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Long-time behavior of solutions to a nonlinear hyperbolic relaxation system ☆

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Abstract

We study the large time behavior of solutions of a one-dimensional hyperbolic relaxation system that may be written as a nonlinear damped wave equation. First, we prove the global existence of a unique solution and their decay properties for sufficiently small initial data. We also show that for some large initial data, solutions blow-up in finite time. For quadratic nonlinearities, we prove that the large time behavior of solutions is given by the fundamental solution of the viscous Burgers equation. In some other cases, the convection term is too weak and the large time behavior is given by the linear heat kernel.

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1. Introduction

This paper is devoted to study the large time behavior of solutions of the following system:

$$\begin{cases} u_t + v_x = 0, \\ v_t + u_x = f(u) - v, \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1}$$

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f being a nonlinear function from $\mathbb{R}$ to $\mathbb{R}$. System (1) is a typical example of hyperbolic system of conservation laws with relaxation and arises in many physical systems such as nonequilibrium gas dynamics, flood flow with friction, viscoelasticity, magnetohydrodynamics, etc. (see Whitham [16]).

The study of the behavior of solutions to hyperbolic relaxation systems has been the object of intensive work. The interested reader is referred to [2,6–8,10,12,17] and the references cited therein for a more detailed account of relaxation of hyperbolic systems. Recently, Liu and Natalini [9] studied the large time behavior of the solutions of (1) when $f(u) = ku^2$ by means of scaling arguments. First, they obtained the needed uniform bounds on the scaled solutions by means of entropy-estimates, valid under the so-called subcharacteristic condition ($|f'(u)| < a$) and assuming that the initial data belong to a positively invariant domain. The main result of [9] shows that, under the assumptions above and for bounded initial data of finite mass, the first component of system (1) decays towards the fundamental solution of the viscous Burgers equation in the L^p-norm, at a rate faster than $t^{-(p-1)/2p}$ with $1 < p < \infty$.

In this article, we consider Eq. (1) as a nonlinear damped wave equation:

$$u_{tt} + u_t - u_{xx} + \partial_x\big(f(u)\big) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2}$$

and initial conditions $u(x,0) = u_0(x)$, $u_t(x,0) = -\partial_x v_0(x)$ in $\mathbb{R}$, (u_0, v_0) being the initial data of (1). We see (2) as a hyperbolic perturbation of the convective heat equation

$$u_t - u_{xx} + \partial_x\big(f(u)\big) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{3}$$

In fact, considering the linear equation associated with (2),

$$u_{tt} + u_t - u_{xx} = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty), \tag{4}$$

the rescaling function $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ satisfies

$$\lambda^{-2} u_{\lambda,tt} + u_{\lambda,t} - \partial_x^2 u_\lambda = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

This indicates that, the term involving second order in time derivatives becomes negligible. According to this formal argument it is natural to expect the asymptotic behavior of the solutions of (2) to be the same as that of the convection–diffusion equation (3).

Equation (3) is a simple model combining both diffusive and convective effects and has been studied in [4,5,18], among others, for nonlinearities satisfying $f(u) = |u|^{q-1}u$ and $q \geqslant 2$. It is by now well known that, when $q = 2$, the asymptotic behavior of the solutions of (3) with initial data in L^1 is given by a one-parameter family of self-similar solutions (see [5]). When $q > 2$, the convective term $\partial_x(f(u))$ is too weak and disappears as $t \to \infty$, the large time behavior being given in a first approximation by the linear heat equation. Finally, when the exponent is subcritical $q < 2$, the solutions of (3) behave like the entropy solutions of the following convective equation (see [4]):

$$u_t + \partial_x\big(f(u)\big) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{5}$$

In this paper we will not address the case $q < 2$ that cannot be handled with the techniques we develop.

In our study of the solutions of (2), we first develop a Fourier splitting method that allows decomposing the semigroup of the linear problem associated to (2) into a exponentially decaying one and a slowly decaying one. For the second one, which corresponds to the low frequency components, the decay properties are those of the semigroup generated by the heat equation. Based on this Fourier splitting and fixed point arguments, we get global in time solutions and the decay estimates of u, u_x and u_t in the $L^2(\mathbb{R})$-norm for small initial data $(u_0, -v_{0,x})$ in $V \equiv H^1 \cap L^1(\mathbb{R}) \times L^2 \cap L^1(\mathbb{R})$. This smallness condition differs from previously used in the existing literature [2,9]. Note that [9] imposes one-sided inequalities on the derivatives of the initial data while we impose smallness in $L^2 \cap L^1$.

We also show that large solutions may blow-up in finite time. This justifies the smallness condition on the initial data. This blow-up result is also extended for even nonlinearities of the form $f(u) = |u|^q$. The proof of the blow-up result uses in a critical way the finite speed of propagation and a construction of explicit solutions in separated variables. At this respect it is worth mentioning that solutions of (3) are global in time. Thus, the finite speed of propagation plays a key role on the blow-up mechanism.

We also obtain sharp decay properties as $t \to \infty$ of the global solutions of (2). This yields uniform estimates on the scaled functions u_λ. Then, compactness arguments yield the asymptotic form of solutions. In this step we need to assume that the initial data belong to the weighted space $L^2(1+|x|; \mathbb{R})$. This is a natural condition in the study of the asymptotic behavior of solutions of heat and convection–diffusion equations (see [3,18]) where, usually, the first momentum of the initial data is assumed to be finite.

An advantage of the method we develop in this article is that it applies also to multidimensional problems and that it does not require of the existence of invariant sets obtained though the maximum principle. Therefore, it applies to multi-dimensional problems as well. Consider for instance the following dissipative wave equation with a nonlinear convective term

$$\begin{cases} u_{tt} + u_t - \Delta u + \mathbf{a} \cdot \nabla f(u) = 0, & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \end{cases} \tag{6}$$

where $\mathbf{a} \in \mathbb{R}^N$ is a fixed vector. When analyzing (6) the first difficulty to overcome is related with the local existence and uniqueness of solutions. We look for solutions in the class $C([0,\tau]; H^k(\mathbb{R}^N)) \cap C^1([0,\tau]; H^{k-1}(\mathbb{R}^N))$ with $k > 0$ such that $2k > N$ so that $H^k(\mathbb{R}^N)$ becomes an algebra because of the continuous embedding $H^k(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. Thus, assuming that f is of class C^{k+1} and its derivatives up to order $k+1$ are uniformly bounded, i.e., $f \in BC^{k+1}(\mathbb{R}^N)$, then classical methods yield local existence and uniqueness of solutions in this space for every pair of initial data $(u_0, u_1) \in H^k(\mathbb{R}^N) \times H^{k-1}(\mathbb{R}^N)$. The method of this paper allows showing that for sufficiently small initial data such that, moreover, $u_0, u_1 \in L^1(\mathbb{R}^N)$, solutions are global in time and decay as $t \to \infty$ with the same velocity as the solutions of the heat equation with initial data in $L^1(\mathbb{R}^N)$, i.e., with the rate

$$\|u(t)\|_{L^p(\mathbb{R}^N)} \leqslant c(1+t)^{-\frac{N}{2}(1-\frac{1}{p})}, \quad \forall t > 0,\ 1 \leqslant p \leqslant \infty.$$

The methods of this paper allow showing that when the initial data belong to $L^2(1 + |x|^{(N+1)/2}; \mathbb{R}^N)$ and (7) is satisfied with $a = 1$ and $q \geqslant (N+1)/N$ it then follows that

$$(1+t)^{-\frac{N}{2}(1-\frac{1}{p})} \|u(t) - v(t)\|_{L^p(\mathbb{R}^N)} \to 0, \quad \text{as } t \to \infty,\ 1 \leqslant p \leqslant \infty,$$

where, for $q = (N+1)/N$, v is the solution of

$$v_t - \Delta v + \mathbf{a} \cdot \nabla\big(|v|^{q-1}v\big) = 0, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+,$$

and for $q > (N+1)/N$ is solution of the heat equation. The same techniques allow obtaining similar results for the hyperbolic version of the Navier–Stokes equations analyzed in [1].

The rest of the paper is organized as follows. In the following section we present the main results of this work. In Section 3, we give some results on the asymptotic behavior in $L^p(\mathbb{R})$ of the solutions of (4). In particular, we develop the Fourier splitting method. The proof of these estimates is given in Appendix A. Section 4 is devoted to prove the existence and uniqueness of the solutions of (2) with the appropriate decay properties as $t \to \infty$. In Section 5, we obtain some compactness results on the family $\{u_\lambda\}$. In Section 6, the limit of the family $\{u_\lambda\}$ when $\lambda \to \infty$ is identified and the proof of the asymptotic behavior of global solutions is finished. Section 7 is devoted to prove the blow-up result.

2. Main results

We write system (1) in the form (2). A very large class of nonlinearities $f(u)$ can be considered satisfying that f and f' are locally Lipschitz real functions with $f(0) = 0$ and for which there exists $q \geqslant 2$ such that

$$\lim_{u\to 0} \frac{f(u)}{|u|^{q-1}u} = a, \quad \text{or} \quad \lim_{u\to 0} \frac{f(u)}{|u|^{q}} = a, \quad a \in \mathbb{R} - \{0\}. \tag{7}$$

To simplify the presentation we consider power-like nonlinearities $f(u) = |u|^{q-1}u$. The equation under consideration reads:

$$\begin{cases} u_{tt} + u_t - u_{xx} + (|u|^{q-1}u)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x), & u_t(x,0) = u_1(x). \end{cases} \tag{8}$$

Recall that $(u_0, u_1) = (u_0, -v_{0,x})$, (u_0, v_0) being the initial data of (1).

