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J. Differential Equations 265 (2018) 98–156

Journal of Differential Equations

www.elsevier.com/locate/jde

Competition in periodic media: II – Segregative limit of pulsating fronts and “Unity is not Strength”-type result ☆

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Received 10 November 2016; revised 13 December 2017
Available online 26 February 2018

Abstract

This paper is concerned with the limit, as the interspecific competition rate goes to infinity, of pulsating front solutions in space-periodic media for a bistable two-species competition–diffusion Lotka–Volterra system. We distinguish two important cases: null asymptotic speed and non-null asymptotic speed. In the former case, we show the existence of a segregated stationary equilibrium. In the latter case, we are able to uniquely characterize the segregated pulsating front, and thus full convergence is proved. The segregated pulsating front solves an interesting free boundary problem. We also investigate the sign of the speed as a function of the parameters of the competitive system. We are able to determine it in full generality, with explicit conditions depending on the various parameters of the problem. In particular, if one species is sufficiently more motile or competitive than the other, then it is the invader. This is an extension of our previous work in space-homogeneous media.

MSC: 35B40; 35K57; 35R35; 92D25

Keywords: Pulsating fronts; Periodic media; Competition–diffusion system; Segregation; Wave speed; Free boundary

☆ The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 321186 – ReaDi – Reaction–Diffusion Equations, Propagation and Modelling held by Henri Berestycki.

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https://doi.org/10.1016/j.jde.2018.02.026
0022-0396/

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Introduction

This is the second part of a sequel to our previous article [24]. In the prequel, we studied the sign of the speed of bistable traveling wave solutions of the following competition–diffusion problem:

$$\begin{cases} \partial_t u_1 - \partial_{xx} u_1 = u_1(1-u_1) - k u_1 u_2 & \text{in } (0,+\infty)\times\mathbb{R} \\ \partial_t u_2 - d\partial_{xx} u_2 = r u_2(1-u_2) - \alpha k u_1 u_2 & \text{in } (0,+\infty)\times\mathbb{R}. \end{cases}$$

We proved that, as $k \to +\infty$, the speed of the traveling wave connecting $(1,0)$ to $(0,1)$ converges to a limit which has exactly the sign of $\alpha^2 - rd$. In particular, if $\alpha = r = 1$ and if k is large enough, the more motile species is the invader: this is what we called the "Unity is not strength" result.

In view of this result, it would seem natural to try to generalize it in heterogeneous spaces, that is to systems with non-constant coefficients. Is the more motile species still the invading one?

Competition–diffusion problems in bounded heterogeneous spaces with various boundary conditions have been widely studied during the past decades. Dockery, Hutson, Mischaikow and Pernarowski [16] showed (in particular) that for the heterogeneous system:

$$\begin{cases} \partial_t u_1 - d_1 \Delta_x u_1 = a_1(x)\, u_1 - u_1^2 - u_1 u_2 & \text{in } (0,+\infty)\times\Omega \\ \partial_t u_2 - d_2 \Delta_x u_2 = a_2(x)\, u_2 - u_2^2 - u_1 u_2 & \text{in } (0,+\infty)\times\Omega \end{cases}$$

with a_1 and a_2 non-constant functions, d_1 and d_2 constant, Ω a bounded open subset of some Euclidean space and homogeneous Neumann boundary conditions, the persistent species is actually the less motile one. The interspecific competition rate of this system is equal to 1 and the system is therefore monostable. On the contrary, as soon as the competition rate is large enough, the system is bistable. We wonder whether this qualitative change might be sufficient to reverse their conclusion. If we are able to extend in some satisfying way our space-homogeneous result, then the conclusion will be reversed indeed.

In the first part [23] of this sequel, the first author studied the existence of bistable pulsating front solutions for the following problem:

$$\begin{cases} \partial_t u_1 = \partial_{xx} u_1 + u_1 f_1(u_1, x) - k u_1 u_2 & \text{in } (0,+\infty)\times\mathbb{R} \\ \partial_t u_2 = d\partial_{xx} u_2 + u_2 f_2(u_2, x) - \alpha k u_1 u_2 & \text{in } (0,+\infty)\times\mathbb{R}. \end{cases}$$

Here, the non-linearities $(u,x) \mapsto u f_i(u,x)$, $i \in \{1,2\}$, are of "KPP"-type and, most importantly, are spatially periodic. Thanks to Fang–Zhao's theorem [21], it was showed that, provided k is large enough and (f_1, f_2) satisfies a high-frequency algebraic hypothesis (we highlight that the condition was algebraic and not asymptotic), there exists indeed such a pulsating front.

While the forthcoming main ideas might be generalizable to systems with periodic diffusion and interspecific competition rates, an existence result is lacking. Therefore we naturally stick with the aforementioned system. Let us recall moreover that the fully heterogeneous problem (non-periodic non-constant coefficients) is, as far as we know, still completely open at this time.

Let us recall as well that several important results about scalar reaction–diffusion equations in periodic media have been established recently (about "KPP"-type, see [4,5,28–30]; about "ignition"-type and monostable non-linearities, see [3]; about bistable non-linearities, see [15,14,31]). The first author used extensively the results about "KPP"-type equations in [23]. In the forthcoming work, we will use the whole collection of results. Especially, we will use several times, in slightly different contexts, the sliding method of Berestycki–Hamel [3].

Integration over a bounded domain with Neumann boundary conditions and over a periodicity cell are somehow similar operations and thus Neumann and periodic boundary conditions yield in general analogous results. The periodic extension of the persistence result by Dockery and his collaborators seems in fact quite straightforward and, conversely, it should be possible to adapt the forthcoming ideas to determine the persistent species in a bistable space-heterogeneous Neumann problem with large competition rate. The comparison is therefore even more meaningful.

The competition-induced segregation phenomenon highlighted by Dancer, Terracini and others (see for instance [8–12]) has been one of our main tools in the preceding pair of articles [23,24] and will still be a cornerstone here. In particular, segregation in two or more dimensions generically yields free boundary problems and this will be a major difference between the space-homogeneous case and this study: here, we will need to dedicate a few pages to the natural free boundary problem induced by the segregation of pulsating fronts. Thanks to the specific setting of pulsating fronts (monotonicity in time, spatial periodicity of the profile, limiting conditions, etc.), we will be able to prove that the free boundary is the graph of a strictly monotonic, bijective and continuous function without resorting to blow-up arguments or monotonicity formulas. We believe that our approach of the free boundary has interest of its own and that the ideas presented here might fond applications in other frameworks.

The following pages will be organized as follows: in the first section, the core hypotheses and framework will be precisely formulated and the main results stated. The second section will focus on the so-called "segregative limit" and will finally lead us to the third section and the statement of the periodic extension of the "Unity is not strength" theorem.

1. Preliminaries and main results

Remark. Subsections 1.1 and 1.3 are mostly a repetition of the preliminaries of the first author's article [23] where the existence of competitive pulsating fronts was investigated. A reader well aware of this article may safely skip these. On the contrary, Subsections 1.2 and 1.4 respectively state the main results of this article and highlight the differences between the present set of technical hypotheses and that of the first author's article [23].

Let $d, k, \alpha, L > 0$, $C = (0, L) \subset \mathbb{R}$ and $(f_1, f_2) : [0, +\infty) \times \mathbb{R} \to \mathbb{R}^2$ L-periodic with respect to its second variable. For any $u : \mathbb{R}^2 \to [0, +\infty)$ and $i \in \{1, 2\}$, we refer to $(t, x) \mapsto f_i(u(t, x), x)$ as $f_i[u]$. Our interest lies in the following competition–diffusion problem:

$$\begin{cases} \partial_t u_1 = \partial_{xx} u_1 + u_1 f_1[u_1] - k u_1 u_2, \\ \partial_t u_2 = d\partial_{xx} u_2 + u_2 f_2[u_2] - \alpha k u_1 u_2. \end{cases} \qquad (\mathcal{P}_k)$$

1.1. Preliminaries

1.1.1. Redaction conventions

- Mirroring the definition of $f_1[u]$ and $f_2[u]$, for any function of two real variables f and any real-valued function u of two real variables, $f[u]$ will refer to $(t, x) \mapsto f(u(t, x), x)$. For any real-valued function u of one real variable, $f[u]$ will refer to $x \mapsto f(u(x), x)$. For any function f of one real variable and any real-valued function u of one or two real variables, $f[u]$ will simply refer to $f \circ u$.
- For the sake of brevity, although we could index everything ($(\mathcal{P})$, u_1, $u_2 \ldots$) on k and d, the dependencies on k or d will mostly be implicit and will only be made explicit when it definitely facilitates the reading.
- Since we consider the limit of this system when $k \to +\infty$, many (but finitely many) results will only be true when "k is large enough". Hence, we define by induction the positive number $k^\star$, whose value is initially 1 and is updated each time a statement is only true when "k is large enough" in the following way: if the statement is true for any $k \geq k^\star$, the value of $k^\star$ is unchanged; if, conversely, there exists $K > k^\star$ such that the statement is true for any $k \geq K$ but false for any $k \in [k^\star, K)$, the value of $k^\star$ becomes that of K. In the text, we will indifferently write "for k large enough" or "provided $k^\star$ is large enough". Moreover, when k indexes appear, they *a priori* indicate that we are considering families indexed on $[k^\star, +\infty)$, but for the sake of brevity, when sequential arguments involve sequences indexed themselves

on increasing elements of $[k^\star, +\infty)^{\mathbb{N}}$, we will not explicitly define these sequences of indexes and will simply stick with the indexes k, reindexing along the course of the proof the considered objects. In such a situation, the statement "as $k \to +\infty$" should be understood unambiguously.

- Periodicity will always implicitly mean L-periodicity (unless explicitly stated otherwise). For any functional space X on $\mathbb{R}$, X_{per} denotes the subset of L-periodic elements of X.
- We will use the classical partial order on the space of functions from any $\Omega \subset \mathbb{R}^N$ to $\mathbb{R}$: $g \leq h$ if for any $x \in \Omega$ $g(x) \leq h(x)$ and $g < h$ if $g \leq h$ and $g \neq h$. We recall that when $g < h$, there might still exists $x \in \Omega$ such that $g(x) = h(x)$. If, for any $x \in \Omega$, $g(x) < h(x)$, we use the notation $g \ll h$. In particular, if $g \geq 0$, we say that g is non-negative, if $g > 0$, we say that g is non-negative non-zero, and if $g \gg 0$, we say that g is positive. Finally, if $g_1 \leq h \leq g_2$, we write $h \in [g_1, g_2]$, if $g_1 < h < g_2$, we write $h \in (g_1, g_2)$, and if $g_1 \ll h \ll g_2$, we write $h \in \langle g_1, g_2 \rangle$.
- We will also use the partial order on the space of vector functions $\Omega \to \mathbb{R}^{N'}$ naturally derived from the preceding partial order. It will involve similar notations.
- Functions f of two or more real variables will sometimes be identified with the maps $t \mapsto (x \mapsto f(t, x))$. This is quite standard in parabolic theory but we stress that the variable of the map will always be the first variable of f, even if this variable is not called t: we will use indeed functions of the pair of variables $(\xi, x) \in \mathbb{R}^2$ and then the maps will be $\xi \mapsto (x \mapsto f(\xi, x))$. So for instance if we say that a function f of (ξ, x) is an element of a functional space $X(\mathbb{R}, Y)$, the latter should be understood unambiguously.

1.1.2. Hypotheses on the reaction

For any $i \in \{1, 2\}$, we have in mind functions f_i such that the reaction term $uf_i[u]$ is of logistic type (also known as "KPP"-type). At least, we want to cover the largest possible class of $(u, x) \mapsto \mu(x)(a - u)$. This is made precise by the following assumptions.

$(\mathcal{H}_1)$ f_i is in $\mathcal{C}^1([0, +\infty) \times \mathbb{R})$.
$(\mathcal{H}_2)$ There exists a constant $m_i > 0$ such that $f_i[0] \geq m_i$.
$(\mathcal{H}_3)$ f_i is decreasing with respect to its first variable and there exists $a_i > 0$ such that, for any $x \in \mathbb{R}$, $f_i(a_i, x) = 0$.

Remark. If f_i is in the class of all $(u, x) \mapsto \mu(x)(a - u)$, then $\mu \in \mathcal{C}^1_{per}(\mathbb{R})$, $\mu \gg 0$ and $a > 0$. More generally, from $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and the periodicity of $f_i[0]$, it follows immediately that there exists a constant $M_i > m_i$ such that $f_i[0] \leq M_i$. Without loss of generality, we assume that m_i and M_i are optimal, that is $m_i = \min_{\overline{C}} f_i[0]$ and $M_i = \max_{\overline{C}} f_i[0]$.

1.1.3. Extinction states

The periodic principal eigenvalues of $\frac{\mathrm{d}^2}{\mathrm{d}x^2} + f_1[0]$ and $d\frac{\mathrm{d}^2}{\mathrm{d}x^2} + f_2[0]$ are negative (as proved by the first author in [23]). Recall (from Berestycki–Hamel–Roques [4] for instance) that the periodic principal eigenvalue of $\mathcal{L}$ is the unique real number λ such that there exists a periodic function $\varphi \gg 0$ satisfying:

$$\begin{cases} -\mathcal{L}\varphi = \lambda\varphi \text{ in } \mathbb{R} \\ \|\varphi\|_{L^\infty(C)} = 1 \end{cases}$$

From this observation, it follows from Berestycki–Hamel–Roques [4] that a_1 (respectively a_2) is the unique periodic non-negative non-zero solution of $-z'' = zf_1[z]$ (resp. $-dz'' = zf_2[z]$).

The states $(a_1, 0)$ and $(0, a_2)$ are clearly periodic stationary states of $(\mathcal{P}_k)$ (for any $k > k^\star$) and are referred to as the *extinction states* of $(\mathcal{P}_k)$ (remark that they are the unique periodic stationary states with one null component and the other one positive, so that it makes sense to call them "the" extinction states). Provided $k^\star$ is large enough, they are moreover locally asymptotically stable (again, as proved in [23]).

We recall also that, for any $k > k^\star$, by virtue of the scalar parabolic comparison principle, any solution (u_1, u_2) of $(\mathcal{P}_k)$ with initial condition $(0,0) < \left(u_{1,0}, u_{2,0}\right) < (a_1, a_2)$ satisfies $(0,0) \ll (u_1, u_2) \ll (a_1, a_2)$.

1.1.4. Pulsating front solutions of $(\mathcal{P})$

Let us add a necessary existence hypothesis.

$(\mathcal{H}_{exis})$ There exists $k^\star > 0$ such that, for any $k > k^\star$, there exists $c_k \in \mathbb{R}$ and $\left(\varphi_{1,k}, \varphi_{2,k}\right) \in \mathcal{C}^2\left(\mathbb{R}^2\right)^2$ such that the following properties hold.

- $\left(u_{1,k}, u_{2,k}\right) : (t, x) \mapsto \left(\varphi_{1,k}, \varphi_{2,k}\right)(x - c_k t, x)$ is a classical solution of $(\mathcal{P}_k)$.
- $\varphi_{1,k}$ and $\varphi_{2,k}$ are respectively non-increasing and non-decreasing with respect to their first variable, generically noted ξ.
- $\varphi_{1,k}$ and $\varphi_{2,k}$ are periodic with respect to their second variable, generically noted x.
- As $\xi \to -\infty$,

$$\max_{x\in[0,L]} \left|\left(\varphi_{1,k}, \varphi_{2,k}\right)(\xi, x) - (a_1, 0)\right| \to 0.$$

- As $\xi \to +\infty$,

$$\max_{x\in[0,L]} \left|\left(\varphi_{1,k}, \varphi_{2,k}\right)(\xi, x) - (0, a_2)\right| \to 0.$$

The pair $\left(u_{1,k}, u_{2,k}\right)$ is referred to as a *pulsating front solution* of $(\mathcal{P}_k)$ with *speed* c_k and *profile* $\left(\varphi_{1,k}, \varphi_{2,k}\right)$.

Before going any further, it is natural to wonder if such a solution is unique.

Conjecture. *Let $k > k^\star$. Let $\left(\hat{\varphi}_1, \hat{\varphi}_2\right)$ and $\hat{c}$ be respectively the profile and the speed of a pulsating front solution $\left(\hat{u}_1, \hat{u}_2\right)$ of $(\mathcal{P})$. Then $\hat{c} = c_k$ and there exists $\hat{\xi} \in \mathbb{R}$ such that $\left(\hat{\varphi}_1, \hat{\varphi}_2\right)$ coincides with:*

$$(\xi, x) \mapsto \left(\varphi_{1,k}, \varphi_{2,k}\right)\left(\xi - \hat{\xi}, x\right).$$

This conjecture is due to the following observation: in most (if not all) problems concerned with bistable traveling or pulsating fronts, the front is unique (in the same sense as above: two fronts have the same speed and have the same profile up to translation).

We refer to Gardner [22], Kan-On [27], Berestycki–Hamel [3] or Ding–Hamel–Zhao [15] for proofs of this type of result in slightly different settings.

Because the proof of such a result:

- would involve precise estimates of the exponential decay of the profiles as $\xi \to \pm\infty$ that cannot be obtained briefly (in the scalar case, see Hamel [25]) and have no additional interest in the forthcoming work,
- would be strongly analogous to the proofs of the preceding collection of references,

we choose to leave this as an open question here for the sake of brevity. We might address this question in a future sequel.

Still, it is useful to have this uniqueness in mind because it clearly motivates our study of $\lim_{k\to+\infty} c_k$.

1.2. "Unity is not strength" theorem for periodic media

In the forthcoming theorem, the parameters d, α, f_1 and f_2 may vary (in some sense which is made precise), but immediately after that they are fixed again (at least up to Section 3).

Theorem 1.1. *["Unity is not strength", periodic case] Assume that there exists an open connected set $\mathfrak{P}$ of parameters:*

$$(d, \alpha, f_1, f_2) \in (0, +\infty)^2 \cap \mathcal{C}\left([0, +\infty), \mathcal{C}_{per}(\mathbb{R})\right)^2$$

in which $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_{exis})$ are satisfied.

The sequence $((d, \alpha, f_1, f_2) \in \mathfrak{P} \mapsto c_k)_{k>k^\star}$ converges pointwise as $k \to +\infty$ to some continuous function $(d, \alpha, f_1, f_2) \in \mathfrak{P} \mapsto c_\infty$. If the function $(d, \alpha, f_1, f_2) \in \mathfrak{P} \mapsto k^\star$ is locally bounded, then this convergence is in fact locally uniform in $\mathfrak{P}$.

Furthermore, for any $(d, \alpha, f_1, f_2) \in \mathfrak{P}$, there exist $\overline{r} > 0$, $\underline{r} \in (0, \overline{r}]$ (both dependent on (f_1, f_2) only) and a non-empty closed interval $\mathcal{R}^0 \subset \left[\underline{r}, \overline{r}\right]$ (dependent on (d, f_1, f_2) only) such that the sign of c_∞ satisfies the following properties.

1. *$c_\infty > 0$ if and only if $\frac{\alpha^2}{d} > \max \mathcal{R}^0$.*
2. *$c_\infty < 0$ if and only if $\frac{\alpha^2}{d} < \min \mathcal{R}^0$.*
3. *If, for any $i \in \{1, 2\}$, f_i has the particular form $(u, x) \mapsto \mu_i(x)(1 - u)$, then:*
   (a) *c_∞ is null or has the sign of:*

$$\alpha^2 - d\frac{\|\mu_2\|_{L^1(C)}}{\|\mu_1\|_{L^1(C)}};$$

   (b) *$\left(\underline{r}, \overline{r}\right)$ satisfies:*

$$\frac{\min_{\overline{C}}(\mu_2)}{\max_{\overline{C}}(\mu_1)} \leq \underline{r} \leq \overline{r} \leq \frac{\max_{\overline{C}}(\mu_2)}{\min_{\overline{C}}(\mu_1)}.$$

The objects $\overline{r}$, $\underline{r}$ and $\mathcal{R}^0$ are respectively defined by formulas $(\mathfrak{F}_{\overline{r}})$, $\left(\mathfrak{F}_{\underline{r}}\right)$ and $\left(\mathfrak{F}_{\mathcal{R}^0}\right)$ (see page 147).

Remark. We emphasize the interest of $\underline{r}$ and $\overline{r}$, which are upper and lower bounds for $\mathcal{R}^0$ which are uniform with respect to d.

We will explain in Section 3 that if $(\mathcal{H}_{exis})$ is derived from the existence result of the first author [23], then a set $\mathfrak{P}$ exists: the main assumption of our theorem makes sense indeed.

The strategy of the proof is as follows.

We will begin with some compactness estimates uniform with respect to k so that a limiting speed and an associated limiting solution, possibly non-unique at this point, can be extracted. This will require a crucial distinction between two cases: limiting speed null or not.

Regarding the first case, we will give some regularity properties of the corresponding solution, that will be called a *segregated stationary equilibrium*. It is unclear whether the segregated stationary equilibrium is unique but this is not surprising: the null speed case is known to be quite degenerate (see for instance Ding–Hamel–Zhao [15]).

On the contrary, the second case will be fully characterized: the corresponding solution, the *segregated pulsating front*, is actually unique (up to translation). Such a uniqueness result will require several intermediary results and in particular a (possibly not complete but already quite thorough) study of its intrinsic free boundary problem.

Subsequently, the uniqueness of the segregated pulsating front will follow from a sliding argument which will also provide us with an exclusion result: there exists a segregated stationary equilibrium for a particular choice of parameters (d, α, f_1, f_2) if and only if there does not exist a segregated pulsating front. Thanks to this result, the uniqueness of the limiting speed will be deduced even though the null case is still degenerate.

We will then obtain a necessary and sufficient condition on (d, α, f_1, f_2) for the existence of a segregated stationary equilibrium thanks to its regularity at the interface (which is, in some sense, the counterpart to the free boundary problem leading to the uniqueness of the segregated pulsating front) and finally, thanks to a classical integration by parts, obtain the sign of the speed provided it is already known to be non-zero.

1.3. A few more preliminaries

1.3.1. Compact embeddings of Hölder spaces

Proposition 1.2. *Let $(a, a') \in (0, +\infty)^2$ and n, n', β, β' such that $(a, a') = (n + \beta, n' + \beta')$, n and n' are non-negative integers and β and β' are in $(0, 1]$.*

If $a \le a'$, then the canonical embedding $i : \mathcal{C}^{n',\beta'}(C) \hookrightarrow \mathcal{C}^{n,\beta}(C)$ is continuous and compact.

