) = o(\ln r)$ and $ru'(r,s) = o(1)$ as $r \to \infty$. Since r^{1-2N_2} is integrable near ∞, by integrating $(ru')' = r^{1-2aN_2}f_1(v,a)$, we deduce that for $r > 1$,

$$u(r,s) = u(1,s) - \int_1^r \frac{1}{t}\int_t^\infty y^{1-2aN_2} f_1(v(y,s),a)dydt = O(1).$$

So, $r^{1-2aN_2}f_1\big(v(r,s),a\big) = O\big(r^{1-2aN_2-2N_2+\beta}\big)$ for all large r such that

$$u(r,s)=I+\int_r^\infty \frac{1}{t}\int_t^\infty y^{1-2aN_2}f_1(v(y,s),a)dydt=I+O\big(r^{2-2aN_2-2N_2+\beta}\big),$$

where

$$I=u(1,s)-\int_1^\infty \frac{1}{t}\int_t^\infty y^{1-2aN_2}f_1(v(y,s),a)dydt.$$

On the other hand, by letting $s_*=\hat{I}_*$ and $I_*=\hat{s}_*$, we deduce from (3.29) that $\big[rv'(r,s_*)\big]'\to r^{1-2aN_2}f(I_*,a)$ as $r\to\infty$. Hence, a similar argument as above implies that as $r\to\infty$,

$$v(r,s_*)=I_*+\int_r^\infty \frac{1}{t}\int_y^\infty y^{1-2aN_2}f_1(v(y,s_*),a)dydt=I_*+O(r^{2-2aN_2}),$$

where I_* is explicitly given by

$$I_*=v(1,s_*)-\int_1^\infty \frac{1}{t}\int_y^\infty y^{1-2aN_2}f_1(v(y,s_*),a)dydt.$$

This completes the proof of Lemma 3.5. □

The following proposition is the final topic of this section, the case $aN_2\ge 2$.

Proposition 3.6. *If $aN_2\ge 2$, $v(r,s)$ is a type I solution of (1.12) for every $s\in\mathbb{R}$. The function $\beta:\mathbb{R}\to(0,4/a)$ is continuous, bijective and strictly decreasing. Moreover, we have (1.19) for the asymptotic behavior of solutions.*

Proof. We know from Lemma 3.2 that $v(r,s)$ is a type I solution of (1.12) for every $s\in\mathbb{R}$ and $\beta:\mathbb{R}\to(0,4/a)$. Let $\hat{a}$ be any positive number and $\hat{M}=(2-aN_2)/\hat{a}\le 0$. Given a number $\hat{N}\in(N_*,N_2)$ with $N_*=(aN_2-2)/a\ge 0$, consider the following equation:

$$\begin{cases}\hat{v}''+\dfrac{1}{r}\hat{v}'=r^{-2\hat{a}\hat{M}}f_1(\hat{v},a), & r>0,\\ \hat{v}(r,\hat{s})=2\hat{N}\ln r+\hat{s}+o(1), & \text{near}\quad r=0.\end{cases}\tag{3.30}$$

This equation is a kind of the equation (4.1) with the hypothesis (A2) in the next section. By Proposition 4.1 in the next section, it follows that

$$\hat{v}(r,\hat{s})=\big[2\hat{N}+\hat{\beta}(\hat{s})\big]\ln r+\hat{I}+o(1)\quad \text{as}\quad r\to\infty,$$

where

$$\hat{\beta}(\hat{s}) = \int_0^\infty r^{1-2\hat{a}\hat{M}} f_1\big(\hat{v}(r,\hat{s}), a\big)dr \ : \ (-\infty,\infty) \to \big(\max\{\tilde{\beta}_{\hat{a},\hat{N},a,\hat{M}}, 0\}, \infty\big)$$

is a bijective function with

$$\tilde{\beta}_{\hat{a},\hat{N},a,\hat{M}} = \frac{4-4(\hat{a}\hat{M}+a\hat{N})}{a} = \frac{4(aN_2 - a\hat{N} - 1)}{a}.$$