Proposition 2.1. *Let the exponent q in* (8) *be such that $q \geqslant 2$ and $(u_0, u_1) \in V$ where $V = [H^1 \cap L^1(\mathbb{R})] \times [L^2 \cap L^1(\mathbb{R})]$. We assume that $\|(u_0,u_1)\|_V \leqslant \delta$, with δ sufficiently small. Then, there exists a unique global solution $u \in BC([0,\infty); H^1(\mathbb{R})) \cap BC^1([0,\infty); L^2(\mathbb{R}))$ of* (8). *Moreover, it satisfies*

$$\|u(t)\|_2 \leqslant c(1+t)^{-\frac{1}{4}}, \qquad \|u_x(t)\|_2 \leqslant c'(1+t)^{-\frac{3}{4}}, \quad \forall t \geqslant 0, \tag{9}$$

where the constants c, c' are proportional to the norm of the initial data in V.

The proof of this proposition is given in Section 4.

It is important to note that the smallness condition on the initial data is necessary since solutions for large data may blow-up in finite time. Indeed, let us look for solutions of (8) with $q = 2$ such that $u(x,t) = xa(t)$. We observe that the initial data of u do not belong to V. However, truncating the support of the initial data and thanks to the finite speed of propagation, the blow-up of solutions of the form $u(x,t) = xa(t)$ leads to blow-up results for data in V too.

Note that $u = xa(t)$ with $x < 0$ solves (8) if and only if a satisfies the differential equation: $a_{tt} + a_t - 2|a|^{q-1}a = 0$. The solution of this equation for suitable initial data blows-up in finite time (see [15] or Section 7). These blow-up result justifies the smallness condition of the previous global existence result.

Blow-up also occurs for even nonlinearities of the form $f(u) = |u|^q$ with $q > 1$. More precisely, for system

$$\begin{cases} u_{tt} + u_t - u_{xx} + (|u|^q)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{cases} \tag{10}$$

the following holds:

Theorem 2.2. *Let u be the solution of* (10) *and*

$$a(t) = \int x u(x,t)\,dx. \tag{11}$$

We assume that the initial data u_0 and u_1 have support in $[K_1, K_2]$ $(K_1 > 0)$. Assume also that $a_0 \equiv a(0) > 0$ and $a_1 \equiv a_t(0) > 0$. If

$$a_0 \geqslant \max\left\{ \left(\frac{q+1}{8\rho}\right)^{\frac{1}{q-1}} e^{\frac{2}{q-1}}, (2q-1)\left(\frac{e^2(2q+2)}{K_1^2\rho(q-1)^2}\right)^{\frac{1}{q-1}} \right\},$$

$$\left(\frac{1}{2}a_0 + a_1\right)^2 \geqslant \frac{1}{4}a_0^2 + \rho\frac{2a_0^{q+1}}{q+1} \quad \textit{with} \quad \rho = (K_1+K_2)^{1-2q}\left(\frac{q-1}{2q-1}\right)^{1-q},$$

then, a blows-up in finite time $t_b < K_1$.

Let us now address the problem of the asymptotic behavior for $t \to \infty$ of global solutions. We note that the smooth solutions of (8) with compact support (or decaying as $|x| \to \infty$) satisfy the conservation law that the mass of $u_t + u$ is conserved along time:

$$m(u_t + u) = \int_{\mathbb{R}} (u_1 + u_0)\,dx = M. \tag{12}$$

Thus, M is a relevant parameter to describe the large time behavior of solutions. The following result shows that, actually, M fully determines the asymptotic behavior.

Theorem 2.3. *Assume that the hypotheses of Proposition* 2.1 *are satisfied, that $q \geqslant 2$ and that $u_0, u_1 \in L_1^2(\mathbb{R}) = \{v \in L^2(\mathbb{R}) \mid (1 + |\cdot|)v \in L^2(\mathbb{R})\}$. Let u be the unique solution of* (8). *Then, there exists a self-similar function $\theta = t^{-1/2} f(x/\sqrt{t})$ such that*

$$\lim_{t\to\infty} t^{\frac{p-1}{2p}} \left\| u(\cdot,t) - \theta(\cdot,t) \right\|_p = 0, \tag{13}$$

for $1 \leqslant p \leqslant \infty$. *When* $q = 2$, θ *is the source type solution of the Burgers equation with the Dirac delta as initial datum*

$$\begin{cases} \theta_t - \theta_{xx} + (|\theta|\theta)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ \theta(x,0) = M\delta_0, \end{cases} \tag{14}$$

M *defined in* (12). *When* $q > 2$, θ *is the fundamental solution of the heat equation*

$$\begin{cases} \theta_t - \theta_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ \theta(x,0) = M\delta_0. \end{cases} \tag{15}$$

In both cases $M = \int_{\mathbb{R}} (u_0 + u_1)\,dx$. *In both cases* M *is defined by* (12).

The nature of the assumptions on the initial data in Proposition 2.1 and Theorem 2.3 is different from [9]. The initial data (u_0, u_1) in (8) coincide with $(u_0, -v_{0,x})$, where (u_0, v_0) are the initial data in (1). Thus, according to hypotheses of [9], u_0, $u_{0,x}$ and u_1 are assumed to be of finite mass and to satisfy a one-sided inequality. However, in this work we only need the smallness condition to be satisfied in the energy space V. By the contrary in Theorem 2.3 we assume the initial data to be in the weighted space $L^2_1(\mathbb{R})$.

It is also interesting to observe that when $a = 0$ in (7) the asymptotic behavior of small solutions is given by the fundamental solution of the linear heat equation.

3. Preliminaries on the linear problem

Now we present some results on the asymptotic behavior of the solutions of

$$\begin{cases} u_{tt} - \Delta u + u_t = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = \varphi^0(x), \qquad u_t(x,0) = \varphi^1(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{16}$$

Note that this is the linear equation involved in the nonlinear equation (1) we analyze. The results in this section play a key role when performing the Fourier splitting of the semigroup.

The well-posedness of (16) can be easily obtained writing it as an abstract evolution equation in the energy space $H = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Hille–Yosida–Phillip's theorem guarantees that (16) generates a semigroup of contractions denoted by $\{S(t)\}_{t \geqslant 0}$. Thus, for any initial data $(\varphi^0, \varphi^1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, (16) has a unique weak solution $u \in C^1(\mathbb{R}^+, H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}^+, L^2(\mathbb{R}^N))$.

The following estimates on this linear semigroup are well known (see [11]):

Lemma 3.1. *Let* u *be the solution of* (16), $k \in \mathbb{N}$ *and* $\alpha \in \mathbb{R}^N_+$. *Then, there exist* $c, c' > 0$ *such that for any initial data*

$$\varphi^0 \in H^{k+|\alpha|} \cap L^a(\mathbb{R}^N), \qquad \varphi^1 \in H^{k+|\alpha|-1} \cap L^a(\mathbb{R}^N), \quad 1 \leqslant a \leqslant 2,$$

it holds, for any $t \geqslant 0$,

$$\begin{aligned} &\left\| \partial_t^k D_x^\alpha u(\cdot,t) \right\|_2 \\ &\quad \leqslant c(1+t)^{-\frac{N}{2}(\frac{1}{a}-\frac{1}{2}) - \frac{|\alpha|+2k}{2}} \left(\left\|\varphi^0\right\|_{k+|\alpha|,2} + \left\|\varphi^0\right\|_a + \left\|\varphi^1\right\|_{k+|\alpha|-1,2} + \left\|\varphi^1\right\|_a \right). \end{aligned} \tag{17}$$

Here and in the sequel $\|\cdot\|_{m,p}$ denotes the norm in $W^{m,p}(\mathbb{R}^N)$. To estimate the asymptotic behavior of the solutions of (16) we introduce the homogeneous Sobolev spaces:

$$\dot{H}^m(\mathbb{R}^N) = \{f\colon |\cdot|^m \hat{f} \in L^2(\mathbb{R}^N)\} \quad \text{and} \quad \|f\|_{\dot{H}^m(\mathbb{R}^N)} = \||\cdot|^m \hat{f}\|_2.$$

Lemma 3.2. *Let $k = 0, 1$ and $\alpha \in \mathbb{R}^N_+$. Then, there exist positive constants $\omega, c, c' > 0$ such that $\forall t \geqslant 0$*

$$\begin{aligned}\|\partial_t^k D_x^\alpha u(\cdot,t)\|_2 &\leqslant ce^{-\omega t}\left(\|\varphi^0\|_{\dot{H}^{k+|\alpha|}} + \|\varphi^1\|_{\dot{H}^{k+|\alpha|-1}}\right)\\ &\quad + c'\left(\|\varphi^0\|_a + \|\varphi^1\|_a\right)(1+t)^{-\frac{N}{2}(\frac{1}{a}-\frac{1}{2})}(1+t)^{-\frac{|\alpha|+2k}{2}}, \end{aligned} \tag{18}$$

when $\varphi^0 \in H^{k+|\alpha|} \cap L^a(\mathbb{R}^N)$, $\varphi^1 \in H^{k+|\alpha|-1} \cap L^a(\mathbb{R}^N)$ with $1 \leqslant a \leqslant 2$. If $k + |\alpha| < 1$, $\dot{H}^{k+|\alpha|-1}$ must be replaced by $H^{k+|\alpha|-1}$ in (18).