It will be clear later on that this problem naturally involves uniform bounds in $\mathcal{C}^{0,1/2}$. Therefore, we fix once and for all $\beta \in \left(0, \frac{1}{2}\right)$ and we will use systematically the compact embeddings $\mathcal{C}^{n,1/2} \hookrightarrow \mathcal{C}^{n,\beta}$, meaning that uniform bounds in $\mathcal{C}^{n,1/2}$ yield relative compactness in $\mathcal{C}^{n,\beta}$.

1.3.2. Additional notations regarding the pulsating fronts

Let $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. For any $k > k^\star$, $(c_k, \varphi_{1,k}, \varphi_{2,k})$ satisfies the following system:

$$\begin{cases} -\mathrm{div}\left(E\nabla\varphi_{1,k}\right) - c_k\partial_\xi\varphi_{1,k} = \varphi_{1,k} f_1\left[\varphi_{1,k}\right] - k\varphi_{1,k}\varphi_{2,k} \\ -d\,\mathrm{div}\left(E\nabla\varphi_{2,k}\right) - c_k\partial_\xi\varphi_{2,k} = \varphi_{2,k} f_2\left[\varphi_{2,k}\right] - \alpha k\varphi_{1,k}\varphi_{2,k}. \end{cases} \qquad \left(\mathcal{PF}_{sys,k}\right)$$

Remark. Be aware that, since $\mathrm{sp}E = \{0, 2\}$, the differential operator:

$$\mathrm{div}\,(E\nabla) = \partial_{\xi\xi} + \partial_{xx} + 2\partial_{\xi x}$$

is only degenerate elliptic. This will trigger difficulties unknown in the space-homogeneous case. Most regularity results will come from the parabolic system $(\mathcal{P})$ and we will need to go back and forth a lot between the so-called "parabolic coordinates" (t, x) and the so-called "traveling coordinates" (ξ, x). This will be possible if and only if the propagation speed is non-zero, whence a necessary distinction of cases.

For any $k > k^\star$, let:

$$\psi_{d,k} = \alpha\varphi_{1,k} - d\varphi_{2,k},$$

$$\psi_{1,k} = \alpha\varphi_{1,k} - \varphi_{2,k},$$

$$v_{d,k} = \alpha u_{1,k} - du_{2,k},$$

$$v_{1,k} = \alpha u_{1,k} - u_{2,k}.$$

A linear combination of the equations of $\left(\mathcal{PF}_{sys,k}\right)$ yields:

$$-\mathrm{div}\left(E\nabla\psi_{d,k}\right) - c_k\partial_\xi\psi_{1,k} = \alpha\varphi_{1,k} f_1\left[\varphi_{1,k}\right] - \varphi_{2,k} f_2\left[\varphi_{2,k}\right] \quad (\mathcal{PF}_k)\,.$$

$(\mathcal{PF}_k)$ does not depend explicitly on k.

$\left(u_{1,k}, u_{2,k}, v_{d,k}, v_{1,k}\right)$ is isomorphic to $\left(\varphi_{1,k}, \varphi_{2,k}, \psi_{d,k}, \psi_{1,k}\right)$ if and only if $c_k \neq 0$. In parabolic coordinates, $(\mathcal{PF}_k)$ becomes:

$$\partial_t v_{1,k} - \partial_{xx} v_{d,k} = \alpha u_{1,k} f_1\left[u_{1,k}\right] - u_{2,k} f_2\left[u_{2,k}\right].$$

As $k \to +\infty$, the following function will naturally appear:

$$\eta : (z, x) \mapsto f_1\left(\frac{z}{\alpha}, x\right) z^+ - \frac{1}{d} f_2\left(-\frac{z}{d}, x\right) z^-,$$

where $z^+ = \max(z, 0)$ and $z^- = -\min(z, 0)$ so that $z = z^+ - z^-$.

We will also denote g_i the partial derivative of $(u, x) \mapsto u f_i(u, x)$ with respect to u:

$$g_i : (u, x) \mapsto f_i(u, x) + u\partial_1 f_i(u, x) \text{ for all } i \in \{1, 2\}.$$

1.4. Comparison between the first and the second part

In addition to the new notations introduced in the preceding subsection ($\left(\mathcal{PF}_{sys}\right)$, $(\mathcal{PF})$, "parabolic coordinates", "traveling coordinates", ψ_d, ψ_1, v_d, v_1), the following differences are pointed out.

- In the first part [23], f_1 and f_2 were only assumed to be Hölder-continuous with respect to x, whereas here we need them to be at least continuously differentiable. Thanks to this technical hypothesis, it is then possible to differentiate with respect to x the various equations and systems involved. In particular, continuous pulsating front solutions of $(\mathcal{P})$ are in fact in $\mathcal{C}^2_{loc}\left(\mathbb{R}^2\right)$. This will similarly yield a stronger regularity at the limit. Nevertheless, we think that Hölder-continuity might actually suffice to obtain most of the forthcoming results.
- The positive zero of $u \mapsto f_i(u,x)$ cannot depend on x anymore. Consequently, while, in the first part [23], the unique positive solution of $-z'' = zf_1[z]$, $\tilde{u}_1$, and the unique positive solution of $-dz'' = zf_2[z]$, $\tilde{u}_2$, were periodic functions of x, here they are the constants a_1 and a_2. This restriction is standard in bistable pulsating front problems (see for instance [15,14,32]) and is especially related to the method generically used to determine the sign of the speed of the pulsating fronts. Still, most of the forthcoming pages is easily generalized (actually, many results need no adaptation at all). We will highlight where this hypothesis is truly needed and will give some indications regarding the non-constant case. In the end, it should be clear why we conjecture that "Unity is not strength" holds true even in the non-constant case.
- A trade-off to these more restrictive assumptions is that here we do not assume *a priori* the high-frequency hypothesis:

$$L < \pi\left(\frac{1}{\sqrt{M_1}} + \sqrt{\frac{d}{M_2}}\right). \quad \left(\mathcal{H}_{freq}\right)$$

  We merely assume existence of pulsating fronts, this hypothesis being referred to as $(\mathcal{H}_{exis})$. It was proved in the first part that if $\left(\mathcal{H}_{freq}\right)$ is satisfied, then so is $(\mathcal{H}_{exis})$.

2. Asymptotic behavior: the infinite competition limit

2.1. Existence of a limiting speed

In order to prove that $(c_k)_{k>k^\star}$ has at least one limit point, we recall an important result from the Fisher–KPP scalar case (see Berestycki–Hamel–Roques [5]).

Theorem 2.1. *For any $\delta \in \{1,d\}$ and $i \in \{1,2\}$, there exists $c^\star[\delta,i] > 0$ such that, for any $s \in \mathbb{R}$, there exists in $\mathcal{C}^2\left(\mathbb{R}^2\right)$ a pulsating front solution of:*

$$\partial_t z - \delta\partial_{xx} z = zf_i[z]$$

connecting a_i to 0 at speed s if and only if $s \geq c^\star[\delta,i]$.

Lemma 2.2. *Provided $k^\star$ is large enough, for any $k > k^\star$ and any pulsating front solution of $(\mathcal{P}_k)$, its speed c satisfies:*

$$-c^\star[d,2] < c < c^\star[1,1].$$

In particular, the family $(c_k)_{k>k^\star}$ is uniformly bounded with respect to k.

Remark. Here, the assumption that k is large enough might in fact be redundant with the underlying assumption of bistability. Indeed, this proof does not use any limiting behavior but only requires that:

$$k > \max\left\{\frac{1}{a_2}\max_{\overline{C}}\left(f_1[0]\right), \frac{1}{\alpha a_1}\max_{\overline{C}}\left(f_2[0]\right)\right\}.$$

In the space-homogeneous logistic case, this condition reduces to $k > \max\left\{1, \alpha^{-1}\right\}$, that is precisely the necessary and sufficient condition for the system to be bistable. In the space-periodic case, according to the proof of [23, Proposition 2.1], both a_i are stable if the condition above is satisfied. Yet an optimal threshold should involve periodic principal eigenvalues instead of these maxima. Furthermore, the instability of any other periodic steady state has only been established for (really) large k (see [23, Theorem 1.2]) and when $\left(\mathcal{H}_{freq}\right)$ holds true. Even for arbitrarily large k, it is unclear whether stable coexistence periodic steady states might exist when $\left(\mathcal{H}_{freq}\right)$ does not hold.

We point out that the following proof provides us with an instance of a detailed proof using the sliding method [3] that will be referred to later on.

Proof. Assume by contradiction that there exists $k > 0$ such that there exists a pulsating front solution (z_1, z_2) of $(\mathcal{P}_k)$ with a speed $c \notin (-c^\star[d,2], c^\star[1,1])$ and a profile (φ_1, φ_2). For instance, assume $c \geq c^\star[1,1]$ (the other case being obviously symmetric), and let $\underline{c} = c^\star[1,1] \leq c$. By virtue of Theorem 2.1, $\underline{c} > 0$ and there exists a pulsating front solution z of:

$$\partial_t z - \partial_{xx} z = z f_1[z]$$

with speed $\underline{c}$ and profile φ.

Now we are in position to use the sliding method to compare z and z_1. This will finally lead to a contradiction.

Step 1: existence of a translation of the profile associated with the higher speed such that it is locally below the other profile.

Fix $\zeta \in \mathbb{R}$. Then let $\zeta_1 \in \mathbb{R}$ such that:

$$\max_{x\in\overline{C}} \varphi_1(\zeta_1, x) < \min_{x\in\overline{C}} \varphi(\zeta, x).$$

Let:

$$\tau = \zeta - \zeta_1,$$
$$\varphi_1^\tau : (\xi, x) \mapsto \varphi_1(\xi - \tau, x),$$
$$\Phi^\tau = \varphi - \varphi_1^\tau,$$

so that:

$$\min_{x\in\overline{C}} \Phi^\tau(\zeta, x) = \min_{x\in\overline{C}} \left(\varphi(\zeta, x) - \varphi_1(\zeta_1, x)\right) > 0.$$

Step 2: up to some extra term, this ordering is global on the left.

Let $\mathcal{U} = (-\infty, \zeta) \times \overline{C}$. Since $\varphi \gg 0$ in $\mathcal{U}$ and $\varphi_1^\tau \in L^\infty(\mathcal{U})$, there exists $\kappa > 0$ such that:

$$\kappa\varphi - \varphi_1^\tau \geq 0 \text{ in } \mathcal{U}.$$

Notice that, since $\Phi^\tau(\xi, x) \to 0$ as $\xi \to \pm\infty$ (uniformly with respect to x), any such κ is larger than or equal to 1.

Step 3: this extra term is actually unnecessary, thanks to the maximum principle.

Let:

$$\kappa^\star = \inf\left\{\kappa > 1 \mid \inf_{\mathcal{U}}\left(\kappa\varphi - \varphi_1^\tau\right) > 0\right\}$$

and let us prove that $\kappa^\star = 1$. We assume by contradiction that $\kappa^\star > 1$ and we take a sequence $(\kappa_n)_{n\in\mathbb{N}} \in (1, \kappa^\star)^{\mathbb{N}}$ which converges to $\kappa^\star$ from below.

There exists a sequence $((\xi_n, x_n)) \in \mathcal{U}^{\mathbb{N}}$ such that for any $n \in \mathbb{N}$,

$$\kappa_n\varphi(\xi_n, x_n) < \varphi_1^\tau(\xi_n, x_n).$$

Since $\kappa_n > 1$, the limits when $\xi \to -\infty$ prove that (ξ_n) is bounded from below, and since it is also bounded from above by ζ, we can extract a convergent subsequence with limit $\xi^\star \in (-\infty, \zeta]$. Similarly, we can extract a convergent subsequence of $(x_n) \in \overline{C}^{\mathbb{N}}$ with limit $x^\star \in \overline{C}$. By continuity, $\left(\kappa^\star\varphi - \varphi_1^\tau\right)(\xi^\star, x^\star) = 0$ and, necessarily, $\xi^\star < \zeta$.

Back to parabolic variables, recall that $\underline{c} > 0$ and let:

$$t^\star = \frac{x^\star - \xi^\star}{\underline{c}},$$

$$\hat{z}_i^\tau : (t, x) \mapsto \varphi_i\left(x - \underline{c}t - \tau, x\right) \text{ for any } i \in \{1, 2\},$$

$$v^\star = \kappa^\star z - \hat{z}_1^\tau,$$

$$f : (t, x) \mapsto -\left(c - \underline{c}\right)\left(\partial_\xi\varphi_1^\tau\right)\left(x - \underline{c}t, x\right)$$

$$E = \left\{(t, x) \in \mathbb{R}^2 \mid x - \underline{c}t < \zeta\right\}.$$

By virtue of $(\mathcal{H}_3)$ and $\kappa^\star > 1$:

$$\kappa^\star z f_1[z] > \kappa^\star z f_1\left[\kappa^\star z\right] \text{ in } E,$$

and moreover:

$$\partial_t v^\star - \partial_{xx} v^\star = \kappa^\star z f_1[z] - \hat{z}_1^\tau f_1\left[\hat{z}_1^\tau\right] + k\hat{z}_1^\tau\hat{z}_2^\tau + f \text{ in } E,$$

$$f \geq 0 \text{ in } E.$$

Now, from the Lipschitz-continuity of f_1 with respect to its first variable, it follows that of $(u, x) \mapsto u f_1(u, x)$, whence there exists $q \in L^\infty(E)$ such that:

$$\partial_t v^\star - \partial_{xx} v^\star \geq q v^\star \text{ in } E.$$

In the end, $v^\star$ is a non-negative super-solution which vanishes at some interior point: by virtue of the parabolic strong minimum principle, it is identically null in $((-\infty, t^\star] \times \mathbb{R}) \cap E$.

But in such an unbounded set, it is always possible to construct an element of $\{\zeta\} \times \overline{C}$, which contradicts:

$$\min_{x \in \overline{C}} \left(\kappa^\star \varphi - \varphi_1^\tau\right)(\zeta, x) > 0.$$

Therefore $\kappa^\star = 1$,

$$\kappa^\star \varphi - \varphi_1^\tau = \Phi^\tau \geq 0 \text{ in } \mathcal{U}$$

and then by periodicity and, once more, by virtue of the parabolic strong minimum principle:

$$\Phi^\tau \gg 0 \text{ in } (-\infty, \zeta) \times \mathbb{R}.$$

Step 4: up to some (possibly different) extra term, this ordering is global on the right.

Near $+\infty$ (in $(\zeta, +\infty) \times \mathbb{R}$), on the contrary, multiplying φ by some $\kappa \gg 1$ is not going to yield a clear ordering anymore since we are interested in the behavior as $\varphi \sim 0$ and $\varphi_1 \sim 0$ (and replacing φ and φ_1^τ by respectively $a_1 - \varphi$ and $a_1 - \varphi_1^\tau$ will not suffice since the monostability has no underlying symmetry).

But it is natural, for instance, to replace this multiplication by the addition of some $\varepsilon \geq 0$ and to prove in the next step that $\varepsilon^\star = 0$. This is actually what was done originally by Berestycki–Hamel [3].

Step 5: this (possibly different) extra term is also unnecessary.

We define $\varepsilon^\star$ as the following quantity:

$$\varepsilon^\star = \inf \left\{ \varepsilon > 0 \mid \inf_{(\zeta, +\infty) \times \overline{C}} \left(\varphi - \varphi_1^\tau + \varepsilon\right) > 0 \right\}.$$

We assume by contradiction that $\varepsilon^\star > 0$ and this yields as before a contact point $(\xi^\star, x^\star) \in (\zeta, +\infty) \times \overline{C}$.

Now the main difficulty is that $u \mapsto u f_1[u]$ is increasing near 0, so that we really cannot hope to have:

$$z f_1[z] \geq (z + \varepsilon) f_1[z + \varepsilon].$$

Still, it is possible to assume without loss of generality that, during the construction of τ, ζ_1 has also been chosen so that:

$$\frac{a_2}{2} \leq \varphi_2(\xi, x) \leq a_2 \text{ for any } (\xi, x) \in [\zeta_1, +\infty) \times \overline{C}.$$

It follows that:

$$\varphi_1^\tau \left(f_1\left[\varphi_1^\tau\right] - k\varphi_2^\tau\right) \leq \varphi_1^\tau \left(f_1\left[\varphi_1^\tau\right] - k\frac{a_2}{2}\right) \text{ in } [\zeta, +\infty) \times \overline{C}.$$

By virtue of the hypotheses $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, provided $k^\star$ is large enough, for any $K > k^\star$, the following non-linearity:

$$u \mapsto u\left(f_1[u] - K\frac{a_2}{2}\right)$$

is decreasing in a neighborhood of 0 (in fact, it is decreasing in $[0, +\infty)$). Then, in addition to this monotonicity, it suffices to use:

$$\varphi f_1[\varphi] \geq \varphi f_1[\varphi] - k\frac{a_2}{2}\varphi$$

and the Lipschitz-continuity of f_1 to conclude this step.

Step 6: thanks to the maximum principle again, the speeds are equal and the profiles are equal up to some translation.

Thus in fact:

$$\Phi^\tau \gg 0 \text{ in } \mathbb{R}^2.$$

Now, let:

$$\tau^\star = \sup\left\{\tau \in \mathbb{R} \mid \Phi^\tau \geq 0 \text{ in } \mathbb{R}^2\right\}.$$

The limits as $\xi \to \pm\infty$ of φ and φ_1 ensure that $\tau^\star < +\infty$. By continuity,

$$\Phi^{\tau^\star} \geq 0.$$

Let us verify quickly that, by virtue of the maximum principle, either $\Phi^{\tau^\star} = 0$ and $c = \underline{c}$, either $\Phi^{\tau^\star} \gg 0$. For instance, assume that $\left(\Phi^{\tau^\star}\right)^{-1}(\{0\})$ is non-empty, so that $\Phi^{\tau^\star} \gg 0$ does not hold. Then there exists $(\xi^\star, x^\star) \in \mathbb{R}^2$ such that $\Phi^{\tau^\star}(\xi^\star, x^\star) = 0$. Once more, we introduce:

$$t^\star = \frac{x^\star - \xi^\star}{\underline{c}},$$

$$v^{\tau^\star}(t, x) = \Phi^{\tau^\star}\left(x - \underline{c}t, x\right),$$

and using the parabolic linear inequality satisfied by $v^{\tau^\star}$, we verify that $v^{\tau^\star}$ is a non-negative super-solution which vanishes at $(t^\star, x^\star)$. Then, by the strong parabolic maximum principle and periodicity with respect to x of $\Phi^{\tau^\star}$, it is actually deduced that $\Phi^{\tau^\star} = 0$, which in turn implies (reinserting $v^{\tau^\star} = 0$ into the original non-linear equation satisfied by $v^{\tau^\star}$ and considering the function f which has been defined earlier) that $c = \underline{c}$.

Finally, assume by contradiction that $\Phi^{\tau^\star} \gg (0, 0)$, i.e. assume that for any $B > 0$,

$$\min_{[-B,B]\times\overline{C}} \Phi^{\tau^\star} > (0, 0).$$

Fix $B > 0$. By continuity, there exists $\epsilon > 0$ such that:

$$\min_{[-B,B]\times\overline{C}} \Phi^{\tau} > (0,0) \text{ for any } \tau \in [\tau^\star, \tau^\star + \epsilon).$$

We can now repeat Steps 2, 3, 4, 5 to show that, for any such τ:

$$\Phi^{\tau} \gg 0 \text{ in } (\mathbb{R}\backslash(-B,B)) \times \mathbb{R}.$$

The maximality of $\tau^\star$ being contradicted, this ends this step.

Step 7: the contradiction.

If $c = \underline{c}$ and $z = z_1$, then thanks to the equations satisfied by z and z_1, $z_2 = 0$ in $\mathbb{R}^2$. This contradicts the limit of φ_2 as $\xi \to +\infty$. $\square$

Corollary 2.3. *$(c_k)_{k>k^\star}$ has a limit point $c_\infty \in \left[-c^\star[d,2], c^\star[1,1]\right]$.*

Remark. Similarly, we do expect that $c_\infty \notin \{-c^\star[d,2], c^\star[1,1]\}$ but will not address this question for the sake of brevity.

2.2. *Existence of a limiting density provided the speed converges*

In this subsection, we fix a sequence $(c_k)_{k>k^\star}$ such that it converges to c_∞.

Then we prove the relative compactness of the associated sequence of pulsating front solutions $\left((u_{1,k}, u_{2,k})\right)_{k>k^\star}$, which will follow from classical parabolic estimates similar to those used by Dancer and his collaborators (see for instance [10]) supplemented by some estimates specific to the pulsating front setting. This supplement will lead indeed to a stronger compactness result than the one presented in the aforementioned work.

If $c_\infty \neq 0$, we will see that $\left((u_{1,k}, u_{2,k})\right)_{k>k^\star}$ is relatively compact if and only if $\left((\varphi_{1,k}, \varphi_{2,k})\right)_{k>k^\star}$ is relatively compact. Moreover, we will show that the compactness result can be improved further thanks to additional pulsating front estimates.

2.2.1. *Normalization*

Before going any further, we point out that, at this point, for any $k > k^\star$, (φ_1, φ_2) is fixed completely arbitrarily among the one-dimensional family of translated profiles. By monotonicity of the profiles with respect to ξ, this choice can in fact be normalized. In the space-homogeneous problem [24], the normalization was used to guarantee that the extracted limit point had no null component. It should be clear that this part of the proof will be strongly analogous. Therefore we choose now normalizations reminiscent to the space-homogeneous ones.

- On one hand, if $c_\infty \leq 0$, we fix without loss of generality for any $k > k^\star$ the normalization:

$$0 = \inf\left\{\xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_{1,k}(\xi, x) < \frac{a_1}{2}\right\}.$$

- On the other hand, if $c_\infty > 0$, we fix without loss of generality for any $k > k^\star$ the normalization:

$$0 = \sup\left\{\xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_{2,k}(\xi, x) < \frac{a_2}{2}\right\}.$$

Remark also that $\left(\left(u_{1,k},u_{2,k}\right)\right)_{k>k^\star}$ is normalized (in the sense that its value at some arbitrary initial time is entirely prescribed) if and only if $\left(\left(\varphi_{1,k},\varphi_{2,k}\right)\right)_{k>k^\star}$ is normalized.

