Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 244 (2008) 599–629
Journal of Differential Equations
www.elsevier.com/locate/jde

Lower and upper solutions for elliptic problems in nonsmooth domains

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Received 31 January 2007; revised 6 July 2007

Available online 22 October 2007

Abstract

In this paper we prove some existence results of semilinear Dirichlet problems in nonsmooth domains in presence of lower and upper solutions well-ordered or not. We first prove existence results in an abstract setting using degree theory. We secondly apply them for domains with conical points.

MSC: 35J25; 35J60

Keywords: Lower and upper solutions; Nonsmooth domains; Semilinear Dirichlet problem

1. Introduction

There is a large literature concerning the existence and localization of a solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.08.008

in presence of lower and upper solutions α and β satisfying $\alpha \leqslant \beta$. This result goes back to G. Scorza Dragoni in 1931 for the one-dimensional Dirichlet problem [25] and to M. Nagumo [24] in 1954 for the multi-dimensional one. In 1972, H. Amann [2] proved his three solutions theorem in presence of two pairs of lower and upper solutions satisfying $\alpha_1 \leqslant \beta_1 \leqslant \beta_2$, $\alpha_1 \leqslant \alpha_2 \leqslant \beta_2$ and $\alpha_2 \nleqslant \beta_1$, by using the relation between lower–upper solutions and degree theory.

In 1972, D.H. Sattinger [26] presents as an open problem the question of the solvability of (1) in presence of lower and upper solutions without ordering. It was pointed out by an example in [3] that these conditions are not sufficient to guarantee the solvability of (1). This example is essentially of the type

$$\begin{cases} -\Delta u = \lambda_m u + \varphi_m(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where λ_m is an eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ different from the first one and φ_m is the corresponding nonzero eigenfunction. It is easy to see that (2) has no solution even if we can construct lower and upper solutions α and β as multiples of the first eigenfunction φ_1 satisfying $\beta < 0 < \alpha$. Hence to have existence of a solution in presence of non-ordered lower and upper solutions, we have to avoid the interference of the nonlinearity with the higher part of the spectrum.

The first important contribution in this direction is due to H. Amann, A. Ambrosetti and G. Mancini [4] in 1978 who assume

$$\sup_{\Omega \times \mathbb{R}} \left| f(x, u) - \lambda_1 u \right| < \infty.$$

More recently, this kind of result was generalized to consider unbounded perturbations of $\lambda_1 u$ (see C. De Coster and M. Henrard [8] and the references therein, as well as [9] for other types of extensions in the framework of parabolic problems).

Up to now, these results have been proved in a regular context (i.e. for domains with a smooth boundary) and the main scope of these papers is the generalization of the conditions on the nonlinearity f. Moreover the proofs of these results (see for instance [8]) use deeply the fact that the solution of (1) with $f(\cdot, u(\cdot)) \in L^p(\Omega)$ is in $W^{2,p}(\Omega)$ with $p > N$ (N being the dimension of Ω, in such a way that $W^{2,p}(\Omega) \hookrightarrow \mathcal{C}^1(\overline{\Omega})$) which is no more true even for non-convex polygonal domains for example. The aim of this paper is to extend Amann–Ambrosetti–Mancini's result in what concerns the regularity of Ω.

This paper is organized in the following way. In the first section, we give general hypotheses and technical results with minimal regularity on the boundary of the domain. Then, we prove some existence results in presence of lower and upper solutions in an abstract setting. In the non-well-ordered case we consider only situations where "f is a bounded perturbation of the first eigenvalue" as in [4]. Extension of the asymptotic behaviour can be done arguing as in [8,9]. Using an argument of [7], we prove also an extension of Amann's three solutions theorem asking less ordering relations between the lower and upper solutions. In Sections 4, 5 and 6, we show how the abstract results can be interpreted in case of a regular domain, a polygonal domain of $\mathbb{R}^2$ and a domain of $\mathbb{R}^n$, for $n \geqslant 3$, with a conical point. The last section is mainly devoted to the construction of the lower and upper solutions for some particular nonlinearities f.

2. Preliminaries

The theory of lower and upper solutions is based on maximum principles due to J.-M. Bony [5] and P.-L. Lions [20]. Here we use them in the following forms (see [11, Theorem 9.6, Lemma 3.4] or [28, Theorem 3.27, Lemma 3.26]).

Theorem 2.1 *(Maximum Principle). Let Ω_1 be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega_1$ of class $\mathcal{C}^{1,1}$, $p > N$ and $\lambda \geqslant 0$. If $u \in W^{2,p}(\Omega_1)$ satisfies*

$$-\Delta u + \lambda u \leqslant 0, \quad \text{in } \Omega_1,$$

then u cannot achieve a maximum $M \geqslant 0$ in Ω_1 unless u is constant.