We also need the asymptotic behavior as $t \to \infty$ of solutions of Eq. (16) in the weighted space $L^2_1(\mathbb{R})$. To do that we perform the change of variables $v(x,t) = xu(x,t)$. Given u solution of (16), then v is the unique solution of the Cauchy problem:

$$\begin{cases} v_{tt} + v_t - v_{xx} + 2u_x = 0 & \text{in } \mathbb{R} \times (0,\infty),\\ v(x,0) = x\varphi^0(x), \qquad v_t(x,0) = x\varphi^1(x) & \text{in } \mathbb{R}. \end{cases} \tag{19}$$

Using the variation of constants formula and applying the decay estimates of $S(t)$ in L^2 (Lemmas 3.1 and 3.2) in this identity, we get:

Lemma 3.3. *Let u be solution of* (16). *Then, there exist $\omega, c, c_1, c_2, c_3 > 0$ such that for any initial data $(\varphi^0, \varphi^1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with*

$$\varphi^0 \in L^2_1 \cap L^b_1 \cap L^a(\mathbb{R}), \qquad \varphi^0_x \in L^2_1(\mathbb{R}), \qquad \varphi^1 \in L^2_1 \cap L^b_1 \cap L^a(\mathbb{R}), \quad a, b \in [1,2],$$

it holds

$$\begin{aligned}\|u\|_{\dot{L}^2_1} &\leqslant ce^{-\omega t}\left(\|x\varphi^0\|_2 + \|x\varphi^1\|_{-1,2}\right) + c_1(1+t)^{-\frac{1}{2}}\left(\|\varphi^0\|_2 + \|\varphi^1\|_{-1,2}\right)\\ &\quad + c_2(1+t)^{\frac{1}{4}-\frac{1}{2b}}\left(\|x\varphi^0\|_b + \|x\varphi^1\|_b\right) + c_3(1+t)^{\frac{3}{4}-\frac{1}{2a}}\left(\|\varphi^0\|_a + \|\varphi^1\|_a\right), \end{aligned} \tag{20}$$

$$\begin{aligned}\|u_x\|_{\dot{L}^2_1} &\leqslant ce^{-\omega t}\left(\|x\varphi^0\|_{\dot{H}^1} + \|x\varphi^1\|_2\right) + c_1(1+t)^{-\frac{1}{4}-\frac{1}{2b}}\left(\|x\varphi^0\|_b + \|x\varphi^1\|_b\right)\\ &\quad + c_2(1+t)^{-\frac{1}{2}}\left(\|\varphi^0\|_{\dot{H}^1} + \|\varphi^1\|_2\right) + c_3(1+t)^{\frac{1}{4}-\frac{1}{2a}}\left(\|\varphi^0\|_a + \|\varphi^1\|_a\right). \end{aligned} \tag{21}$$

Here and in the sequel $\|\cdot\|_{\dot{L}^a_s}$ denotes the semi-norm in the weighted space $L^a_s(\mathbb{R}^N)$, i.e.,

$$\|f\|^a_{\dot{L}^a_s} = \int_{\mathbb{R}^N} |x|^{as}|f(x)|^a\,dx.$$

The proof of these lemmas are given in Appendix A.

4. Global existence and decay estimates

This section is devoted to prove Proposition 2.1.

Fix initial data $(u_0, u_1) \in V$. Using the semigroup notation and the variation of constants formula, we introduce the function

$$\big[\Phi(u)\big](t) = S(t)[u_0, u_1] - \int_0^t S(t-s)\big[f\big(u(s)\big)_x\big]\,ds, \tag{22}$$

where we denote, to simplify the presentation, $S(t-s)[0, f(u(s))_x]$ by $S(t-s)[f(u(s))_x]$. Recall that $f(u) = |u|^{q-1}u$. With this notation, Eq. (8) can be written as $u(t) = [\Phi(u)](t)$. In other words, the problem is reduced to the obtention of fixed points of Φ. To do this we apply the Banach fixed point theorem in the Banach space

$$X \equiv \big\{u \in C\big([0,\infty); H^1(\mathbb{R})\big) \cap BC^1\big([0,\infty); L^2(\mathbb{R})\big) \text{ s.t.}$$
$$(1+t)^{\frac{1}{4}}u, (1+t)^{\frac{3}{4}}u_x \in L^\infty\big([0,\infty); L^2(\mathbb{R})\big)\big\}$$

with norm

$$\|u\|_X \equiv \big\|(1+t)^{\frac{1}{4}}u\big\|_{L^\infty([0,\infty);L^2(\mathbb{R}))} + \big\|(1+t)^{\frac{3}{4}}u_x\big\|_{L^\infty([0,\infty);L^2(\mathbb{R}))} + \|u_t\|_{L^\infty([0,\infty);L^2(\mathbb{R}))}.$$

Define the ball $B_R = \{u \in X \text{ s.t. } \|u\|_X \leqslant R\}$. We have the following lemma.

Lemma 4.1. *There exist $c, c' > 0$ so that $\Phi(u) \in X$ and $\|\Phi(u)(t)\|_X \leqslant c'\|(u_0, u_1)\|_V + cR^q$, for any $u \in B_R$ and $t \geqslant 0$.*

We assume for the moment that this lemma is true. Since $q > 1$, choosing $R > 0$ such that $c'\|(u_0, u_1)\|_V \leqslant R/6$ and sufficiently small we obtain that $\Phi(B_R) \subset B_R$. Note that the smallness condition on R imposes a smallness condition on the size of the initial data (on $\|(u_0, u_1)\|_V$) too. Thus, from now on, the initial data (u_0, u_1) are assumed to be small in V. Now, we are going to see that Φ is a contraction.

Lemma 4.2. *Let Φ be defined as in* (22)*. Then, for any $u, v \in B_R$ and $t \geqslant 0$, there exists a constant $c > 0$ so that $\|\Phi(u) - \Phi(v)\|_X \leqslant cR^{q-1}\|u - v\|_X$.*

Assuming this lemma, since $q > 1$, for R sufficiently small, Φ is a strict contraction in B_R. Applying the Banach theorem, there exists a unique solution u of (8) in B_R so that $u \in BC([0,\infty); H^1(\mathbb{R}))$ and $u \in BC^1([0,\infty); L^2(\mathbb{R}))$. Moreover, since $\|u\|_X \leqslant R$, we get for any $t \geqslant 0$ that $\|u(t)\|_2 \leqslant R(1+t)^{-1/4}$ and $\|u_x(t)\|_2 \leqslant R(1+t)^{-3/4}$. This confirms that the constants c, c' in (9) are proportional to the norm of the initial data in V.

The same argument shows that for any initial data $(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, for $\tau > 0$ sufficiently small, there exists a unique solution u of (8) so that $u \in C([0,\tau); H^1(\mathbb{R})) \cap C^1([0,\tau); L^2(\mathbb{R}))$. This allows showing that the solution we have built in B_R is in fact the only solution of (8).

Now, we are going to prove Lemmas 4.1 and 4.2. First of all, we show some estimates that we shall use in the proofs:

$$\int_0^t (1+t-s)^a (1+s)^b \, ds \leqslant \begin{cases} c(1+t)^{a+b+1}, & a, b > -1, \\ c(1+t)^a, & a \geqslant -1 \text{ and } b < -1, \\ c(1+t)^b, & a < -1, b \geqslant -1, \\ c(1+t)^{\max(a,b)}, & a, b < -1, a, b \neq -1. \end{cases} \tag{23}$$

Let us now write estimates (17) and (18) for the solution of (16) with initial data $\varphi^0 = 0$ and $\varphi^1 = g$. For $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+^N$, we have, by (17), that

$$\left\| \partial_t^k D_x^\alpha S(t)[g] \right\|_2 \leqslant c(1+t)^{-\frac{N}{2}(\frac{1}{a}-\frac{1}{2})-\frac{|\alpha|+2k}{2}} \left(\|g\|_{k+|\alpha|-1,2} + \|g\|_a \right). \tag{24}$$

For $k = 0, 1$ and $\alpha \in \mathbb{R}_+^N$, by (18) we have that

$$\left\| \partial_t^k D_x^\alpha S(t)[g] \right\|_2 \leqslant c e^{-\omega t} \|g\|_{\dot{H}^{k+|\alpha|-1}} + c' \|g\|_a (1+t)^{-\frac{N}{2}(\frac{1}{a}-\frac{1}{2})-\frac{|\alpha|+2k}{2}}. \tag{25}$$

Moreover, the following interpolation inequalities hold for any $u \in H^1(\mathbb{R})$,

$$\|u\|_\infty \leqslant \sqrt{2} \|u\|_2^{\frac{1}{2}} \|u_x\|_2^{\frac{1}{2}}, \qquad \|u\|_{\dot{H}^a(\mathbb{R})} \leqslant \|u\|_2^{1-a} \|u_x\|_2^a, \quad a \in (0,1). \tag{26}$$

Proof of Lemma 4.1. First, we use estimates (17) in (22). Thus, it is sufficient to prove the following estimates:

$$\int_0^t \left\| S_x(t-s)\left[f\big(u(s)\big) \right] \right\|_2 ds \leqslant c(1+t)^{-\frac{1}{4}} R^q, \quad \forall u \in B_R, \tag{27}$$

$$\int_0^t \left\| S_x(t-s)\left[f\big(u(s)\big)_x \right] \right\|_2 ds \leqslant c(1+t)^{-\frac{3}{4}} R^q, \quad \forall u \in B_R, \tag{28}$$

$$\int_0^t \left\| S_t(t-s)\left[f\big(u(s)\big)_x \right] \right\|_2 ds \leqslant c R^q, \quad \forall u \in B_R. \tag{29}$$

Again, using estimate (25) with $a = 1$, $k = 0$ and $\alpha = 1$ in (27) and (28) and with $a = 1$, $k = 1$ and $\alpha = 0$ in (29), we have

$$\int_0^t \left\| S_x(t-s)\left[f\big(u(s)\big) \right] \right\|_2 ds \leqslant c \int_0^t (1+t-s)^{-\frac{3}{4}} \left[\left\| f\big(u(s)\big) \right\|_2 + \left\| f\big(u(s)\big) \right\|_1 \right] ds,$$

$$\int_0^t \left\| S_x(t-s)\left[f\big(u(s)\big)_x \right] \right\|_2 ds \leqslant c \int_0^t (1+t-s)^{-\frac{3}{4}} \left[\left\| f\big(u(s)\big)_x \right\|_2 + \left\| f\big(u(s)\big)_x \right\|_1 \right] ds,$$

$$\int_0^t \left\| S_t(t-s)\left[f\big(u(s)\big)_x\right]\right\|_2 ds \leqslant c \int_0^t (1+t-s)^{-\frac{5}{4}}\left[\left\|f\big(u(s)\big)_x\right\|_2 + \left\|f\big(u(s)\big)_x\right\|_1\right] ds.$$