2.2.2. Compactness results

Proposition 2.4. *The following collection of properties holds independently of the sign of c_∞.*

(1) *[Segregation] $\left(\varphi_{1,k}\varphi_{2,k}\right)_{k>k^\star}$ converges to 0 in $L^1_{loc}\left(\mathbb{R}\times C\right)$.*
(2) *[Persistence] $(0,0)$ is not a limit point of $\left(\left(\varphi_{1,k},\varphi_{2,k}\right)\right)_{k>k^\star}$ in $L^1_{loc}\left(\mathbb{R}^2,\mathbb{R}^2\right)$.*
(3) *[Uniform bound in the diagonal direction] For any $n\in\mathbb{N}$, $\left(\left(\partial_x+\partial_\xi\right)\left(\varphi_{1,k},\varphi_{2,k}\right)\right)_{k>k^\star}$ is uniformly bounded with respect to k in $L^2\left((-n,n)\times C,\mathbb{R}^2\right)$.*
(4) *[Uniform bound in the ξ direction] For any $k>k^\star$ and any $x\in\overline{C}$,*

$$\int_{\mathbb{R}}\partial_\xi\varphi_{1,k}\left(\zeta,x\right)d\zeta=-\int_{\mathbb{R}}\left|\partial_\xi\varphi_{1,k}\left(\zeta,x\right)\right|d\zeta=-a_1$$

and

$$\int_{\mathbb{R}}\partial_\xi\varphi_{2,k}\left(\zeta,x\right)d\zeta=\int_{\mathbb{R}}\left|\partial_\xi\varphi_{2,k}\left(\zeta,x\right)\right|d\zeta=a_2.$$

(5) *[Uniform bound in the x direction] For any $T>0$, $\left(\left(u_{1,k},u_{2,k}\right)\right)_{k>k^\star}$ is uniformly bounded with respect to k in $L^2\left((-T,T),H^1\left(C,\mathbb{R}^2\right)\right)$.*
(6) *[Uniform bound in the t direction] For any $T>0$, $\left(\partial_t v_{1,k}\right)_{k>k^\star}$ is uniformly bounded with respect to k in $L^2\left((-T,T),\left(H^1\left(C\right)\right)'\right)$.*
(7) *[Compactness in traveling coordinates] $\left(\left(\varphi_{1,k},\varphi_{2,k}\right)\right)_{k>k^\star}$ is relatively compact in the topology of $L^1_{loc}\left(\mathbb{R}^2,\mathbb{R}^2\right)$.*
(8) *[Compactness in parabolic coordinates] There exists:*

$$\left(u_{1,\infty},u_{2,\infty}\right)\in\left(L^\infty\left(\mathbb{R}^2\right)\cap L^2\left((-T,T),H^1\left((0,L)\right)\right)\right)^2$$

such that:
   (a) $\partial_t v_{1,\infty}\in L^2\left((-T,T),\left(H^1\left((0,L)\right)\right)'\right)$;
   (b) *$\left(u_{1,\infty},u_{2,\infty}\right)$ is a limit point of $\left(\left(u_{1,k},u_{2,k}\right)\right)_{k>k^\star}$ in the topology of $L^1_{loc}\left(\mathbb{R}^2,\mathbb{R}^2\right)$;*
   (c) *$u_{1,\infty}$ and $u_{2,\infty}$ are in $\mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$;*
   (d) *$u_{1,\infty}=\alpha^{-1}v^+_{d,\infty}=\alpha^{-1}v^+_{1,\infty}$ and $u_{2,\infty}=d^{-1}v^-_{d,\infty}=v^-_{1,\infty}$.*

Proof. The segregation property comes directly from an integration of, say, the first equation of $\left(\mathcal{PF}_{sys,k}\right)$ over some $(-n,n)\times C$. The persistence of at least one component is a consequence of the choice of normalization: for instance, if $c_\infty\leq 0$, necessarily $\left(\varphi_{1,k}\right)_{k>k^\star}$ does not vanish.

To get the uniform bound in the diagonal direction, we introduce a cut-off function. For any $n\in\mathbb{N}$, there exists a non-negative non-zero function $\chi\in\mathcal{D}\left(\mathbb{R}^2\right)$ such that, for any $x\in\overline{C}$, $\chi\left(\xi,x\right)=0$ if $\xi\notin[-n-1,n+1]$ and $\chi\left(\xi,x\right)=1$ if $\xi\in[-n,n]$.

Let $k > k^\star$. Multiplying the first equation of $\left(\mathcal{PF}_{sys}\right)$ by $\varphi_{1,k}\chi$ and integrating by parts in $\mathbb{R} \times C$, we obtain:

$$\int \left(\partial_\xi \varphi_{1,k}\right)^2 \chi - \frac{1}{2}\int \varphi_{1,k}^2 \partial_{\xi\xi}\chi + \int \chi \left(\partial_x \varphi_{1,k}\right)^2 + 2\int \chi\, \partial_\xi \varphi_{1,k}\partial_x \varphi_{1,k}$$
$$\leq \int M_1 \varphi_{1,k}^2 \chi - c\int \frac{\varphi_{1,k}^2}{2}\partial_\xi \chi.$$

(The integrals being implicitly over $\mathbb{R} \times C$.)

Using $\chi \geq \mathbf{1}_{[-n,n]}$, the k-uniform L^∞-bound for $\left(\varphi_{1,k}\right)_{k>k^\star}$ and:

$$|c| \leq \max\left\{c^\star[d,2], c^\star[1,1]\right\},$$

we deduce the existence of a constant R_n independent on k such that:

$$\int_{[-n,n]\times C} \left|\partial_\xi \varphi_{1,k} + \partial_x \varphi_{1,k}\right|^2 \leq R_n.$$

The same proof holds for $\varphi_{2,k}$. Finally, the same computation in parabolic coordinates gives immediately the uniform bound in the x direction.

The uniform bound in the ξ direction is a straightforward result. Provided the uniform bound in the x direction, the uniform bound in the t direction comes from an integration over $(0,T)\times C$ of $(\mathcal{PF})$ multiplied by some test function in $L^2\left((0,T), H^1(C)\right)$.

The relative compactness in both systems of coordinates follows from the embedding:

$$L^2_{loc}\left(\mathbb{R}^2\right) \hookrightarrow L^1_{loc}\left(\mathbb{R}^2\right)$$

and the compact embedding:

$$W^{1,1}_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right) \hookrightarrow L^1_{loc}\left(\mathbb{R}^2\right).$$

To obtain the continuity of $u_{1,\infty}$ and $u_{2,\infty}$, we consider a convergent subsequence. Since the convergence occurs a.e. up to extraction, the limit point is actually in $L^\infty\left(\mathbb{R}^2, \mathbb{R}^2\right)$, whence:

$$v_{1,\infty} \in L^\infty(\mathbb{R}\times(0,L)) \cap L^2\left((-T,T), H^1((0,L))\right),$$
$$\partial_t v_{1,\infty} \in L^2\left((-T,T), \left(H^1((0,L))\right)'\right).$$

It follows from a standard regularity result that $v_{1,\infty} \in \mathcal{C}\left([-T,T], L^2((0,L))\right)$ (see for instance Evans [20, 5.9.2]).

Then, we pass the parabolic version of $(\mathcal{PF})$ to the limit in $\mathcal{D}'(\mathbb{R})$ and we can apply DiBenedetto's theory [13]: $v_{1,\infty}$ is a locally bounded weak solution of the following parabolic equation:

$$\partial_t z - \partial_x\left(\left(\mathbf{1}_{z>0} + d\mathbf{1}_{z<0}\right)\partial_x z\right) = f_1\left[\frac{z}{\alpha}\right]z^+ - f_2[-z]z^-.$$

In a large class of degenerate parabolic equations which contains in particular this equation, locally bounded weak solutions are, for any $\delta \in (0,1)$, spatially $\mathcal{C}^{0,\delta}_{loc}$ and temporally $\mathcal{C}^{0,\delta/2}_{loc}$, whence *a fortiori* $v_{1,\infty} \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$ (with $\delta = 2\beta \in (0,1)$).

Finally, by virtue of the segregation property:

$$u_{1,\infty} = \alpha^{-1} v^+_{1,\infty} \text{ a.e.,}$$
$$u_{2,\infty} = v^-_{1,\infty} \text{ a.e..}$$

From this, it follows that $v_{1,\infty}$ is in $\mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$ if and only if $u_{1,\infty}$ and $u_{2,\infty}$ are themselves in $\mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$, whence

$$\left(u_{1,\infty}, u_{2,\infty}\right) \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right). \quad \square$$

Remark. At this point, we do not know if the limit points in parabolic coordinates and in traveling coordinates are related. Yet, when $c_\infty \neq 0$, we can improve the preceding results and relate the limit points indeed.

Proposition 2.5. *Assume $c_\infty \neq 0$. The following additional collection of properties holds.*

(1) *[Improved uniform bound in the ξ direction] Provided $k^\star$ is large enough, $\left(\partial_\xi\left(\varphi_{1,k}, \varphi_{2,k}\right)\right)_{k>k^\star}$ is uniformly bounded with respect to k in $L^2\left(\mathbb{R}\times C, \mathbb{R}^2\right)$.*
(2) *[Improved compactness] There exists $\left(\varphi_{1,seg}, \varphi_{2,seg}\right) \in L^\infty\left(\mathbb{R}^2, \mathbb{R}^2\right) \cap H^1_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$ such that, up to extraction:*
   (a) *$\left(\left(\varphi_{1,k}, \varphi_{2,k}\right)\right)_{k>k^\star}$ converges to $\left(\varphi_{1,seg}, \varphi_{2,seg}\right)$ strongly in $L^2_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$ and a.e.;*
   (b) *$\left(\left(\nabla\varphi_{1,k}, \nabla\varphi_{2,k}\right)\right)_{k>k^\star}$ converges to $\left(\nabla\varphi_{1,seg}, \nabla\varphi_{2,seg}\right)$ weakly in $L^2_{loc}\left(\mathbb{R}^2, \mathbb{R}^4\right)$;*
   (c) *$\left(\left(u_{1,k}, u_{2,k}\right)\right)_{k>k^\star}$ converges to:*

$$\left(u_{1,seg}, u_{2,seg}\right) : (t,x) \mapsto \left(\varphi_{1,seg}, \varphi_{2,seg}\right)(x - c_\infty t, x)$$

   strongly in $L^2_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$, a.e., and $\left(\left(\nabla u_{1,k}, \nabla u_{2,k}\right)\right)_{k>k^\star}$ converges weakly in $L^2_{loc}\left(\mathbb{R}^2, \mathbb{R}^4\right)$.

Proof. Since $c_\infty \neq 0$, we assume without loss of generality that $k^\star$ is sufficiently large to ensure that $c_k \neq 0$ for any $k > k^\star$.

We start by showing that the uniform boundedness in $L^2(\mathbb{R}\times C)$ of $\left(\partial_\xi\varphi_{1,k}\right)_{k>k^\star}$ is equivalent to that of $\left(\partial_\xi\varphi_{2,k}\right)_{k>k^\star}$ and to that of $\left(\partial_\xi\psi_{d,k}\right)_{k>k^\star}$.

- First step of the equivalence: assume that $\left(\|\partial_\xi\varphi_{1,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded. Let $k > k^\star$. Multiply $(\mathcal{PF}_k)$ by $\partial_\xi\psi_{d,k}$, remark that:

$$\partial_\xi\psi_{1,k} = \frac{1}{d}\left(\alpha(d-1)\,\partial_\xi\varphi_{1,k} + \partial_\xi\psi_{d,k}\right)$$

  and integrate by parts over $(-n,n)\times C$ with some $n\in\mathbb{N}$. By classical parabolic estimates, the terms involving E vanish as $n\to+\infty$. By change of variable, for any $i\in\{1,2\}$,

$$\int_C \int_{-n}^{n} \varphi_{i,k} f_i \left[\varphi_{i,k}\right] \partial_\xi \varphi_{i,k} = \int_C \int_{\varphi_i(-n,x)}^{\varphi_i(+n,x)} z f_i\left(z,x\right) \mathrm{d}z \mathrm{d}x,$$

whence as $n \to +\infty$:

$$\int_C \int_{-n}^{n} \varphi_{1,k} f_1 \left[\varphi_{1,k}\right] \partial_\xi \varphi_{1,k} \to -\int_C \int_{0}^{a_1} z f_1\left(z,x\right) \mathrm{d}z \mathrm{d}x,$$

$$\int_C \int_{-n}^{n} \varphi_{2,k} f_2 \left[\varphi_{2,k}\right] \partial_\xi \varphi_{2,k} \to \int_C \int_{0}^{a_2} z f_1\left(z,x\right) \mathrm{d}z \mathrm{d}x.$$

It follows that:

$$\begin{aligned}\left(-\frac{c_k}{d}\right) \int_{\mathbb{R}\times C} \left(\alpha\left(d-1\right)\partial_\xi \varphi_{1,k} + \partial_\xi \psi_{d,k}\right)\partial_\xi \psi_{d,k} = & -\alpha \int_C \int_0^{a_1} z f_1\left(z,x\right) \mathrm{d}z \mathrm{d}x \\ & + \alpha \int_{\mathbb{R}\times C} \varphi_{1,k} f_1\left[\varphi_{1,k}\right]\left(-d\partial_\xi \varphi_{2,k}\right) \\ & + \int_{\mathbb{R}\times C} \left(-\varphi_{2,k}\right) f_2\left[\varphi_{2,k}\right]\left(\alpha \partial_\xi \varphi_{1,k}\right) \\ & + d \int_C \int_0^{a_2} z f_1\left(z,x\right) \mathrm{d}z \mathrm{d}x.\end{aligned}$$

Dividing by $-\frac{c_k}{d}$ which stays away from 0, the result reduces to:

$$\begin{aligned}\alpha\left(d-1\right)\int \partial_\xi \varphi_{1,k}\partial_\xi \psi_{d,k} + \int \left|\partial_\xi \psi_d\right|^2 = & \frac{\alpha d}{c_k}\int d\varphi_{1,k} f_1\left[\varphi_{1,k}\right]\partial_\xi \varphi_{2,k} \\ & + \frac{\alpha d}{c_k}\int \varphi_{2,k} f_2\left[\varphi_{2,k}\right]\partial_\xi \varphi_{1,k} \\ & + \frac{\alpha d}{c_k}\int_C \int_0^{a_1} z f_1\left(z,x\right) \mathrm{d}z \mathrm{d}x \\ & - \frac{d^2}{c_k}\int_C \int_0^{a_2} z f_1\left(z,x\right) \mathrm{d}z \mathrm{d}x.\end{aligned}$$

Using the boundedness in L^∞ of $\varphi_{i,k} f_i\left[\varphi_{i,k}\right]$ and the relations:

$$\int \left|\partial_\xi \varphi_{1,k}\right| = La_1,$$

$$\int \left|\partial_\xi \varphi_{2,k}\right| = La_2,$$

we obtain that the right-hand side is uniformly bounded. Since $\partial_\xi \varphi_{1,k}$ and $\partial_\xi \psi_{d,k}$ are both non-positive non-zero, if $d \geq 1$, the uniform boundedness of $\left(\int \left|\partial_\xi \psi_{d,k}\right|^2\right)_{k>k^\star}$ follows. Otherwise, there exists $R > 0$ such that:

$$\begin{aligned}\int \left|\partial_\xi \psi_{d,k}\right|^2 &\leq R + |\alpha (d-1)| \int \partial_\xi \varphi_{1,k} \partial_\xi \psi_{d,k} \\ &\leq R + |\alpha (d-1)| \left(\int \left|\partial_\xi \varphi_{1,k}\right|^2\right)^{1/2} \left(\int \left|\partial_\xi \psi_{d,k}\right|^2\right)^{1/2}.\end{aligned}$$

This shows that $\left(\int \left|\partial_\xi \psi_{d,k}\right|^2\right)^{1/2}$, which is positive, is also smaller than or equal to the largest zero of the following polynomial:

$$X^2 - |\alpha (d-1)| \, \|\partial_\xi \varphi_{1,k}\|_{L^2(\mathbb{R}\times C)} X - R$$

(which is itself positive and uniformly bounded).

- Second step of the equivalence: assume that $\left(\|\partial_\xi \varphi_{2,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded. A slight adaptation of the first step (using $\partial_\xi \psi_1 = \partial_\xi \psi_d + (d-1)\, \partial_\xi \varphi_2$) shows that the third statement is implied indeed.
- Third step of the equivalence: assume that $\left(\|\partial_\xi \psi_{d,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded. Since, for any $k > k^\star$:

$$\|\partial_\xi \psi_d\|_{L^2}^2 = \alpha^2 \|\partial_\xi \varphi_1\|_{L^2}^2 + d^2 \|\partial_\xi \varphi_2\|_{L^2}^2 - 2\alpha d \left\langle \partial_\xi \varphi_1, \partial_\xi \varphi_2 \right\rangle_{L^2},$$

with a positive third term, the first and the second statements are immediately implied.

Now that the equivalence is established, we simply show that if $c_\infty > 0$, $\left(\|\partial_\xi \varphi_{1,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded, and conversely if $c_\infty < 0$, $\left(\|\partial_\xi \varphi_{2,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded. Multiplying the first equation of $\left(\mathcal{PF}_{sys}\right)$ by $\partial_\xi \varphi_1$, integrating over $\mathbb{R}\times C$, and using the sign of $\partial_\xi \varphi_1$ and classical parabolic estimates at $\pm\infty$, the result reduces to:

$$\begin{aligned}c \int_{\mathbb{R}\times(0,L)} \left|\partial_\xi \varphi_1\right|^2 &= k \int_{\mathbb{R}\times(0,L)} \varphi_1 \varphi_2 \partial_\xi \varphi_1 + \int_0^L \int_0^{a_1} z f_1(z,x)\, \mathrm{d}z \mathrm{d}x \\ &\leq \int_0^L \int_0^{a_1} z f_1(z,x)\, \mathrm{d}z \mathrm{d}x.\end{aligned}$$

Similarly, we obtain:

$$c \int\limits_{\mathbb{R}\times(0,L)} \left|\partial_\xi \varphi_2\right|^2 = \alpha k \int\limits_{\mathbb{R}\times(0,L)} \varphi_1 \varphi_2 \partial_\xi \varphi_2 - \int\limits_0^L \int\limits_0^{a_2} z f_2(z,x)\,\mathrm{d}z\mathrm{d}x$$
$$\geq - \int\limits_0^L \int\limits_0^{a_2} z f_2(z,x)\,\mathrm{d}z\mathrm{d}x.$$

The improved uniform bound in the ξ direction immediately follows.

The improved relative compactness of $\left(\left(\varphi_{1,k}, \varphi_{2,k}\right)\right)_{k>k^\star}$ is a straightforward consequence of the previous lemmas, of Sobolev's embeddings and of Banach–Alaoglu's theorem. For the relative compactness of $\left(\left(u_{1,k}, u_{2,k}\right)\right)_{k>k^\star}$, let $[s] : (t,x) \mapsto (x - st, x)$, so that for any $k > k^\star$ $(u_1, u_2) = (\varphi_1, \varphi_2) \circ [c]$. For any $i \in \{1,2\}$:

$$\|u_i - u_{i,seg}\|_{L^2_{loc}} \leq \|\varphi_i \circ [c] - \varphi_i \circ [c_\infty]\|_{L^2_{loc}} + \|\varphi_i \circ [c_\infty] - \varphi_{i,seg} \circ [c_\infty]\|_{L^2_{loc}}.$$

Then, by virtue of Fréchet–Kolmogorov's theorem, the right-hand side vanishes as $k \to +\infty$. The same argument holds for the weak convergence of the derivatives. □

Remark. We point out that the preceding result is specific to the case of constant a_1 and a_2 (without this assumption, one term due to E does not vanish after the integration by parts). In the general case, we do not know if the bounds of Proposition 2.4 can be improved.

Corollary 2.6. *If $c_\infty \neq 0$, the parabolic limit point $\left(u_{1,seg}, u_{2,seg}\right)$ obtained with the improved compactness result from Proposition 2.5 is also a limit point $\left(u_{1,\infty}, u_{2,\infty}\right)$ in the sense of Proposition 2.4. In particular, $\left(u_{1,seg}, u_{2,seg}\right) \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$, whence $\left(\varphi_{1,seg}, \varphi_{2,seg}\right) \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$ as well.*

Remark. The case $c_\infty = 0$ is somehow degenerate and does not really correspond to what intuition calls a "pulsating" front. Moreover, we will need quite different techniques to handle the two cases and, even in the very end, there will be no clear common framework. Therefore, hereafter, we call the case $c_\infty = 0$ "segregated stationary equilibrium" whereas the case $c_\infty \neq 0$ is referred to as "segregated pulsating front". These terms will be precisely defined in a moment.

2.3. Characterization of the segregated stationary equilibrium

In this subsection, we assume $c_\infty = 0$ and we use Proposition 2.4 to get an extracted convergent subsequence of pulsating fronts, still denoted $\left(\left(u_{1,k}, u_{2,k}\right)\right)_{k>k^\star}$, with limit $\left(u_{1,\infty}, u_{2,\infty}\right)$. Up to an additional extraction, we assume a.e. convergence of $\left(u_{1,k}, u_{2,k}, u_{1,k}u_{2,k}\right)$ to $\left(u_{1,\infty}, u_{2,\infty}, 0\right)$.

Obviously, since $c_\infty = 0$, we expect that $\left(u_{1,\infty}, u_{2,\infty}\right)$ does not depend on t. This will be true indeed, so that it makes sense to refer to this case as "stationary equilibrium". To stress this particularity, we fix t_{cv} such that $\left((u_1, u_2)_{|\{t_{cv}\}\times\mathbb{R}}\right)_{k>k^\star}$ converges a.e. and we define $e = \left(v_{d,\infty}\right)_{|\{t_{cv}\}\times\mathbb{R}}$, so that if $\left(u_{1,\infty}, u_{2,\infty}\right)$ is constant with respect to t, $\left(\alpha u_{1,\infty}, d u_{2,\infty}\right)(t,x) = \left(e^+, e^-\right)(x)$ for any $(t,x) \in \mathbb{R}^2$.

