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Semilinear Schrödinger equations with a potential of some critical inverse-square type

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Abstract

We solve semilinear Schrödinger equations with a singular potential of inverse-square scale, especially the threshold of nonnegativity and selfadjointness.

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Keywords: Semilinear Schrödinger equations; Hartree equations; Energy methods; Inverse-square potentials; Strichartz estimates; Radial-spherical decomposition

1. Introduction

In this article, we consider the Cauchy problems for following semilinear Schrödinger equations with a singular potential of inverse-square type

$$\begin{cases} i\,\dfrac{\partial u}{\partial t} = (-\Delta + V)u + g(u) \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u(0) = u_0, \end{cases} \tag{NLS}$$

where $i = \sqrt{-1}$, $N \geq 3$, and $V \in C(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfies the *inverse-square condition* (the scales of inverse-square)

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$$V(\mu x) = \mu^{-2} V(x), \quad \mu > 0.$$

Let $P := -\Delta + V$. Then P is scale-symmetric operator, that is, $P[u(\mu x)] = \mu^2 (Pu)(\mu x)$ for $\mu > 0$. One of the important subjects of modern physics is analyzing many aspects of physical phenomena with singular poles (at origin). To end this, we ordinarily describe the Schrödinger operators perturbed by singular potentials such as the scales of inverse-square. For example, the Efimov states (the circumstances that the two-particle attraction is so weak that any two bosons can not form a pair, but the three bosons can be stable bound states): see e.g., [22,23]; effects on dipole-bound anions in polar molecules: see e.g., [32,2,8]; capture of matter by black holes (via near-horizon limits): see e.g., [25,10]; the motions of cold neutral atoms interacting with thin charged wires (falling in the singularity or scattering): see e.g., [20,6]; the dynamics of a dipole in a cosmic string background: see e.g., [4]; the renormalization group of limit cycle in nonrelativistic quantum mechanics: see e.g., [5,7]; and so on. Moreover, the Calogero–Moser models (see [13,14,37]), typical example of integrable dynamical systems, are expressed by the following Hamiltonian.

$$\sum_{j=1}^{N} \Big(-\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j^2} \Big) = -\Delta + \sum_{j=1}^{N} k_j \, x_j^{-2} =: -\Delta + V_{\mathrm{CM}}.$$

The potentials V_{CM} are the scales of inverse-square but it appears a great deal of singularities: not only the origin but also the family of hyperplanes $x_1 x_2 \cdots x_N = 0$. Thus we examine the simplified singularity as like $a|x|^{-2}$ (the pole is only the origin).

These are certainly linear problems. But the nonlinear ones also are disclosed. Applying the mean-field approximation to the intractable nonrelativistic many-body problem, the problems are turned into the nonlinear evolution equations: Hartree-Fock equations. Moreover, if we describe the nonlinear effects as like the Kerr effects, we need to analyze the nonlinear equations (usually the semilinear evolution equations).

In view of the Cauchy problems **(NLS)**, $g(u)$ is a nonlinear term so that the conservation laws are held, for example, $g(u) := \lambda |u|^{p-1} u$ (pure power nonlinearity; local type) and $g(u) := \lambda u \, (|x|^{-\gamma} * |u|^2)$ (usual Hartree nonlinearity; nonlocal type) with $\lambda \in \mathbb{R}$. A typical example of V is $a|x|^{-2}$ $(a \in \mathbb{R})$. The Hardy inequality

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} \, dx \le \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx \quad N \ge 3 \tag{1.1}$$

implies the presence of threshold for nonnegativity and selfadjointness (in the sense of form-sum, known as the Friedrichs extensions) of $P_a := -\Delta + a|x|^{-2}$. Note that the coefficient $(N-2)^2/4$ is optimal. P_a is nonnegative and selfadjoint in $L^2(\mathbb{R}^N)$ if $a \ge a(N) := -(N-2)^2/4$.

The contraction methods are the simple, powerful, and useful methods for constructing unique solution of nonlinear Schrödinger equations. We can solve **(NLS)** with $V(x) = a|x|^{-2}$ via the contraction principle since the Strichartz estimates are established by Burq–Planchon–Stalker–Tahvildar-Zadeh [11]. But Okazawa–Suzuki–Yokota [39] showed the global unique existence with unsatisfactory conditions for **(NLS)** with local nonlinearities.

Proposition 1.1 *(Via the contraction methods, [39, Theorem 1.2]). Let $N \ge 3$, $V(x) = a|x|^{-2}$, and $u_0 \in H^1(\mathbb{R}^N)$. Assume*

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$$g(u) = |u|^{p-1}u,\ 1 \le p < \frac{N+2}{N-2}, \quad a > a(N) + \left[\frac{N(p-1)}{2(p+1)}\right]^2.$$

Then there exists a unique solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ *to* **(NLS)**.

Recently, the restriction of a can be relaxed even in the contraction methods. Specifically, Killip–Miao–Visan–Zhang–Zheng [30] proved

$$\|(-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^N)} \le C_1 \|P_a^{s/2} f\|_{L^q(\mathbb{R}^N)},$$
$$\text{if } \frac{1}{2} - \frac{1-s+\nu(a)}{N} < \frac{1}{q} < \min\left\{1, \frac{1}{2} + \frac{1+\nu(a)}{N}\right\};$$
$$\|P_a^{s/2} f\|_{L^q(\mathbb{R}^N)} \le C_2 \|(-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^N)},$$
$$\text{if } \max\left\{\frac{s}{N}, \frac{1}{2} - \frac{1+\nu(a)}{N}\right\} < \frac{1}{q} < \min\left\{1, \frac{1}{2} + \frac{1+\nu(a)}{N}\right\},$$

where $\nu(a) := [a + (N-2)^2/4]^{1/2}$. Unfortunately, in view of the above inequalities, it is hard to remove fully the restriction of a by applying the contraction methods. The unsatisfactory condition is removed in Okazawa–Suzuki–Yokota [40] by applying the *energy methods* (see also Section 5):

Proposition 1.2 *(Via the energy methods, [40, Theorem 5.1]). Let* $N \ge 3$, $V(x) = a|x|^{-2}$, $a > a(N)$, *and* $u_0 \in H^1(\mathbb{R}^N)$. *Assume that* $g(u) := |u|^{p-1}u$ $(1 \le p < (N+2)/(N-2))$. *Then there exists a unique solution* $u \in C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ *to* **(NLS)**.

Note that if $a = a(N)$, the energy space $X^1(\mathbb{R}^N) := D((1 + P_{a(N)})^{1/2})$ does not coincide with $H^1(\mathbb{R}^N) = D((1-\Delta)^{1/2}) = D((1+P_a)^{1/2})$ $(a > a(N))$. This is the consequence of the optimality for (1.1). Here $X^1(\mathbb{R}^N)$ is not just equal but almost equal to $H^1(\mathbb{R}^N)$. In fact, Suzuki [48] showed for $N \ge 2$ and $0 < s < 1$

$$\big\||x|^{-s} f\big\|_{L^2(\mathbb{R}^N)} \le \frac{\Gamma((1-s)/2)}{2^s\,\Gamma((1+s)/2)} \|P_{a(N)}^{s/2} f\|_{L^2(\mathbb{R}^N)}, \tag{1.2}$$

$$\big\|(-\Delta)^{s/2} f\big\|_{L^2(\mathbb{R}^N)} \le \frac{\Gamma((N+2s)/4)\,\Gamma((1-s)/2)}{\Gamma((N-2s)/4)\,\Gamma((1+s)/2)} \|P_{a(N)}^{s/2} f\|_{L^2(\mathbb{R}^N)}. \tag{1.3}$$

The contraction methods do not work well, while the energy methods still work. See Suzuki [48] for the following results.

Proposition 1.3 *(Critical case of a, [48, Theorem 4.1]). Let* $N \ge 3$, $a = a(N)$, *and* $u_0 \in X^1(\mathbb{R}^N)$. *Assume that* $g(u) := |u|^{p-1}u$ $(1 \le p < (N+2)/(N-2))$. *Then there exists a unique solution* $u \in C(\mathbb{R}; X^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; X^{-1}(\mathbb{R}^N))$ *to* **(NLS)**.

Recently, Azzam [1] analyzed the resolvents for P_a with $a = -1/4$ in the spatial dimension $N = 3$. The above results treated the energy-subcritical nonlinearities. Actually, **(NLS)** with $V(x) = a|x|^{-2}$ are partially solved under the energy-critical nonlinearities. Killip–Miao–Visan–Zhang–Zheng [29] cleverly applied the contraction principle and global existence is proved for

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the spatial dimension $N = 3$ and $g(u) = |u|^4 u$ global existence and scattering $|u|^4 u$ under the condition $a > -1/4 + 1/25$. Miao–Murphy–Zheng [34] extended the results to $N = 3$ with $g(u) = \pm|u|^4 u$ for $a > -1/4 + 1/25$, and $N = 4$ with $g(u) = \pm|u|^2 u$ for $a > -1 + 1/9$. Here the case of plus sign nonlinearities implies the global existence, the case of minus sign nonlinearities implies blowing up in finite time. Since we see the global and unique existence of solution to **(NLS)** of $V(x) = a|x|^{-2}$, we can consider the global analysis of solutions, especially, asymptotic behavior.

The scattering problems for **(NLS)** with $V(x) = a|x|^{-2}$ and the pure power type are partially solved in $H^1(\mathbb{R}^N)$. Zhang–Zheng [55] proved the Morawetz inequality and constructed the wave operators in $a \geq 0$ if $N = 3$, and in

$$a > a(N) + \frac{4}{(p+1)^2}, \ 1 + \frac{4}{N} < p < 1 + \frac{4}{N-2} \text{ if } N \geq 4.$$

In this context, they research the threshold between the global existence and blowing up in finite time. Killip–Murphy–Visan–Zheng [31] considered the dichotomy between scattering and blowing up for the spatial dimension $N = 3$ and cubic power nonlinearity $-|u|^2 u$. Later, Zheng [56] investigated the dichotomy for $N \geq 3$ and $g(u) = -|u|^{p-1} u$ under the radial solutions. On the one hand, Dinh [21] studied threshold between the global existence and blowing up in finite time for the same dimension and nonlinearities. See also Murphy [38] for recently works for the dichotomy.

On the other hand, Lu–Miao–Murphy [33] considered the profile decomposition and constructed the wave operators in $a > a(N)$ and $7/3 < p \leq 3$ if $N = 3$, and in

$$a > a(N) + \left(\frac{N-2}{2} - \frac{1}{p+1}\right)^2, \ \max\left\{1 + \frac{4}{N}, 1 + \frac{2}{N-2}\right\} < p < 1 + \frac{4}{N-2}$$

if $3 \leq N \leq 6$. Note that the contraction methods are applied to the both of studies. Moreover, Trachanas–Zographopoulos [51] considered the orbital stability of standing waves for **(NLS)** with $V(x) = a|x|^{-2}$ and the pure power type nonlinearity $-|u|^{p-1} u$ in the radially symmetric settings. They applied the energy methods in Cazenave [15, Chapter 3] for radial solutions. This is also considered in Bensouilah–Dihn–Zhu [3] without the radial settings, partially. Csobo–Genoud [18] considered the minimal blow-up solutions for L^2-critical ($p = 1 + 4/N$) of **(NLS)** with $V(x) = a|x|^{-2}$ ($a(N) < a < 0$).

Now we consider the general inverse-square potential V. In particular, $V(x) := (b \cdot x)|x|^{-3}$ ($b \in \mathbb{R}^N$); the Hamiltonian perturbed the potential arises from the study of electron capture by polar molecules (see e.g., Leblond [32]). Here the threshold between the selfadjointness and the bound states of $-\Delta + kx_3|x|^{-3}$ in $L^2(\mathbb{R}^3)$ is $k \approx k_0 = 1.27863$. Note that $k_0 > |a(3)| = 1/4$ and the selfadjointness is followed beyond the bounds of the Hardy inequality. Now we consider the condition

$$\int_{S^{N-1}} \left[|\nabla_S u|^2 + \left(V + \frac{(N-2)^2}{4}\right)|u|^2\right] d\sigma_{S^{N-1}} \geq \delta_V \int_{S^{N-1}} |u|^2 \, d\sigma_{S^{N-1}}, \tag{1.4}$$

where ∇_S is the spherical gradient on S^{N-1}. Here δ_V is settled in the optimal constant:

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$$\delta_V := \inf\Big\{ \int_{S^{N-1}} \Big[|\nabla_S u|^2 + \Big(V + \frac{(N-2)^2}{4}\Big)|u|^2 \Big] d\sigma_{S^{N-1}};\ \|u\|_{L^2(S^{N-1})} = 1 \Big\}.$$

We see that $\delta_V \geq 0$ yields the nonnegativity and selfadjointness of $P := -\Delta + V$ in the sense of form-sum (see also (2.2)). Assume that V is not critical, that is, $\delta_V > 0$ in (1.4). Then Burq–Planchon–Stalker–Tahvildar-Zadeh [12] proved the Strichartz estimates for $\exp(-itP)$:

$$\|e^{-itP}\varphi\|_{L^\tau(\mathbb{R};L^\rho(\mathbb{R}^N))} \leq C_\tau \|\varphi\|_{L^2(\mathbb{R}^N)},$$

where (τ, ρ) is the *admissible pair* which satisfies

$$\frac{2}{\tau} + \frac{N}{\rho} = \frac{N}{2}, \quad \tau \geq 2,\ \rho \geq 2.$$

Note that the *endpoint* $(2, 2N/(N-2))$ is included. The energy methods and the Strichartz estimates follow the unique existence of local in time solution $u \in C([-T,T]; H^1(\mathbb{R}^N)) \cap C^1([-T,T]; H^{-1}(\mathbb{R}^N))$ to **(NLS)**.

This paper is concerned with the critical case $\delta_V = 0$ in (1.4). In particular, we try to establish the energy estimates like (1.2) and (1.3) and the Strichartz estimates for $\exp(-itP)$, and solve **(NLS)**. In more detail, we show the following three theorems. First, the energy estimates are put together as follows.

Theorem 1.4. *Let $N \geq 2$ and $s \in [0,1)$. Assume that V satisfies (1.4) with $\delta_V = 0$. Then*

$$\big\|(-\Delta)^{s/2}u\big\|_{L^2(\mathbb{R}^N)} \leq \frac{\Gamma((1+s+a(V))/2)\,\Gamma((1-s)/2)}{\Gamma((1-s+a(V))/2)\,\Gamma\big((1+s)/2\big)} \|P^{s/2}u\|_{L^2(\mathbb{R}^N)}, \tag{1.5}$$

where $a(V) := (-\min\{V(x); x \in S^{N-1}\})^{1/2} > 0$. If $N = 2$ and $\min\{V(x); x \in S^{N-1}\} = 0$, then $a(V) = 0$ is admitted.

Secondly, the Strichartz estimates for $\exp(-itP)$ can be established as follows.

Theorem 1.5. *Let $N \geq 3$. Assume that V satisfies (1.4) with $\delta_V = 0$. Then for any admissible pairs (τ_j, ρ_j) $(j = 0,1,2)$ without the endpoint $(2, 2N/(N-2))$*

$$\|e^{-itP}\varphi\|_{L^{\tau_0}(\mathbb{R};L^{\rho_0}(\mathbb{R}^N))} \leq C_{\tau_0}(V)\, \|\varphi\|_{L^2(\mathbb{R}^N)}, \tag{1.6}$$

$$\Big\| \int_0^t e^{-i(t-s)P} F(s)\,ds \Big\|_{L^{\tau_2}(\mathbb{R};L^{\rho_2}(\mathbb{R}^N))} \leq C_{\tau_1,\tau_2}(V)\, \|F\|_{L^{\tau_1'}(\mathbb{R};L^{\rho_1'}(\mathbb{R}^N))}. \tag{1.7}$$

Finally we solve the nonlinear problems **(NLS)**. To end with, we apply the energy methods established by Okazawa–Suzuki–Yokota [40]. For giving much information about the energy methods, we consider the Hartree–Choquard type nonlinear terms

$$g(u) := \lambda |u(x)|^{p-1}u(x) \int_{\mathbb{R}^N} \frac{|u(y)|^{p+1}}{|x|^\alpha\, |x-y|^\beta\, |y|^\alpha}\, dy. \tag{1.8}$$

We denote $\mathcal{D} := D((1+P)^{1/2})$ and $\mathcal{D}^* := D((1+P)^{-1/2})$, a dual of $\mathcal{D}$.

Theorem 1.6. *Let $N \geq 3$, $\lambda \in \mathbb{R}$, $\alpha \geq 0$, $\beta \geq 0$, $2\alpha + \beta < (N \wedge 4)$, and $1 \leq p < 1 + (4 - \alpha - \beta)/(N-2)$. Assume that V satisfies (1.4) with $\delta_V = 0$. Then for any $u_0 \in \mathcal{D}$ there uniquely exists a local solution $u \in C([-T,T];\mathcal{D}) \cap C^1([-T,T];\mathcal{D}^*)$ to (NLS) with (1.8). Moreover, u satisfies*

$$\|u(t)\|_{L^2(\mathbb{R}^N)} = \|u_0\|_{L^2(\mathbb{R}^N)}, \quad E(u(t)) = E(u_0),$$

where

$$E(\varphi) := \frac{1}{2}\|(1+P)^{1/2}\varphi\|^2_{L^2(\mathbb{R}^N)} + \frac{\lambda}{2(p+1)} \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^{p+1}\,|\varphi(y)|^{p+1}}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dx\,dy.$$

Furthermore, if u_0 is also belongs to $D(|x|)$, then $u \in C([-T,T];D(|x|))$.

