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Nonautonomous bifurcation patterns for one-dimensional differential equations ☆

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Abstract

Although, bifurcation theory of ordinary differential equations with autonomous and periodic time dependence is a major object of research in the study of dynamical systems since decades, the notion of a nonautonomous bifurcation is not yet established. In this article, two different approaches are discussed which are based on special notions of attractivity and repulsivity. Generalizations of the well-known one-dimensional transcritical and pitchfork bifurcation are obtained.

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1. Introduction

Dynamical systems are mathematical objects used to model phenomena whose state changes over time. Since these models appear in many applications, e.g., in physics, biology or economy, the theory of dynamical systems has become very popular. Dynamical systems often depend on certain parameters, and it is a main object of bifurcation theory to describe qualitative changes in case these parameters are varied.

In many cases, the notion of dynamical system is not general enough to model real world phenomena, since there are often good reasons to suppose that the underlying rules are time-

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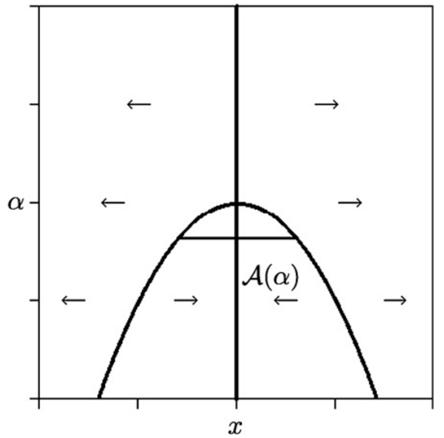

Fig. 1. Pitchfork bifurcation.

dependent. For biological processes, for instance, it is more realistic to take evolutionary adaptations into account, and sometimes it is unavoidable to consider random perturbations such as white noise or to model the control of a process by a human being. The appropriate class to treat such problems are the so-called *nonautonomous dynamical systems*. Another reason to consider nonautonomous systems is given by the fact that the investigation of states of dynamical systems which are nonconstant in time leads to nonautonomous problems in form of the equation of perturbed motion.

Bifurcation theory for dynamical systems has been a central object of research since decades, but there is a lack of a general theory for nonautonomous dynamical systems. However, in the last twenty years, active research was done in case of quasi-periodic and strictly ergodic time-dependence, cf. [9,11,13–16].

In this paper, two different concepts of a nonautonomous bifurcation are introduced for ordinary differential equations. Please note that we make no special assumptions concerning the time dependence. Such a general situation has already been considered in [17,19,20]. The bifurcation concept used in these papers, however, differs to some extent from the notion of a nonautonomous bifurcation we use here (see the discussion at the end of Section 3). We obtain nonautonomous generalizations of the well-known one-dimensional transcritical and pitchfork bifurcation which are formulated in terms of Taylor coefficients for the right-hand side. The authors of [20] also discussed the occurrence of nonautonomous one-dimensional bifurcation scenarios, but there conditions are of a quite different form than the results obtained in this paper. This subject is discussed at the end of Section 5.

Since the concept of a nonautonomous bifurcation here is based on phenomenological observations from the autonomous bifurcation theory, it is useful to look at an autonomous bifurcation scenario. For a real parameter α, consider the ordinary differential equation $\dot{x} = x(\alpha + x^2)$, which is a prototype of a pitchfork bifurcation as indicated in Fig. 1. For $\alpha \geqslant 0$, there is only one equilibrium, which is given by zero and which is repulsive. By letting the parameter α pass through zero in negative direction, this equilibrium becomes attractive, and two other repulsive equilibria, given by $\pm\sqrt{-\alpha}$, are bifurcating.

In order to establish a nonautonomous bifurcation theory, consider this scenario in the following way: for $\alpha < 0$, the trivial solution is attractive, and the domain of attraction $\mathcal{A}(\alpha)$ is given by the open interval between the two other equilibria. Now, the main point is that this domain of attraction undergoes a qualitative change from a nontrivial to a trivial object in the limit $\alpha \nearrow 0$. Moreover, the closure of $\mathcal{A}(\alpha)$ is also a repeller, and thus, also a repeller changes qualitatively for $\alpha \nearrow 0$. We call the shrinking of a domain of attraction (repulsion, respectively) a *bifurcation*, whereas the case of a changing repeller (attractor, respectively) is denoted as a *transition*.

To implement this idea in the nonautonomous context, locally defined notions of attractive and repulsive solutions, domains of attractivity and repulsivity, as well as attractor and repeller are needed. We distinguish between three points of view concerning different time domains. The concepts are introduced for the entire time (all-time attractivity, repulsivity, bifurcation and transition), the past (past attractivity, repulsivity, bifurcation and transition) and the future (future attractivity, repulsivity, bifurcation and transition).

Finally, please note that, although the applications in this paper are of dimension one, the concepts of bifurcation and transition also apply in a higher-dimensional setting, since the definitions of attractivity and repulsivity are given in a very general form. The main tool for the analysis of such systems is the method of center manifold reduction (see, e.g., [4]). The basic idea is to detect a bifurcation of the system restricted to a center manifold. For instance, consider again the above mentioned motivating example $\dot{x} = x(\alpha + x^2)$ with an additional second equation, given by $\dot{y} = \lambda y$. In case $\lambda > 0$, the trivial solution is not attractive for $\alpha < 0$, in contrast to the one-dimensional system, and therefore, we have no bifurcation of attraction areas but only a transition of repellers. For $\lambda < 0$, the trivial solution is attractive, and thus, the two-dimensional system admits a bifurcation of attraction areas, but no longer a repeller transition. Restricting the attention to the lower-dimensional invariant manifold $\mathbb{R} \times \{0\}$, however, yields the original one-dimensional system, and for this system we obtain both a bifurcation and a transition.

2. Preliminaries

We denote by $\mathbb{R}$ the set containing all reals and write $\mathbb{R}_\kappa^+ := [\kappa, \infty)$ and $\mathbb{R}_\kappa^- := (-\infty, \kappa]$ for given $\kappa \in \mathbb{R}$; $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. The set of real $M \times N$ matrices is denoted by $\mathbb{R}^{M \times N}$. The Euclidean space $\mathbb{R}^N$ is equipped with the Euclidean norm $\|\cdot\|$, and we write $U_\varepsilon(x_0) = \{x \in \mathbb{R}^N\colon \|x - x_0\| < \varepsilon\}$ for the ε-neighborhood of some point $x_0 \in \mathbb{R}^N$. For arbitrary nonempty sets $A, B \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, let $d(x, A) := \inf\{d(x, y)\colon y \in A\}$ be the *distance* of x to A and $d(A|B) := \sup\{d(x, B)\colon x \in A\}$ be the *Hausdorff semi-distance* of A and B.

Let $g : X \to Y$ be a function from a set X to a set Y. Then the *graph* of g is defined by $\operatorname{graph} g := \{(x, y) \in X \times Y\colon y = g(x)\}$.

Given a differentiable function $g : X \subset \mathbb{R}^N \to \mathbb{R}^M$, we write $Dg : X \to \mathbb{R}^{M \times N}$ for its derivative and $D_i g : X \to \mathbb{R}^M$ for its partial derivative with respect to the ith variable, $i \in \{1, \ldots, N\}$. Higher order derivatives $D^n g$ or $D_i^n g$ are defined inductively.

In this article, we consider unbounded intervals of the form $\mathbb{I} = \mathbb{R}$, $\mathbb{I} = \mathbb{R}_\kappa^-$ or $\mathbb{I} = \mathbb{R}_\kappa^+$, respectively. Given a continuous function $f : \mathbb{I} \times \mathbb{R}^N \to \mathbb{R}^N$, we use the notation

$$\dot{x} = f(t, x) \tag{2.1}$$

to denote the ordinary differential equation $\dot{x}(t) = f(t, x(t))$. We assume that f fulfills conditions for the local existence and uniqueness of solutions. Let λ stand for the *general solution* of (2.1), i.e., $\lambda(\cdot, \tau, \xi)$ is the unique noncontinuable solution of (2.1) satisfying the initial condition $\lambda(\tau, \tau, \xi) = \xi$. For arbitrary nonempty sets $M \subset \mathbb{R}^N$, we define $\lambda(t, \tau, M) := \bigcup_{\xi \in M} \lambda(t, \tau, \xi)$.

The *transition operator* $\Phi : \mathbb{I} \times \mathbb{I} \to \mathbb{R}^{N\times N}$ of a linear differential equation

$$\dot{x} = A(t)x \tag{2.2}$$

with a function of matrices $A : \mathbb{I} \to \mathbb{R}^{N\times N}$ is defined by $\Phi(t,\tau)x := \lambda(t,\tau,x)$ for all $t,\tau \in \mathbb{I}$ and $x \in \mathbb{R}^N$, where λ is the general solution of (2.2).