We start with an important particular case.

Lemma 2.7. *Assume that, provided $k^\star$ is large enough, $(c_k)_{k>k^\star} = 0$. Then:*

- *for any $k > k^\star$, (u_1, u_2) reduces to:*

$$(t, x) \mapsto (\varphi_1, \varphi_2)(x, x),$$

- *for any $(t, x) \in \mathbb{R}^2$:*

$$\left(\alpha u_{1,\infty}, du_{2,\infty}\right)(t, x) = \left(e^+, e^-\right)(x),$$

- *the convergence of $\left((\alpha u_1, du_2)_{|\{t_{cv}\}\times\mathbb{R}}\right)_{k>k^\star}$ to $\left(e^+, e^-\right)$ actually occurs in $\mathcal{C}^{0,\beta}_{loc}(\mathbb{R})$,*
- *the convergence of $\left((v_d)_{|\{t_{cv}\}\times\mathbb{R}}\right)_{k>k^\star}$ to e actually occurs in $\mathcal{C}^{2,\beta}_{loc}(\mathbb{R})$,*
- *e satisfies:*

$$-e'' = \eta[e].$$

Proof. The system $(\mathcal{P})$ reduces to an elliptic system. It is then easy to deduce the locally uniform convergence, the time-independence and the limiting equation. We refer, for instance, to [23] for details. $\square$

Some of the preceding results can be extended.

Lemma 2.8. *The properties:*

- *for any $(t, x) \in \mathbb{R}^2$, $\left(\alpha u_{1,\infty}, du_{2,\infty}\right)(t, x) = \left(e^+, e^-\right)(x)$;*
- *$e \in \mathcal{C}^2(\mathbb{R})$ and $-e'' = \eta[e]$;*

hold true regardless of any sign assumption on the sequence $(c_k)_{k>k^\star}$.

Proof. The two statements are actually quite easy to verify. Let $\left(t, t', x\right) \in \mathbb{R}^3$ such that, for any $i \in \{1, 2\}$ and any $\tau \in \left\{t, t'\right\}$, $u_{i,k}(\tau, x) \to u_{i,\infty}(\tau, x)$ as $k \to +\infty$. Recalling that:

$$\int_{\mathbb{R}} \partial_t u_{i,k} = -c_k \int_{\mathbb{R}} \partial_\xi \varphi_{i,k} \to 0$$

as $k \to +\infty$ is sufficient to show that in the following inequality:

$$\begin{aligned}\left|u_{i,\infty}(t, x) - u_{i,\infty}\left(t', x\right)\right| \le & \left|u_{i,\infty}(t, x) - u_{i,k}(t, x)\right| \\ & + \left|\int_t^{t'} \partial_t u_{i,k}(\tau, x)\,\mathrm{d}\tau\right| \\ & + \left|u_{i,k}\left(t', x\right) - u_{i,\infty}\left(t', x\right)\right|\end{aligned}$$

the right-hand side converges to 0 as $k \to +\infty$. Therefore the left-hand side is 0, whence $u_{i,\infty}$ is constant with respect to the time variable in a dense subset of $\mathbb{R}^2$, and then by continuity, it holds *a fortiori* everywhere in $\mathbb{R}^2$.

As for the regularity and limiting equation, the equation is satisfied *a priori* in the distributional sense, then in the classical sense by elliptic regularity. $\square$

Lemma 2.9. *For any $x \in \mathbb{R}$, the sequence $(e(x+nL))_{n\in\mathbb{N}}$ is non-increasing.*

Proof. By monotonicity with respect to ξ and periodicity with respect to x, for any $(t,x) \in \mathbb{R}^2$ and any $k > k^\star$:

$$\begin{aligned} v_d(t, x+L) - v_d(t,x) &\leq \psi_d(x - ct + L, x + L) - \psi_d(x - ct, x) \\ &\leq \psi_d(x - ct + L, x) - \psi_d(x - ct, x) \\ &\leq 0. \end{aligned}$$

In particular, for any $(t,x) \in \mathbb{R}^2$ and any $k > k^\star$, the sequence $\left(v_{d,k}(t, x+nL)\right)_{n\in\mathbb{N}}$ is non-increasing, and then, passing to the limit as $k \to +\infty$, the sequence $(e(x+nL))_{n\in\mathbb{N}}$ is non-increasing. This holds for any x in a dense subset of $\mathbb{R}$ and then for any $x \in \mathbb{R}$ by continuity of e. $\square$

Lemma 2.10. *e is non-zero and sign-changing. Moreover:*

$$\inf e^{-1}((-\infty, 0)) > -\infty.$$

Proof. The normalization:

$$0 = \inf\left\{\xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_{1,k}(\xi, x) < \frac{a_1}{2}\right\},$$

implies that $u_{1,\infty} \neq 0$, whence $e \neq 0$. It shows also that the set:

$$\left\{n \in \mathbb{Z} \mid \exists x \in \overline{C} \quad \varphi_{1,k}(x + nL, x + nL) < \frac{a_1}{2}\right\}$$

is uniformly bounded with respect to k from below. In particular, it has a minimum $\underline{n}_k \in \mathbb{Z}$. Then let:

$$x_k = \inf\left\{x \in \overline{C} \mid \varphi_{1,k}\left(x + \underline{n}_k L, x + \underline{n}_k L\right) < \frac{a_1}{2}\right\},$$

so that:

$$\varphi_{1,k}(x, x) > \frac{a_1}{2} \text{ for any } x < x_k + \underline{n}_k L.$$

By monotonicity, we deduce:

$$\varphi_{1,k}(\xi, x) > \frac{a_1}{2} \text{ for any } \xi < x < \underline{n}_k L.$$

If (up to extraction) $\underline{n}_k \to +\infty$ as $k \to +\infty$, then the definition of the normalization is contradicted by the preceding inequality evaluated at $\xi = 0$ and $x \in [L, 2L]$, whence $\left(\underline{n}_k\right)_{k>k^\star}$ is uniformly bounded from above as well. In particular, up to extraction, $\left(\underline{n}_k\right)_{k>k^\star}$ converges to a finite limit. The finiteness of $\inf \{x \in \mathbb{R} \mid e(x) < 0\}$ follows immediately.

By uniqueness, if $e > 0$, $e = \alpha a_1$. This is discarded by the finiteness of $\lim\limits_{k\to+\infty} \underline{n}_k$, whence e is sign-changing. $\square$

Remark. If, instead of the normalization sequence:

$$0 = \inf \left\{ \xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_1(\xi, x) < \frac{a_1}{2} \right\} \text{ for any } k > k^\star,$$

we choose:

$$0 = \sup \left\{ \xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_2(\xi, x) < \frac{a_2}{2} \right\} \text{ for any } k > k^\star,$$

and if we consider once again the case $c_\infty = 0$, the preceding results hold apart from $\inf e^{-1}((-\infty, 0)) > -\infty$, which is naturally replaced by:

$$\sup e^{-1}((0, +\infty)) < +\infty.$$

In view of these results, we state the following definition.

Definition 2.11. A function $z \in \mathcal{C}^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is called a *segregated stationary equilibrium* if:

(1) $-z'' = \eta[z]$;
(2) for any $x \in \overline{C}$, $(z(x + nL))_{n\in\mathbb{N}}$ is non-increasing;
(3) z is non-zero and sign-changing;
(4) $\inf z^{-1}((-\infty, 0)) > -\infty$ or $\sup z^{-1}((0, +\infty)) < +\infty$.

Corollary 2.12. *e is a segregated stationary equilibrium.*

Let us derive some properties necessarily satisfied by any segregated stationary equilibrium. The first one is obvious but will be useful.

Proposition 2.13. *If z is a segregated stationary equilibrium, then for any $n \in \mathbb{Z}$, $x \mapsto z(x + nL)$ is a segregated stationary equilibrium as well.*

The following one is easily derived from the second order necessary conditions satisfied at a local extremum.

Proposition 2.14. *Let z be a segregated stationary equilibrium. Then $-da_2 < z < \alpha a_1$.*

The following one highlights some difficulties which are intrinsic to the null speed limit.

Proposition 2.15. *Let z be a segregated stationary equilibrium and*

$$\mathcal{Z}(z) = z^{-1}(\{0\}).$$

The set $\mathcal{Z}(z)$ is a discrete set. If it is a finite set, its cardinal is odd. Moreover, it has a minimum or a maximum.

Proof. The fact that $\mathcal{Z}(z)$ is a discrete set follows easily from Hopf's lemma and the regularity of z. Provided finiteness of the set, the monotonicity of $(z(x+nL))_{n\in\mathbb{N}}$ for any $x \in C$ yields the parity of $\#\mathcal{Z}(z)$. Finally, the existence of an extremum comes from the definition of the segregated stationary equilibrium. $\square$

Remark. Under the more restrictive assumption $(\mathcal{H}_{freq})$ presented by the first author in [23], it is possible to prove that every segregated stationary equilibrium has a unique zero. It is basically deduced from the fact that, when there are multiple zeros, the segregated stationary equilibrium restricted to any interval delimited by two consecutive zeros is the unique solution of a semi-linear Dirichlet problem. The monotonicity of $(e(x+nL))_{n\in\mathbb{N}}$ ensures that the distance between these consecutive zeros is smaller than L and then, considering the next zero and using $(\mathcal{H}_{freq})$, a contradiction arises. We do not detail this proof here.

Proposition 2.16. *Let z be a stationary segregated equilibrium.*

If $z^{-1}(\{0\})$ has a minimum, as $n \to +\infty$,

$$\|z - \alpha a_1\|_{\mathcal{C}^2([-(n+1)L,-nL])} \to 0.$$

If $z^{-1}(\{0\})$ has a maximum, as $n \to +\infty$,

$$\|z - d a_2\|_{\mathcal{C}^2([nL,(n+1)L])} \to 0.$$

Proof. We assume that $z^{-1}(\{0\})$ has a minimum, the other case being similar. Since, for any $x \in [0, L)$, $(z(x-nL))_{n\in\mathbb{N}}$ is bounded and non-decreasing, it converges to a limit $z_{-\infty}(x)$. Using Lipschitz-continuity of z, we are able to prove that $z_{-\infty}$ is Lipschitz-continuous in $\overline{C}$. Using elliptic regularity, the distributional equation:

$$-z''_{-\infty} = z_{-\infty} f_1\left[\frac{z_{-\infty}}{\alpha}\right]$$

and Arzela–Ascoli's theorem, we are able to prove in fact that $z_{-\infty} \in \mathcal{C}^{2,\beta}\left(\overline{C}\right)$ and that the convergence occurs in $\mathcal{C}^{2,\beta}\left(\overline{C}\right)$. This proves that $z_{-\infty}$ also satisfies in the classical sense the equation. Moreover,

$$|z(x-(n+1)L) - z(x-nL)| \to 0$$

as $n \to +\infty$ and, this proves that $z_{-\infty}$ is periodic. Since it is also positive, by uniqueness, $z_{-\infty} = \alpha a_1$. $\square$

2.4. Characterization of the segregated pulsating fronts

In this subsection, we assume $c_\infty \neq 0$ and we use Proposition 2.5 to get an extracted convergent subsequence of profiles, still denoted $\left(\left(\varphi_{1,k}, \varphi_{2,k}\right)\right)_{k>k^\star}$, with limit $\left(\varphi_{1,seg}, \varphi_{2,seg}\right)$. Up to an additional extraction, we assume a.e. convergence of $\left(\varphi_{1,k}, \varphi_{2,k}, \varphi_{1,k}\varphi_{2,k}\right)$ to $\left(\varphi_{1,seg}, \varphi_{2,seg}, 0\right)$. We define $\phi = \alpha\varphi_{1,seg} - d\varphi_{2,seg}$ and $w = \alpha u_{1,seg} - du_{2,seg}$ (that is, (ϕ, w) is the limit of $\left(\left(\psi_{d,k}, v_{d,k}\right)\right)_{k>k^\star}$).

Here, parabolic limit points and traveling limit points are naturally related by the isomorphism $(t,x) \mapsto (x - c_\infty t, x)$. Therefore we can freely use the more convenient system of variables.

2.4.1. Definitions and asymptotics

Hereafter,

$$\sigma : z \mapsto \mathbf{1}_{z>0} + \frac{1}{d}\mathbf{1}_{z<0},$$

$$\hat{\sigma} : z \mapsto \mathbf{1}_{z>0} + d\mathbf{1}_{z<0}.$$

Remark. Clearly, for any $z \in \mathcal{C}\left(\mathbb{R}^2\right)$:

- $\sigma[z]$ and $\hat{\sigma}[z]$ are in $L^\infty\left(\mathbb{R}^2\right)$;
- $\sigma[z]$ and $\hat{\sigma}[z]$ vanish if and only if z vanish;
- $\sigma[z]\hat{\sigma}[z] = 1$ in $\mathbb{R}^2$ apart from the zero set of z;
- $\sigma[z]z$ and $\hat{\sigma}[z]z$ are in $\mathcal{C}\left(\mathbb{R}^2\right)$; furthermore, if $z \in W^{1,\infty}\left(\mathbb{R}^2\right)$, then they are Lipschitz-continuous.

Lemma 2.17. *The equalities:*

$$\sigma[w](t,x) = \sigma[\phi](x - c_\infty t, x),$$

$$\hat{\sigma}[w](t,x) = \hat{\sigma}[\phi](x - c_\infty t, x),$$

hold for all $(t,x) \in \mathbb{R}^2$.

Furthermore, the following equalities hold in $L^2_{loc}\left(\mathbb{R}^2\right)$*:*

$$\partial_t(\sigma[w]w) = \sigma[w]\partial_t w,$$

$$\partial_x w = \hat{\sigma}[w]\partial_x(\sigma[w]w),$$

$$\partial_\xi(\sigma[\phi]\phi) = \sigma[\phi]\partial_\xi\phi,$$

$$\partial_x\phi = \hat{\sigma}[\phi]\partial_x(\sigma[\phi]\phi).$$

Proof. The equalities between the weak derivatives are derived easily from the weak formulation of $(\mathcal{PF})$ (recall the proof of Proposition 2.4). When passing to the limit $k \to +\infty$, it is possible to obtain equivalently all these equations (we restrict ourselves here to parabolic coordinates, the equalities in traveling coordinates being obtained analogously):

$$\sigma[w]\partial_t w - \partial_{xx} w = \eta[w],$$

$$\partial_t (\sigma [w] w) - \partial_{xx} w = \eta [w] ,$$
$$\partial_t (\sigma [w] w) - \partial_x \left(\hat{\sigma} [w] \partial_x (\sigma [w] w)\right) = \eta [w] . \quad \square$$

Definition 2.18. Let $s \in \mathbb{R}\setminus\{0\}$ and $\mathcal{C}_0^1\left(\mathbb{R}^2\right)$ be the subset of compactly supported elements of $\mathcal{C}^1\left(\mathbb{R}^2\right)$.

We say that $\varphi \in \mathcal{C}\left(\mathbb{R}^2\right) \cap H^1_{loc}\left(\mathbb{R}^2\right)$ is a weak solution of:

$$-\mathrm{div}\,(E\nabla\varphi) - s\partial_\xi (\sigma [\varphi] \varphi) = \eta [\varphi] \quad (\mathcal{SPF}[s])$$

if, for any test function $\zeta \in \mathcal{C}_0^1\left(\mathbb{R}^2\right)$:

$$\int E\nabla\varphi.\nabla\zeta + s\int \sigma [\varphi] \varphi \partial_\xi \zeta = \int \eta [\varphi] \zeta.$$

Lemma 2.19. *ϕ is a weak solution of $(\mathcal{SPF}[c_\infty])$.*

Proof. This is merely the traveling formulation of the limiting equation obtained *a priori* in $\mathcal{D}'\left(\mathbb{R}^2\right)$ and *a fortiori* holding in the weak sense. $\square$

Remark. Since $c_\infty\sigma [\phi] \partial_\xi\phi$ and $\eta [\phi]$ are in $L^2_{loc}\left(\mathbb{R}^2\right)$, $-\mathrm{div}\,(E\nabla\phi)$ is actually in $L^2_{loc}\left(\mathbb{R}^2\right)$ as well and we can also consider test functions in $L^2_{loc}\left(\mathbb{R}^2\right)$, but then we cannot integrate by parts as in the equality above.

Proposition 2.20. *Let $s \in \mathbb{R}\setminus\{0\}$. If φ is a weak solution of $(\mathcal{SPF}[s])$, then $z : (t, x) \mapsto \varphi (x - st, x)$ is a weak solution of:*

$$\partial_t (\sigma [z] z) - \partial_{xx} z = \eta [z] ,$$

in the sense that for any $\zeta \in \mathcal{C}_0^1\left(\mathbb{R}^2\right)$, the following holds:

$$\int (\sigma [z] z\partial_t\zeta - \partial_x z\partial_x\zeta + \eta [z] \zeta) = 0.$$

Remark. Similarly, we can restrict ourselves regarding this weak parabolic equation to test functions $\zeta \in L^2_{loc}\left(\mathbb{R}^2\right)$ but then we cannot integrate by parts.

Lemma 2.21. *ϕ is periodic with respect to x and non-increasing with respect to ξ.*

Proof. Thanks to the a.e. convergence, periodicity with respect to x and monotonicity with respect to ξ are preserved a.e., that is at least in a dense subset of $\mathbb{R}^2$. Continuity extends these behaviors everywhere. $\square$

Lemma 2.22. *ϕ is non-zero and sign-changing.*

Remark. This statement holds if and only if both $\varphi_{1,seg}$ and $\varphi_{2,seg}$ are non-zero (or equivalently non-negative non-zero).

Proof. Assume for example $c_\infty < 0$. The normalization gives immediately $\varphi_{1,seg} \neq 0$. If $\varphi_{2,seg} = 0$, $u_{1,seg}$ is a non-negative solution in $\mathbb{R}^2$ of:

$$\partial_t z - \partial_{xx} z = z f_1 [z] .$$

By the parabolic strong minimum principle, $u_{1,seg} \gg 0$, and by parabolic regularity, $u_{1,seg}$ is regular. By classical parabolic estimates, as $\xi \to -\infty$, $\varphi_{1,seg}$ converges uniformly in x to a positive periodic solution of:

$$-\partial_{xx} z = z f_1 [z] ,$$

that is to a_1. Similarly, $\varphi_{1,seg}$ converges to 0 as $\xi \to +\infty$.

Thus $\varphi_{1,seg}$ is a pulsating front connecting a_1 to 0 at speed $c_\infty < 0$. This is a contradiction (see Theorem 2.1).

A symmetric proof discards the case $c_\infty > 0$. □

In view of these results, we state the following definition.

Definition 2.23. Let:

$$s \in \mathbb{R} \setminus \{0\} ,$$

$$z \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right) \cap H^1_{loc}\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right),$$

$$\varphi : (\xi, x) \mapsto z\left(\frac{x-\xi}{s}, x\right).$$

z is called a *segregated pulsating front with speed s and profile φ* if:

(1) φ is a weak solution of $(\mathcal{SPF}[s])$;
(2) φ is non-increasing with respect to ξ;
(3) φ is periodic with respect to x;
(4) φ is non-zero and sign-changing.

Corollary 2.24. *w is a segregated pulsating front with speed c_∞ and profile ϕ.*

Proposition 2.25. *Let z be a segregated pulsating front with profile φ. As $\xi \to +\infty$,*

$$\max_{x \in \overline{C}} |\varphi(-\xi, x) - \alpha a_1| + \max_{x \in \overline{C}} |\varphi(\xi, x) + d a_2| \to 0.$$

Proof. It follows from classical parabolic estimates and the monotonicity of φ with respect to ξ. □

2.4.2. The intrinsic free boundary problem

We intend to conclude the characterization of the segregated pulsating front with a uniqueness result. Our proof will use a sliding argument and the continuity of $\partial_x z$. Obviously, in $\mathbb{R}^2 \backslash z^{-1}(\{0\})$, classical parabolic regularity applies and the regularity of a segregated pulsating front is only limited by that of η. On the contrary, the regularity of z at the free boundary $z^{-1}(\{0\})$ is a tough problem and, as usual in free boundary problems, requires a detailed study of the regularity of the free boundary itself. This study is the object of the following pages.

Let us stress here that our interest does not lie in the most general study of the free boundaries of the solutions of $(\mathcal{SPF}[s])$. To show that $\partial_x z$ is continuous, Lipschitz-continuity of the free boundary is sufficient, and we are able to prove such a regularity only using the monotonicity properties of the segregated pulsating fronts as well as the parabolic maximum principle. We believe that this proof has interest of its own. Yet, at the end of this subsection, we will explain why we expect the free boundary to actually be $\mathcal{C}^1$ and $\partial_t z$ to be continuous without any additional assumption.

Up to the next subsection, let z be a segregated pulsating front with speed $s \neq 0$ and profile φ and let:

$$\Gamma = \left\{(t,x) \in \mathbb{R}^2 \mid z(t,x) = 0\right\},$$
$$\Omega_+ = \left\{(t,x) \in \mathbb{R}^2 \mid z(t,x) > 0\right\},$$
$$\Omega_- = \left\{(t,x) \in \mathbb{R}^2 \mid z(t,x) < 0\right\}.$$

Before going any further, let us state precisely the results of this subsection in the following proposition.

Theorem 2.26. *There exists a continuous bijection $\Xi : \mathbb{R} \to \mathbb{R}$ such that Γ is the graph of Ξ and such that:*

$$\begin{cases} \Omega_+ = \left\{(t,x) \in \mathbb{R}^2 \mid x < \Xi(t)\right\} \\ \Omega_- = \left\{(t,x) \in \mathbb{R}^2 \mid x > \Xi(t)\right\}. \end{cases}$$

Moreover, $\partial_x z \in \mathcal{C}^{0,\beta}\left(\mathbb{R}^2\right)$ and $(\partial_x z)_{|\Gamma} \ll 0$.

Remark. Of course, this type of result is strongly reminiscent of the celebrated paper by Angenent [1] about the number of zeros of a solution of a parabolic equation. We stress that this result cannot be applied here because of the non-linearity due to $\sigma[z]$. It will be clearly established during the proof that this lack of regularity is compensated here by the monotonicity of z.

