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Journal of Differential Equations 280 (2021) 754–804

Journal of Differential Equations

www.elsevier.com/locate/jde

Global well-posedness for the energy-critical focusing nonlinear Schrödinger equation on $\mathbb{T}^4$

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Received 30 June 2020; revised 13 December 2020; accepted 24 January 2021
Available online 4 February 2021

Abstract

In this paper, we consider the global (in time) well-posedness for the focusing cubic nonlinear Schrödinger equation (NLS) on 4-dimensional tori –either rational or irrational– and with initial data in H^1. We prove that if a maximal-lifespan solution of the focusing cubic NLS $u : I \times \mathbb{T}^4 \to \mathbb{C}$ satisfies $\sup_{t\in I} \|u(t)\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}$, then it is a global solution. W denotes the ground state on Euclidean space, which is a stationary solution of the corresponding focusing equation in $\mathbb{R}^4$. As a consequence, we also construct the global solution with some threshold conditions related to the modified energy of the initial data which is the energy modified by the mass of the initial data and the best constants of Sobolev embedding on $\mathbb{T}^4$.

MSC: 35Q55; 49K40

Keywords: Periodic nonlinear Schrödinger equation; Focusing; Global well-posedness; Ground state

1. Introduction

In this paper, we consider the cubic nonlinear Schrödinger equation (NLS) in the periodic setting $x \in \mathbb{T}^4_\lambda$

$$(i\partial_t + \Delta)u = \mu u|u|^2, \tag{1.1}$$

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https://doi.org/10.1016/j.jde.2021.01.031

where $\mu = \pm 1$ (-1: the focusing case, $+1$: the defocusing case). And $u : \mathbb{R} \times \mathbb{T}^4_\lambda \to \mathbb{C}$ is a complex-valued function of time space $\mathbb{R}$ and spatial space $\mathbb{T}^4_\lambda$, a general rectangular tori, i.e.

$$\mathbb{T}^4_\lambda := \mathbb{R}^4/(\prod_{i=1}^{4} \lambda_i \mathbb{Z}), \qquad \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

where $\lambda_i \in (0, \infty)$ for $i = 1, 2, 3, 4$. Specifically, if the ratio of arbitrary two $\lambda_i' s$ in $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is an irrational number, then $\mathbb{T}^4_\lambda$ is called an irrational torus, otherwise $\mathbb{T}^4_\lambda$ is called a rational torus. Since our proof does not change no matter either rational or irrational tori. For the convenience, we use $\mathbb{T}^4 := \mathbb{T}^4_\lambda$ hence-forth in the paper.

Solutions of (1.1) conserve in both the mass of u:

$$M(u)(t) := \int_{\mathbb{T}^4} |u(t)|^2 \, dx \tag{1.2}$$

and the energy of u:

$$E(u)(t) := \frac{1}{2} \int_{\mathbb{T}^4} |\nabla u(t)|^2 \, dx + \frac{1}{4} \mu \int_{\mathbb{T}^4} |u(t)|^4 \, dx. \tag{1.3}$$

On $\mathbb{R}^d$, the scaling symmetry plays an important role in the well-posedness (existence, uniqueness and continuous dependence of the data to solution map) problem of initial value problem (IVP) for NLS:

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1} u, & p > 1 \\ u(0, x) = u_0(x) \in \dot{H}^s(\mathbb{R}^d). \end{cases} \tag{1.4}$$

The IVP (1.4) is scale invariant in the Sobolev norm $\dot{H}^{s_c}$, where $s_c := \frac{d}{2} - \frac{2}{p-1}$ is called the scaling critical regularity.

For H^s data with $s > s_c$ (sub-critical regime), the local well-posedness of the IVP (1.4) in sub-critical regime was proven by Cazenave-Weissler [13]. For H^s data with $s = s_c$ (critical regime), Bourgain [4] first proved the large data global well-posedness and scattering for the defocusing energy-critical ($s_c = 1$) NLS in $\mathbb{R}^3$ with the radially symmetric initial data in $\dot{H}^1$ by introducing an induction method on the size of energy and a refined Morawetz inequality. A different proof of the same result was given by Grillakis in [33]. Then a breakthrough was made by Colliander-Keel-Staffilani-Takaoka-Tao [14]. Their work extended the results of Bourgain [4] and Grillakis [33]. They proved global well-posedness and scattering of the energy-critical problem in $\mathbb{R}^3$ for general large data in $\dot{H}^1$. Then similar results were proven by Ryckman-Vişan [56] and Vişan [58] on the higher dimension $\mathbb{R}^d$ spaces. Furthermore, Dodson proved mass-critical ($s_c = 0$) global well-posedness results for $\mathbb{R}^d$ in his series of papers [19,21,22].

For the corresponding problems on the tori, the Strichartz estimates on rational tori $\mathbb{T}^d$ (see [32,59,46] for the Strichartz estimates in the Euclidean spaces $\mathbb{R}^d$), which prove the local well-posedness of the periodic NLS, was initially developed by Bourgain [3]. In [3], the number theoretical related lattice counting arguments were used, hence this method worked better in the rational tori than irrational tori. Recently Bourgain-Demeter's work [7] proved the optimal Strichartz estimates on both rational and irrational tori via a totally different approach which doesn't depend on the lattice counting lattice. Also there are other important references [5,15,12,6,34,51,16,18,28] on the Strichartz estimates on the tori and global existence of solution of the Cauchy problem in sub-critical regime. On the general compact manifolds, Burq-Gerard-Tzvetkov derived the Strichartz type estimates and applied these estimates to the global well-posedness of NLS on compact manifolds in a series of their papers [8,9,11,10]. We also refer to [62,31,36,37] and references therein for the other results of global existence sub-critical NLS on compact manifolds.

In the critical regime, Herr-Tataru-Tzvetkov [40] studied the global existence of the energy-critical NLS on $\mathbb{T}^3$ and first proved the global well-posedness with small initial data in H^1. They used a crucial trilinear Strichartz type estimates in the context of the critical atomic spaces U^p and V^p, which were originally developed in unpublished work on wave maps by Tararu. These atomic spaces were systematically formalized by Hadac-Herr-Koch [35] (see also [52][41]) and now the atomic spaces U^p and V^p are widely used in the field of the critical well-posedness theory of nonlinear dispersive equations. The large data global well-posedness result of the energy-critical NLS on rational $\mathbb{T}^3$ was proven by Ionescu-Pausader [43], which is the first large data critical global well-posedness result of NLS on a compact manifold. In a series of papers, Ionescu-Pausader [43][44] and Ionescu-Pausader-Staffilani [45] developed a method to obtain energy-critical large data global well-posedness in more general manifolds ($\mathbb{T}^3$, $\mathbb{T}^3 \times \mathbb{R}$, and $\mathbb{H}^3$) based on the corresponding results on the Euclidean spaces in the same dimension. So far, their method has been successfully applied to other manifolds in several following papers [55,57,60,61].

In this paper, we prove the global well-posedness result of focusing energy-critical NLS below the ground state[1] on the both rational and irrational tori in the 4-dimension. To the best of the author's knowledge, this is the first result establishing global well-posedness for the focusing energy-critical NLS below the ground state on a compact domain.

1.1. The main result

In the focusing case ($\mu = -1$), we prove global well-posedness (GWP) when we have kinetic energy of the solution is a priori all time bounded by the kinetic energy of the ground state W in $\mathbb{R}^4$. Moreover,

$$W(x) = W(x,t) = \frac{1}{1 + \frac{|x|^2}{8}} \qquad \text{in } \dot{H}^1(\mathbb{R}^4) \tag{1.5}$$

which is a stationary solution of the focusing case of (1.1) and also solves the elliptic equation in $\mathbb{R}^4$

$$\Delta W + |W|^2 W = 0. \tag{1.6}$$

Then we define a constant C_4 by using the stationary solution W. And also C_4 is the best constant in Sobolev embedding (see Remark 2.2).

[1] Here we say 'below the ground state' in the sense of (1.10) in Theorem 1.1.

$$\|W\|_{\dot{H}^1(\mathbb{R}^4)}^2 = \|W\|_{L^4(\mathbb{R}^4)}^4 := \frac{1}{C_4^4} \qquad \text{and then} \qquad E_{\mathbb{R}^4}(W) = \frac{1}{4C_4^4}, \tag{1.7}$$

where $E_{\mathbb{R}^4}(W)$ is the energy of W in the Euclidean space $\mathbb{R}^4$:

$$E_{\mathbb{R}^4}(W) := \frac{1}{2}\int_{\mathbb{R}^4} |\nabla W(x)|^2\, dx - \frac{1}{4}\int_{\mathbb{R}^4} |W(x)|^4\, dx. \tag{1.8}$$

Theorem 1.1 *(GWP of the focusing NLS). Assume $u_0 \in H^1(\mathbb{T}^4)$. Suppose that a maximal-lifespan solution $u : I \times \mathbb{T}^4 \to \mathbb{C}$ to the initial value problem*

$$(i\partial_t + \Delta)u = -u|u|^2, \qquad u(0) = u_0 \tag{1.9}$$

obeys

$$\sup_{t\in I} \|u(t)\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}. \tag{1.10}$$

Then for any $T \in [0, +\infty)$, u is a solution in $C([-T, T] : H^1(\mathbb{T}^4))$.

In the above theorem, in particular, the solution space we used in the proof is $X^1([-T, T]) \subset C([-T, T] : H^1(\mathbb{T}^4))$. It is an adapted atomic space (see Definition 3.5).

Remark 1.2. It is worth mentioning that the proof of Theorem 1.1 is general for both rational and irrational tori. The only two places in the whole proof, which may be effected by the rationality/irrationality of tori, are the Strichartz estimates (Lemma 3.18) and the extinction lemma (Lemma 5.4). The Strichartz estimates[2] (Lemma 3.18) are proved by the breakthrough work of Bourgain-Demeter [7] for both rational and irrational tori. The whole construction of the local well-posedness and stability theory (in particular, the bilinear estimate (Lemma 4.2) and the refined nonlinear estimate (Proposition 4.4)) essentially relies on the Strichartz estimates and hence it is general for both rational and irrational tori. Also the proof of the extinction lemma (Lemma 5.4) does not depend on the rationality/irrationality of tori.

In fact, Deng-Germain-Guth proved Strichartz estimates over larger time scales for the Schrödinger equation on generic[3] irrational tori than the Strichartz estimates in Lemma 3.18. Based on these Strichartz estimates over larger time scales, Deng [17] established the polynomial growth of Sobolev norms for the energy-critical NLS on generic irrational tori in three dimensions.

Remark 1.3. The analog result of Theorem 1.1 for the energy-critical NLS on 3-dimensional tori $\mathbb{T}^3$ can also be expected providing the corresponding GWP for the focusing energy-critical NLS in $\mathbb{R}^3$. However, the GWP for the focusing energy-critical NLS in $\mathbb{R}^3$ with the non-radial data below the ground state is still an open problem.

[2] Note that the scale invariant version of Strichartz estimates (see [51] for references) is used in our paper.

[3] The generic irrational tori means that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ works outside of a certain null set.

For some technical reason[4] in the focusing case, we shall introduce two modified energies of u:

$$E_*(u)(t) := \frac{1}{2}(\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|u(t)\|^2_{L^2(\mathbb{T}^4)}) - \frac{1}{4}\|u(t)\|^4_{L^4(\mathbb{T}^4)}, \tag{1.11}$$

and

$$E_{**}(u)(t) := \frac{1}{2}(\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|u(t)\|^2_{L^2(\mathbb{T}^4)}) - \frac{1}{4}\|u(t)\|^4_{L^4(\mathbb{T}^4)} + \frac{c_*^2 C_4^4}{4}\|u(t)\|^4_{L^2(\mathbb{T}^4)}, \tag{1.12}$$

where c_* is a fixed constant determined by the Sobolev embedding on $\mathbb{T}^4$ in Lemma 2.1. By the definitions (1.11)(1.12), $E_*(u)(t)$ and $E_{**}(u)(t)$ are conserved in time.

We also introduce $\|u\|_{H^1_*(\mathbb{T}^4)}$ as a modified inhomogeneous Sobolev norm:

$$\|u\|^2_{H^1_*(\mathbb{T}^4)} = \|u\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|u\|^2_{L^2(\mathbb{T}^4)} \tag{1.13}$$

Obviously, $H^1_*(\mathbb{T}^4)$-norm and $H^1(\mathbb{T}^4)$-norm are two comparable norms ($\|u\|_{H^1_*(\mathbb{T}^4)} \simeq \|u\|_{H^1(\mathbb{T}^4)}$). Using the above modified energies and modified Sobolev norm, we can prove the following corollary about GWP of the focusing NLS under some conditions of initial data.

Corollary 1.4. *Assume that $u_0 \in H^1(\mathbb{T}^4)$ satisfying*

$$\|u_0\|_{H^1_*(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}, \quad E_*(u_0) < E_{\mathbb{R}^4}(W); \tag{1.14}$$

or

$$\|u_0\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}, \quad E_{**}(u_0) < E_{\mathbb{R}^4}(W), \tag{1.15}$$

where $E_*(u_0)$ and $E_{**}(u_0)$ are two modified Energies defined in (1.11) and (1.12), and $E_{\mathbb{R}^4}(W)$ is the Energy in the Euclidean space defined in (1.8). Then for any $T \in [0,\infty)$, there exists a unique global solution $u \in X^1([-T,T])$ of the initial value problem (1.9). In addition, the mapping $u_0 \to u$ extends to a continuous mapping from $H^1(\mathbb{T}^4)$ to $X^1([-T,T])$ for any $T \in [0,\infty)$.

Remark 1.5. In the defocusing case ($\mu = +1$), we could also achieve the similar global well-posedness result for (1.1) as Theorem 1.1 but with arbitrary large initial data in $H^1(\mathbb{T}^4)$ by a similar proof. In particular, in the defocusing case the local theory (Section 4), Euclidean profiles (Section 5) and profile decomposition (Section 6) are the same as in the focusing case, however, we should run an induction on the energy and mass of u instead of $\|u\|_{L^\infty_t \dot{H}^1}$ in Section 7.

The main parts in the proof of Theorem 1.1 will follow the concentration-compactness framework of Kenig-Merle [47], which is a deep and broad road map to deal with critical problems (see also in [48][49]). Our first step is to obtain the critical local well-posedness theory and the stability theory of (1.1) in $\mathbb{T}^4$. For that purpose, we follow Herr-Tataru-Tzvetkov's idea [40][41]

[4] See Section 2 for details.

and introduce the adapted critical spaces X^s and Y^s, which are frequency localized modification of atomic spaces U^p and V^p, as our solution spaces and nonlinear spaces. Applied Lemma 3.18 to the atomic spaces with strip decomposition technique in time-space frequency space, we obtain a crucial bilinear estimate and then the local well-posedness of (1.1). Then we measure the solution in a weaker critical space-time norm Z, which plays a similar role as $L^{10}_{x,t}$ norm in [14]. On one hand, equipped with Z-norm, we obtain the refined bilinear estimate (Lemma 4.2) and hence it is proven that the solution stays regular as long as Z-norm stays finite (i.e. global well-posedness with a priori Z-norm bound).

On the other hand, we show that concentration of a large amount of the Z-norm in finite time is self-defeating. The reason is as follows. Concentration of a large amount the Z-norm in finite time can only happen around a space-time point, which can be considered as a Euclidean-like solution. To implement this, by a contradiction argument, we construct a sequence of initial data which implies a sequence of solutions and leads the Z-norm towards infinity. Then following the profile decomposition idea (firstly by Gerard [30] in Sobolev embedding and Merle-Vega [54] in the Schrödinger equation), we perform a linear profile decomposition of the sequence of initial data with one Scale-1-profile and a series of Euclidean profiles that concentrate at space-time points. We get nonlinear profiles by running the linear profiles along the nonlinear Schrödinger flow as initial data. By the contradiction condition, the scattering properties of nonlinear Euclidean profiles and the defect of interaction between different profiles show that there is actually at most one profile which is the Euclidean profile. And the corresponding nonlinear Euclidean profile is just the Euclidean-like solution we want. Euclidean-like solution can be interpreted in some sense as solutions in the Euclidean space $\mathbb{R}^4$, however, this kind of concentration as a Euclidean-like solution is prevented by the global well-posedness and scattering theory on the Euclidean space $\mathbb{R}^4$ in Dodson's work [20].

Comparing with the defocusing case, to achieve the above concentration-compactness process in the focusing case under the ground state (as in Theorem 1.1), we need to introduce a new ingredient which is the almost orthogonality of nonlinear profiles (Lemma 6.6). In Section 7, we rely on Lemma 6.6 heavily to make sure the induction process on the norm $\|u\|_{L_t^\infty \dot{H}^1}$ runs properly. Note that in the defocusing case, one should run the induction on the energy and mass of u which are conserved, so this almost orthogonality of nonlinear profiles is not necessary.

In the focusing case, the global well-posedness result for arbitrary initial data in H^1 is usually not excepted. A natural question that arises is what initial conditions ensure the global solution of the focusing NLS on tori. In order to answer this, we need one more new ingredient: the energy trapping lemma on tori (see Section 2). Recall that in the energy-critical focusing NLS on $\mathbb{R}^d$, Kenig-Merle [47] first proved the global well-posedness and scattering with initial data below a ground state threshold ($E_{\mathbb{R}^d}(u_0) < E_{\mathbb{R}^d}(W)$ and $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$) in the radial case ($d \geq 3$). And then the corresponding results without the radial conditions were proven by Killp-Vişan [50] ($d \geq 5$) and Dodson [20] ($d = 4$). We also refer to [25,42,29,26,53,23,24] for more focusing NLS results. As is known that the conditions in $\mathbb{R}^d$ are $E_{\mathbb{R}^d}(u_0) < E_{\mathbb{R}^d}(W)$ and $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ which is tightly related to the Sobolev embedding with the best constant in $\mathbb{R}^d$, however the sharp version of Sobolev embedding (Lemma 2.1) is quite different. So compared to the conditions for initial data in Euclidean space $\mathbb{R}^d$, the conditions in Corollary 1.4 are also modified on tori. On $\mathbb{T}^d$, we modify the energy and Sobolev norm with some L^2 norm based on the best constants of Sobolev embedding (Lemma 2.1) in $\mathbb{T}^d$, so that the modified conditions together with Sobolev embedding derive the energy trapping property which controls the Sobolev norm globally in time. Note that similar modified energies are also introduced in the focusing NLS on the hyperbolic space (see [2][27]). In Section 2, we will discuss the Sobolev embedding and the energy trapping lemma in detail.

1.2. Outline of the following paper

The rest of the paper is organized as follows. In Section 2, we prove the energy trapping property for the focusing NLS on tori. In Section 3, we introduce the adapted atomic spaces X^s, Y^s and Z norm and provide some corresponding embedding properties of the spaces. In Section 4, we use Herr-Tataru-Tzvetkov's method and Bourgain-Demeter's sharp Strichartz estimate to develop a large-data local well-posedness and stability theory for (1.1). In Section 5, we study the behavior of Euclidean-like solutions to the linear and nonlinear equation concentrating to a point in space and time. In Section 6, we introduce a profile decomposition to measure the defects of compactness in the Strichartz inequality and in particular we also prove the almost orthogonality of nonlinear profiles. In Section 7, we prove the main theorem (Theorem 1.1) except for a lemma. In Section 8, we prove the remaining lemma about approximate solutions.

Acknowledgments

The author is greatly indebted to his advisor, Andrea R. Nahmod, for suggesting this problem and her patient guidance and warm encouragement over the past years. The author would like to thank Benoît Pausader for prompting the author to study the focusing case. The author also would like to thank Chenjie Fan for his helpful discussions on the focusing case of this paper. The author acknowledges support from the National Science Foundation through Andrea R. Nahmod's grants NSF-DMS 1201443 and NSF-DMS 1463714.

2. Energy trapping for the focusing NLS

Before proceeding to the proof of main theorem (Theorem 1.1), we explain how Theorem 1.1 implies Corollary 1.4 in the focusing case by using the energy trapping argument. In this section, we prove the energy trapping argument in $\mathbb{T}^4$ which is different from the energy trapping argument in the Euclidean spaces.

Lemma 2.1 *(Sobolev embedding with best constants by [1][39][38]). Let $f \in H^1(\mathbb{T}^4)$, then there exists a positive constant c_*, such that*

$$\|f\|^2_{L^4(\mathbb{T}^4)} \leq C_4^2(\|f\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|f\|^2_{L^2(\mathbb{T}^4)}), \tag{2.1}$$

where C_4 is the best constant of this inequality.

Remark 2.2. C_4 is the same constant as expressed in (1.7), because C_4 is also the best constant in Sobolev embedding in $\mathbb{R}^4$, $\|f\|^2_{L^4(\mathbb{R}^4)} \leq C_4^2\|f\|^2_{\dot{H}^1(\mathbb{R}^4)}$, and the function $W(x)$ holds the Sobolev embedding with the best constant C_4 in $\mathbb{R}^4$.

Remark 2.3. Since $\|u\|^2_{H^1_*(\mathbb{T}^4)} = \|u\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|u\|^2_{L^2(\mathbb{T}^4)}$, the Sobolev embedding (Lemma 2.1) can be also written in the form:

$$\|f\|_{L^4(\mathbb{T}^4)}^2 \leq C_4^2 \|f\|_{H_*^1(\mathbb{T}^4)}^2.$$

Suppose $c_{opt} := \inf\{c_* : c_* \text{ holds } (2.1)\}$. By taking $f = 1$, it's easy to check that $c_{opt} \geq C_4^{-2}(\text{volume of } \mathbb{T}^4)^{-1/2}$.