By the first interpolation inequality (26) and $q \geqslant 2$, we get

$$\begin{aligned}
&\left\|f\big(u(s)\big)\right\|_2 \leqslant c\|u\|_2^{\frac{q+1}{2}}\|u_x\|_2^{\frac{q-1}{2}}, \qquad && \left\|f\big(u(s)\big)\right\|_1 \leqslant c'\|u\|_2^{\frac{2+q}{2}}\|u_x\|_2^{\frac{q-2}{2}}, \\
&\left\|f\big(u(s)\big)_x\right\|_2 \leqslant c\|u\|_2^{\frac{q-1}{2}}\|u_x\|_2^{\frac{q+1}{2}}, \qquad && \left\|f\big(u(s)\big)_x\right\|_1 \leqslant c\|u\|_2^{\frac{q}{2}}\|u_x\|_2^{\frac{q}{2}}.
\end{aligned} \tag{30}$$

Then, as $u \in B_R$, we obtain, for any $s \geqslant 0$:

$$\begin{aligned}
&\left\|f\big(u(s)\big)\right\|_2 \leqslant cR^q(1+s)^{-\frac{2q-1}{4}}, \qquad && \left\|f\big(u(s)\big)\right\|_1 \leqslant cR^q(1+s)^{-\frac{q-1}{2}}, \\
&\left\|f\big(u(s)\big)_x\right\|_2 \leqslant cR^q(1+s)^{-\frac{2q+1}{4}}, \qquad && \left\|f\big(u(s)\big)_x\right\|_1 \leqslant cR^q(1+s)^{-\frac{q}{2}}.
\end{aligned} \tag{31}$$

We prove (27) and (29) using estimates (31) and thanks to (23). On the other hand, we obtain (28) with $q > 2$ using estimates (31) on $[f(u(s))]_x$ in L^2 and L^1, and, applying (23). Now, we show (28) in the case $q = 2$. Let $\alpha \in (0,1)$ and apply (25) with $a = 2$. We get

$$\begin{aligned}
&\int_0^t \left\|\partial_x^{2-\alpha} S(t-s)\left[\partial_x^\alpha f\big(u(s)\big)\right]\right\|_2 ds \\
&\qquad \leqslant cR^2 \int_0^t e^{-\omega(t-s)}(1+s)^{-\frac{5}{4}} ds + cR^2 \int_0^t (1+t-s)^{-\frac{2-\alpha}{2}}(1+s)^{-\frac{3+2\alpha}{4}} ds.
\end{aligned}$$

We note that, by (31), for $q = 2$, $\|\partial_x^\alpha f(u(s))\|_{\dot{H}^{1-\alpha}} \leqslant cR^2(1+s)^{-5/4}$. On the other hand, by the second estimate of (26) and by (31) for $q = 2$ we have $\|\partial_x^\alpha f(u(s))\|_2 \leqslant cR^2(1+s)^{-(3+2\alpha)/4}$. Then, taking $\alpha \in (0, 1/2)$ and thanks to (23), we prove (28) for $q = 2$.

Finally, we observe that $\Phi(u)$ and $\Phi(u)_t$ are continuous in time with values in $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$, respectively. Let us check the continuity of $\Phi(u)$ ($\Phi(u)_t$ can be treated in a similar way). Indeed, given $t, t_0 > 0$ with $t \geqslant t_0$ (the case $t \leqslant t_0$ is equivalent), by (22) we have

$$\begin{aligned}
\Phi(u)(t) - \Phi(u)(t_0) = S(t)[u_0, u_1] - S(t_0)[u_0, u_1] - \int_{t_0}^t S(t-s)\left[f\big(u(s)\big)_x\right] ds \\
- \int_0^{t_0} \left(S(t-s)\left[f\big(u(s)\big)_x\right] - S(t_0 - s)\left[f\big(u(s)\big)_x\right]\right) ds.
\end{aligned} \tag{32}$$

Taking into account that $(u_0, u_1) \in V$, the continuity property of the semigroup $S(t)$ and that $f(u(s))$ with $u \in B_R$ is bounded in $H^1(\mathbb{R})$ and $W^{1,1}(\mathbb{R})$ and using (25), we only have to study the limit of the last term of (32). First, thanks to (24) and since $u \in B_R$, we see that $\|S(t-s) \times [f(u(s))_x]\|_{1,2}$ and $\|S(t_0-s)[f(u(s))_x]\|_{1,2}$ are bounded above by a function in $L^1(0,t_0)$ depending on s which is independent of t. On the other hand, by the continuity of the semigroup $S(t)$, we get, for any $s \in (0,t_0)$,

$$\lim_{t\to t_0} \big\| \big[S(t-s) - S(t_0-s)\big]\big[f\big(u(s)\big)_x\big]\big\|_{1,2} = 0,$$

because $f(u(s)) \in H^1(\mathbb{R})$. Then, by the dominated convergence theorem, the last term of (32) goes to zero when $t \to t_0$ and we conclude the continuity of $\Phi(u)$. $\square$

Proof of Lemma 4.2. Given $u, v \in B_R$, let be $w(t) = \Phi(u)(t) - \Phi(v)(t)$. We define

$$\varphi(t) = \sup_{0\leqslant s\leqslant t} \big\{(1+s)^{\frac{1}{4}} \big\|u(s)-v(s)\big\|_2\big\}, \qquad \phi(t) = \sup_{0\leqslant s\leqslant t} \big\{(1+s)^{\frac{3}{4}} \big\|u_x(s)-v_x(s)\big\|_2\big\}. \tag{33}$$

Thus, using (24) with $a=1$,

$$\big\|w(t)\big\|_2 \leqslant c\int_0^t (1+t-s)^{-\frac{3}{4}}\big[\big\|f\big(u(s)\big)-f\big(v(s)\big)\big\|_2 + \big\|f\big(u(s)\big)-f\big(v(s)\big)\big\|_1\big]\,ds,$$

$$\big\|w_x(t)\big\|_2 \leqslant c\int_0^t (1+t-s)^{-\frac{3}{4}}\big[\big\|f\big(u(s)\big)_x-f\big(v(s)\big)_x\big\|_2 + \big\|f\big(u(s)\big)_x-f\big(v(s)\big)_x\big\|_1\big]\,ds,$$

$$\big\|w_t(t)\big\|_2 \leqslant c\int_0^t (1+t-s)^{-\frac{5}{4}}\big[\big\|f\big(u(s)\big)_x-f\big(v(s)\big)_x\big\|_2 + \big\|f\big(u(s)\big)_x-f\big(v(s)\big)_x\big\|_1\big]\,ds.$$

In view of the definition of φ and ϕ in (33), using that $u, v \in B_R$ and the first estimate of (26), we get, for any $t \leqslant s$,

$$\begin{aligned}
\big\|f\big(u(s)\big)-f\big(v(s)\big)\big\|_2 &\leqslant cR^{q-1}(1+s)^{-\frac{2q-1}{4}}\varphi(t),\\
\big\|f\big(u(s)\big)-f\big(v(s)\big)\big\|_1 &\leqslant cR^{q-1}(1+s)^{-\frac{q-1}{2}}\varphi(t),\\
\big\|f\big(u(s)\big)_x-f\big(v(s)\big)_x\big\|_2 &\leqslant cR^{q-1}(1+s)^{-\frac{2q+1}{4}}\big[\phi(t)+\varphi(t)^{\frac{1}{2}}\phi(t)^{\frac{1}{2}}\big],\\
\big\|f\big(u(s)\big)_x-f\big(v(s)\big)_x\big\|_1 &\leqslant cR^{q-1}(1+s)^{-\frac{q}{2}}\big[\phi(t)+\varphi(t)\big].
\end{aligned}$$

Using these inequalities we prove $\forall u, v \in B_R$, $t \geqslant 0$,

$$\begin{aligned}
(1+t)^{\frac{1}{4}}\big\|\Phi(u)(t)-\Phi(v)(t)\big\|_2 &\leqslant cR^{q-1}\varphi(t),\\
(1+t)^{\frac{3}{4}}\big\|\Phi(u)_x(t)-\Phi(v)_x(t)\big\|_2 &\leqslant cR^{q-1}\big[\varphi(t)+\phi(t)\big],\\
\big\|\Phi(u)_t(t)-\Phi(v)_t(t)\big\|_2 &\leqslant cR^{q-1}\big[\varphi(t)+\phi(t)\big]
\end{aligned}$$

proceeding like in the proof of Lemma 4.1. $\square$

The proof of Proposition 2.1 is now complete. We now obtain some extra estimates on the behavior of u that will be useful in the sequel.

Proposition 4.3. *Let u be the solution of* (8) *under the hypotheses of Proposition* 2.1. *Then, for any $t \geqslant 0$, we get*

$$\left\|u_t(t)\right\|_2 \leqslant \begin{cases} c(1+t)^{-\frac{2q+5}{8}} & \text{for } 2 \leqslant q < \frac{5}{2}, \\ c(1+t)^{-\frac{5}{4}} & \text{for } q \geqslant \frac{5}{2}, \end{cases} \tag{34}$$

with a constant c that depends on the norm of the initial data in V and the exponent q.