The proof of Theorem 2.26 begins with a couple of lemmas leading to the existence of Ξ.

Lemma 2.27. *The quantities:*

$$\Xi_+(t) = \sup\{x \in \mathbb{R} \mid z(t,x) > 0\},$$
$$\Xi_-(t) = \inf\{x \in \mathbb{R} \mid z(t,x) < 0\},$$

are well-defined and finite.

Proof. By Proposition 2.25, for any $(t,x) \in \mathbb{R}^2$:

$$\lim_{n\to+\infty} \max_{x\in\overline{C}} |\varphi(x+nL-st,x)+da_2| = 0,$$
$$\lim_{n\to+\infty} \max_{x\in\overline{C}} |\varphi(x-nL-st,x)-\alpha a_1| = 0.$$

By periodicity with respect to x:

$$\begin{aligned}\varphi(x\pm nL-st,x) &= \varphi(x\pm nL-st,x\pm nL)\\ &= z(t,x\pm nL)\end{aligned}$$

and thus $x \mapsto z(t,x)$ is negative at $+\infty$, positive at $-\infty$, whence $\Xi_+(t)$ and $\Xi_-(t)$ are well-defined and finite. □

Lemma 2.28. *Let* $(t,x) \in \mathbb{R}^2$.

(1) *If* $s > 0$ *and* $z(t,x) \leq 0$, *then for any* $y > x$, $z(t,y) < 0$.
(2) *If* $s < 0$ *and* $z(t,x) \geq 0$, *then for any* $y < x$, $z(t,y) > 0$.

Proof. Let us show for instance the first statement, the other one being symmetric.

By Lemma 2.27, there exists $X > x$ such that $z(t,X) < 0$. Since φ is non-increasing with respect to ξ, z is non-decreasing with respect to t, whence for any $t' < t$, $z(t',x) \leq 0$ and $z(t',X) < 0$. Moreover, by Proposition 2.25, there exists $T > 0$ such that:

$$z(t-T,y) < 0 \text{ for any } y \in [x,X].$$

By continuity of z, there exists $\tau > 0$ such that:

$$z \ll 0 \text{ in } [t-T,t-T+\tau]\times[x,X].$$

Let:

$$\tau^\star = \sup\{\tau\in(0,T) \mid z \ll 0 \text{ in } [t-T,t-T+\tau]\times(x,X)\}$$

and let us check that $\tau^\star = T$.

If $\tau^\star < T$, then there exists $y \in (x,X)$ such that $z(t-T+\tau^\star,y) = 0$. But in the parabolic cylinder $[t-T,t-T+\tau^\star]\times[x,X]$, $z<0$ satisfies a regular parabolic equation and satisfies also the strong parabolic maximum principle, which immediately contradicts the strict sign of z at $t-T$.

Thus $\tau^\star = T$ and then, if there exists $y \in (x,X)$ such that $z(t,y) = 0$, applying once more the strong parabolic maximum principle gives the same contradiction.

The proof is ended by passing to the limit $X \to +\infty$. □

Corollary 2.29. *For any* $t \in \mathbb{R}$, *the zero of* $x \mapsto z(t,x)$ *is unique, or equivalently,* $\Xi_+(t) = \Xi_-(t)$.

Lemma 2.30. *For any $t \in \mathbb{R}$, let $\Xi(t)$ be the unique zero of $x \mapsto z(t,x)$.*

Then $\Xi : \mathbb{R} \to \mathbb{R}$ is unbounded, non-decreasing if $s > 0$ and non-increasing if $s < 0$, and continuous.

Furthermore, Γ is exactly the graph of Ξ,

$$\Omega_- = \left\{(t,x) \in \mathbb{R}^2 \mid x > \Xi(t)\right\},$$
$$\Omega_+ = \left\{(t,x) \in \mathbb{R}^2 \mid x < \Xi(t)\right\}.$$

Proof. Assume for instance and up to the end of the proof $s > 0$ (the case $s < 0$ is similar).

Since φ is non-increasing with respect to ξ, z is non-decreasing with respect to t. Assume by contradiction that there exists $t, t' \in \mathbb{R}$ such that $t' < t$ and $\Xi(t) < \Xi(t')$. By Lemma 2.28, for any $x > \Xi(t)$, $z(t,x) < 0$, whence in particular $z\left(t, \Xi\left(t'\right)\right) < 0$, whence by monotonicity of z, $z\left(t', \Xi\left(t'\right)\right) < 0$, which contradicts the definition of $\Xi\left(t'\right)$. Thus Ξ is non-decreasing.

The unboundedness is straightforward: considering the limiting signs of $t \mapsto z(t,x)$ shows by continuity that this function has at least one zero for any $x \in \mathbb{R}$. But if Ξ was bounded, thanks to Lemma 2.28 once again, it would be possible to build a counter-example.

Finally, continuity is also straightforward, since it is well-known that a monotonic function admits left-sided and right-sided limits at every point and that every discontinuity it has is a jump discontinuity. The existence of such a discontinuity, that is of a segment $\{t^\star\} \times \left[x^\star, x^\star + X\right]$ included in the free boundary, would immediately contradict Lemma 2.28. □

Corollary 2.31. *Both Ω_+ and Ω_- have a Lipschitz boundary.*

Proof. It suffices to recall that every point of the graph of a monotone function satisfies an interior cone condition and that such a condition characterizes Lipschitz boundaries. □

In view of this regularity of $\Omega_\pm$ and by means of easy integration by parts, we are now able to generalize to any segregated pulsating front a property that was immediately satisfied by w (Lemma 2.17).

Corollary 2.32. *The following equalities hold in $L^2_{loc}\left(\mathbb{R}^2\right)$:*

$$\partial_t (\sigma[z]z) = \sigma[z]\partial_t z,$$
$$\partial_x z = \hat{\sigma}[z]\partial_x(\sigma[z]z),$$
$$\partial_\xi (\sigma[\varphi]\varphi) = \sigma[\varphi]\partial_\xi \varphi,$$
$$\partial_x \varphi = \hat{\sigma}[\varphi]\partial_x(\sigma[\varphi]\varphi).$$

Proof. Let us show for instance the first one. Let $(\zeta_n)_{n\in\mathbb{N}} \in \left(\mathcal{D}\left(\mathbb{R}^2\right)\right)^{\mathbb{N}}$ such that (ζ_n) converges in L^2_{loc} to some test function $\zeta \in L^2_{loc}$. For any $n \in \mathbb{N}$, we have:

$$\begin{aligned}\int \partial_t(\sigma[z]z)\zeta_n &= -\int \sigma[z]z\partial_t\zeta_n \\ &= -\int_{\Omega_+} z\partial_t\zeta_n - \int_{\Omega_-}\frac{1}{d}z\partial_t\zeta_n.\end{aligned}$$

Since $\Omega_\pm$ have a Lipschitz boundary, we can integrate by parts once again (recalling that, by definition, $z_{|\Gamma} = 0$):

$$\int \partial_t \left(\sigma\left[z\right] z\right) \zeta_n = \int_{\Omega_+} \partial_t z \zeta_n + \int_{\Omega_-} \frac{1}{d} \partial_t z \zeta_n$$
$$= \int \sigma\left[z\right] \partial_t z \zeta_n.$$

Passing to the limit $n \to +\infty$ ends the proof. □

More interestingly, we are now closer to an explicit free boundary condition. The following three lemmas are dedicated to this question.

Lemma 2.33. *Let Ξ be defined as in Lemma 2.30.*

Then the traces $\left(\partial_x z^+\right)_{|\partial\Omega_+}$ and $\left(\partial_x z^-\right)_{|\partial\Omega_-}$ are well-defined in $L^2_{loc}\left(\partial\Omega_+\right)$ and $L^2_{loc}\left(\partial\Omega_-\right)$ respectively.

Proof. Since $\partial\Omega_+$ (respectively $\partial\Omega_-$) is a Lipschitz boundary, let us prove that $\left(\partial_x\left(z^+\right)\right)_{|\Omega_+}$ (resp.$\left(\partial_x\left(z^-\right)\right)_{|\Omega_-}$) is in $H^1_{loc}\left(\Omega_+\right)$ (resp. $H^1_{loc}\left(\Omega_-\right)$). It is already established that it is in $L^2_{loc}\left(\mathbb{R}^2\right)$. Considering the equation satisfied by z then shows immediately that $\left(\partial_{xx}\left(z^+\right)\right)_{|\Omega_+}$ (resp. $\left(\partial_{xx}\left(z^-\right)\right)_{|\Omega_-}$) is in $L^2_{loc}\left(\mathbb{R}^2\right)$ as well. To conclude, it remains to prove that $\left(\partial_{tx}\left(z^+\right)\right)_{|\Omega_+}$ (resp. $\left(\partial_{tx}\left(z^-\right)\right)_{|\Omega_-}$) is in $L^2_{loc}\left(\Omega_+\right)$ (resp. $L^2_{loc}\left(\Omega_-\right)$).

Let $t_1, t_2, x_1, x_2 \in \mathbb{R}$ such that $t_1 < t_2$, $x_1 < x_2$ and $[t_1, t_2] \times [x_1, x_2] \subset \Omega_+$. Let $\chi \in \mathcal{D}\left(\mathbb{R}^2\right)$ be a non-negative non-zero function identically equal to 1 in $[t_1, t_2] \times [x_1, x_2]$. From the following equation, satisfied in the classical sense in Ω_+:

$$\partial_t\left(\partial_t z\right) - \partial_{xx}\left(\partial_t z\right) = g_1\left[\frac{z}{\alpha}\right] \partial_t z,$$

multiplied by $\partial_t z\chi$ and integrated over $\mathbb{R}^2$, we deduce:

$$-\int \frac{1}{2} \left|\partial_t z\right|^2 \partial_t \chi + \int \left|\partial_{xt} z\right|^2 \chi - \frac{1}{2} \int \left|\partial_t z\right|^2 \partial_{xx} \chi = \int g_1\left[\frac{z}{\alpha}\right] \left|\partial_t z\right|^2 \chi.$$

It follows that there exists a constant $R > 0$ such that:

$$\|\partial_{xt} z\|^2_{L^2([t_1,t_2]\times[x_1,x_2])} \le R \|\partial_t z\|_{L^2([t_1,t_2]\times[x_1,x_2])} \|\chi\|_{H^2(\mathbb{R}^2)},$$

whence $\partial_{tx} z^+ \in L^2_{loc}\left(\Omega_+\right)$ indeed.

Similarly, $\partial_{tx}\left(z^-\right) \in L^2_{loc}\left(\Omega_-\right)$.

In the end, Ω_+ and Ω_- are Lipschitz domains, $\left(\partial_x\left(z^+\right)\right)_{|\Omega_+} \in H^1_{loc}\left(\Omega_+\right)$ and $\left(\partial_x\left(z^-\right)\right)_{|\Omega_-} \in H^1_{loc}\left(\Omega_-\right)$, whence their traces can be rigorously defined in $L^2_{loc}\left(\partial\Omega_+\right)$ and $L^2_{loc}\left(\partial\Omega_-\right)$ respectively. □

Lemma 2.34. *Let Ξ be defined as in Lemma 2.30.*

For any non-negative test function with compact support $\zeta \in \mathcal{C}_0^1\left(\mathbb{R}^2\right)$, the following equalities hold:

$$\int\limits_{\Omega_+} \left(\sigma\left[z\right] z \partial_t \zeta - \partial_x z \partial_x \zeta + \eta\left[z\right] \zeta\right) = \int\limits_{\partial\Omega_+} \partial_x z \zeta,$$

$$\int\limits_{\Omega_-} \left(\sigma\left[z\right] z \partial_t \zeta - \partial_x z \partial_x \zeta + \eta\left[z\right] \zeta\right) = \int\limits_{\partial\Omega_-} \partial_x z \zeta.$$

Proof. We prove the equality concerning Ω_+, the other one being similar.

First, it is straightforward that:

$$\left(\sigma\left[z\right]\right)_{|\Omega_+} = 1.$$

Let $\varepsilon > 0$ and:

$$\Omega_+^\varepsilon = \left\{(t,x) \in \mathbb{R}^2 \mid \Xi\left(t\right) - \varepsilon \leq x < \Xi\left(t\right)\right\}.$$

Then:

$$\int\limits_{\Omega_+} \left(\sigma\left[z\right] z \partial_t \zeta - \partial_x z \partial_x \zeta\right) = \int\limits_{\Omega_+^\varepsilon} \left(z \partial_t \zeta - \partial_x z \partial_x \zeta\right) + \int\limits_{\Omega_+ \backslash \Omega_+^\varepsilon} \left(z \partial_t \zeta - \partial_x z \partial_x \zeta\right)$$

Let

$$\tau_\varepsilon : x \mapsto \inf\left\{t \in \mathbb{R} \mid \Xi\left(t\right) = x + \varepsilon\right\}.$$

This function is increasing, piecewise-continuous, measurable and satisfies the following equality:

$$\mathbf{1}_{\Omega_+ \backslash \Omega_+^\varepsilon} = \mathbf{1}_{\left\{(t,x) \in \mathbb{R}^2 \mid \tau_\varepsilon(x) \leq t\right\}}.$$

By integration by parts and using the equation satisfied by z in $\Omega_+ \backslash \Omega_+^\varepsilon$:

$$\begin{aligned}\int\limits_{\Omega_+ \backslash \Omega_+^\varepsilon} \left(z \partial_t \zeta - \partial_x z \partial_x \zeta\right) = &- \int\limits_{\Omega_+ \backslash \Omega_+^\varepsilon} \eta\left[z\right] \zeta \\ &- \int\limits_{\mathbb{R}} \partial_x z\left(t, \Xi\left(t\right) - \varepsilon\right) \zeta\left(t, \Xi\left(t\right) - \varepsilon\right) \mathrm{d}t \\ &- \int\limits_{\mathbb{R}} z\left(\tau_\varepsilon\left(x\right), x\right) \zeta\left(\tau_\varepsilon\left(x\right), x\right) \mathrm{d}x.\end{aligned}$$

By the Cauchy–Schwarz inequality and dominated convergence, as $\varepsilon \to 0$:

$$\int_{\Omega_+^\varepsilon} (z\partial_t\zeta - \partial_x z\partial_x\zeta) \to 0,$$

$$\int_{\Omega_+ \backslash \Omega_+^\varepsilon} \eta[z]\zeta \to \int_{\Omega_+} \eta[z]\zeta,$$

$$\int_{\mathbb{R}} z(\tau_\varepsilon(x), x)\,\zeta(\tau_\varepsilon(x), x)\,\mathrm{d}x \to 0.$$

Therefore, the following convergence holds as $\varepsilon \to 0$:

$$-\int_{\mathbb{R}} \partial_x z(t, \Xi(t) - \varepsilon)\,\zeta(t, \Xi(t) - \varepsilon)\,\mathrm{d}t \to \int_{\Omega_+} (\sigma[z]z\partial_t\zeta - \partial_x z\partial_x\zeta) + \int_{\Omega_-} \eta[z]\zeta.$$

Lemma 2.33 indicates that the trace of $\partial_x z\zeta$ at $\partial\Omega_+$ is well-defined in L^2. Therefore, it remains to show that:

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \partial_x z(t, \Xi(t) - \varepsilon)\,\zeta(t, \Xi(t) - \varepsilon)\,\mathrm{d}t = -\int_{\partial\Omega_+} \partial_x z\zeta$$

Define, for any $\varepsilon > 0$:

$$z_\varepsilon : (t, x) \mapsto z(t, x - \varepsilon),$$

$$\zeta_\varepsilon : (t, x) \mapsto \zeta(t, x - \varepsilon).$$

It is clear that the trace of $\partial_x z_\varepsilon\zeta_\varepsilon$ is well-defined in L^2 as well and satisfies:

$$\int_{\mathbb{R}} \partial_x z(t, \Xi(t) - \varepsilon)\,\zeta(t, \Xi(t) - \varepsilon)\,\mathrm{d}t = \int_{\partial\Omega_+} (-1)\,\partial_x z_\varepsilon\zeta_\varepsilon.$$

Now, by virtue of the trace's theorem, there exists a constant $R > 0$ such that:

$$\|\partial_x z_\varepsilon\zeta_\varepsilon - \partial_x z\zeta\|_{L^2(\partial\Omega_+)} \le R\|\partial_x z_\varepsilon\zeta_\varepsilon - \partial_x z\zeta\|_{H^1(\Omega_+)}.$$

Integrating by parts and using the continuity of z and $\partial_x\zeta$, it is easily deduced that the right-hand side converges to 0 as $\varepsilon \to 0$. Hence the claimed result follows. □

We can now prove that Ξ is bijective and that a free boundary condition is satisfied in a weak sense.

Lemma 2.35. *Let Ξ be defined as in Lemma 2.30.*

Then Ξ is bijective and the functions:

$$z_{x,-} : t \mapsto (\partial_x z)_{|\partial\Omega_-} (t, \Xi (t)),$$

$$z_{x,+} : t \mapsto (\partial_x z)_{|\partial\Omega_+} (t, \Xi (t)),$$

where $(\partial_x z)_{|\partial\Omega_\pm}$ are the traces of $\partial_x z$ at each side of Γ, are in $L^2_{loc}(\mathbb{R})$ and are equal a.e..

Furthermore, if $s > 0$, $z_{x,-} \ll 0$, and if $s < 0$, $z_{x,+} \ll 0$.

Proof. Assume for instance $s > 0$, the other case being similar.

First, we prove the a.e. equality of $z_{x,+}$ and $z_{x,-}$, as well as the sign of $z_{x,-}$.

Let $\zeta \in \mathcal{C}^1_0(\mathbb{R}^2)$ be any non-negative test function and let $\zeta_\Gamma : t \mapsto \zeta(t, \Xi(t))$. By Lemma 2.34:

$$\int_{\partial\Omega_+} \partial_x z \zeta + \int_{\partial\Omega_-} \partial_x z \zeta = 0$$

where the unit vector normal to $\partial\Omega_+$ is the opposite of the one normal to $\partial\Omega_-$, whence we obtain:

$$\int_{\mathbb{R}} z_{x,+}\zeta_\Gamma = \int_{\mathbb{R}} z_{x,-}\zeta_\Gamma.$$

That is, for a.e. t, $z_{x,+}(t) = z_{x,-}(t)$, or, in other words, for a.e. $t \in \mathbb{R}$, $x \mapsto \partial_x z(t, x)$ is continuous. The sign of $z_{x,-}(t)$ follows directly from Hopf's lemma applied at the vertex $(t, \Xi(t))$ of the smooth parabolic cylinder $(t-1, t) \times (\Xi(t), \Xi(t)+1)$.

Then, it is clear that a continuous unbounded real-valued function is necessarily surjective, whence Ξ is bijective if and only if it is injective (or equivalently if and only if it is strictly monotonic). We are going to prove directly that Ξ is injective.

Differentiating (firstly in the distributional sense) the equation satisfied by z with respect to t in $\mathbb{R}^2 \backslash \Gamma$ yields the following regular and linear parabolic equations:

$$\begin{cases} \partial_t(\partial_t z) - \partial_{xx}(\partial_t z) - \alpha g_1\left[\frac{z}{\alpha}\right]\partial_t z = 0 & \text{in } \Omega_+ \\ \partial_t(\partial_t z) - d\partial_{xx}(\partial_t z) + d g_2\left[-\frac{z}{d}\right]\partial_t z = 0 & \text{in } \Omega_-. \end{cases}$$

Let $x \in \mathbb{R}$. Assume that $\Xi^{-1}(\{x\})$ is not a singleton. By (large) monotonicity, it is then a segment, say $[t_1, t_2]$. Applying classical parabolic regularity on this system of equations in $(t_1, t_2) \times (x, x+1)$ shows that $\partial_t z$ is $\mathcal{C}^1$ with respect to t and $\mathcal{C}^2$ with respect to x up to $(t_1, t_2) \times \{x\}$. Moreover, $\partial_t z = 0$ along $(t_1, t_2) \times \{x\}$. By classical parabolic regularity and Hopf's lemma, for any $t \in (t_1, t_2)$, the right-sided and the left-sided limit of $\partial_x \partial_t z(t, y)$ as $y \to x$ exists and have opposite sign.

Remark that, away from Γ, the equations satisfied by z, $\partial_t z$ and $\partial_x z$ suffice to show that $z \in \mathcal{C}^2(\Omega_+) \cap \mathcal{C}^2(\Omega_-)$. Therefore Schwarz' theorem can be applied away from Γ.

Thus, for any $t, t' \in (t_1, t_2)$ and some $\varepsilon > 0$ small enough, we get:

$$\partial_x z (t, x \pm \varepsilon) - \partial_x z \left(t', x \pm \varepsilon\right) = \int_{t'}^{t} \partial_t \partial_x z (\tau, x \pm \varepsilon) \, \mathrm{d}\tau$$

$$= \int_{t'}^{t} \partial_x \partial_t z (\tau, x \pm \varepsilon) \, \mathrm{d}\tau.$$

These two integrals have an opposite strict sign: with respect to t, $\partial_x z$ is decreasing on one side of $(t_1, t_2) \times \{x\}$ and increasing on the other. This contradicts the fact that, for a.e. $t \in \mathbb{R}$, $x \mapsto \partial_x z (t, x)$ is continuous (see the first step of the proof). Therefore for any $x \in \mathbb{R}$, $\mathbb{R} \times \{x\} \cap \Gamma$ is a singleton, whence Ξ is bijective. $\square$

Corollary 2.36. *The function $x \mapsto x - s\Xi^{-1}(x)$ is continuous and periodic. Furthermore,*

$$\left\{\left(x - s\Xi^{-1}(x), x\right) \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\} = \varphi^{-1}(\{0\}).$$

Proof. The periodicity comes from the periodicity with respect to x of φ. $\square$

Remark. This corollary confirms that, roughly speaking, the free boundary is located near the straight line of equation $x = st + \Xi(0)$. In other words, Ξ can be represented as the sum of $t \mapsto st$ and a $\frac{L}{s}$-periodic function Ξ_{per}.