Lemma 2.4. *(i) Suppose $f \in H^1(\mathbb{T}^4)$ and $\delta_0 > 0$ satisfying*

$$\|f\|_{H_*^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)} \qquad \text{and} \qquad E_*(f) < (1-\delta_0)E_{\mathbb{R}^4}(W), \tag{2.2}$$

then there exists $\bar{\delta} = \bar{\delta}(\delta_0) > 0$ such that

$$\|f\|_{H_*^1(\mathbb{T}^4)}^2 < (1-\bar{\delta})\|W\|_{\dot{H}^1(\mathbb{R}^4)}^2 \tag{2.3}$$

$$\|f\|_{H_*^1(\mathbb{T}^4)}^2 - \|f\|_{L^4(\mathbb{T}^4)}^4 \geq \bar{\delta}\|f\|_{H_*^1(\mathbb{T}^4)}^2, \tag{2.4}$$

and in particular

$$E_*(f) \geq \frac{1}{4}(1+\bar{\delta})\|f\|_{H_*^1(\mathbb{T}^4)}^2. \tag{2.5}$$

(ii) Suppose $f \in H^1(\mathbb{T}^4)$ and $\delta_0 > 0$ satisfying

$$\|f\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)} \qquad \text{and} \qquad E_{**}(f) < (1-\delta_0)E_{\mathbb{R}^4}(W), \tag{2.6}$$

then there exists $\bar{\delta} = \bar{\delta}(\delta_0) > 0$ such that

$$\|f\|_{\dot{H}^1(\mathbb{T}^4)}^2 < (1-\bar{\delta})\|W\|_{\dot{H}^1(\mathbb{R}^4)}^2 \tag{2.7}$$

$$\|f\|_{\dot{H}^1(\mathbb{T}^4)}^2 - \|f\|_{L^4(\mathbb{T}^4)}^4 + 2c_*\|f\|_{L^2(\mathbb{T}^4)} + c_*^2 C_4^4 \|f\|_{L^2(\mathbb{T}^4)}^4 \geq \bar{\delta}\|f\|_{\dot{H}^1(\mathbb{T}^4)}^2, \tag{2.8}$$

and in particular

$$E_{**}(f) \geq \frac{1}{4}(1+\bar{\delta})\|f\|_{\dot{H}^1(\mathbb{T}^4)}^2. \tag{2.9}$$

Proof. In the proof of part (i), we almost identically follow the proof of Lemma 3.4 in Kenig-Merle's paper [47], but use $H_*^1(\mathbb{T}^4)$-norm instead of $\dot{H}^1(\mathbb{T}^4)$-norm. Consider a quadratic function $g_1 = \frac{1}{2}y - \frac{C_4^4}{4}y^2$, and plug in $\|f\|_{H_*^1(\mathbb{T}^4)}^2$, by Sobolev embedding (Lemma 2.1) and the assumption (2.2), we have that

$$\begin{aligned} g_1(\|f\|_{H_*^1}^2) &= \frac{1}{2}\|f\|_{H_*^1}^2 - \frac{C_4^4}{4}\|f\|_{H_*^1}^4 \\ &\leq \frac{1}{2}\|f\|_{H_*^1}^2 - \frac{1}{4}\|f\|_{L^4}^4 = E_*(f) \\ &< (1-\delta_0)E_{\mathbb{R}^4}(W) = (1-\delta_0)g_1(\|W\|_{\dot{H}^1(\mathbb{R}^4)}^2). \end{aligned} \tag{2.10}$$

It is easy to know $\|f\|^2_{H^1_*(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}$, from (2.10) and the property of quadratic function g_1, where $\bar{\delta} \sim \delta_0^{\frac{1}{2}}$.

Then choose $g_2(y) = y - C_4^4 y^2$, if plug in $\|f\|^2_{H^1_*(\mathbb{T}^4)}$, by Sobolev embedding (Lemma 2.1), we have that

$$g_2(\|f\|^2_{H^1_*(\mathbb{T}^4)}) = \|f\|^2_{H^1_*(\mathbb{T}^4)} - C_4^4\|f\|^4_{H^1_*(\mathbb{T}^4)} \le \|f\|^2_{H^1_*(\mathbb{T}^4)} - \|f\|^4_{L^4(\mathbb{T}^4)}. \tag{2.11}$$

Since $g_2(0) = 0$, $g_2''(y) = -2C_4^4 < 0$ and $\|f\|^2_{H^1_*(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}$, by Jensen's inequality and (1.7),

$$g_2(\|f\|^2_{H^1_*(\mathbb{T}^4)}) > g_2((1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)})\frac{\|f\|^2_{H^1_*(\mathbb{T}^4)}}{(1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}} = \bar{\delta}\|f\|^2_{H^1_*(\mathbb{T}^4)}. \tag{2.12}$$

Together (2.11) and (2.12), we get (2.4).

By (2.4), we get (2.5)

$$E_*(f) = \frac{1}{4}\|f\|^2_{H^1_*(\mathbb{T}^4)} + \frac{1}{4}(\|f\|^2_{H^1_*(\mathbb{T}^4)} - \|f\|^4_{L^4(\mathbb{T}^4)}) \ge \frac{1}{4}(1+\bar{\delta})\|f\|^2_{H^1_*(\mathbb{T}^4)}.$$

The proof of part (ii) would be similar with part (i). Under the assumptions (2.6) of part (ii), by squaring Sobolev embedding (Lemma 2.1), we have that

$$C_4^4\|f\|^4_{\dot{H}^1(\mathbb{T}^4)} \ge \|f\|^4_{L^4} - 2c_*\|f\|^2_{L^2(\mathbb{T}^4)} - c_*^2C_4^4\|f\|^4_{L^2(\mathbb{T}^4)} \tag{2.13}$$

Plugging $\|f\|^2_{\dot{H}^1(\mathbb{T}^4)}$ into g_1, by (2.13), we hold that

$$\begin{aligned} g_1(\|f\|^2_{\dot{H}^1}) &= \frac{1}{2}\|f\|^2_{\dot{H}^1} - \frac{C_4^4}{4}\|f\|^4_{\dot{H}^1} \\ &\le \frac{1}{2}\|f\|^2_{\dot{H}^1} - \frac{1}{4}\|f\|^4_{L^4} + \frac{c_*}{2}\|f\|^2_{L^2(\mathbb{T}^4)} + \frac{c_*^2C_4^4}{4}\|f\|^4_{L^2(\mathbb{T}^4)} = E_{**}(f) \\ &< (1-\delta_0)E_{\mathbb{R}^4}(W) = (1-\delta_0)g_1(\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}). \end{aligned} \tag{2.14}$$

It is easy to know $\|f\|^2_{\dot{H}^1(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}$, from (2.14) and the property of quadratic function g_1, where $\bar{\delta} \sim \delta_0^{\frac{1}{2}}$. Similarly, we can also hold (2.8)(2.9) under the assumption (2.6). □

Theorem 2.5 *(Energy trapping). (i) Let u be a solution of IVP (1.9), such that for $\delta_0 > 0$*

$$\|u_0\|_{H^1_*(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}, \quad E_*(u_0) < (1-\delta_0)E_{\mathbb{R}^4}(W); \tag{2.15}$$

Let $I \ni 0$ be the maximal interval of existence, then there exists $\bar{\delta} = \bar{\delta}(\delta_0) > 0$ such that for all $t \in I$

$$\|u(t)\|^2_{H^1_*(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|_{\dot{H}^1(\mathbb{R}^4)}, \tag{2.16}$$

$$\|u(t)\|^2_{H^1_*(\mathbb{T}^4)} - \|u(t)\|^4_{L^4(\mathbb{T}^4)} \geq \bar{\delta}\|u(t)\|^2_{H^1_*(\mathbb{T}^4)}, \tag{2.17}$$

and in particular

$$E_*(u)(t) \geq \frac{1}{4}(1+\bar{\delta})\|u(t)\|^2_{H^1_*(\mathbb{T}^4)}. \tag{2.18}$$

(ii) Let u be a solution of IVP (1.9), such that for $\delta_0 > 0$

$$\|u_0\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}, \quad E_{**}(u_0) < (1-\delta_0)E_{\mathbb{R}^4}(W); \tag{2.19}$$

Let $I \ni 0$ be the maximal interval of existence, then there exists $\bar{\delta} = \bar{\delta}(\delta_0) > 0$ such that for all $t \in I$

$$\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|_{\dot{H}^1(\mathbb{R}^4)}, \tag{2.20}$$

$$\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)} - \|u(t)\|^4_{L^4(\mathbb{T}^4)} + 2c_*\|u(t)\|^2_{L^2(\mathbb{T}^4)} + c_*^2 C_4^4\|u(t)\|^4_{L^2(\mathbb{T}^4)} \geq \bar{\delta}\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)}, \tag{2.21}$$

and in particular

$$E_{**}(u)(t) \geq \frac{1}{4}(1+\bar{\delta})\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)}. \tag{2.22}$$

Proof. By the conservation of energy and mass, this theorem directly from Lemma 2.4 by the continuity argument. □

Remark 2.6. The energy trapping lemma (Theorem 2.5) shows that if the initial data satisfies the condition (1.14) or (1.15) then the solution $u(t)$ satisfies $\|u(t)\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}$ for all t in the lifespan of the solution. So Theorem 1.1 implies Corollary 1.4. In particular, we also obtain that $E_*(u)(t) \simeq \|u(t)\|^2_{H^1_*(\mathbb{T}^4)}$ under the assumption (2.15) and $E_{**}(u)(t) \simeq \|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)}$ under the assumption (2.19) by Theorem 2.5.

3. Adapted function spaces

In this section, we introduce X^s and Y^s spaces which are based on the atomic space U^p and V^p which were firstly applied to PDEs in [35][40][41], while we'll use the X^s and Y^s spaces in the proof of global well-posedness. $\mathcal{H}$ is a separable Hilbert space on $\mathbb{C}$, and $\mathcal{Z}$ denotes the set of finite partitions $-\infty = t_0 < t_1 < \ldots < t_K = \infty$ of the real line, with the convention that $v(\infty) := 0$ for any function $v : \mathbb{R} \to \mathcal{H}$.

Definition 3.1 *(Definition 2.1 in [40]).* Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset \mathcal{H}$ with $\sum_{k=0}^K \|\phi_k\|^p_{\mathcal{H}} = 1$ and $\phi_0 = 0$. A U^p-atom is a piecewise defined function $a : \mathbb{R} \to \mathcal{H}$ of the form

$$a = \sum_{k=1}^{K} \mathbb{1}_{[t_{k-1},t_k)} \phi_{k-1}.$$

The atomic Banach space $U^p(\mathbb{R}, \mathcal{H})$ is then defined to be the set of all functions $u : \mathbb{R} \to \mathcal{H}$ such that

$$u = \sum_{j=1}^{\infty} \lambda_j a_j, \qquad \text{for } U^p\text{-atoms } a_j, \qquad \{\lambda_j\}_j \in \ell^1,$$

with the norm

$$\|u\|_{U^p} := \inf\{\sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j,\ \lambda_j \in \mathbb{C} \text{ and } a_j \text{ an } U^p \text{ atom}\}.$$

Here $\mathbb{1}_I$ denotes the indicator function over the time interval I.

Definition 3.2 *(Definition 2.2 in [40]).* Let $1 \leq p < \infty$. The Banach space $V^p(\mathbb{R}, \mathcal{H})$ is defined to be the set of all functions $v : \mathbb{R} \to \mathcal{H}$ with $v(\infty) := 0$ and $v(-\infty) := \lim_{t \to -\infty} v(t)$ exists, such that

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^{K} \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{\mathcal{H}}^p\right)^{\frac{1}{p}} \quad \text{is finite.}$$

Likewise, let V_-^p denote the closed subspace of all $v \in V^p$ with $\lim_{t \to -\infty} v(t) = 0$. $V_{-,rc}^p$ means all right-continuous V_-^p functions.

Remark 3.3 *(Some embedding properties).* Note that for $1 \leq p \leq q < \infty$,

$$U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow U^q(\mathbb{R}, \mathcal{H}) \hookrightarrow L^\infty(\mathbb{R}, \mathcal{H}), \tag{3.1}$$

and functions in $U^p(\mathbb{R}, \mathcal{H})$ are right continuous, and $\lim_{t \to -\infty} u(t) = 0$ for each $u \in U^p(\mathbb{R}, \mathcal{H})$. Also note that,

$$U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow V_{-,rc}^p(\mathbb{R}, \mathcal{H}) \hookrightarrow U^q(\mathbb{R}, \mathcal{H}). \tag{3.2}$$

Definition 3.4 *(Definition 2.5 in [40]).* For $s \in \mathbb{R}$, we let $U_\Delta^p H^s$, respectively $V_\Delta^p H^s$, be the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that $t \mapsto e^{-it\Delta} u(t)$ is in $U^p(\mathbb{R}, H^s)$, respectively in $V^p(\mathbb{R}, H^s)$ with norm

$$\|u\|_{U^p(\mathbb{R},H^s)} := \|e^{-it\Delta} u(t)\|_{U^p(\mathbb{R},H^s)}, \quad \|u\|_{V^p(\mathbb{R},H^s)} := \|e^{-it\Delta} u(t)\|_{V^p(\mathbb{R},H^s)}.$$

Definition 3.5 *(Definition 2.6 in [40]).* For $s \in \mathbb{R}$, we define X^s as the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that for every $n \in \mathbb{Z}^d$, the map $t \mapsto e^{it|n|^2} \widehat{u(t)}(n)$ is in $U^2(\mathbb{R}, \mathcal{C})$, and with the norm

$$\|u\|_{X^s} := (\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \|e^{it|n|^2}\widehat{u(t)}(n)\|^2_{U^2_t})^{\frac{1}{2}} \quad \text{is finite.} \tag{3.3}$$

Definition 3.6 *(Definition 2.7 in [40]).* For $s \in \mathbb{R}$, we define Y^s as the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that for every $n \in \mathbb{Z}^d$, the map $t \mapsto e^{it|n|^2}\widehat{u(t)}(n)$ is in $V^2_{rc}(\mathbb{R}, \mathcal{C})$, and with the norm

$$\|u\|_{Y^s} := (\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \|e^{it|n|^2}\widehat{u(t)}(n)\|^2_{V^2_t})^{\frac{1}{2}} \quad \text{is finite.} \tag{3.4}$$

Note that

$$U^2_\Delta H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V^2_\Delta H^s. \tag{3.5}$$

Proposition 3.7 *(Proposition 2.10 in [35]). Suppose $u := e^{it\Delta}\phi$ which is a free Schrödinger solution, then we obtain that*

$$\|u\|_{X^s([0,\delta])} \leq \|\phi\|_{H^s}.$$

Proof. Since $u := e^{it\Delta}\phi$, then $\|u\|_{X^s} = (\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \|\widehat{\phi}(n)\|^2_{U^2_t})^{\frac{1}{2}} \leq \|\phi\|_{H^s}$. $\square$

Remark 3.8. Compared with Bourgain's $X^{s,b}$ first introduced in Bourgain,

$$\|v\|_{X^{s,b}} = \|e^{-it\Delta}v\|_{H^b_t H^s_x},$$
$$\|v\|_{U^p_\Delta H^s} = \|e^{-it\Delta}v\|_{U^p_t H^s_x},$$
$$\|v\|_{V^p_\Delta H^s} = \|e^{-it\Delta}v\|_{V^p_t H^s_x}.$$

And also later we will see the atomic spaces enjoy the similar duality and transfer principle properties with $X^{s,b}$.

Remark 3.9. Follow the definitions, it's easy to check the following embedding properties:

$$U^2_\Delta H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V^2_\Delta H^s \hookrightarrow L^\infty(\mathbb{R}, H^s). \tag{3.6}$$

Definition 3.10 *(X^s and Y^s restricted to a time interval I).* For intervals $I \subset \mathbb{R}$, we define $X^s(I)$ and $Y^s(I)$ as following

$$X^s(I) := \{v \in C(I : H^s) : \|v\|_{X^s(I)} := \sup_{J\subset I,\ |J|\leq 1} \inf_{\tilde{v}|_J = v} \|\tilde{v}\|_{X^s} < \infty\},$$

and

$$Y^s(I) := \{v \in C(I : H^s) : \|v\|_{Y^s(I)} := \sup_{J\subset I,\ |J|\leq 1} \inf_{\tilde{v}|_J = v} \|\tilde{v}\|_{Y^s} < \infty\}.$$

We will consider our solution in $X^1(I)$ spaces, and then let's introduce nonlinear norm $N(I)$.

Definition 3.11 *(Nonlinear norm $N(I)$).* Let $I = [0, T]$, then

$$\|f\|_{N(I)} := \left\| \int_0^t e^{i(t-t')\Delta} f(t')dt' \right\|_{X^1(I)}$$

Proposition 3.12 *(Proposition 2.11 in [41]). Let $s > 0$. For $f \in L^1(I, H^1(\mathbb{T}^4))$ we have*

$$\|f\|_{N(I)} \le \sup_{v \in Y^{-1}(I)} \left| \int_I \int_{\mathbb{T}^4} f(t,x)\overline{v(t,x)}dxdt \right|. \tag{3.7}$$

Now, we will need a weaker norm Z, which plays a similar role as $L^{10}_{t,x}$ norm in [14].

Definition 3.13.

$$\|v\|_{Z(I)} := \sup_{J \subset I,\ |J| \le 1} \left(\sum_{N \in 2^{\mathbb{Z}}} N^2 \|P_N v\|^4_{L^4(\mathbb{T}^4 \times J)} \right)^{\frac{1}{4}}.$$

Remark 3.14. $\|v\|_{Z(I)}$ actually can be considered as

$$\sum_{p \in \{p_1, p_2, \cdots, p_k\}} \sup_{J \subset I,\ |J| \le 1} \left(\sum_{N \in 2^{\mathbb{Z}}} N^{6-p} \|P_N v\|^p_{L^p(\mathbb{T}^4 \times J)} \right)^{\frac{1}{p}},$$

and $\{p_1, p_2, \cdots, p_k\}$ should be the L^p estimates that we need to use in the proof of nonlinear estimate. In our case, we only need $\|P_N u\|_{L^4(\mathbb{T}^4 \times I)} \lesssim \|P_N u\|_{Z(I)}$ in the proof of the nonlinear estimates, so we choose $\{p_1, p_2, \cdots, p_k\} = \{4\}$.

The following property shows us that $Z(I)$ is a weaker norm than $X^1(I)$.

Proposition 3.15.

$$\|v\|_{Z(I)} \lesssim \|v\|_{X^1(I)}.$$

Proof. By the definition of $Z(I)$ and the following Strichartz type estimates Proposition 3.19, we obtain that

$$\begin{aligned} \sup_{J \subset I,\ |J| \le 1} \left(\sum_{N \text{ dyadic number}} N^2 \|P_N v\|^4_{L^4(\mathbb{T}^4 \times J)} \right)^{\frac{1}{4}} &\lesssim \sup_{J \subset I,\ |J| \le 1} \left(\sum_{N \text{ dyadic number}} N^4 \|P_N v\|^4_{X^0(J)} \right)^{\frac{1}{4}} \\ &\lesssim \|v\|_{X^1(I)}. \quad \square \end{aligned}$$

Proposition 3.16 *(Proposition 2.10 in [35]). Suppose $u := e^{it\Delta}\phi$ which is a free Schrödinger solution, then we obtain that*

$$\|u\|_{X^s([0,T))} \leq \|\phi\|_{H^s}.$$

Proof. Since $u := e^{it\Delta}\phi$, then $\|u\|_{X^s} = (\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \|\widehat{\phi}(n)\|^2_{U_t^2})^{\frac{1}{2}} \leq \|\phi\|_{H^s}$. □

Finally we state a 'transfer principle' proposition about the atomic space U^p_Δ which is firstly introduced and proved in [35].

Proposition 3.17 *(Proposition 2.19 in [35]). Let $T_0 : L^2 \times \cdots \times L^2 \to L^1_{loc}$ be m-linear operator. Assume that for some $1 \leq p,\ q \leq \infty$*

$$\|T_0(e^{it\Delta}\phi_1, \cdots, e^{it\Delta}\phi_m)\|_{L^p(\mathbb{R}, L^q_x(\mathbb{T}^d))} \lesssim \prod_{i=1}^m \|\phi_i\|_{L^2(\mathbb{T}^d)}. \tag{3.8}$$

Then, there exists an extension $T : U^p_\Delta \times \cdots \times U^p_\Delta \to L^p(\mathbb{R}, L^q(\mathbb{T}^d))$ satisfying

$$\|T(u_1, \cdots, u_m)\|_{L^p(\mathbb{R}, L^q(\mathbb{T}^d))} \lesssim \prod_{i=1}^m \|u_i\|_{U^p_\Delta}; \tag{3.9}$$

and such that $T(u_1, \cdots, u_m)(t, \cdot) = T_0(u_1(t), \cdots, u_m(t))(\cdot)$, a.e.

Lemma 3.18 *(Strichartz type estimates [3][7]). If $p > 3$, then*

$$\|P_N e^{it\Delta} f\|_{L^p_{t,x}([-1,1]\times\mathbb{T}^4)} \lesssim_p N^{2-\frac{6}{p}} \|f\|_{L^2_x}$$

and

$$\|P_C e^{it\Delta} f\|_{L^p_{t,x}([-1,1]\times\mathbb{T}^4)} \lesssim_p N^{2-\frac{6}{p}} \|f\|_{L^2_x}$$

where C is a cube of side length N.