Thanks to the energy methods, we can also consider other nonlinearities as like $|u|^{p-1}u$ and

$$g(u) = u(x)\int\limits_{\mathbb{R}^N} k(x,y)|u(y)|^2\,dy$$

(see Examples 5.7–5.9 for details).

To carry out these above, we divide the paper into six sections. In Section 2, we introduce the radial-spherical decomposition respect to P. In Section 3, we evaluate the energy spaces for (NLS). Theorem 1.4 is confirmed in this section. In particular, we show $D((1+P)^{1/2}) \subset H^s(\mathbb{R}^N) = D((1-\Delta)^{s/2})$. The Strichartz estimates for $\exp(-itP)$ are confirmed in Section 4 via the local smoothing estimates. Thus Theorem 1.5 is verified in this section. Well-posedness, that is, unique existence for (NLS) is proved in Section 5 by applying energy methods. The nonlinearity is acceptable in the energy methods for both the pure power nonlinearity (local type) and the usual Hartree nonlinearity (nonlocal type). In this paper, we choose the Hartree-Choquard type nonlinearity (Theorem 1.6). Finally, we conclude with some remarks in Section 6.

1.1. Notations and preliminaries

Let $m(r)$ be a nonnegative and measurable function on $\mathbb{R}_+ = (0,\infty)$. The family $L^2(\mathbb{R}_+; m(r)\,dr)$ is the Hilbert spaces with norm

$$\|u\|_{L^2(\mathbb{R}_+;m(r)\,dr)} := \Big[\int_0^\infty |u(r)|^2\,m(r)\,dr\Big]^{1/2}.$$

Let $I \subset \mathbb{R}$ be an open interval, and Y be a reflexive Banach space. Then $C(\bar{I};Y)$ is a family of the continuous Y-valued function on $\bar{I}$ (the closure of I; closed interval). On the other hand, the vector-valued Lebesgue space $L^p(I;Y)$ is equipped with norm

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$$\|u\|_{L^p(I;Y)} := \big\| \|u(\cdot)\|_Y \big\|_{L^p(I)} < \infty.$$

Moreover, the vector-valued Sobolev space $W^{1,p}(I;Y)$ can be also defined:

$$W^{1,p}(I;Y) := \{u \in L^p(I;Y); \|u'\|_{L^p(I;Y)} < \infty\}.$$

Here u' denotes the weak derivative of u respect to time variable $t \in I$. It is well-known that $W^{1,p}(I;Y) \subset C(\bar{I};Y)$ $(1 \le p \le \infty)$ is continuous (see [16, Corollary 1.4.36]).

Also we denote $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$.

2. The radial-spherical decomposition

We consider the generalization of spherical harmonics decomposition (see e.g., [45, Lemma IV.2.18]). Applying the polar coordinate system to $P := -\Delta + V$, we see

$$P = -\partial_r^2 - \frac{N-1}{r}\partial_r + \frac{1}{r^2}(-\Delta_S + V),$$

where $\Delta_S = \Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on $S^{N-1} := \{x \in \mathbb{R}^N; |x| = 1\}$. Since S^{N-1} is a compact Riemannian manifold, we can analyze the spectrum (obviously, eigenvalues) of $-\Delta_S + V$ in $L^2(S^{N-1})$.

Lemma 2.1. *Let $N \ge 2$ and $V \in C(S^{N-1};\mathbb{R})$. Then there exists a Hilbert basis $\{Y_{m,j}\}_{m,j}$ of $L^2(S^{N-1})$ and a sequence $\{\lambda_m\}_m \subset \mathbb{R}$ such that*

$$\begin{aligned}
&\lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots, \quad \lambda_m \to \infty \text{ as } m \to \infty,\\
&Y_{m,j} \in H^2(S^{N-1}) \cap C^{1,\alpha}(S^{N-1}) \setminus \{0\} \ (0 < \alpha < 1),\\
&(-\Delta_S + V)Y_{m,j} = \lambda_m Y_{m,j}.
\end{aligned}$$

Moreover, $\mathcal{Y}_m := \{Y; (-\Delta_S + V)Y = \lambda_m Y\} \neq \{0\}$, the m-th eigenspace of $-\Delta_S + V$, is a finite dimensional vector space, whose orthogonal normalized basis $\{Y_{m,j}\}_j$ exists.

Here we remark the regularity of eigenfunctions. We see

$$Y_{m,j} = (-\Delta_S + 1)^{-1}(\lambda_m + 1 - V)Y_{m,j}.$$

Thus if $Y_{m,j} \in L^p(S^{N-1})$ $(1 < p < \infty)$, then $Y_{m,j} \in W^{2,p}(S^{N-1})$ by the elliptic regularity lemmas. On the other hand, it follows from the variational principle that $Y_{m,j} \in H^1(S^{N-1})$. The Sobolev embeddings $H^1(S^{N-1}) \subset L^{2^*}(S^{N-1})$ $(2^* := (N-1)/(N-3)$ if $N \ge 4$, and $2^* < \infty$ if $N = 3)$ imply that $Y_{m,j}$ has higher regularity. By repeating this step by step, we can get $Y_{m,j} \in W^{2,p}(S^{N-1}) \subset C(S^{N-1})$ $(p > (N-1)/2)$. Note that it follows from the Sobolev embeddings $W^{2,p}(S^{N-1}) \subset C^{1,\alpha}(S^{N-1})$ $(N-1 < p < \infty)$ that $Y_{m,j}$ belongs also to $C^1(S^{N-1})$. Here we need only the regularity as $Y_{m,j} \in H^1(S^{N-1}) \cap L^\infty(S^{N-1})$ in our arguments. In particular, we see $|\nabla_x u|^2 = |\partial_r u|^2 + r^{-2}|\nabla_S u|^2$ and if $\partial_r f, r^{-1} f \in L^2(\mathbb{R}_+; r^{N-1}\,dr)$ and $Y \in H^1(S^{N-1})$, then $u(x) := f(|x|)Y(x/|x|)$ belongs to $H^1(\mathbb{R}^N)$:

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$$|\nabla_x u|^2 = |\partial_r f|^2 |Y|^2 + |r^{-1} f|^2 |\nabla_S Y|^2 \in L^2(\mathbb{R}^N).$$

Note that $L^2(\mathbb{R}^N) = L^2(\mathbb{R}_+; r^{N-1}\, dr) \otimes L^2(S^{N-1})$. Here the Hardy type inequalities are available for $L^2(\mathbb{R}_+; r^{N-1}\, dr)$.

$$\begin{aligned} &\int_0^\infty \frac{|f(r)|^2}{r^2}\, r^{N-1}\, dr \le \frac{4}{(N-2)^2} \int_0^\infty |\partial_r f(r)|^2\, r^{N-1}\, dr, \quad N \ge 3, \\ &\int_0^\infty \frac{|f(r)|^2}{r^2}\, (1+2\sigma r) e^{-\sigma r}\, dr \le 4 \int_0^\infty |\partial_r f(r)|^2\, (1+2\sigma r) e^{-\sigma r}\, dr, \\ &\qquad \sigma \ge 0,\ f(0) = 0. \end{aligned} \tag{2.1}$$

Thus $\partial_r f \in L^2(\mathbb{R}_+; r^{N-1}\, dr)$ yields that $r^{-1} f \in L^2(\mathbb{R}_+; r^{N-1}\, dr)$ if $N \ge 3$.

It is remarkable that if V satisfies (1.4) with $\delta_V \ge 0$, then

$$\begin{aligned} c_1 \, \|(1-\Delta)^{1/2} u\|^2_{L^2(\mathbb{R}^N)} &\le \|(1+P)^{1/2} u\|^2_{L^2(\mathbb{R}^N)} \\ &\le c_2 \, \|(1-\Delta)^{1/2} u\|^2_{L^2(\mathbb{R}^N)}, \end{aligned} \tag{2.2}$$

$$c_1 := \frac{\delta_V}{\delta_V + \|V_-\|_{L^\infty(S^{N-1})}} \ge 0, \quad c_2 := 1 + \frac{4\|V_+\|_{L^\infty(S^{N-1})}}{(N-2)^2} > 0,$$

where $V_\pm(x) := (\pm V(x)) \vee 0$ (double-sign corresponds); see [12, Proposition 1.3]. This concludes the nonnegativity and selfadjointness of P in the sense of form-sum. Moreover, if $\delta_V > 0$, then we find $D((1+P)^{1/2}) = H^1(\mathbb{R}^N)$.

Now we define the projections on $L^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}_+; r^{N-1}\, dr)$ as

$$\begin{aligned} U_{m,j} f(x) &:= Y_{m,j}(x/|x|)(T_{m,j} f)(|x|), \\ (T_{m,j} f)(r) &:= \int_{S^{N-1}} f(r\widehat{x})\overline{Y_{m,j}(\widehat{x})}\, d\sigma_{S^{N-1}}. \end{aligned}$$

The subclass of $L^2(\mathbb{R}^N)$ related to $P = -\Delta + V$ can be considered as

$$L^2_{=m}(\mathbb{R}^N) := \Big\{ \sum_j f_{m,j}(|x|) Y_{m,j}(x/|x|);\ f_{m,j} \in L^2(\mathbb{R}_+; r^{N-1}\, dr) \Big\}.$$

Especially, since the first eigenvalue is simple ($\dim \mathcal{Y}_1 = 1$), we see

$$L^2_{=1}(\mathbb{R}^N) = \Big\{ f_{1,1}(|x|) Y_{1,1}(x/|x|);\ f_{1,1} \in L^2(\mathbb{R}_+; r^{N-1}\, dr) \Big\}.$$

Thus $L^2_{=1}(\mathbb{R}^N)$ is isomorphic to $L^2_{\mathrm{rad}}(\mathbb{R}^N)$, the family of radially symmetric functions of $L^2(\mathbb{R}^N)$. It follows from Lemma 2.1 that $L^2(\mathbb{R}^N) = \bigoplus_{m=1}^\infty L^2_{=m}(\mathbb{R}^N)$ and

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$$\|f\|^2_{L^2(\mathbb{R}^N)} = \sum_{m=1}^{\infty}\sum_j \|U_{m,j}f\|^2_{L^2(\mathbb{R}^N)} = \sum_{m=1}^{\infty}\sum_j \|T_{m,j}f\|^2_{L^2(\mathbb{R}_+;r^{N-1}\,dr)}.$$

Thus the *radial-spherical decomposition* of f respect to P is described as

$$f(x) = \sum_{m=1}^{\infty}\sum_j T_{m,j}f(|x|)\,Y_{m,j}(x/|x|) = \sum_{m,j} f_{m,j}(|x|)Y_{m,j}(x/|x|).$$

Let $f = \sum_{m,j} f_{m,j}Y_{m,j}$ and $g = \sum_{m,j} g_{m,j}Y_{m,j}$ be the radial-spherical decompositions of f and g respect to P. Then we see

$$\int_{\mathbb{R}^N} f(x)\overline{g(x)}\,dx = \sum_{m=1}^{\infty}\sum_j \int_0^{\infty} f_{m,j}(r)\overline{g_{m,j}(r)}\,r^{N-1}\,dr.$$

On the one hand, P maps from $D(P)\cap L^2_{=m}(\mathbb{R}^N)$ to $L^2_{=m}(\mathbb{R}^N)$. In fact, if $f \in D(P)\cap L^2_{=m}(\mathbb{R}^N)$, then P can be represented by the operators which consists only of multiplication and differentiation by r:

$$Pf = A_{\nu_m}f := \Big(-\partial_r^2 - \frac{N-1}{r}\partial_r + \frac{\nu_m^2 - (N-2)^2/4}{r^2}\Big)f,$$

where $\nu_m := \sqrt{\lambda_m + (N-2)^2/4} \geq 0$. We see $\nu_1 = 0$ if (1.4) with $\delta_V = 0$. We denote χ as the projection on $L^2_{=1}(\mathbb{R}^N)$: $\chi f = U_{1,1}f$, and define $\chi^\perp := 1-\chi$. Note that χ and $\chi^\perp$ are commutative to P and $|x|$. For example, $\chi\Phi(P) = \Phi(P)\chi$ and $\chi^\perp\Phi(|x|) = \Phi(|x|)\chi^\perp$ are completed.

3. Evaluation of energy spaces

We prove $(H^1(\mathbb{R}^N)\subset)D((1+P)^{1/2}) \subset H^s(\mathbb{R}^N)$ $(s<1)$ by following Suzuki [48] which is considered the typical case $V(x) = -[(N-2)^2/4]\,|x|^{-2}$. First we introduce the *Mellin transform*:

$$\mathcal{M}[f(r)](z) := \int_0^{\infty} r^{z-1}f(r)\,dr.$$

Note that

$$\mathcal{M}[r^\alpha f(r)](z) = \mathcal{M}[f(r)](z+\alpha), \qquad \alpha\in\mathbb{R},$$
$$\mathcal{M}[\partial_r f(r)](z) = -(z-1)\mathcal{M}[f(r)](z-1), \qquad \operatorname{Re} z > 0.$$

Applying the Mellin transform to $A_\nu f$ $(\nu \geq 0)$ for $f \in L^2(\mathbb{R}_+; r^{N-1}\,dr)$, we see

$$\mathcal{M}[A_\nu f](z) = -(z-\kappa-2+\nu)(z-\kappa-2-\nu)\mathcal{M}[f](z-2),$$

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where

$$\kappa := \frac{N-2}{2}.$$

The linear operator A_ν has another representation. To carry out this we consider the *Hankel transform of order* ν of f:

$$(\mathcal{H}_\nu f)(r) := \int_0^\infty (r\rho)^{-\kappa} J_\nu(r\rho) f(\rho)\rho^{N-1} d\rho,$$

where J_ν is the Bessel functions of the first kind of order ν. Using the Hankel transforms, we see $A_\nu = \mathcal{H}_\nu r^2 \mathcal{H}_\nu$. In fact, the Mellin transform of $A_\nu^{s/2} f = \mathcal{H}_\nu r^s \mathcal{H}_\nu f$ ($\nu \geq 0$) is described as

$$\begin{aligned} &\mathcal{M}[A_\nu^{s/2} f](z) \\ &= 2^s \frac{\Gamma\big((z-\kappa+\nu)/2\big)\Gamma\big(1-(z-s-\kappa-\nu)/2\big)}{\Gamma\big((z-s-\kappa+\nu)/2\big)\Gamma\big(1-(z-\kappa-\nu)/2\big)} \mathcal{M}[f](z-s), \end{aligned} \tag{3.1}$$

where Γ is the Gamma function. The Mellin transform plays similar as the Laplace and Fourier transforms.

Lemma 3.1 *([49, Lemma 2.5]).* (i) *It holds the Plancherel type equality for the Mellin transform*:

$$\int_0^\infty f(s)\overline{g(s)} s^{N-1} ds = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{M}[f]\Big(\frac{N}{2}+iy\Big)\overline{\mathcal{M}[g]\Big(\frac{N}{2}+iy\Big)}\, dy. \tag{3.2}$$

(ii) *Assume that the linear operator B in $L^2(\mathbb{R}_+; r^{N-1}\, dr)$ satisfies*

$$\mathcal{M}[Bf](z) = F(z)\mathcal{M}[f](z), \quad C_F := \sup_y \Big|F\Big(\frac{N}{2}+iy\Big)\Big| < +\infty.$$

Then B is bounded. Moreover,

$$\|Bf\|_{L^2(\mathbb{R}_+; r^{N-1}\, dr)} \leq C_F \|f\|_{L^2(\mathbb{R}_+; r^{N-1}\, dr)}.$$

Next lemma is the evaluation of maximum of the products of the Gamma functions.

Lemma 3.2 *([49, Lemma 2.2]). Let $\alpha, \beta \in C([0,\infty); \mathbb{R}) \cap C^1((0,\infty); \mathbb{R})$ and $c > 0$ such that*

$$\alpha(x) - c > \beta(x) > 0 \quad \forall x \geq 0; \qquad 0 \leq \alpha'(x) \leq \beta'(x) \quad \forall x > 0.$$

Define the function γ as

$$\gamma(x) := \frac{\Gamma(\alpha(x))\,\Gamma(\beta(x))}{\Gamma(\alpha(x)-c)\,\Gamma(\beta(x)+c)}.$$

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Then γ is decreasing in $x \ge 0$ and hence $\gamma(x) \le \gamma(0)$ for $x \ge 0$.

Now we verify the evaluation of the energy space, especially, we prove Theorem 1.4: $D((1+P)^{1/2}) \subset H^s(\mathbb{R}^N)$ ($s<1$), in a way similar to [49, Theorem 3.2] (see also Yafaev [54]).