We note that $2N_2 - 2\hat{N} > 0$ and

$$2N_2 - 2\hat{N} - \tilde{\beta}_{\hat{a},\hat{N},a,\hat{M}} = \frac{4-2aN_2+2a\hat{N}}{a} > \frac{4-2aN_2+2aN_*}{a} = 0.$$

So, there exists $\hat{s}_0 = \hat{s}_0(\hat{N}_2) \in \mathbb{R}$ such that $\hat{\beta}(\hat{s}_0) = 2N_2 - 2\hat{N}$ and

$$\hat{v}(r,\hat{s}_0) = 2N_2 \ln r + \hat{I}_0 + o(1) \quad \text{for some} \quad \hat{I}_0 = \hat{I}_0(\hat{N}) \quad \text{as} \quad r \to \infty.$$

If we set $v(r,\hat{I}_0) = \hat{v}(r^{-1},\hat{s}_0)$, then $v(r,\hat{I}_0)$ satisfies

$$\begin{cases} v'' + \dfrac{1}{r}v' = r^{-2aN_2} f_1(v,a) \quad \text{for} \quad r > 0, \\ v(r,\hat{I}_0) = -2N_2 \ln r + \hat{I}_0 + o(1) \quad \text{near } r = 0, \\ v(r,\hat{I}_0) = \big[-2N_2 + \hat{\beta}(\hat{s}_0)\big] \ln r + \hat{s}_0 + o(1) \quad \text{as } r \to \infty. \end{cases} \tag{3.31}$$

Hence, $v(r,\hat{I}_0)$ is a solution of (1.12). We write the third line of (3.31) as follows:

$$v(r,\hat{I}_0) = [-2N_2 + \beta(\hat{I}_0)] \ln r + \hat{s}_0 + o(1) \quad \text{as } r \to \infty,$$

where $\beta(\hat{I}_0) = \hat{\beta}(\hat{s}_0) = 2N_2 - 2\hat{N}$.

Since

$$\lim_{\hat{N} \nearrow N_2} \beta(\hat{I}_0) = 0, \qquad \lim_{\hat{N} \searrow N_*} \beta(\hat{I}_0) = \frac{4}{a},$$

β is onto. In addition, arguing as in Step 3 of the proof of Lemma 3.4, one can see that β is one-to-one. Finally, following exactly the same argument of Step 3 in Lemma 3.5, we deduce that $\hat{I}_0 \to \infty$ as $\hat{N} \nearrow N_2$ and thus $\beta(s) = \hat{\beta}(\hat{s}_0) \to 0$ as $\hat{I}_0 \to \infty$. As a consequence, β is strictly decreasing and bijective. The asymptotic behavior of $v(r,s)$ as $r \to \infty$ follows from the similar argument as Step 5 in the proof of Lemma 3.5. □

4. A generalized version for the case $N_1 \geq 0$ and $N_2 = 0$

In this section, we study the equation (1.13) and prove Theorem 1.2. In particular, we will treat a generalized version of (1.13):

$$\begin{cases} v'' + \frac{1}{r}v' = r^{-2aM} f_1(v, b), & r > 0, \\ v(r) = 2N \ln r + s + o(1) \quad \text{near} & r = 0. \end{cases} \tag{4.1}$$

Here, we assume one of the following:

(A1) N and M are nonnegative real numbers, and a and b are positive real numbers such that $0 \leq aM < 1$,
(A2) N, a, b are a positive real numbers and M is a nonpositive real number.

Regarding the hypothesis (A1), we have three examples in this paper as follows. First, if $N > 0$, $a = b$, and $M = 0$, (4.1) corresponds to (1.13). Second, if $N = M > 0$, then (4.1) leads to the equation (3.20) in the proof of Lemma 3.5. Third, if $M > 0$ and $N = 0$, (4.1) represents the equation (3.27) in the proof of Lemma 3.5. On the other hand, the hypothesis (A2) was used in the equation (3.30) so that $N > 0$ and $M \leq 0$.