By the 'transfer principle' proposition (Proposition 3.17) and Strichartz type estimate Lemma 3.18, we obtain the following corollary:

Corollary 3.19. *If $p > 3$, then*

$$\|P_N v\|_{L^p([-1,1]\times\mathbb{T}^4)} \lesssim_p N^{2-\frac{6}{p}} \|v\|_{U^p_\Delta([-1,1])},$$

and

$$\|P_C v\|_{L^p([-1,1]\times\mathbb{T}^4)} \lesssim_p N^{2-\frac{6}{p}} \|v\|_{U^p_\Delta([-1,1])},$$

where C is a cube of side length N.

4. Local well-posedness and stability theory

In this section, we present large-data local well-posedness, and stability results. Although Herr, Tataru, and Tzvetkov's idea [41] together with Bourgain and Demeter's result [7] gives the local well-posedness of (1.9), to obtain the stability results, we need a refined nonlinear estimate and the corresponding refined local well-posedness result.

Definition 4.1 *(Definition of solutions).* Given an interval $I \subseteq \mathbb{R}$, we call $u \in C(I : H^1(\mathbb{T}^4))$ a strong solution of (1.9) if $u \in X^1(I)$ and u satisfies that for all $t, s \in I$,

$$u(t) = e^{it\Delta}u_0 + i\int_s^t e^{i(t-t')\Delta}u(t')|u(t')|^2 dt'.$$

First, we need to introduce

$$\|u\|_{Z'(I)} := \|u\|_{Z(I)}^{\frac{3}{4}}\|u\|_{X^1(I)}^{\frac{1}{4}}. \tag{4.1}$$

Lemma 4.2 *(Bilinear estimates in [41]). Assuming $|I| \leq 1$ and $N_1 \geq N_2$, there holds that*

$$\|P_{N_1}u_1 P_{N_2}u_2\|_{L^2_{x,t}(\mathbb{T}^4\times I)} \lesssim (\frac{N_2}{N_1} + \frac{1}{N_2})^{\kappa}\|P_{N_1}u_1\|_{Y^0(I)}\|P_{N_2}u_2\|_{Y^1(I)} \tag{4.2}$$

for some $\kappa > 0$.

Remark 4.3. This Bilinear estimate is Proposition 2.8 in [41]. The proof of Lemma 4.2 relies on L^p estimates in Lemma 3.18 (for some $p < 4$). In the proof not only we need the decoupling properties for spatial frequency, but also we need further trip partitions to apply the decoupling properties for time frequency.

Let's introduce an refined nonlinear estimate.

Proposition 4.4 *(Refined nonlinear estimate). For $u_k \in X^1(I)$, $k = 1, 2, 3$, $|I| \leq 1$, we hold the estimate*

$$\|\prod_{k=1}^{3}\widetilde{u_k}\|_{N(I)} \lesssim \sum_{\{i,j,k\}=\{1,2,3\}} \|u_i\|_{X^1(I)}\|u_j\|_{Z'(I)}\|u_k\|_{Z'(I)} \tag{4.3}$$

where $\widetilde{u_k} = u_k$ or $\widetilde{u_k} = \overline{u_k}$ for $k = 1, 2, 3$.

In particular, if there exist constants $A, B > 0$, such that $u_1 = P_{>A}u_1$, $u_2 = P_{>A}u_2$ and $u_3 = P_{<B}u_3$, then we obtain that

$$\|\prod_{k=1}^{3}\widetilde{u_k}\|_{N(I)} \lesssim \|u_1\|_{X^1(I)}\|u_2\|_{Z'(I)}\|u_3\|_{Z'(I)} + \|u_2\|_{X^1(I)}\|u_1\|_{Z'(I)}\|u_3\|_{Z'(I)}. \tag{4.4}$$

Proof. By the Proposition 3.12, and suppose N_0, N_1, N_2, N_3 are dyadic, and WLOG we assume $N_1 \geq N_2 \geq N_3$.

$$\|\prod_{k=1}^{3} \widetilde{u_k}\|_{N(I)} \lesssim \sup_{\|u_0\|_{Y^{-1}}} |\int_{\mathbb{T}^4\times I} \overline{u_0} \prod_{k=1}^{3} \widetilde{u_k}\, dxdt|$$

$$\leq \sup_{\|u_0\|_{Y^{-1}}} \sum_{N_0, N_1\geq N_2\geq N_3} |\int_{\mathbb{T}^4\times I} \overline{P_{N_0}u_0} \prod_{k=1}^{3} P_{N_k}\widetilde{u_k}\, dxdt|$$

Then we know $N_1 \sim \max(N_2, N_0)$ by the spatial frequency orthogonality. There are two cases:

1. $N_0 \sim N_1 \geq N_2 \geq N_3$;
2. $N_0 \leq N_2 \sim N_1 \geq N_3$.

Case 1: $N_0 \sim N_1 \geq N_2 \geq N_3$
By Cauchy-Schwartz inequality and Lemma 4.2, we have that

$$\begin{aligned} &|\int \overline{P_{N_0}u_0} P_{N_1}\widetilde{u_1} P_{N_2}\widetilde{u_2} P_{N_3}\widetilde{u_3}\, dxdt| \leq \|P_{N_0}u_0 P_{N_2}u_2\|_{L^2_{x,t}} \|P_{N_1}u_1 P_{N_3}u_3\|_{L^2_{x,t}} \\ \lesssim &(\frac{N_3}{N_1} + \frac{1}{N_3})^{\kappa}(\frac{N_2}{N_0} + \frac{1}{N_2})^{\kappa} \|P_{N_0}u_0\|_{Y^0(I)} \|P_{N_1}u_1\|_{Y^0(I)} \|P_{N_2}u_2\|_{X^1(I)} \|P_{N_3}u_3\|_{X^1(I)} \end{aligned} \tag{4.5}$$

Assume $\{C_j\}$ is a cube partition of size N_2, and $\{C_k\}$ is a cube partition of size N_3. By $\{P_{C_j} P_{N_0}u_0 P_{N_2}u_2\}_j$ and $\{P_{C_k} P_{N_1}u_1 P_{N_3}u_3\}_k$ are both almost orthogonality, Corollary 3.19 and definition of Z norm, we obtain that

$$\begin{aligned} &|\int \overline{P_{N_0}u_0} P_{N_1}\widetilde{u_1} P_{N_2}\widetilde{u_2} P_{N_3}\widetilde{u_3}\, dxdt| \leq \|P_{N_0}u_0 P_{N_2}u_2\|_{L^2_{x,t}} \|P_{N_1}u_1 P_{N_3}u_3\|_{L^2_{x,t}} \\ \lesssim &(\sum_{C_j} \|P_{C_j} P_{N_0}u_0 P_{N_2}u_2\|^2_{L^2_{x,t}})^{\frac{1}{2}} (\sum_{C_k} \|P_{C_k} P_{N_1}u_1 P_{N_3}u_3\|^2)^{\frac{1}{2}} \\ \lesssim &(\sum_{C_j} \|P_{C_j} P_{N_0}u_0\|^2_{L^4_{x,t}} \|P_{N_2}u_2\|^2_{L^4_{x,t}})^{\frac{1}{2}} (\sum_{C_k} \|P_{C_k} P_{N_1}u_1\|^2_{L^4_{x,t}} \|P_{N_3}u_3\|^2_{L^4_{x,t}})^{\frac{1}{2}} \\ \lesssim &(\sum_{C_j} \|P_{C_j} P_{N_0}u_0\|^2_{Y^0(I)} (N_2^{\frac{1}{2}} \|P_{N_2}u_2\|_{L^4_{x,t}})^2)^{\frac{1}{2}} (\sum_{C_k} \|P_{C_k} P_{N_1}u_1\|^2_{Y^0(I)} (N_3^{\frac{1}{2}} \|P_{N_3}u_3\|_{L^4_{x,t}})^2)^{\frac{1}{2}} \\ \lesssim &\|P_{N_0}u_0\|_{Y^0(I)} \|P_{N_1}u_1\|_{Y^0(I)} \|P_{N_2}u_2\|_{Z(I)} \|P_{N_2}u_2\|_{Z(I)}. \end{aligned} \tag{4.6}$$

Interpolating (4.5) with (4.6) we obtain that

$$\begin{aligned} &|\int \overline{P_{N_0}u_0} P_{N_1}\widetilde{u_1} P_{N_2}\widetilde{u_2} P_{N_3}\widetilde{u_3}\, dxdt| \\ \lesssim &(\frac{N_3}{N_1} + \frac{1}{N_3})^{\kappa_1}(\frac{N_2}{N_0} + \frac{1}{N_2})^{\kappa_1} \|P_{N_0}u_0\|_{Y^{-1}(I)} \|P_{N_1}u_1\|_{X^1(I)} \|P_{N_2}u_2\|_{Z'(I)} \|P_{N_2}u_2\|_{Z'(I)}. \end{aligned} \tag{4.7}$$

Sum (4.7) over all $N_0 \sim N_1 \geq N_2 \geq N_3$,

$$\sum_{N_0 \sim N_1 \geq N_2 \geq N_3} (\frac{N_3}{N_1} + \frac{1}{N_3})^{\kappa_1} (\frac{N_2}{N_0} + \frac{1}{N_2})^{\kappa_1} \|P_{N_0} u_0\|_{Y^{-1}(I)} \|P_{N_1} u_1\|_{X^1(I)} \|P_{N_2} u_2\|_{Z'(I)} \|P_{N_2} u_2\|_{Z'(I)}$$
$$\lesssim \|u_0\|_{Y^{-1}(I)} \|u_1\|_{X^1 I} \|u_2\|_{Z'(I)} \|u_3\|_{Z'(I)}.$$

Case 2: $N_0 \leq N_2 \sim N_1 \geq N_3$

Similarly we have that

$$\begin{aligned} &|\int \overline{P_{N_0} u_0} P_{N_1} \widetilde{u_1} P_{N_2} \widetilde{u_2} P_{N_3} \widetilde{u_3}\, dxdt| \\ &\lesssim (\frac{N_3}{N_1} + \frac{1}{N_3})^{\kappa} (\frac{N_0}{N_2} + \frac{1}{N_0})^{\kappa} \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} u_1\|_{Y^0(I)} \|P_{N_2} u_2\|_{X^1(I)} \|P_{N_3} u_3\|_{X^1(I)}. \end{aligned} \tag{4.8}$$

Similar with (4.6), we obtain that:

$$\begin{aligned} &|\int \overline{P_{N_0} u_0} P_{N_1} \widetilde{u_1} P_{N_2} \widetilde{u_2} P_{N_3} \widetilde{u_3}\, dxdt| \\ &\lesssim \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} u_1\|_{Y^0(I)} \|P_{N_2} u_2\|_{Z(I)} \|P_{N_3} u_3\|_{Z(I)}. \end{aligned} \tag{4.9}$$

We interpolate (4.8) with (4.9) and sum over $N_0 \leq N_2 \sim N_1 \geq N_3$. Then we have that

$$\begin{aligned} &\sum_{N_0 \leq N_2 \sim N_1 \geq N_3} |\int \overline{P_{N_0} u_0} P_{N_1} \widetilde{u_1} P_{N_2} \widetilde{u_2} P_{N_3} \widetilde{u_3}\, dxdt| \\ &\lesssim \|P_{N_0} u_0\|_{Y^{-1}(I)} \|P_{N_1} u_1\|_{X^1(I)} \|P_{N_2} u_2\|_{Z'(I)} \|P_{N_3} u_3\|_{Z'(I)}. \end{aligned}$$

Next we summarize these two cases and similarly consider $N_1 \geq N_3 \geq N_2$, $N_2 \geq N_1 \geq N_3$, $N_2 \geq N_3 \geq N_1$, $N_3 \geq N_1 \geq N_2$, and $N_3 \geq N_2 \geq N_1$, we can get the desired estimate (4.3).

In particular, if there exist constants $A, B > 0$, such that $u_1 = P_{>A} u_1$, $u_2 = P_{>A} u_2$ and $u_3 = P_{<B} u_3$, then we only consider the sum when $N_1 \geq N_2 \gtrsim N_3$ and $N_2 \geq N_1 \gtrsim N_3$. So we get the estimate (4.4). □

Proposition 4.5 *(Local well-posedness). Assume that $E > 0$ is fixed. There exists $\delta_0 = \delta_0(E)$ such that if*

$$\|e^{it\Delta} u_0\|_{Z'(I)} < \delta$$

for some $\delta \leq \delta_0$, some interval $0 \in I$ with $|I| \leq 1$ and some function $u_0 \in H^1(\mathbb{T}^4)$ satisfying $\|u_0\|_{H^1} \leq E$, then there exists a unique strong solution to (1.1) $u \in X^1(I)$ such that $u(0) = u_0$. Besides

$$\|u - e^{it\Delta} u_0\|_{X^1(I)} \leq \delta^{\frac{5}{3}}. \tag{4.10}$$

Proof. Consider the set

$$S = \{u \in X^1(I) : \|u\|_{X^1(I)} \le 2E, \qquad \|u\|_{Z'(I)} \le a\},$$

the mapping

$$\Phi(v) = e^{it\Delta}u_0 - i\mu \int_0^t e^{i(t-s)\Delta} v(s)|v(s)|^2 \, ds.$$

For $u, v \in S$, by Proposition 4.4, there exists a constant $C > 0$, we have that

$$\begin{aligned} &\|\Phi(u) - \Phi(v)\|_{X^1(I)} \\ \le & C\left(\|u\|_{X^1(I)} + \|v\|_{X^1(I)}\right)\left(\|u\|_{Z'(I)} + \|v\|_{Z'(I)}\right)\|u - v\|_{X^1(I)} \\ \le & CEa\|u - v\|_{X^1(I)} \end{aligned}$$

Similarly, using Proposition 3.16 and nonlinear estimate Proposition 4.4, we also obtain that

$$\begin{aligned} \|\Phi(u)\|_{X^1(I)} &\le \|\Phi(0)\|_{X^1(I)} + \|\Phi(u) - \Phi(0)\|_{X^1(I)} \\ &\le \|u_0\|_{H^1} + CEa^2 \end{aligned}$$

and

$$\begin{aligned} \|\Phi(u)\|_{Z'(I)} &\le \|\Phi(0)\|_{Z'(I)} + \|\Phi(u) - \Phi(0)\|_{Z'(I)} \\ &\le \delta + CEa^2. \end{aligned}$$

Now, we choose $a = 2\delta$ and we let $\delta_0 = \delta_0(E)$ be small enough. We see that Φ is a contraction on S, so we have a fixed point u. And it's easy to check (4.10) and uniqueness in $X^1(I)$. □

As a consequence, we also get a global well-posedness result with small initial data which will be used in Section 7. Note that the proof of the following proposition is analogous to the proof in Herr-Tataru-Tzvetkov [40], hence I will skip the proof of the following proposition.

Proposition 4.6 *(Small data global well-posedness). If $\|\phi\|_{H^1(\mathbb{T}^4)} = \delta \le \delta_0$, then the unique strong solution with initial data ϕ is global and satisfies*

$$\|u\|_{X^1([-2,2])} \le 2\delta$$

and moreover

$$\|u - e^{it\Delta}\phi\|_{X^1([-2,2])} \lesssim \delta^2.$$

Lemma 4.7 *(Z-norm controls the global existence). Assume that $I \subseteq \mathbb{R}$ is a bounded open interval.*

1. *If E is a nonnegative finite number, that u is a strong solution of (1.1) and*

$$\|u\|_{L_t^\infty(I,H^1)} \leq E.$$

Then, if

$$\|u\|_{Z(I)} < +\infty$$

there exists an open interval J with $\bar{I} \subset J$ such that u can be extended to a strong solution of (1.1) on J, besides

$$\|u\|_{X^1(I)} \leq C(E, \|u\|_{Z(I)}).$$

2. *(GWP with a priori bound) Assume C is some positive finite number and we have a priori bound $\|u\|_{Z(I)} < C$, for any solution u of (1.1) in the interval I, then this IVP (1.9) is well-posedness on I. (In particular, if u blows up in finite time, then u blows up in the Z-norm.)*

Proof. Suppose $I = (0, T)$. For any $\varepsilon > 0$, there exists $T_1 < T$ such that $\|u\|_{Z(T_1,T)} \leq \varepsilon$.

By the continuity arguments of $h(s) = \|e^{i(t-T_1)\Delta}u(T_1)\|_{Z'(T_1,T_1+s)}$ where $T_1 \geq T - 1$ such that $\|u\|_{Z(T_1,T)} \leq \varepsilon$. Then combined the part *(1)* and Proposition 4.5, it's clear to show the part *(2)*. □

Proposition 4.8 *(Stability). Assume I is an open bounded interval, $\mu \in [-1, 1]$, and $\tilde{u} \in X^1(I)$ satisfies the approximate Schrödinger equation*

$$(i\partial_t + \Delta)\tilde{u} = \mu\tilde{u}|\tilde{u}|^2 + e, \qquad \text{on } \mathbb{T}^4 \times I. \tag{4.11}$$

Assume in addition that

$$\|\tilde{u}\|_{Z(I)} + \|\tilde{u}\|_{L_t^\infty(I,H^1(\mathbb{T}^4))} \leq M, \tag{4.12}$$

for some $M \in [1, \infty]$. Assume $t_0 \in I$ and $u_0 \in H^1(\mathbb{T}^4)$ is such that the smallness condition:

$$\|u_0 - \tilde{u}(0)\|_{H^1(\mathbb{T}^4)} + \|e\|_{N(I)} \leq \varepsilon \tag{4.13}$$

holds for some $0 < \varepsilon < \varepsilon_1$, where $\varepsilon_1 \leq 1$. $\varepsilon_1 = \varepsilon_1(M) > 0$ is a small constant.

Then there exists a strong solution $u \in X^1(I)$ of the NLS

$$(i\partial_t + \Delta)u = \mu u|u|^2,$$

such that $u(t_0) = u_0$ and

$$\begin{aligned} \|u\|_{X^1(I)} + \|\tilde{u}\|_{X^1(I)} &\leq C(M), \\ \|u - \tilde{u}\|_{X^1(I)} &\leq C(M)\varepsilon. \end{aligned} \tag{4.14}$$

Proof. First, we need to show the short time Stability, which follows a similar proof as the proof of Proposition 4.5. Then, by using Lemma 4.7, we extend to the entire time interval. □

5. Euclidean profiles

In this section, we introduce the Euclidean profiles which is linear and nonlinear solutions on $\mathbb{T}^4$ concentrated at a point. The Euclidean profiles perform similar with the solutions in the Euclidean space $\mathbb{R}^4$ and hence Euclidean profiles hold some similar well-posedness and scattering properties by using the theory for NLS in Euclidean space $\mathbb{R}^4$ as a black box and Dodson [20].

We fix a spherically symmetric function $\eta \in C_0^\infty(\mathbb{R}^4)$ supported in the ball of radius 2 and equal to 1 in the ball of radius 1. Given $\phi \in \dot{H}^1(\mathbb{R}^4)$ and a real number $N \geq 1$ we define

$$\begin{aligned} Q_N\phi &\in H^1(\mathbb{R}^4), \qquad && (Q_N\phi)(x) = \eta(x/N^{\frac{1}{2}})\phi(x), \\ \phi_N &\in H^1(\mathbb{R}^4), && \phi_N(x) = N(Q_N\phi)(Nx), \\ f_N &\in H^1(\mathbb{T}^4), && f_N(y) = \phi_N(\Psi^{-1}(y)), \end{aligned} \tag{5.1}$$

where $\Psi : \{x \in \mathbb{R}^4 : |x| < 1\} \to O_0 \subseteq \mathbb{T}^4$, $\Psi(x) = x$.

The cutoff function $\eta(\frac{x}{N^{1/2}})$ is useful to concentrate our focus on the range of a point, and the choice of the order $1/2$ actually can be chosen any number between $1/2$ and 1.

Thus $Q_N\phi$ is a compactly supported modification of profile ϕ. ϕ_N is a $\dot{H}^1$-invariant rescaling of $Q_N\phi$, and f_N is the function obtained by transferring ϕ_N to a neighborhood of 0 in $\mathbb{T}^4$.