Proof of Theorem 1.4. We divide the proof into 2 steps.
Step 1. First we show that for any $a>0$

$$\begin{aligned}&\|(P+a|x|^{-2})^{s/2}u\|_{L^2(\mathbb{R}^N)}\\ &\le \frac{\Gamma((1+s+\sqrt{a})/2)\,\Gamma((1-s)/2)}{\Gamma((1-s+\sqrt{a})/2)\,\Gamma\big((1+s)/2\big)}\,\|P^{s/2}u\|_{L^2(\mathbb{R}^N)}.\end{aligned} \tag{3.3}$$

Set the indexes

$$\nu := \nu_m = \sqrt{\lambda_m + \frac{(N-2)^2}{4}},\ \mu := \mu_m = \sqrt{a+\lambda_m + \frac{(N-2)^2}{4}},$$

where λ_m is the m-th eigenvalue of $-\Delta_S + V$. Note that $\mu > \nu \ge 0$. It is important that if Y_m is an eigenfunction of $-\Delta_S + V$ corresponding to the value λ_m, then Y_m is also that of $-\Delta_S + V + a$ corresponding to $\lambda_m + a$. Thus the radial-spherical decompositions respect to $P = -\Delta + V$ and ones respect to $P + a|x|^{-2}$ are exactly same. We consider the operator $A_\mu^{s/2} A_\nu^{-s/2}$. The Mellin transform of $A_\mu^{s/2} A_\nu^{-s/2} f$ ($f \in L^2(\mathbb{R}_+; r^{N-1}dr)$) is $\mathcal{M}[A_\mu^{s/2} A_\nu^{-s/2} f](z) = F_m(z)\mathcal{M}[f](z)$, where

$$\begin{aligned}F_m(z) &:= \frac{\Gamma\big((z-\kappa+\mu)/2\big)\Gamma\big(1-(z-s-\kappa-\mu)/2\big)}{\Gamma\big((z-s-\kappa+\mu)/2\big)\Gamma\big(1-(z-\kappa-\mu)/2\big)}\\ &\quad\times \frac{\Gamma\big((z-s-\kappa+\nu)/2\big)\Gamma\big(1-(z-\kappa-\nu)/2\big)}{\Gamma\big((z-\kappa+\nu)/2\big)\Gamma\big(1-(z-s-\kappa-\nu)/2\big)}.\end{aligned}$$

We evaluate $|F_m((N/2)+iy)|$ to apply Lemma 3.1. It follows from the Euler limit formula of the Gamma function that

$$|F_m(N/2+iy)| = \frac{\Gamma\big((1+s+\mu)/2\big)\Gamma\big((1-s+\nu)/2\big)}{\Gamma\big((1-s+\mu)/2\big)\Gamma\big((1+s+\nu)/2\big)}\Big[\prod_{k=0}^{\infty} R_k(y)\Big]^{1/2},$$

where $M_k := 1+\mu+2k$, $N_k := 1+\nu+2k$, and

$$\begin{aligned}R_k(y) &= \frac{\big(1+y^2/(N_k+s)^2\big)\big(1+y^2/(M_k-s)^2\big)}{\big(1+y^2/(N_k-s)^2\big)\big(1+y^2/(M_k+s)^2\big)}\\ &= 1 - y^2\frac{4s[N_k/(N_k^2-s^2)^2 - M_k/(M_k^2-s^2)^2]}{\big(1+y^2/(N_k-s)^2\big)\big(1+y^2/(M_k+s)^2\big)}\\ &\quad - y^4\frac{4s(M_k-N_k)(N_kM_k-s^2)}{(M_k^2-s^2)^2(N_k^2-s^2)^2\big(1+y^2/(N_k-s)^2\big)\big(1+y^2/(M_k+s)^2\big)}\end{aligned}$$

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$$\leq 1 \quad \forall\, y \in \mathbb{R}.$$

Note that $\varphi(t) := t/(t^2 - s^2)^2$ is decreasing in $t > s$ if $M_k > N_k$ ($\mu > \nu$). Thus we have

$$\left| F_m\Big(\frac{N}{2} + iy\Big)\right| \leq \frac{\Gamma\big((1+s+\mu)/2\big)\Gamma\big((1-s+\nu)/2\big)}{\Gamma\big((1-s+\mu)/2\big)\Gamma\big((1+s+\nu)/2\big)} =: c_m \quad \forall\, y \in \mathbb{R}$$

and hence it follows from Lemma 3.1 (ii) that

$$\| A_\mu^{s/2} f\|_{L^2(\mathbb{R}_+; r^{N-1}\,dr)} \leq c_m \, \| A_\nu^{s/2} f\|_{L^2(\mathbb{R}_+; r^{N-1}\,dr)}.$$

Apply Lemma 3.2 with $\alpha(x) := (\sqrt{x+a} + 1 + s)/2$, $\beta(x) := (\sqrt{x} + 1 - s)/2$, and $c := s > 0$. Since $c_m = \gamma(\lambda_m + \kappa^2)$, we see

$$c_m \leq \gamma(\lambda_1 + \kappa^2) = \gamma(0) = \frac{\Gamma\big((1+s+\sqrt{a})/2\big)\Gamma\big((1-s)/2\big)}{\Gamma\big((1-s+\sqrt{a})/2\big)\Gamma\big((1+s)/2\big)}.$$

Let $u = \sum_{m,j} u_{m,j} Y_{m,j}$ be a radial-spherical decomposition respect to P (and $P + a|x|^{-2}$). Then we have

$$\begin{aligned}
\|(P + a|x|^{-2})^{s/2} u\|^2_{L^2(\mathbb{R}^N)} &= \sum_{m,j} \| A_\mu^{s/2} u_{m,j}\|^2_{L^2(\mathbb{R}_+; r^{N-1}\,dr)} \\
&\leq \sum_{m,j} c_m^2 \, \| A_\nu^{s/2} u_{m,j}\|^2_{L^2(\mathbb{R}_+; r^{N-1}\,dr)} \\
&\leq \sum_{m,j} c_1^2 \, \| A_\nu^{s/2} u_{m,j}\|^2_{L^2(\mathbb{R}_+; r^{N-1}\,dr)} \\
&= c_1^2 \, \| P^{s/2} u\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

This is nothing but (3.3).

Step 2. Set $a > 0$ so that $a \geq -\min\{V(x); x \in S^{N-1}\}$. Then $V(x) + a|x|^{-2} \geq 0$ a.a. $x \in \mathbb{R}^N$. Thus we see

$$\|(-\Delta)^{1/2} u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2\,dx$$

$$\leq \int_{\mathbb{R}^N} [|\nabla u|^2 + (V + a|x|^{-2})|u|^2]\,dx = \|(P + a|x|^{-2})^{1/2} u\|^2_{L^2(\mathbb{R}^N)}.$$

The Heinz-Kato inequality [26, Theorem 2] yields

$$\|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)} \leq \|(P + a|x|^{-2})^{s/2} u\|_{L^2(\mathbb{R}^N)} \quad \forall\, s \in [0,1].$$

Combining (3.3) into this we conclude (1.5). $\square$

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Remark 3.3. Let $N \geq 2$ and $s \in [0, 1)$. Assume that V satisfies (1.4) with $\delta_V = 0$. In a way similar to Theorem 1.4, we can show the Hardy type inequality respect to P.

$$\big\||x|^{-s}u\big\|_{L^2(\mathbb{R}^N)} \leq \frac{\Gamma((1-s)/2)}{2^s\,\Gamma((1+s)/2)}\|P^{s/2}u\|_{L^2(\mathbb{R}^N)}. \tag{3.4}$$

See e.g., [48, Theorem 3.1].

Remark 3.4. Let $N \geq 2$ and $s \in [0, 1]$. Assume that V satisfies (1.4) with $\delta_V > 0$. In a way similar to Theorem 1.4, we can prove the following inequalities.

$$\begin{aligned}
&\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^N)} \leq C_{s,V}\,\|P^{s/2}u\|_{L^2(\mathbb{R}^N)},\\
&\qquad C_{s,V} := \frac{\Gamma\big((1+s+\sqrt{a+\delta_V})/2\big)\Gamma\big((1-s+\sqrt{\delta_V})/2\big)}{\Gamma\big((1-s+\sqrt{a+\delta_V})/2\big)\Gamma\big((1+s+\sqrt{\delta_V})/2\big)};\\
&\big\||x|^{-s}u\big\|_{L^2(\mathbb{R}^N)} \leq \frac{\Gamma\big((1-s+\sqrt{\delta_V})/2\big)}{2^s\,\Gamma\big((1+s+\sqrt{\delta_V})/2\big)}\|P^{s/2}u\|_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Putting $s := 1$ and $a := \|V_-\|_{L^\infty(S^{N-1})}$ in the above first inequality, we get justly the first inequality in (2.2).

The key tools in the proof of Theorem 1.4 are the radial-spherical decompositions and the Mellin transforms. We can also apply the above technics to the following Cauchy-Schwarz type inequality (see e.g., [49]). These inequalities are the key ingredients for showing the continuity of the weighted energy spaces for **(NLS)** (see also Section 5).

Proposition 3.5. *Let $N \geq 2$ and φ be a real-valued and radially symmetric function. Assume that V satisfies (1.4) with $\delta_V \geq 0$. Then*

$$\Big|\int_{\mathbb{R}^N}\varphi\,\overline{u}\,x\cdot\nabla v\,dx\Big| \leq \|x\,\varphi\,u\|_{L^2(\mathbb{R}^N)}\,\|P^{1/2}v\|_{L^2(\mathbb{R}^N)} + \frac{N}{2}\Big|\int_{\mathbb{R}^N}\varphi\,\overline{u}\,v\,dx\Big|. \tag{3.5}$$

Especially,

$$\Big|\mathrm{Im}\int_{\mathbb{R}^N}\varphi\,\overline{u}\,x\cdot\nabla u\,dx\Big| \leq \|P^{1/2}u\|_{L^2(\mathbb{R}^N)}\,\|x\,\varphi\,u\|_{L^2(\mathbb{R}^N)}. \tag{3.6}$$

Proof. Let $u = \sum_{m,j}u_{m,j}Y_{m,j}$ and $v = \sum_{m,j}v_{m,j}Y_{m,j}$ be radial-spherical decompositions respect to P. Then we see that

$$\int_{\mathbb{R}^N}\varphi\,\overline{u}\,x\cdot\nabla v\,dx = \sum_{m,j}\int_0^\infty\varphi\,\overline{u_{m,j}(r)}\,r\,(\partial_r v_{m,j})(r)\,r^{N-1}\,dr.$$

By using (3.2), we obtain that

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$$\begin{aligned}
&\sum_{m,j}\int_0^\infty \overline{r\varphi u_{m,j}(r)}\,(\partial_r v_{m,j})(r)\,r^{N-1}\,dr\\
&=\sum_{m,j}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{M}[\partial_r v_{m,j}]\Big(\frac{N}{2}+iy\Big)\overline{\mathcal{M}[r\varphi u_{m,j}]\Big(\frac{N}{2}+iy\Big)}\,dy\\
&=\frac{N}{2}\sum_{m,j}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{M}[v_{m,j}]\Big(\frac{N}{2}-1+iy\Big)\overline{\mathcal{M}[\varphi u_{m,j}]\Big(\frac{N}{2}+1+iy\Big)}\,dy\\
&+i\sum_{m,j}\frac{1}{2\pi}\int_{-\infty}^{\infty}y\mathcal{M}[v_{m,j}]\Big(\frac{N}{2}-1+iy\Big)\overline{\mathcal{M}[\varphi u_{m,j}]\Big(\frac{N}{2}+1+iy\Big)}\,dy\\
&=:\frac{N}{2}I_1+iI_2.
\end{aligned}$$

Here I_1 is calculated by (3.2) as follows:

$$\begin{aligned}
I_1&=\sum_{m,j}\int_0^\infty r^{-1}v_{m,j}\overline{r\varphi u_{m,j}}\,r^{N-1}\,dr=\sum_{m,j}\int_0^\infty \varphi\,\overline{u_{m,j}}\,v_{m,j}\,r^{N-1}\,dr\\
&=\int_{\mathbb{R}^N}\varphi\,\overline{u}\,v\,dx.
\end{aligned}$$

On the other hand, we see from (3.1) that

$$\begin{aligned}
&\mathcal{M}[A_\nu^{1/2}f]\Big(\frac{N}{2}+iy\Big)\\
&=(\nu-iy)\frac{\Gamma\big((\nu-iy)/2\big)\Gamma\big((\nu+1+iy)/2\big)}{\Gamma\big((\nu+iy)/2\big)\Gamma\big((\nu+1-iy)/2\big)}\mathcal{M}[f]\Big(\frac{N}{2}-1+iy\Big).
\end{aligned}$$

Applying (3.2), we obtain

$$\int_0^\infty |A_\nu^{1/2}f|^2\,r^{N-1}\,dr=\frac{1}{2\pi}\int_{-\infty}^{\infty}(\nu^2+y^2)\Big|\mathcal{M}[f]\Big(\frac{N}{2}-1+iy\Big)\Big|^2\,dy. \tag{3.7}$$

Set $\nu=\nu_m$. Applying the Cauchy–Schwarz inequality and (3.7), we calculate

$$\begin{aligned}
|I_2|&\le\Big[\sum_{m,j}\int_0^\infty |A_{\nu_m}^{1/2}v_{m,j}|^2\,r^{N-1}\,dr\Big]^{1/2}\Big[\sum_{m,j}\int_0^\infty |r\,\varphi\,u_{m,j}|^2\,r^{N-1}\,dr\Big]^{1/2}\\
&=\|P^{1/2}v\|_{L^2(\mathbb{R}^N)}\,\|x\,\varphi\,u\|_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

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Therefore we have (3.5). If $v = u$, then $I_1 \in \mathbb{R}$, and hence $\operatorname{Im} I_1 = 0$. Thus (3.6) is also proved. $\square$

4. Strichartz estimates

We need to consider the Lorentz spaces to derive the Strichartz estimates for $\exp(-itP)$. The definition of Lorentz spaces is fully written in Bergh-Löfström [9], Triebel [52], and others. Let $f \in L^1_{\text{loc}}(\Omega)$ $(\Omega \subset \mathbb{R}^N)$. Then the distribution function of f is defined as

$$m(f;s) := \mu(\{x \in \Omega; |f(x)| > s\}),$$

where $\mu(A)$ is the Lebesgue measure of $A \subset \Omega$. Since $m(f;s)$ is decreasing in s, we can consider the decreasing rearrangement of f as

$$f^*(t) := \inf\{s; m(f,s) \le t\}.$$

Now Lorentz spaces $L^{p,q}(\Omega)$ $(p, q \in [1,\infty])$ is specified the family of f such that the following quasi-norm $|f|_{L^{p,q}(\Omega)}$ is finite:

$$|f|_{L^{p,q}(\Omega)} := \begin{cases} \left(\displaystyle\int_0^\infty [t^{1/p} f^*(t)]^q \, t^{-1} dt\right)^{1/q} & 1 \le q < \infty, \\ \displaystyle\sup_{t>0} t^{1/p} f^*(t) & q = \infty. \end{cases}$$

Also Lorentz spaces $L^{p,q}(\Omega)$ is characterized in the real interpolation between the usual Lebesgue spaces:

$$L^{p,q}(\Omega) = \big(L^{p_0}(\Omega), L^{p_1}(\Omega)\big)_{\theta,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ 0 < \theta < 1.$$

Typical example belonging to Lorentz spaces is $|x|^{-1} \in L^{N,\infty}(\mathbb{R}^N)$ $(N \ge 2)$. $L^{p,\infty}(\Omega)$ is also called *L^p-type weak Lebesgue space*. Note that $L^{p,p}(\Omega) = L^p(\Omega)$ and the continuous inclusions $L^{p,1}(\Omega) \subset L^{p,q_1}(\Omega) \subset L^{p,q_2}(\Omega) \subset L^{p,\infty}(\Omega)$ for $1 \le q_1 \le q_2 \le \infty$. If $p, q \in [1,\infty)$, then the dual space of $L^{p,q}(\Omega)$ is $L^{p',q'}(\Omega)$ (p' is the Hölder conjugate of p: $p' = p/(p-1)$).

Note that $|f|_{L^{p,p}(\Omega)} = \|f\|_{L^p(\Omega)}$. This concludes $L^{p,p}(\Omega) = L^p(\Omega)$ $(1 \le p \le \infty)$. Define the average function of f as

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s)\, ds.$$

Then the following is an equivalent norm of $|\cdot|_{L^{p,q}(\Omega)}$ if $p \in (1,\infty)$

$$\|f\|_{L^{p,q}(\Omega)} := \begin{cases} \left(\displaystyle\int_0^\infty [t^{1/p} f^{**}(t)]^q \, t^{-1} dt\right)^{1/q} & 1 \le q < \infty, \\ \displaystyle\sup_{t>0} t^{1/p} f^{**}(t) & q = \infty. \end{cases}$$

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In fact, the norm equivalence is verified:

$$|f|_{L^{p,q}(\Omega)} \leq \|f\|_{L^{p,q}(\Omega)} \leq \frac{p}{p-1}|f|_{L^{p,q}(\Omega)}$$

Here $L^{p,q}(\Omega)$ ($p \in (1,\infty)$) is a Banach space with the above norm $\|\cdot\|_{L^{p,q}(\Omega)}$.