Theorem 2.2 *(Hopf Boundary Point Lemma). Let Ω_1 be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega_1$ of class $\mathcal{C}^{1,1}$, $p > N$, $\lambda \geqslant 0$, $x_0 \in \partial\Omega_1$ and b_0 such that $(b_0|\nu(x_0)) > 0$ for $\nu(x_0)$ the outward normal at x_0. If $u \in W^{2,p}(\Omega_1)$ satisfies $-\Delta u + \lambda u \leqslant 0$ in Ω_1 and u achieves a strict local maximum $M \geqslant 0$ at x_0, then*

$$\big(b_0 \big| \nabla u(x_0)\big) > 0.$$

A simple consequence of Theorem 2.1 is the following corollary.

Corollary 2.3. *Let Ω be a bounded domain in $\mathbb{R}^N$, $p > N$ and $\lambda \geqslant 0$. If $u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfies*

$$\begin{cases} -\Delta u + \lambda u \leqslant 0, & \text{in } \Omega, \\ u \leqslant 0, & \text{on } \partial\Omega, \end{cases}$$

then $u \leqslant 0$ in $\overline{\Omega}$.

As in [27], in our purpose, the first eigenfunction plays a crucial role. We here recall some useful properties (see [11, Section 8.12] or [10,21]).

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then the eigenvalue problem: find $\lambda \in \mathbb{R}$ and $u \in H^1_0(\Omega) \setminus \{0\}$ such that*

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

has a sequence of solutions $(\lambda_n, \varphi_n)_{n\geqslant 1}$ such that

(i) $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ *and* $\lim_{n\to\infty} \lambda_n = \infty$;
(ii) $\varphi_n \in \mathcal{C}^\infty(\Omega)$;
(iii) *λ_1 is simple and the corresponding eigenfunction φ_1 can be chosen such that $\varphi_1(x) > 0$ in Ω*;
(iv) $\lambda_1 = \min\{\int_\Omega |\nabla u|^2\,dx \mid u \in H^1_0(\Omega),\ \int_\Omega u^2\,dx = 1\}$;
(v) *if $u \in H^1_0(\Omega)$ is such that $\int_\Omega |\nabla u|^2\,dx = \lambda_1 \int_\Omega u^2\,dx$, then there exists $C \in \mathbb{R}$ with $u = C\varphi_1$.*

Following [29] we introduce the following space.

Definition 2.1. We define the space

$$\mathcal{C}_{\varphi_1} = \{u \in \mathcal{C}(\overline{\Omega}) \mid \exists a > 0,\ \forall x \in \Omega,\ |u(x)| \leqslant a\varphi_1(x)\}.$$

This space is a Banach space endowed with the norm

$$\|u\|_{\varphi_1} = \inf\{a > 0 \mid \forall x \in \Omega,\ |u(x)| \leqslant a\varphi_1(x)\}.$$

We denote the open ball in that space

$$B_{\varphi_1}(0, R) := \{u \in \mathcal{C}_{\varphi_1} \mid \|u\|_{\varphi_1} < R\}.$$

Remark 2.1. In case $\varphi_1 \in \mathcal{C}(\overline{\Omega})$, there exists $C > 0$ such that

$$\forall u \in \mathcal{C}_{\varphi_1}, \quad \|u\|_\infty \leqslant C\|u\|_{\varphi_1}.$$

Definition 2.2. Given functions $u, v : \overline{\Omega} \to \mathbb{R}$, we write

- $u \leqslant v$ if $u(x) \leqslant v(x)$ in $\overline{\Omega}$;
- $u < v$ if $u \leqslant v$ and $u \neq v$;
- $u \ll v$ if there exists $\epsilon > 0$ such that $u + \epsilon\varphi_1 \leqslant v$ with φ_1 given by Proposition 2.4;
- $[u, v] = \{w \in \mathcal{C}_{\varphi_1} \mid u \leqslant w \leqslant v\}$.

Remark 2.2. Defining in $\mathcal{C}_{\varphi_1}$, the order cone $K_{\varphi_1} = \{v \in \mathcal{C}_{\varphi_1} \mid v \geqslant 0\}$, we observe that $u \ll v$ if and only if $v - u \in \operatorname{int}(K_{\varphi_1})$.

The regularity assumptions on the domain we use are the following

Assumption (H-1). There exist $p > N$ and a normed space $\mathcal{A} \subset L^p_{\mathrm{loc}}(\Omega)$ such that, for every $h \in \mathcal{A}$, the problem

$$\begin{cases} -\Delta u = h, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

admits a unique solution $u \in \mathcal{C}_{\varphi_1}$.

Moreover we ask that

- the cone $K = \{w \in \mathcal{A} \mid w \geqslant 0 \text{ a.e. in } \Omega\}$ is normal;
- $\mathcal{C}_{\varphi_1}$ is continuously imbedded in $\mathcal{A}$;
- the operator $T : \mathcal{A} \to \mathcal{C}_{\varphi_1} : h \mapsto u$, with u the unique solution of (4), is compact.

Remark 2.3. Observe that, as $h \in L^p_{\mathrm{loc}}(\Omega)$, the local regularity theory for the Laplacian implies that $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ (see for example [11] or [22]).