Theorem 5.1 *(GWP of the focusing cubic NLS in $\mathbb{R}^4$ [20]). Assume $\phi \in \dot{H}^1(\mathbb{R}^4)$, under the assumption that*

$$\sup_{t \in \text{lifespan of } v} \|v(t)\|_{\dot{H}^1(\mathbb{R}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)},$$

then there is a unique global solution $v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4))$ of the initial-value problem

$$(i\partial_t + \Delta)v = -v|v|^2, \qquad v(0) = \phi, \tag{5.2}$$

and

$$\|\nabla_{\mathbb{R}^4} v\|_{(L_t^\infty L_x^2 \cap L_t^2 L_x^4)(\mathbb{R}\times\mathbb{R}^4)} \leq C(\|\phi\|_{\dot{H}^1(\mathbb{R}^4)}, E_{\mathbb{R}^4}(\phi)) < +\infty. \tag{5.3}$$

Moreover, this solution scatters in the sense that there exists $\phi^{\pm\infty} \in \dot{H}^1(\mathbb{R}^4)$, such that

$$\|v(t) - e^{it\Delta}\phi^{\pm\infty}\|_{\dot{H}^1(\mathbb{R}^4)} \to 0, \text{ as } t \to \pm\infty. \tag{5.4}$$

Besides, if $\phi \in H^5(\mathbb{R}^4)$ then $v \in C(\mathbb{R} : H^5(\mathbb{R}^4))$ and

$$\sup_{t\in\mathbb{R}} \|v(t)\|_{H^5(\mathbb{R}^4)} \lesssim_{\|\phi\|_{H^5(\mathbb{R}^4)}} 1. \tag{5.5}$$

Remark 5.2 *(Persistence of regularity).* Consider $\phi \in H^5(\mathbb{R}^4)$, and $v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4))$ is a solution of (1.1) with $v(0) = \phi$ and satisfying

$$\|\nabla_{\mathbb{R}^4} v\|_{(L_t^\infty L_x^2 \cap L_t^2 L_x^4)(\mathbb{R}\times\mathbb{R}^4)} < +\infty.$$

So we can have a finite partition $\{I_k\}_{k=1}^K$ of $\mathbb{R}$, ($I_k = [t_{k-1}, t_k)$ and $t_K = \infty$) s.t. $\|\nabla_{\mathbb{R}^4} v\|_{L_t^4 L_x^{8/3}} < \frac{1}{2}$, for each k, we have that

$$\begin{aligned}
&\|v(t)\|_{L_t^\infty(I_k:H^5(\mathbb{R}^4))} \leq \|e^{i(t-t_{k-1})\Delta} v(t_{k-1})\|_{H^5} + \|\langle\nabla\rangle^5 |v(t)|^2 v(t)\|_{L_t^2 L_x^{4/3}(I_k)} \\
\leq &\|v(t_{k-1})\|_{H_x^5} + \|\langle\nabla\rangle^5 v\|_{L_t^\infty L_x^2(I_k)} \|v(t)\|^2_{L_t^4 L_x^8(I_k)} \\
\leq &\|v(t_{k-1})\|_{H^5} + \frac{1}{4}\|v\|_{L_t^\infty(I_k:H^5(\mathbb{R}^4))}
\end{aligned}$$

which implies $\|v(t)\|_{L_t^\infty(I_k:H^5(\mathbb{R}^4))} \leq \frac{4}{3}\|v(t_{k-1})\|_{H^5}$ for each $1 \leq k \leq K$, so $\|v(t)\|_{L_t^\infty(\mathbb{R}:H_x^5(\mathbb{R}^4))} < \infty$.

Theorem 5.3. *Assume $T_0 \in (0, \infty)$, and $\mu \in \{-1, 0, 1\}$ are given, and define f_N as (5.1) above. Suppose that if v is a solution of (5.2) then v satisfies*

$$\sup_{t \in \text{lifespan of } v} \|v(t)\|_{\dot{H}^1(\mathbb{R}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}, \qquad \text{when } \mu \in \{-1\}.$$

Then the following conclusions hold:

1. *There is $N_0 = N_0(\phi, T_0)$ sufficiently large such that for any $N \geq N_0$ there is a unique solution $U_N \in C((-T_0N^{-2}, T_0N^{-2}) : H^1(\mathbb{T}^4))$ of the initial value problem*

$$(i\partial_t + \Delta)U_N = -U_N|U_N|^2, \qquad U_N(0) = f_N. \tag{5.6}$$

   Moreover, for any $N \geq N_0$,

$$\|U_N\|_{X^1(-T_0N^{-2}, T_0N^{-2})} \lesssim_{E_{\mathbb{R}^4}(\phi),\, \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}} 1. \tag{5.7}$$

2. *Assume $\varepsilon_1 \in (0, 1]$ is sufficiently small (depending on only $E_{\mathbb{R}^4}\phi$), $\phi' \in H^5(\mathbb{R}^4)$, and $\|\phi - \phi'\|_{\dot{H}^1(\mathbb{R}^4)} \leq \varepsilon_1$. Let $v' \in C(\mathbb{R} : H^5(\mathbb{R}^4))$ denote the solution of the initial value problem*

$$(i\partial_t + \Delta)v' = -v'|v'|^2, \qquad v'(0) = \phi'. \tag{5.8}$$

   For R, $N \geq 1$, we define

$$\begin{aligned}
v'_R(x,t) &= \eta(x/R)v'(x,t), && (x,t) \in \mathbb{R}^4 \times (-T_0, T_0) \\
v'_{R,N}(x,t) &= N v'_R(Nx, N^2 t), && (x,t) \in \mathbb{R}^4 \times (-T_0N^{-2}, T_0N^{-2}) \\
V_{R,N}(y,t) &= v'_{R,N}(\Psi^{-1}(y), t), && (y,t) \in \mathbb{T}^4 \times (-T_0N^{-2}, T_0N^{-2}).
\end{aligned} \tag{5.9}$$

   Then there exists $R_0 \geq 1$ (depending on T_0, ϕ' and ε_1), for any $R \geq R_0$, we obtain that

$$\limsup_{N\to\infty} \|U_N - V_{R,N}\|_{X^1(-T_0N^{-2}, T_0N^{-2})} \lesssim_{E_{\mathbb{R}^4}(\phi),\, \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}} \varepsilon_1. \tag{5.10}$$

$V_{R,N}$ can be considered as solve NLS firstly, then cutoff and scaling, while U_N can be considered as cutoff and scaling firstly, then solve NLS.

Proof. We show Part (1) and Part (2) together, by Proposition 4.8 (stability).

Using Theorem 5.1, we know v' globally exists and satisfying

$$\|\nabla_{\mathbb{R}^4} v'\|_{(L_t^\infty L_x^2 \cap L_t^2 L_x^4)(\mathbb{R}\times\mathbb{R}^4)} \lesssim 1,$$

and

$$\sup_{t\in\mathbb{R}} \|v'(t)\|_{H^5(\mathbb{R}^4)} \lesssim_{\|\phi'\|_{H^5(\mathbb{R}^4)}} 1. \tag{5.11}$$

Let's consider $v'_R(x,t) = \eta(x/R)v'(x,t)$.

$$(i\partial_t + \Delta_{\mathbb{R}^4})v'_R = (i\partial_t + \Delta_{\mathbb{R}^4})(\eta(x/R)v'(x,t))$$

$$= \eta(x/R)(i\partial_t + \Delta_{\mathbb{R}^4})v'(x,t) + R^{-2}v'(x,t)(\Delta_{\mathbb{R}^4}\eta)(x/R) + 2R^{-1}\sum_{j=1}^4 \partial_j v'(x,t)\partial_j \eta(x/R),$$

which implies

$$(i\partial_t + \Delta_{\mathbb{R}^4})v'_R = \lambda|v'_R|^2 v'_R + e_R(x,t),$$

where $e_R(x,t) = \mu(\eta(x/R) - \eta^3(x/R))v'|v'|^2 + R^{-2}v'(x,t)(\Delta_{\mathbb{R}^4}\eta)(x/R) + 2R^{-1}\sum_{j=1}^4 \partial_j v'(x,t)\partial_j\eta(x/R)$. After scaling, we get

$$(i\partial_t + \Delta_{\mathbb{R}^4})v'_{R,N} = \mu|v'_{R,N}|^2 v'_R + e_{R,N}(x,t),$$

where $e_{R,N}(x,t) = N^3 e_R(Nx, N^2 t)$. With $V_{R,N}(y,t) = v'_{R,N}(\Phi^{-1}(y),t)$ and taking $N \geq 10R$, we obtain that

$$(i\partial_t + \Delta_{\mathbb{R}^4})V_{R,N}(y,t) = \mu|V_{R,N}|^2 V_{R,N} + E_{R,N}(y,t), \tag{5.12}$$

where $E_{R,N}(y,t) = e_{R,N}(\Phi^{-1}(y),t)$. By Proposition 4.8, we need following conditions:

1. $\|V_{R,N}\|_{L_t^\infty([-T_0N^{-2},T_0N^{-2}]:H^1(\mathbb{T}^4))} + \|V_{R,N}\|_{Z([-T_0N^{-2},T_0N^{-2}])} \leq M$;
2. $\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{T}^4)} \leq \varepsilon$;
3. $\|E_{R,N}\|_{N([-T_0N^{-2},T_0N^{-2}])} \leq \varepsilon$.

We prove all 3 conditions as follows:

Case 1: $\|V_{R,N}\|_{L_t^\infty([-T_0N^{-2},T_0N^{-2}]:H^1(\mathbb{T}^4))} + \|V_{R,N}\|_{Z([-T_0N^{-2},T_0N^{-2}])} \leq M$.

Since $v'(x,t)$ globally exists, $V_{R,N}(y,t)$ also globally exists. Given $T_0 \in (0,\infty)$,

$$
\begin{aligned}
&\sup_{t\in[-T_0N^{-2},T_0N^{-2}]} \|V_{R,N}(t)\|_{H^1(\mathbb{T}^4)} \le \sup_{t\in[-T_0N^{-2},T_0N^{-2}]} \|v'_{R,N}(t)\|_{H^1(\mathbb{R}^4)} \\
&= \sup_{t\in[-T_0N^{-2},T_0N^{-2}]} \frac{1}{N}\|v'_R(N^2t)\|_{L^2(\mathbb{R}^4)} + \|v'_R(N^2t)\|_{\dot H^1(\mathbb{R}^4)} \\
&\le \sup_{t\in[-T_0,T_0]} \|v'_R\|_{H^1(\mathbb{R}^4)} = \sup_{t\in[-T_0,T_0]} \|\eta(x/R)v'(x,t)\|_{H^1(\mathbb{R}^4)} \\
&\lesssim \|v'(x,t)\|_{H^1(\mathbb{R}^4)} \le \|\phi'(t)\|_{H^5(\mathbb{R}^4)}.
\end{aligned}
$$

By Littlewood-Paley theorem and Sobolev embedding, we obtain that

$$
\begin{aligned}
&\|V_{R,N}\|_{Z([-T_0N^{-2},T_0N^{-2}])} = \sup_{J\subset[-T_0N^{-2},T_0N^{-2}]} \Big(\sum_{M\, dyadic} M^2\|P_MV_{R,N}\|^4_{L^4(J\times\mathbb{T}^4)}\Big)^{\frac14} \\
&\le \sup_{J\subset[-T_0N^{-2},T_0N^{-2}]} \|(\sum_M (\|\langle 1-\Delta\rangle^{\frac14}P_MV_{R,N})^2)^{\frac12}\|_{L^4(J\times\mathbb{T}^4)} \\
&\lesssim \sup_{J\subset[-T_0N^{-2},T_0N^{-2}]} \|\langle 1-\Delta\rangle^{\frac14}V_{R,N}\|_{L^4(J\times\mathbb{T}^4)} \\
&\le \sup_{J\subset[-T_0N^{-2},T_0N^{-2}]} \|\langle 1-\Delta\rangle^{\frac12}V_{R,N}\|_{L^4_t(J)L^{\frac83}_x(\mathbb{T}^4)} \\
&\lesssim \|v'_R\|_{L^4_tL^{\frac83}_x([-T_0,T_0]\times\mathbb{R}^4)} + \|\nabla_{\mathbb{R}^4}v'_R\|_{L^4_tL^{\frac83}_x([-T_0,T_0]\times\mathbb{R}^4)}.
\end{aligned}
$$

Since $\|v'_R\|_{L^4_tL^{\frac83}_x([-T_0,T_0]\times\mathbb{R}^4)} + \|\nabla_{\mathbb{R}^4}v'_R\|_{L^4_tL^{\frac83}_x([-T_0,T_0]\times\mathbb{R}^4)} \lesssim \sup_t \|v'(t)\|_{H^5}$, by (5.11) we obtain

$$
\|V_{R,N}\|_{Z([-T_0N^{-2},T_0N^{-2}])} \lesssim_{\|\phi'\|_{H^5(\mathbb{R}^4)}} 1.
$$

Case 2: $\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{T}^4)} \le \varepsilon$.
By Hölder inequality, we have that

$$
\begin{aligned}
\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{T}^4)} &\le \|\phi_N(\Psi^{-1}(y)) - \phi'_{R,N}(\Psi^{-1}(y))\|_{\dot H^1(\mathbb{T}^4)} \\
&\le \|Q_N\phi - \phi'_R\|_{\dot H^1(\mathbb{R}^4)} \\
&\le \|\eta(\frac{x}{N^{\frac12}})\phi(x) - \phi(x)\|_{\dot H^1(\mathbb{R}^4)} + \|\phi-\phi'\|_{\dot H^1(\mathbb{R}^4)} \\
&\quad + \|\eta(\frac{x}{N^{\frac12}})\phi'(x) - \phi'(x)\|_{\dot H^1(\mathbb{R}^4)}.
\end{aligned}
$$

With $N \ge 10R$, and $R > R_0$, R_0 large enough, we have that

$$
\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{T}^4)} \le 2\varepsilon_1.
$$

Case 3: $\|E_{R,N}\|_{N([-T_0N^{-2},T_0N^{-2}])} \le \varepsilon$.
Next, by Proposition 3.12 and scaling invariance, we obtain that

$$
\begin{aligned}
\|E_{R,N}\|_{N([-T_0N^{-2},T_0N^{-2}])} &= \left\| \int_0^t e^{i(t-s)\Delta} E_{R,N}(s)\, ds \right\|_{X^1([-T_0N^{-2},T_0N^{-2}])} \\
&\leq \sup_{\|u_0\|_{Y^{-1}}=1} \||\nabla|^{-1}u_0\|_{L_t^\infty L_x^2} \||\nabla| E_{R,N}\|_{L_t^1 L_x^2} \\
&\leq \sup_{\|u_0\|_{Y^{-1}}=1} \|u_0\|_{Y^{-1}} \||\nabla| E_{R,N}\|_{L_t^1 L_x^2([-T_0N^{-2},T_0N^{-2}]\times\mathbb{T}^4)} \\
&= \|\nabla_{\mathbb{R}^4} e_R\|_{L_t^1 L_x^2([-T_0,T_0]\times\mathbb{R}^4)}.
\end{aligned}
$$

For the $|\nabla_{\mathbb{R}^4} e_R(x,t)|$, we have the following estimate

$$
\begin{aligned}
&|\nabla_{\mathbb{R}^4} e_R(x,t)| = |\nabla_{\mathbb{R}^4}(\mu(\eta(x/R) - \eta^3(x/R)))v'(x,t)|v'(x,t)|^2 \\
&\quad + R^{-2} v'(x,t)(\Delta_{\mathbb{R}^4}\eta)(\frac{x}{R}) + 2R^{-1}(\sum_{j=1}^4 \partial_j v'(x,t)\partial_j \eta(x/R)| \\
&\leq |\nabla_{\mathbb{R}^4}(\eta(\frac{x}{R}) - \eta(\frac{x}{R})^3)v'(x,t)|v'(x,t)|^2| + 3|(\eta(\frac{x}{R}) - \eta(\frac{x}{R})^3)\nabla_{\mathbb{R}^4} v'(x,t)|v'(x,t)|^2| \\
&\quad + R^{-3}|v'(x,t)\nabla_{\mathbb{R}^4}\Delta_{\mathbb{R}^4}\eta(\frac{x}{R})| + R^{-2}|\nabla_{\mathbb{R}^4} v'(x,t)(\Delta_{\mathbb{R}^4}\eta)(\frac{x}{R})| + R^{-1}|\Delta_{\mathbb{R}^4} v'(x,t)\nabla_{\mathbb{R}^4}\eta(\frac{x}{R})| \\
&\lesssim_{\|\phi'\|_{H^5(\mathbb{R}^4)}} \mathbb{1}_{[R,2R]}(|x|)\left(|v'(x,t)| + |\nabla_{\mathbb{R}^4} v'(x,t)|\right) + \frac{1}{R}\left(|\langle\nabla_{\mathbb{R}^4}\rangle^2 v'(x,t)|\right).
\end{aligned}
$$

Since $\|\nabla^2_{\mathbb{R}^4} v'(x,t)\|_{L_x^\infty} \lesssim_{\|\phi'\|_{H^5}} 1$, $\|\nabla_{\mathbb{R}^4} v'(x,t)\|_{L_x^\infty} \lesssim_{\|\phi'\|_{H^5}} 1$, and $\|v'(x,t)\|_{L_x^\infty} \lesssim_{\|\phi'\|_{H^5}} 1$ (by Sobolev embedding), we obtain that

$$
\begin{aligned}
&\|\nabla_{\mathbb{R}^4} e_R\|_{L_t^1 L_x^2([-T_0,T_0]\times\mathbb{R}^4)} = \int_{-T_0}^{T_0} (\int_{\mathbb{R}^4} |\nabla_{\mathbb{R}^4} e_R|\, dx)^{\frac{1}{2}} dt \\
&\leq \int_{-T_0}^{T_0} \left(\int_{\mathbb{R}^4} \mathbb{1}_{[R,2R]}(|x|)(|v'(x,t)|^2 + |\nabla_{\mathbb{R}^4} v'(x,t)|^2)\, dx + \frac{1}{R^2}\int_{\mathbb{R}^4} |\langle\nabla_{\mathbb{R}^4}\rangle^2 v'(x,t)|^2\, dx \right)^{\frac{1}{2}} dt \\
&\lesssim_{\|\phi'\|_{H^5}} 2T_0 \left(\int_{\mathbb{R}^4} \mathbb{1}_{[R,2R]}(|x|)\langle\nabla_{\mathbb{R}^4}\rangle^2 v'(x,t)|^2\, dx \right)^{\frac{1}{2}} + \frac{1}{R} \to 0, \text{ as } R \to \infty.
\end{aligned}
$$

So we can obtain that

$$
\|\nabla E_{R,N}\|_{L_t^1 L_x^2([-T_0N^{-2},T_0N^{-2}]\times\mathbb{T}^4)} < \varepsilon_1,
$$

where $R > R_0$, and R_0 large enough. By checking all three conditions above, we have the desired result. $\square$

Next, we prove an extinction lemma as Ionescu and Pausader [43] did in their paper about energy critical NLS on $\mathbb{T}^3$. Here we prove the corresponding extinction lemma on $\mathbb{T}^4$ which is one of the essential ingredients of the whole proof.

Lemma 5.4 *(Extinction lemma). Let $\phi \in \dot{H}^1(\mathbb{R}^4)$, and define f_N as in (5.1). For any $\varepsilon > 0$, there exist $T = T(\phi, \varepsilon)$ and $N_0(\phi, \varepsilon)$ such that for all $N \geq N_0$, there holds that*

$$\|e^{it\Delta} f_N\|_{Z([TN^{-2}, T^{-1}])} \lesssim \varepsilon.$$

Proof. For $M \geq 1$, we define

$$K_M(x,t) = \sum_{\xi \in \mathbb{Z}^4} e^{-i[t|\xi|^2 + x\cdot\xi]\eta(\xi/M)} = e^{it\Delta} P_{\leq M}\delta_0.$$

We know from [Lemma 3.18, Bourgain [3]] that K_M satisfies

$$|K_M(x,t)| \lesssim \prod_{i=1}^{4} \left(\frac{M}{\sqrt{q_i}(1 + M|t/(\lambda_i) - a_i/q_i|^{\frac{1}{2}})} \right), \tag{5.13}$$

if a_i and q_i satisfying $\frac{t}{\lambda_i} = \frac{a_i}{q_i} + \beta_i$, where $q_i \in \{1, \cdots, M\}$, $a_i \in \mathbb{Z}$, $(a_i, q_i) = 1$ and $|\beta_i| \leq (Mq_i)^{-1}$ for all $i = 1, 2, 3, 4$.