Corollary 2.37. *The monotonicity of z with respect to t is strict. Equivalently, φ is decreasing with respect to ξ.*

Proof. Just apply the strong maximum principle to the equations satisfied by $\partial_t z$ in each component of $\mathbb{R}^2 \backslash \Gamma$ to get that, in $\mathbb{R}^2 \backslash \Gamma$, $\partial_t z \gg 0$ if $s > 0$ and $\partial_t z \ll 0$ if $s < 0$, which is sufficient to obtain strict monotonicity since the measure of Γ (as a measurable subset of $\mathbb{R}^2$) is zero. $\square$

Now, thanks to a technique developed by Aronson for the porous media equation [2], we are able to prove the continuity of $\partial_x z$.

Lemma 2.38. *Let Ξ be defined as in Lemma 2.30 and $z_{x,+}$ and $z_{x,-}$ be defined as in Lemma 2.35. If $s > 0$ (respectively $s < 0$), $z_{x,+}(t)$ (resp. $z_{x,-}(t)$) is actually defined for any $t \in \mathbb{R}$. Moreover, the function $z_{x,+}$ (resp. $z_{x,-}$) is non-positive and locally uniformly bounded from below.*

Proof. We only prove the result in the case $s > 0$, the other one being symmetric.

Let $t \in \mathbb{R}$ and $x, x' \in \mathbb{R}$ such that $x < x' < \Xi(t)$. For any $\tilde{x} \in \left(x, x'\right)$,

$$\partial_{xx} z (t, \tilde{x}) = \partial_t z (t, \tilde{x}) - z (t, \tilde{x}) f_1 (z (t, \tilde{x}), \tilde{x}).$$

On one hand, the term $z(t, \tilde{x}) f_1 (z (t, \tilde{x}), \tilde{x})$ is bounded from below by 0 and from above by a constant R independent on $\tilde{x}$. On the other hand, $\partial_t z (t, \tilde{x}) > 0$. Thus:

$$\partial_{xx} z\left(t, \tilde{x}\right) \geq -R.$$

Integrating this inequality, we obtain:

$$\partial_x z\left(t, x'\right) \geq \partial_x z\left(t, x\right) - R\left(x' - x\right).$$

It follows that:

$$\liminf_{x' \to \Xi(t)} \partial_x z\left(t, x'\right) \geq \partial_x z\left(t, x\right) - R\left(\Xi\left(t\right) - x\right),$$

and then:

$$\liminf_{x' \to \Xi(t)} \partial_x z\left(t, x'\right) \geq \limsup_{x \to \Xi(t)} \partial_x z\left(t, x\right).$$

Hence:

$$\lim_{x \to 0, x > 0} \partial_x z\left(t, \Xi\left(t\right) - x\right)$$

exists. From the sign of z in Ω_+, it is clear that it is non-positive. Using once more the inequality:

$$\liminf_{x' \to \Xi(t)} \partial_x z\left(t, x'\right) \geq \partial_x z\left(t, x\right) - R\left(\Xi\left(t\right) - x\right)$$

together with the local boundedness of $\partial_x z$ in Ω_+, it follows that the limit is locally uniformly bounded from below. Finally, it necessarily coincides with $z_{x,+}\left(t\right)$. □

Corollary 2.39. $\partial_x z \in L^\infty\left(\mathbb{R}^2\right)$.

Lemma 2.40. *We have* $\partial_x z \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$.

Proof. Let $\zeta \in \mathcal{C}^2_0\left(\mathbb{R}^2\right)$. Choosing as test functions in the weak formulation in L^2_{loc} of:

$$\sigma\left[z\right]\partial_t z - \partial_{xx} z = \eta\left[z\right]$$

a sequence of smooth functions converging in $L^2_{loc}\left(\mathbb{R}^2\right)$ to $\hat{\sigma}\left[z\right]\partial_x \zeta$, we obtain:

$$\int \partial_t z \partial_x \zeta - \int \hat{\sigma}\left[z\right]\partial_{xx} z \partial_x \zeta = \int \hat{\sigma}\left[z\right]\eta\left[z\right]\partial_x \zeta.$$

Remarking the following equalities:

$$\begin{aligned} \int \partial_t z \partial_x \zeta &= -\int z \partial_t\left(\partial_x \zeta\right) \\ &= -\int z \partial_x\left(\partial_t \zeta\right) \\ &= \int \partial_x z \partial_t \zeta, \end{aligned}$$

$$\int \hat{\sigma}[z]\eta[z]\partial_x\zeta = -\int \partial_x\left(\hat{\sigma}[z]\eta[z]\right)\partial_x\zeta$$
$$= -\int \hat{\sigma}[z]\partial_x(\eta[z])\partial_x\zeta,$$

(where, by virtue of $(\mathcal{H}_1)$, $\partial_x(\eta[z])$ is piecewise-continuous and *a fortiori* is in $L^\infty\left(\mathbb{R}^2\right)$), we deduce:

$$-\int \partial_x z\partial_t\zeta + \int \hat{\sigma}[z]\partial_{xx}z\partial_x\zeta = \int \hat{\sigma}[z]\partial_x(\eta[z])\zeta.$$

Hence we can once more apply DiBenedetto's theory [13]: $\partial_x z$, which is both in $L^\infty\left(\mathbb{R}^2\right)$ and in $\mathcal{C}_{loc}\left(\mathbb{R}, L^2_{loc}(\mathbb{R})\right)$ (by classical parabolic estimates similar to those detailed previously in the proof of Proposition 2.4), is a locally bounded weak solution of:

$$\partial_t Z - \partial_x\left(\hat{\sigma}[z]\partial_x Z\right) = \hat{\sigma}[z]\partial_x(\eta[z])$$

and therefore is locally Hölder-continuous indeed. □

Remark. Let us explain here why $\partial_t z$ is very likely to be continuous as well (equivalently, Ξ is very likely to be continuously differentiable). There are in fact some articles related to this free boundary problem and although none of them is exactly what we need here, they strongly lead to this conjecture (let us cite for instance Evans [19], Cannon–Yin [7] and Jensen [26]).

Roughly speaking, the idea would be to regularize $(\mathcal{SPF}[s])$, to show the uniqueness of the weak solution of the problem written in divergence form, to prove thanks to the maximum principle that the regularization of $\| (\partial_t z)(\partial_x z)^{-1} \|_{L^\infty}$ is bounded uniformly with respect to the regularization, to obtain consequently that Ξ is Lipschitz-continuous, and then to deduce from Caffarelli's classical results about one–phase Stefan problems [6] that $\Xi \in \mathcal{C}^1(\mathbb{R})$, whence finally $\partial_t z \in \mathcal{C}\left(\mathbb{R}^2\right)$.

Since we do not need such results to conclude this study about pulsating fronts, we choose not to investigate further in this direction. Nevertheless, the rigorous proof of the continuity of $\partial_t z$ in the more general framework of weak solutions of $(\mathcal{SPF}[s])$ might be the object of a future follow-up to this article.

Let us conclude this subsection with the following corollary, which takes into account the previous remark and gives an interesting formula.

Corollary 2.41. *If $d = 1$, then $\partial_t z, \partial_{xx} z \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$ and $\Xi \in \mathcal{C}^1(\mathbb{R})$.*

If $d \neq 1$ and if $\partial_t z \in L^\infty\left(\mathbb{R}^2\right)$, then $\Xi \in \mathcal{C}^1(\mathbb{R})$, $\partial_t z \in \mathcal{C}\left(\mathbb{R}^2\right)$, $\hat{\sigma}[z]\partial_{xx}z \in \mathcal{C}\left(\mathbb{R}^2\right)$ and the following equality holds for any $t \in \mathbb{R}$:

$$\Xi'(t) = \frac{d}{1-d}\frac{\lim_{\varepsilon\to 0,\varepsilon>0}\left(\partial_{xx}z(t,\Xi(t)-\varepsilon) - \partial_{xx}z(t,\Xi(t)+\varepsilon)\right)}{\partial_x z(t,\Xi(t))}.$$

Proof. Regularity in the symmetrical case $d = 1$ follows from classical parabolic regularity.

Provided $d \neq 1$ and global boundedness of $\partial_t z$, let $\varepsilon > 0$ small enough so that the implicit function theorem can be applied at the level set $z^{-1}(\{\pm\varepsilon\})$. There exists $\Xi_{\pm\varepsilon} \in \mathcal{C}^1(\mathbb{R})$ such that $\Xi_{+\varepsilon} \ll \Xi \ll \Xi_{-\varepsilon}$ and such that:

$$\Xi'_{\pm\varepsilon}(t) = -\frac{\partial_t z(t, \Xi_{\pm\varepsilon}(t))}{\partial_x z(t, \Xi_{\pm\varepsilon}(t))}.$$

Passing to the limit $\varepsilon \to 0$, we deduce that Ξ is Lipschitz-continuous. Then, by Caffarelli [6], $\partial_t z, \partial_{xx} z \in \mathcal{C}\left(\overline{\Omega_+}\right) \cap \mathcal{C}\left(\overline{\Omega_-}\right)$ and $\Xi \in \mathcal{C}^1(\mathbb{R})$. Thus $\Xi_{\pm\varepsilon} \to \Xi$ in $\mathcal{C}^1_{loc}(\mathbb{R})$ as $\varepsilon \to 0$, whence $\partial_t z$ is moreover continuous at Γ. Then, since $\hat{\sigma}[z]\,\partial_{xx} z = \partial_t z - \hat{\sigma}[z]\,\eta[z]$, $\hat{\sigma}[z]\,\partial_{xx} z$ is continuous in $\mathbb{R}^2$ as well. Finally, the formula relating Ξ' to the jump discontinuity of $\partial_{xx} z$ is easily obtained:

$$\begin{aligned}
&\lim_{\varepsilon\to 0, \varepsilon>0} \left(\partial_{xx} z(t, \Xi(t) - \varepsilon) - \partial_{xx} z(t, \Xi(t) + \varepsilon)\right) \\
&= \lim_{\varepsilon\to 0, \varepsilon>0} \left(\partial_t z(t, \Xi(t) - \varepsilon) - \frac{1}{d}\partial_t z(t, \Xi(t) + \varepsilon)\right) \\
&\quad + \lim_{\varepsilon\to 0, \varepsilon>0} \left(\eta[z](t, \Xi(t) - \varepsilon) - \eta[z](t, \Xi(t) + \varepsilon)\right) \\
&= \left(1 - \frac{1}{d}\right)\partial_t z(t, \Xi(t)) \\
&= -\left(1 - \frac{1}{d}\right)\Xi'(t)\,\partial_x z(t, \Xi(t)). \quad \square
\end{aligned}$$

2.4.3. Uniqueness

We are now able to end our characterization.

Theorem 2.42. *Let z_1 and z_2 be segregated pulsating fronts with respective speeds $s_1 \neq 0$ and $s_2 \neq 0$ and respective profiles φ_1 and φ_2.*

Then $s_1 = s_2$ and there exists $\tau \in \mathbb{R}$ such that $\varphi_1^\tau = \varphi_2$, where $\varphi_1^\tau : (\xi, x) \mapsto \varphi_1(\xi - \tau, x)$.

In other words, the speed is unique and the profile is unique up to translation with respect to ξ.

Proof. We are going to use once more the sliding method. Remark that, up to the free boundary, this is the most simple case: bistable scalar equation. Therefore we refer to the proof of Lemma 2.2 for the details and only point out here some technical differences due to the presence of the free boundary.

Step 1: existence of a translation of the profile associated with the highest speed such that it is locally below the other profile.

Here it is useful to additionally require that, at ζ, the upper profile is positive (uniformly with respect to x) whereas the lower profile is negative (uniformly as well). This will simplify some arguments in Steps 2, 3, 4 and 5 since it is now clear that the contact points $(\xi^\star, x^\star)$ are necessarily located away from the free boundary, whence the arguments of the usual sliding method for regular pulsating fronts (Berestycki–Hamel [3]) apply straightforwardly.

Step 2: up to some extra term, this ordering is global on the left.

No new idea here: multiply the upper profile by some $\kappa \geq 1$.

Step 3: this extra term is actually unnecessary, thanks to the maximum principle.

Similarly, there is no new idea here as well and it follows easily that $\kappa^\star = 1$.

Step 4: up to some (possibly different) extra term, this ordering is global on the right.

Thanks to the underlying symmetry due to the bistable structure, the proof of this step is much simpler here: just change every profile into its opposite and repeat straightforwardly Step 2.

Step 5: this (possibly different) extra term is also unnecessary.

Similarly, repeat Step 3 to prove that $\kappa^\star = 1$.

Step 6: thanks to the maximum principle again, the speeds are equal and the profiles are equal up to some translation.

This is the step which requires additional care because of the free boundary. To this end, let us introduce some notations.

We assume that $s_1 \leq s_2$. Let:

$$v_2 : (t,x) \mapsto \varphi_2 (x - s_1 t, x),$$
$$v_1^{\tau^\star} : (t,x) \mapsto \varphi_1 \left(x - s_1 t - \tau^\star, x\right),$$
$$v = v_2 - v_1^{\tau^\star},$$

where $\tau^\star$ is defined as in Lemma 2.2.

At this step of the proof, it is established that $v \geq 0$. Let $\mathcal{Z} = v^{-1}(\{0\})$. With the same argument as in Lemma 2.2, we can discard the possibility $\mathcal{Z} = \emptyset$. Now there are basically three cases.

(1) There exists $(t^\star, x^\star) \in \mathcal{Z}$ such that $v_2(t^\star, x^\star) > 0$. Then by virtue of the usual parabolic strong maximum principle, $\left(v_1^{\tau^\star}\right)^+ = (v_2)^+$ in some parabolic cylinder whose final time is $t^\star$ and whose spatial center is $x^\star$. Thus v is identically null in this cylinder, whence by strict monotonicity (see Corollary 2.37) of φ_2 with respect to ξ, $s_1 = s_2$, $v_2 = z_2$ in this cylinder, and then by periodicity of $\varphi_1 - \varphi_2$ with respect to x, $\left(v_1^{\tau^\star}\right)^+ = v_2^+$ in $\mathbb{R}^2$ and their free boundaries (i.e. zero sets) coincide. Thus there exists a unique bijection Ξ such that this free boundary is the graph of Ξ. By continuity of $\partial_x v_1^{\tau^\star}$ and $\partial_x v_2$ (see Proposition 2.26), $\partial_x v = 0$ on the other side of the free boundary, whence by virtue of Hopf's lemma the equality $v_1^{\tau^\star} = v_2$ extends everywhere.

(2) There exists $(t^\star, x^\star) \in \mathcal{Z}$ such that $v_2(t^\star, x^\star) < 0$. Then, by the exact same argument (this is once more due to the underlying symmetry), $v_1^{\tau^\star} = v_2$ in $\mathbb{R}^2$.

(3) Every $(t^\star, x^\star) \in \mathcal{Z}$ is such that $v_1^{\tau^\star}(t^\star, x^\star) = v_2(t^\star, x^\star) = 0$. Thanks to Hopf's lemma again, this case is actually contradictory. On one hand, since $\partial_x v \in \mathcal{C}\left(\mathbb{R}^2\right)$ and v is non-negative non-zero in $\mathbb{R}^2$, for any $(t^\star, x^\star) \in \mathcal{Z}$, $\partial_x v(t^\star, x^\star) = 0$. On the other hand, although the free boundaries of $v_1^{\tau^\star}$ and v_2 are here *a priori* distinct, we can still apply Hopf's lemma at $(t^\star, x^\star)$ in a suitable parabolic cylinder and get a strict sign for $\partial_x v(t^\star, x^\star)$. $\square$

Remark. At this point, it would be tempting to notice that this kind of proof can be easily generalized if one of the two speeds is zero (in this case, the argument is usually referred to as a "quenching" or "blocking" argument) and then to use it to show that a segregated stationary equilibrium cannot coexist with a segregated pulsating front. Unfortunately, this is not possible. A segregated stationary equilibrium is *a priori* a much more general notion than what could be defined as a "segregated pulsating front with null speed" (the basic reason being that, when $c_\infty = 0$, the change of variables $(t, x) \mapsto (x - c_\infty t, x)$ is not an isomorphism anymore).

Nevertheless, it is still possible to use some kind of more elaborated quenching argument, as shows the following theorem.

Theorem 2.43. *If there exists a segregated pulsating front, there does not exist a segregated stationary equilibrium.*

Proof. Assume that there exist both a segregated pulsating front z with speed $s \neq 0$ and profile φ and a segregated stationary equilibrium e.

Assume for instance that $s > 0$ and that e has a smallest zero:

$$x_1 = \min e^{-1}(\{0\}) \in \mathbb{R}.$$

As in the usual sliding method, we construct (and do not detail these constructions) $\tau \in \mathbb{R}$ and $\kappa > 1$ such that:

$$(\xi, x) \mapsto \kappa e(\xi) - \varphi(\xi - \tau, x)$$

is positive everywhere in $(-\infty, x_1) \times \mathbb{R}$, with a fixed gap at $\{x_1\} \times \mathbb{R}$ (constructing for instance τ such that $\max\limits_{x \in \overline{C}} \varphi(x_1 - \tau, x) = -\frac{da_2}{2}$). Then we define $\kappa^\star$ as the infimum of these κ, we assume by contradiction that $\kappa^\star > 1$ and we construct consequently the first contact point $(\xi^\star, x^\star)$ with $\xi^\star < x_1$. By virtue of Proposition 2.14, $\xi^\star > -\infty$. Let $t^\star = \frac{x^\star - \xi^\star}{s}$.

Notice that there exists a neighborhood of $(\xi^\star, x^\star)$ such that $\varphi \gg 0$ in this neighborhood. Consequently, there exists $\varepsilon > 0$ such that both functions:

$$x \mapsto \varphi\left(x - st^\star - \tau, x\right),$$
$$v_{\tau,\kappa^\star} : x \mapsto \kappa^\star e\left(x + \xi^\star - x^\star\right) - \varphi\left(x - st^\star - \tau, x\right),$$

are non-negative non-zero everywhere in $\left[x^\star - \varepsilon, x^\star + \varepsilon\right]$. Moreover, $v_{\tau,\kappa^\star}(x^\star) = 0$. Thanks to the inequality:

$$\kappa\eta[e] \geq \kappa\eta[\kappa e] \text{ in } \left(x^\star - \varepsilon, x^\star + \varepsilon\right),$$

we get:

$$-\kappa e''\left(x + \xi^\star - x^\star\right) \geq \kappa\eta\left(\kappa e\left(x + \xi^\star - x^\star\right), x + \xi^\star - x^\star\right) \text{ for any } x \in \left(x^\star - \varepsilon, x^\star + \varepsilon\right),$$

whence, since $\partial_t z > 0$, $v_{\tau,\kappa^\star}$ satisfies:

$$-v''_{\tau,\kappa^\star}(x) > q_{\kappa^\star}(x)\, v_{\tau,\kappa^\star}(x) \text{ for any } x \in \left(x^\star - \varepsilon, x^\star + \varepsilon\right),$$

where $q_{\kappa^\star} \in L^\infty(\mathbb{R})$ is defined as:

$$q_{\kappa^\star} : x \mapsto \begin{cases} \frac{\eta\left(\kappa^\star e\left(x+\xi^\star-x^\star\right), x+\xi^\star-x^\star\right) - \eta\left(\varphi\left(x-st^\star-\tau,x\right),x\right)}{v_{\tau,\kappa^\star}} & \text{if } v_{\tau,\kappa^\star}(x) \neq 0 \\ 1 & \text{if } v_{\tau,\kappa^\star}(x) = 0. \end{cases}$$

The function $v_{\tau,\kappa^\star}$ is a non-negative non-zero super-solution of some elliptic problem. Since the elliptic strong maximum principle contradicts the existence of $\xi^\star$, $\kappa^\star = 1$ indeed.

Repeating the argument near $\xi = +\infty$ with some $\kappa \leq 1$ then proves that (up to some increase of τ) $e(\xi) - \varphi(\xi - \tau, x) \gg 0$ actually holds in $\mathbb{R}^2$. Note that in this case, the proof is simpler, since the negativity of φ in $(\xi^\star, +\infty) \times \mathbb{R}$ follows from its normalization and monotonicity. We point out that, *a priori*, there are two cases, depending on the existence of $\max e^{-1}(\{0\})$. But in fact these two cases do not require different arguments.

Now, just as usual, we can define:

$$\tau^\star = \sup\left\{\tau \in \mathbb{R} \mid e(\xi) - \varphi(\xi - \tau, x) \geq 0 \text{ for any } (\xi, x) \in \mathbb{R}^2\right\}.$$

Assume by contradiction that:

$$\min_{[-B,B]\times\mathbb{R}} \left(e(\xi) - \varphi\left(\xi - \tau^\star, x\right)\right) > 0$$

for any $B > 0$ such that:

$$e(B) < 0,$$
$$\min_{x\in\mathbb{R}} \varphi\left(-B - \tau^\star, x\right) > 0.$$

By continuity, we then obtain for $\tau > \tau^*$ close enough,

$$\min_{[-B,B]\times\mathbb{R}} \left(e(\xi) - \varphi(\xi - \tau, x)\right) > 0,$$
$$\min_{x\in\mathbb{R}} \varphi(-B - \tau, x) > 0.$$

It follows from the same type of arguments as those presented at the beginning of this proof that:

$$e(\xi) - \varphi(\xi - \tau, x) \gg 0 \text{ in } (\mathbb{R}\backslash(-B, B)) \times \mathbb{R},$$

thus contradicting the maximality of τ^*.

Hence, there exists $B > 0$ such that:

$$\min_{[-B,B]\times\mathbb{R}} \left(e(\xi) - \varphi\left(\xi - \tau^\star, x\right)\right) = 0,$$

i.e. there exists $(\xi^\star, x^\star) \in [-B, B] \times \mathbb{R}$ such that:

$$e\left(\xi^\star\right) - \varphi\left(\xi^\star - \tau^\star, x^\star\right) = 0.$$

Let:

$$t^\star = \frac{x^\star - \xi^\star}{s},$$
$$v : (t, x) \mapsto e\left(x + \xi^\star - x^\star\right) - \varphi\left(x - st - \tau^\star, x\right)$$

and notice that:

$$v(t,x) > 0 \text{ for any } (t,x) \in [t^\star - 1, t^\star) \times \mathbb{R},$$
$$v\left(t^\star, x^\star\right) = 0.$$

Now, we need to distinguish two cases, as in the proof of Theorem 2.42:

- if $\xi^\star \notin e^{-1}(\{0\})$, using the continuity of v and the strong parabolic maximum principle in some parabolic cylinder $\left[t^\star - \varepsilon, t^\star\right] \times \left[x^\star - \varepsilon, x^\star + \varepsilon\right]$ (with a small enough ε so that the signs of $e(x + \xi^\star - x^\star)$ and of $\varphi(x - st - \tau^\star, x)$ do not change in this cylinder), we get a contradiction;
- if $x^\star \in e^{-1}(\{0\})$, using the continuity of e' and $\partial_x z$ and Hopf's lemma at the vertex $(t^\star, x^\star)$ of the parabolic cylinder $\left[t^\star - 1, t^\star\right] \times \left[x^\star, x^\star + 1\right]$, we get a contradiction as well.