Now we use the following Hölder inequality in Lorentz spaces (see e.g., O'Neil [41]).

Lemma 4.1. *Let $f \in L^{p_1,q_1}(\Omega)$ and $g \in L^{p_2,q_2}(\Omega)$. Assume that $p \in (1,\infty)$, $q \in [1,\infty]$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then $fg \in L^{p,q}(\Omega)$. Moreover, the Hölder inequality in Lorentz space is fulfilled:

$$\|fg\|_{L^{p,q}(\Omega)} \leq 3p' \|f\|_{L^{p_1,q_1}(\Omega)} \|g\|_{L^{p_2,q_2}(\Omega)}. \tag{4.1}$$

By definition of the Lorentz norms, we see

$$\|fg\|_{L^{p,q}(\Omega)} \leq \|f\|_{L^{\infty}(\Omega)} \|g\|_{L^{p,q}(\Omega)}.$$

Moreover, if $u \in L^{p,q}(\Omega)$, then $\left||u|^\alpha\right|_{L^{p/\alpha,q/\alpha}(\Omega)} = |u|^\alpha_{L^{p,q}(\Omega)}$ for $0 < \alpha < (p \wedge q)$. Thus we see

$$\left\||u|^\alpha\right\|_{L^{p/\alpha,q/\alpha}(\Omega)} \leq C\,\|u\|^\alpha_{L^{p,q}(\Omega)} \quad (1 < p < \infty).$$

In connection with Lorentz spaces, we can consider the real interpolation between the space-time (vector-valued) Lebesgue-Lorentz spaces (see [52, Theorem 1.18.4] and [9, Theorem 5.3.1]):

$$\left(L^{p_0}(I; L^{q_0,r_0}(\Omega)), L^{p_1}(I; L^{q_1,r_1}(\Omega))\right)_{\theta,p} = L^p(I; L^{q,p}(\Omega)),$$

where $p_0,\ p_1 \in [1,\infty)$, $q_0,\ q_1 \in [1,\infty]$, $r_0,\ r_1 \in [1,\infty]$, $\theta \in (0,1)$ and

$$p_0 \neq p_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

As notion in the introduction, [12, Section 3.2.1] and [42, Theorem 2] establish the Strichartz estimates for the super-critical case for V.

Proposition 4.2. *Let $N \geq 3$. Assume that V satisfies (1.4) with $\delta_V > 0$. Then for any admissible pairs (τ_j, ρ_j) $(j = 0, 1, 2)$ one has*

$$\|e^{-itP}\varphi\|_{L^{\tau_0}(\mathbb{R}; L^{\rho_0,2}(\mathbb{R}^N))} \leq C_{\tau_0}(V)\,\|\varphi\|_{L^2(\mathbb{R}^N)}, \tag{4.2}$$

$$\left\|\int_0^t e^{-i(t-s)P} F(s)\,ds\right\|_{L^{\tau_2}(\mathbb{R}; L^{\rho_2,2}(\mathbb{R}^N))} \leq C_{\tau_1,\tau_2}(V)\,\|F\|_{L^{\tau_1'}(\mathbb{R}; L^{\rho_1',2}(\mathbb{R}^N))}. \tag{4.3}$$

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We show the following estimates. These ensure the Strichartz estimates for $\exp(-itP)$ (Theorem 1.5).

Proposition 4.3. *Let $N \geq 3$. Assume that V satisfies (1.4) with $\delta_V = 0$. Then for any admissible pairs (τ_j, ρ_j) $(j = 0, 1, 2)$*

$$\|\chi^{\perp} e^{-itP}\varphi\|_{L^{\tau_0}(\mathbb{R}; L^{\rho_0,2}(\mathbb{R}^N))} \leq C_{\tau_0}(V)\, \|\varphi\|_{L^2(\mathbb{R}^N)}, \tag{4.4}$$

$$\Big\| \int_0^t \chi^{\perp} e^{-i(t-s)P} F(s)\, ds \Big\|_{L^{\tau_2}(\mathbb{R}; L^{\rho_2,2}(\mathbb{R}^N))} \leq C_{\tau_1,\tau_2}(V)\, \|F\|_{L^{\tau_1'}(\mathbb{R}; L^{\rho_1',2}(\mathbb{R}^N))}, \tag{4.5}$$

$$\|\chi e^{-itP}\varphi\|_{L^2(\mathbb{R}; L^{2N/(N-2),\infty}(\mathbb{R}^N))} \leq C_2(V)\, \|\varphi\|_{L^2(\mathbb{R}^N)}. \tag{4.6}$$

Here χ is the projection onto $L^2_{=1}(\mathbb{R}^N)$ and $\chi^{\perp} = 1 - \chi$ (see the last part of Section 2).

Local smoothing estimates produced by Kato [27] is the key ingredient of the confirming the Strichartz estimates. Set X and Y be (complex) Hilbert spaces. Let S be a self-adjoint operator on X. Suppose that A is a closed and densely defined operator from $D(A) \subset X$ to Y. Then A is called *S-smooth* if the following ingredient is finite:

$$\begin{aligned}
&\|A\|^2_{\mathrm{sm}(S)} \\
&:= \sup\Big\{ \frac{1}{4\pi^2} \int_{-\infty}^{\infty} [\|A(S-\lambda+i\varepsilon)^{-1}u\|_Y^2 + \|A(S-\lambda-i\varepsilon)^{-1}u\|_Y^2]\, d\lambda; \\
&\qquad \varepsilon > 0, u \in X \text{ with } \|u\|_X = 1 \Big\}.
\end{aligned}$$

If the operator $A : D(A) \subset X \to Y$ satisfies

$$\|A(S-\lambda \pm i\varepsilon)^{-1} A^* u\|_Y \leq C_A \|u\|_Y \tag{4.7}$$

($A^* : D(A^*) \subset Y \to X$ is the adjoint operator of A). Then A is S-smooth and $\|A\|^2_{\mathrm{sm}(S)} \leq C_A/\pi$ (see [27, Theorem 5.1] or Corollary to [43, Theorem XIII.25]). Moreover,

$$\|Ae^{-itS}\varphi\|_{L^2(\mathbb{R};Y)} \leq \sqrt{2\pi \|A\|^2_{\mathrm{sm}(S)}}\, \|\varphi\|_X \leq \sqrt{2C_A}\, \|\varphi\|_X. \tag{4.8}$$

Furthermore, applying the Fourier and Laplace transforms to (4.7), we see that

$$\Big\| \int_0^t Ae^{-i(t-\tau)S} A^* F(\tau)\, d\tau \Big\|_{L^2(\mathbb{R};Y)} \leq C_A \|F\|_{L^2(\mathbb{R};Y)}. \tag{4.9}$$

See Mochizuki [36, Proposition 3] and D'Ancona [19, Theorem 2.3] for details.

To begin with the Strichartz estimates, we consider the evaluation like (4.7).

Lemma 4.4. *Let V satisfy (1.4) with $\delta_V = 0$. Then there exists $C > 0$ such that*

$$\sup_{\zeta \in \mathbb{C}\setminus\mathbb{R}} \|(\chi^{\perp}|x|^{-1})(P+\zeta)^{-1}(|x|^{-1}\chi^{\perp})f\|_{L^2(\mathbb{R}^N)} \le C\,\|f\|_{L^2(\mathbb{R}^N)}. \tag{4.10}$$

This concludes $\chi^{\perp}|x|^{-1}$ is P-smooth. Here $\chi^{\perp}$ is defined in Section 2.

Proof. Step 1. In a view of [12, Theorem 2.1], we have for $\nu > 0$

$$\|r^{-1}(A_\nu + z^2)^{-1}r^{-1}f\|_{L^2(0,\infty;r^{N-1}\,dr)} \le C(\nu)\|f\|_{L^2(0,\infty;r^{N-1}\,dr)}. \tag{4.11}$$

The verifications are strictly carried out in [12], however, the coefficient seems not to be suitable (see Remark 4.6). In fact, we see $C(\nu) \le (\nu^{-2} \vee \sqrt{2}\nu^{-1})$. We consider the same as in [12] to derive the sufficient coefficient. Let $u \in L^2(\mathbb{R}_+; r^{N-1}\,dr)$ be a solution to $A_\nu u + z^2 u = f$. Putting $v(r) = u(r)r^{(N-1)/2}e^{zr}$, we see

$$-v''(r) + 2zv'(r) + \frac{4\nu^2-1}{4r^2}v(r) = r^{(N-1)/2}e^{zr}f(r).$$

Multiply $re^{-2\sigma r}\overline{v'(r)}$ ($\sigma := \operatorname{Re} z > 0$), integrate by parts, and apply the Cauchy-Schwarz inequality. Thus we have

$$\begin{aligned}
&\int_0^\infty \frac{1+2\sigma r}{2}e^{-2\sigma r}|v'(r)|^2\,dr + \int_0^\infty \frac{1+2\sigma r}{2}e^{-2\sigma r}\frac{4\nu^2-1}{4r^2}|v(r)|^2\,dr \\
&= \operatorname{Re}\int_0^\infty e^{(z-2\sigma)r}\overline{v'(r)}\,rf(r)r^{(N-1)/2}\,dr \\
&\le \frac{\delta}{2}\int_0^\infty e^{-2\sigma r}|v'(r)|^2\,dr + \frac{1}{2\delta}\int_0^\infty |rf(r)|^2 r^{N-1}\,dr.
\end{aligned}$$

Transposing the first term of the right hand side, we obtain

$$\begin{aligned}
&\frac{1-\delta}{2}\int_0^\infty (1+2\sigma r)e^{-2\sigma r}|v'(r)|^2\,dr + \frac{4\nu^2-1}{8}\int_0^\infty (1+2\sigma r)e^{-2\sigma r}\frac{|v(r)|^2}{r^2}\,dr \\
&\le \frac{1}{2\delta}\int_0^\infty |rf(r)|^2 r^{N-1}\,dr.
\end{aligned}$$

Applying the weighed Hardy inequality ([12, Lemma 2.2]; see (2.1)), we have

$$\int_0^\infty (1+2\sigma r)e^{-2\sigma r}\frac{|v(r)|^2}{r^2}\,dr \le \frac{4}{\delta(4\nu^2-\delta)}\int_0^\infty |rf(r)|^2 r^{N-1}\,dr.$$

Optimizing $\delta \in (0, 1]$ with $\delta < 4\nu^2$, we conclude that

$$\int_0^\infty \frac{|v(r)|^2}{r^2}\,dr \le C_1(\nu)\int_0^\infty |rf(r)|^2 r^{N-1}\,dr,$$

where $C_1(\nu) := \nu^{-4}$ if $\nu \le 1/\sqrt{2}$ and $C_1(\nu) := 4/(4\nu^2-1)$ if $\nu > 1/\sqrt{2}$. Note that $C_1(\nu) \le \nu^{-4} \vee 2\nu^{-2}$. (4.11) follows from $|v(r)|^2 = r^{N-1}|u(r)|^2 e^{2\sigma r} \ge r^{N-1}|u(r)|^2$.

Step 2. Note that $\nu_1 = 0$. Sum (4.11) over from $m = 2$ to ∞.

$$\begin{aligned}
&\|(\chi^\perp |x|^{-1})(P+z^2)^{-1}(|x|^{-1}\chi^\perp)f\|^2_{L^2(\mathbb{R}^N)}\\
&= \sum_{m=2}^\infty \sum_k \|r^{-1}(A_{\nu_m}+z^2)^{-1}r^{-1}f_{m,k}\|^2_{L^2(\mathbb{R}_+;r^{N-1}\,dr)}\\
&\le \sum_{m=2}^\infty \sum_k \Big(\frac{1}{\nu_m^4} \vee \frac{2}{\nu_m^2}\Big)\|f_{m,k}\|^2_{L^2(\mathbb{R}_+;r^{N-1}\,dr)}\\
&\le \Big(\frac{1}{\nu_2^4} \vee \frac{2}{\nu_2^2}\Big)\sum_{m=2}^\infty \sum_k \|f_{m,k}\|^2_{L^2(\mathbb{R}_+;r^{N-1}\,dr)}\\
&\le \Big(\frac{1}{\nu_2^4} \vee \frac{2}{\nu_2^2}\Big)\|f\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

This is nothing but (4.10). Note that we can select C in (4.10) as $(\nu_2^{-2}) \vee (\sqrt{2}\nu_2^{-1})$. □

Remark 4.5. We can explicitly denote the operator $(A_\nu + z^2)^{-1}$ ($\operatorname{Re} z > 0$) as

$$(A_\nu + z^2)^{-1}f(r) = \int_0^\infty g_\nu(r,s)f(s)s^{N-1}\,ds,$$

$$g_\nu(r,s) := (rs)^{-(N-2)/2}\begin{cases} I_\nu(zr)K_\nu(zs) & r < s,\\ K_\nu(zr)I_\nu(zs) & r > s,\end{cases}$$

where I_ν and K_ν are modified Bessel functions (of order ν) of the first and second kinds. This is a Hilbert-Schmidt type operator. Thus if we show (4.11), it is sufficient to calculate $\|r^{-1}g(r,s)s^{-1}\|_{L^2(\mathbb{R}_+^2;r^{N-1}\,dr\otimes s^{N-1}\,ds)}$. To end this, we evaluate

$$\begin{aligned}
&\|r^{-1}g_\nu(r,s)s^{-1}\|^2_{L^2(\mathbb{R}_+^2;r^{N-1}\,dr\otimes s^{N-1}\,ds)} \qquad (4.12)\\
&= \iint_{\mathbb{R}_+\times\mathbb{R}_+} \frac{\big|I_\nu\big(z(r\wedge s)\big)K_\nu\big(z(r\vee s)\big)\big|^2}{rs}\,dr\,ds.
\end{aligned}$$

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Unfortunately, we cannot explicitly compute the above integral (4.12). Thus we need to another evaluation of (4.11) as Step 1 of the proof of Lemma 4.4. Here we can see that $\nu = 0$ is excluded in (4.11). Let $\nu > 0$. Then

$$I_\nu(zx) \sim \left(\frac{zx}{2}\right)^\nu \ (0 \le x \ll 1), \quad K_\nu(zx) \sim \sqrt{\frac{\pi}{2zx}} e^{-zx} \ (x \gg 1).$$

On the other hand, if $\nu = 0$, then

$$I_0(zx) \sim 1 \ (0 \le x \ll 1), \quad K_0(zx) \sim \sqrt{\frac{\pi}{2zx}} e^{-zx} \ (x \gg 1).$$

Thus the integral (4.12) diverges.

Set $S := P$ and $A := \chi^\perp |x|^{-1}$ in (4.7). By virtue of (4.8) and (4.9), we conclude that

$$\|\chi^\perp |x|^{-1} e^{-itP}\varphi\|_{L^2(\mathbb{R};L^2(\mathbb{R}^N))} \le (\sqrt{2}\nu_2^{-1} \vee 2^{3/4}\nu_2^{-1/2})\|\varphi\|_{L^2(\mathbb{R}^N)}, \tag{4.13}$$

$$\left\| \int_0^t \chi^\perp |x|^{-1} e^{-i(t-\tau)P} |x|^{-1}\chi^\perp F(\tau)\, d\tau \right\|_{L^2(\mathbb{R};L^2(\mathbb{R}^N))} \tag{4.14}$$
$$\le (\nu_2^{-4} \vee 2\nu_2^{-2})\, \|F\|_{L^2(\mathbb{R};L^2(\mathbb{R}^N))}.$$

Remark 4.6. In a view of [11, Theorem 1] and [44], we see

$$\|r^{-1} e^{-itA_\nu} f\|_{L^2(\mathbb{R};L^2(\mathbb{R}_+;r^{N-1}\,dr))} = \sqrt{\frac{\pi}{2\nu}} \|f\|_{L^2(\mathbb{R}_+;r^{N-1}\,dr)} \quad (\nu > 0).$$

Hence we can obtain the refinement of (4.13):

$$\|\chi^\perp |x|^{-1} e^{-itP} f\|_{L^2(\mathbb{R};L^2(\mathbb{R}^N))} \le \sqrt{\frac{\pi}{2\nu_2}} \|f\|_{L^2(\mathbb{R}^N)}.$$

Note that $\pi/(2\nu) < 2\nu^{-2}$ if $\nu \le 1/\sqrt{2}$ and $\pi/(2\nu) < 4/\sqrt{4\nu^2 - 1} < 2\sqrt{2}\nu^{-1}$ if $\nu > 1/\sqrt{2}$.