From this, we conclude that for any $1 \leq S \leq M$,

$$\|K_M(x,t)\|_{L^\infty_{x,t}(\mathbb{T}^4 \times [SM^{-2}, S^{-1}])} \lesssim S^{-2}M^4. \tag{5.14}$$

This follows directly from (5.13) and Dirichlet's approximation lemma which is stated as follows: *For any real number α, and any positive integer N, there exist integers p and q such $1 \leq q \leq N$ and $|q\alpha - p| < \frac{1}{N}$.*

Assume that $|t| \leq \frac{1}{S}$. For each $i \in \{1, 2, 3, 5\}$, $\frac{t}{\lambda_i} = \frac{a_i}{q_i} + \beta_i$ and $|\beta_i| \leq \frac{1}{Mq_i} \leq \frac{1}{M} \leq \frac{1}{S}$. So we obtain that

$$\left|\frac{a_i}{q_i}\right| \leq \frac{2}{S} \quad \Longrightarrow \quad q_i \geq \frac{a_i}{2}.$$

Therefore we have that either $q_i \geq \frac{1}{2}S \quad (a_i \geq 1)$ or $a_i = 0$ for each i. If $q_i \geq \frac{1}{2}S \quad (a_i \geq 1)$, then

$$\frac{M}{\sqrt{q_i}(1 + M|t/(\lambda_i) - a_i/q_i|^{\frac{1}{2}})} \lesssim \frac{M}{\sqrt{q_i}} \lesssim S^{-\frac{1}{2}}M.$$

If $a_i = 0$, then

$$\frac{M}{\sqrt{q_i}(1 + M|t/(\lambda_i) - a_i/q_i|^{\frac{1}{2}})} \lesssim \frac{M}{\sqrt{q_i}M|t|^{1/2}} \lesssim |t|^{-\frac{1}{2}} \leq S^{-\frac{1}{2}}M.$$

So we have that $|K_M(x,t)| \lesssim S^{-2}M^4$. By the definition as in (5.1), to prove the extinction lemma, we may assume that $\phi \in C_0^\infty(\mathbb{R}^4)$, we claim that

$$\begin{aligned}\|f_N\|_{L^1(\mathbb{T}^4)} &\lesssim_\phi N^{-3}\\ \|P_K f_N\|_{L^2(\mathbb{T}^4)} &\lesssim_\phi \left(1+\frac{K}{N}\right)^{-10} N^{-1}.\end{aligned} \tag{5.15}$$

Let's consider the bound of $\|f_N\|_{L^1(\mathbb{T}^4)}$:

$$\|f_N\|_{L^1(\mathbb{T}^4)} = \|\phi_N(x)\|_{L^1(\mathbb{R}^4)} \le \frac{1}{N^3}\|\phi\|_{L^1(\mathbb{R}^4)}.$$

Then we consider the bound of $\|P_K f_N\|_{L^2(\mathbb{T}^4)}$:

$$\|P_K f_N\|_{L^2(\mathbb{T}^4)} == \frac{1}{N}\|P_{\frac{K}{N}}(\eta(\frac{x}{N^{\frac{1}{2}}})\phi(x))\|_{L^2(\mathbb{R}^4)} \le \frac{1}{N}\left(1+\frac{K}{N}\right)^{-10}\|\phi\|_{H^{10}}.$$

By Lemma 3.18, for $p>3$ we obtain that

$$\|e^{it\Delta}P_K f_N\|_{L^p_{x,t}(\mathbb{T}^4\times[-1,1])} \le K^{2-\frac{6}{p}}\left(1+\frac{K}{N}\right)^{-10} N^{-1}\|\phi\|_{H^{10}}. \tag{5.16}$$

Then let's estimate $\|e^{it\Delta}f_N\|_{Z([TN^{-2},T^{-1}])}$. We know that

$$\|e^{it\Delta}f_N\|_{Z([TN^{-2},T^{-1}])} = \sup_{J\subset[TN^{-2},T^{-1}]}\left(\sum_K K^2\|P_K e^{it\Delta}f_N\|^4_{L^4(J\times\mathbb{T}^4)}\right)^{\frac{1}{4}}$$

To estimate it, we decompose the sum above into three parts:

$$\left(\sum_{K\le NT^{-\frac{1}{100}}} + \sum_{K\ge NT^{\frac{1}{100}}} + \sum_{NT^{-\frac{1}{100}}\le K\le NT^{\frac{1}{100}}}\right) K^2\|P_K e^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)}.$$

Case 1: $K\le NT^{-\frac{1}{100}}$:
By (5.16), we obtain that

$$\sum_{K\le NT^{-\frac{1}{100}}} K^2\|P_K e^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)} \lesssim_\phi (NT^{-\frac{1}{100}})^4 N^{-4} = T^{-\frac{1}{25}}.$$

Case 2: $K\ge NT^{\frac{1}{100}}$:
By (5.16), we obtain that

$$\begin{aligned}\sum_{K\ge NT^{\frac{1}{100}}} K^2\|P_K e^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)} &\le \sum_{K\ge NT^{\frac{1}{100}}} K^4\left(1+\frac{K}{N}\right)^{-40} N^{-4}\|\phi\|_{H^{10}}\\ &\lesssim_\phi T^{-\frac{4}{100}}.\end{aligned}$$

Case 3: $NT^{-\frac{1}{100}} \le K \le NT^{\frac{1}{100}}$:
Let's consider $K \in [NT^{-\frac{1}{100}}, NT^{\frac{1}{100}}]$ and set $M \sim \max(K, N)$ and $S \sim T$.

$$\begin{aligned} \|e^{it\Delta}P_K f_N\|_{L^\infty_{x,t}(\mathbb{T}^4\times[TN^{-2},T^{-1}])} &= \|K_M * f_N\|_{L^\infty_{x,t}(\mathbb{T}^4\times[TN^{-2},T^{-1}])} \\ &\le \|K_M\|_{L^\infty_{x,t}(\mathbb{T}^4\times[TN^{-2},T^{-1}])}\|f_N\|_{L^1_{x,t}(\mathbb{T}^4)} \\ &\lesssim_\phi T^{-2}K^4N^{-3} \le T^{-2+\frac{1}{25}}N. \end{aligned} \tag{5.17}$$

$$\begin{aligned} \|e^{it\Delta}P_N f_N\|_{L^3_{x,t}(\mathbb{T}^4\times[TN^{-2},T^{-1}])} &\lesssim_\phi K^\varepsilon\left(1+\frac{K}{N}\right)^{-10}N^{-1} \\ &\le N^{-1+\varepsilon}T^{-\frac{\varepsilon}{100}}. \end{aligned} \tag{5.18}$$

By interpolating (5.17) with (5.18), we have that

$$\|e^{it\Delta}P_K f_N\|_{L^4_{x,t}([TN^{-2},T^{-1}])} \lesssim_\phi \left(N^{-1+\varepsilon}T^{\frac{\varepsilon}{100}}\right)^{\frac{3}{4}}\left(T^{-2+\frac{1}{25}}N\right)^{\frac{1}{4}} \le N^{-\frac{1}{4}}T^{-\frac{1}{100}}. \tag{5.19}$$

Summing $K^2\|P_Ke^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)}$ over K, we obtain that

$$\begin{aligned} \sum_{NT^{-\frac{1}{100}}\le K\le NT^{\frac{1}{100}}} K^2\|P_Ke^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)} &\le \sum_{NT^{-\frac{1}{100}}\le K\le NT^{\frac{1}{100}}} K^2(N^{-\frac{1}{4}}T^{-\frac{1}{100}})^4 \\ &\le \sum_{NT^{-\frac{1}{100}}\le K\le NT^{\frac{1}{100}}} K^2N^{-2}T^{-\frac{1}{25}} \\ &\le T^{-\frac{1}{50}}. \end{aligned}$$

Summarizing all three cases by setting T large enough, we hold the estimate. $\square$

Let's now consider $f \in L^2(\mathbb{T}^4)$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{T}^4$,

$$\begin{aligned} (\pi_{x_0}f)(x) &:= f(x-x_0), \\ (\Pi_{t_0,x_0})f(x) &:= (\pi_{x_0}e^{-it_0\Delta}f)(x). \end{aligned}$$

As in (5.1), given $\phi \in \dot{H}^1(\mathbb{R}^4)$ and $N \ge 1$, we define

$$T_N\phi(x) := N\tilde{\phi}(N\Psi^{-1}(x)),\ \text{where } \tilde{\phi}(y) := \eta(y/N^{\frac{1}{2}})\phi(y)$$

and we have that $T_N : \dot{H}^1(\mathbb{R}^4) \to H^1(\mathbb{R}^4)$ is a linear operator with $\|T_N\phi\|_{H^1(\mathbb{T}^4)} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}$.

Definition 5.5. Let $\widetilde{\mathcal{F}}_e$ denote the set of renormalized Euclidean frames

$$\begin{aligned} \widetilde{\mathcal{F}}_e :=& \{(N_k, t_k, x_k)_{k\ge1} : N_k \in [1,\infty),\ t_k \to 0,\ x_k \in \mathbb{T}^4,\ N_k \to \infty \\ & \text{and either } t_k = 0 \text{ for any } k \ge 1 \text{ or } \lim_{k\to\infty} N_k^2|t_k| = \infty\}. \end{aligned}$$

Proposition 5.6 *(Euclidean profiles). Assume that $\mathcal{O} = (N_k, t_k, x_k)_k \in \widetilde{\mathcal{F}}_e$ and $\phi \in \dot{H}^1(\mathbb{R}^4)$. Suppose that if v is a solution of (5.2) with $v(0) = \phi$ then v satisfies*

$$\sup_{t \in \text{lifespan of } v} \|v(t)\|_{\dot{H}^1(\mathbb{R}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}.$$

Then

1. *there exists $\tau = \tau(\phi)$ such that for k large enough (depending only on ϕ and $\mathcal{O}$) there is a nonlinear solution $U_k \in X^1(-\tau, \tau)$ of the initial value problem (1.9) with initial data $U_k(0) = \Pi_{t_k,0}(T_{N_k}\phi)$ and*

$$\|U_k\|_{X^1(-\tau,\tau)} \lesssim_{E_{\mathbb{R}^4}(\phi),\, \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}} 1; \tag{5.20}$$

2. *there exists a Euclidean solution $u \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4))$ of*

$$(i\partial_t + \Delta_{\mathbb{R}^4})u = -u|u|^2$$

with scattering data $\phi^{\pm\infty}$ defined as Theorem 5.1 such that the following holds, up to a subsequence: for any $\varepsilon > 0$, there exists $T(\phi, \varepsilon)$ such that for all $T \geq T(\phi, \varepsilon)$, there exists $R(\phi, \varepsilon, T)$ such that for all $R \geq R(\phi, \varepsilon, T)$, there holds that

$$\|U_k - \widetilde{u_k}\|_{X^1(\{|t-t_k| \leq T N_k^{-2}\} \cap \{|t| < T^{-1}\})} \leq \varepsilon, \tag{5.21}$$

for k large enough, where

$$(\pi_{-x_k}\widetilde{u_k})(x,t) = N_k \eta(N_k \Psi^{-1}(x)/R) u(N_k \Psi^{-1}(x), N_k^2(t - t_k)). \tag{5.22}$$

In addition, up to a subsequence,

$$\|U_k(t) - \Pi_{t_k - t, x_k} T_{N_k} \phi^{\pm\infty}\|_{X^1(\{\pm(t-t_k) \geq \pm T N_k^{-2}\} \cap \{|t| < T^{-1}\})'} \leq \varepsilon, \tag{5.23}$$

for k large enough (depending on ϕ, ε, T, and R).

Proof. By the statement, it is equivalent to prove the case when $x_k = 0$.
Part (1): First, for k large enough, we can make

$$\|\phi - \eta(\frac{x}{N^{\frac{1}{2}}})\phi\|_{\dot{H}^1(\mathbb{R}^4)} \leq \varepsilon_1.$$

For each N_k, we choose $T_{0,N_k} = \tau N_k^2$ (T_{0,N_k} is the coefficient in Lemma 5.3). For each T_{0,N_k}, we make R_k large enough to make Theorem 5.3 work. (Note: in this case, R_k determined by T_{0,N_k} as in the proof of Theorem 5.3.)
Part (2): Let's consider first case in Euclidean frame: $t_k = 0$ for all k. (5.20) is directly from Theorem 5.3, by choosing k, R for any fixed T large enough.

To prove (5.21), we need to choose $T(\phi,\delta)$ large enough, to make sure

$$\|\nabla_{\mathbb{R}^4} u\|_{L^3_{x,t}(\mathbb{R}^4\times\{|t|>T(\phi,\delta)\})} \leq \delta.$$

By Theorem 5.1, we obtain that

$$\|u(\pm T(\phi,\delta)) - e^{\pm iT(\phi,\delta)\Delta}\phi^{\pm\infty}\|_{\dot{H}^1(\mathbb{R}^4)} \leq \delta,$$

which implies

$$\|U_{N_k}(\pm T N_k^{-2}) - \Pi_{-\pm T, x_k} T_{N_k}\phi^{\pm\infty}\|_{H^1(\mathbb{T}^4)} \leq \delta. \tag{5.24}$$

By Proposition 3.15 and Proposition 3.16, we have

$$\|e^{it\Delta}\left(U_{N_k}(\pm T N_k^{-2}) - \Pi_{-\pm T, x_k} T_{N_k}\phi^{\pm\infty}\right)\|_{X^1(|t|<T^{-1})} \leq \delta. \tag{5.25}$$

By Proposition 4.5, we obtain that

$$\|U_{N_k} - e^{it\Delta}U_{N_k}(\pm T N_k^{-2})\|_{X^1} \leq \delta, \tag{5.26}$$

and combining (5.25) and (5.26), we have

$$\|U_{N_k} - \Pi_{-t,x_k} T_{N_k}\phi^{\pm\infty}\|_{X^1(\{\pm t\geq \pm T N_k^{-2}\}\cap\{|t|<T^{-1}\})} \leq \varepsilon,$$

when we choose δ small enough.

The second case: $N_k^2|t_k| \to \infty$.

$$\begin{aligned} U_k(0) &= \Pi_{t_k,0}(T_{N_k}\phi) \\ &= e^{-it_k\Delta}\left(N_k^{\frac{1}{2}}\widetilde{\phi}(N_k\Psi^{-1}(x))\right) \\ &= e^{-it_k\Delta}\left(N_k^{\frac{1}{2}}\eta(N_k^{\frac{1}{2}}\Psi^{-1}(x))\phi(N_k\Psi^{-1}(x))\right). \end{aligned}$$

By existence of wave operator of NLS, we know the following initial value problem is global well-posed, so there exists v satisfying:

$$\begin{cases} (i\partial_t + \Delta_{\mathbb{R}^4})v = \mu v|v|^2, \\ \lim_{t\to-\infty}\|v(t) - e^{it\Delta}\phi\|_{\dot{H}^1(\mathbb{R}^4)} = 0. \end{cases} \tag{5.27}$$

We set

$$\widetilde{v_k}(t) = N_k^{\frac{1}{2}}\eta(N_k\Psi^{-1}(x)/R)v(N_k\Psi^{-1}(x), N_k^2 t),$$

so we have $\widetilde{v_k}(-t_k) = N_k^{\frac{1}{2}}\eta(N_k\Psi^{-1}(x)/R)v(N_k\Psi^{-1}(x), -N_k^2 t_k)$.

For k and R large enough,

$$\begin{aligned}&\|\widetilde{v_k}(-t_k)-e^{-it_k\Delta}N_k^{\frac{1}{2}}\eta(N_k^{\frac{1}{2}}\Psi^{-1}(x))\phi(N_k\Psi^{-1}(x))\|_{\dot{H}^1(\mathbb{T}^4)}\\ \leq&\|\eta(\frac{x}{N_k^{\frac{1}{2}}})v(x,-N_k^2t_k)-e^{it_kN_k^2\Delta}\eta(\frac{x}{N_k^{\frac{1}{2}}})\phi(x)\|_{\dot{H}^1(\mathbb{R}^4)}\\ \leq&\varepsilon.\end{aligned}$$

So $V_k(t)$ solves initial value problem (1.9) in $\mathbb{T}^4$, with initial data $V_k(0)=\widetilde{V}_k(0)$, which implies $V_k(t)$ exists in $[-\delta,\delta]$, and $\|V_k(t)-\widetilde{V}_k(t)\|_{X^1([-\delta,\delta])}\lesssim\varepsilon$.

By the stability property (Proposition 4.8), $\|U_k-V_k\|_{X^1([-\delta,\delta])}\to 0,$ as $k\to\infty$. $\square$

The following corollary (Corollary 5.7) decomposes the nonlinear Euclidean profiles U_k defined in the Proposition 5.6. This corollary follows closely in a part of the proof of Lemma 6.2 in [43]. I state it here as a corollary because the almost orthogonality of nonlinear profiles (Lemma 6.6) heavily relies on this decomposition lemma (Corollary 5.7).

Corollary 5.7 *(Decomposition of the nonlinear Euclidean profiles U_k). Consider U_k is the nonlinear Euclidean profiles w.r.t. $\mathcal{O}=(N_k,t_k,x_k)_k\in\widetilde{\mathcal{F}_e}$ defined above. For any $\theta>0$, there exist T_θ^0 sufficiently large such that for all $T_\theta\geq T_\theta^0$ and R_θ sufficiently large such that for all k large enough (depending on R_θ) we can decompose U_k as following:*

$$\mathbb{1}_{(-T_\theta^{-1},T_\theta^{-1})}(t)U_k=\omega_k^{\theta,-\infty}+\omega_k^{\theta,+\infty}+\omega_k^\theta+\rho_k^\theta,$$

and $\omega_k^{\theta,\pm\infty}$, ω_k^θ, and ρ_k^θ satisfy the following conditions:

$$\begin{aligned}\|\omega_k^{\theta,\pm\infty}\|_{Z'(-T_\theta^{-1},T_\theta^{-1})}+\|\rho_k^\theta\|_{X^1(-T_\theta^{-1},T_\theta^{-1})}&\leq\theta,\\ \|\omega_k^{\theta,\pm\infty}\|_{X^1(-T_\theta^{-1},T_\theta^{-1})}+\|\omega_k^\theta\|_{X^1(-T_\theta^{-1},T_\theta^{-1})}&\lesssim 1,\\ \omega_k^{\theta,\pm\infty}&=P_{\leq R_\theta N_k}\omega_k^{\theta,\pm\infty}\\ |\nabla_x^m\omega_k^\theta|+(N_k)^{-2}\mathbb{1}_{S_k^\theta}|\partial_t\nabla_x^m\omega_k^\theta|&\leq R_\theta(N_k)^{|m|+1}\mathbb{1}_{S_k^\theta},\ 0\leq|m|\leq 10,\end{aligned}\tag{5.28}$$

where

$$S_k^\theta:=\{(x,t)\in\mathbb{T}^4\times(-T_\theta,T_\theta):|t-t_k|<T_\theta(N_k)^{-2},\ |x-x_k|\leq R_\theta(N_k)^{-1}\}.$$

Proof. By Proposition 5.6, there exists $T(\phi,\frac{\theta}{4})$, such that for all $T\geq T(\phi,\frac{\theta}{4})$, there exists $R(\phi,\frac{\theta}{4},T)$ such that for all $R\geq R(\phi,\frac{\theta}{2},T)$, there holds that

$$\|U_k-\widetilde{u_k}\|_{X^1(\{|t-t_k|\leq T(N_k)^{-2}\}\cap\{|t|<T^{-1}\})}\leq\frac{\theta}{2},$$

for k large enough, where

$$\left(\pi_{-x_k}\widetilde{u_k}\right)(x,t) = N_k\eta(N_k\Psi^{-1}(x)/R)u(N_k\Psi^{-1}(x), N_k^2(t-t_k)),$$

where u is a solution of (1.1) with scattering data $\phi^{\pm\infty}$.

In addition, up to subsequence,

$$\|U_k - \Pi_{t_k-t,x_k}T_{N_k}\phi^{\pm\infty}\|_{X^1(\{\pm(t-t_k)\geq T(N_k)^{-2}\}\cap\{|t|\leq T^{-1}\})} \leq \frac{\theta}{4},$$

for k large enough (depending on ϕ, θ, T, and R).

Choose a sufficiently large $T_\theta > T(\phi, \frac{\theta}{4})$ based on the extinction lemma (Lemma 5.4), such that

$$\|e^{it\Delta}\Pi_{t_k,x_k}T_{N_k}\phi^{\pm\infty}\|_{Z(T_\theta(N_k)^{-2},T_\theta^{-1})} \leq \frac{\theta}{4}$$

when k large enough.

And then we choose $R_\theta = R(\phi, \frac{\theta}{2}, T_\theta)$.

Denote:

1. $\omega_k^{\theta,\pm\infty} := \mathbb{1}_{\{\pm(t-t_k)\geq T_\theta(N_k)^{-2},|t|\leq T_\theta^{-1}\}}\left(\Pi_{t_k-t,x_k}T_{N_k}\phi^{\theta,\pm\infty}\right)$,
   where
   $$\|\phi^{\theta,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^4)} \lesssim 1,\ \phi^{\theta,\pm\infty} = P_{\leq R_\theta}(\phi^{\theta,\pm\infty}),$$
   which implies $\omega_k^{\theta,\pm\infty} = P_{\leq R_\theta N_\theta}\omega_k^{\theta,\pm\infty}$.
2. $\omega_k^\theta := \widetilde{u_k}\cdot\mathbb{1}_{S_k^\theta}$, where $S_k^\theta := \{(x,t)\in\mathbb{T}^4\times(-T_\theta,T_\theta) : |t-t_k| < T_\theta(N_k)^{-2},\ |x-x_k| \leq R_\theta(N_k)^{-1}\}$.
   By the stability property (Proposition 4.8) and Theorem 5.3, we can adjust ω_k^θ and $\omega_k^{\theta,\pm\infty}$, with an acceptable error, to make
   $$|\nabla_x^m\omega_k^\theta| + (N_k)^{-2}\mathbb{S}_{\mathbb{k}}^{\alpha,\theta}|\partial_t\nabla_x^m\omega_k^\theta| \leq R_\theta(N_k)^{|m|+1}\mathbb{1}_{S_k^\theta},\ 0\leq|m|\leq 10.$$
3. $\rho_k := \mathbb{1}_{(-T_\theta^{-1},T_\theta^{-1})}(t)U_k^\alpha - \omega_k^\theta - \omega^{\theta,+\infty} - \omega^{\theta,-\infty}$.

By (5.21) and (5.23), we obtain that

$$\|\rho_k^\theta\|_{X^1(\{|t|<T_\theta^{-1}\})} \leq \frac{\theta}{2},$$

and then we have

$$\|\omega_k^{\theta,\pm\infty}\|_{Z'(-T_\theta^{-1},T_\theta^{-1})} + \|\rho_k^\theta\|_{X^1(-T_\theta^{-1},T_\theta^{-1})} \leq \theta,$$
$$\|\omega_k^{\theta,\pm\infty}\|_{X^1(-T_\theta^{-1},T_\theta^{-1})} + \|\omega_k^\theta\|_{X^1(-T_\theta^{-1},T_\theta^{-1})} \lesssim 1. \quad \square$$

6. Profile decomposition

In this section, we construct the profile decomposition on $\mathbb{T}^4$ for linear Schrödinger equations. The arguments and propositions in this section are almost identical to those in the Section 5 of [44], except for one more lemma (Lemma 6.6) about almost orthogonality of nonlinear profiles which is useful in the focusing case.

As in the previous section, given $f \in L^2(\mathbb{R}^4)$, $t_0 \in \mathbb{R}$, and $x_0 \in \mathbb{T}^4$, we define:

$$(\Pi_{t_0,x_0}) f(x) := (e^{-it_0\Delta} f)(x - x_0)$$
$$T_N\phi(x) := N\widetilde{\phi}(N\Psi^{-1}(x)),$$

where $\widetilde{\phi}(y) := \eta(\frac{y}{N^{\frac{1}{2}}})\phi(y)$.

Observe that $T_N : \dot{H}^1(\mathbb{R}^4) \to H^1(\mathbb{T}^4)$ is a linear operator with $\|T_N\phi\|_{H^1(\mathbb{T}^4)} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}$.