Proof. Using the variation of constants formula, by (17) and (25) with $a \in [1,2]$, we get

$$\begin{aligned} \left\|u_t(t)\right\|_2 &\leqslant c(1+t)^{-\frac{5}{4}}\left(\|u_0\|_{1,2} + \|u_0\|_1 + \|u_1\|_2 + \|u_1\|_1\right) \\ &\quad + c\int_0^t e^{-\omega(t-s)}\left\|f\left(u(s)\right)_x\right\|_2 ds + c'\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{1}{2a}}\left\|f\left(u(s)\right)_x\right\|_a ds. \end{aligned} \tag{35}$$

Using (30) and estimates (9), we get C depending on the initial data and $q \geqslant 2$ such that

$$\int_0^t e^{-\omega(t-s)}\left\|f\left(u(s)\right)_x\right\|_2 ds \leqslant C(1+t)^{-\frac{5}{4}}.$$

Now, we study the second integral of (35) distinguishing two cases: $q \geqslant 5/2$ and $q \in [2, 5/2)$. Taking $a = 1$ and using (30) with (9), we prove (34), for $q \geqslant 5/2$,

$$\int_0^t (1+t-s)^{-\frac{5}{4}}\left\|f\left(u(s)\right)_x\right\|_1 ds \leqslant C(1+t)^{-\frac{5}{4}}.$$

Now, for $q \in [2, 5/2)$, we use the Hölder inequality and since $a \in [1,2)$, then $2a(q-1)/(2-a) \geqslant 2$, thanks to (26) and (9), we obtain

$$\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{1}{2a}}\left\|f\left(u(s)\right)_x\right\|_a ds \leqslant c\left((1+t)^{\frac{1}{2a}-\frac{q+1}{2}} + (1+t)^{-\frac{3}{4}-\frac{1}{2a}}\right),$$

by (23) since $(3/4 + 1/2a) > 1$ and $(q+1)/2 - 1/2a > 1$. We choose $a \in (1,2)$ depending on $q \in [2, 5/2)$ so that the decay estimate is optimal, i.e., $(q+1)/2 - 1/(2a) = 3/4 + 1/(2a)$. Then $a = 4/(2q-1) \in (1, 4/3]$, and we conclude the proof of (34) for $q \in [2, 5/2)$. $\square$

5. Compactness

The goal of this section is to obtain a compactness result in $L^2((0,T)\times\mathbb{R})$ for the family $\{u_\lambda\}_{\lambda>0}$, defined by

$$u_\lambda(x,t)=\lambda u\big(\lambda x,\lambda^2 t\big). \tag{36}$$

This fact will be later used in the proof of Theorem 2.3. For any $\lambda>0$ the function u_λ solves

$$\begin{cases} \lambda^{-2}u_{\lambda,tt}+u_{\lambda,t}-u_{\lambda,xx}+\lambda^{2-q}(|u_\lambda|^{q-1}u_\lambda)_x=0, & (x,t)\in\mathbb{R}\times(0,\infty),\\ u_\lambda(x,0)=u_{0,\lambda}(x)=\lambda u_0(\lambda x), & u_{\lambda,t}(x,0)=u_{1,\lambda}(x)=\lambda^3 u_1(\lambda x). \end{cases} \tag{37}$$

On the other hand, thanks to (9), we have, for any $t\geqslant 0$

$$\big\|u_\lambda(t)\big\|_2\leqslant c\lambda^{\frac12}\big(1+\lambda^2 t\big)^{-\frac14},\qquad \big\|u_{\lambda,x}(t)\big\|_2\leqslant c\lambda^{\frac32}\big(1+\lambda^2 t\big)^{-\frac34}. \tag{38}$$

By (38) the norm of u_λ in $L^2((0,T)\times\mathbb{R})$ is bounded when $\lambda\to\infty$. Then, by extracting subsequences (that we denote with the same subindex λ to simplify the notation), u_λ converges weakly in $L^2((0,T)\times\mathbb{R})$. Now, we proceed in several steps to conclude the proof of compactness in $L^2((0,T)\times\mathbb{R})$: (1) We establish that the sequence $\{u_\lambda\}$ is relatively compact in $C([t_0,T];L^p_{\mathrm{loc}}(\mathbb{R}))$ for $1\leqslant p\leqslant\infty$ and $t_0>0$. (2) We obtain the compactness in $C([t_0,T];L^p(\mathbb{R}))$ for $t_0>0$ and any $p\in[1,\infty]$. To do it, we prove the existence of $k_0>0$ for any $\varepsilon>0$ such that, for any $k\geqslant k_0$ and $\lambda\geqslant 1$,

$$\int\limits_{|x|\geqslant k}\big|u_\lambda(x,t)\big|\,dx\leqslant\varepsilon,\quad \forall t\in[t_0,T]. \tag{39}$$

(3) We conclude proving that for any $\varepsilon>0$, there exists $t_0>0$ sufficiently small such that, for $\lambda\geqslant 1$, it holds

$$\int\limits_0^{t_0}\int\limits_{\mathbb{R}}\big|u_\lambda(x,t)\big|^2\,dx\leqslant\varepsilon. \tag{40}$$

First, we have the following local compactness result.

Proposition 5.1. *Let u be the unique solution of* (8) *under the hypotheses of Proposition* 2.1. *Let $\{u_\lambda\}$ be the family defined in* (36) *for $N=1$. Then, for any $t_0>0$, $\{u_\lambda\}$ is relatively compact in $C([t_0,T];L^p_{\mathrm{loc}}(\mathbb{R}))$ for $1\leqslant p\leqslant\infty$ and $t_0>0$.*

To prove this result, we use a variant of the Aubin–Lions compactness lemma. The following estimates on the time derivative are needed:

Lemma 5.2. *Let $0<t_0<T$. Then, under the hypotheses of Proposition* 5.1, *$\{u_{\lambda,t}\}_{\lambda\geqslant 1}$ is uniformly bounded in $L^2(t_0,T;H^{-1}(\mathbb{R}))$.*

Proof. We consider the function

$$v_\lambda(x,t) = \lambda^{-2} e^{\lambda^2 t} u_{\lambda,t}(x,t), \tag{41}$$

that satisfies $v_{\lambda,t} = e^{\lambda^2 t}[\lambda^{-2} u_{\lambda,tt} + u_{\lambda,t}]$. Since u_λ verifies (37),

$$\|v_{\lambda,t}\|_{-1,2} \leqslant c e^{\lambda^2 t}\big[\|u_{\lambda,x}\|_2 + \lambda^{2-q}\big\|f(u_\lambda)\big\|_2\big].$$

On the one hand, we get

$$\int_{\mathbb{R}} \lambda^{4-2q}\big|f(u_\lambda)\big|^2\,dx \leqslant c\lambda^3 \int_{\mathbb{R}} \big|u\big(y,\lambda^2 t\big)\big|^{2q}\,dy \leqslant c\lambda^3\big(1+\lambda^2 t\big)^{-\frac{2q-1}{2}},$$

by (26) and estimates (9). With this estimate and (38) we have, for any $q \geqslant 2$,

$$\|v_{\lambda,t}\|_{-1,2} \leqslant c e^{\lambda^2 t}\lambda^{\frac{3}{2}}\big(1+\lambda^2 t\big)^{-\frac{3}{4}} \leqslant c e^{\lambda^2 t}, \quad \forall t \geqslant t_0 > 0.$$

Using the Cauchy–Schwarz inequality and the previous estimate, we have

$$\partial_t\big(\big\|v_\lambda(t)\big\|_{-1,2}^2\big) \leqslant c e^{\lambda^2 t}\big\|v_\lambda(t)\big\|_{-1,2} \quad \text{for any } t \geqslant t_0.$$

Then, $\partial_t(\|v_\lambda(t)\|_{-1,2}) \leqslant c e^{\lambda^2 t}$. Integrating in time, we get

$$\big\|v_\lambda(t)\big\|_{-1,2} \leqslant \big\|v_\lambda(t_0)\big\|_{-1,2} + c\lambda^{-2}\big[e^{t\lambda^2} - e^{t_0\lambda^2}\big] \quad \text{for any } t \geqslant t_0.$$

Since v_λ is defined by (41), $\|v_\lambda(t)\|_{-1,2} = \lambda^{-2} e^{\lambda^2 t}\|u_{\lambda,t}(t)\|_{-1,2}$. Thus, we get

$$\big\|u_{\lambda,t}(t)\big\|_{-1,2} \leqslant e^{(t_0-t)\lambda^2}\big\|u_{\lambda,t}(t_0)\big\|_{-1,2} + 2c\big[1-e^{(t_0-t)\lambda^2}\big] \quad \text{for any } t \geqslant t_0.$$

Thanks to (34), we have the following decay estimates for $u_{\lambda,t}$:

$$\big\|u_{\lambda,t}(t)\big\|_2 \leqslant \begin{cases} c\lambda^{\frac{5}{2}}(1+\lambda^2 t)^{-\frac{2q+5}{8}} & \text{for } 2 \leqslant q < \frac{5}{2}, \\ c\lambda^{\frac{5}{2}}(1+\lambda^2 t)^{-\frac{5}{4}} & \text{for } q \geqslant \frac{5}{2}, \end{cases} \tag{42}$$

and, in view of the continuity of the embedding $L^2(\mathbb{R}) \hookrightarrow H^{-1}(\mathbb{R})$, we get that, by (42), $\|u_{\lambda,t}(t_0)\|_{-1,2} \leqslant c\lambda^{1/4}$ for any $q \geqslant 2$ and $t_0 > 0$. Thus, we have

$$\big\|u_{\lambda,t}(t)\big\|_{-1,2} \leqslant c e^{(t_0-t)\lambda^2}\lambda^{\frac{1}{4}} + 2c\big[1-e^{(t_0-t)\lambda^2}\big] \quad \text{for any } t \geqslant t_0,$$

and we conclude the proof. □

Proof of Proposition 5.1. We observe that $H^1_{\mathrm{loc}}(\mathbb{R})$ is included in $L^p_{\mathrm{loc}}(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$ with compact embedding. Moreover, $L^2_{\mathrm{loc}}(\mathbb{R})$ is included in $H^{-1}_{\mathrm{loc}}(\mathbb{R})$ and $L^p_{\mathrm{loc}}(\mathbb{R})$ in $L^2_{\mathrm{loc}}(\mathbb{R})$ for $2 \leqslant p \leqslant \infty$ with continuous embeddings.