Remark 2.4. In the practical situations, $\mathcal{A} = L^p_w(\Omega) := \{f \in L^p_{\mathrm{loc}}(\Omega)\colon\ w^{1/p} f \in L^p(\Omega)\}$, for some $w \in L^1_{\mathrm{loc}}(\Omega)$, $w > 0$, where w depends on the regularity of the domain and $\|f\|_{\mathcal{A}} = (\int_\Omega w|f|^p\,dx)^{1/p}$. In that case, the cone K is obviously normal.

We can now give the following results concerning the first eigenfunction. We use the space

$$\mathcal{C}_0(\overline{\Omega}) = \{u \in \mathcal{C}(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}.$$

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume that the first eigenfunction φ_1 defined in Proposition* 2.4 *satisfies $\varphi_1 \in \mathcal{C}_0(\overline{\Omega})$. Let $p > N$ and $v \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ with $\|v\|_{\varphi_1} = 1$ be a solution of* (3) *with $\lambda = \lambda_1$. Then $v = \pm\varphi_1$.*

Remark 2.5. Observe that, by [11, Corollary 8.28 and the remark after Theorem 8.29], if $N < 6$ and Ω satisfies an exterior cone condition (which is in particular the case if $\partial\Omega$ is Lipschitz), then $\varphi_1 \in \mathcal{C}_0(\overline{\Omega})$.

Remark 2.6. This proposition is not a direct consequence of Proposition 2.4 as v is not a priori in $H^1_0(\Omega)$. Note further that this result is closely related to the Krein–Rutman theorem (see [29]) but the strong positiveness of T in $\mathcal{C}_{\varphi_1}$ is not easy to establish since we make no regularity assumptions on the boundary of Ω. We then give here a simple and direct proof.

Proof of Proposition 2.5. As $v \in \mathcal{C}(\overline{\Omega})$, by Lax–Milgram theorem, there exists a unique $u \in H^1_0(\Omega)$ such that

$$\forall \xi \in H^1_0(\Omega), \qquad \int_\Omega \nabla u\, \nabla \xi\, dx = \lambda_1 \int_\Omega v\xi\, dx. \tag{5}$$

By the local regularity theory, we know that $u \in \mathcal{C}(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$.

As $v - \varphi_1 \leqslant 0$ in Ω, applying (5) with $\xi = (u - \varphi_1)^+$, we obtain

$$\begin{aligned}\int_\Omega \left|\nabla (u-\varphi_1)^+\right|^2 dx &= \int_\Omega \nabla(u-\varphi_1)\,\nabla(u-\varphi_1)^+\,dx\\ &= \lambda_1 \int_\Omega (v-\varphi_1)(u-\varphi_1)^+\,dx \leqslant 0.\end{aligned}$$

Hence $(u - \varphi_1)^+ = 0$, i.e. $u \leqslant \varphi_1$, a.e. in Ω. In the same way we prove that $u \geqslant -\varphi_1$ a.e. in Ω. By continuity of u and φ_1 we conclude that

$$-\varphi_1 \leqslant u \leqslant \varphi_1, \quad \text{in } \Omega.$$

As $\varphi_1 \in \mathcal{C}_0(\overline{\Omega})$ we deduce that $u \in \mathcal{C}(\overline{\Omega})$ and hence $u \in \mathcal{C}_{\varphi_1}$. We then have $u - v \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ satisfying

$$\begin{cases} -\Delta(u-v) = 0, & \text{in } \Omega,\\ u - v = 0, & \text{on } \partial\Omega.\end{cases}$$

By Corollary 2.3, $v = u \in H^1_0(\Omega)$ and we conclude by Proposition 2.4. □

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that Assumption* (H-1) *is satisfied. Let (λ_1, φ_1) be the first eigenvalue and eigenfunction defined in Proposition* 2.4. *Assume that $\varphi_1 \in \mathcal{C}_0(\overline{\Omega})$. Let $p > N$, $\gamma \in \mathcal{A}$, $d \in \mathcal{A} \cap L^\infty(\Omega)$ with $d > 0$. Then there exists a unique $w \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ solution of*

$$\begin{cases} -\Delta w = (\lambda_1 - d)w + \gamma, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Proof. We know that $w \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ is a solution of (6) if and only if w is a solution of

$$w = T\big((\lambda_1 - d)w\big) + T(\gamma).$$

Hence, by the Fredholm alternative, the result will be proved if we show that

$$w = T\big((\lambda_1 - d)w\big)$$

has only the trivial solution in $\mathcal{C}_{\varphi_1}$.