The pair (z, e) cannot exist.

If $s < 0$, we change $v_{\tau,\kappa^\star}$ into $-v_{\tau,\kappa^\star}$ so that $\partial_t z < 0$ yields a negative sub-solution and we deduce similarly $e(\xi) - \varphi(\xi - \tau, x) \gg 0$. The end of the proof is carried on similarly.

If $\min e^{-1}(\{0\})$ does not exist, then $\max e^{-1}(\{0\})$ does: it suffices to change the roles of e and φ, in the sense that now we have to show that $\varphi(\xi - \tau, x) - e(\xi) \gg 0$. Near $\xi = -\infty$, the studied quantity is $\kappa e - \varphi$ with $\kappa \leq 1$, and near $\xi = +\infty$, the studied quantity is $\kappa e - \varphi$ with $\kappa \geq 1$. Once $\kappa^\star = 1$ is established, the end of the proof is exactly the same. □

Remark. The preceding proof only works in the case of constant a_1 and a_2. In the case of non-constant extinction states, this type of quenching argument does not hold anymore because Proposition 2.14 is not true anymore and therefore we cannot prove that $\xi^\star < -\infty$ when trying to prove that $\kappa^\star = 1$. We do not know how to prove the theorem in such a case and we stress that this is really unsatisfying. Still, we think it is natural to make the following conjecture.

Conjecture 2.44. *Theorem 2.43 still holds true in the non-constant case.*

2.5. Uniqueness of the asymptotic speed

From now on, $(c_k)_{k>k^\star}$ refers to the general family indexed on $(k^\star, +\infty)$ instead of an *a priori* extracted convergent sequence. In the following, we will prove that $(c_k)_{k>k^\star}$ converges indeed to c_∞ as $k \to +\infty$.

Definition 2.45. We say that $s \in \mathbb{R}$ *satisfies Property* $(\mathcal{E}(d, \alpha, f_1, f_2))$ if one of the following holds:

- $s = 0$ and there exists a segregated stationary equilibrium;
- $s \neq 0$ and there exists a segregated pulsating front with speed s.

The set of all $s \in \mathbb{R}$ satisfying Property $(\mathcal{E}(d, \alpha, f_1, f_2))$ is referred to as $\Sigma_{(d,\alpha,f_1,f_2)}$.

Remark. This set does not depend at all on $k^\star$.

Following Theorems 2.42 and 2.43, we deduce the following uniqueness result.

Corollary 2.46. *There is at most one $s \in \mathbb{R}$ satisfying Property* $(\mathcal{E}(d, \alpha, f_1, f_2))$.

To conclude about the convergence of the speeds, it suffices to recall that c_∞ satisfies of course Property $(\mathcal{E}(d, \alpha, f_1, f_2))$.

Proposition 2.47. *The limit at $+\infty$ of the function $k \mapsto c_k$ is well-defined.*

Remark. If a_1 and a_2 are non-constant, as explained before, the quenching argument cannot be used and we do not have the uniqueness in $\mathbb{R}$ of the elements satisfying Property $(\mathcal{E}(d, \alpha, f_1, f_2))$. Still, we have the uniqueness in $\mathbb{R}\setminus\{0\}$, whence in particular the countability of the limit points of $k \mapsto c_k$ as $k \to +\infty$. Therefore, using the intermediate value theorem, we can still prove that the limit of the continuous function $k \mapsto c_k$ as $k \to +\infty$ is well-defined. In other words, the convergence of (c_k) can be proved even without proving Conjecture 2.44.

2.6. Conclusion of this section

The function $k \mapsto c_k$ converges at $+\infty$.

If its limit c_∞ is non-zero, then both families $\left((u_{1,k}, u_{2,k})\right)_{k>k^\star}$ and $\left((\varphi_{1,k}, \varphi_{2,k})\right)_{k>k^\star}$ have a unique limit point (which are respectively the segregated pulsating front w traveling with speed c_∞ and its profile ϕ), and therefore the functions $k \mapsto \left(\varphi_{1,k}, \varphi_{2,k}\right)$ and $k \mapsto \left(u_{1,k}, u_{2,k}\right)$ converge as well as $k \to +\infty$.

If $c_\infty = 0$, then $\left((u_{1,k}, u_{2,k})\right)_{k>k^\star}$ might have multiple limit points, each one of them being a segregated stationary equilibrium.

3. Sign of the asymptotic speed depending on the parameters

In this final section, we investigate the sign of c_∞ as a function of (d, α), which is consequently not considered as fixed anymore ($L > 0$ and (f_1, f_2) are still fixed nevertheless).

We assume the existence of $D_{exis} \geq 0$ such that, for any $d > D_{exis}$ and any $\alpha > 0$, $(\mathcal{H}_{exis})$ is satisfied.

Once $(d, \alpha) \in (D_{exis}, +\infty) \times (0, +\infty)$ is given, c_∞ is naturally defined. If $c_\infty \neq 0$, ϕ and w are well-defined as well.

Remark. These assumptions are natural in view of the existence result under the hypothesis $\left(\mathcal{H}_{freq}\right)$ exhibited by the first author [23]. Indeed, if $\left(\mathcal{H}_{freq}\right)$ is assumed, then it implies $(\mathcal{H}_{exis})$ and the existence of an explicit D_{exis}:

$$D_{exis} = \begin{cases} M_2\left(\frac{L}{\pi} - \frac{1}{\sqrt{M_1}}\right)^2 & \text{if } L\sqrt{M_1} > \pi \\ 0 & \text{if } L\sqrt{M_1} \leq \pi. \end{cases}$$

3.1. Necessary and sufficient conditions on the parameters for the asymptotic speed to be zero

Here the idea is to follow what we did in the space-homogeneous case [24] to deduce a free boundary condition satisfied by any segregated stationary equilibrium. To this end, we need the following result, which shares some similarities with Proposition 4.1 of Du–Lin [17,18] but is, on one hand, restricted to the null speeds and, on the other hand, extended to the space-periodic non-linearities.

Proposition 3.1. *Let $x_0 \in \mathbb{R}$ and $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, periodic with respect to x and satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. The following problem:*

$$\begin{cases} -z'' = zf[z] & \text{in } (x_0, +\infty) \\ z(x_0) = 0 \end{cases}$$

admits a unique non-negative non-zero solution $z_{x_0,f} \in \mathcal{C}^2([x_0, +\infty))$.

Furthermore, the function

$$\Theta : (x_0, f) \mapsto z'_{x_0,f}(x_0)$$

(that is the right-sided derivative of $z_{x_0,f}$ at x_0) satisfies:

(1) $\Theta \gg 0$;
(2) Θ *is continuous with respect to the canonical topology of* $\mathbb{R} \times \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$;
(3) Θ *is periodic with respect to its first variable;*
(4) *for any* $\kappa > 0$,

$$\Theta\left(x_0, (z, x) \mapsto f\left(\frac{z}{\kappa}, x\right)\right) = \kappa\Theta(x_0, f).$$

Proof. Firstly, let us point out that Du–Lin's proposition [17,18] is readily extended to generic "KPP"-type non-linearities which do not depend on the spatial variable. We do not detail this extension here.

Thus, let $\overline{f} : z \mapsto \max_{y \in \overline{C}} f(z, y)$. It can be checked that $z \mapsto z\overline{f}[z]$ is indeed a KPP-type non-linearity (mostly, it reduces to the proof of the fact that $\overline{f}$ is decreasing and negative after some fixed value). Then, let $\overline{z}$ be the solution given by (the aforementioned extension of) Du–Lin's proposition of:

$$\begin{cases} -z''(x) = z\overline{f}[z] & \text{in } (x_0, +\infty) \\ z(x_0) = 0. \end{cases}$$

Similarly, let $\underline{f} : z \mapsto \min_{y \in \overline{C}} f(z, y)$ and $\underline{z}$ be the solution of:

$$\begin{cases} -z''(x) = z\underline{f}[z] & \text{in } (x_0, +\infty) \\ z(x_0) = 0. \end{cases}$$

We intend to prove that $\overline{z}$ and $\underline{z}$ form an ordered pair of super- and sub-solution for the problem at hand.

Let a be the positive constant given by $(\mathcal{H}_3)$ such that $f(a, x) = 0$ for all $x \in \overline{C}$. By standard elliptic estimates,

$$\lim_{+\infty} \overline{z} = \lim_{+\infty} \underline{z} = a.$$

By Du–Lin's proposition, we know that $\overline{z}'(x_0)$ and $\underline{z}'(x_0)$ (understood as right-sided derivatives) are finite, whence there exists $\kappa > 0$ such that:

$$\kappa\overline{z} - \underline{z} \geq 0 \text{ in } (x_0, +\infty).$$

Let:

$$\kappa^\star = \inf\left\{\kappa > 0 \mid \kappa\overline{z} - \underline{z} \gg 0 \text{ in } (x_0, +\infty)\right\}$$

and assume by contradiction that $\kappa^\star > 1$. We can fix a sequence $(\kappa_n)_{n\in\mathbb{N}} \in (1, \kappa^\star)^{\mathbb{N}}$ which converges to $\kappa^\star$ from below. There exists a sequence $(x_n)_{n\in\mathbb{N}} \in (x_0, +\infty)^{\mathbb{N}}$ such that:

$$\left(\kappa_n\overline{z} - \underline{z}\right)(x_n) < 0.$$

Since $\lim\limits_{+\infty}\left(\kappa_n\overline{z} - \underline{z}\right) = (\kappa_n - 1)\, a > 0$, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded and then convergent up to extraction.

If x_∞ is the limit of (x_n), then by continuity:

$$\kappa^\star\overline{z}\,(x_\infty) = \underline{z}\,(x_\infty).$$

Now, remarking that:

$$\kappa^\star\overline{z}\overline{f}\,[\overline{z}] \geq \kappa^\star\overline{z}\overline{f}\left[\kappa^\star\overline{z}\right]$$

by monotonicity of $\overline{f}$, it follows by Lipschitz-continuity of $\overline{f}$ that $\kappa^\star\overline{z} - \underline{z}$ is a positive super-solution of some linear elliptic problem which vanishes at x_∞. Provided $x_\infty \neq x_0$, this contradicts the elliptic strong minimum principle and the strict ordering at $+\infty$.

But if $x_\infty = x_0$, then Hopf's lemma implies that:

$$\left(\kappa^\star\overline{z} - \underline{z}\right)'(x_0) > 0.$$

From this inequality, the optimality of $\kappa^\star$ is easily contradicted.

Hence $\kappa^\star = 1$, that is $\overline{z}$ and $\underline{z}$ are indeed a pair of ordered super- and sub-solution of the problem. Since f depends on x (the special case of f constant with respect to x, that is Du–Lin's case, can be discarded here without loss of generality), they are not solutions themselves, whence their ordering is strict:

$$\underline{z} \ll \overline{z} \text{ in } (x_0, +\infty).$$

Finally, by virtue of classical existence–comparison results for semi-linear elliptic problems, there exists a solution of the problem $z_{x_0,f}$ satisfying furthermore:

$$\underline{z} \ll z_{x_0,f} \ll \overline{z}.$$

The uniqueness of $z_{x_0,f}$ follows from similar arguments.

The positivity of Θ easily follows from $z_{x_0,f} \gg \underline{z}$. Its continuity comes from the uniqueness of $z_{x_0,f}$ and classical compactness arguments. Its periodicity with respect to x comes from the uniqueness of $z_{x_0,f}$ and the periodicity of f with respect to x. The last property comes from the following easy fact. Let $\kappa > 0$ and $Z = \kappa z_{x_0,f}$. It is easily verified that:

$$-Z'' = Zf\left[\frac{Z}{\kappa}\right] \text{ in } (x_0, +\infty)$$

and then by uniqueness $Z = z_{x_0, f_\kappa}$ where $f_\kappa : (z, x) \mapsto f\left(\frac{z}{\kappa}, x\right)$. $\square$

Before going any further, we recall that it suffices to choose different normalization sequences to deduce that, if $c_\infty = 0$, there exists at least one segregated stationary equilibrium e_1 satisfying:

$$\inf e_1^{-1}((-\infty, 0)) > -\infty$$

and at least one segregated stationary equilibrium e_2 satisfying:

$$\sup e_2^{-1}((0, +\infty)) < +\infty.$$

If $c_\infty = 0$, we define consequently $x_1 = \min e_1^{-1}(\{0\})$ and $x_2 = \max e_2^{-1}(\{0\})$. Recall that, without loss of generality, we can assume that $(x_1, x_2) \in [0, L)^2$.

Lemma 3.2. *Let $(d, \alpha) \in (D_{exis}, +\infty) \times (0, +\infty)$, $f_{1,x_1} : (z, x) \mapsto f_1(z, 2x_1 - x)$ and Θ be defined as in Proposition 3.1. Assume $c_\infty = 0$.*

Then:

$$\alpha\Theta\left(x_1, f_{1,x_1}\right) \geq d\Theta\left(x_1, \frac{1}{d}f_2\right),$$

$$\alpha\Theta\left(x_2, f_{1,x_2}\right) \leq d\Theta\left(x_2, \frac{1}{d}f_2\right).$$

Proof. We prove the first inequality, the second one being proved similarly (using e_2 instead of e_1).

First, if:

$$e_1^{-1}(\{0\}) \setminus \{x_1\} = \emptyset,$$

then e_1 has a unique zero. Now, consider the problems satisfied by the functions:

$$z_1 : x \mapsto e_1^+(2x_1 - x),$$

$$z_2 : x \mapsto e_1^-(x).$$

It is clear that:

$$(z_1, z_2) = \left(z_{x_1, (z,x)\mapsto f_1\left(\frac{z}{\alpha}, 2x_1 - x\right)}, z_{x_1, (z,x)\mapsto \frac{1}{d}f_2\left(\frac{z}{d}, x\right)}\right).$$

Since $e_1 \in \mathcal{C}^2(\mathbb{R})$, $z_1'\left(x_1^+\right) = z_2'\left(x_1^+\right)$ is necessary. From the relations:

$$\Theta\left(x_1, (z, x) \mapsto f_1\left(\frac{z}{\alpha}, 2x_1 - x\right)\right) = \alpha\Theta\left(x_1, f_{1,x_1}\right),$$

$$\Theta\left(x_1, (z,x) \mapsto \frac{1}{d} f_2\left(\frac{z}{d}, x\right)\right) = d\Theta\left(x_1, \frac{1}{d} f_2\right),$$

we see that we are in the case of equality.

Next, if:

$$e_1^{-1}(\{0\}) \setminus \{x_1\} \neq \emptyset,$$

then let:

$$y_1 = \min e_1^{-1}(\{0\}) \setminus \{x_1\}.$$

Clearly, $z_3 = \left(e_1^-\right)_{|(x_1,y_1)}$ is the unique non-negative non-zero solution of:

$$\begin{cases} -dz'' = zf_2\left[\frac{z}{d}\right] & \text{in } (x_1, y_1) \\ z(x_1) = z(y_1) = 0. \end{cases}$$

Now it can be easily verified that z_3 is a sub-solution for the problem satisfied by $z_{x_1,(z,x)\mapsto \frac{1}{d} f_2\left(\frac{z}{d},x\right)}$. The inequality follows. □

Remark. We explained previously that, if $\left(\mathcal{H}_{freq}\right)$ [23] is assumed, each segregated stationary equilibrium has a unique zero x_e. In such a case, we have equality:

$$\alpha\Theta\left(x_e, f_{1,x_e}\right) = d\Theta\left(x_e, \frac{1}{d} f_2\right).$$

Let $(d, \alpha) \in (0, +\infty)^2$. With the same notations as before, we define the following sets:

$$X^+_{(d,\alpha)} = \left\{x \in [0, L) \mid \alpha\Theta\left(x, f_{1,x}\right) \geq d\Theta\left(x, \frac{1}{d} f_2\right)\right\},$$
$$X^-_{(d,\alpha)} = \left\{x \in [0, L) \mid \alpha\Theta\left(x, f_{1,x}\right) \leq d\Theta\left(x, \frac{1}{d} f_2\right)\right\}.$$

Clearly, from the preceding corollary, if $c_\infty = 0$,

$$\begin{cases} X^+_{(d,\alpha)} \neq \emptyset \\ X^-_{(d,\alpha)} \neq \emptyset. \end{cases}$$

Proposition 3.3. *Let $(d, \alpha) \in (0, +\infty)^2$, $f_{1,x} : (z, y) \mapsto f_1(z, 2x - y)$, Θ be defined as in Proposition 3.1 and:*

$$A_d : x \mapsto \frac{d\Theta\left(x, \frac{1}{d} f_2\right)}{\Theta\left(x, f_{1,x}\right)}.$$

The function A_d is continuous, positive and periodic, does not depend on α and satisfies the following properties.

- *If there exists* $x \in X^{+}_{(d,\alpha)}$*, then* $\alpha \geq A_d(x)$*.*
- *If there exists* $x \in X^{-}_{(d,\alpha)}$*, then* $\alpha \leq A_d(x)$*.*
- *It has a global minimum and a global maximum.*

Consequently, provided $d > D_{exis}$*,* $\alpha \in [\min A_d, \max A_d]$ *if and only if* $c_\infty = 0$*.*

Proof. Everything is straightforward apart maybe the following implication: if $\alpha \in \left[\min A_d, \max A_d\right]$, then $c_\infty = 0$. In fact, if there exists $x_e \in [0, L)$ such that $\alpha = A(x_e)$, then the following function:

$$z : y \mapsto \begin{cases} z_{x_e,(z,x)\mapsto f_1\left(\frac{z}{\alpha},2x_e-x\right)}(2x_e - y) & \text{if } y < x_e, \\ -z_{x_e,(z,x)\mapsto \frac{1}{d} f_2\left(\frac{z}{d},x\right)}(y) & \text{if } y \geq x_e, \end{cases}$$

is a segregated stationary equilibrium, which implies by uniqueness (see Theorem 2.43) that $c_\infty = 0$. $\square$

Remark. The preceding proposition characterizes sharply $\{\alpha > 0 \mid c_\infty = 0\}$. Moreover, it also gives an implicit characterization of the diffusion rates such that $c_\infty = 0$. With this in mind, understanding whether A_d is constant or not would be of great interest.

Let us recall that if a_1 and a_2 are not constant, we do not know how to prove Theorem 2.43. Therefore in such a case the preceding sharpness is lost and we might still have a non-zero c_∞ for some $\alpha \in [\min A_d, \max A_d]$. This pathological situation seems highly unlikely (recall Conjecture 2.44).

From this result, we can also deduce an explicit estimate for the range of parameters (d, α), as indicated by the following statement.

Proposition 3.4. *Let* $\Lambda \subset \mathbb{R}^2$ *be the following set:*

$$\left\{(d,\alpha) \in (0,+\infty)^2 \mid X^{+}_{(d,\alpha)} \neq \emptyset \text{ and } X^{-}_{(d,\alpha)} \neq \emptyset\right\}.$$

There exists $\underline{r} > 0$ *and* $\overline{r} \geq \underline{r}$*, defined by formulas* $\left(\mathfrak{F}_{\underline{r}}\right)$ *and* $(\mathfrak{F}_{\overline{r}})$ *which only depend on* (f_1, f_2)*, such that, for any* $(d, \alpha) \in \Lambda$*,*

$$\underline{r} \leq \frac{\alpha^2}{d} \leq \overline{r}.$$

Remark. Although these estimates do not depend on d, they are also less precise than the previous statement. Indeed, we will see in the course of the proof that, for any $d > 0$:

$$\sqrt{\underline{r}d} \leq \min A_d,$$
$$\max A_d \leq \sqrt{\overline{r}d},$$

and furthermore it should be expected that these inequalities are actually strict. Thus the interest of this proposition lies mostly in the fact that $\underline{r}$ and $\overline{r}$ do not depend on d.