On the other hand, recall that, according to Lemma 5.2, $\{u_{\lambda,t}\}_{\lambda\geqslant 1}$ is uniformly bounded in $L^2(t_0,T;H^{-1}(\mathbb{R}))$ for $0<t_0<T$. Moreover, by (38), $\{u_\lambda\}$ is uniformly bounded in $L^\infty(t_0,T;H^1(\mathbb{R}))$, for any $0<t_0<T$. Then, using classical compactness results (see [14, Corollary 4, p. 85]), the family $\{u_\lambda\}$ is relatively compact in $C([t_0,T];L^p(K))$ for any compact set $K\subset\mathbb{R}$ and $2\leqslant p\leqslant\infty$. This concludes the proof. □

The proof of (39) is a consequence of the following lemma whose proof is given later.

Lemma 5.3. *Let u be solution of* (8) *given by Proposition* 2.1. *Moreover, assume that $u_0,u_1\in L^2_1(\mathbb{R})$. Then, we have*

$$\int_{\mathbb{R}} |x|^2\big|u(x,t)\big|^2\,dx\leqslant c(1+t)^{\frac{1}{2}}. \tag{43}$$

Indeed, by the Cauchy–Schwarz inequality and the definition of u_λ,

$$\int_{|x|\geqslant k}\big|u_\lambda(x,t)\big|\,dx\leqslant\sqrt{2}(\lambda k)^{-\frac{1}{2}}\big\|u\big(\lambda^2 t\big)\big\|_{\dot L^2_1},$$

and, thanks to (43), we prove (39):

$$\int_{|x|\geqslant k}\big|u_\lambda(x,t)\big|\,dx\leqslant c(\lambda k)^{-\frac{1}{2}}\big(1+\lambda^2 t\big)^{\frac{1}{4}}\leqslant ck^{-\frac{1}{2}}(1+T)^{\frac{1}{4}}. \tag{44}$$

Now, thanks to estimate (38) and choosing t_0 sufficiently small, we get (40). Then, as a simple consequence, we have:

Proposition 5.4. *Under the hypotheses of Lemma* 5.3, *the family $\{u_\lambda\}$ is relatively compact in $C([t_0,T];L^p(\mathbb{R}))$ for $0<t_0<T<\infty$ and any $p\in[1,\infty]$.*

Proof. First, by Proposition 5.1, (44) and Vitali's theorem readily imply the relative compactness $\{u_\lambda\}$ in $C([t_0,T];L^1(\mathbb{R}))$. The L^p analogue now follows from (38) by the Gagliardo–Nirenberg inequality. □

Proof of Lemma 5.3. We use the variation of constants formula. Applying Lemma 3.3 with $b=2$ and $a=1$ to estimate $S(t)[u_0,u_1]$ and with $b=1$ and $a=1$ to estimate $\partial_x S(t-s)\times[f(u(s))]$, we have

$$\begin{aligned}\big\|u(t)\big\|_{\dot L^2_1}&\leqslant c'(1+t)^{\frac{1}{4}}\big(\|u_0\|_{L^1}+\|u_1\|_{L^1}+\|u_0\|_{L^2_1}+\|u_1\|_{L^2_1}\big)\\&\quad+c\int_0^t e^{-\omega(t-s)}\big\|f\big(u(s)\big)\big\|_{\dot L^2_1}\,ds+c_1\int_0^t(1+t-s)^{-\frac{3}{4}}\big\|f\big(u(s)\big)\big\|_{\dot L^1_1}\,ds\\&\quad+c_2\int_0^t(1+t-s)^{-\frac{1}{2}}\big\|f\big(u(s)\big)\big\|_2\,ds+c_3\int_0^t(1+t-s)^{-\frac{1}{4}}\big\|f\big(u(s)\big)\big\|_1\,ds.\end{aligned} \tag{45}$$

Now, we study the integral terms. Since $f(u)=|u|^{q-1}u$, using the decay estimates (9), we obtain immediately:

$$\|f(u(s))\|_{\dot{L}^2_1} \leqslant c(1+s)^{-\frac{q-1}{2}}\|u\|_{\dot{L}^2_1}, \qquad \|xf(u(s))\|_1 \leqslant c(1+s)^{-\frac{2q-3}{4}}\|u\|_{\dot{L}^2_1},$$
$$\|f(u(s))\|_2 \leqslant c(1+s)^{-\frac{2q-1}{4}}, \qquad \|f(u(s))\|_1 \leqslant c(1+s)^{-\frac{q-1}{2}}.$$

Therefore, applying these estimates in the integrals appearing in (45), we deduce that, for any $t \geqslant 0$,

$$\|u(t)\|_{\dot{L}^2_1} \leqslant c(1+t)^{\frac{1}{4}} + \tilde{c}\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{2q-3}{4}}\|u(s)\|_{\dot{L}^2_1}\,ds. \tag{46}$$

Now, we define $\alpha(t)=\sup\{\beta(s):\ s\in[0,t]\}$ with $\beta(s)=(1+s)^{-1/4}\|u(s)\|_{\dot{L}^2_1}$. First, since $(1+t-s)^{-3/4}(1+s)^{-(2q-3)/4} \leqslant 1$, we have that $\alpha(t)$ is finite applying the Gronwall lemma in (46). Now, we consider $\varepsilon \in (0,1/2)$ and from (46) we have

$$\begin{aligned}
\beta(t) &\leqslant c + \tilde{c}\alpha(t)(1+t)^{-\frac{1}{4}}\big(1+(1-\varepsilon)t\big)^{-\frac{3}{4}}\int_0^{\varepsilon t}(1+s)^{-\frac{q-2}{2}}\,ds \\
&\quad + \tilde{c}\alpha(t)(1+t)^{-\frac{1}{4}}(1+\varepsilon t)^{-\frac{q-2}{2}}\int_{\varepsilon t}^{t}(1+t-s)^{-\frac{3}{4}}\,ds \\
&\leqslant c + \tilde{c}\alpha(t)\big(K_1\varepsilon + K_2(1+\varepsilon t)^{-\frac{q-2}{2}}\big),
\end{aligned} \tag{47}$$

with $K_1, K_2 > 0$ and independent of ε. We distinguish two cases:

1. Case $q>2$. We choose ε such that $\tilde{c}K_1\varepsilon \leqslant 1/2$ and t_0 such that $\tilde{c}K_2(\varepsilon t_0)^{-(q-2)/2} \leqslant 1/4$. Then, from (47) we obtain that, for any $t \geqslant 0$, $\alpha(t) \leqslant \alpha(t_0)+c+3\alpha(t)/4$, from which we get (43) for $q>2$.

2. Case $q=2$. From (47) we have $\alpha(t) \leqslant c+\tilde{c}(K_1\varepsilon+K_2)\alpha(t)$. The constant $\tilde{c}>0$ is proportional to the norm of the initial data in V. Therefore, as the norm of (u_0,u_1) is small in V, we can assume that $\tilde{c}(K_1\varepsilon+K_2) \leqslant 3/4$. Thus, we obtain (43) for $q=2$. □

6. Identification of the limit and asymptotic behavior

First, we are going to identify the limit of the sequence $\{u_\lambda\}_{\lambda>0}$. As $\lambda \to \infty$, Eq. (37) formally reduces to the heat equation for $q>2$ and to the Burgers equation for $q=2$. In particular, we prove the following proposition.

Proposition 6.1. *Under the hypotheses of Proposition* 2.1 *and Lemma* 5.3*, the sequence* $\{u_\lambda\}_{\lambda>0}$ *converges in* $L^2((0,T)\times\mathbb{R})$ *to* θ *where, if the exponent* $q>2$*,* θ *is the solution of the heat equation* (15)*, and, if the exponent* $q=2$*,* θ *is the solution of the Burgers equation* (14).

Proof. To do it, we use the weak formulation of solutions. We consider the following space of test functions $\mathcal{D}(T) = \{\varphi \in C([0,T]; H^2(\mathbb{R})) \cap C^1([0,T]; L^2(\mathbb{R})) \colon \varphi(T,x) = 0\}$. The solutions $\{u_\lambda\}$ of the rescaled problem (37) satisfy:

$$\int_0^T \int_{\mathbb{R}} u_\lambda(\varphi_t + \varphi_{xx})\,dx\,dt + \lambda^{2-q} \int_0^T \int_{\mathbb{R}} |u_\lambda|^{q-1} u_\lambda \varphi_x\,dx\,dt$$
$$+ \lambda \int_{\mathbb{R}} \bigl[u_0(\lambda x) + u_1(\lambda x)\bigr]\varphi(x,0)\,dx + \lambda^{-2} \int_0^T \int_{\mathbb{R}} u_{\lambda,t}\varphi_t\,dx\,dt = 0, \quad \varphi \in \mathcal{D}(T). \tag{48}$$

Since u_λ is relatively compact in $L^2((0,T)\times\mathbb{R})$, there exists a subsequence (denoted with the same subscript λ) u_λ converging in $L^2((0,T)\times\mathbb{R})$ to some function θ. Then,

$$\lim_{\lambda\to\infty} \int_0^T \int_{\mathbb{R}} u_\lambda(\varphi_t + \varphi_{xx}) = \int_0^T \int_{\mathbb{R}} \theta(\varphi_t + \varphi_{xx}), \quad \forall \varphi \in \mathcal{D}(T). \tag{49}$$

By the change of variable $y = \lambda x$ and applying the dominated convergence theorem, we get

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}} \bigl[u_0(y) + u_1(y)\bigr]\varphi(y/\lambda, 0)\,dy = M\varphi(0,0), \quad \forall \varphi \in \mathcal{D}(T). \tag{50}$$