Let $w \in \mathcal{C}_{\varphi_1}$ be a solution of $w = T((\lambda_1 - d)w)$, i.e. $w \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ is a solution of

$$\begin{cases} -\Delta w = (\lambda_1 - d)w, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{7}$$

As $(\lambda_1 - d)w \in L^2(\Omega)$, by Lax–Milgram theorem, there exists a unique $\tilde{w} \in H^1_0(\Omega)$ such that, for all $\xi \in H^1_0(\Omega)$,

$$\int_\Omega \nabla \tilde{w}\, \nabla \xi \, dx = \int_\Omega (\lambda_1 - d) w \xi \, dx.$$

Let C be such that $|w| \leqslant C\varphi_1$. Then we have, for $D > \max(C, \frac{(\|d\|_{L^\infty(\Omega)} + \lambda_1)C}{\lambda_1})$,

$$\begin{aligned} \int_\Omega \big|\nabla(\tilde{w} - D\varphi_1)^+\big|^2 dx &= \int_\Omega \nabla(\tilde{w} - D\varphi_1)\, \nabla(\tilde{w} - D\varphi_1)^+ \, dx \\ &= \int_\Omega \big[(\lambda_1 - d)w - D\lambda_1\varphi_1\big](\tilde{w} - D\varphi_1)^+ \, dx \\ &\leqslant \int_\Omega \varphi_1\big(|\lambda_1 - d|\, C - D\lambda_1\big)(\tilde{w} - D\varphi_1)^+ \, dx \leqslant 0. \end{aligned}$$

It follows that $(\tilde{w} - D\varphi_1)^+ = 0$, i.e. $\tilde{w} \leqslant D\varphi_1$, a.e. in Ω. In the same way we prove that $\tilde{w} \geqslant -D\varphi_1$ a.e. in Ω. By local regularity, we have that $\tilde{w} \in \mathcal{C}(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ and as $|\tilde{w}| \leqslant D\varphi_1$ in Ω, we have $\tilde{w} \in \mathcal{C}(\overline{\Omega})$ and hence $\tilde{w} \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$.

We then have $\tilde{w} - w \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ satisfying

$$\begin{cases} -\Delta(\tilde{w} - w) = 0, & \text{in } \Omega, \\ \tilde{w} - w = 0, & \text{on } \partial\Omega. \end{cases}$$

By Corollary 2.3, $\tilde{w} = w$ and hence, for all $\xi \in H^1_0(\Omega)$,

$$\int_\Omega \nabla w \, \nabla \xi \, dx = \int_\Omega (\lambda_1 - d) w \xi \, dx.$$

In particular for $\xi = w$ and using the variational characterization of the first eigenvalue (Proposition 2.4(iv)) we obtain

$$0 \leqslant \int_\Omega |\nabla w|^2 \, dx - \lambda_1 \int_\Omega w^2 \, dx = -\int_\Omega d w^2 \, dx \leqslant 0.$$

We have then

$$\int_\Omega |\nabla w|^2 \, dx = \lambda_1 \int_\Omega w^2 \, dx \quad \text{and} \quad \int_\Omega d w^2 \, dx = 0.$$

From the first identity, we deduce from Proposition 2.4 that $w = C\varphi_1$ for some C, and by the second, we have $C = 0$ as $d > 0$ and hence $w = 0$. Hence $w = 0$ as $d > 0$. $\square$

On the nonlinearity f we assume the following regularity

Assumption (H-2). For the space $\mathcal{A} \subset L^p_{\mathrm{loc}}(\Omega)$ of Assumption (H-1), we assume that the Nemytskii operator

$$N : \mathcal{C}_{\varphi_1} \to \mathcal{A} : u \mapsto f(x, u)$$

is continuous.

Remark 2.7. Observe that in case $\mathcal{A} = L^p_w(\Omega) := \{f \in L^p_{\mathrm{loc}}(\Omega)\colon\ w^{1/p} f \in L^p(\Omega)\}$, for some $w \in L^1_{\mathrm{loc}}(\Omega)$, $w > 0$, Assumption (H-2) will be satisfied in particular if $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{A}$-Carathéodory according to the following definition.

Definition 2.3. A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be $\mathcal{A}$-*Carathéodory* if

(i) for a.e. $x \in \Omega$, the function $f(x, \cdot)$ is continuous;
(ii) for all $z \in \mathbb{R}$, the function $f(\cdot, z)$ is measurable;
(iii) for all $R > 0$, there exists $h_R \in \mathcal{A}$ such that, for all $u \in B_{\varphi_1}(0, R)$, $|f(x, u(x))| \leqslant h_R(x)$ a.e. on Ω.

3. Abstract formulation

Definition 3.1. A function $\alpha \in \mathcal{C}(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ is a *lower solution of* (1) if

(i) for a.e. $x \in \Omega$, $-\Delta\alpha(x) \leqslant f(x, \alpha(x))$;
(ii) for all $x \in \partial\Omega$, $\alpha(x) \leqslant 0$.

Similarly, a function $\beta \in \mathcal{C}(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ is an *upper solution of* (1) if

(i) for a.e. $x \in \Omega$, $-\Delta\beta(x) \geqslant f(x, \beta(x))$;
(ii) for all $x \in \partial\Omega$, $\beta(x) \geqslant 0$.