Proof of Proposition 4.3. We divide the proof into 3 steps.
Step 1. We show (4.4). Set $u(t) := \chi^\perp \exp(-itP)\varphi = \exp(-itP)\chi^\perp\varphi$. The Duhamel formula implies

$$u(t) = e^{it\Delta}\chi^\perp\varphi - i\int_0^t e^{i(t-s)\Delta} V\, u(s)\, ds.$$

Here the Strichartz estimates for $\exp(it\Delta)$ ($V \equiv 0$) are also verified (see also Keel–Tao [28]). Thus we see

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$$\begin{aligned}
&\|\chi^{\perp} e^{-itP}\varphi\|_{L^\tau(\mathbb{R};L^{\rho,2}(\mathbb{R}^N))} \\
&\le \|e^{it\Delta}\chi^{\perp}\varphi\|_{L^\tau(\mathbb{R};L^{\rho,2}(\mathbb{R}^N))} + \Big\| \int_0^t e^{i(t-s)\Delta} V\,u(s)\,ds \Big\|_{L^\tau(\mathbb{R};L^{\rho,2}(\mathbb{R}^N))} \\
&\le C_\tau(0)\,\|\varphi\|_{L^2(\mathbb{R}^N)} + C_{2,\tau}(0)\,\|V u(t)\|_{L^2(\mathbb{R};L^{2N/(N+2),2}(\mathbb{R}^N))}.
\end{aligned}$$

Here (4.1) and (4.13) yield

$$\begin{aligned}
&\|V u(t)\|_{L^2(\mathbb{R};L^{2N/(N+2),2}(\mathbb{R}^N))} \\
&= \big\| |x|^2 V\,|x|^{-1}\,\chi^{\perp}|x|^{-1}e^{-itP}\varphi \big\|_{L^2(\mathbb{R};L^{2N/(N+2),2}(\mathbb{R}^N))} \\
&\le C_N \,\big\||x|^2 V\big\|_{L^\infty(\mathbb{R}^N)}\,\big\||x|^{-1}\big\|_{L^{N,\infty}(\mathbb{R}^N)}\,\big\|\chi^{\perp}|x|^{-1}e^{-itP}\varphi\big\|_{L^2(\mathbb{R};L^2(\mathbb{R}^N))} \\
&\le C_N'\,\|\varphi\|_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Combining into this we obtain (4.4).

Step 2. We prove (4.5). Let $a \ge 0$. We define

$$\Psi_a F(t) := \int_0^t e^{-i(t-s)(P+a|x|^{-2})}\,F(s)\,ds.$$

Define $u(t) := -i\Psi_a F(t)$. Then u satisfies $i\,u_t = Pu + a|x|^{-2}u + F$. The Duhamel formula implies

$$\begin{aligned}
u(t) &= -i\int_0^t e^{-i(t-s)P}\,[a|x|^{-2}u(s) + F(s)]\,ds \\
&= -i\Psi_0[a|x|^{-2}u(t) + F(t)] = -i\Psi_0 F(t) - a\Psi_0|x|^{-2}\Psi_a F(t).
\end{aligned}$$

Thus we see $\Psi_a F(t) = \Psi_0 F(t) - ia\Psi_0|x|^{-2}\Psi_a F(t)$. In a way similar to this (with seeing $V = (V + a|x|^{-2}) - a|x|^{-2}$), we also obtain $\Psi_0 F(t) = \Psi_a F(t) + ia\Psi_a|x|^{-2}\Psi_0 F(t)$. These conclude that $\Psi_0|x|^{-2}\Psi_a = \Psi_a|x|^{-2}\Psi_0$ (commutative property). Hence we see

$$\begin{aligned}
\Psi_0 F(t) &= \Psi_a F(t) + ia\Psi_a|x|^{-2}\Psi_0 F(t) \\
&= \Psi_a F(t) + ia\Psi_a|x|^{-2}[\Psi_a F(t) + ia\Psi_a|x|^{-2}\Psi_0 F(t)] \\
&= \Psi_a F(t) + ia\Psi_a|x|^{-2}\Psi_a F(t) - a^2\Psi_a|x|^{-2}\Psi_0|x|^{-2}\Psi_a F(t).
\end{aligned}$$

Now we consider

$$\Big\| \int_0^\infty \chi^{\perp} e^{-i(t-s)P}\,F(s)\,ds \Big\|_{L^\tau(\mathbb{R};L^{\rho,2}(\mathbb{R}^N))}$$

$$
\begin{aligned}
&= \|\Psi_0 \chi^\perp F(t)\|_{L^\tau(\mathbb{R}; L^{\rho,2}(\mathbb{R}^N))} \\
&\le \|\Psi_a \chi^\perp F(t)\|_{L^\tau(\mathbb{R}; L^{\rho,2}(\mathbb{R}^N))} + a \left\| \Psi_a |x|^{-2} \Psi_a \chi^\perp F(t) \right\|_{L^\tau(\mathbb{R}; L^{\rho,2}(\mathbb{R}^N))} \\
&+ a^2 \left\| \Psi_a |x|^{-2} \Psi_0 |x|^{-2} \Psi_a \chi^\perp F(t) \right\|_{L^\tau(\mathbb{R}; L^{\rho,2}(\mathbb{R}^N))} \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

Since $\delta_{V+a|x|^{-2}} = a + \delta_V = a > 0$, (4.3) implies

$$
I_1 \le C_{2,\tau}(V_a) \|F\|_{L^2(\mathbb{R}; L^{2N/(N+2),2}(\mathbb{R}^N))}, \qquad V_a := V + a|x|^{-2}.
$$

Applying (4.3) twice, we can evaluate I_2:

$$
\begin{aligned}
|I_2| &\le a\, C_{2,\tau}(V_a) \left\| |x|^{-2} \Psi_a \chi^\perp F(t) \right\|_{L^2(\mathbb{R}; L^{2N/(N+2),2}(\mathbb{R}^N))} \\
&\le a\, C_{2,\tau}(V_a)\, C_N \left\| |x|^{-2} \right\|_{L^{N/2,\infty}(\mathbb{R}^N)} \|\Psi_a \chi^\perp F(t)\|_{L^2(\mathbb{R}; L^{2N/(N-2),2}(\mathbb{R}^N))} \\
&\le a\, C_{2,\tau}(V_a)^2\, C_N \|F\|_{L^2(\mathbb{R}; L^{2N/(N+2),2}(\mathbb{R}^N))}.
\end{aligned}
$$

For I_3 we also apply (4.14), not only (4.3).

$$
\begin{aligned}
|I_3| &\le a^2 C_{2,\tau}(V_a) \left\| |x|^{-2} \Psi_0 |x|^{-2} \Psi_a \chi^\perp F(t) \right\|_{L^2(\mathbb{R}; L^{2N/(N+2),2}(\mathbb{R}^N))} \\
&\le a^2 C_{2,\tau}(V_a)\, C_N \left\| |x|^{-1} \right\|_{L^{N,\infty}(\mathbb{R}^N)} \\
&\times \left\| \chi^\perp |x|^{-1} \Psi_0 |x|^{-1} \chi^\perp |x|^{-1} \Psi_a F(t) \right\|_{L^2(\mathbb{R}; L^2(\mathbb{R}^N))} \\
&\le a^2 C_{2,\tau}(V_a)\, C_N'\, C_{\chi^\perp |x|^{-1}} \left\| |x|^{-1} \Psi_a F(t) \right\|_{L^2(\mathbb{R}; L^2(\mathbb{R}^N))} \\
&\le a^2 C_{2,\tau}(V_a)\, C_N''\, C_{\chi^\perp |x|^{-1}} \left\| |x|^{-1} \right\|_{L^{N,\infty}(\mathbb{R}^N)} \|\Psi_a F(t)\|_{L^2(\mathbb{R}; L^{2N/(N-2),2}(\mathbb{R}^N))} \\
&\le a^2 C_{2,\tau}(V_a)^2\, C_N'''\, C_{\chi^\perp |x|^{-1}} \|F\|_{L^2(\mathbb{R}; L^{2N/(N+2),2}(\mathbb{R}^N))}.
\end{aligned}
$$

Combining these we conclude (4.5) with $(\tau_1, \rho_1) = (2, 2N/(N-2))$ (endpoint).

Next we obtain (4.5) for any admissible pairs. Set $T : L^2(\mathbb{R}^N) \to L^\tau(\mathbb{R}; L^{\rho,2}(\mathbb{R}^N))$ as $T\varphi := \chi^\perp \exp(-itP)\varphi$. Then the linear mapping

$$
\begin{aligned}
TT^* F(t) &= \chi^\perp e^{-itP} \int_{-\infty}^{\infty} e^{isP} \chi^\perp F(s)\, ds \\
&= \int_{-\infty}^{\infty} \chi^\perp e^{-i(t-s)P} F(s)\, ds
\end{aligned}
$$

is bounded from $L^{\tau_1'}(\mathbb{R}; L^{\rho_1',2}(\mathbb{R}^N))$ to $L^{\tau_2}(\mathbb{R}; L^{\rho_2,2}(\mathbb{R}^N))$. The Christ–Kiselev lemma [17] with $\tau_1' < \tau_2$ ensures that Ψ_0 is bounded linear mapping from $L^{\tau_1'}(\mathbb{R}; L^{\rho_1',2}(\mathbb{R}^N))$ to $L^{\tau_2}(\mathbb{R}; L^{\rho_2,2}(\mathbb{R}^N))$.

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Step 3. Next we consider (4.6). Note that the endpoint Strichartz estimate like (4.4) is not established since we cannot obtain the Kato smoothness bounds (see Remark 4.5). Now we follow the Mizutani approach as in [35]. Let Δ_2 be the Laplacian on $\mathbb{R}^2$. Tao [50, Corollary 1.4] shows the endpoint Strichartz estimate under the radial symmetry:

$$\|\exp(it\Delta_2)\varphi\|_{L^2(\mathbb{R};L^\infty_{\mathrm{rad}}(\mathbb{R}^2))} \le C_{\mathrm{end}}\,\|\varphi\|_{L^2_{\mathrm{rad}}(\mathbb{R}^2)}. \tag{4.15}$$

Since the first eigenvalue of $-\Delta_S + V$ in $L^2(S^{N-1})$ is simple, there uniquely exists a positive eigenfunction $Y_1 \in L^\infty(S^{N-1}) \cap H^2(S^{N-1})$ such that $\|Y_1\|_{L^2(S^{N-1})} = 1$. Define the operator $U_Y : L^2_{\mathrm{rad}}(\mathbb{R}^2) \to L^2_{=1}(\mathbb{R}^N)$ as

$$U_Y\varphi(x) := \sqrt{2\pi}\,|x|^{-(N-2)/2}\varphi(|x|)Y_1(|x|^{-1}x).$$

Then U_Y is a unitary operator:

$$\begin{aligned}\|U_Y\varphi\|^2_{L^2(\mathbb{R}^N)} &= \int_0^\infty\Big[\int_{S^{N-1}} |\sqrt{2\pi}r^{-(N-2)/2}\varphi(r)Y_1(y)|^2\,d\sigma(y)\Big]r^{N-1}\,dr \\ &= \int_0^\infty 2\pi|\varphi(r)|^2\Big[\int_{S^{N-1}} |Y_1(y)|^2\,d\sigma(y)\Big]r\,dr = \|\varphi\|^2_{L^2_{\mathrm{rad}}(\mathbb{R}^2)}.\end{aligned}$$

Moreover, we have

$$U_Y\exp(it\Delta_2) = e^{-itP}U_Y. \tag{4.16}$$

To show this, first we see

$$\begin{aligned}U_Y(-\Delta_2\varphi)(ry) &= U_Y\Big(-\partial_r^2\varphi - \frac{1}{r}\partial_r\varphi\Big) \\ &= \sqrt{2\pi}\,Y_1(y)\,r^{-(N-2)/2}\Big(-\partial_r^2\varphi - \frac{1}{r}\partial_r\varphi\Big).\end{aligned}$$

On the one hand,

$$\begin{aligned}&P(U_Yf)(ry) \\ &= \Big(-\partial_r^2 - \frac{N-1}{r}\partial_r + \frac{1}{r^2}(-\Delta_S + V)\Big)(\sqrt{2\pi}\,r^{-(N-2)/2}\varphi(r)Y_1(y)) \\ &= \sqrt{2\pi}\,Y_1(y)\Big(-\partial_r^2 - \frac{N-1}{r}\partial_r - \frac{(N-2)^2}{4r^2}\Big)[r^{-(N-2)/2}\varphi(r)] \\ &= (\sqrt{2\pi}Y_1(y))r^{-(N-2)/2}\Big(-\partial_r^2 - \frac{1}{r}\partial_r\Big)\varphi(r).\end{aligned}$$

Thus we see $U_Y(-\Delta_2) = PU_Y$ and (4.16).

Now we prove (4.6). It follows from (4.16) and (4.15) that

$$\begin{aligned}
&\|\chi e^{-itP}\varphi\|_{L^2(\mathbb{R};L^{2N/(N-2),\infty}(\mathbb{R}^N))}\\
&=\left\|\sqrt{2\pi}Y_1(|x|^{-1}x)|x|^{-(N-2)/2}U_Y^{-1}e^{-itP}\chi\varphi\right\|_{L^2(\mathbb{R};L^{2N/(N-2),\infty}(\mathbb{R}^N))}\\
&\le\left\|\sqrt{2\pi}Y_1(|x|^{-1}x)|x|^{-(N-2)/2}\right\|_{L^{2N/(N-2),\infty}(\mathbb{R}^N)}\\
&\times\|\exp(it\Delta_2)U_Y^{-1}\chi\varphi\|_{L^2(\mathbb{R};L^\infty_{\mathrm{rad}}(\mathbb{R}^2))}\\
&\le C_{N,V}\,C_{\mathrm{end}}\,\|U_Y^{-1}\chi\varphi\|_{L^2_{\mathrm{rad}}(\mathbb{R}^2)}\le C_{N,V}\,C_{\mathrm{end}}\,\|\varphi\|_{L^2(\mathbb{R}^N)}. \qquad\square
\end{aligned}$$

Proof of Theorem 1.5. (4.6) and the selfadjointness of P ensure that for $\theta\in(0,1)$

$$\begin{aligned}
\left\||\chi e^{-itP}\varphi|^{1-\theta}\right\|_{L^\infty(\mathbb{R};L^{2/(1-\theta),2/(1-\theta)}(\mathbb{R}^N))}&=\|\chi\varphi\|^{1-\theta}_{L^2(\mathbb{R}^N)}\le\|\varphi\|^{1-\theta}_{L^2(\mathbb{R}^N)},\\
\left\||\chi e^{-itP}\varphi|^{\theta}\right\|_{L^{2/\theta}(\mathbb{R};L^{2N/(\theta(N-2)),\infty}(\mathbb{R}^N))}&\le C_2(V)^\theta\,\|\varphi\|^\theta_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Applying the Hölder inequality (4.1) we see that

$$\|\chi e^{-itP}\varphi\|_{L^{2/\theta}(\mathbb{R};L^{2N/(N-2\theta),2/(1-\theta)}(\mathbb{R}^N))}\le C\,\|\varphi\|_{L^2(\mathbb{R}^N)}.$$

Thus $\chi\exp(-itP)$ is a bounded linear map from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R};L^{2N/(N-2),\infty}(\mathbb{R}^N))$ and from $L^2(\mathbb{R}^N)$ to $L^{2/\theta}(\mathbb{R};L^{2N/(N-2\theta),2/(1-\theta)}(\mathbb{R}^N))$. The real interpolation of space-time Lebesgue-Lorentz spaces implies that $\chi\exp(-itP)$ is bounded linear map from $L^2(\mathbb{R}^N)$ to $L^\tau(\mathbb{R};L^{\rho,\tau}(\mathbb{R}^N))$, where

$$\frac{1}{\tau}=\frac{1-\omega(1-\theta)}{2},\qquad\frac{1}{\rho}=\frac{1}{2}-\frac{1-\omega(1-\theta)}{N},\qquad\omega\in(0,1).$$

Here (τ,ρ) is admissible pair without the endpoint.