Thanks to (42), since $\min(5/4, (2q+5)/8) > 1$ for $q \geqslant 2$, we get

$$\lim_{\lambda\to\infty} \lambda^{-2} \int_0^T \int_{\mathbb{R}} u_{\lambda,t}\varphi_t = 0. \tag{51}$$

Now, we study the integral in (48) involving the term $|u_\lambda|^{q-1}u_\lambda$. We distinguish two cases: $q > 2$ and $q = 2$. Using (26) and estimates (38), we obtain for $0 < T < \infty$:

$$\lim_{\lambda\to\infty} \lambda^{2-q} \int_0^T \int_{\mathbb{R}} \bigl||u_\lambda|^{q-1} u_\lambda \varphi_x\bigr|\,dx\,dt \leqslant \lim_{\lambda\to\infty} c\lambda \int_0^T \bigl(1 + \lambda^2 t\bigr)^{-\frac{q-1}{2}}\,dt = 0, \tag{52}$$

provided $q > 2$ since

$$\int_0^T \bigl(1 + \lambda^2 t\bigr)^{-\frac{q-1}{2}}\,dt \leqslant \begin{cases} c\lambda^{-2} & \text{with } q > 3, \\ c\lambda^{-2}\ln(1 + \lambda^2 T) & \text{with } q = 3, \\ c\lambda^{-2}(1 + \lambda^2 T)^{\frac{3-q}{2}} & \text{with } q \in [2,3). \end{cases}$$

Therefore, (49)–(52) guarantee that the limit θ of u_λ in $L^2((0,T)\times\mathbb{R})$ satisfies the weak formulation of the heat equation (15) when the exponent $q > 2$.

Now, since the sequence $\{u_\lambda\}$ converges to θ in $L^2((0,T)\times\mathbb{R})$, we get, thanks to the fact that $\varphi_x \in C((0,T)\times L^\infty(\mathbb{R}))$,

$$\lim_{\lambda\to\infty}\int_0^T\int_{\mathbb{R}} |u_\lambda|u_\lambda\varphi_x\,dx\,dt = \int_0^T\int_{\mathbb{R}} |\theta|\theta\varphi_x\,dx\,dt. \tag{53}$$

Thus, thanks to (49)–(51) and (53), the limit θ of the sequence u_λ satisfies the weak formulation of the Burgers equation (14) when the exponent $q=2$.

As a final point, we note that the uniqueness of the weak solutions of (14) and (15) in $L^2((0,T)\times\mathbb{R})$ is well known thanks to the classical transposition method. This fact guarantees that the limit θ is unique and that it is actually the limit of the whole family $\{u_\lambda\}$. $\square$

Now, we conclude this section deriving the asymptotic behavior of the solutions of (8).

Proof of Theorem 2.3. We know by Proposition 5.4 that, when $\lambda\to\infty$, there exists θ such that the sequence $\{u_\lambda\}$ converges to θ in $L^p(\mathbb{R})$ $(1\leqslant p\leqslant\infty)$ for $t=1$

$$\lim_{\lambda\to\infty}\big\|u_\lambda(\cdot,1)-\theta(\cdot,1)\big\|_p = 0. \tag{54}$$

We note that, by Proposition 6.1, this convergence holds for the whole sequence $\{u_\lambda\}$ and that, if the exponent $q>2$ (respectively $q=2$), θ is the solution of the heat equation (15) (respectively Burgers equation (14)). In both cases, it is well be known that the solutions are self-similar: $\theta(x,t)=t^{-1/2}f_M(x/\sqrt{t})$, for a suitable profile f_M (see, for example, [5]). Thus, θ is invariant under the rescaling transformation (36), i.e., $\theta_\lambda\equiv\theta$, and

$$\int_{\mathbb{R}}\big|u_\lambda(x,1)-\theta(x,1)\big|^p\,dx = \lambda^{p-1}\int_{\mathbb{R}}\big|u\big(y,\lambda^2\big)-\theta\big(y,\lambda^2\big)\big|^p\,dy.$$

We choose $\lambda^2=t$ and thanks to (54), we get the convergence result (13). $\square$

7. Blow-up

In [15] the global nonexistence of nondecreasing positive solutions for the differential inequalities of the type $a_{tt}+a_t\geqslant\alpha a^q$ is studied. In particular, denoting $T^*<\infty$ the maximal existence time, $a(t)$ with initial data $a_0>0$, $a_1\geqslant 0$ blows-up in the sense that: $\|u\|_{W^{2,1}(0,T^*)}=+\infty$. Now, we show a more general nonexistence result with an explicit estimate on the blow-up time using different arguments than in [15].

Lemma 7.1. *Let $a=a(t)$ be a solution of*

$$a_{tt}+a_t=\alpha|a|^{q-1}a,\quad t>0, \tag{55}$$

with $q>1$, $\alpha>0$ and initial data (a_0,a_1) such that $a_0,a_1>0$ or $a_0,a_1<0$. Then, if

$$|a_0|\geqslant\left(\frac{q+1}{8\alpha}\right)^{\frac{1}{q-1}}e^{\frac{2}{q-1}}\quad\textit{and}\quad\left(\frac{1}{2}a_0+a_1\right)^2\geqslant\frac{1}{4}a_0^2+\alpha\frac{2|a_0|^{q+1}}{q+1}, \tag{56}$$

a blows-up in finite time $t_b\leqslant|a_0|^{(1-q)/2}e\sqrt{2q+2}/\sqrt{\alpha}(q-1)$.

Proof. First, we note that $a, a_t > 0$ for all $t > 0$ when $a_0, a_1 > 0$. Indeed, assume that there exists $\tau < \infty$ such that $a(\tau) = 0$ and $a(t) > 0$ for any $t < \tau$. Then, from (55), $a_t(t) \geqslant a_1 e^{-t} > 0$ for any $t < \tau$ and, therefore, a being positive and increasing in $(0, \tau)$ it may not vanish in time $t = \tau$. Furthermore, from (55) we deduce that a_t is positive as well. Analogously, we have $a, a_t < 0$ for all $t > 0$ when $a_0, a_1 < 0$.

Now, let $b(t) = e^{t/2} a(t)$. Then (55) becomes

$$b_{tt} - \frac{b}{4} = \alpha e^{-\frac{q-1}{2} t} |b|^{q-1} b, \quad t > 0, \tag{57}$$

with initial data $b(0) = a_0$ and $b'(0) = a_0/2 + a_1$. Now, we multiply (57) by b_t, integrate and again by parts, we obtain

$$\frac{1}{2} b_t^2 \geqslant \frac{1}{2}\left(\frac{1}{2} a_0 + a_1\right)^2 - \frac{1}{8} a_0^2 + \alpha e^{-\frac{q-1}{2} t} \frac{|b(t)|^{q+1}}{q+1} - \alpha \frac{|a_0|^{q+1}}{q+1}.$$

Given an arbitrary a_0 we assume that a_1 is sufficiently large such that (56) holds. Considering $\tau > 0$ arbitrary, we have $b_t^2 \geqslant c^2 |b|^{q+1}$, for $0 \leqslant t \leqslant \tau$ with $c^2 = 2\alpha e^{-(q-1)\tau/2}/(q+1)$. When $a, a_t > 0$, then $b_t \geqslant c b^{(q+1)/2}$ and after integration we obtain that $b(t) \geqslant (a_0^{(1-q)/2} - c(q-1) \times t/2)^{-2/(q-1)}$, and thus, when $q > 1$, the solution b blows-up in time $t_b \leqslant 2a_0^{(1-q)/2}/c(q-1)$ and, $t_b \leqslant \tau$, if

$$|a_0| \geqslant \left(\frac{2(q+1)}{\alpha(q-1)^2}\right)^{\frac{1}{q-1}} \tau^{-\frac{2}{q-1}} e^{\frac{\tau}{2}}, \quad \tau > 0. \tag{58}$$

Now, we consider the function $h(\tau) = \tau^{-2/(q-1)} e^{\tau/2}$ in the right-hand side of (58). The minimum critical point of this function in $(0, \infty)$ is $\tau = 4/(q-1)$. Taking this value in (58), a_0 has to satisfy (56). Consequently, if a_0 and a_1 satisfy (56), a blows-up in finite time $t_b \leqslant a_0^{(1-q)/2} e\sqrt{2q+2}/\sqrt{\alpha}(q-1)$. (In the case $a, a_t < 0$, then $b_t \leqslant -c|b|^{(q+1)/2}$ and, we conclude as above.) □

As we mentioned in Section 2, the blow-up result for the ODE (55) with exponent $q = 2$ implies the existence of blowing-up solutions of the form $u(x,t) = xa(t)$ for the PDE (8) with $q = 2$. The corresponding initial data are of the form $u_0(x) = xa_0$, $u_1(x) = xa_1$. Obviously they do not belong to the space V. Note however that, due to the finite speed of propagation ($= 1$ in model (8)) one can modify the solution so that it blows-up and has compact support in x. Indeed, let a be solution of (55) blowing-up in time T. Let $\varphi \in C_c^\infty(\mathbb{R})$ be such that $\varphi(x) = x$ for all $x \in [-3T, 0]$. Let the initial data for (8) be $u_0(x) = a_0\varphi(x)$, $u_1(x) = a_1\varphi(x)$. Then, the solution of (8) in $[-3T + t, -t]$ is of the form $u \equiv xa(t)$. Consequently, it blows-up in time T for all $x \in [-2T, -T]$.

Now, we prove the blow-up result for even nonlinearities.