A *solution of* (1) is a function $u \in \mathcal{C}(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ which is both a lower and an upper solution of (1).

Definition 3.2. A lower solution α of (1) is said *strict* if, for all u solution of (1) with $u \geqslant \alpha$, we have $u \gg \alpha$.

In a similar way, an upper solution β of (1) is said *strict* if, for all u solution of (1) with $u \leqslant \beta$, we have $u \ll \beta$.

Our first result concerns the well-ordered case, i.e. the case $\alpha \leqslant \beta$.

Theorem 3.1. *Let Ω be a bounded domain and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that Assumptions* (H-1) *and* (H-2) *are satisfied. Assume that there exist $\alpha, \beta \in \mathcal{C}(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$, respectively lower and upper solutions of* (1)*, such that $\alpha \leqslant \beta$. Moreover assume that there exists $h \in \mathcal{A}$ such that, for all $u \in [\alpha, \beta]$, $|f(x, u(x))| \leqslant h(x)$ a.e. in Ω.*

Then problem (1) *has at least one solution $u \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ such that*

$$\alpha \leqslant u \leqslant \beta.$$

Moreover if α and β are strict, then there exists $R > 0$ such that

$$\deg\bigl(I - T \circ N, \mathcal{S} \cap B_{\varphi_1}(0, R)\bigr) = 1,$$

where

$$\mathcal{S} = \{u \in \mathcal{C}_{\varphi_1}\colon\ \alpha \ll u \ll \beta\}.$$

Remark 3.1.

(i) If α and β are strict then there exists $\epsilon > 0$ such that $\beta - \alpha \geqslant \epsilon\varphi_1$.
(ii) If in Assumptions (H-1) and (H-2), $\mathcal{C}_{\varphi_1}$ is replaced by $\mathcal{C}(\overline{\Omega})$, we obtain a similar result except that the solution is now in $\mathcal{C}(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$.

Proof of Theorem 3.1. *Step 1. Existence of a solution $u \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ of* (1) *with $\alpha \leqslant u \leqslant \beta$*. Let $\gamma : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by $\gamma(x, u) = \max\{\alpha(x), \min\{u, \beta(x)\}\}$. Observe that $\Gamma\colon \mathcal{C}_{\varphi_1} \to \mathcal{C}_{\varphi_1} : u \mapsto \gamma(\cdot, u)$ is continuous.

We study the modified problem

$$\begin{cases} -\Delta u = f\big(x, \gamma(x,u)\big), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Claim 1. *Every solution $u \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ of* (8) *is such that $\alpha \leqslant u \leqslant \beta$.*

We prove that $\alpha \leqslant u$; the other part is proved in a similar way. By contradiction, assume that $\max_{x\in\overline{\Omega}}(\alpha(x) - u(x)) = M > 0$. As $\alpha - u \leqslant 0$ on $\partial\Omega$, we can find $\Omega_1 \subset \Omega$ with $\partial\Omega_1$ of class $\mathcal{C}^{1,1}$ and $x_0, x_1 \in \Omega_1$ such that $\alpha(x_0) - u(x_0) = M$, $\alpha(x_1) - u(x_1) < M$ and $\alpha(x) - u(x) \geqslant 0$ on Ω_1. This contradicts the maximum principle (Theorem 2.1) as for a.e. $x \in \Omega_1$

$$-\Delta(\alpha - u)(x) \leqslant f\big(x, \alpha(x)\big) - f\big(x, \alpha(x)\big) = 0.$$

Claim 2. *The problem* (8) *has at least one solution $u \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$.*

By Assumptions (H-1) and (H-2), the operator $T \circ N \circ \Gamma : \mathcal{C}_{\varphi_1} \to \mathcal{C}_{\varphi_1}$ is completely continuous. Moreover, by assumption, there exists $R > 0$ such that, for every $u \in \mathcal{C}_{\varphi_1}$, $\|T \circ N \circ \Gamma(u)\|_{\varphi_1} < R$. Hence, for all $\lambda \in [0,1]$

$$\begin{aligned} \deg\big(I - T \circ N \circ \Gamma, B_{\varphi_1}(0,R)\big) &= \deg\big(I - \lambda T \circ N \circ \Gamma, B_{\varphi_1}(0,R)\big) \\ &= \deg\big(I, B_{\varphi_1}(0,R)\big) = 1, \end{aligned}$$

and (8) has at least one solution.

Claim 3. *The problem* (1) *has at least one solution $u \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ satisfying $\alpha \leqslant u \leqslant \beta$.*

By Claim 2, (8) has at least one solution u. By Claim 1, this solution satisfies $\alpha \leqslant u \leqslant \beta$ and hence, is a solution of (1).