Definition 6.1 *(Euclidean frames).*

1. We define a Euclidean frame to be a sequence $\mathcal{F}_e = (N_k, t_k, x_k)_k$ with $N_k \geq 1$, $N_k \to +\infty$, $t_k \in \mathbb{R}$, $t_k \to 0$, $x_k \in \mathbb{T}^4$. We say that two frames, $(N_k, t_k, x_k)_k$ and $(M_k, s_k, y_k)_k$ are orthogonal if

$$\lim_{k\to+\infty} \left(\ln\left|\frac{N_k}{M_k}\right| + N_k^2 |t_k - s_k| + N_k |x_k - y_k| \right) = \infty.$$

Two frames that are not orthogonal are called equivalent.

2. If $\mathcal{O} = (N_k, t_k, x_k)_k$ is a Euclidean frame and if $\phi \in \dot{H}^1(\mathbb{R}^4)$, we define the Euclidean profile associated to $(\phi, \mathcal{O})$ as the sequence $\widetilde{\phi}_{\mathcal{O}_k}$:

$$\widetilde{\phi}_{\mathcal{O}_k} := \Pi_{t_k,x_k}(T_{N_k}\phi).$$

Proposition 6.2 *(Equivalence of frames [44]). (1) If $\mathcal{O}$ and $\mathcal{O}'$ are equivalent Euclidean frames, then there exists an isometry $T : \dot{H}^1(\mathbb{R}^4) \to \dot{H}^1(\mathbb{R}^4)$ such that for any profile $\widetilde{\phi}_{\mathcal{O}'_k}$, up to a subsequence there holds that*

$$\limsup_{k\to\infty} \|\widetilde{T\phi}_{\mathcal{O}_k} - \widetilde{\phi}_{\mathcal{O}'_k}\|_{H^1(\mathbb{T}^4)} = 0.$$

(2) If $\mathcal{O}$ and $\mathcal{O}'$ are orthogonal Euclidean frames and $\widetilde{\phi}_{\mathcal{O}_k}$, $\widetilde{\phi}_{\mathcal{O}'_k}$ are corresponding profiles, then, up to a subsequence:

$$\lim_{k\to\infty} \langle \widetilde{\phi}_{\mathcal{O}_k}, \widetilde{\phi}_{\mathcal{O}'_k} \rangle_{H^1 \times H^1(\mathbb{T}^4)} = 0; \tag{6.1}$$

$$\lim_{k\to\infty} \langle |\widetilde{\phi}_{\mathcal{O}_k}|^2, |\widetilde{\phi}_{\mathcal{O}'_k}|^2 \rangle_{L^2 \times L^2(\mathbb{T}^4)} = 0. \tag{6.2}$$

The following proposition is the main statement of this section. We omit the proof of this proposition because it is similar to [44, Proposition 5.5].

Proposition 6.3 *(Profile decompositions). Consider $\{f_k\}_k$ a sequence of functions in $H^1(\mathbb{T}^4)$ and $0 < A < \infty$ satisfying*

$$\limsup_{k\to+\infty} \|f_k\|_{H^1(\mathbb{T}^4)} \leq A$$

and a sequence of intervals $I_k = (-T_k, T^k)$ such that $|I_k| \to 0$ as $k \to \infty$. Up to passing to a subsequence, assume that $f_k \rightharpoonup g \in H^1(\mathbb{T}^4)$. There exists $J^* \in \{0, 1, ...\} \cup \{\infty\}$, and a sequence of profile $\widetilde{\psi}_k^\alpha := \widetilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha$ associated to pairwise orthogonal Euclidean frames $\mathcal{O}^\alpha$ and $\psi^\alpha \in H^1(\mathbb{R}^4)$ such that extracting a subsequence, for every $0 \leq J \leq J^*$, we have

$$f_k = g + \sum_{1\leq\alpha\leq J} \widetilde{\psi}_k^\alpha + R_k^J \tag{6.3}$$

where R_k^J is small in the sense that

$$\limsup_{J\to J^*} \limsup_{k\to\infty} \|e^{it\Delta} R_k^J\|_{Z(I_k)} = 0. \tag{6.4}$$

Besides, we also have the following orthogonality relations:

$$\begin{aligned} \|f_k\|_{L^2}^2 &= \|g\|_{L^2}^2 + \|R_k^J\|_{L^2}^2 + o_k(1). \\ \|\nabla f_k\|_{L^2}^2 &= \|\nabla g\|_{L^2}^2 + \sum_{\alpha\leq J} \|\nabla_{\mathbb{R}^4}\psi^\alpha\|_{L^2(\mathbb{R}^4)}^2 + \|\nabla R_k^J\|_{L^2}^2 + o_k(1). \\ \lim_{J\to J^*} \limsup_{k\to\infty} &\left| \|f_k\|_{L^4}^4 - \|g\|_{L^4}^4 - \sum_{\alpha\leq J} \|\widetilde{\psi}_k^\alpha\|_{L^4}^4 \right| = 0. \end{aligned} \tag{6.5}$$

Remark 6.4. g and $\widetilde{\psi}_k^\alpha$ for all α are called profiles. In addition, we call g is Scale-1-profile, and $\widetilde{\psi}_k^\alpha$ are called Euclidean profiles.

Remark 6.5 *(Almost orthogonality of the energy).* By (5.1), we have that $\|\widetilde{\psi}_k^\alpha\|_{L^2(\mathbb{T}^4)} \leq \frac{1}{N_k}\|\psi^\alpha\|_{L^2(\mathbb{R}^4)} \to 0$ as $k \to \infty$ and $\|\widetilde{\psi}_k^\alpha\|_{\dot{H}^1(\mathbb{T}^4)}^2 = \frac{1}{N}\|\nabla\eta(\frac{\cdot}{N^{\frac{1}{2}}})\psi^\alpha\|_{L^2(\mathbb{R}^4)}^2 + \|\eta(\frac{\cdot}{N^{\frac{1}{2}}})\psi^\alpha\|_{\dot{H}^1(\mathbb{R}^4)}^2$. Then above and (6.5), we know that

$$\lim_{J\to J^*} \lim_{k\to\infty} \left(\sum_{1\leq\alpha\leq J} E(\widetilde{\psi}_k^\alpha) + E(R_k^J) + E(g) - E(f_k) \right) = 0.$$

Lemma 6.6 *(Almost orthogonality of nonlinear profiles). Define U_k^α, U_k^β as the maximal lifespan I_k solutions of (1.1) with initial data $U_k^\alpha(0) = \widetilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha$, $U_k^\beta(0) = \widetilde{\psi}_{\mathcal{O}_k^\beta}^\beta$, where $\mathcal{O}^\alpha$ and $\mathcal{O}^\beta$ are orthogonal. And define G to be the solution of the maximal lifespan I_0 of (1.1) with initial data $G(0) = g$. And $0 \in I_k$ and $\lim_{k\to\infty} |I_k| = 0$. Then*

$$\lim_{k\to\infty} \sup_{t\in I_k} \langle U_k^\alpha(t), U_k^\beta(t)\rangle_{\dot{H}^1\times\dot{H}^1} = 0, \qquad \lim_{k\to\infty} \sup_{t\in I_k\cap I_0} \langle U_k^\alpha(t), G(t)\rangle_{\dot{H}^1\times\dot{H}^1} = 0. \tag{6.6}$$

Proof. Set $U_k^0(0) = g$ and $U_k^0 = G$ for all k, such that U_k^0 can be considered as a nonlinear profile with a trivial frame $\mathcal{O} = (1, 0, 0)_k$.

For any $\theta > 0$, by the decomposition of the nonlinear profiles U^α and U^β (Corollary 5.7), there exist $T_{\theta,\alpha}$, $R_{\theta,\alpha}$, $T_{\theta,\beta}$, $R_{\theta,\beta}$ sufficiently large

$$U_k^\alpha = \omega_k^{\alpha,\theta,-\infty} + \omega_k^{\alpha,\theta,+\infty} + \omega_k^{\alpha,\theta} + \rho_k^{\alpha,\theta},$$
$$U_k^\beta = \omega_k^{\beta,\theta,-\infty} + \omega_k^{\beta,\theta,+\infty} + \omega_k^{\beta,\theta} + \rho_k^{\beta,\theta}.$$

For U_k^0, set $U_k^0 := \omega_k^{\alpha,\theta,-\infty} + \omega_k^{\alpha,\theta,+\infty} + \omega_k^{\alpha,\theta} + \rho_k^{\alpha,\theta}$ where $\rho_k^{0,\theta} = \omega_k^{0,\theta} = 0$ and $\omega^{0,\theta,+\infty} = \omega^{0,\theta,-\infty} = \frac{1}{2}G$. And by taking $T_{\theta,0}$ large, it is easy to make $\|G\|_{Z'(-T_{\theta,0},T_{\theta,0})} \leq \theta$. So $\langle U_k^\alpha(t), G(t)\rangle_{\dot{H}^1\times\dot{H}^1}$ can be considered as a special case of $\langle U_k^\alpha(t), U_k^\beta(t)\rangle_{\dot{H}^1\times\dot{H}^1}$ when $\beta = 0$.

Since $\rho_k^{\alpha,\theta}$, $\rho_k^{\beta,\theta}$ are the small terms with the X^1-norm less than θ, for any fixed $t \in I_k$, it will suffice to consider the following three terms:

1. $\langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta,\pm\infty}\rangle_{\dot{H}^1\times\dot{H}^1}$;
2. $\langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta}\rangle_{\dot{H}^1\times\dot{H}^1}$;
3. $\langle \omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta}\rangle_{\dot{H}^1\times\dot{H}^1}$.

Case (1): $\langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta,\pm\infty}\rangle_{\dot{H}^1\times\dot{H}^1}$.
By the constructions of $\omega_k^{\alpha,\theta,\pm\infty}$, $\omega_k^{\beta,\theta,\pm\infty}$ in the proof of Lemma 5.7, we obtain that

$$\omega_k^{\alpha,\theta,\pm\infty} := \mathbb{1}_{\{\pm(t-t_k^\alpha)\geq T_{\alpha,\theta}(N_k^\alpha)^{-2}, |t|\leq T_{\alpha,\theta}^{-1}\}} \left(\Pi_{t_k^\alpha - t, x_k^\alpha} T_{N_k^\alpha} \phi^{\alpha,\theta,\pm\infty}\right), \tag{6.7}$$

$$\omega_k^{\alpha,\theta,\pm\infty} := \mathbb{1}_{\{\pm(t-t_k^\beta)\geq T_{\beta,\theta}(N_k^\beta)^{-2}, |t|\leq T_{\beta,\theta}^{-1}\}} \left(\Pi_{t_k^\beta - t, x_k^\beta} T_{N_k^\beta} \phi^{\beta,\theta,\pm\infty}\right). \tag{6.8}$$

For any fixed $t \in I_k$, we obtain that

$$\langle \omega_k^{\alpha,\theta,\pm\infty}(t), \omega_k^{\beta,\theta,\pm\infty}(t)\rangle_{\dot{H}^1\times\dot{H}^1} = \langle \phi_{\mathcal{O}_k^\alpha}^{\alpha,\theta,\pm\infty}, \phi_{\mathcal{O}_k^\beta}^{\beta,\theta,\pm\infty}\rangle_{\dot{H}^1\times\dot{H}^1}.$$

By (6.1) of Proposition 6.2, we obtain that

$$\lim_{k\to\infty} \sup_t \langle \omega_k^{\alpha,\theta,\pm\infty}(t), \omega_k^{\beta,\theta,\pm\infty}(t)\rangle_{\dot{H}^1\times\dot{H}^1} = 0.$$

Case (2): $\langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta}\rangle_{\dot{H}^1\times\dot{H}^1}$.
By the constructions of $\omega_k^{\alpha,\theta,\pm\infty}$, $\omega_k^{\beta,\theta,\pm\infty}$ in the proof of Lemma 5.7, we obtain that

$$\omega_k^{\beta,\theta} := \widetilde{u_k}^\beta \cdot \mathbb{1}_{S_k^{\beta,\theta}},$$

where $S_k^{\beta,\theta} := \{(x,t) \in \mathbb{T}^4 \times (-T_{\beta,\theta}, T_{\beta,\theta}) : |t - t_k^\beta| < T_{\beta,\theta}(N_k^\beta)^{-2},\ |x - x_k^\beta| \leq R_{\beta,\theta}(N_k^\beta)^{-1}\}$ and $\widetilde{u_k}^\beta$ is defined in (2). Following a similar proof of the **Case 4** in the proof of (8.1) in Lemma 7.3, we have that $\lim_{k\to\infty} \sup_t \langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta}\rangle_{\dot{H}^1\times\dot{H}^1} = 0$.

Case (3): $\langle \omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta} \rangle_{\dot{H}^1 \times \dot{H}^1}$.

For $\varepsilon > 0$ small.

If $N_k^\alpha / N^\beta k + N_k^\beta / N_k^\alpha \leq \varepsilon^{-1000}$ and k is large enough then $S_k^{\alpha,\theta} \cap S_k^{\beta,\theta} = \emptyset$. (By the definition of orthogonality of frames, $N_k^\alpha / N_k^\beta + N_k^\beta / N_k^\alpha \leq \varepsilon^{-1000}$ implies $(N_k^\alpha)^2 |t_k^\alpha - t_k^\beta| \to \infty$ or $N_k^\alpha |x_k^\alpha - x_k^\beta| \to \infty$, so $S_k^{\alpha,\theta} \cap S_k^{\beta,\theta} = \emptyset$.) In this case, $\omega_k^{\alpha,\theta} \omega_k^{\beta,\theta} \equiv 0$.

If $N_k^\alpha / N_k^\beta \geq \varepsilon^{-1000}/2$. Denote that

$$\omega_k^{\alpha,\theta} \omega_k^{\beta,\theta} = \omega_k^{\alpha,\theta} \widetilde{\omega}_k^{\beta,\theta} := \omega_k^{\alpha,\theta} \cdot (\omega_k^{\beta,\theta} \mathbb{1}_{(t_k^\alpha - T_{\alpha,\theta}(N_k^\alpha)^{-2}, t_k^\alpha + T_{\alpha,\theta}(N_k^\alpha)^{-2})}(t)).$$

By $\varepsilon^{10} N_k^\alpha >> \varepsilon^{-10} N_k^\beta$ and the Claim † in the proof of Lemma 8.2, we obtain that

$$\begin{aligned} \langle \omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta} \rangle_{\dot{H}^1 \times \dot{H}^1} \leq & \langle P_{\leq \varepsilon^{10} N_k^\alpha} \omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta} \rangle_{\dot{H}^1 \times \dot{H}^1} + \langle P_{> \varepsilon^{10} N_k^\alpha} \omega_k^{\alpha,\theta}, P_{> \varepsilon^{-10} N_k^\beta} \omega_k^{\beta,\theta} \rangle_{\dot{H}^1 \times \dot{H}^1} \\ & + \langle P_{> \varepsilon^{10} N_k^\alpha} \omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta} \rangle_{\dot{H}^1 \times \dot{H}^1} \\ & \lesssim \varepsilon. \quad \square \end{aligned}$$

7. Proof of the main theorems

It suffices to prove the solutions remain bounded in Z-norm on intervals of length at most 1. To obtain this, we run the induction on $\|u\|_{L_t^\infty \dot{H}^1}$ (in the focusing case $\mu = -1$).

Definition 7.1. Define

$$\Lambda(L, \tau) = \sup_{\substack{u \text{ is a solution} \\ \text{of } (1.9)}} \{\|u\|_{Z(I)} : \sup_{t \in I} \|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)} < L,\ |I| \leq \tau\},$$

where u is any strong solution of (1.9) with initial data u_0 in interval I of length $|I| \leq \tau$.

It is easy to see that Λ is an increasing function of both L and τ, and moreover, by the definition we have the sublinearity of Λ in τ: $\Lambda(L, \tau + \sigma) \leq \Lambda(L, \tau) + \Lambda(L, \sigma)$. Hence we define

$$\Lambda_0(L) = \lim_{\tau \to 0} \Lambda(L, \tau),$$

and for all τ, we have that $\Lambda(L, \tau) < +\infty \Leftrightarrow \Lambda_0(L) < +\infty$. Finally, we define

$$E_{max} = \sup\{L : \Lambda_0(L) < +\infty\}.$$

Theorem 7.2. *Consider E_{max} defined above. $E_{max} \geq \|W\|^2_{\dot{H}^1(\mathbb{R}^4)}$.*

Then it is easy to check that Theorem 1.1 is true by Theorem 7.2. Hence it will suffice to prove Theorem 7.2.

Proof of Theorem 7.2. Suppose for the contradiction argument that $E_{max} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}$. By the definition of E_{max}, there exists a sequence of solutions u_k such that

$$\sup_{t\in[-T_k,T^k]} \|u(t)\|_{\dot{H}^1(\mathbb{T}^4)} \to E_{max}, \quad \|u_k\|_{Z(-T_k,0)},\ \|u_k\|_{Z(0,T^k)} \to +\infty, \tag{7.1}$$

for some $T_k,\ T^k \to 0$ as $k \to +\infty$. For the simplicity of notations, set

$$L(\phi) := \sup_{t\in[-T_k,T^k]} \|u_\phi(t)\|^2_{\dot{H}^1(\mathbb{T}^4)},$$

where $u_\phi(t)$ is the solution of (1.1) with initial data $u_\phi(0) = \phi$. By the Proposition 6.3, after extracting a subsequence, (7.1) gives a sequence of profiles $\widetilde{\psi}^\alpha_k$, where $\alpha, k = 1, 2, \cdots$, and a decomposition

$$u_k(0) = g + \sum_{1\le\alpha\le J} \widetilde{\psi}^\alpha_k + R^J_k,$$

satisfying

$$\limsup_{J\to\infty} \limsup_{k\to\infty} \|e^{it\Delta} R^J_k\|_{Z(I_k)} = 0. \tag{7.2}$$

And moreover the almost orthogonality in the Proposition 6.3 and the almost orthogonality of nonlinear profiles (Lemma 6.6), we obtain that

$$\begin{aligned} &L(\alpha) := \lim_{k\to+\infty} L(\widetilde{\psi}^\alpha_{\mathcal{O}^\alpha_k}) \in [0, E_{max}], \\ &\lim_{J\to J^*} \left(\sum_{1\le\alpha\le J} L(\alpha) + \lim_{k\to\infty} L(R^J_k) \right) + L(g) = E_{max}. \end{aligned} \tag{7.3}$$

Case 1: $g \neq 0$ and no any Euclidean profiles.
There is no any Euclidean profiles, and by Remark 2.6, $\|g\|_{H^1(\mathbb{T}^4)} \lesssim L(g) \le E_{max}$. Then, by $I_k \to 0$ as $k \to \infty$, there exist, $\eta > 0$, s.t. for k large enough

$$\|e^{it\Delta} u_k(0)\|_{Z(-T_k,T^k)} \le \|e^{it\Delta} g\|_{Z(-\eta,\eta)} + \varepsilon \le \delta_0$$

where δ_0 is given by the local theory in Proposition 4.5. In this case, we conclude that $\|u_k\|_{Z(-T_k,T^k)} \lesssim 2\delta_0$ which contradicts (7.1).
Case 2: $g = 0$ and only one Euclidean profile $\widetilde{\psi}^1_k$ such that $L(1) = E_{max}$.
By Remark 6.5 and (7.3), we obtain that $L(\widetilde{\psi}^1_k) \le E_{max}$ which implies $\|\psi\|_{\dot{H}^1(\mathbb{R}^4)} < \infty$ (if $\mu = +1$) or $\sup_t \|u_\psi\|_{\dot{H}^1(\mathbb{R}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}$ (if $\mu = -1$). Denote U^1_k is the solution of (1.1) with $U^1_k(0) = \widetilde{\psi}^1_k$. In this case, we use the part (1) of Proposition 5.6 and Remark 2.6. Given some $\epsilon > 0$, for k large enough, we have that

$$\|U^1_k\|_{X^1(-T_k,T^k)} \le \|U^1_k\|_{X^1(-\delta,\delta)} \lesssim 1, \qquad \text{and} \qquad \|U^1_k(0) - u_k(0)\|_{H^1(\mathbb{T}^4)} \le \epsilon. \tag{7.4}$$

By (7.4) and Proposition 4.8, we obtain that

$$\|u_k\|_{Z(I_k)} \lesssim \|u_k\|_{X^1(I_k)} \lesssim 1,$$

which contradicts (7.1).

Case 3: At least two of all profiles are nonzero.

By (7.3), $L(g) < E_{max}$ and $L(\alpha) < E_{max}$ for any $\alpha = 1, 2, \cdots$. By almost orthogonality and relabeling the profiles, we can assume that for all α,

$$L(\alpha) \le L(1) < E_{max} - \eta, \ L(g) < E_{max} - \eta, \text{ for some } \eta > 0.$$

Define U_k^α as the maximal life-span solution of (1.1) with initial data $U_k^\alpha(0) = \widetilde{\psi}_k^\alpha$ and G to be the maximal life-span solution of (1.1) with initial data $G(0) = g$.

By the definition of Λ and the hypothesis $E_{max} < \infty$ (if $\mu = +1$) and $E_{max} < E_W$ (if $\mu = -1$), we have

$$\|G\|_{Z(-1,1)} + \lim_{k\to\infty} \|U_k^\alpha\|_{Z(-1,1)} \le 2\Lambda(E_{max} - \eta/2, 2) \lesssim 1.$$

By Proposition 4.7, it follows that for any α and any $k > k_0(\alpha)$ sufficient large,

$$\|G\|_{X^1(-1,1)} + \|U_k^\alpha\|_{X^1(-1,1)} \lesssim 1.$$

For $J, k \ge 1$, we define

$$U_{prof,k}^J := G + \sum_{\alpha=1}^J U_k^\alpha = \sum_{\alpha=0}^J U_k^\alpha,$$

where we set that $U_k^0 := G$.