Let us denote the solution of (4.1) by $v(r, s, a, N, b, M)$, or simply $v(r, s)$. Since $(rv')' > 0$, v is strictly increasing. For any fixed $r_0 > 0$, integrating $(rv')' > 0$ twice on $[r_0, r]$, we get

$$v(r, s) > v(r_0, s) + r_0 v'(r_0, s) \ln \frac{r}{r_0}, \quad \forall r > r_0.$$

As a consequence, $\lim_{r\to\infty} v(r, s) = \infty$ for any $s \in \mathbb{R}$. Hence, $v(r, s)$ is a type II solution for any $s \in \mathbb{R}$. We note that $v(r, s)$ is a solution of (1.13) if and only if

$$v(r, s) = 2N \ln r + s + \int_0^r \frac{1}{t} \int_0^t y^{1-2aM} f_1\big(v(y, s), b\big) dy dt, \quad r > 0. \tag{4.2}$$

As before, we define

$$\beta(s) = \int_0^\infty r^{1-2aM} f_1\big(v(r, s), b\big) dr \; > \; 0. \tag{4.3}$$

As in Section 2, one can see that $\beta(s)$ is always finite for each s and

$$v(r, s) = \big[2N + \beta(s)\big] \ln r + O(1)$$

as $r \to \infty$. Since $\beta(s) < \infty$, we see that

$$\beta(s) \geq \max\left\{\frac{2 - 2(aM + bN)}{b}, 0\right\}.$$

On the other hand, we obtain the Pohozaev type identities:

$$\beta(\beta+4N) = -(4-4aM)\int_0^\infty r^{1-2aM} G\big(v(r,s),b\big)\,dr, \tag{4.4}$$

$$\beta\big(\beta-\tilde\beta_{a,N,b,M}\big) = \frac{16-16aM}{b\varepsilon^2}\int_0^\infty \frac{r^{1-2aM}}{(1+e^v)^{1+b}}\,dr, \tag{4.5}$$

where

$$\tilde\beta = \tilde\beta_{a,N,b,M} = \frac{4-4(aM+bN)}{b}. \tag{4.6}$$

As a consequence, we obtain a more accurate range of $\beta(s)$ such that $\beta(s) > \max\{\tilde\beta, 0\}$. Now, the main result of this section is the following.

Proposition 4.1. *Assume the hypothesis (A1) or (A2). Then,*

$$\beta : (-\infty,\infty) \to \big(\max\{\tilde\beta_{a,N,b,M},0\},\infty\big)$$

is bijective and strictly decreasing. Furthermore,

$$v(r,s) = [2N+\beta]\ln r + I + O(r^{2-2(aM+bN)-b\beta}) \quad as \quad r\to\infty,$$

where $I = I(a,N,b,M,s)$ is a constant.

Proof. For the surjectivity of γ, we prove that

$$\lim_{s\to\infty}\beta(s) = \max\{\tilde\beta,0\}, \qquad \lim_{s\to-\infty}\beta(s)=\infty. \tag{4.7}$$

To see this, let $\lambda>0$ be given. Let r_s be a unique number such that $\lambda = v(r_s,s) \ge 2N\ln r_s + s$. Hence, $r_s\to 0$ as $s\to\infty$ and

$$0<\beta(\beta-\tilde\beta) = \Big(\int_0^{r_s} + \int_{r_s}^\infty\Big)\frac{(16-16aM)r^{1-aM}}{b\varepsilon^2\big(1+e^{v(r,s)}\big)^{1+b}}\,dr \;\le\; o(1) + \frac{4e^{-\lambda}(1-aM)}{b}\beta$$

as $s\to\infty$. So,

$$\limsup_{s\to\infty}\left[\beta(s)\Big(\beta(s)-\tilde\beta-\frac{4e^{-\lambda}(1-aM)}{b}\Big)\right]\le 0.$$

Letting $\lambda\to\infty$, we obtain the first limit of (4.7).

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On the other hand, let $s_n \to -\infty$ and assume that $\beta(s_n) \le B$ for some $B > 0$. Let R_n and T_n be unique numbers such that $v(R_n, s_n) = 0$ and $v(T_n, s_n) = 1$. Then, we infer from the solution formula (4.2) that $R_n \to \infty$ as $s_n \to -\infty$. Since $rv'(r,s) \le (2N + B)$, we have $T_n/R_n \ge e^{1/(2N+B)}$. Then, as $s_n \to -\infty$,

$$B > \beta(s_n) \ge \int_{R_n}^{T_n} r^{1-2aM} f_1\big(v(r,s), b\big)dr$$

$$\ge \Big(\inf_{0\le v\le 1} f_1(v,b)\Big) \cdot \frac{R_n^2}{2-2aM} \cdot \big(e^{\frac{2-2aM}{2N_1+B}} - 1\big) \quad \to \quad \infty,$$

a contradiction. Thus, the second limit of (4.7) follows.