A subset M of the extended phase space $\mathbb{I} \times \mathbb{R}^N$ is called a *nonautonomous set* if for all $t \in \mathbb{I}$, the so-called *t-fibers* $M(t) := \{x \in \mathbb{R}^N\colon\ (t,x) \in M\}$ are nonempty. We call M *compact* if all t-fibers are compact. M is said to be invariant if $\lambda(t,\tau,M(\tau)) = M(t)$ for all $t,\tau \in \mathbb{I}$.

Whenever the term "all-time (past, future, respectively)" is used in this paper, we mean "all-time or past or future, respectively."

3. Notions of attractivity and repulsivity

In this section, new concepts of attractivity and repulsivity are introduced. In our nonautonomous situation, we distinguish between the analysis of the entire time, the past and the future.

The definitions of attractivity are local forms of established concepts which have been developed since the 1990s. This relationship is discussed in more detail after the statement of the definitions.

We begin with the definitions for the entire time. First, note that an all-time attractor is a local form of a uniform attractor as discussed, e.g., in [7].

Definition 3.1 *(All-time attractivity and repulsivity).* Let $\mathbb{I} = \mathbb{R}$, A and R be compact and invariant nonautonomous sets and $\mu : \mathbb{R} \to \mathbb{R}^N$ be a solution of (2.1).

(i) A is called *all-time attractor* if there exists an $\eta > 0$ with

$$\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} d\bigl(\lambda\bigl(t+\tau,\tau,U_\eta\bigl(A(\tau)\bigr)\bigr)\big|A(t+\tau)\bigr) = 0.$$

The supremum of all positive η with this property is denoted by $\mathcal{A}_A^{\pm}$ and called *all-time attraction radius* of A.

(ii) μ is called *all-time attractive* if graph μ is an all-time attractor.

(iii) R is called *all-time repeller* if there exists an $\eta > 0$ with

$$\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} d\bigl(\lambda\bigl(\tau-t,\tau,U_\eta\bigl(R(\tau)\bigr)\bigr)\big|R(\tau-t)\bigr) = 0.$$

The supremum of all positive η with this property is denoted by $\mathcal{R}_R^{\pm}$ and called *all-time repulsion radius* of R.

(iv) μ is called *all-time repulsive* if graph μ is an all-time repeller.

In the following definition, the notions of past attractivity and repulsivity are explained. Note that a past attractor is a local form of a pullback attractor (see, e.g., [5]), i.e., it attracts a neighborhood of itself in the sense of pullback attraction.

Definition 3.2 *(Past attractivity and repulsivity).* Let $\mathbb{I} = \mathbb{R}_\kappa^-$, A and R be compact and invariant nonautonomous sets and $\mu : \mathbb{I} \to \mathbb{R}^N$ be a solution of (2.1).

(i) A is called *past attractor* if there exists an $\eta > 0$ with

$$\lim_{t\to\infty} d\big(\lambda\big(\tau, \tau - t, U_\eta\big(A(\tau - t)\big)\big)\big|A(\tau)\big) = 0 \quad \text{for all } \tau \in \mathbb{I}.$$

The supremum of all positive η with this property is denoted by $\mathcal{A}_A^-$ and called *past attraction radius* of A.

(ii) μ is called *past attractive* if graph μ is a past attractor.

(iii) R is called *past repeller* if there exists an $\eta > 0$ with

$$\lim_{t\to\infty} d\big(\lambda\big(\tau - t, \tau, U_\eta\big(R(\tau)\big)\big)\big|R(\tau - t)\big) = 0 \quad \text{for all } \tau \in \mathbb{I}.$$

The supremum of all $\eta > 0$ such that there exists a $\hat{\kappa} \in \mathbb{I}$ with

$$\lim_{t\to\infty} d\big(\lambda\big(\tau - t, \tau, U_\eta\big(R(\tau)\big)\big)\big|R(\tau - t)\big) = 0 \quad \text{for all } \tau \leqslant \hat{\kappa}$$

is denoted by $\mathcal{R}_R^-$ and called *past repulsion radius* of R.

(iv) μ is called *past repulsive* if graph μ is a past repeller.

Finally, the notions of future attractivity and repulsivity are introduced.

Definition 3.3 *(Future attractivity and repulsivity).* Let $\mathbb{I} = \mathbb{R}_\kappa^+$, A and R be compact and invariant nonautonomous sets and $\mu : \mathbb{I} \to \mathbb{R}^N$ be a solution of (2.1).

(i) A is called *future attractor* if there exists an $\eta > 0$ with

$$\lim_{t\to\infty} d\big(\lambda\big(\tau + t, \tau, U_\eta\big(A(\tau)\big)\big)\big|A(\tau + t)\big) = 0 \quad \text{for all } \tau \in \mathbb{I}.$$

The supremum of all $\eta > 0$ such that there exists a $\hat{\kappa} \in \mathbb{I}$ with

$$\lim_{t\to\infty} d\big(\lambda\big(\tau + t, \tau, U_\eta\big(A(\tau)\big)\big)\big|A(\tau + t)\big) = 0 \quad \text{for all } \tau \geqslant \hat{\kappa}$$

is denoted by $\mathcal{A}_A^+$ and called *future attraction radius* of A.

(ii) μ is called *future attractive* if graph μ is a future attractor.

(iii) A is called *future repeller* if there exists an $\eta > 0$ with

$$\lim_{t\to\infty} d\big(\lambda\big(\tau, \tau + t, U_\eta\big(R(\tau + t)\big)\big)\big|R(\tau)\big) = 0 \quad \text{for all } \tau \in \mathbb{I}.$$

The supremum of all positive η with this property is denoted by $\mathcal{R}_R^+$ and called *future repulsion radius* of R.

(iv) μ is called *future repulsive* if graph μ is a future repeller.

Having the new definitions at hand, a few historical remarks are in order.

Since the 1990s, the attractivity of nonautonomous sets is intensively discussed. In particular, the notions of *pullback attractor* and *forward attractor* have been introduced (see, e.g., [5,6]). Closely related to pullback attractors are the so-called *random attractors* (see, e.g., [2,8]). Pullback and forward attractors whose attraction rate is uniform with respect to the time are called *uniform attractors* (such attractors are discussed in the monograph [7]). The attractors introduced in this paper are local versions of uniform, pullback and forward attractors, respectively.

Please note that the notion of a past attractor is a special case of the most general form of a pullback attractor (see, e.g., [2, Definition 9.3.1, p. 483]). The so-called *attraction universe* of such a pullback attractor has to be chosen so that it contains a neighborhood of the attractor itself. Another form of a local pullback attractor is introduced in [19,20] (see also the discussion at the end of this section). In the literature, often global pullback attractors are considered. In this case, the attraction universe is supposed to contain all fiber-wise constant and compact nonautonomous sets.

Remark 3.4.

(i) Every all-time attractor (repeller, respectively) is both a past attractor (repeller, respectively) and a future attractor (repeller, respectively).

(ii) Every future attractive solution is uniformly asymptotically stable in the sense of Lyapunov.

(iii) The notions of future attractivity and repulsivity can be derived from the concept of past attractivity and repulsivity via the differential equation under time reversal, given by

$$\dot{x} = -f(-t, x). \tag{2.1^{-1}}$$

A past attractor (repeller, respectively) of (2.1) corresponds to a future repeller (attractor, respectively) of $(2.1)^{-1}$.

(iv) Due to the continuity of the general solution, one can derive the following equivalent characterizations: a compact and invariant nonautonomous set A is a past attractor if and only if there exists an $\eta > 0$ with

$$\lim_{t\to\infty} d\big(\lambda\big(\kappa, \kappa - t, U_\eta\big(A(\kappa - t)\big)\big)\big|A(\kappa)\big) = 0.$$

A compact and invariant nonautonomous set R is a future repeller if and only if there exists an $\eta > 0$ with

$$\lim_{t\to\infty} d\big(\lambda\big(\kappa, \kappa + t, U_\eta\big(R(\kappa + t)\big)\big)\big|R(\kappa)\big) = 0.$$

Such a reduction is not possible for past repellers and future attractors.

(v) The notions of Definitions 3.2 and 3.3 (including the attraction and repulsion radii) do not depend on the choice of $\kappa \in \mathbb{R}$, since the behavior of (2.1) on finite time intervals has no effect on the attractivity or repulsivity of a nonautonomous set. This fact is also due to continuity of the general solution.

(vi) The Hausdorff semi-distance d in Definitions 3.1, 3.2 and 3.3 can be replaced by the Hausdorff distance d_H, which for nonempty sets $A, B \subset X$ is defined by $d_H(A, B) := \max\{d(A|B), d(B|A)\}$.

We are able to give a complete classification of the attractivity and repulsivity of one-dimensional linear equations.