Step 2. Degree computation in case α and β are strict. By Claim 1, we know that every fixed point u of $T \circ N \circ \Gamma$ is such that $\alpha \leqslant u \leqslant \beta$ and is a fixed point of $T \circ N$. Moreover, if α and β are strict, u satisfies $\alpha \ll u \ll \beta$ and $u \in \mathcal{S}$. By the excision property of the degree and as $T \circ N \circ \Gamma = T \circ N$ on $\mathcal{S}$, we obtain

$$\begin{aligned} \deg\big(I - T \circ N, \mathcal{S} \cap B_{\varphi_1}(0,R)\big) &= \deg\big(I - T \circ N \circ \Gamma, \mathcal{S} \cap B_{\varphi_1}(0,R)\big) \\ &= \deg\big(I - T \circ N \circ \Gamma, B_{\varphi_1}(0,R)\big) = 1. \qquad \square \end{aligned}$$

Remark 3.2. Observe that R is such that, for all $k \in \mathcal{A}$ with $|k| \leqslant h$, we have $\|T(k)\|_{\varphi_1} < R$.

Our next result extends Amann's three solutions theorem and gives the existence of three solutions in presence of two pairs of lower and upper solutions with order relations.

Theorem 3.2. *Let Ω be a bounded domain and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be such that Assumptions* (H-1) *and* (H-2) *are satisfied. Assume that there exist $\alpha_1,\alpha_2\in\mathcal{C}(\overline{\Omega})\cap W^{2,p}_{\mathrm{loc}}(\Omega)$, lower solutions and $\beta_1,\beta_2\in\mathcal{C}(\overline{\Omega})\cap W^{2,p}_{\mathrm{loc}}(\Omega)$, upper solutions of* (1) *such that*

$$\alpha_1\leqslant\beta_1,\qquad \alpha_1\leqslant\beta_2,\qquad \alpha_2\leqslant\beta_2,$$

and there exists $x_0\in\Omega$ with

$$\alpha_2(x_0)>\beta_1(x_0).$$

Suppose further that β_1 and α_2 are strict.

Moreover assume that there exists $h\in\mathcal{A}$ such that, for all $u\in[\alpha_1,\beta_1]\cup[\alpha_2,\beta_2]\cup[\alpha_1,\beta_2]$, $|f(x,u(x))|\leqslant h(x)$ a.e. in Ω.

Then the problem (1) *has at least three solutions $u_1,u_2,u_3\in\mathcal{C}_{\varphi_1}\cap W^{2,p}_{\mathrm{loc}}(\Omega)$ such that*

$$\alpha_1\leqslant u_1\ll\beta_1,\qquad \alpha_2\ll u_2\leqslant\beta_2$$

and there exist $x_1,x_2\in\Omega$ with

$$u_3(x_1)>\beta_1(x_1),\qquad u_3(x_2)<\alpha_2(x_2).$$

Notice that the condition $u_3(x_1)>\beta_1(x_1)$ and $u_3(x_2)<\alpha_2(x_2)$ is a localization condition that implies that $u_3\neq u_1$ and $u_3\neq u_2$.

Proof. Define, for $i,j\in\{1,2\}$, $\gamma_{i,j}(x,u)=\max\{\alpha_i(x),\min\{u,\beta_j(x)\}\}$ and $\Gamma_{i,j}:\mathcal{C}_{\varphi_1}\to\mathcal{C}_{\varphi_1}: u\mapsto\gamma_{i,j}(\cdot,u)$. Observe that $\Gamma_{i,j}$ is continuous and consider the modified problem

$$\begin{cases}-\Delta u=f\bigl(x,\gamma_{1,2}(x,u)\bigr), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega.\end{cases}\tag{9}$$

Let us choose k so that $\beta_1\leqslant\beta_2+k$ and $\alpha_1-k\leqslant\alpha_2$ and let R be such that, for every $k\in\mathcal{A}$ with $|k|\leqslant h$, $\|T(k)\|_{\varphi_1}<R$.

Step 1. Computation of $\deg(I-T\circ N\circ\Gamma_{1,2},\mathcal{S}_{1,1}\cap B_{\varphi_1}(0,R))$*, where*

$$\mathcal{S}_{1,1}=\{u\in\mathcal{C}_{\varphi_1}\mid \alpha_1-k\ll u\ll\beta_1\}.$$

Define the alternative modified problem

$$\begin{cases}-\Delta u=\bar f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases}\tag{10}$$

where

$$\bar f(t,u)=\max\bigl\{f\bigl(x,\gamma_{1,1}(x,u)\bigr),f\bigl(x,\gamma_{1,2}(x,u)\bigr)\bigr\}.$$

Observe that $\bar N:\mathcal{C}_{\varphi_1}\to\mathcal{A}:u\to\bar f(\cdot,u)$ is continuous. For any $\lambda\in[0,1]$, we consider then the homotopy $\lambda T\circ\bar N+(1-\lambda)T\circ N\circ\Gamma_{1,2}$.

Claim 1. *If $\lambda \in [0,1]$ and u is a fixed point of $\lambda\, T \circ \bar{N} + (1-\lambda)\, T \circ N \circ \Gamma_{1,2}$, we have $\alpha_1 \leqslant u \leqslant \beta_2$.*

This result follows from the usual maximum principle argument as in Claim 1 of the proof of Theorem 3.1.