Claim. *There is a constant Q such that*

$$\|U_{prof,k}^J\|_{X^1(-1,1)}^2 + \sum_{\alpha=0}^J \|U_k^\alpha\|_{X^1(-1,1)}^2 \le Q^2, \tag{7.5}$$

uniformly on J.

From (7.2) we know that there are only finite many profiles such that $L(\alpha) \ge \frac{\delta_0}{2}$. We may assume that for all $\alpha \ge A$, $L(\alpha) \le \delta_0$. Consider U_k^α for $k \ge A$, by small data GWP result (Proposition 4.6), we have that

$$\begin{aligned}
&\|U_{prof,k}^J\|_{X^1(-1,1)} = \|\sum_{0\le\alpha\le J} U_k^\alpha\|_{X^1(-1,1)} \\
&\le \sum_{0\le\alpha\le A} \|U_k^\alpha\|_{X^1(-1,1)} + \|\sum_{A\le\alpha\le J} (U_k^\alpha - e^{it\Delta}U_k^\alpha(0))\|_{X^1(-1,1)} + \|e^{it\Delta} \sum_{A\le\alpha\le J} U_k^\alpha(0)\|_{X^1(-1,1)}
\end{aligned}$$

$$\lesssim (A+1) + \sum_{A\le\alpha\le J} \|U_k^\alpha(0)\|_{H^1}^2 + \|\sum_{A\le\alpha\le J} U_k^\alpha(0)\|_{H^1}$$
$$\lesssim (A+1) + \sum_{A\le\alpha\le J} L(\alpha) + E_{max}^{\frac{1}{2}}$$
$$\lesssim 1.$$

And also similarly, we have that

$$\begin{aligned}\sum_{\alpha=0}^{J} \|U_k^\alpha\|_{X^1(-1,1)}^2 &= \sum_{\alpha=0}^{A-1} \|U_k^\alpha\|_{X^1(-1,1)}^2 + \sum_{A\le\alpha\le J} \|U_k^\alpha\|_{X^1(-1,1)}^2 \\ &\lesssim A + \sum_{A\le\alpha\le J} L(\alpha) \\ &\lesssim 1.\end{aligned}$$

We denote that

$$U_{app,k}^J = \sum_{0\le\alpha\le J} U_k^\alpha + e^{it\Delta} R_k^J$$

is a solution of the approximation equation (4.11) with the error term:

$$\begin{aligned} e &= (i\partial_t + \Delta) U_{app,k}^J - F(U_{app,k}^J) \\ &= \sum_{0\le\alpha\le J} F(U_k^\alpha) - F(\sum_{0\le\alpha\le J} U_k^\alpha + e^{it\Delta} R_k^J),\end{aligned}$$

where $F(u) = u|u|^2$.

From (7.5) we know $\|U_{app,k}^J\|_{X^1(-1,1)} \le Q$.

By Lemma 7.3 (proven later), we obtain that

$$\limsup_{k\to\infty} \|e\|_{N(I_k)} \le \varepsilon/2, \text{ for } J \ge J_0(\varepsilon).$$

We use the stability proposition (Proposition 4.8) to conclude that u_k satisfies

$$\|u_k\|_{X^1(I_k)} \lesssim \|U_{app,k}^J\|_{X^1(I_k)} \le \|U_{prof,k}^J\|_{X^1(-1,1)} \|e^{it\Delta} R_k^J\|_{X^1(-1,1)} \lesssim 1,$$

which contradicts (7.1). □

Lemma 7.3. *With the same notation, we obtain that*

$$\limsup_{J\to\infty} \limsup_{k\to\infty} \|\sum_{0\le\alpha\le J} F(U_k^\alpha) - F(\sum_{0\le\alpha\le J} U_k^\alpha + e^{it\Delta} R_k^J)\|_{N(I_k)} = 0. \tag{7.6}$$

8. Proof of Lemma 7.3

Consider

$$\begin{aligned}&\|\sum_{0\le\alpha\le J} F(U_k^\alpha) - F(U^J_{prof,k} + e^{it\Delta}R_k^J)\|_{N(I_k)}\\ \le&\|F(U^J_{prof,k} + e^{it\Delta}R_k^J)) - F(U^J_{prof,k})\|_{N(I_k)} + \|F(U^J_{prof,k}) - \sum_{0\le\alpha\le J} F(U_k^\alpha)\|_{N(I_k)}.\end{aligned}$$

It will suffice that we can prove

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\|F(U^J_{prof,k} + e^{it\Delta}R_k^J)) - F(U^J_{prof,k})\|_{N(I_k)} = 0, \tag{8.1}$$

and

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\|F(U^J_{prof,k}) - \sum_{0\le\alpha\le J} F(U_k^\alpha)\|_{N(I_k)} = 0. \tag{8.2}$$

Before proving (8.1) and (8.2), we need several lemmas.

Denote that $\mathfrak{D}_{p,q}(a,b)$ stands for a $p+q$ - linear expression with p factors consisting of either $\overline{a}$ or a and q factors consisting of either $\overline{b}$ or b.

Lemma 8.1 *(a high-frequency linear solution does not interact significantly with a low-frequency profile). Assume that $B, N \ge 2$, and dyadic numbers, and assume that $\omega : \mathbb{T}^4 \times (-1,1) \to \mathbb{C}$ is a function satisfying*

$$|\nabla^j \omega| \le N^{j+1}\mathbb{1}_{|x|\le N^{-1},|t|\le N^{-2}},\ \ j = 0,1.$$

Then we hold that

$$\|\mathfrak{D}_{2,1}(\omega, e^{it\Delta}P_{>BN}f)\|_{N(-1,1)} \lesssim (B^{-1/200} + N^{-1/200})\|f\|_{H^1(\mathbb{T}^4)}.$$

Proof. We may assume that $\|f\|_{H^1(\mathbb{T}^4)} = 1$ and $f = P_{>BN}f$. By Proposition 3.12, we obtain that

$$\begin{aligned}&\|\mathfrak{D}_{2,1}(\omega, e^{it\Delta}P_{>BN}f)\|_{N(I)}\\ \lesssim&\|\mathfrak{D}_{2,1}(\omega, \nabla e^{it\Delta}f)\|_{L^1((-1,1),L^2)} + \|e^{it\Delta}f\|_{L_t^\infty L_x^2}\|\omega\|_{L_t^2L_x^\infty}\||\nabla\omega| + |\omega|\|_{L_t^2L_x^\infty}\\ \lesssim&\|\mathfrak{D}_{2,1}(\omega, \nabla e^{it\Delta}f)\|_{L^1((-1,1),L^2)} + B^{-1}.\end{aligned}$$

(It's easy to check that $\|\omega\|_{L_t^2L_x^\infty} \le \left(\int_{-N^{-2}}^{N^{-2}}(N)^2\,dt\right)^{\frac{1}{2}}$, $\|\nabla\omega\|_{L_t^2L_x^\infty} \le \left(\int_{-N^{-2}}^{N^{-2}}N^4\,dt\right)^{\frac{1}{2}} = N$, and $\|P_{>BN}f\|_{L^2} \le \frac{1}{BN}\|f\|_{H^1}$.)

Now we let $G(x,t) := N^4 \eta_{\mathbb{R}^4}(N\Psi^{-1}(x))\eta_{\mathbb{R}}(N^2 t)$,

$$\|\mathfrak{D}_{2,1}(\omega, \nabla e^{it\Delta} f)\|^2_{L^1((-1,1),L^2)} \lesssim \sum_{j=1}^{4} \int_{-1}^{1} \langle \partial_j f, [\int_{-1}^{1} e^{-it\Delta} G e^{it\Delta}\, dt]\rangle_{L^2 \times L^2}.$$

It remains to prove that

$$\|K\|_{L^2(\mathbb{T}^4)\to L^2(\mathbb{T}^4)} \lesssim N^2 (B^{-\frac{1}{100}} + N^{-\frac{1}{100}}),$$

where $K = P_{>BN} \int_{\mathbb{R}} e^{-it\Delta} G e^{it\Delta} P_{>BN}\, dt$. We then compute the Fourier coefficients of K as follows:

$$\begin{aligned}
c_{p,q} &= \langle e^{ipx}, K e^{iqx}\rangle \\
&= \int_{\mathbb{T}^4} \overline{P_{>BN} e^{ipx}} \int_{\mathbb{R}} e^{-it\Delta} G e^{it\Delta} P_{>BN}\, dt dx \\
&= (1-\eta_{\mathbb{R}^4})(p/BN)(1-\eta_{\mathbb{R}^4})(q/BN) \int_{\mathbb{T}^4\times[-1,1]} \overline{e^{-it|p|^2+ipx}} G(t,x) e^{-it|q|^2+iqx}\, dx dt.
\end{aligned}$$

Hence, we obtain that

$$|c_{p,q}| \lesssim N^{-2} \left(1 + \frac{\left||p|^2 - |q|^2\right|}{N^2}\right)^{-10} \left(1 + \frac{|p-q|}{N}\right)^{-10} \mathbb{1}_{\{|p|\geq BN\}} \mathbb{1}_{\{|q|\geq BN\}}.$$

Using Schur's lemma, we have that

$$\|K\|_{L^2(\mathbb{T}^4)\to L^2(\mathbb{T}^4)} \lesssim \sup_{p\in\mathbb{Z}^4} \sum_{q\in\mathbb{Z}^4} |c_{p,q}| + \sup_{q\in\mathbb{Z}^4} \sum_{p\in\mathbb{Z}^4} |c_{p,q}|.$$

It suffices to prove that

$$N^{-4} \sup_{|p|\geq BN} \sum_{v\in\mathbb{Z}^4} \left(1 + \frac{\left||p|^2 - |p+v|^2\right|}{N^2}\right)^{-10} \left(1 + \frac{|v|}{N}\right)^{-10} \lesssim B^{-\frac{1}{100}} + N^{-\frac{1}{100}}. \tag{8.3}$$

We will separate (8.3) into the following 3 sums:

$$\sum_{|v|\geq NB^{\frac{1}{100}}}, \quad \sum_{\substack{|v|\leq NB^{\frac{1}{100}} \\ |v\cdot p|\geq N^2 B^{\frac{1}{10}}}} \quad \text{and} \quad \sum_{\substack{|v|\leq NB^{\frac{1}{100}} \\ |p\cdot v|\leq N^2 B^{\frac{1}{10}}}}.$$

Then we discuss case by case.

Case 1:

$$\sum_{|v|\geq NB^{\frac{1}{100}}}\left(1+\frac{\left||p|^2-|p+v|^2\right|}{N^2}\right)^{-10}\left(1+\frac{|v|}{N}\right)^{-10}\lesssim\sum_{|v|\geq NB^{\frac{1}{100}}}\left(1+\frac{|v|}{N}\right)^{-10}\lesssim B^{-\frac{6}{100}}.$$

Case 2:

$$\begin{aligned}\sum_{\substack{|v|\leq NB^{\frac{1}{100}}\\|v\cdot p|\geq N^2B^{\frac{1}{10}}}}\left(1+\frac{\left||p|^2-|p+v|^2\right|}{N^2}\right)^{-10}\left(1+\frac{|v|}{N}\right)^{-10}&\lesssim\sum_{\substack{|v|\leq NB^{\frac{1}{100}}\\|v\cdot p|\geq N^2B^{\frac{1}{10}}}}\left(1+\frac{2|v\cdot p|}{N^2}\right)^{-10}\\&\lesssim B^{-\frac{6}{10}}.\end{aligned}$$

Case 3:
Denote $\hat{p}=\frac{p}{|p|}$

$$\begin{aligned}&N^{-4}\sup_{|p|\geq BN}\sum_{\substack{|v|\leq NB^{\frac{1}{100}}\\|p\cdot v|\leq N^2B^{\frac{1}{10}}}}\left(1+\frac{\left||p|^2-|p+v|^2\right|}{N^2}\right)^{-10}\left(1+\frac{|v|}{N}\right)^{-10}\\\leq&N^{-4}\sup_{|p|\geq BN}\#\{v:|v|\leq NB^{\frac{1}{100}},\ |\hat{p}\cdot v|\leq NB^{-\frac{9}{10}}\}\\=&N^{-4}(NB^{\frac{1}{100}})^3NB^{-\frac{9}{10}}\\\leq&B^{-\frac{87}{100}}.\quad\square\end{aligned}$$

Lemma 8.2. *Assume that $\mathfrak{O}_\alpha=(N_{k,\alpha},t_{k,\alpha},x_{k,\alpha})_k\in\mathcal{F}_e,\ \alpha\in\{1,2\}$, are two orthogonal frames, $I\subseteq(-1,1)$ is a fixed open interval, $0\in I$, and $T_1,\ T_2,\ R\in[1,\infty)$ are fixed numbers, $R\geq T_1+T_2$. For k large enough, for $\alpha\in\{1,2\}$*

$$|\nabla_x^m\omega_k^{\alpha,\theta}|+(N_{k,\alpha})^{-2}\mathbb{1}_{S_k^{\alpha,\theta}}|\partial_t\nabla_x^m\omega_k^{\alpha,\theta}|\leq R_{\theta,\alpha}(N_k^\alpha)^{|m|+1}\mathbb{1}_{S_k^{\alpha,\theta}},\ 0\leq|m|\leq 10,$$

where

$$S_k^{\alpha,\theta}:=\{(x,t)\in\mathbb{T}^4\times I:|t-t_{k,\alpha}|<T_\alpha(N_{k,\alpha})^{-2},\ |x-x_{k,\alpha}|\leq R(N_{k,\alpha})^{-1}\}.$$

And assume that $(\omega_{k,1},w_{k,2},f_k)_k$ are 3 sequences of functions with properties $\|f_k\|_{X^1(I)}\leq 1$ for all k large enough, then

$$\limsup_{k\to\infty}\|\omega_{k,1}\,\omega_{k,2}\,f_k\|_{N(I)}=0$$

Proof. For $\varepsilon > 0$ small. If $N_{k,1}/N_{k,2} + N_{k,2}/N_{k,1} \le \varepsilon^{-1000}$ and k is large enough then $S_{k,1} \cap S_{k,2} = \emptyset$. (By the definition of orthogonality of frames, $N_{k,1}/N_{k,2} + N_{k,2}/N_{k,1} \le \varepsilon^{-1000}$ implies $N_{k,1}^2|t_{k,1} - t_{k,2}| \to \infty$ or $N_{k,1}|x_{k,1} - x_{k,2}| \to \infty$, so $S_{k,1} \cap S_{k,2} = \emptyset$.) In this case, $\omega_{k,1}\,\omega_{k,2}\, f_k \equiv 0$.

If $N_{k,1}/N_{k,2} \ge \varepsilon^{-1000}/2$. Denote that

$$\omega_{k,1}\omega_{k,2} = \omega_{k,1}\widetilde{\omega}_{k,2} := \omega_{k,1} \cdot (w_{k,2}\mathbb{1}_{(t_{k,1}-T_1 N_{k,1}^{-2}, t_{k,1}+T_1 N_{k,1}^{-2})}(t)).$$

Claim †. *For k large enough,*

1. $\|\widetilde{\omega}_{k,2}\|_{X^1(I)} \lesssim_R 1$;
2. $\|P_{>\varepsilon^{-10}N_{k,2}}\widetilde{\omega}_{k,2}\|_{X^1(I)} \lesssim_R \varepsilon$;
3. $\|\widetilde{\omega}_{k,2}\|_{Z(I)} \lesssim_R \varepsilon$;
4. $\|\omega_{k,1}\|_{X^1(I)} \lesssim_R 1$;
5. $\|P_{\le \varepsilon^{10}}\omega_{k,1}\|_{X^1(I)} \lesssim_R \varepsilon$.

By this Claim †, Proposition 4.4, and $\varepsilon^{10}N^1 >> \varepsilon^{-10}N_2$ we obtain that

$$\begin{aligned}\|\omega_{k,1}\omega_{k,2} f_k\|_{N(I)} \le & \|(P_{\le \varepsilon^{10}N_{k,1}}\omega_{k,1})(\widetilde{\omega}_{k,2}) f_k\|_{N(I)} + \|(P_{>\varepsilon^{10}N_{k,1}}\omega_{k,1})(P_{>\varepsilon^{-10}N_{k,2}}\widetilde{\omega}_{k,2}) f_k\|_{N(I)} \\ & + \|(P_{>\varepsilon^{10}N_{k,1}}\omega_{k,1})(P_{\le\varepsilon^{-10}N_{k,2}}\widetilde{\omega}_{k,2}) f_k\|_{N(I)} \\ \lesssim_R & \varepsilon.\end{aligned}$$

More detail about the Claim †:

(1): We consider $\widetilde{\omega}_{k,2} w_{k,2}\mathbb{1}_{(t_{k,1}-T_1 N_{k,1}^{-2}, t_{k,1}+T_1 N_{k,1}^{-2})}(t)$.

$$\begin{aligned}\|\widetilde{\omega}_{k,2}\|_{X^1(I)} \lesssim & \left(\int\limits_{|x-x_{k,2}|\le RN_{k,2}^{-1}} |\langle\nabla\rangle\widetilde{\omega}_{k,2}(0)|^2\, dx \right)^{\frac{1}{2}} \\ & + \left(\sum_N \left(\int_I dt \|P_N(\partial_t\widetilde{\omega}_{k,2})\|_{H^1} + \|P_N\Delta\widetilde{\omega}_{k,2}\|_{H^1} \right)^2 \right)^{\frac{1}{2}} \\ \lesssim & (R^2 N_{k,2}^4 R^4 N_{k,2}^{-4})^{\frac{1}{2}} + \int_I (\|\partial_t\widetilde{\omega}_{\alpha,k}\|_{H^1} + \|\Delta\widetilde{\omega}_{k,2}\|_{H^1})\, dt \\ \lesssim & 1.\end{aligned}$$

(2): We consider the high frequency part of $\widetilde{\omega}_{k,2}$.

$$\|P_{>\varepsilon^{-10}N_{k,2}}\widetilde{\omega}_{k,2}\|_{X^1(I)}$$

$$\leq \left(\int_{|x-x_{k,2}|\leq RN_{k,2}^{-1}} |P_{>\varepsilon^{-10}N_{k,2}} \langle \nabla \rangle \widetilde{\omega}_{k,2}(0)|^2 \, dx \right)^{\frac{1}{2}} + \int \| P_{>\varepsilon^{-10}N_{k,2}} (i\partial_t + \Delta)\widetilde{\omega}_{k,2} \|_{H^1} \, dt$$

$$\leq \left(\int_{|x-x_{k,2}|\leq RN_{k,2}^{-1}} \left(\frac{\varepsilon^{10}}{N_{k,2}} \right)^2 |P_{>\varepsilon^{-10}N_{k,2}} \langle \nabla \rangle^2 \widetilde{\omega}_{k,2}(0)|^2 \, dx \right)^{\frac{1}{2}}$$

$$+ \int_{|t-t_{k,2}|<N_{k,2}^{-2}R} \frac{\varepsilon^{10}}{N_{k,2}} \| (i\partial_t + \Delta)\widetilde{\omega}_{k,2} \|_{H^2} \, dt$$

$$\leq \varepsilon^{10} R^3 + N_{k,2}^{-2} R \frac{\varepsilon^{10}}{N_{k,2}} (R^4 N_{k,2}^{-2} R^2 N_{k,2}^{1} 0)^{\frac{1}{2}}$$

$$\lesssim \varepsilon^{10} R^4.$$

(3): We consider the Z-norm of $\widetilde{\omega}_{k,2}$.

$$\begin{aligned} \|\widetilde{\omega}_{k,2}\|_{Z(I)} &\leq \left(\sum_N N^2 \| P_N \widetilde{\omega}_{k,2} \|^4_{L^4(\mathbb{T}^4 \times (t_{k,1}-RN_{k,1}^{-2}, t_{k,1}+RN_{k,1}^{-2}))} \right)^{1/4} \\ &\lesssim \left\| \left(\sum_N |\nabla^{1/2} P_N \widetilde{\omega}_{k,2}|^2 \right)^{1/2} \right\|_{L^4} \\ &\lesssim \| \nabla^{1/2} \widetilde{\omega}_{k,2} \|_{L^4(\mathbb{T}^4 \times (t_{k,1}-RN_{k,1}^{-2}, t_{k,1}+RN_{k,1}^{-2}))} \\ &\lesssim R^{\frac{9}{4}} \left(\frac{N_{k,2}}{N_{k,1}} \right)^{\frac{1}{2}} \leq R^{\frac{9}{4}} \varepsilon^{500}. \end{aligned}$$

(4): Similar with (1).
(5):

$$\begin{aligned} & \| P_{\leq \varepsilon^{10} N_{k,1}} \omega_{k,1} \|_{X^1(I)} \\ \lesssim & \varepsilon^{10} N_{k,1} \left(\| P_{\leq \varepsilon^{10} N_{k,1}} \omega_{k,1}(0) \|_{L^2} + \int \| P_{\leq \varepsilon^{10} N_{k,1}} (i\partial_t + \Delta)\omega_{k,1} \|_{L^2} \, dt \right) \\ \lesssim & \varepsilon^{10} R^4. \quad \square \end{aligned} \tag{8.4}$$

Proof of (8.1).

$$\begin{aligned} & F(U^J_{prof,k} + e^{it\Delta} R^J_k) - F(U^J_{prof,k}) \\ = & \mathfrak{D}_{2,1}(U^J_{prof,k}, e^{it\Delta} R^J_k) + \mathfrak{D}_{1,2}(U^J_{prof,k}, e^{it\Delta} R^J_k) + |e^{it\Delta} R^J_k|^2 e^{it\Delta} R^J_k \end{aligned}$$

First, by the nonlinear estimate (Proposition 4.4), we have

$$\begin{aligned}&\||e^{it\Delta}R_k^J|^2e^{it\Delta}R_k^J\|_{N(I_k)}\\ \lesssim&\|e^{it\Delta}R_k^J\|_{Z'(I_k)}^2\|e^{it\Delta}R_k^J\|_{X^1(I_k)}\end{aligned}$$

Since $\|e^{it\Delta}R_k^J\|_{Z'(I_k)}\to 0$ as $J,k\to\infty$, and $\|e^{it\Delta}R_k^J\|_{X^1(I_k)}\lesssim 1$,

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\||e^{it\Delta}R_k^J|^2e^{it\Delta}R_k^J\|_{N(I_k)}=0.$$

Second, also by the nonlinear estimate Proposition 4.4 and Proposition 3.15,

$$\begin{aligned}&\|\mathfrak{D}_{1,2}(U_{prof,k}^J,e^{it\Delta}R_k^J)\|_{N(I_k)}\\ \lesssim&\|U_{prof,k}^J\|_{X^1(I_k)}\|e^{it\Delta}R_k^J\|_{X^1(I_k)}\|e^{it\Delta}R_k^J\|_{Z'(I_k)}\to 0,\end{aligned}$$

as $k,J\to\infty$.