Example 3.5 *(Attractivity and repulsivity of scalar linear equations).* Consider the linear nonautonomous differential equation

$$\dot{x} = a(t)x$$

with a continuous function $a : \mathbb{R} \to \mathbb{R}$. It is easy to see that every invariant and compact nonautonomous set $M \subset \mathbb{R} \times \mathbb{R}$ is an

- all-time attractor if and only if $\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+t} a(s)\,ds = -\infty$,
- all-time repeller if and only if $\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+t} a(s)\,ds = \infty$,
- past attractor if and only if $\lim_{t\to-\infty} \int_t^0 a(s)\,ds = -\infty$,
- past repeller if and only if $\lim_{t\to-\infty} \int_t^0 a(s)\,ds = \infty$,
- future attractor if and only if $\lim_{t\to\infty} \int_0^t a(s)\,ds = -\infty$,
- future repeller if and only if $\lim_{t\to\infty} \int_0^t a(s)\,ds = \infty$.

In all cases, the attraction or repulsion radii are ∞, respectively.

In order to obtain a first example of a nonautonomous bifurcation, we generalize the autonomous example from the introduction.

Example 3.6. Given a real parameter α and a nonautonomous differential equation

$$\dot{x} = \alpha a(t)x + b(t)x^3 = x\big(\alpha a(t) + b(t)x^2\big) \tag{3.1}$$

with continuous functions $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}^+_\kappa$ for some $\kappa > 0$. For simplicity, we define

$$w(\alpha, t) := \sqrt{-\alpha \frac{a(t)}{b(t)}} \quad \text{for all } t \in \mathbb{R} \text{ with } \alpha a(t) < 0.$$

Then, for fixed $t \in \mathbb{R}$ with $\alpha a(t) < 0$, the zero set of the right-hand side is $\{0, \pm w(t)\}$; for all $t \in \mathbb{R}$ with $\alpha a(t) \geqslant 0$, this zero set is the singleton $\{0\}$. An elementary discussion of the sign of the right-hand side of this equation yields that the trivial solution is

- all-time attractive with
$$\inf_{t\in\mathbb{R}} w(\alpha, t) \leqslant \mathcal{A}_0^\pm \leqslant \sup_{t\in\mathbb{R}} w(\alpha, t)$$
if $\inf_{t\in\mathbb{R}} -\alpha \frac{a(t)}{b(t)} > 0$,
- all-time repulsive with $\mathcal{R}_0^\pm = \infty$ if $-\alpha \frac{a(t)}{b(t)} \leqslant 0$ for all $t \in \mathbb{R}$,
- past attractive with
$$\liminf_{t\to-\infty} w(\alpha, t) \leqslant \mathcal{A}_0^- \leqslant \limsup_{t\to-\infty} w(\alpha, t)$$
if $\liminf_{t\to-\infty} -\alpha \frac{a(t)}{b(t)} > 0$,

- past repulsive with $\mathcal{R}_0^- = \infty$ if $\limsup_{t\to-\infty} -\alpha\frac{a(t)}{b(t)} \leqslant 0$,
- future attractive with

$$\liminf_{t\to\infty} w(\alpha,t) \leqslant \mathcal{A}_0^+ \leqslant \limsup_{t\to\infty} w(\alpha,t)$$

  if $\liminf_{t\to\infty} -\alpha\frac{a(t)}{b(t)} > 0$,
- future repulsive with $\mathcal{R}_0^+ = \infty$ if $\limsup_{t\to\infty} -\alpha\frac{a(t)}{b(t)} \leqslant 0$.

Since $\lim_{\alpha\to 0} w(\alpha,t) = 0$, this means that (3.1) admits a

- supercritical all-time bifurcation at $\alpha = 0$ if

$$\inf_{t\in\mathbb{R}} -\frac{a(t)}{b(t)} > 0 \quad \text{and} \quad \sup_{t\in\mathbb{R}} -\frac{a(t)}{b(t)} < \infty,$$

- subcritical all-time bifurcation at $\alpha = 0$ if

$$\inf_{t\in\mathbb{R}} \frac{a(t)}{b(t)} > 0 \quad \text{and} \quad \sup_{t\in\mathbb{R}} \frac{a(t)}{b(t)} < \infty,$$

- supercritical past bifurcation at $\alpha = 0$ if

$$\liminf_{t\to-\infty} -\frac{a(t)}{b(t)} > 0 \quad \text{and} \quad \limsup_{t\to-\infty} -\frac{a(t)}{b(t)} < \infty,$$

- subcritical past bifurcation at $\alpha = 0$ if

$$\liminf_{t\to-\infty} \frac{a(t)}{b(t)} > 0 \quad \text{and} \quad \limsup_{t\to-\infty} \frac{a(t)}{b(t)} < \infty,$$

- supercritical future bifurcation at $\alpha = 0$ if

$$\liminf_{t\to\infty} -\frac{a(t)}{b(t)} > 0 \quad \text{and} \quad \limsup_{t\to\infty} -\frac{a(t)}{b(t)} < \infty,$$

- subcritical future bifurcation at $\alpha = 0$ if

$$\liminf_{t\to\infty} \frac{a(t)}{b(t)} > 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{a(t)}{b(t)} < \infty.$$

A further generalization of this differential equation is discussed in Theorem 6.1. It is also shown there that this example also admits attractor and repeller transitions.

We conclude this section by a discussion of the relationship of our concept to the ideas used in [17,19,20].

In [17], a nonautonomous bifurcation is understood as a (continuous or discontinuous) transition from a nontrivial (global) pullback attractor to a trivial pullback attractor. A transition in our sense, however, concerns *local* attractors and repellers. Moreover, in many cases, a *continuous* transition in the sense of [17] implies the bifurcation of the domain of repulsion of a past repulsive solution in the interior of the pullback attractor.

In [19,20], several notions of stability, instability, attractivity and repulsivity are introduced for nonautonomous differential equations. As in our case, these notions reflect the *local* behavior of the system, but—although there are similarities—they do not coincide with the definitions used in this paper. Using these special definitions, in [19,20], a nonautonomous bifurcation is understood as a merging process of two distinct solutions with different stability behavior. In dimension one, the three authors found conditions concerning the Taylor coefficients of the right-hand side which guarantee the existence of such bifurcations. These conditions are formulated in a natural way using explicitly solvable models. In Section 5, we compare this with our results in case of the transcritical bifurcation.

4. Linearized attractivity and repulsivity

Let x^* be an equilibrium of an autonomous differential equation $\dot{x} = g(x)$ with a C^1-function $g: D \to \mathbb{R}^N$, $D \subset \mathbb{R}^N$ an open set. It is well known that x^* is exponentially asymptotically stable if all eigenvalues of $Dg(x^*)$ have negative real part, and x^* is repulsive if all eigenvalues of this derivative have positive real part. In this section, we want to derive similar criteria which correspond to the notions of attractivity and repulsivity introduced in the previous section.

For similar considerations in the autonomous case, we refer to [12, Section III.6]; for the nonautonomous situation, see also [3, Lemma 3.4, p. 70].

Theorem 4.1 *(Linearized attractivity and repulsivity). Consider an unbounded interval $\mathbb{I}$ of the form $\mathbb{R}$, $\mathbb{R}_\kappa^-$ or $\mathbb{R}_\kappa^+$, respectively, and let*

$$\dot{x} = A(t)x + F(t, x) \tag{4.1}$$

be a nonautonomous differential equation with continuous functions $A: \mathbb{I} \to \mathbb{R}^{N \times N}$ and $F: \mathbb{I} \times U \to \mathbb{R}^N$, $U \subset \mathbb{R}^N$ a neighborhood of 0, such that $F(t, 0) = 0$ for all $t \in \mathbb{I}$. Let λ denote the general solution of (4.1) and $\Phi: \mathbb{I} \times \mathbb{I} \to \mathbb{R}^{N \times N}$ denote the transition operator of the linearized equation $\dot{x} = A(t)x$. Then the following statements are fulfilled:

(i) *In case there exist $\beta < 0$, $K \geqslant 1$ and $\delta > 0$ such that*

$$\|\Phi(t, s)\| \leqslant K e^{\beta(t-s)} \quad \textit{for all } t \geqslant s$$

and

$$\|F(t, x)\| \leqslant \frac{-\beta}{2K} \|x\| \quad \textit{for all } t \in \mathbb{I} \textit{ and } x \in U_\delta(0), \tag{4.2}$$

we have

$$d\big(\lambda\big(t, \tau, U_{\delta/K}(0)\big) \big| \{0\}\big) \leqslant \delta e^{\frac{\beta}{2}(t-\tau)} \quad \textit{for all } \tau, t \in \mathbb{I} \textit{ with } \tau \leqslant t,$$

i.e., the trivial solution of (4.1) *is all-time* (*past, future, respectively*) *attractive.*