Proof of Theorem 2.2. Due to the finite speed of propagation and applying Hölder's inequality, we get for any $t \leqslant K_1$

$$a(t)^q \leqslant (K_1 + K_2)^{2q-1} \left(\frac{q-1}{2q-1}\right)^{q-1} \int |u|^q \, dx.$$

Now, multiplying Eq. (8) by x, integrating the equation and thanks to the previous inequality, the function $a(t)$, defined in (11), satisfies the inequality $a_{tt} + a_t \geqslant \rho a^q$ with $\rho = (K_1 + K_2)^{1-2q}((q-1)/(2q-1))^{1-q}$. Under the hypotheses of Theorem 2.2, the initial data a_0 and a_1 of a are positive. With these initial data a is positive and it satisfies (55) replacing α with ρ. Then, applying Lemma 7.1, the function a (11) blows-up in finite time $t_b \leqslant a_0^{(1-q)/2} e\sqrt{2q+2}/(\sqrt{\rho}(q-1))$, if the constants (a_0, a_1) satisfy (56) with $\alpha = \rho$. Finally, by the definition of ρ, $t_b \leqslant K_1$ under the hypotheses of Theorem 2.2. □

Appendix A. Linear estimates

In this appendix we prove Lemmas 3.2 and 3.3. The proofs can be carried out by means of a careful analysis of the Fourier transform of solutions. Equation (16) can be written as

$$\begin{cases} \hat{u}_{tt} + |\xi|^2 \hat{u} + \hat{u}_t = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ \hat{u}(\xi, 0) = \hat{\varphi}^0(\xi), & \hat{u}_t(\xi, 0) = \hat{\varphi}^1(\xi). \end{cases} \tag{A.1}$$

Define $\hat{v}(\xi, t) = \hat{u}(\xi, t)\chi_{|\xi| \leqslant 1/4}$ and $\hat{w}(\xi, t) = \hat{u}(\xi, t)\chi_{|\xi| > 1/4}$, where χ stands for the characteristic function. Denoting respectively by v and w the inverse Fourier transform of $\hat{v}$ and $\hat{w}$, one obtains a decomposition of the solution u of (16) itself $u = v + w$.

Lemma A.1. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}^N_+$. Then,*

$$\left\| \partial_x^\alpha \partial_t^k w(\cdot, t) \right\|_2 \leqslant c e^{-\omega t} \left[\left\| \varphi^0 \right\|_{\dot{H}^{k+|\alpha|}} + \left\| \varphi^1 \right\|_{\dot{H}^{k+|\alpha|-1}} \right], \quad \forall t \geqslant 0. \tag{A.2}$$

The proof is easily obtained using a Lyapunov function as in [13].

Now, we can decompose $\hat{v}$ as $\hat{v}(\xi, t) = \hat{p}(\xi, t) + \hat{q}(\xi, t)$, where, for $|\xi| < 1/4$,

$$\hat{p}(\xi, t) = \left[\hat{\varphi}^0(\xi)\left(\frac{1}{2} + r(\xi) \right) + \hat{\varphi}^1(\xi) \right] \frac{e^{-\frac{t}{2}}}{2r(\xi)} e^{r(\xi)t},$$

$$\hat{q}(\xi, t) = \left[\hat{\varphi}^0(\xi)\left(r(\xi) - \frac{1}{2} \right) - \hat{\varphi}^1(\xi) \right] \frac{e^{-\frac{t}{2}}}{2r(\xi)} e^{-r(\xi)t},$$

with $r(\xi) = \sqrt{1/4 - |\xi|^2}$, for $|\xi| \leqslant 1/4$. Let p and q be the inverse Fourier transforms of $\hat{p}$ and $\hat{q}$, respectively. We have:

Lemma A.2. *Let $k = 0, 1$ and $\alpha \in \mathbb{R}^N_+$. Then,*

$$\left\| \partial_x^\alpha \partial_t^k q(\cdot, t) \right\|_2 \leqslant c e^{-\frac{t}{2}} \left[\left\| \varphi^0 \right\|_{\dot{H}^{|\alpha|+k}} + \left\| \varphi^1 \right\|_{\dot{H}^{|\alpha|-1+k}} \right], \quad \forall t \geqslant 0. \tag{A.3}$$

However, when $k + |\alpha| < 1$, $\dot{H}^{k+|\alpha|-1}$ must be replaced by $H^{k+|\alpha|-1}$ in (A.3).

Lemma A.3. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}^N_+$. Then, $t \geqslant 0$,*

$$\left\| \partial_t^k \partial_x^\alpha p(\cdot, t) \right\|_2 \leqslant c(1+t)^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{2})}(1+t)^{-\frac{2k+|\alpha|}{2}} \left[\left\| \varphi^0 \right\|_a + \left\| \varphi^1 \right\|_a \right]. \tag{A.4}$$

Proof of Lemma A.2. We observe that for $|\xi| \leqslant 1/4$, $r(\xi)$ is positive, and

$$\left|\partial_t^k \hat{q}(\xi,t)\right| \leqslant \left|\hat{\varphi}^0(\xi)\left(\frac{1}{2}-r(\xi)\right)+\hat{\varphi}^1(\xi)\right|\frac{1}{2r(\xi)}\left(\frac{1}{2}+r(\xi)\right)^k e^{-\frac{t}{2}}.$$

Thus, q decays exponentially. Since $|\xi| \leqslant 1/4$, we get $2 \leqslant 1/r(\xi) \leqslant 4/\sqrt{3}$ and $3/4 \leqslant r(\xi)+1/2 \leqslant 1$. Moreover, since $3(1/2-r(\xi))/4 \leqslant (1/2-r(\xi))(1/2+r(\xi)) = |\xi|^2$, then $(1/2-r(\xi)) \leqslant 4|\xi|^2/3$ for any $|\xi| \leqslant 1/4$. Using these inequalities and Parseval's identity, (A.3) is obtained immediately. $\square$

Proof of Lemma A.3. First, it is easy to check that

$$\int\limits_{|\xi|\leqslant\delta} e^{-t\omega|\xi|^2}|\xi|^k\,d\xi \leqslant c(\omega,\delta,k)(1+t)^{-\frac{k+N}{2}}. \tag{A.5}$$

Now, since $|\xi| \leqslant 1/4$, we know that $-3|\xi|^2/4 \leqslant r(\xi)-1/2 \leqslant -|\xi|^2$. Thus,

$$\left|\partial_t^k \hat{p}(\xi,t)\right| \leqslant c\left(\left|\hat{\varphi}^0(\xi)\right|+\left|\hat{\varphi}^1(\xi)\right|\right)|\xi|^{2k}e^{-|\xi|^2 t}.$$

Using Parseval's identity, Hölder's inequality and (A.5), we get

$$\left\|\partial_x^\alpha\partial_t^k p(\cdot,t)\right\|_2 \leqslant c\left(\left\|\hat{\varphi}^0\right\|_{2p}+\left\|\hat{\varphi}^1\right\|_{2p}\right)(1+t)^{-\frac{N}{4p'}}(1+t)^{-\frac{2k+|\alpha|}{2}}, \quad \forall t \geqslant 0.$$

Choosing $1 \leqslant a \leqslant 2$ such that $1/a + 1/(2p) = 1$, taking into account that $1/2p' = 1/a - 1/2$, and by the Hausdorff–Young inequality, we prove (A.4). $\square$

Combining this estimate, (A.2) and (A.3), we conclude the proof of Lemma 3.2.

Finally, we prove Lemma 3.3.

Proof of Lemma 3.3. Using the variation of constants formula to define v, we have by Lemma 3.2

$$\begin{aligned}\|v\|_2 \leqslant{}& ce^{-\omega t}\left(\left\|x\varphi^0\right\|_2+\left\|x\varphi^1\right\|_{-1,2}\right)+c'(1+t)^{\frac{1}{4}-\frac{1}{2b}}\left(\left\|x\varphi^0\right\|_b+\left\|x\varphi^1\right\|_b\right)\\ &+2\int_0^t\left\|S_x(t-s)\left[0,u(s)\right]\right\|_2\,ds.\end{aligned} \tag{A.6}$$

By estimate (17) with ($\varphi^0=0$, $\varphi^1=u(s)$) and ($a=2$, $k=0$, $\alpha=1$), and estimate (18) for ($\alpha=0$, $k=0$), we obtain

$$\begin{aligned}&\int_0^t\left\|S_x(t-s)\left[0,u(s)\right]\right\|_2\,ds\\ &\quad\leqslant c\left(\left\|\varphi^0\right\|_2+\left\|\varphi^1\right\|_{-1,2}\right)(1+t)^{-\frac{1}{2}}+c'\left(\left\|\varphi^0\right\|_a+\left\|\varphi^1\right\|_a\right)(1+t)^{\frac{3}{4}-\frac{1}{2a}},\end{aligned}$$

thanks to (14). Then, returning to (A.6), we prove (20). Note that $xu_x = v_x - u$. Since the decay of $\|u(t)\|_2$ is known by (17), we only need to obtain the behavior of v_x to get (21). Using estimate (18), we get

$$\|v_x\|_2 \leqslant ce^{-\omega t}\left(\|x\varphi^0\|_{\dot{H}^1} + \|x\varphi^1\|_2\right) + c'(1+t)^{-\frac{1}{4}-\frac{1}{2b}}\left(\|x\varphi^0\|_b + \|x\varphi^1\|_b\right) \\ + 2\int_0^t \left\|S_x(t-s)\left[0, u_x(s)\right]\right\|_2 ds. \tag{A.7}$$

Applying (17) with ($a = 2$, $k = 0$, $\alpha = 1$), (18) with ($k = 0$, $\alpha = 1$) and using (23), we get

$$\int_0^t \left\|S_x(t-s)\left[0, u_x(s)\right]\right\|_2 ds \\ \leqslant c\left(\|\varphi^0\|_{\dot{H}^1} + \|\varphi^1\|_2\right)(1+t)^{-\frac{1}{2}} + c'\left(\|\varphi^0\|_a + \|\varphi^1\|_a\right)(1+t)^{\frac{1}{4}-\frac{1}{2a}}.$$

Then, coming back to (A.7) we obtain (21). □

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