Claim 2. *If $\lambda \in [0,1]$ and $u \in \bar{\mathcal{S}}_{1,1}$ is a fixed point of $\lambda\, T \circ \bar{N} + (1-\lambda) T \circ N \circ \Gamma_{1,2}$, we have $u \ll \beta_1$.*

Assume there exists $x_0 \in \overline{\Omega}$ such that $u(x_0) = \beta_1(x_0)$. We deduce from Claim 1 that $\alpha_1 \leqslant u \leqslant \beta_2$ so that u solves (1). As further β_1 is a strict upper solution, the claim follows.

Claim 3. $\deg(I - T \circ N \circ \Gamma_{1,2}, \mathcal{S}_{1,1}) = 1$.

It follows from the above claims that $\alpha_1 - k$ and β_1 are strict lower and upper solutions of (10) and we deduce from Theorem 3.1 and the properties of the degree that

$$\begin{aligned}
&\deg\bigl(I - T \circ N \circ \Gamma_{1,2}, \mathcal{S}_{1,1} \cap B_{\varphi_1}(0,R)\bigr)\\
&\quad = \deg\bigl(I - \bigl(\lambda T \circ \bar{N} + (1-\lambda) T \circ N \circ \Gamma_{1,2}\bigr), \mathcal{S}_{1,1} \cap B_{\varphi_1}(0,R)\bigr)\\
&\quad = \deg\bigl(I - T \circ \bar{N}, \mathcal{S}_{1,1} \cap B_{\varphi_1}(0,R)\bigr) = 1.
\end{aligned}$$

Step 2. $\deg(I - T \circ N \circ \Gamma_{1,2}, \mathcal{S}_{2,2} \cap B_{\varphi_1}(0,R)) = 1$, *where*

$$\mathcal{S}_{2,2} = \{u \in \mathcal{C}_{\varphi_1} \mid \alpha_2 \ll u \ll \beta_2 + k\}.$$

The proof of this result parallels the proof of Step 1.

Step 3. There exist three solutions u_i $(i = 1,2,3)$ of (1) such that

$$\alpha_1 \leqslant u_1 \ll \beta_1, \qquad \alpha_2 \ll u_2 \leqslant \beta_2, \qquad \alpha_1 \leqslant u_i \leqslant \beta_2, \quad \textit{for } i = 1,2,3,$$

and there exist $x_1, x_2 \in \Omega$ with

$$u_3(x_1) > \beta_1(x_1), \qquad u_3(x_2) < \alpha_2(x_2).$$

The first two solutions are obtained from the fact that

$$\deg\bigl(I - T \circ N \circ \Gamma_{1,2}, \mathcal{S}_{1,1} \cap B_{\varphi_1}(0,R)\bigr) = 1$$

and

$$\deg\bigl(I - T \circ N \circ \Gamma_{1,2}, \mathcal{S}_{2,2} \cap B_{\varphi_1}(0,R)\bigr) = 1.$$

Define

$$\mathcal{S}_{1,2} = \{u \in \mathcal{C}_{\varphi_1} \mid \alpha_1 - k \ll u \ll \beta_2 + k\}.$$

We have

$$\begin{aligned}1&=\deg\big(I-T\circ N\circ\Gamma_{1,2},\mathcal{S}_{1,2}\cap B_{\varphi_1}(0,R)\big)\\&=\deg\big(I-T\circ N\circ\Gamma_{1,2},\mathcal{S}_{1,1}\cap B_{\varphi_1}(0,R)\big)+\deg\big(I-T\circ N\circ\Gamma_{1,2},\mathcal{S}_{2,2}\cap B_{\varphi_1}(0,R)\big)\\&\quad+\deg\big(I-T\circ N\circ\Gamma_{1,2},\big(\mathcal{S}_{1,2}\setminus(\bar{\mathcal{S}}_{1,1}\cup\bar{\mathcal{S}}_{2,2})\big)\cap B_{\varphi_1}(0,R)\big),\end{aligned}$$

which implies

$$\deg\big(I-T\circ N\circ\Gamma_{1,2},\big(\mathcal{S}_{1,2}\setminus(\bar{\mathcal{S}}_{1,1}\cup\bar{\mathcal{S}}_{2,2})\big)\cap B_{\varphi_1}(0,R)\big)=-1$$

and the existence of $u_3\in\mathcal{S}_{1,2}\setminus(\bar{\mathcal{S}}_{1,1}\cup\bar{\mathcal{S}}_{2,2})$ follows.