(ii) *In case there exist* $\beta > 0$, $K \geqslant 1$ *and* $\delta > 0$ *such that*

$$\|\Phi(t,s)\| \leqslant K e^{\beta(t-s)} \quad \text{for all } t \leqslant s$$

and

$$\|F(t,x)\| \leqslant \frac{\beta}{2K}\|x\| \quad \text{for all } t \in \mathbb{I} \text{ and } x \in U_\delta(0),$$

we have

$$d\big(\lambda\big(t,\tau,U_{\delta/K}(0)\big)\big|\{0\}\big) \leqslant \delta e^{\frac{\beta}{2}(t-\tau)} \quad \text{for all } \tau, t \in \mathbb{I} \text{ with } t \leqslant \tau,$$

i.e., the trivial solution of (4.1) *is all-time* (*past, future, respectively*) *repulsive.*

Proof. We only prove (i), since (ii) can be shown analogously. Given $\tau \in \mathbb{I}$ and $\xi \in U_\delta(0)$, we now prove an estimate for the general solution under the additional assumption

$$\lambda(t,\tau,\xi) \in U_\delta(0) \quad \text{for all } t \geqslant \tau. \tag{4.3}$$

The solution $\lambda(\cdot,\tau,\xi)$ of (4.1) is also a solution of inhomogeneous linear differential equation

$$\dot{x} = A(t)x + F\big(t, \lambda(t,\tau,\xi)\big).$$

Thus, the variation of constants formula implies

$$\lambda(t,\tau,\xi) = \Phi(t,\tau)\xi + \int_\tau^t \Phi(t,s)F\big(s,\lambda(s,\tau,\xi)\big)\,ds \quad \text{for all } t \geqslant \tau,$$

and hence,

$$\begin{aligned}\|\lambda(t,\tau,\xi)\| &\leqslant \|\Phi(t,\tau)\|\|\xi\| + \int_\tau^t \|\Phi(t,s)\|\big\|F\big(s,\lambda(s,\tau,\xi)\big)\big\|\,ds \\ &\overset{(4.2)}{\leqslant} K e^{\beta(t-\tau)}\|\xi\| + \int_\tau^t K e^{\beta(t-s)}\frac{-\beta}{2K}\|\lambda(s,\tau,\xi)\|\,ds \quad \text{for all } t \geqslant \tau\end{aligned}$$

is fulfilled. This implies

$$e^{-\beta t}\|\lambda(t,\tau,\xi)\| \leqslant K e^{-\beta\tau}\|\xi\| + \frac{-\beta}{2}\int_\tau^t e^{-\beta s}\|\lambda(s,\tau,\xi)\|\,ds \quad \text{for all } t \geqslant \tau.$$

Hence, Gronwall's inequality (cf. [1, Theorem 4.1.7, p. 242]) yields the estimate

$$\|\lambda(t,\tau,\xi)\| \leqslant K e^{\frac{\beta}{2}(t-\tau)}\|\xi\| \quad \text{for all } t \geqslant \tau. \tag{4.4}$$

We define $\eta := \frac{\delta}{K}$. Since $\frac{\beta}{2} < 0$, the assumption (4.3) is fulfilled for all $\tau \in \mathbb{I}$ and $\xi \in U_\eta(0)$, and thus, (4.4) holds for such τ and ξ. This implies

$$d\big(\lambda\big(t, \tau, U_\eta(0)\big)\big|\{0\}\big) \leqslant K\eta e^{\frac{\beta}{2}(t-\tau)} \quad \text{for all } \tau, t \in \mathbb{I} \text{ with } \tau \leqslant t.$$

From this inequality, the required conditions for the all-time (past, future, respectively) attractivity are easily obtained. □

5. Nonautonomous transcritical bifurcation

This section is devoted to a nonautonomous generalization of the classical transcritical bifurcation.

Theorem 5.1 *(Nonautonomous transcritical bifurcation). Let $x_- < 0 < x_+$ and $\alpha_- < \alpha_+$ be in $\overline{\mathbb{R}}$ and $\mathbb{I}$ be an unbounded interval of the form $\mathbb{R}$, $\mathbb{R}_\kappa^-$ or $\mathbb{R}_\kappa^+$, respectively, and consider the nonautonomous differential equation*

$$\dot{x} = a(t, \alpha)x + b(t, \alpha)x^2 + r(t, x, \alpha) \tag{5.1}$$

with continuous functions $a : \mathbb{I} \times (\alpha_-, \alpha_+) \to \mathbb{R}$, $b : \mathbb{I} \times (\alpha_-, \alpha_+) \to \mathbb{R}$ and $r : \mathbb{I} \times (x_-, x_+) \times (\alpha_-, \alpha_+) \to \mathbb{R}$ fulfilling $r(\cdot, 0, \cdot) \equiv 0$. Let $\Phi_\alpha : \mathbb{I} \times \mathbb{I} \to \mathbb{R}$ denote the transition operator of the linearized equation $\dot{x} = a(t, \alpha)x$, and assume, there exists an $\alpha_0 \in (\alpha_-, \alpha_+)$ such that the following hypotheses hold:

(i) Hypothesis on linear part. *There exist two functions $\beta_1, \beta_2 : (\alpha_-, \alpha_+) \to \mathbb{R}$ which are either both monotone increasing or both monotone decreasing and $K \geqslant 1$ such that $\lim_{\alpha \to \alpha_0} \beta_1(\alpha) = \lim_{\alpha \to \alpha_0} \beta_2(\alpha) = 0$ and*

$$\Phi_\alpha(t, s) \leqslant K e^{\beta_1(\alpha)(t-s)} \quad \textit{for all } \alpha \in (\alpha_-, \alpha_+) \textit{ and } t, s \in \mathbb{I} \textit{ with } t \geqslant s,$$
$$\Phi_\alpha(t, s) \leqslant K e^{\beta_2(\alpha)(t-s)} \quad \textit{for all } \alpha \in (\alpha_-, \alpha_+) \textit{ and } t, s \in \mathbb{I} \textit{ with } t \leqslant s.$$

(ii) Hypothesis on nonlinearity. *The quadratic term either fulfills*

$$0 < \liminf_{\alpha \to \alpha_0} \inf_{t \in \mathbb{I}} b(t, \alpha) \leqslant \limsup_{\alpha \to \alpha_0} \sup_{t \in \mathbb{I}} b(t, \alpha) < \infty \tag{5.2}$$

or

$$-\infty < \liminf_{\alpha \to \alpha_0} \inf_{t \in \mathbb{I}} b(t, \alpha) \leqslant \limsup_{\alpha \to \alpha_0} \sup_{t \in \mathbb{I}} b(t, \alpha) < 0, \tag{5.3}$$

and the remainder satisfies

$$\lim_{x \to 0} \sup_{\alpha \in (\alpha_0 - |x|, \alpha_0 + |x|)} \sup_{t \in \mathbb{I}} \frac{|r(t, x, \alpha)|}{|x|^2} = 0 \tag{5.4}$$

Table 1

Trivial solution	$\alpha \in (\hat{\alpha}_-, \alpha_0)$	$\alpha \in (\alpha_0, \hat{\alpha}_+)$
β_1, β_2 incr.	attractive, $\lim_{\alpha \nearrow \alpha_0} \mathcal{A}_0^\alpha = 0$	repulsive, $\lim_{\alpha \searrow \alpha_0} \mathcal{R}_0^\alpha = 0$
β_1, β_2 decr.	repulsive, $\lim_{\alpha \nearrow \alpha_0} \mathcal{R}_0^\alpha = 0$	attractive, $\lim_{\alpha \searrow \alpha_0} \mathcal{A}_0^\alpha = 0$

and

$$\limsup_{\alpha \to \alpha_0} \limsup_{x \to 0} \sup_{t \in \mathbb{I}} \frac{2K|r(t,x,\alpha)|}{|x| \max\{-\beta_1(\alpha), \beta_2(\alpha)\}} < 1. \tag{5.5}$$

Then there exist $\hat{\alpha}_- < 0 < \hat{\alpha}_+$ such that the following statements are fulfilled:

(i) *In case the functions β_1 and β_2 are monotone increasing, the trivial solution is all-time* (*past, future, respectively*) *attractive for $\alpha \in (\hat{\alpha}_-, \alpha_0)$ and all-time* (*past, future, respectively*) *repulsive for $\alpha \in (\alpha_0, \hat{\alpha}_+)$. The differential equation* (5.1) *admits an all-time* (*past, future, respectively*) *bifurcation, since the corresponding radii of all-time* (*past, future, respectively*) *attraction and repulsion satisfy*

$$\lim_{\alpha \nearrow \alpha_0} \mathcal{A}_0^\alpha = 0 \quad \textit{and} \quad \lim_{\alpha \searrow \alpha_0} \mathcal{R}_0^\alpha = 0.$$

(ii) *In case the functions β_1 and β_2 are monotone decreasing, the trivial solution is all-time* (*past, future, respectively*) *repulsive for $\alpha \in (\hat{\alpha}_-, \alpha_0)$ and all-time* (*past, future, respectively*) *attractive for $\alpha \in (\alpha_0, \hat{\alpha}_+)$. The differential equation* (5.1) *admits an all-time* (*past, future, respectively*) *bifurcation, since the corresponding radii of all-time* (*past, future, respectively*) *repulsion and attraction satisfy*

$$\lim_{\alpha \nearrow \alpha_0} \mathcal{R}_0^\alpha = 0 \quad \textit{and} \quad \lim_{\alpha \searrow \alpha_0} \mathcal{A}_0^\alpha = 0.$$

The two cases of the preceding theorem are gathered in Table 1.