Proof. Recalling from Proposition 3.1 the definition of $z_{x_0,f}$, we define for any $d>0$ and any $y\in\overline{C}$ the following functions:

$$z_{1,y}=z_{y,f_{1,y}},$$
$$z_{2,y}:x\mapsto z_{y,\frac{1}{d}f_2}\left(\sqrt{d}x+y\right).$$

Most importantly, $z_{2,y}$ satisfies:

$$\begin{cases}-z_{2,y}''(x)=z_{2,y}(x)\,f_2\left(z_{2,y}(x),\sqrt{d}x+y\right) & \text{for any } x\in(0,+\infty),\\ z_{2,y}(0)=0.\end{cases}$$

Let $\overline{f_2}:z\mapsto\max\limits_{x\in\overline{C}}f_2(z,x)$ and $\overline{z}$ be the solution of:

$$\begin{cases}-z''=z\overline{f_2}[z] & \text{in } (0,+\infty)\\ z(0)=0.\end{cases}$$

Similarly, let $\underline{f_2}:z\mapsto\min\limits_{x\in\overline{C}}f_2(z,x)$ and $\underline{z}$ be the solution of:

$$\begin{cases}-z''=z\underline{f_2}[z] & \text{in } (0,+\infty)\\ z(0)=0.\end{cases}$$

It can easily be checked (see the proof of Proposition 3.1) that the solutions $\underline{z}$ and $\overline{z}$ form a pair of sub-solution and super-solution for the problem satisfied by $z_{2,y}$. By uniqueness, $\underline{z}\le z_{2,y}\le\overline{z}$. Since $\sqrt{d}\Theta\left(y,\frac{1}{d}f_2\right)=z_{2,y}'(0)$, consequently:

$$\underline{z}'(0)\le\sqrt{d}\Theta\left(y,\frac{1}{d}f_2\right)\le\overline{z}'(0).$$

Then, for any $(d,\alpha)\in\Lambda$, we deduce from the preceding estimate and from the definitions of $X^+_{(d,\alpha)}$ and $X^-_{(d,\alpha)}$ that there exists $(x_1,x_2)\in[0,L)^2$ such that:

$$\alpha\Theta\left(x_1,f_{1,x_1}\right)\ge\sqrt{d}\underline{z}'(0),$$
$$\alpha\Theta\left(x_2,f_{1,x_2}\right)\le\sqrt{d}\overline{z}'(0).$$

The conclusion follows from the following definitions:

$$\underline{r}=\left(\frac{\underline{z}'(0)}{\max\limits_{x\in\overline{C}}\Theta\left(x,f_{1,x}\right)}\right)^2,\qquad\left(\mathfrak{F}_{\underline{r}}\right)$$
$$\overline{r}=\left(\frac{\overline{z}'(0)}{\min\limits_{x\in\overline{C}}\Theta\left(x,f_{1,x}\right)}\right)^2.\qquad\left(\mathfrak{F}_{\overline{r}}\right)\qquad\square$$

Corollary 3.5. *Assume that, for any $i \in \{1,2\}$, f_i has the particular form $(u,x) \mapsto \mu_i(x)(1-u)$ with $\mu_i \in \mathcal{C}^1_{per}(\mathbb{R})$, $\mu_i \gg 0$.*

Then:

$$\frac{\min\limits_{\overline{C}}(\mu_2)}{\max\limits_{\overline{C}}(\mu_1)} \le \underline{r} \le \overline{r} \le \frac{\max\limits_{\overline{C}}(\mu_2)}{\min\limits_{\overline{C}}(\mu_1)}.$$

Proof. In such a case, the functions $\overline{f_2}$ and $\underline{f_2}$ defined in the proof of Proposition 3.4 reduce to:

$$\overline{f_2} : z \mapsto \max_{\overline{C}}(\mu_2)(1-z),$$

$$\underline{f_2} : z \mapsto \min_{\overline{C}}(\mu_2)(1-z).$$

Define analogously:

$$\overline{f_1} : z \mapsto \max_{\overline{C}}(\mu_1)(1-z),$$

$$\underline{f_1} : z \mapsto \min_{\overline{C}}(\mu_1)(1-z).$$

Denoting the functions $\overline{z}$ and $\underline{z}$ defined in the proof of Proposition 3.4 as $\overline{z}_2$ and $\underline{z}_2$, the definitions of $\underline{r}$ and $\overline{r}$ read:

$$\underline{r} = \left(\frac{\underline{z}_2'(0)}{\max\limits_{x\in\overline{C}} \Theta\left(x, f_{1,x}\right)}\right)^2,$$

$$\overline{r} = \left(\frac{\overline{z}_2'(0)}{\min\limits_{x\in\overline{C}} \Theta\left(x, f_{1,x}\right)}\right)^2.$$

Defining analogously the functions $\overline{z}_1$ and $\underline{z}_1$, we obtain by a super- and sub-solution argument similar to that of Proposition 3.4 the following estimates:

$$\underline{z}_1'(0) \le \min_{x\in\overline{C}} \Theta\left(x, f_{1,x}\right) \le \max_{x\in\overline{C}} \Theta\left(x, f_{1,x}\right) \le \overline{z}_1'(0),$$

which leads subsequently to:

$$\underline{r} \ge \left(\frac{\underline{z}_2'(0)}{\overline{z}_1'(0)}\right)^2,$$

$$\overline{r} \le \left(\frac{\overline{z}_2'(0)}{\underline{z}_1'(0)}\right)^2.$$

Now let us determine $\Theta(0, z \mapsto r(1-z))$ for any constant $r > 0$. Multiplying the equality satisfied by $z = z_{0,z\mapsto r(1-z)}$ by z', we find:

$$-\left(\frac{(z')^2}{2}\right)' = r\left(\frac{z^2}{2}\right)' - r\left(\frac{z^3}{3}\right)'.$$

Integrating between 0 and $+\infty$, it follows $\left(z'(0)\right)^2 = \frac{r}{6}$, that is:

$$\Theta(0, z \mapsto r(1-z)) = \sqrt{\frac{r}{6}}.$$

Applying this equality with $r = \max_{\overline{C}}(\mu_2)$, $r = \min_{\overline{C}}(\mu_2)$, $r = \max_{\overline{C}}(\mu_1)$ and $r = \min_{\overline{C}}(\mu_1)$, the claimed estimates for $\underline{r}$ and $\overline{r}$ follow directly. □

Thanks to the existence of $\underline{r}$ and $\overline{r}$, we now know that the quantity $\frac{\alpha^2}{d}$ plays a particular role (and this is obviously reminiscent of the space-homogeneous case [24]). Therefore, we also state the following (immediate) proposition.

Proposition 3.6. *For any $d \in (0, +\infty)$, let:*

$$\mathcal{R}_d^0 = \left[\frac{(\min A_d)^2}{d}, \frac{(\max A_d)^2}{d}\right]. \quad \left(\mathfrak{F}_{\mathcal{R}^0}\right)$$

The set $\mathcal{R}_d^0$ is a non-empty, closed, subinterval of $\left[\underline{r}, \overline{r}\right]$.

Assume moreover that $d > D_{exis}$. Then $c_\infty = 0$ if and only if $\frac{\alpha^2}{d} \in \mathcal{R}_d^0$.

Remark. Once more, in the case of non-constant a_1 and a_2, one implication is lacking, but proving Conjecture 2.44 would be sufficient to recover it.

The length of $\mathcal{R}_d^0$ is a very interesting open question (which is obviously equivalent to that of the constancy of A_d). Recall that in the space-homogeneous case [24], $\mathcal{R}_d^0 = \left\{\frac{f_2[0]}{f_1[0]}\right\}$ is a singleton which does not depend on d.

3.2. Sign of a non-zero asymptotic speed

Proposition 3.7. *Let $(d, \alpha) \in (0, +\infty)^2$. Let z be a segregated pulsating front with speed $s \neq 0$ and profile φ.*

Then s has the sign of:

$$\int_0^L \int_{-da_2}^{\alpha a_1} \eta(z, x)\, dz dx = \int_0^L \left(\alpha^2 \int_0^{a_1} z f_1(z, x)\, dz - d \int_0^{a_2} z f_2(z, x)\, dz\right) dx.$$

Remark. In view of well-known results about bistable scalar traveling waves, and more recently pulsating fronts (see for instance Ding–Hamel–Zhao [15]), such a result was to be expected.

It could be tempting to try to get rid of the *a priori* condition $s \neq 0$ and to show that the existence of a segregated stationary equilibrium implies:

$$\int_C \int_{-da_2}^{\alpha a_1} \eta(z,x)\,\mathrm{d}z\mathrm{d}x = 0.$$

But Zlatos [32] showed on the contrary that it is possible to build counter-examples of pure bistable non-linearities F of positive integral such that:

$$\partial_t z - \partial_{xx} z = F[z]$$

does not admit any transition front with non-zero speed. Therefore we do not investigate further in this direction.

Proof. We have justified previously that in the equation $(\mathcal{SPF}[s])$, every term $(\operatorname{div}(E\nabla\varphi)$, $\partial_\xi\varphi$ and $\eta[\varphi])$ is well-defined in $L^2_{loc}(\mathbb{R}^2)$. Thus we consider the test function $\partial_\xi\varphi\mathbf{1}_{[-B,B]\times\overline{C}} \in L^2_{loc}(\mathbb{R}^2)$ for some large enough $B > 0$. By large, we mean here that we assume the following:

$$\min_{x\in\overline{C}} \varphi(\xi,x) > 0 \text{ for any } \xi < -B,$$

$$\max_{x\in\overline{C}} \varphi(\xi,x) < 0 \text{ for any } \xi > B.$$

Hence the subset of the free boundary $\left\{(\xi,x)\in\mathbb{R}\times\overline{C} \mid \varphi(\xi,x)=0\right\}$ is included in $(-B,B)\times\overline{C}$.

Multiplying $(\mathcal{SPF}[s])$ by $\partial_\xi\varphi$ and integrating over $(-B,B)\times\overline{C}$ yield:

$$\int_{-B}^{B}\int_0^L \operatorname{div}(E\nabla\varphi)\,\partial_\xi\varphi + s\int_{-B}^{B}\int_0^L \sigma[\varphi]\left(\partial_\xi\varphi\right)^2 = -\int_{-B}^{B}\int_0^L \eta[\varphi]\,\partial_\xi\varphi.$$

First, by change of variable, Lipschitz-continuity of the free boundary (see Proposition 2.26) and definition of η:

$$\begin{aligned}
-\int_0^L\int_{-B}^{B} \eta[\varphi]\,\partial_\xi\varphi &= \int_0^L \int_{\varphi(B,x)}^{\varphi(-B,x)} \eta(z,x)\,\mathrm{d}z\mathrm{d}x \\
&= \int_0^L \left(\alpha^2 \int_0^{\varphi(-B,x)/\alpha} zf_1(z,x)\,\mathrm{d}z - d\int_0^{-\varphi(B,x)/d} zf_2(z,x)\,\mathrm{d}z\right)\mathrm{d}x.
\end{aligned}$$

Then, since we do not know that $\partial_\xi \varphi$ is continuous, the term $\int_{-B}^{B} \int_0^L \operatorname{div}(E\nabla\varphi)\,\partial_\xi\varphi$ is dealt with a standard mollification procedure. There exists a sequence of non-negative non-zero mollifiers $(\theta_n)_{n\in\mathbb{N}} \in \mathcal{D}(\mathbb{R})$. For any $n \in \mathbb{N}$, let:

$$\varphi_n : (\xi, x) \mapsto \int \varphi(\xi - \zeta, x)\,\theta_n(\zeta)\,\mathrm{d}\zeta.$$

On one hand, for any $n \in \mathbb{N}$, it is clear that all the terms $\partial_{\xi\xi}\varphi_n$, $\partial_{xx}\varphi_n$, $\partial_{\xi x}\varphi_n$ are classically defined. By periodicity and integration by parts, we easily obtain:

$$\int_{-B}^{B}\int_0^L \operatorname{div}(E\nabla\varphi_n)\,\partial_\xi\varphi_n = \frac{1}{2}\int_0^L \left(\left[\left(\partial_\xi\varphi_n\right)^2(\xi,x) \right]_{-B}^{B} - \left[(\partial_x\varphi_n)^2(\xi,x) \right]_{-B}^{B} \right) \mathrm{d}x.$$

It can be easily verified that if both sets:

$$\pm B + 2\operatorname{supp}\theta_1 = \pm B + 2\bigcup_{n\in\mathbb{N}} \operatorname{supp}\theta_n$$

do not intersect the free boundary, that is if B is large enough indeed, then as $n \to +\infty$:

$$\max_{x\in\overline{C}} \left|\partial_\xi\varphi_n(\pm B, x) - \partial_\xi\varphi(\pm B, x)\right| + \max_{x\in\overline{C}} \left|\partial_x\varphi_n(\pm B, x) - \partial_x\varphi(\pm B, x)\right| \to 0.$$

It follows that:

$$\begin{aligned} &\frac{1}{2}\int_0^L \left(\left[\left(\partial_\xi\varphi_n\right)^2(\xi,x) \right]_{-B}^{B} - \left[(\partial_x\varphi_n)^2(\xi,x) \right]_{-B}^{B} \right) \mathrm{d}x \\ &\quad \to \frac{1}{2}\int_0^L \left(\left[\left(\partial_\xi\varphi\right)^2(\xi,x) \right]_{-B}^{B} - \left[(\partial_x\varphi)^2(\xi,x) \right]_{-B}^{B} \right) \mathrm{d}x. \end{aligned}$$

On the other hand:

$$\int_{-B}^{B}\int_0^L \operatorname{div}(E\nabla\varphi)\,\partial_\xi(\varphi - \varphi_n) \leq \|\operatorname{div}(E\nabla\varphi)\|_{L^2((-B,B)\times C)} \|\partial_\xi(\varphi - \varphi_n)\|_{L^2((-B,B)\times C)},$$

$$\int_{-B}^{B}\int_0^L \operatorname{div}(E\nabla(\varphi - \varphi_n))\,\partial_\xi\varphi_n \leq \|\operatorname{div}(E\nabla(\varphi - \varphi_n))\|_{L^2((-B,B)\times C)} \sup_{n\in\mathbb{N}} \|\partial_\xi\varphi_n\|_{L^2((-B,B)\times C)},$$

and, once more by standard mollification theory, $\|\partial_\xi(\varphi - \varphi_n)\|_{L^2((-B,B)\times C)}$ and $\|\operatorname{div}(E\nabla(\varphi - \varphi_n))\|_{L^2((-B,B)\times C)}$ converge to 0 as $n \to +\infty$.

Therefore, passing to the limit $n \to +\infty$, we obtain the expected equality:

$$\int_{-B}^{B}\int_{0}^{L} \operatorname{div}(E\nabla\varphi)\,\partial_\xi\varphi = \frac{1}{2}\int_0^L \left(\left[\left(\partial_\xi\varphi\right)^2(\xi,x)\right]_{-B}^{B} - \left[\left(\partial_x\varphi\right)^2(\xi,x)\right]_{-B}^{B}\right)\mathrm{d}x.$$

Finally, using these computations to pass to the limit $B \to +\infty$ in the equality:

$$\int_{-B}^{B}\int_{0}^{L} \operatorname{div}(E\nabla\varphi)\,\partial_\xi\varphi + s\int_{-B}^{B}\int_{0}^{L}\sigma[\varphi]\left(\partial_\xi\varphi\right)^2 = -\int_{-B}^{B}\int_{0}^{L}\eta[\varphi]\,\partial_\xi\varphi$$

it follows:

$$s\int_{\mathbb{R}\times C}\sigma[\varphi]\left(\partial_\xi\varphi\right)^2 = \int_0^L\left(\alpha^2\int_0^{a_1} zf_1(z,x)\,\mathrm{d}z - d\int_0^{a_2} zf_2(z,x)\,\mathrm{d}z\right)\mathrm{d}x,$$

and since:

$$0 < \min\left\{1,\frac{1}{d}\right\}\|\partial_\xi\varphi\|^2_{L^2(\mathbb{R}\times C)} \le \int_{\mathbb{R}\times C}\sigma[\varphi]\left(\partial_\xi\varphi\right)^2,$$

the claimed relationship between s and $\int_0^L\int_{-da_2}^{\alpha a_1}\eta(z,x)\,\mathrm{d}z\mathrm{d}x$ follows. □

Corollary 3.8. *Let* $(d,\alpha) \in (D_{exis},+\infty)\times(0,+\infty)$. *Then:*

(1) *if* $\frac{\alpha^2}{d} > \max\mathcal{R}_d^0$, $c_\infty > 0$;
(2) *if* $\frac{\alpha^2}{d} < \min\mathcal{R}_d^0$, $c_\infty < 0$.

Proof. It suffices to remark that, for any $i \in \{1,2\}$, $\int_0^L\int_0^{a_i} zf_i(z,x)\,\mathrm{d}z\mathrm{d}x > 0$. □

Remark. We recall that, in the proof of Proposition 3.7, the fact that a_1 and a_2 are constant is crucial. This issue has already been encountered (see the remark following Proposition 2.5). Therefore, in the general setting, it is not possible to obtain such an explicit formula for the sign of c_∞. Nevertheless, let us point out that the results of Corollary 3.8 should still hold in this case:

- there still exists $\overline{r} \ge \underline{r} > 0$ such that $0 \notin \Sigma_{(d,\alpha,f_1,f_2)}$ if (d,α) does not satisfy $\underline{r} \le \frac{\alpha^2}{d} \le \overline{r}$, since the whole subsection 3.1 can be easily generalized (even though:
  - we cannot prove that $c_\infty = 0$ if $\alpha \in [\min A_d, \max A_d]$, i.e. if $\frac{\alpha^2}{d} \in \mathcal{R}_d^0$ (but recall Conjecture 2.44);
  - additional care is needed since a non-constant a_2 would *a priori* depend on d);
- we will prove in the next section that $(d,\alpha) \mapsto c_\infty$ is continuous at least in $\left\{(d,\alpha) \in (D_{exis},+\infty)\times(0,+\infty) \mid \frac{\alpha^2}{d} \notin \mathcal{R}_d^0\right\}$;

- the study of the limit of the segregated pulsating front as $\alpha \to 0$ or $\alpha \to +\infty$ (which can be rigorously done since D_{exis} does not depend on α) should easily yield the sign of the speed at such limits:
  - formally, as $\alpha \to 0$, the positive part of w vanishes and we are left with a Fisher–KPP pulsating front connecting 0 to $-da_2$, consequently with a negative speed;
  - formally, as $\alpha \to +\infty$, the negative part of $\frac{w}{\alpha}$ vanishes and we are left with a Fisher–KPP pulsating front connecting a_1 to 0, consequently with a positive speed;
- hence, by connectedness and continuity, Corollary 3.8 would be recovered indeed.

To conclude, let us highlight an important particular case.

Corollary 3.9. *Assume that, for any $i \in \{1, 2\}$, f_i has the particular form $(u, x) \mapsto \mu_i(x)(1-u)$ with $\mu_i \in \mathcal{C}^1_{per}(\mathbb{R})$, $\mu_i \gg 0$.*

Let:

$$r = \frac{\|\mu_2\|_{L^1(C)}}{\|\mu_1\|_{L^1(C)}}.$$

If $c_\infty \neq 0$, then it has the sign of $\alpha^2 r - d$.

Proof. In such a case, for any $i \in \{1, 2\}$, $a_i = 1$ and:

$$\int_0^L \int_0^1 z f_i(z, x)\, \mathrm{d}z \mathrm{d}x = \frac{1}{6} \int_0^L \mu_i(x)\, \mathrm{d}x. \quad \square$$

3.3. Continuity of the asymptotic speed with respect to the parameters

In this final subsection, we even allow (f_1, f_2) to vary in the set $\mathcal{F}$ of all L-periodic $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, equipped with the canonical topology of $\mathcal{C}^1\left(\mathbb{R}^2, \mathbb{R}\right)$.

Proposition 3.10. *Assume that for any $(f_1, f_2) \in \mathcal{F}^2$, there exists a non-negative $D_{exis} = D^{f_1, f_2}_{exis}$ as defined before.*

Let:

$$\mathfrak{P} = \left\{ (d, \alpha, f_1, f_2) \in (0, +\infty)^2 \times \mathcal{F}^2 \mid d > D^{f_1, f_2}_{exis} \right\}.$$

The function:

$$\begin{array}{ccc} \mathfrak{P} & \to & \mathbb{R} \\ (d, \alpha, f_1, f_2) & \mapsto & c_\infty \end{array}$$

is well-defined and continuous.

Assume moreover that the function $(d, \alpha, f_1, f_2) \in \mathfrak{P} \mapsto k^\star$ is locally bounded. Then the convergence of $((d, \alpha, f_1, f_2) \in \mathfrak{P} \mapsto c_k)_{k > k^\star}$ to $(d, \alpha, f_1, f_2) \in \mathfrak{P} \mapsto c_\infty$ is locally uniform.

Remark. If $(\mathcal{H}_{exis})$ follows from $\left(\mathcal{H}_{freq}\right)$ [23] and if:

$$D_{exis} = \begin{cases} M_2\left(\frac{L}{\pi} - \frac{1}{\sqrt{M_1}}\right)^2 & \text{if } L\sqrt{M_1} > \pi, \\ 0 & \text{if } L\sqrt{M_1} \leq \pi, \end{cases}$$

then $\left(f_1, f_2\right) \mapsto D_{exis}^{f_1,f_2}$ is indeed well-defined (and actually continuous) in $\mathcal{F}^2$.

Proof. Just verify (with the same integrations by parts than those used in the course of the proofs of Propositions 2.4 and 2.5) that:

- all families of segregated pulsating fronts satisfy some locally uniform estimates (with respect to (d, α, f_1, f_2)) in $\mathcal{C}_{loc}\left(\mathbb{R}, L^2_{loc}(\mathbb{R})\right) \cap L^2_{loc}\left(\mathbb{R}, H^1_{loc}(\mathbb{R})\right)$ and therefore, by virtue of DiBenedetto's theory [13], in $\mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$;
- all families of segregated stationary equilibrium satisfy some locally uniform estimates (with respect to (d, α, f_1, f_2)) in $\mathcal{C}^{2,\beta}_{loc}(\mathbb{R})$.

The continuity of c_∞ is then a classical consequence of Theorems 2.42 and 2.43 and of compactness arguments.

The locally uniform convergence is proved with similar compactness arguments, this time using the fact that the compactness estimates of Propositions 2.4 and 2.5 are locally uniform. □

Remark. We recall that in the case of non-constant a_1 and a_2, we cannot prove Theorem 2.43. Therefore it is not possible to prove complete continuity of c_∞. In the whole subset:

$$\left\{(d, \alpha, f_1, f_2) \in \mathfrak{P} \mid \frac{\alpha^2}{d} \in \mathcal{R}^0_{d,f_1,f_2}\right\},$$

c_∞ might not be continuous and jump between 0 and some non-zero values. Still, it is not possible to jump directly from a positive value to a negative one, whence the zero set is in any case non-empty. Moreover, we recall that these issues are completely subordinated to Conjecture 2.44.

3.3.1. As a conclusion: what about monotonicity?

Regarding the monotonicity of $\alpha \mapsto c_\infty$: it should be easily established, via super- and sub-solutions, that $\alpha \mapsto c_\infty$ is non-decreasing (a proof that we do not detail here for the sake of brevity). Recall moreover that we already suggested in the previous subsection that $c_\infty \to -c^\star[d,2]$ as $\alpha \to 0$ and $c_\infty \to c^\star[1,1]$ as $\alpha \to +\infty$, whence $\alpha \mapsto c_\infty$ would in fact be from $(0, +\infty)$ onto $(-c^\star[d,2], c^\star[1,1])$.

Regarding the monotonicity of $d \mapsto c_\infty$: on the contrary, such a result should in general not be expected. We recall that:

- the dependency of the speed of a bistable front on its diffusion coefficient is in general unclear;
- even for the Fisher–KPP equation, as long as heterogeneity is introduced, the monotonicity of the minimal speed as a function of the diffusion coefficient is in general lost (for instance, in space-time periodic media, a counter example has been exhibited by the second author [29]).

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