Here the case $\tau\le\rho$ is a good estimate because of $L^{\rho,\tau}(\mathbb{R}^N)\subset L^\rho(\mathbb{R}^N)$. The case is written down as $2<\tau\le 2+4/N$ (or $2+4/N\le\rho<2N/(N-2)$). If the bad case $\tau>2+4/N$ (or $2\le\rho<2+4/N$), we consider the usual Hölder inequality.

$$\begin{aligned}
&\|\chi e^{-itP}\varphi\|_{L^\tau(\mathbb{R};L^\rho(\mathbb{R}^N))}\\
&\le\|\chi e^{-itP}\varphi\|^{1-\theta_*}_{L^\infty(\mathbb{R};L^2(\mathbb{R}^N))}\|\chi e^{-itP}\varphi\|^{\theta_*}_{L^{2+4/N}(\mathbb{R};L^{2+4/N}(\mathbb{R}^N))}\\
&\le C_{2+4/N}(V)^{\theta_*}\|\varphi\|_{L^2(\mathbb{R}^N)},
\end{aligned}$$

where

$$\frac{1}{\tau}=\frac{1-\theta_*}{\infty}+\frac{\theta_*}{2+4/N},\qquad\frac{1}{\rho}=\frac{1-\theta_*}{2}+\frac{\theta_*}{2+4/N}.$$

Combining this into (4.4), we obtain (1.6) (see also Suzuki [49, Lemma 4.3 and Proposition 4.8] for the above arguments). Finally, (1.7) is followed from TT^* arguments as in the latter of **Step 2** in Proposition 4.3. $\square$

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5. Local and global existence of the semilinear Schrödinger equations with a critical inverse-square potentials

Now we solve **(NLS)**, that is, we prove Theorem 1.6. We refer [48, Theorem 4.1] for one of the most simple cases $g(u) := \lambda\,|u|^{p-1}u$ (power type; local nonlinearity) for **(NLS)** (see Example 5.7 for details). We remark $\mathcal{D} := D((1+P)^{1/2})$ and $\mathcal{D}^* := D((1+P)^{-1/2})$, a dual of $\mathcal{D}$. We see the Gelfand triplet $\mathcal{D} \subset L^2(\mathbb{R}^N) \subset \mathcal{D}^*$. Note that $\mathcal{D} \supsetneq H^1(\mathbb{R}^N)$ by (1.5) and

$$\|u\|_{\mathcal{D}} := \|(1+P)^{1/2}u\|_{L^2(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} [|u|^2 + |\nabla u|^2 + V(x)|u|^2]\,dx\Big)^{1/2},$$

$$\|u\|_{\mathcal{D}^*} := \|(1+P)^{-1/2}u\|_{L^2(\mathbb{R}^N)}.$$

Note that if $\lambda > 0$, or $\lambda < 0$ and $1 < p < 1 + (2 - 2\alpha - \beta)/N$, then the local solution u to **(NLS)** with (1.8) can be extended globally in time (see also Remark 5.5). We give the proof of Theorem 1.6 for the case $\alpha > 0$ and $\beta > 0$; the cases $\alpha = 0$, and $\beta = 0$ are more simple. The proof of Theorem 1.6 is divided into 3 steps:

Step 1. Check the conditions for confirming the existence of solution to (1.8);

Step 2. Verification of uniqueness of solution to (1.8);

Step 3. Showing the continuity of solution in $D(|x|)$.

Theorem 1.6 is owing to the energy methods established by Okazawa–Suzuki–Yokota [40]. We consider the abstract Cauchy problems for semilinear Schrödinger equations. Let S be a nonnegative selfadjoint operator in a complex Hilbert space X. Put $X_S := D((1+S)^{1/2})$ and $X_S^* := D((1+S)^{-1/2})$, the dual of X_S.

$$\begin{cases} i\dfrac{du}{dt} = Su + g(u), \\ u(0) = u_0, \end{cases} \tag{ACP}$$

where $i = \sqrt{-1}$, $g : X_S \to X_S^*$ is a nonlinear operator under the following conditions. For simple notation we use $B_M := \{u \in X_S;\ \|u\|_{X_S} = \|(1+S)^{1/2}u\|_X \le M\}$.

(G1) Existence of energy functional: there exists $G \in C^1(X_S; \mathbb{R})$ such that $d_{X_S}G = g$, that is, given $u \in X_S$, for every $\varepsilon > 0$ there exists $\delta = \delta(u,\varepsilon) > 0$ such that

$$|G(u+v) - G(u) - \mathrm{Re}\,\langle g(u), v\rangle_{X_S^*, X_S}| \le \varepsilon\,\|v\|_{X_S} \quad \forall\, v \in B_\delta;$$

(G2) Local Lipschitz continuity: for all $M > 0$ there exists $C(M) > 0$ such that

$$\|g(u) - g(v)\|_{X_S^*} \le C(M)\|u - v\|_{X_S} \quad \forall\, u,\ v \in B_M;$$

(G3) Hölder-like continuity of energy functional: given $M > 0$, for all $\delta > 0$ there exists a constant $C_\delta(M) > 0$ such that

$$|G(u) - G(v)| \le \delta + C_\delta(M)\|u - v\|_X \quad \forall\, u,\ v \in B_M;$$

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(G4) Gauge condition: $\mathrm{Re}\,\langle g(u), i\,u\rangle_{X_S^*,X_S} = 0\ \forall\, u \in X_S$;

(G5) Closedness condition: given a bounded open interval $I \subset \mathbb{R}$, let $\{w_n\}_n$ be any bounded sequence in $L^\infty(I; X_S)$ such that

$$\begin{cases} w_n(t) \to w(t)\ (n \to \infty) & \text{weakly in } X_S \text{ a.a. } t \in I, \\ g(w_n) \to f\ (n \to \infty) & \text{weakly}^* \text{ in } L^\infty(I; X_S^*). \end{cases}$$

Then

$$\mathrm{Re} \int_I \langle f(t), i\,w(t)\rangle_{X_S^*,X_S}\,dt = \lim_{n\to\infty} \mathrm{Re} \int_I \langle g(w_n(t)), i\,w_n(t)\rangle_{X_S^*,X_S}\,dt. \tag{5.1}$$

Here $f = g(w)$ is guaranteed if $w_n(t) \to w(t)$ $(n \to \infty)$ strongly in X a.a. $t \in I$.

Proposition 5.1 *(Energy methods [40, Theorems 2.1 and 2.2]). Assume that $g : X_S \to X_S^*$ satisfies* **(G1)–(G5)***. Then for every $u_0 \in X_S$ with $\|u_0\|_{X_S} \le M$ there exist $T_M > 0$ and a local solution u to* **(ACP)** *on $(-T_M, T_M)$. Moreover, u belongs to $C_{\mathrm{w}}([-T_M, T_M]; X_S) \cap W^{1,\infty}(-T_M, T_M; X_S^*)$ and satisfies*

$$\|u(t)\|_X = \|u_0\|_X, \quad E(u(t)) \le E(u_0) \quad \forall\, t \in [-T_M, T_M], \tag{5.2}$$

where $E(\cdot)$ is the energy functional given by

$$E(\varphi) := \frac{1}{2}\|(1+S)^{1/2}\varphi\|_X^2 + G(\varphi), \quad \varphi \in X_S.$$

Furthermore, if the uniqueness of local solutions to **(ACP)** *is verified, then the local solution u to* **(ACP)** *belongs to $C([-T_M, T_M]; X_S) \cap C^1([-T_M, T_M]; X_S^*)$ and the inequality of (5.2) is just equality.*

Now we prepare for the proof of Theorem 1.6. One of the key ingredients is the Hardy-Littlewood-Sobolev inequality

$$\big\| |x|^{-\gamma} * F \big\|_{L^q(\mathbb{R}^N)} \le C_{\mathrm{HLS}}(q, r)\, \|F\|_{L^r(\mathbb{R}^N)}, \tag{5.3}$$
$$1 < q < r < \infty,\ \frac{1}{q} + \frac{\gamma}{N} = 1 + \frac{1}{r}.$$

Moreover, the fractional Sobolev embeddings are verified (see e.g., Remark 3 in Section 2.8.1 of [52]):

$$\|u\|_{L^{2N/(N-2s)}(\mathbb{R}^N)} \le C_s\, \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}.$$

It follows form (1.5) that for $0 < s < 1$

$$\|u\|_{L^{2N/(N-2s)}(\mathbb{R}^N)} \le C_s'\, \|P^{s/2} u\|_{L^2(\mathbb{R}^N)} \le C_s'\, \|u\|_{L^2(\mathbb{R}^N)}^{1-s} \|u\|_{\mathcal{D}}^{s} \le C_s'\, \|u\|_{\mathcal{D}}. \tag{5.4}$$

Moreover, the Rellich compactness lemma is preserved. If $\Omega \subset \mathbb{R}^N$ is bounded domain, then $\mathcal{D}|_\Omega := \{u|_\Omega \in L^2(\Omega); u \in \mathcal{D}\} \subset L^q(\Omega)$ $(2 \le q < 2N/(N-2))$ is compact (see also [48, Lemma 4.5]).

Proof of Theorem 1.6 (Step 1. Verifying the conditions (G1)–(G5)). Applying the Hardy-Littlewood-Sobolev inequality (5.3), we see that

$$\Big| \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_1(x)|^{p-1} u_2(x) \overline{u_3(x)}\, |v_1(y)|^{p-1} v_2(y) \overline{v_3(y)}}{|x|^\alpha \, |x-y|^\beta \, |y|^\alpha} \, dxdy \Big| \tag{5.5}$$
$$\le C_{\mathrm{HLS}} \, \|u_1\|_{\mathcal{E}}^{p-1} \, \|u_2\|_{\mathcal{E}} \, \|u_3\|_{\mathcal{E}} \, \|v_1\|_{\mathcal{E}}^{p-1} \, \|v_2\|_{\mathcal{E}} \, \|v_3\|_{\mathcal{E}},$$

where

$$\begin{aligned}
\|u\|_{\mathcal{E}} &:= \big\| |x|^{-\alpha/(p+1)} u \big\|_{L^{2N(p+1)/(2N-\beta)}(\mathbb{R}^N)}, \\
C_{\mathrm{HLS}} &:= C_{\mathrm{HLS}}(2N/\beta, 2N/(2N-\beta)), \\
s &:= \frac{N(p-1)+2\alpha+\beta}{2(p+1)} < 1, \quad \theta := \frac{2\alpha}{N(p-1)+2\alpha+\beta} \in (0,1).
\end{aligned}$$

The Hölder inequality, (5.4), and (3.4) imply that

$$\begin{aligned}
\|u\|_{\mathcal{E}} &\le \big\| |x|^{-s} u \big\|_{L^2(\mathbb{R}^N)}^{\theta} \, \|u\|_{L^{2N/(N-2s)}(\mathbb{R}^N)}^{1-\theta} \le C_{ss} \, \|P^{s/2} u\|_{L^2(\mathbb{R}^N)} \\
&\le C_{ss} \, \|u\|_{\mathcal{D}}^{s} \|u\|_{L^2(\mathbb{R}^N)}^{1-s} \le C_{ss} \, \|u\|_{\mathcal{D}}.
\end{aligned} \tag{5.6}$$

Simple calculations imply the following inequalities for $A, B, \xi \in \mathbb{C}$.

$$\big| |A+\xi|^{p+1} - |A|^{p+1} - (p+1) \mathrm{Re}\, |A|^{p-1} A \overline{\xi} \big| \tag{5.7}$$
$$\le p(p+1)(|A|+|\xi|)^{p-1} |\xi|^2,$$

$$\big| |A|^{p-1}A - |B|^{p-1}B \big| \le p(|A|^{p-1} + |B|^{p-1})|A-B|, \tag{5.8}$$

$$\big| |A|^{p+1} - |B|^{p+1} \big| \le (p+1)(|A|^p + |B|^p)|A-B|. \tag{5.9}$$

Define the energy functional of $g(u)$ as

$$G(u) := \frac{\lambda}{2(p+1)} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^{p+1} \, |u(y)|^{p+1}}{|x-y|^\gamma} \, dxdy.$$

To derive **(G1)**, we evaluate

$$G(u+v) - G(u) - \mathrm{Re}\, \langle g(u), v \rangle_{\mathcal{D}^*, \mathcal{D}} = \frac{\lambda}{2(p+1)} I_1 + \frac{\lambda}{2(p+1)} I_2 + \frac{\lambda}{2} I_3,$$

where

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$$I_j := \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f_j(x,y)}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dx\,dy \quad (j=1,\,2,\,3),$$

$$\begin{aligned} f_1(x,y) &:= [|(u+v)(x)|^{p+1} - |u(x)|^{p+1} - (p+1)\mathrm{Re}\,|u(x)|^{p-1}u(x)\overline{v(x)}] \\ &\qquad \times |(u+v)(y)|^{p+1}, \\ f_2(x,y) &:= |(u+v)(y)|^{p+1} - |u(y)|^{p+1} - (p+1)\mathrm{Re}\,|u(y)|^{p-1}u(y)\overline{v(y)}, \\ f_3(x,y) &:= |u(x)|^{p-1}\mathrm{Re}\{u(x)\overline{v(x)}\}\,[|(u+v)(y)|^{p+1} - |u(y)|^{p+1}]. \end{aligned}$$

Applying (5.7) and (5.5), we can evaluate

$$\begin{aligned} |I_1| &\le \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{p(p+1)[|u(x)|+|v(x)|]^{p-1}|v(x)|^2\,[|u(y)|+|v(y)|]^{p+1}}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dx\,dy \\ &\le p(p+1)C_{\mathrm{HLS}}\,(\|u\|_{\mathcal{E}} + \|v\|_{\mathcal{E}})^{p-1}\|v\|_{\mathcal{E}}^2(\|u\|_{\mathcal{E}} + \|v\|_{\mathcal{E}})^{p+1} \\ &= p(p+1)C_{\mathrm{HLS}}\,(\|u\|_{\mathcal{E}} + \|v\|_{\mathcal{E}})^{2p}\|v\|_{\mathcal{E}}^2. \end{aligned}$$

In a way similar to the above, we also obtain

$$\begin{aligned} |I_2| &\le \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^{p+1}\,p(p+1)[|u(y)|+|v(y)|]^{p-1}|v(y)|^2}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dx\,dy \\ &\le p(p+1)C_{\mathrm{HLS}}\,\|u\|_{\mathcal{E}}^{p+1}(\|u\|_{\mathcal{E}} + \|v\|_{\mathcal{E}})^{p-1}\|v\|_{\mathcal{E}}^2 \\ &\le p(p+1)C_{\mathrm{HLS}}\,(\|u\|_{\mathcal{E}} + \|v\|_{\mathcal{E}})^{2p}\|v\|_{\mathcal{E}}^2. \end{aligned}$$

Moreover, (5.9) and (5.5) yield

$$\begin{aligned} |I_3| &\le \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p|v(x)|\,(p+1)[|(u+v)(y)|^p + |u(y)|^p]|v(y)|}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dx\,dy \\ &\le (p+1)C_{\mathrm{HLS}}\,\|u\|_{\mathcal{E}}^p\|v\|_{\mathcal{E}}[\|u\|_{\mathcal{E}}^p + (\|u\|_{\mathcal{E}} + \|v\|_{\mathcal{E}})^p]\|v\|_{\mathcal{E}} \\ &\le 2p(p+1)C_{\mathrm{HLS}}\,(\|u\|_{\mathcal{E}} + \|v\|_{\mathcal{E}})^{2p}\|v\|_{\mathcal{E}}^2. \end{aligned}$$

Thus we calculate

$$|G(u+v) - G(u) - \mathrm{Re}\,\langle g(u), v\rangle_{\mathcal{D}^*,\mathcal{D}}| \le |\lambda|(2p+1)C_{\mathrm{HLS}}\,(\|u\|_{\mathcal{E}} + \|v\|_{\mathcal{E}})^{2p}\|v\|_{\mathcal{E}}^2.$$

Applying (5.6), we obtain for every $u,\ v \in \mathcal{D}$

$$|G(u+v) - G(u) - \mathrm{Re}\,\langle g(u), v\rangle_{\mathcal{D}^*,\mathcal{D}}| \le C(p,N)\,[\|u\|_{\mathcal{D}} + \|v\|_{\mathcal{D}}]^{2p}\|v\|_{\mathcal{D}}^2,$$

where $C(p,N) := |\lambda|(2p+1)C_{\mathrm{HLS}}\,C_{ss}^{2p+2}$. Now let $M > 0$ and $\varepsilon > 0$. Then we see that

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$$|G(u+v)-G(u)-\mathrm{Re}\,\langle g(u),v\rangle_{\mathcal{D}^*,\mathcal{D}}| \le C(p,N)\,(M+1)^{2p}\|v\|_{\mathcal{D}}^2 \quad \forall\, u,\ v\in\mathcal{D} \text{ with } \|u\|_{\mathcal{D}}\le M,\ \|v\|_{\mathcal{D}}\le 1.$$

Hence by setting $\delta=\delta(u,\varepsilon)>0$ as $\delta := 1\wedge(\varepsilon/[C(p,N)\,(M+1)^{2p}])$, we conclude that $|G(u+v)-G(u)-\mathrm{Re}\,\langle g(u),v\rangle_{\mathcal{D}^*,\mathcal{D}}| \le \varepsilon\,\|v\|_{\mathcal{D}}$ for $v\in\mathcal{D}$ with $\|v\|_{\mathcal{D}}\le\delta$. This is nothing but **(G1)**.