Proof. First of all, we assume w.l.o.g. that $K > 1$. Let λ_α denote the general solution of (5.1). We will only prove assertion (i), since the proof of (ii) is similar. Hence, the functions β_1 and β_2 are monotone increasing. W.l.o.g., we only treat the case (5.2). We choose $\hat{\alpha}_- < \alpha_0 < \hat{\alpha}_+$ such that

$$0 < \inf_{\alpha \in (\hat{\alpha}_-, \hat{\alpha}_+),\, t \in \mathbb{I}} b(t,\alpha) \leqslant \sup_{\alpha \in (\hat{\alpha}_-, \hat{\alpha}_+),\, t \in \mathbb{I}} b(t,\alpha) < \infty \tag{5.6}$$

(cf. (5.2)) and

$$\limsup_{x \to 0} \sup_{t \in \mathbb{I}} \frac{|r(t,x,\alpha)|}{|x|} < \frac{-\min\{\beta_1(\alpha), -\beta_2(\alpha)\}}{2K} \quad \text{for all } \alpha \in (\hat{\alpha}_-, \hat{\alpha}_+)$$

(cf. (5.5)). Because of these two relations, Theorem 4.1 can be applied, and the attractivity and repulsivity of the trivial solutions as stated in the theorem follows. Assume to the contrary that

$$\eta := \limsup_{\alpha \nearrow \alpha_0} \mathcal{A}_0^\alpha > 0$$

holds. Due to (5.6) and (5.4), there exist $\tilde{\alpha}_- \in (\hat{\alpha}_-, \alpha_0)$, $\xi \in (0, \eta)$ and $L \in (0, \frac{\xi}{4K})$ with

$$b(t,\alpha)x^2 + r(t,x,\alpha) > L \quad \text{for all } t \in \mathbb{I},\ \alpha \in (\tilde{\alpha}_-, \alpha_0) \text{ and } x \in \left[\frac{\xi}{2K^2}, \xi\right]. \tag{5.7}$$

We fix $\hat{\alpha} \in (\tilde{\alpha}_-, \alpha_0)$ such that $\mathcal{A}_0^{\hat{\alpha}} > \xi$ and $\beta_2(\hat{\alpha}) \geqslant \beta := -\frac{2KL}{\xi} > -\frac{1}{2}$. For arbitrary $\tau \in \mathbb{I}$, the solution $\mu_\tau(\cdot) := \lambda_{\hat{\alpha}}(\cdot, \tau, \xi)$ of (5.1) is also a solution of the inhomogeneous linear differential equation

$$\dot{x} = a(t,\hat{\alpha})x + b(t,\hat{\alpha})\big(\mu_\tau(t)\big)^2 + r\big(t, \mu_\tau(t), \hat{\alpha}\big). \tag{5.8}$$

Since $\mathcal{A}_0^{\hat{\alpha}} > \xi = \mu_t(t)$ for all $t \in \mathbb{I}$, there exist $\tau, \tau_2 \in \mathbb{I}$, $\tau < \tau_2$, with $\mu_\tau(\tau_2) \leqslant \frac{\xi}{2K^2}$. We choose τ_2 minimal with this property, i.e., $\mu_\tau(t) \geqslant \frac{\xi}{2K^2}$ for all $t \in [\tau, \tau_2]$. Furthermore, we choose $\tau_1 \in [\tau, \tau_2]$ such that

$$\mu_\tau(\tau_1) = \frac{\xi}{2K} \quad \text{and} \quad \mu_\tau(t) \in \left[\frac{\xi}{2K^2}, \xi\right] \quad \text{for all } t \in [\tau_1, \tau_2].$$

Therefore, and due to (5.7) and the variation of constants formula, applied to (5.8), the relation

$$\begin{aligned}
\mu_\tau(\tau_2) &= \Phi_{\hat{\alpha}}(\tau_2, \tau_1)\mu_\tau(\tau_1) + \int_{\tau_1}^{\tau_2} \Phi_{\hat{\alpha}}(\tau_2, t)\Big(b(t,\hat{\alpha})\big(\mu_\tau(t)\big)^2 + r\big(t, \mu_\tau(t), \hat{\alpha}\big)\Big)\, dt \\
&> \frac{\xi}{2K^2} e^{\beta(\tau_2 - \tau_1)} + \frac{L}{K} \int_{\tau_1}^{\tau_2} e^{\beta(\tau_2 - t)}\, dt \\
&= e^{\beta(\tau_2-\tau_1)} \underbrace{\left(\frac{\xi}{2K^2} + \frac{L}{K\beta}\right)}_{=0} - \frac{L}{K\beta} = \frac{\xi}{2K^2}
\end{aligned}$$

holds ($K > 1$ implies $\tau_1 < \tau_2$). This is a contradiction and proves $\lim_{\alpha \nearrow \alpha_0} \mathcal{A}_0^\alpha = 0$. Analogously, one can show $\lim_{\alpha \searrow \alpha_0} \mathcal{R}_0^\alpha = 0$ and treat the case (5.3). □

Remark 5.2.

(i) In the limit $\alpha \to \alpha_0$, the attractivity or repulsivity of the trivial solution is only lost in one direction. For instance, in case the functions β_1, β_2 are monotone increasing and (5.2) is satisfied, there exists a $\gamma < 0$ such that $(\gamma, 0]$ is attracted by the trivial solution of (5.4) for $\alpha \in (\hat{\alpha}_-, \alpha_0)$ in the sense of past, future or all-time attractivity, respectively. Since the loss of stability is only one-sided, we have no transition phenomena connected with the transcritical bifurcation, contrary to the pitchfork bifurcation in the next section.

(ii) The hypothesis on the linear part implies that the all-time (past, future, respectively) dichotomy spectrum of the linearization $\dot{x} = a(t,\alpha)x$ converges to $\{0\}$ in Hausdorff distance in the limit $\alpha \to \alpha_0$ (see [21]).

(iii) Condition (5.5) is only used to obtain the attractivity or repulsivity of the trivial solution by applying Theorem 4.1. Alternatively, one can directly postulate that the trivial solution changes their stability at the parameter value α_0 from, say, attractivity to repulsivity.
(iv) Please note that the above bifurcation result is essentially the combination of two scenarios which are independent of each other. This means that it is possible to consider (5.1) only for $\alpha > \alpha_0$ or $\alpha < \alpha_0$, respectively, in order to obtain the results which apply for these parameter values.

The following example shows that Theorem 5.1 is indeed a nonautonomous generalization of the well-known autonomous result.

Example 5.3. Let $x_- < 0 < x_+$ and $\alpha_- < 0 < \alpha_+$ be in $\overline{\mathbb{R}}$, and consider the autonomous differential equation

$$\dot{x} = f(x, \alpha), \tag{5.9}$$

where the C^4-function $f : (x_-, x_+) \times (\alpha_-, \alpha_+) \to \mathbb{R}$ satisfies the following assumptions:

(i) $f(0, \alpha) = 0$ for all $\alpha \in (\alpha_-, \alpha_+)$,
(ii) $D_1 f(0, 0) = 0$,
(iii) $D_1 D_2 f(0, 0) \neq 0$,
(iv) $D_1^2 f(0, 0) \neq 0$.