Third, consider

$$\|\mathfrak{D}_{2,1}(U_{prof,k}^J,e^{it\Delta}R_k^J)\|_{N(I_k)},$$

assume $\varepsilon>0$ is fixed, there exists $A=A(\varepsilon)$ sufficiently large, such that for all $J\geq A$ and $k\geq k_0(J)$

$$\|U_{prof,k}^J-U_{prof,k}^A\|_{X^1(-1,1)}\leq\varepsilon.$$

Then

$$\begin{aligned}&\|\mathfrak{D}_{2,1}(U_{prof,k}^J,e^{it\Delta}R_k^J)\|_{N(I_k)}\\ \leq&\|\mathfrak{D}_{2,1}(U_{prof,k}^A,e^{it\Delta}R_k^J)\|_{N(I_k)}+\|\mathfrak{D}_{1,1,1}(U_{prof,k}^A,U_{prof,k}^J-U_{prof,k}^A,e^{itDD}R_k^J)\|_{N(I_k)}\\ &+\|\mathfrak{D}_{2,1}(U_{prof,k}^J-U_{prof,k}^A,e^{it\Delta}R_k^J)\|_{N(I_k)}\to\|\mathfrak{D}_{2,1}(U_{prof,k}^A,e^{it\Delta}R_k^J)\|_{N(I_k)}+\varepsilon,\end{aligned}$$

as $k,J\to\infty$.

It remains to prove that

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\|\mathfrak{D}_{2,1}(U_{prof,k}^A,e^{it\Delta}R_k^J)\|_{N(I_k)}\lesssim\varepsilon.$$

By the definition of $U_{prof,k}^A$, it suffices to prove that for any $\alpha_1,\alpha_2\in\{0,1,\cdots,A\}$.

Fix $\theta=\varepsilon A^{-2}/10$, apply the decomposition in Lemma 5.7 to all nonlinear profiles U_k^α, $\alpha=1,2,\cdots,A$. We assume that

$$T_{\theta,\alpha}=T_\theta,\quad\text{and } R_{\theta,\alpha}=R_\theta,$$

for any $\alpha=1,2,\cdots,A$.

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\|\mathfrak{D}_{1,1,1}(U_k^{\alpha_1}, U_k^{\alpha_2}, e^{it\Delta}R_k^J)\|_{N(I_k)} \lesssim \varepsilon A^{-2}. \tag{8.5}$$

Case 1: $\alpha_1 = 0$ or $\alpha_2 = 0$.
Without loss of generality, suppose $\alpha_2 = 0$.

Since $\|U_k^0\|_{X^1(-1,1)} = \|G\|_{X^1(-1,1)} \lesssim 1$, for any k large enough such that $\|G\|_{Z'(I_k)} \lesssim \varepsilon A^{-2}$, and $\|G\|_{X^1(I_k)} \lesssim 1$.

By the nonlinear estimate Proposition 4.4 and Proposition 3.15,

$$\begin{aligned}
&\|\mathfrak{D}_{1,1,1}(G, U_k^{\alpha_2}, e^{it\Delta}R_k^J)\|_{N(I_k)} \\
\lesssim&\|G\|_{Z'(I_k)}\|U_k^{\alpha_2}\|_{Z'(I_k)}\|e^{it\Delta}R_k^J\|_{X^1(I_k)} + \|G\|_{Z'(I_k)}\|U_k^{\alpha_2}\|_{X^1(I_k)}\|e^{it\Delta}R_k^J\|_{Z'(I_k)} \\
&+ \|G\|_{X^1(I_k)}\|U_k^{\alpha_2}\|_{Z'(I_k)}\|e^{it\Delta}R_k^J\|_{Z'(I_k)} \\
\lesssim&\varepsilon A^{-2},
\end{aligned}$$

when taking k, J large enough.
Case 2: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_1 = \alpha_2$.

Taking k large enough, we have $I_k \subset (-T_\theta^{-1}, T_\theta^{-1})$

$$\mathbb{1}_{I_k}(t)U_k^\alpha = \omega_k^{\alpha,\theta,-\infty} + \omega_k^{\alpha,\theta,+\infty} + \omega_k^{\alpha,\theta} + \rho_k^{\alpha,\theta}.$$

By the nonlinear estimate Proposition 4.4, (5.28) and Lemma 8.2 (since $\|e^{it\Delta}R_k^J\|_{X^1(I_k)} \lesssim 1$ uniformly for both k and J), we obtain that

$$\begin{aligned}
\|\mathfrak{D}_{1,1,1}(U_k^{\alpha_1}, U_k^{\alpha_2}, e^{it\Delta}R_k^J)\|_{N(I_k)} &\lesssim \frac{1}{2}A^{-2}\varepsilon + \|\mathfrak{D}_{1,1,1}(\omega_k^{\alpha_1,\theta,+\infty}, \omega_k^{\alpha_1,\theta,-\infty}, e^{it\Delta}R_k^J)\|_{N(I_k)} \\
&\lesssim A^{-2}\varepsilon,
\end{aligned}$$

when k large enough.
Case 3: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_1 \neq \alpha_2$.
Using Lemma 8.1, and set B sufficiently large and k sufficiently large, we obtain that,

$$\begin{aligned}
\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, P_{>BN_{k,\alpha}}e^{it\Delta}R_k^J)\|_{N(I_k)} &\lesssim (\frac{1}{B^{1/200}} + \frac{1}{N_{k,\alpha}^{1/200}})\|R_k^J\|_{H^1} \\
&\lesssim \frac{\varepsilon}{4}A^{-2}.
\end{aligned} \tag{8.6}$$

We may also assume that B is sufficiently large such that, for k large enough, by a similar estimate as (8.4), we obtain that

$$\|P_{\leq B^{-1}N_{k,\alpha}}\omega_k^{\alpha,\theta}\|_{X^1(I_k)} \leq \frac{\varepsilon}{4}A^{-2}. \tag{8.7}$$

Using the modified nonlinear estimate (4.4) of Lemma 4.4 and bounds (8.6), (8.7), it remains to prove that

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\|\mathfrak{D}_{2,1}(P_{>B^{-1}N_{k,\alpha}}\omega_k^{\alpha,\theta}, P_{\leq BN_{k,\alpha}}e^{it\Delta}R_k^J)\|_{N(I_k)} = 0. \quad \square$$

Proof of (8.2).

$$F(U^J_{prof,k})-\sum_{0\le\alpha\le J}F(U^\alpha_k)=\sum_{\substack{0\le\alpha_1,\alpha_2,\alpha_3\le J\\ \alpha_1\ne\alpha_2 \text{ or } \alpha_1\ne\alpha_3 \text{ or } \alpha_2\ne\alpha_3}}\mathfrak{D}_{1,1,1}(U^{\alpha_1}_k,U^{\alpha_2}_k,U^{\alpha_3}_k)$$

By (7.5), we choose $A(\theta)$ large enough, such that $\sum_{A\le\alpha\le J}\|U_\alpha\|^2_{X^1(-1,1)}\le\theta$.

So we have

$$\begin{aligned}&\|\sum_{\substack{0\le\alpha_1,\alpha_2,\alpha_3\le J\\ \alpha_1\ne\alpha_2 \text{ or } \alpha_1\ne\alpha_3 \text{ or } \alpha_2\ne\alpha_3}}\mathfrak{D}_{1,1,1}(U^{\alpha_1}_k,U^{\alpha_2}_k,U^{\alpha_3}_k)\|_{N(I_k)}\\ \le&\|\sum_{\substack{0\le\alpha_1,\alpha_2,\alpha_3\le A\\ \alpha_1\ne\alpha_2 \text{ or } \alpha_1\ne\alpha_3 \text{ or } \alpha_2\ne\alpha_3}}\mathfrak{D}_{1,1,1}(U^{\alpha_1}_k,U^{\alpha_2}_k,U^{\alpha_3}_k)\|_{N(I_k)}+\theta.\end{aligned}$$

Using Lemma 5.7,

$$\begin{aligned}&\|\sum_{\substack{0\le\alpha_1,\alpha_2,\alpha_3\le A\\ \alpha_1\ne\alpha_2 \text{ or } \alpha_1\ne\alpha_3 \text{ or } \alpha_2\ne\alpha_3}}\mathfrak{D}_{1,1,1}(U^{\alpha_1}_k,U^{\alpha_2}_k,U^{\alpha_3}_k)\|_{N(I_k)}\\ \le&\|\sum_F\mathfrak{D}_{1,1,1}(W^1_k,W^2_k,W^3_k)\|_{N(I_k)},\end{aligned}$$

where

$$\begin{aligned}F:=&\{(W^1_k,W^2_k,W^3_k):W^i_k\in\{\omega^{\alpha,\theta,+\infty}_k,\omega^{\alpha,\theta,-\infty}_k,\omega^{\alpha,\theta}_k,\rho^{\alpha,\theta}_k\},\\ &\text{for } 0\le\alpha\le A,\ \text{and each } i, \text{at least two different } \alpha\}\end{aligned}$$

and $\#F<A^3$.

Consider the following several cases:

Case 1: the terms containing one error component $\rho^{\alpha,\theta}_k$.

By the nonlinear estimate (Proposition 4.4),

$$\|\mathfrak{D}_{1,1,1}(W^1_k,W^2_k,\rho^{\alpha,\theta}_k)\|_{N(I_k)}\le\|\rho^{\alpha,\theta}_k\|_{X^1(I_k)}\|W^1_k\|_{X^1(I_k)}\|W^2_k\|_{X^1(I_k)}\lesssim\theta,$$

for k large enough.

Case 2: the terms containing two scattering components $\omega^{\alpha,\theta,\pm\infty}_k$ and $\omega^{\beta,\theta,\pm\infty}_k$ (maybe $\alpha=\beta$ or not).

$$\begin{aligned}&\|\mathfrak{D}_{1,1,1}(\omega^{\alpha,\theta,\pm\infty}_k,\omega^{\beta,\theta,\pm\infty}_k,W^3_k)\|_{N(I_k)}\\ \le&\|W^3_k\|_{X^1(I_k)}(\|\omega^{\alpha,\theta,\pm\infty}_k\|_{X^1(I_k)}+\|\omega^{\beta,\theta,\pm\infty}_k\|_{X^1(I_k)})(\|\omega^{\alpha,\theta,\pm\infty}_k\|_{Z'(I_k)}+\|\omega^{\beta,\theta,\pm\infty}_k\|_{Z'(I_k)})\\ \lesssim&\theta,\end{aligned}$$

for k large enough.

Case 3: the terms containing two different cores $\omega_k^{\alpha,\theta}$ and $\omega_k^{\beta,\theta}$ with $\alpha \neq \beta$.
By Lemma 8.2, for k large enough, we obtain that

$$\|\mathfrak{D}_{1,1,1}(\omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta}, W_k^3)\|_{N(I_k)} \lesssim \theta.$$

Case 4: the others: $\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta,\pm\infty})$ with $\alpha \neq \beta$.

Case 4.1: $\limsup_{k\to\infty} \frac{N_{k,\beta}}{N_{k,\alpha}} = +\infty$.
By Lemma 8.1, and choosing B and k large enough,

$$\|\mathfrak{D}_{2,1}(\omega^{\alpha,\theta}, P_{>BN_{k,\alpha}}\omega^{\beta,\theta,\pm\infty})\|_{N(I_k)} \lesssim (B^{-1/200} + N_{k,\alpha}^{-1/200}) \lesssim \theta. \tag{8.8}$$

And for the other part,

$$\begin{aligned}
\|P_{\le BN_{k,\alpha}}\omega_k^{\beta,\theta,\pm\infty}\|_{X^1(I_k)} &= \|P_{\le BN_{k,\beta}\frac{N_{k,\alpha}}{N_{k,\beta}}}\omega_k^{\beta,\theta,\pm\infty}\|_{X^1(I_k)} \\
&= \|P_{\le BN_{k,\beta}\frac{N_{k,\alpha}}{N_{k,\beta}}}\pi_{x_k^\beta}T_{N_{k,\beta}}(\phi^{\beta,\theta,\pm\infty})\|_{H^1(\mathbb{T}^4)} \\
&= \|P_{\le B\frac{N_{k,\alpha}}{N_{k,\beta}}}\phi^{\beta,\theta,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^4)} \to 0, \text{ as } k\to\infty.
\end{aligned}$$

So for k large enough, we obtain that

$$\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, P_{\le BN_{k,\alpha}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \lesssim \|P_{\le BN_{k,\alpha}}\omega_k^{\beta,\theta,\pm\infty}\|_{X^1(I_k)}\|\omega_k^{\alpha,\theta}\|^2_{X^1(I_k)} \lesssim \theta.$$

Case 4.2: $\limsup_{k\to\infty} \frac{N_{k,\alpha}}{N_{k,\beta}} = +\infty$.

We assume that B is sufficiently large such that for k large, by a similar estimate as (8.8), we obtain that

$$\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, P_{>BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \lesssim (B^{-1/200} + N_{k,\beta}^{-1/200}) \lesssim \theta.$$

And by the similar estimate as (8.4), for k large enough, we obtain that

$$\|P_{\le N_{k,\beta}}\omega_k^{\alpha,\theta}\|_{X^1(I_k)} = \|P_{\le N_{k,\alpha}\frac{N_{k,\beta}}{N_{k,\alpha}}}\omega_k^{\alpha,\theta}\|_{X^1(I_k)} \lesssim \theta,$$

and $\|P_{>N_{k,\beta}}\omega_k^{\alpha,\theta}\|_{X^1(I_k)} \lesssim 1$.
Consider the remaining part, by the nonlinear estimate (4.3) and (4.4),

$$\begin{aligned}
&\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, P_{\le BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \\
\lesssim &\|\mathfrak{D}_{2,1}(P_{\le N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\le BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} + \|\mathfrak{D}_{2,1}(P_{>N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\le BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \\
&+ \|\mathfrak{D}_{1,1,1}(P_{>N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\le N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\le BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \\
\lesssim &\|P_{\le N_{k,\beta}}\omega_k^{\alpha,\theta}\|_{X^1(I_k)} + \|\mathfrak{D}_{2,1}(P_{>N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\le BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)}
\end{aligned}$$

$$\lesssim \theta + \|P_{\leq BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty}\|_{Z'(I_k)}$$
$$\lesssim \theta,$$

where $\limsup_{J\to\infty}\limsup_{k\to\infty}\|P_{\leq BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty}\|_{Z'(I_k)} = 0$ (by a similar estimate with (5.28) from extinction lemma).

Case 4.3: $N_{k,\alpha} \approx N_{k,\beta}$ and $N_{k,\alpha}|x_k^\alpha - x_k^\beta| \to \infty$ as $k \to \infty$.

From Proposition 6.2, we can use an equivalent frame of $\mathcal{O}^\alpha$ to adjust $N_{k,\alpha}$ and t_k^α such that $N_{k,\alpha} = N_{k,\beta}$ and $t_k^\alpha = t_k^\beta$.

By the definition of $\omega_k^{\alpha,\theta}$ and $\omega_k^{\beta,\theta,\pm\infty}$, for k large enough, we obtain that $\omega_k^{\beta,\theta}\omega_k^{\alpha,\theta,\pm\infty} \equiv 0$.

Case 4.4: $N_{k,\alpha} \approx N_{k,\beta}$ and $N_{k,\alpha}^2|t_k^\alpha - t_k^\beta| \to \infty$ as $k \to \infty$.

By Proposition 6.2, we can adjust $N_{k,\alpha}$ such that $N_{k,\alpha} = N_{k,\beta} := N_k$.

By the definition of $\omega_k^{\alpha,\theta}$ and $\omega_k^{\beta,\theta,\pm\infty}$, taking k large enough and $N_k^2|t_k^\alpha - t_k^\beta| > T_\theta$, we obtain that

$$\omega_k^{\alpha,\theta}\omega_k^{\beta,\theta,\pm\infty} = \mathbb{1}_{[t_\alpha - \frac{T_\theta}{N_k^2}, t_\alpha + \frac{T_\theta}{N_k^2}]}\omega_k^{\alpha,\theta}\omega_k^{\beta,\theta,\pm\infty},$$

and also $\omega_k^{\alpha,\theta,\pm\infty} = P_{\leq R_\theta N_k}\omega_k^{\alpha,\theta,\pm\infty}$.

By (5.15) and (5.17), for any $T \leq N_k$, we obtain that

$$\begin{aligned} \|\omega_k^{\beta,\theta,\pm\infty}\|_{L^2(\mathbb{T}^4)} &= \|P_{\leq R_\theta N_k}\omega_k^{\beta,\theta,\pm\infty}\|_{L^2(\mathbb{T}^4)} \\ &\lesssim (1+R_\theta)^{-10}\frac{1}{N_k}, \end{aligned} \tag{8.9}$$

and

$$\sup_{|t-t_k^\beta|\in[TN_k^{-2},T^{-1}]}\|w_k^{\beta,\theta,\pm\infty}\|_{L^\infty(\mathbb{T}^4)} \lesssim T^{-2}R_\theta^4 N_k. \tag{8.10}$$

Interpolate (8.9) and (8.10), we can obtain that

$$\sup_{|t-t_k^\beta|\in[TN_k^{-2},T^{-1}]}\|w_k^{\beta,\theta,\pm\infty}\|_{L^p(\mathbb{T}^4)} \lesssim_{R_\theta} T^{\frac{4}{p}-2}N_k^{1-\frac{4}{p}}. \tag{8.11}$$

By choosing $T_k = N_k|t_k^\alpha - t_k^\beta|^{\frac{1}{2}} \to \infty$ as $k \to \infty$ and using (8.11), we obtain that

$$\sup_{t\in[t_k^\alpha - \frac{T_\theta}{N_k^2}, t_k^\alpha + \frac{T_\theta}{N_k^2}]}\|\omega_k^{\beta,\theta,\pm\infty}\|_{L^\infty(\mathbb{T}^4)} \lesssim_{R_\theta} T_k^{-2}N_k, \tag{8.12}$$

and

$$\sup_{t\in[t_k^\alpha - \frac{T_\theta}{N_k^2}, t_k^\alpha + \frac{T_\theta}{N_k^2}]}\|\langle\nabla\rangle\omega_k^{\beta,\theta,\pm\infty}\|_{L^4(\mathbb{T}^4)} \lesssim_{R_\theta} T_k^{-1}N_k. \tag{8.13}$$

So by using of Leibniz rule, (5.28), (8.13) and (8.12), we obtain that

$$
\begin{aligned}
&\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta},\omega_k^{\beta,\theta,\pm\infty})\|_{N([t_k^\alpha-\frac{T_\theta}{N_k^2},t_k^\alpha+\frac{T_\theta}{N_k^2}])}\\
\lesssim&\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta},\omega_k^{\beta,\theta,\pm\infty})\|_{L^1([t_k^\alpha-\frac{T_\theta}{N_k^2},t_k^\alpha+\frac{T_\theta}{N_k^2}],H^1(\mathbb{T}^4))}\\
\lesssim&\int_{t_k^\alpha-\frac{T_\theta}{N_k^2}}^{t_k^\alpha+\frac{T_\theta}{N_k^2}}\Big(\|\mathfrak{D}_2(\langle\nabla\rangle\omega_k^{\alpha,\theta})\|_{L^2(\mathbb{T}^4)}\|\omega_k^{\beta,\theta,\pm\infty}\|_{L^\infty(\mathbb{T}^4)}+\|\mathfrak{D}_2(\omega_k^{\alpha,\theta})\|_{L^4(\mathbb{T}^4)}\|\langle\nabla\rangle\omega_k^{\beta,\theta,\pm\infty}\|_{L^4(\mathbb{T}^4)}\Big)\,dt\\
\lesssim&\int_{t_k^\alpha-\frac{T_\theta}{N_k^2}}^{t_k^\alpha+\frac{T_\theta}{N_k^2}}\Big(N_k^2T_k^{-2}R_\theta^8+N_k^2T_k^{-1}R_\theta^3\Big)\,dt\\
\lesssim&T_k^{-1}T_\theta R_\theta^8\to 0 \text{ as } k\to\infty. \quad\square
\end{aligned}
$$

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