Next we verify **(G2)**. We consider $\langle g(u)-g(v),w\rangle_{\mathcal{D}^*,\mathcal{D}}$ for $u,\ v,\ w\in\mathcal{D}$. We see

$$\begin{aligned}
&\langle g(u)-g(v),w\rangle_{\mathcal{D}^*,\mathcal{D}}\\
&=\lambda \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{[|u(x)|^{p-1}u(x)-|v(x)|^{p-1}v(x)]\overline{w(x)}\,|u(y)|^{p+1}}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dx\,dy\\
&+\lambda \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|v(x)|^{p-1}v(x)\overline{w(x)}\,[|u(y)|^{p+1}-|v(y)|^{p+1}]}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dx\,dy.
\end{aligned}$$

Applying (5.8), (5.9), and (5.5), we calculate for every $w\in\mathcal{D}$

$$\begin{aligned}
&|\langle g(u)-g(v),w\rangle_{\mathcal{D}^*,\mathcal{D}}|\\
&\le |\lambda| C_{\mathrm{HLS}}\, p[\|u\|_{\mathcal{E}}^{p-1}+\|v\|_{\mathcal{E}}^{p-1}]\,\|u-v\|_{\mathcal{E}}\|w\|_{\mathcal{E}}\,\|u\|_{\mathcal{E}}^{p+1}\\
&+|\lambda| C_{\mathrm{HLS}}\,(p+1)\|v\|_{\mathcal{E}}^{p}\|w\|_{\mathcal{E}}[\|u\|_{\mathcal{E}}^{p}+\|v\|_{\mathcal{E}}^{p}]\|u-v\|_{\mathcal{E}}\\
&\le 2(2p+1)|\lambda| C_{\mathrm{HLS}}\,[\|u\|_{\mathcal{E}}\vee\|v\|_{\mathcal{E}}]^{2p}\,\|u-v\|_{\mathcal{E}}\|w\|_{\mathcal{E}}.
\end{aligned}$$

Thus we see from (5.6) that

$$\|g(u)-g(v)\|_{\mathcal{D}^*}\le 2(2p+1)|\lambda| C_{\mathrm{HLS}}\,C_{ss}^{2p+2}\,[\|u\|_{\mathcal{D}}\vee\|v\|_{\mathcal{D}}]^{2p}\,\|u-v\|_{\mathcal{D}}.$$

Hence we obtain **(G2)**.

Next we consider **(G3)**. First we see

$$\begin{aligned}
&G(u)-G(v)\\
&=\frac{\lambda}{2(p+1)}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{[|u(x)|^{p+1}-|v(x)|^{p+1}]\,[|u(y)|^{p+1}+|v(y)|^{p+1}]}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dx\,dy.
\end{aligned}$$

Applying (5.9) and (5.5), we calculate for $u,\ v\in\mathcal{D}$

$$|G(u)-G(v)|\le\frac{|\lambda|}{2}C_{\mathrm{HLS}}\,[\|u\|_{\mathcal{E}}^{p}+\|v\|_{\mathcal{E}}^{p}]\|u-v\|_{\mathcal{E}}\,[\|u\|_{\mathcal{E}}^{p+1}+\|v\|_{\mathcal{E}}^{p+1}].$$

Applying (5.6), we obtain

$$|G(u)-G(v)|\le 2^{1+s}|\lambda| C_{\mathrm{HLS}}\,C_{ss}^{2p+2}\,[\|u\|_{\mathcal{D}}\vee\|v\|_{\mathcal{D}}]^{2p+s}\,\|u-v\|_{L^2(\mathbb{R}^N)}^{1-s}.$$

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The Young inequality yields **(G3)**.

(G4) is immediately verified by the definition of $g(u)$. To verify **(G5)** first we divide $|x|^{-\alpha}$ into $V_1 + V_2$, where $V_1(x) := |x|^{-\alpha}\chi_{|x|\geq 1}(x)$ and $V_2(x) := |x|^{-\alpha}\chi_{|x|<1}(x)$. Note that $V_1 \in L^{N/(\alpha-\varepsilon)}(\mathbb{R}^N)$ and $V_2 \in L^{N/(\alpha+\varepsilon)}(\mathbb{R}^N)$ for sufficiently small $\varepsilon > 0$. Let $\{w_n\}_n$ be a sequence in $L^\infty(I;\mathcal{D})$ satisfying

$$\begin{cases} w_n(t) \to w(t)\ (n\to\infty) & \text{weakly in } \mathcal{D} \quad \text{a.a. } t \in I, \\ g(w_n) \to f\ (n\to\infty) & \text{weakly* in } L^\infty(I;\mathcal{D}^*). \end{cases}$$

Next we define

$$g_{jk}(u) := \lambda|u(x)|^{p-1}u(x)\int_{\mathbb{R}^N} \frac{V_j(x)\,|u(y)|^{p+1}\,V_k(y)}{|x-y|^\beta}\,dy,$$

$$q(jk) := \begin{cases} \dfrac{2N(p+1)}{2N-2\alpha-\beta+2\varepsilon} & (j,k) = (1,1), \\ \dfrac{2N(p+1)}{2N-2\alpha-\beta} & (j,k) = (1,2),\ (2,1), \\ \dfrac{2N(p+1)}{2N-2\alpha-\beta-2\varepsilon} & (j,k) = (2,2). \end{cases}$$

Note that the Hardy-Littlewood-Sobolev inequality (5.3) implies

$$\|g_{jk}(u)\|_{L^{q(jk)'}(\mathbb{R}^N)} \leq |\lambda| C_{\mathrm{HLS}}(jk)\,\|V_j\|\,\|V_k\|\,\|u\|^{2p+1}_{L^{q(jk)}(\mathbb{R}^N)},$$

where $\|V_1\| = \|V_1\|_{L^{N/(\alpha-\varepsilon)}(\mathbb{R}^N)}$, $\|V_2\| = \|V_2\|_{L^{N/(\alpha+\varepsilon)}(\mathbb{R}^N)}$, and

$$\begin{aligned} C_{\mathrm{HLS}}(11) &= C_{\mathrm{HLS}}(22) = C_{\mathrm{HLS}}(2N/\beta, 2N/(2N-\beta)), \\ C_{\mathrm{HLS}}(12) &= C_{\mathrm{HLS}}(2N/(\beta+2\varepsilon), 2N/(2N-\beta+2\varepsilon)), \\ C_{\mathrm{HLS}}(21) &= C_{\mathrm{HLS}}(2N/(\beta-2\varepsilon), 2N/(2N-\beta-2\varepsilon)). \end{aligned}$$

Since $\{w_n\}_n$ is bounded in $L^\infty(I;\mathcal{D})$, $\{g_{jk}(w_n)\}_n$ is bounded in $L^\infty(I;L^{q(\cdot)'}(\mathbb{R}^N))$. Moreover, there exist a subsequence $\{w_{n(l)}\}_l$ of $\{w_n\}_n$ and $f_{jk} \in L^\infty(I;L^{q(\cdot)'}(\mathbb{R}^N))$ such that

$$g_{jk}(w_{n(l)}) \to f_{jk}\ (l\to\infty) \quad \text{weakly* in } L^\infty(I;L^{q(\cdot)'}(\mathbb{R}^N)) \quad j,k = 1,2. \tag{5.10}$$

Note that the weak convergence is also verified in $L^\infty(I;\mathcal{D})$. To confirm (5.1) let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open subset with C^1 boundary. Then

$$\begin{aligned} &\langle f_{jk}(t), w(t)\rangle_{L^{q(jk)'}(\Omega),L^{q(jk)}(\Omega)} \\ &= \langle f_{jk}(t) - g_{jk}(w_{n(l)}(t)), w(t)\rangle_{L^{q(jk)'}(\Omega),L^{q(jk)}(\Omega)} \\ &+ \langle g_{jk}(w_{n(l)}(t)), w(t) - w_{n(l)}(t)\rangle_{L^{q(jk)'}(\Omega),L^{q(jk)}(\Omega)} \end{aligned}$$

$$+ \langle g_{jk}(w_{n(l)}(t)), w_{n(l)}(t)\rangle_{L^{q(jk)'}(\Omega), L^{q(jk)}(\Omega)}$$
$$=: I^1_{jk}(l;t) + I^2_{jk}(l;t) + I^3_{jk}(l;t).$$

The weak convergence (5.10) asserts that

$$\int_I I^1_{jk}(l;t)\,dt \to 0\ (l \to \infty).$$

Next we consider $I^2_{jk}(l;t)$. The Rellich compactness lemma implies that $w_{n(l)}(t) \to w(t)$ $(l \to \infty)$ strongly in $L^{q(jk)}(\Omega)$ a.a. $t \in I$. Hence it follows from the boundedness of $\{g(w_{n(l)}(t))\}_l$ in $L^{q(jk)'}(\Omega)$ a.a. $t \in I$ that $I^2_{jk}(l;t) \to 0$ $(l \to \infty)$ for a.a. $t \in I$. Moreover, the boundedness of $\{w_{n(l)}\}_l$ in $L^\infty(I; L^{q(jk)}(\Omega))$ and $\{g(w_{n(l)})\}_l$ in $L^\infty(I; L^{q(jk)'}(\Omega))$ implies that

$$\int_I I^2_{jk}(l;t)\,dt \to 0\ (l \to \infty).$$

By definition of $g_{jk}(u)$, we can show $\operatorname{Im} I_3(t) = 0$. Since $f = f_{11} + f_{12} + f_{21} + f_{22}$, the former half of **(G5)** is verified.

Next we show that $f = g(w)$ by assuming further that $w_n(t) \to w(t)$ in $L^2(\mathbb{R}^N)$ a.a. $t \in I$. Let $M := \sup_n \|w_n\|_{L^\infty(I;\mathcal{D})}$. It follows from **(G2)** and (5.6) that

$$\begin{aligned}
&\|g(w_n(t)) - g(w(t))\|_{\mathcal{D}^*} \\
&\le 2^{1+s}(2p+1)|\lambda| C_{\mathrm{HLS}}\, C_{ss}^{2p+2} [\|w_n(t)\|_{\mathcal{D}} \vee \|w(t)\|_{\mathcal{D}}]^{2p+s} \|w_n(t) - w(t)\|^{1-s}_{L^2(\mathbb{R}^N)} \\
&\le 2^{2+s}(2p+1)|\lambda| C_{\mathrm{HLS}}\, C_{ss}^{2p+2} M^{2p+s} \|w_n(t) - w(t)\|^{1-s}_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain the strong convergence $g(w_n(t)) \to g(w(t))$ $(n \to \infty)$ in $\mathcal{D}^*$ a.a. $t \in I$. Therefore we conclude that $f = g(w)$ and **(G5)** is completely verified. □

Remark 5.2. Here we can simplify and relax the condition **(G5)** by virtue of **(G4)**:

(G5)′ weak closedness: for any sequence $\{u_n\}_n$ in X_S assume that

$$\begin{cases} u_n \to u\ (n \to \infty) & \text{weakly in } X_S, \\ g(u_n) \to f\ (n \to \infty) & \text{weakly in } X_S^* \end{cases} \Rightarrow\ f = g(u).$$

See [40, Lemma 5.3] for the proof. This condition is applicable in the pure power nonlinearity. Note that we cannot apply to (1.8).

Proof of Theorem 1.6 (Step 2. Uniqueness). We carry out the standard argument. Let $u, v \in L^\infty(-T,T;\mathcal{D})$ be local weak solutions to (1.8) on $(-T,T)$ with initial values $u(0) = u_0$ and $v(0) = v_0$. Then u satisfies the following integral equations:

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$$u(t) = \exp(-itP)u_0 - i\int_0^t \exp(-i(t-s)P)g(u(s))\,ds.$$

Let $(r(jk), q(jk))$ be Schrödinger admissible pairs: $2/r(jk) + N/q(jk) = N/2$, i.e.,

$$r(jk) := \begin{cases} \dfrac{4(p+1)}{N(p-1)+2\alpha+\beta-2\varepsilon} & (j,k) = (1,1), \\ \dfrac{4(p+1)}{N(p-1)+2\alpha+\beta} & (j,k) = (1,2),\ (2,1), \\ \dfrac{4(p+1)}{N(p-1)+2\alpha+\beta+2\varepsilon} & (j,k) = (2,2). \end{cases}$$

Applying the Strichartz estimates (1.6) and (1.7) with admissible pairs $(r(jk), q(jk))$, and the Sobolev embeddings (5.4), we see that

$$\begin{aligned} &\|u-v\|_{L^\tau(-T,T;L^\rho(\mathbb{R}^N))} \\ &\le C_\tau(V)\,\|u_0-v_0\|_{L^2(\mathbb{R}^N)} \\ &+ \sum_{j,k} C_{jk}(2T)^{1-2/r(jk)}\,\|u-v\|_{L^{r(jk)}(-T,T;L^{q(jk)}(\mathbb{R}^N))} \\ &\times [\|u\|_{L^\infty(-T,T;L^{q(jk)}(\mathbb{R}^N))} \vee \|v\|_{L^\infty(-T,T;L^{q(jk)}(\mathbb{R}^N))}]^{2p} \\ &\le C_\tau(V)\,\|u_0-v_0\|_{L^2(\mathbb{R}^N)} \\ &+ \sum_{j,k} C_{jk}\,M_{jk}^{2p}\,(2T)^{1-2/r(jk)}\,\|u-v\|_{L^{r(jk)}(-T,T;L^{q(jk)}(\mathbb{R}^N))}, \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} C_{jk} &:= 2(2p+1)|\lambda| C_{\mathrm{HLS}}(jk)\,\|V_j\|\,\|V_k\|\,C_{r(jk),\tau}(V), \\ M_{jk} &:= C'_{s(jk)}(\|u\|_{L^\infty(-T,T;\mathcal{D})} \vee \|v\|_{L^\infty(-T,T;\mathcal{D})}), \end{aligned}$$

and $s(jk) := N[2^{-1} - q(jk)^{-1}] \in (0,1)$. Set (τ, ρ) as $(r(jk), q(jk))$ $(j,\ k = 1,\ 2)$ and sum up (5.11). Take $T_0 \in (0,T)$ sufficiently small for absorbing a part of the right hand sides into the left hand sides. Thus we see that

$$\sum_{j,k} \|u-v\|_{L^{r(jk)}(-T_0,T_0;L^{q(jk)}(\mathbb{R}^N))} \le 2\sum_{j,k} C_{r(jk)}\|u_0-v_0\|_{L^2(\mathbb{R}^N)}.$$

Here if $u_0 = v_0$, then $u(t) = v(t)$ a.a. $t \in (-T_0, T_0)$. Extending the interval step by step, we conclude the uniqueness of the solution to (1.8). □

Remark 5.3. We can also construct the Lipschitz like inequality.

$$\|u(t)-v(t)\|_{L^2(\mathbb{R}^N)} \le L\,e^{\omega|t|}\,\|u_0-v_0\|_{L^2(\mathbb{R}^N)},$$

where $u(t)$ and $v(t)$ are (local) solutions to **(NLS)** with (1.8) of the initial values $u(0) = u_0$ and $v(0) = v_0$. L and ω are dependent on V, $\|u_0\|_{\mathcal{D}}$, $\|v_0\|_{\mathcal{D}}$, λ, γ, and p. See e.g., [39, Proposition 3.7]. Thus the continuous dependence of initial value for **(NLS)** with (1.8) is induced.

Proof of Theorem 1.6 (Step 3. Continuity in the weighted energy spaces). Assume further $u_0 \in D(|x|)$. Now we consider

$$f_\varepsilon(t) := \frac{1}{2}\left\| \frac{x}{\sqrt{1+\varepsilon|x|^2}} u(t)\right\|^2_{L^2(\mathbb{R}^N)} = \frac{1}{2}\int_{\mathbb{R}^N} \frac{|x|^2}{1+\varepsilon|x|^2}|u(t,x)|^2\,dx.$$

Differentiating in t and integrating by parts, we obtain

$$\begin{aligned}
&f'_\varepsilon(t)\\
&= \mathrm{Re}\int_{\mathbb{R}^N} \frac{|x|^2}{1+\varepsilon|x|^2} u_t(t,x)\overline{u(t,x)}\,dx = \mathrm{Im}\int_{\mathbb{R}^N} -\Delta u(t,x)\frac{|x|^2}{1+\varepsilon|x|^2}\overline{u(t,x)}\,dx\\
&= \mathrm{Im}\int_{\mathbb{R}^N} \overline{u(t,x)}\,\nabla u(t,x)\cdot\nabla\frac{|x|^2}{1+\varepsilon|x|^2}\,dx\\
&= \mathrm{Im}\int_{\mathbb{R}^N} \overline{u(t,x)}\,\nabla u(t,x)\cdot\frac{2x}{(1+\varepsilon|x|^2)^2}\,dx.
\end{aligned}$$

Applying (3.6) with $\varphi(r) = 2/(1+\varepsilon r^2)^2$, we evaluate

$$\begin{aligned}
f'_\varepsilon(t) &\le 2\,\|P^{1/2}u(t)\|_{L^2(\mathbb{R}^N)}\left\|\frac{x\,u(t)}{(1+\varepsilon|x|^2)^2}\right\|_{L^2(\mathbb{R}^N)}\\
&\le 2\,\|P^{1/2}u(t)\|_{L^2(\mathbb{R}^N)}\sqrt{2f_\varepsilon(t)}.
\end{aligned}$$

Thus $u \in L^\infty(I;\mathcal{D})$ implies $f_\varepsilon(t)$ is uniformly bounded in $\varepsilon > 0$. Letting $\varepsilon \to +0$, we conclude $|x|u(t) \in L^2(\mathbb{R}^N)$. Moreover, we see

$$\left|\|xu(t_2)\|^2_{L^2(\mathbb{R}^N)} - \|xu(t_1)\|^2_{L^2(\mathbb{R}^N)}\right| \le 4\left|\int_{t_1}^{t_2} \|P^{1/2}u(s)\|_{L^2(\mathbb{R}^N)}\,ds\right|$$

and hence $|x|u \in C([-T,T];L^2(\mathbb{R}^N))$. □

Remark 5.4. Let $u(t)$ be a solution to **(NLS)** with (1.8). If $u(0) \in \mathcal{D}\cap D(|x|)$, then we see

$$\frac{d}{dt}\|xu(t)\|^2_{L^2(\mathbb{R}^N)} = 4\,\mathrm{Im}\int_{\mathbb{R}^N} x\overline{u(t,x)}\cdot\nabla u(t,x)\,dx.$$

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Remark 5.5. Global solution $u \in C(\mathbb{R}; \mathcal{D}) \cap C^1(\mathbb{R}; \mathcal{D}^*)$, that is, local solution to **(NLS)** with (1.8) can be extended globally in time, is followed under the condition

$$\lambda > 0, \text{ or } \lambda < 0 \text{ with } 1 \le p < 1 + \frac{2 - 2\alpha - \beta}{N}.$$

To end this we show

$$G(u) \ge -C_1(\|u\|_{L^2(\mathbb{R}^N)}) - \frac{1}{2+\varepsilon}\|u\|_{\mathcal{D}}^2. \tag{5.12}$$

See also **(G6)** as in the conditions of the energy methods by Okazawa–Suzuki–Yokota [40]. The case $\lambda > 0$ is simple: $G(u) \ge 0$. Thus we assume $\lambda < 0$. Applying (5.5) and (5.6), we see

$$\begin{aligned} G(u) &\ge -\frac{|\lambda|}{2(p+1)} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^{p+1}\,|u(y)|^{p+1}}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dx\,dy \\ &\ge -\frac{|\lambda|}{2(p+1)} C_{\mathrm{HLS}}\, C_{ss}^{2p+2}\, \|u\|_{L^2(\mathbb{R}^N)}^{2(p+1)(1-s)}\, \|u\|_{\mathcal{D}}^{2(p+1)s}. \end{aligned}$$

Here $p < 1 + (2 - 2\alpha - \beta)/N$ implies $2(p+1)s = N(p-1) + 2\alpha + \beta < 2$. Thus the Young inequality ensures (5.12). The energy conservation and (5.12) follow a priori uniform boundedness estimate for $\|u(t)\|_{\mathcal{D}}$. Obviously, if $\lambda < 0$, $p = 1 + (2 - 2\alpha - \beta)/N$, and $\|u_0\|_{L^2(\mathbb{R}^N)}$ is sufficiently small, then local solution to **(NLS)** with (1.8) also can be extended globally in time.