Please note that (i) implies $D_2^n f(0, \alpha) = 0$ for all $\alpha \in (\alpha_-, \alpha_+)$ and $n \in \mathbb{N}$. Then (5.9) admits an autonomous transcritical bifurcation (see, e.g., [22, p. 265]), i.e., there exist a neighborhood $U \times V$ of $(0, 0)$ in $\mathbb{R}^2$ and a C^1-function $h : U \to V$ with $h(0) = 0$ and

$$f\bigl(x, h(x)\bigr) = 0 \quad \text{for all } x \in U.$$

Except the trivial equilibria and the equilibria described by h, there are no other equilibria in $U \times V$. Now, we will show that this example fulfills the hypotheses of Theorem 5.1. Thereto, we write the second order Taylor expansion of f (see, e.g., [18, p. 349]):

$$f(x, \alpha) = \underbrace{D_1 D_2 f(0, 0)\alpha}_{=:\bar{a}(\alpha)} x + \underbrace{\frac{1}{2} D_1^2 f(0, 0)}_{=:\bar{b}(\alpha)} x^2 + r(x, \alpha),$$

where

$$r(x, \alpha) = \int_0^1 \frac{(1-t)^2}{2} \bigl(D_1^3 f(tx, t\alpha)x^3 + 3D_1^2 D_2 f(tx, t\alpha)x^2\alpha$$
$$+ 3D_1 D_2^2 f(tx, t\alpha)x\alpha^2 + D_2^3 f(tx, t\alpha)\alpha^3\bigr)\, dt.$$

Obviously, the hypothesis on the linear part are fulfilled (with $\beta_1(\alpha) := \beta_2(\alpha) := \bar{a}(\alpha)$ and $K := 1$), and (5.2) or (5.3) holds, since the above defined function $\bar{b}$ is constant. Furthermore, the representation for the remainder implies that

$$\lim_{x \to 0} \sup_{\alpha \in (-|x|,|x|)} \frac{|r(x,\alpha)|}{|x|^2} = 0$$

and

$$\limsup_{x \to 0} \frac{|r(x,\alpha)|}{|x|} \leqslant \alpha^2 \int_0^1 \frac{(1-t)^2}{2} \left(\left| 3 D_1 D_2^2 f(0, t\alpha) \right| + t \left| D_1 D_2^3 f(0, t\alpha)\alpha \right| \right) dt.$$

This means that (5.5) holds, since $\max\{-\beta_1(\alpha), \beta_2(\alpha)\}$ depends linearly in α. Hence, all hypotheses of Theorem 5.1 are fulfilled, and thus, this example shows that Theorem 5.1 is a proper generalization of the well-known autonomous transcritical bifurcation pattern.

We close this section by a short comparison of the results from this section and [20, Section 5]. Since we have outlined the relationship of the basic concepts of bifurcation at the end of Section 3 already, we will here only compare the conditions imposed on the equations which are sufficient for a transcritical bifurcation.

First, the hypothesis on the linear part of Theorem 5.1 is more restrictive than in [20, Theorem 7], since here it is not allowed that the function a is unbounded in the limit $t \to \pm\infty$. In [20, Theorem 7], however, this is possible whenever the second Taylor coefficient is also unbounded in the limit $t \to \pm\infty$ with the same growth rate. Note that in this case the dichotomy spectrum of the linearization does not converge to $\{0\}$ as described in Remark 5.2(ii).

On the other side, however, the conditions on the Taylor coefficients in [20, Theorem 7] are not formulated in the limit $\alpha \to \alpha_0$ as in this paper. Hence, to apply the results, one has to restrict the attention to a sufficiently small neighborhood in which higher order terms are negligible.

6. Nonautonomous pitchfork bifurcation

This section is devoted to a nonautonomous generalization of the classical pitchfork bifurcation. In addition to the bifurcations of attraction and repulsion areas, also transition phenomena are obtained here.

Theorem 6.1 *(Nonautonomous pitchfork bifurcation). Let $x_- < 0 < x_+$ and $\alpha_- < \alpha_+$ be in $\overline{\mathbb{R}}$ and $\mathbb{I}$ be an unbounded interval of the form $\mathbb{R}$, $\mathbb{R}_\kappa^-$ and $\mathbb{R}_\kappa^+$, respectively, and consider the nonautonomous differential equation*

$$\dot{x} = a(t,\alpha)x + b(t,\alpha)x^3 + r(t,x,\alpha) \tag{6.1}$$

with continuous functions $a : \mathbb{I} \times (\alpha_-, \alpha_+) \to \mathbb{R}$, $b : \mathbb{I} \times (\alpha_-, \alpha_+) \to \mathbb{R}$ and $r : \mathbb{I} \times (x_-, x_+) \times (\alpha_-, \alpha_+) \to \mathbb{R}$ fulfilling $r(\cdot, 0, \cdot) \equiv 0$. Let $\Phi_\alpha : \mathbb{I} \times \mathbb{I} \to \mathbb{R}$ be the transition operator of the linearized equation $\dot{x} = a(t,\alpha)x$, and assume, there exists an $\alpha_0 \in (\alpha_-, \alpha_+)$ such that the following hypotheses hold:

(i) Hypothesis on linear part. *There exist two functions* $\beta_1, \beta_2 : (\alpha_-, \alpha_+) \to \mathbb{R}$ *which are either both monotone increasing or both monotone decreasing and* $K \geqslant 1$ *such that* $\lim_{\alpha\to\alpha_0} \beta_1(\alpha) = \lim_{\alpha\to\alpha_0} \beta_2(\alpha) = 0$ *and*

$$\Phi_\alpha(t,s) \leqslant K e^{\beta_1(\alpha)(t-s)} \quad \textit{for all } \alpha \in (\alpha_-, \alpha_+) \textit{ and } t, s \in \mathbb{I} \textit{ with } t \geqslant s,$$
$$\Phi_\alpha(t,s) \leqslant K e^{\beta_2(\alpha)(t-s)} \quad \textit{for all } \alpha \in (\alpha_-, \alpha_+) \textit{ and } t, s \in \mathbb{I} \textit{ with } t \leqslant s.$$

(ii) Hypothesis on nonlinearity. *The cubic term either fulfills*

$$0 < \liminf_{\alpha\to\alpha_0} \inf_{t\in\mathbb{I}} b(t,\alpha) \leqslant \limsup_{\alpha\to\alpha_0} \sup_{t\in\mathbb{I}} b(t,\alpha) < \infty \tag{6.2}$$

or

$$-\infty < \liminf_{\alpha\to\alpha_0} \inf_{t\in\mathbb{I}} b(t,\alpha) \leqslant \limsup_{\alpha\to\alpha_0} \sup_{t\in\mathbb{I}} b(t,\alpha) < 0, \tag{6.3}$$

and the remainder satisfies

$$\lim_{x\to 0} \sup_{\alpha\in(\alpha_0-x^2,\alpha_0+x^2)} \sup_{t\in\mathbb{I}} \frac{|r(t,x,\alpha)|}{|x|^3} = 0 \tag{6.4}$$

and

$$\limsup_{\alpha\to\alpha_0} \limsup_{x\to 0} \sup_{t\in\mathbb{I}} \frac{2K|r(t,x,\alpha)|}{|x| \max\{-\beta_1(\alpha), \beta_2(\alpha)\}} < 1.$$

Then there exist $\hat\alpha_- < 0 < \hat\alpha_+$ *such that the following statements are fulfilled*:

(i) *In case* (6.2) *and the functions* β_1 *and* β_2 *are monotone increasing, the trivial solution is all-time* (*past, future, respectively*) *attractive for* $\alpha \in (\hat\alpha_-, \alpha_0)$ *and all-time* (*past, future, respectively*) *repulsive for* $\alpha \in (\alpha_0, \hat\alpha_+)$. *The differential equation* (6.1) *admits an all-time* (*past, future, respectively*) *bifurcation, since the corresponding radii of all-time* (*past, future, respectively*) *attraction satisfy*

$$\lim_{\alpha\nearrow\alpha_0} \mathcal{A}_0^\alpha = 0.$$

If, in addition, $\mathbb{I} = \mathbb{R}_\kappa^+$, *then, for* $\alpha \in (\hat\alpha_-, \alpha_0)$, *there exists a nontrivial future repeller* $R_\alpha \subset \mathbb{I} \times \mathbb{R}$, *and we have a future repeller transition, since*

$$\lim_{\alpha\nearrow\alpha_0} d_H\big(R_\alpha(t), \{0\}\big) = 0 \quad \textit{for all } t \in \mathbb{I}.$$

(ii) *In case* (6.3) *and the functions* β_1 *and* β_2 *are monotone increasing, the trivial solution is all-time* (*past, future, respectively*) *attractive for* $\alpha \in (\hat\alpha_-, \alpha_0)$ *and all-time* (*past, future, respectively*) *repulsive for* $\alpha \in (\alpha_0, \hat\alpha_+)$. *The differential equation* (6.1) *admits an all-time* *(past, future, respectively) bifurcation, since the corresponding radii of all-time (past, future, respectively) repulsion satisfy*

$$\lim_{\alpha \searrow \alpha_0} \mathcal{R}_0^\alpha = 0.$$

If, in addition, $\mathbb{I} = \mathbb{R}_\kappa^-$*, then, for* $\alpha \in (\alpha_0, \hat{\alpha}_+)$*, there exists a nontrivial past attractor* $A_\alpha \subset \mathbb{I} \times \mathbb{R}$*, and we have a past attractor transition, since*

$$\lim_{\alpha \searrow \alpha_0} d_H\big(A_\alpha(t), \{0\}\big) = 0 \quad \textit{for all } t \in \mathbb{I}.$$