To see the injectivity of β, let $\varphi(r,s) = \partial v(r,s)/\partial s$ be the unique solution of the linearized equation

$$\begin{cases} \varphi'' + \dfrac{1}{r}\varphi' = r^{-2aM} f_1'\big(v(r,s), b\big)\varphi, \\ \varphi(0,s) = 1, \qquad \varphi'(0,s) = 0. \end{cases} \tag{4.8}$$

Due to the Lebesgue Dominated Convergence Theorem, it holds that

$$\beta'(s) = \int_0^\infty r^{1-2aM} f_1'(v(r,s), b)\varphi(r,s)dr = \lim_{r\to\infty} r\varphi'(r,s).$$

Set $w_c(r,s) = rv'(r,s) + c$ for $c \in \mathbb{R}$. Then, w_c satisfies

$$\begin{cases} w_c'' + \dfrac{1}{r}w_c' = r^{-2aM} f_1'\big(v(r,s), b\big)w_c + r^{-2aM}\Psi_c(r,s), \\ w_c(0,s) = 2N + c, \qquad w_c'(0,s) = 0, \end{cases} \tag{4.9}$$

where

$$\Psi_c(r,s) = (2-2aM) f_1\big(v(r,s), b\big) - c f_1'\big(v(r,s), b\big)$$
$$= \frac{4e^{v(r,s)}\Big\{\big[(2-2aM) + cb\big]e^{v(r,s)} + \big[(2-2aM) - c\big]\Big\}}{\varepsilon^2(1+e^{v(r,s)})^{2+b}}.$$

By proceeding in the same way as in the proof of Lemma 2.10, we can deduce that $\varphi(\cdot, s)$ has a unique zero r_1.

We will show that $\lim_{r\to\infty} r\varphi'(r) = \delta \ne 0$. Then, β is strictly decreasing in view of (4.7). To the contrary, let us assume that $\delta = 0$. In the following, we write $v(r) = v(r,s)$ and so on. Let $c_0 = -(2-2aM)/b < 0$. Then, it follows that

$$0 > -\int_0^{r_1} r^{1-2aM}\Psi_{c_0}(r)\varphi(r)dr = r_1\varphi'(r_1)w_{c_0}(r_1),$$

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which implies that $w_{c_0}(r_1) > 0$. Thus, $c_1 = -r_1 v'(r_1) < c_0$ such that $(2 - 2aM) + bc_1 < 0$. Since $v(r)$ is an increasing function and $\Psi_{c_1}(r) \nearrow 0$ as $r \to \infty$, either $\Psi_{c_1}(r)$ has a unique zero or $\Psi_{c_1}(r) < 0$ for all $r > 0$. We note that the latter case happens only when $N = 0$ and $(2 + c_1 b)e^s + (2 - c_1) < 0$. If $\Psi_{c_1}(r)$ has unique zero at r_0, then $r_0 \le r_1$. Otherwise, we have $\Psi_{c_1}(r) > 0$ for all $r < r_1$ such that

$$0 > -\int_0^{r_1} r\Psi_{c_1}(r)\varphi(r)dr = [r\varphi' w_{c_1} - r w_{c_1}' \varphi](r_1) = 0,$$

a contradiction. As a consequence, $\Psi_{c_1}(r) < 0$ on (r_1, ∞). Therefore, since $\delta = 0$,

$$0 > -\int_{r_1}^{\infty} r\Psi_{c_1}(r)\varphi(r)dr = \lim_{r\to\infty} [r\varphi'(r) w_{c_1}(r) - r w_{c_1}'(r)\varphi(r)] = 0,$$

which leads to a contradiction.

The asymptotic behavior in the limit $r \to \infty$ can be proved by a similar argument for the proof of Lemma 2.11. This finishes the proof. $\square$

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