Remark 5.6. We can also prove the well-posedness of **(NLS)** with other nonlinearities under (1.4) with $\delta_V = 0$. See Examples 5.7–5.9 for suitable nonlinearities.

- **Local well-posedness**: for any $u_0 \in \mathcal{D}$ there exists a unique local solution to **(NLS)** $u \in C([-T, T]; \mathcal{D}) \cap C^1([-T, T]; \mathcal{D}^*))$. Moreover, u satisfies

  $$\|u(t)\|_{L^2(\mathbb{R}^N)} = \|u_0\|_{L^2(\mathbb{R}^N)}, \quad E(u(t)) = E(u_0),$$

  where $E(\varphi) := (1/2)\,\|(1+P)^{1/2}\varphi\|_{L^2(\mathbb{R}^N)}^2 + G(\varphi)$. Furthermore, if u_0 is also belongs to $D(|x|)$, then $u \in C([-T, T]; D(|x|))$;
- **Global well-posedness**: the local solution u to **(NLS)** can be extended globally in time $u \in C(\mathbb{R}; \mathcal{D}) \cap C^1(\mathbb{R}; \mathcal{D}^*))$.

Example 5.7 *(See also Okazawa–Suzuki–Yokota [40]).* Let $g : \mathbb{C} \to \mathbb{C}$ be power type nonlinearities so that

(N1) $g(0) = 0$ and there exist $p \in [1, (N+2)/(N-2))$ and $K \ge 0$ such that

$$|g(z_1) - g(z_2)| \le K(1 + |z_1|^{p-1} + |z_2|^{p-1})|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{C};$$

(N2) $g(x) \in \mathbb{R}$ $(x > 0)$ and $g(e^{i\theta}z) = e^{i\theta}g(z)$ $(z \in \mathbb{C}, \theta \in \mathbb{R})$;

(N3) there exist $q \in [1, 1 + 4/N)$ and $K_1, K_2 \geq 0$ such that

$$F(|z|) \geq -K_1|z|^2 - K_2|z|^{q+1} \quad \forall z \in \mathbb{C},$$

where F is the primitive integral of g:

$$F(x) := \int_0^x g(s)\,ds \quad \forall\, x > 0.$$

In such a case, we can define the energy functionals of g as

$$G(u) := \int_{\mathbb{R}^N} F(|u(x)|)\,dx.$$

Suppose **(N1)** and **(N2)**. Then we can ensure the local well-posedness for **(NLS)**. If we assume further **(N3)**, then the global well-posedness for **(NLS)** is achieved.

Example 5.8 *(See also Suzuki [48, Theorem 4.1]).* Let $g(u) := \lambda\,|x|^{-r}|u|^{p-1}u$ $(\lambda \in \mathbb{R})$. In such a case, we can define the energy functionals of g as

$$G(u) := \lambda \int_{\mathbb{R}^N} \frac{|u(x)|^{p+1}}{|x|^r}\,dx.$$

Suppose $0 < r < 2$ and $1 \leq p < (N + 2 - 2r)/(N - 2)$. Then we can ensure the local well-posedness for **(NLS)**. If we assume further that $\lambda > 0$ or $\lambda < 0$ with $p < 1 + (4 - 2r)/N$, then the global well-posedness for **(NLS)** is achieved.

Example 5.9 *(See also Suzuki [46]).* Let $g(u) := u\,K[k](|u|^2)$, where

$$K[k](f) := \int_{\mathbb{R}^N} k(x, y) f(y)\,dy.$$

Here k satisfies the following conditions:

(K1) k is a symmetric real-valued function, that is, $k(x, y) = k(y, x) \in \mathbb{R}$ a.a. $x,\ y \in \mathbb{R}^N$;
(K2) $k \in L_x^\infty(L_y^\infty) + L_x^\beta(L_y^\alpha)$ for some $\alpha,\ \beta \in [1, \infty]$ such that $\alpha \leq \beta$ and $\alpha^{-1} + \beta^{-1} < 4/N$;
(K3) $k_-(x, y) := 0 \vee (-k(x, y))$ belongs to $L_x^\infty(L_y^\infty) + L_x^{\tilde\beta}(L_y^{\tilde\alpha})$ for some $\tilde\alpha,\ \tilde\beta \in [1, \infty]$ such that $\tilde\alpha \leq \tilde\beta$ and $\tilde\alpha^{-1} + \tilde\beta^{-1} < 2/N$.

Note that $L_x^\beta(L_y^\alpha)$ is the family of $k(x, y)$ such that $k(x, \cdot) \in L^\alpha(\mathbb{R}^N)$ a.a. $x \in \mathbb{R}^N$ with $\big\| \|k(x, \cdot)\|_{L^\alpha(\mathbb{R}^N)} \big\|_{L^\beta(\mathbb{R}^N)} < \infty$. In such a case, we can define the energy functionals of g as

$$G(u) := \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} k(x, y)\, |u(x)|^2\, |u(y)|^2\, dx\, dy.$$

Suppose **(K1)** and **(K2)**. Then we can ensure the local well-posedness for **(NLS)**. If we assume further **(K3)**, then the global well-posedness for **(NLS)** is achieved.

6. Concluding remarks

As mentioned in the introduction, local and global well-posedness for **(NLS)** with the non-linear term as in Examples 5.7–5.9 can be also consider under (1.4) with $\delta_V > 0$. The case is more simple since $\mathcal{D}$ is just equal to $H^1(\mathbb{R}^N)$ (see (2.2)). Here the virial identity for the solution **(NLS)** can be considered under (1.4) with both $\delta_V > 0$ and $\delta_V = 0$. To derive the virial identity for **(NLS)** we consider the approximated problems

$$\begin{cases} i\, \dfrac{\partial u_{\varepsilon,\delta,a}}{\partial t} = -\Delta u_{\varepsilon,\delta,a} + V_{\delta,a} u_{\varepsilon,\delta,a} + g_\varepsilon(u_{\varepsilon,\delta,a}) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u(0) = u_0, \end{cases} \qquad \textbf{(NLS)}_{\varepsilon,\delta}$$

where $V_{\delta,a}$ is the approximation of V: $V_{\delta,a} := \rho_\delta * [V + a|x|^{-2}] \in C^\infty(\mathbb{R}^N)$ (if $\delta_V = 0$, then $a > 0$; if $\delta_V > 0$, then we set $a = 0$ beforehand). Here ρ_δ is a Friedrichs mollifier. $g_\varepsilon(u)$ is a suitable approximation of $g(u)$ and $\rho_\delta * v$ is the convolution of ρ_δ and v. For example, if $g(u) := \lambda\, |u|^{p-1} u$, then $g_\varepsilon(u) := \lambda\, \rho_\varepsilon * [|\rho_\varepsilon * u|^{p-1}(\rho_\varepsilon * u)]$. We can solve $\textbf{(NLS)}_{\varepsilon,\delta}$ globally in $H^2(\mathbb{R}^N) = D(1 - \Delta)$ and $H^2(\mathbb{R}^N) \cap D(|x|)$. Thus we can construct the virial identity for $\textbf{(NLS)}_{\varepsilon,\delta}$ in simple calculations. Letting $\delta \to +0$, $a \to +0$ (only the case $\delta_V = 0$), and $\varepsilon \to +0$ (with using the Strichartz estimates) in order, we can also verify the virial identity for **(NLS)**. See Suzuki [47,49] for more precise calculations and demonstrations. For example, let $g(u) := \lambda |u|^{p-1} u$ ($\lambda \in \mathbb{R}$, $1 \le p < 1 + 4/(N-2)$). Then we see the following virial identity:

$$\frac{d^2}{dt^2} \|x\, u(t)\|^2_{L^2(\mathbb{R}^N)} = 8\, \|P^{1/2} u(t)\|^2_{L^2(\mathbb{R}^N)} + \frac{4\lambda N(p-1)}{p+1} \int_{\mathbb{R}^N} |u(t)|^{p+1}\, dx.$$

Let $g(u) := \lambda(|x|^{-\gamma} * |u|^{p+1})|u|^{p-1} u$ ($\lambda \in \mathbb{R}$, $0 < \gamma < (N \wedge 4)$, $1 \le p < 1 + (4-\gamma)/(N-2)$). Then we see

$$\begin{aligned} \frac{d^2}{dt^2} \|x\, u(t)\|^2_{L^2(\mathbb{R}^N)} &= 8\, \|P^{1/2} u(t)\|^2_{L^2(\mathbb{R}^N)} \\ &\quad + \frac{4\lambda[N(p-1)+\gamma]}{p+1} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(t,x)|^{p+1}\, |u(t,y)|^{p+1}}{|x-y|^\gamma}\, dx\, dy. \end{aligned}$$

Our results depend on the eigenvalue problems for $-\Delta_S + V$ in $L^2(S^{N-1})$. Thus the regularity of V can be weaker, for example, $N \ge 3$ and $V|_{S^{N-1}} \in L^\infty(S^{N-1})$ (continuity is disable). Thus we can treat $V(x) := c\, \mathrm{sign}(x_N)|x|^{-2}$ for sufficiently small $c > 0$. More singular case, for example, $V|_{S^{N-1}} \in L^p(S^{N-1})$ can be considered. If $p > (N-1)/2$, then the eigenvalue problems are well-posed. Here the regularity of eigenfunctions Y is $W^{2,p}(S^{N-1})$. Since the Sobolev embeddings $W^{2,p}(S^{N-1}) \subset C(S^{N-1})$ are continuous, we can find $Y \in H^2(S^{N-1}) \cap C(S^{N-1})$. Note that if $p > N-1$, then the eigenfunctions belongs also to $C^1(S^{N-1})$. If $V|_{S^{N-1}} \in L^p(S^{N-1})$ $(p > (N-1)/2)$ satisfies (1.4) with $\delta_V \geq 0$, then (3.4) and (1.5) can be established. Typical examples of more singular V are $V(x) = V(x', x_N) = c\,|x|^{-2(1-a)}|x'|^{-2a}$ $(0 < a < 1)$ and $V(x) = V(x', x_N) = c\,|x|^{-2(1-a)}|x_N|^{-2a}$ $(0 < a < 1/(N-1))$. Let θ be the principal latitude of S^{N-1}. Then

$$\begin{aligned}\Delta_{S^{N-1}} &= \partial_\theta^2 + \frac{(N-2)\cos\theta}{\sin\theta}\partial_\theta + \frac{1}{(\sin\theta)^2}\Delta_{S^{N-2}},\\ d\sigma_{S^{N-1}} &= (\sin\theta)^{N-2}\,d\theta\,d\sigma_{S^{N-2}}.\end{aligned}$$

If $V(x) = c\,|x|^{-2(1-a)}|x'|^{-2a}$ $(0 < a < 1)$, then

$$\|V\|^p_{L^p(S^{N-1})} = |c|^p \int_0^\pi (\sin\theta)^{-2ap}\,(\sin\theta)^{N-2}\,d\theta \int_{S^{N-2}} d\sigma_{S^{N-2}}.$$

The integration is converged if $-2ap + N - 2 > -1$, that is, $2ap < N-1$. Also if $V(x) = c\,|x|^{-2(1-a)}|x_N|^{-2a}$ $(0 < a < 1/(N-1))$, then

$$\|V\|^p_{L^p(S^{N-1})} = |c|^p \int_0^\pi (\cos\theta)^{-2ap}\,(\sin\theta)^{N-2}\,d\theta \int_{S^{N-2}} d\sigma_{S^{N-2}}.$$

The integration is converged if $-2ap > -1$, that is, $2ap < 1$. Here one of the critical case $V = c|x'|^{-2} = c(\sin\theta)^{-2}$ is open for the eigenvalue problems and evaluations of energy spaces. As the key of the consideration, we have the Hardy type inequality on sphere (see Xiao [53])

$$\begin{aligned}&\frac{(N-2)^2}{4}\int_{S^{N-1}} \frac{|u|^2}{(\sin\theta)^2}\,d\sigma_{S^{N-1}}\\ &\leq \int_{S^{N-1}} |\nabla_S u|^2\,d\sigma_{S^{N-1}} + C\int_{S^{N-1}} |u|^2\,d\sigma_{S^{N-1}}.\end{aligned}$$

In a view of the proof of Lemma 4.4, we have proved

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}} \|(\chi(P)|x|^{-1})(P+\zeta)^{-1}(|x|^{-1}\chi(P))f\|_{L^2(\mathbb{R}^N)} \leq C\|f\|_{L^2(\mathbb{R}^N)},$$

where $\chi(P) := [-\Delta_S + V + (N-2)^2/4]^{1/4}$ (derivatives towards the amplitudes). Since ν_m^2 is the m-th eigenvalue of $-\Delta_S + V + (N-2)^2/4$, $\nu_m^{1/2}$ is the m-th eigenvalue of $\chi(P)$. The local smoothing estimates may bring the Strichartz estimates for $\exp(-itP)$ with $V|_{S^{N-1}} \in L^p(S^{N-1})$.

We can deeply study the case $N = 2$: $\Delta_S = \partial_\phi^2$ $(0 \leq \phi \leq 2\pi)$ with the periodic boundary condition $u(0) = u(2\pi)$, $u_\phi(0) = u_\phi(2\pi)$. Thus we have $-\Delta_S + V = -\partial_\phi^2 + V(\cos\phi, \sin\phi)$ with the periodic boundary condition. This is an ordinary differential operator. Note that multiplicity of eigenvalue is at most 2. We can expect the explicit formula of $\exp(-itP)$ as in [24]. Here the

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non-endpoint Strichartz estimates as like (1.6) and (1.7) can be constructed under the condition $\delta_V \geq 0$. See the latter of proof of Theorem 3 in [11] (we also need (4.15)).

Here in a way similar to this article, we may consider the evaluations of energy spaces and the Strichartz estimates for

$$\begin{cases} i\dfrac{\partial u}{\partial t} = (-\Delta + V)u & \text{in } \mathbb{R} \times \mathcal{C}_\Omega, \\ u(0,x) = u_0(x) & \text{on } \mathcal{C}_\Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R} \times \partial\mathcal{C}_\Omega, \end{cases}$$

where V satisfies the inverse-square condition $V(\mu x) = \mu^{-2}V(x)$, Ω is a open subset of S^{N-1}, and $\mathcal{C}_\Omega := \{x \in \mathbb{R}^N \setminus \{0\}; x/|x| \in \Omega\}$ (conic domain). Indeed, we may solve the semilinear Schrödinger equations with inverse-square type potentials on cones. For example, we can consider $\mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^N; x_N > 0\}$. The odd reflection approach implies the Strichartz estimates for $\exp(it\Delta_D)$ are fully verified; Δ_D is the Laplacian in $\mathbb{R}^N_+$ with the Dirichlet condition. After that we follow the arguments as in Sections 2, 3 and 4 (with replacing S^{N-1} by $S^{N-1}_+ := \{\widehat{x} = (x', x_N) \in S^{N-1}; x_N > 0\}$.

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