(iii) *In case* (6.2) *and the functions* β_1 *and* β_2 *are monotone decreasing, the trivial solution is all-time (past, future, respectively) repulsive for* $\alpha \in (\hat{\alpha}_-, \alpha_0)$ *and all-time (past, future, respectively) attractive for* $\alpha \in (\alpha_0, \hat{\alpha}_+)$*. The differential equation* (6.1) *admits an all-time (past, future, respectively) bifurcation, since the corresponding radii of all-time (past, future, respectively) attraction satisfy*

$$\lim_{\alpha \searrow \alpha_0} \mathcal{A}_0^\alpha = 0.$$

If, in addition, $\mathbb{I} = \mathbb{R}^+$*, then, for* $\alpha \in (\alpha_0, \hat{\alpha}_+)$*, there exists a nontrivial future repeller* $R_\alpha \subset \mathbb{I} \times \mathbb{R}$*, and we have a future repeller transition, since*

$$\lim_{\alpha \searrow \alpha_0} d_H\big(R_\alpha(t), \{0\}\big) = 0 \quad \textit{for all } t \in \mathbb{I}.$$

(iv) *In case* (6.3) *and the functions* β_1 *and* β_2 *are monotone decreasing, the trivial solution is all-time (past, future, respectively) repulsive for* $\alpha \in (\hat{\alpha}_-, \alpha_0)$ *and all-time (past, future, respectively) attractive for* $\alpha \in (\alpha_0, \hat{\alpha}_+)$*. The differential equation* (6.1) *admits an all-time (past, future, respectively) bifurcation, since the corresponding radii of all-time (past, future, respectively) repulsion satisfy*

$$\lim_{\alpha \nearrow \alpha_0} \mathcal{R}_0^\alpha = 0.$$

If, in addition, $\mathbb{I} = \mathbb{R}^-$*, then, for* $\alpha \in (\hat{\alpha}_-, \alpha_0)$*, there exists a nontrivial past attractor* $A_\alpha \subset \mathbb{I} \times \mathbb{R}$*, and we have a past attractor transition, since*

$$\lim_{\alpha \nearrow \alpha_0} d_H\big(A_\alpha(t), \{0\}\big) = 0 \quad \textit{for all } t \in \mathbb{I}.$$

The four cases of the preceding theorem are gathered in Table 2.

Proof. The first part of this theorem concerning the bifurcation of the attraction or repulsion radii, respectively, can be proved using the same methods as in the proof of Theorem 5.1. We write $\tilde{\alpha}_-$ and $\tilde{\alpha}_+$ for the constants $\hat{\alpha}_-$ and $\hat{\alpha}_+$ used in this proof. For the proof of the attractor and repeller transitions, w.l.o.g., we only consider the case (ii), i.e., $\mathbb{I} = \mathbb{R}_\kappa^-$, condition (6.3) holds

Table 2

Trivial solution	$\alpha \in (\hat{\alpha}_-, \alpha_0)$	$\alpha \in (\alpha_0, \hat{\alpha}_+)$
β_1, β_2 incr., (6.2)	attractive, $\lim_{\alpha \nearrow \alpha_0} \mathcal{A}_0^\alpha = 0$	repulsive
β_1, β_2 incr. (6.3)	attractive	repulsive, $\lim_{\alpha \searrow \alpha_0} \mathcal{R}_0^\alpha = 0$
β_1, β_2 decr., (6.2)	repulsive	attractive, $\lim_{\alpha \searrow \alpha_0} \mathcal{A}_0^\alpha = 0$
β_1, β_2 decr. (6.3)	repulsive, $\lim_{\alpha \nearrow \alpha_0} \mathcal{R}_0^\alpha = 0$	attractive

and the functions β_1 and β_2 are monotone increasing. We denote the general solution of (6.1) by λ_α and define

$$b_+ := \frac{1}{2} \sup_{t \in \mathbb{I},\, \alpha \in (\tilde{\alpha}_-, \tilde{\alpha}_+)} b(t, \alpha) < 0.$$

Due to (6.4), there exists a $\rho > 0$ such that

$$\left|r(t, x, \alpha)\right| \leqslant -b_+|x|^3 \quad \text{for all } x \in [-\rho, \rho],\ \alpha \in \left(\alpha_0 - x^2, \alpha_0 + x^2\right) \text{ and } t \in \mathbb{I}.$$

The remaining proof is divided into two steps.

Step 1. For given $x_1, x_2, x_3 \leqslant \rho$ such that $0 < x_1 \leqslant x_2 \leqslant \frac{x_3}{2K}$, there exists a uniquely determined constant

$$\alpha^* = \alpha^*(x_1, x_2, x_3) \in \left(\alpha_0, \min\left\{\alpha_0 + x_1^2, \tilde{\alpha}_+\right\}\right]$$

with the following properties:

- for all $\tau \leqslant t \leqslant \kappa$ and $\alpha \in (\alpha_0, \alpha^*)$, we have $\lambda_\alpha(t, \tau, [-x_2, x_2]) \subset (-x_3, x_3)$,
- there exists a constant $T^* > 0$ such that for all $\alpha \in (\alpha_0, \alpha^*)$ and $\tau \leqslant \kappa - T^*$, there exist $t_+, t_- \in [0, T^*]$ with

$$\lambda_\alpha(\tau + t_+, \tau, x_2) = x_1 \quad \text{and} \quad \lambda_\alpha(\tau + t_-, \tau, -x_2) = -x_1,$$

- α^* is chosen maximal, i.e., for all bigger α^*, one of the two above properties is violated.

We will only prove the existence of a constant α^* such that

(a) for all $\tau \leqslant t \leqslant \kappa$ and $\alpha \in (\alpha_0, \alpha^*)$, we have $\lambda_\alpha(t, \tau, x_2) \leqslant x_3$,
(b) there exists a constant $T^* > 0$ such that for all $\alpha \in (\alpha_0, \alpha^*)$ and $\tau \leqslant \kappa - T^*$, there exists a $t_+ \in [0, T^*]$ with $\lambda_\alpha(\tau + t_+, \tau, x_2) = x_1$,

since the extension to the above assertion follows similarly and by taking the supremum of all such α^*. We first note that for arbitrary $\tau \in \mathbb{I}$, the solution $\mu_\tau(\cdot) := \lambda_\alpha(\cdot, \tau, x_2)$ of (6.1) is also a solution of the inhomogeneous linear differential equation

$$\dot{x} = a(t, \alpha)x + b(t, \alpha)\left(\mu_\tau(t)\right)^3 + r\left(t, \mu_\tau(t), \alpha\right). \tag{6.5}$$

Concerning the expression

$$s(\alpha, T) := K e^{\beta_1(\alpha)T} x_2 + \frac{b_+ x_1^3}{K} T \quad \text{for all } \alpha \in (\alpha_0, \tilde{\alpha}_+) \text{ and } T \geqslant 0,$$

there exist $\alpha^* \in (\alpha_0, \min\{\alpha_0 + x_1^2, \tilde{\alpha}_+\}]$ and $T^* > 0$ such that

$$s(\alpha, T^*) < 0 \quad \text{and} \quad s(\alpha, T) \leqslant 2K x_2 \quad \text{for all } \alpha \in (\alpha_0, \alpha^*] \text{ and } T \in [0, T^*].$$

This follows by choosing $T^* > 0$ such that $\frac{b_+ x_1^3}{K} T^* \leqslant -2K x_2$ and α^* such that $\exp(\beta_1(\alpha^*)T^*) \leqslant 2$. Choose $\alpha \in (\alpha_0, \alpha^*]$ and $\tau, \tau^* \leqslant \kappa$ with $\tau \leqslant \tau^*$, and assume that $x_1 \leqslant \mu_\tau(t) \leqslant x_3$ for all $t \in [\tau, \tau^*]$. Then the variation of constants formula, applied to (6.5), yields the relation

$$\begin{aligned}
\mu_\tau(\tau^*) &= \Phi_\alpha(\tau^*, \tau) x_2 + \int_\tau^{\tau^*} \underbrace{\Phi_\alpha(\tau^*, s)}_{\geqslant \frac{1}{K} \exp(\beta_2(\alpha)(\tau^* - s))} \underbrace{\big(b(s,\alpha)\big(\mu_\tau(s)\big)^3 + r\big(s, \mu_\tau(s), \alpha\big)\big)}_{\leqslant b_+ x_1^3 < 0} \, ds \\
&\leqslant K e^{\beta_1(\alpha)(\tau^* - \tau)} x_2 + \int_\tau^{\tau^*} \frac{1}{K} e^{\beta_2(\alpha)(\tau^* - s)} b_+ x_1^3 \, ds \\
&= K e^{\beta_1(\alpha)(\tau^* - \tau)} x_2 + \frac{b_+ x_1^3}{K \beta_2(\alpha)} \big(e^{\beta_2(\alpha)(\tau^* - \tau)} - 1\big) \\
&\leqslant K e^{\beta_1(\alpha)(\tau^* - \tau)} x_2 + \frac{b_+ x_1^3}{K} (\tau^* - \tau) = s(\alpha, \tau^* - \tau).
\end{aligned}$$

Since $s(\alpha, T) \leqslant 2K x_2 \leqslant x_3$ for all $T \in [0, T^*]$, the assumption $\mu_\tau(t) \leqslant x_3$ for all $t \in [\tau, \tau^*]$ is justified. This proves (a). Because of $s(\alpha, T^*) < 0$, also (b) is fulfilled.

Step 2. There exists an $\hat{\alpha}_+ \in (\alpha_0, \tilde{\alpha}_+)$ such that for all $\alpha \in (\alpha_0, \hat{\alpha}_+)$, there exists a nontrivial past attractor $A_\alpha \subset \mathbb{I} \times \mathbb{R}$ of (6.1) which fulfills

$$\lim_{\alpha \searrow \alpha_0} d_H\big(A_\alpha(t), \{0\}\big) = 0 \quad \text{for all } t \in \mathbb{I}.$$

For $x_3 := \frac{\rho}{K}$ and $x_2 := \frac{x_3}{2K}$, we consider the function $\gamma : (0, x_2) \to (\alpha_0, \alpha_+)$, defined by

$$\gamma(x_1) := \alpha^*(x_1, x_2, x_3) \quad \text{for all } x_1 \in (0, x_2),$$

where α^* stems from Step 1. We set $\bar{\alpha} := \gamma(\frac{x_2}{2})$ and define

$$\delta(\alpha) := \inf\big\{x_1 \in (0, x_2):\ \gamma(x_1) \geqslant \alpha\big\} \quad \text{for all } \alpha \in (\alpha_0, \bar{\alpha}].$$

Due to $\alpha_0 < \alpha^*(x_1, x_2, x_3) \leqslant \alpha_0 + x_1^2$, we have $\lim_{x_1 \to 0} \gamma(x_1) = \alpha_0$, and since γ is monotone increasing, this implies that δ is monotone increasing, $\delta(\alpha) > 0$ for all $\alpha \in (\alpha_0, \bar{\alpha}]$ and

$$\lim_{\alpha \searrow \alpha_0} \delta(\alpha) = 0. \tag{6.6}$$

We define

$$\bar{x}_3(\alpha) := 3K\delta(\alpha) \quad \text{and} \quad \bar{x}_2(\alpha) := \bar{x}_1(\alpha) := \frac{3}{2}\delta(\alpha) \quad \text{for all } \alpha \in (\alpha_0, \bar{\alpha}]$$

and consider the function $\bar{\gamma} : (\alpha_0, \bar{\alpha}] \to (\alpha_0, \alpha_+)$, defined by

$$\bar{\gamma}(\alpha) := \alpha^*\big(\bar{x}_1(\alpha), \bar{x}_2(\alpha), \bar{x}_3(\alpha)\big) \quad \text{for all } \alpha \in (\alpha_0, \bar{\alpha}],$$

where α^* is taken from Step 1 again. Moreover, we define

$$M := [-x_2, x_2] \quad \text{and} \quad B_\alpha := \big[-\bar{x}_3(\alpha), \bar{x}_3(\alpha)\big] \quad \text{for all } \alpha \in (\alpha_0, \bar{\alpha}]$$

and fix a $\beta \in (\alpha_0, \bar{\alpha}]$ and an $\alpha \in (\alpha_0, \min\{\bar{\gamma}(\beta), \beta\})$. Since $\alpha \leqslant \beta$ and $x_2 \geqslant \frac{3}{2}\delta(\beta)$, and due to the definition of δ, there exists a $T^* > 0$ such that for all $\tau \leqslant \kappa - T^*$, there exist $t_+, t_- \in [0, T^*]$ with

$$\lambda_\alpha\big(t^+, \tau, x_2\big) = \frac{3}{2}\delta(\beta) = \bar{x}_2(\beta) \quad \text{and} \quad \lambda_\alpha\big(t^-, \tau, -x_2\big) = -\frac{3}{2}\delta(\beta) = -\bar{x}_2(\beta).$$

Moreover, since $\alpha < \bar{\gamma}(\beta)$, for all $\tau \leqslant t \leqslant \kappa$, we have

$$\lambda_\alpha\big(t, \tau, \big[-\bar{x}_2(\beta), \bar{x}_2(\beta)\big]\big) \subset \big(-\bar{x}_3(\beta), \bar{x}_3(\beta)\big).$$

This means that, considering Eq. (6.1), $B_\beta \times \mathbb{I}$ is $\{M \times \mathbb{I}\}$-absorbing (see [10, Definition 3.3]). Then due to [10, Theorem 3.5], there exists a global $\{M \times \mathbb{I}\}$-attractor $A_\alpha \subset B_\beta \times \mathbb{I}$ (see [10, Definition 3.4]), i.e.,

$$\lim_{t\to\infty} d\big(\lambda(\tau, \tau - t, M)\big|A_\alpha(\tau)\big) = 0 \quad \text{for all } \tau \in \mathbb{I}.$$

We have the representation

$$A_\alpha(t) := \bigcap_{\tau^* \leqslant t} \overline{\bigcup_{\tau \leqslant \tau^*} \lambda(t, \tau, B_\beta)} \quad \text{for all } t \in \mathbb{I}.$$

Since M is neighborhood of $B_\beta \supset A_\alpha$, this global $\{M \times \mathbb{I}\}$-attractor is also a past attractor. The limit relation

$$\lim_{\alpha \nearrow \alpha_0} d_H\big(A_\alpha(t), \{0\}\big) = 0 \quad \text{for all } t \in \mathbb{I}$$

follows from $A_\alpha \subset B_\beta \times \mathbb{I}$ for all $\alpha < \min\{\bar{\gamma}(\beta), \beta\}$ and (6.6). By setting $\hat{\alpha}_+ := \bar{\gamma}(\bar{\alpha})$, all assertions of this theorem are proved. □

Remark 6.2.

(i) In the limit $\alpha \to \alpha_0$, the attractivity or repulsivity of the trivial solution is lost in both directions, i.e., no situation as described in Remark 5.2(i) can occur.

(ii) The hypothesis on the linear part implies that the all-time (past, future, respectively) dichotomy spectrum of the linearization $\dot{x} = a(t, \alpha)x$ converges to $\{0\}$ in Hausdorff distance in the limit $\alpha \to \alpha_0$ (see [21]).

(iii) As in Example 5.3, one can show that Theorem 6.1 is a proper generalization of the well-known autonomous pitchfork bifurcation (see, e.g., [22, p. 267]).

(iv) Please note that the above bifurcation result is essentially the combination of two scenarios which are independent of each other. This means that it is possible to consider (6.1) only for $\alpha > \alpha_0$ or $\alpha < \alpha_0$, respectively, in order to obtain the results which apply for these parameter values.

Finally, we now compare case (i) of the above theorem with the equivalent autonomous bifurcation.

Example 6.3. Let $x_- < 0 < x_+$ and $\alpha_- < 0 < \alpha_+$ be in $\overline{\mathbb{R}}$, and consider the autonomous differential equation

$$\dot{x} = f(x, \alpha), \tag{6.7}$$

where the C^4-function $f : (x_-, x_+) \times (\alpha_-, \alpha_+) \to \mathbb{R}$ satisfies the following assumptions:

(i) $f(0, \alpha) = 0$ for all $\alpha \in (\alpha_-, \alpha_+)$,
(ii) $D_1 f(0, 0) = 0$,
(iii) $D_1 D_2 f(0, 0) > 0$,
(iv) $D_1^2 f(0, 0) = 0$,
(v) $D_1^3 f(0, 0) > 0$.

Then (6.7) admits an autonomous pitchfork bifurcation (see, e.g., [22, p. 268], and see Fig. 1 for the bifurcation diagram). There exist a neighborhood $U \times V$ of $(0, 0)$ in $\mathbb{R}^2$ and a C^2-function $h : U \to V$ with $h(0) = 0$ and

$$f\big(x, h(x)\big) = 0 \quad \text{for all } x \in U.$$

Except the trivial equilibria and the equilibria described by h, there are no other equilibria in $U \times V$, and the function h is maximal at $x = 0$. It can be verified that this situation fits into case (i) of Theorem 6.1: the functions β_1 and β_2 can be chosen to be increasing, since $D_1 D_2 f(0, 0) > 0$ by (iii), and (6.2) holds, since $D_1^3 f(0, 0) > 0$ by (v). Due to (iii), the trivial equilibrium of (6.7) is attractive for $\alpha < 0$ and repulsive for $\alpha > 0$, and this carries over to nonautonomous notions of attractivity and repulsivity. The function h describes repulsive equilibria of (6.7), and these equilibria are the boundary of the domain of attraction of the trivial equilibria. Since $\lim_{x \to 0} h(x) = 0$, we have a nonautonomous bifurcation in form of a shrinking domain of attraction.

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