As we know from Claim 1, that the solutions u of (9) are such that

$$\alpha_1\leqslant u\leqslant\beta_2,$$

they are solutions of (1). □

Remark 3.3. Observe that in this theorem

$$u_1\leqslant\min\{\beta_1,\beta_2\}\quad\text{and}\quad u_2\geqslant\max\{\alpha_1,\alpha_2\}.$$

Theorem 3.3. *Let Ω be a bounded domain and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be such that Assumptions* (H-1) *and* (H-2) *are satisfied and moreover $\varphi_1\in\mathcal{C}_0(\overline{\Omega})$.*

Assume that there exist $\alpha,\beta\in\mathcal{C}(\overline{\Omega})\cap W^{2,p}_{\mathrm{loc}}(\Omega)$ lower and upper solutions of (1) *such that, for some $C>0$,*

$$\alpha\ll C\varphi_1,\qquad -C\varphi_1\ll\beta,$$

and there exists $x_0\in\Omega$ with

$$\alpha(x_0)>\beta(x_0).$$

Moreover suppose that, for every $R>C$, there exists $h_R\in\mathcal{A}$ such that, for all $u\in[\alpha,R\varphi_1]\cup[-R\varphi_1,\beta]\cup[-R\varphi_1,R\varphi_1]$, $|f(x,u(x))|\leqslant h_R(x)$ a.e. in Ω.

Assume further that there exists $\gamma\in\mathcal{A}$ such that, for all $(x,u)\in\Omega\times\mathbb{R}$,

$$\big|f(x,u)-\lambda_1 u\big|\leqslant\gamma(x).$$

Then the problem (1) *has at least one solution $u\in\mathcal{C}_{\varphi_1}\cap W^{2,p}_{\mathrm{loc}}(\Omega)$ such that $u\in\overline{\mathcal{O}}$ where*

$$\mathcal{O}=\big\{u\in\mathcal{C}_{\varphi_1}\ \big|\ \min(u-\alpha)<0<\max(u-\beta)\big\}.$$

Moreover, if α and β are strict, there exists $R>0$ such that

$$\deg\big(I-T\circ N,\mathcal{O}\cap B_{\varphi_1}(0,R)\big)=-1.$$

Remark 3.4. Observe that if α and β are strict, every solution $u\in\overline{\mathcal{O}}$ satisfies $u\in\mathcal{O}$.

Proof of Theorem 3.3. In the course of this proof, we relabel α and β as $\alpha = \alpha_2$ and $\beta = \beta_1$ in order to apply Theorem 3.2.

For every $r > 1$, define the function $f_r : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} f_r(x,u) &= f(x,u) && \text{if } |u| < r \\ &= \big(|u| - r\big)\left(\lambda_1 - \frac{1}{r}d\right)u + \big(r + 1 - |u|\big) f(x,u) && \text{if } r \leqslant |u| \leqslant r+1 \\ &= \left(\lambda_1 - \frac{1}{r}d\right)u && \text{if } r+1 < |u|, \end{aligned}$$

where $d \in \mathcal{A} \cap L^\infty(\Omega)$, $d > 0$.

For every $r > 1$, consider the modified problem

$$\begin{cases} -\Delta u = f_r(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Observe that we can decompose $f_r(x,u) = p_r(x,u)u + q_r(x,u)$ such that, for all $(x,u) \in \Omega \times \mathbb{R}$,

$$\lambda_1 - \frac{1}{r}d \leqslant p_r(x,u) \leqslant \lambda_1,$$
$$\big|q_r(x,u)\big| \leqslant \gamma(x).$$

Claim. *There exists $K > 1$ such that, for all $r > K$ and for all $u \in \overline{\mathcal{O}}$, solution of* (11)*, we have $\|u\|_{\varphi_1} < K$.*

Otherwise, for all $n \geqslant 1$, there exist $r_n > n$ and $u_n \in \overline{\mathcal{O}}$ solution of (11) for $r = r_n$ with $\|u_n\|_{\varphi_1} \geqslant n$. Then $v_n = u_n/\|u_n\|_{\varphi_1}$ satisfies

$$\begin{cases} -\Delta v_n = p_{r_n}(x,u_n)v_n + \dfrac{q_{r_n}(x,u_n)}{\|u_n\|_{\varphi_1}}, & \text{in } \Omega, \\ v_n = 0, & \text{on } \partial\Omega. \end{cases}$$

As $\{p_{r_n}(x,u_n)v_n + \frac{q_{r_n}(x,u_n)}{\|u_n\|_{\varphi_1}} \mid n \in \mathbb{N}\}$ is bounded in $\mathcal{A}$, we deduce from Assumption (H-1) that, up to a subsequence, $v_{n_k} \to v$ in $\mathcal{C}_{\varphi_1}$. It is then easy to see that $p_{r_{n_k}}(x,u_{n_k})v_{n_k} \to \lambda_1 v$ in $\mathcal{A}$ and $\frac{q_{r_{n_k}}(x,u_{n_k})}{\|u_{n_k}\|_{\varphi_1}} \to 0$ in $\mathcal{A}$. Passing to the limit in

$$v_{n_k} = T\left(p_{r_{n_k}}(x,u_{n_k})v_{n_k} + \frac{q_{r_{n_k}}(x,u_{n_k})}{\|u_{n_k}\|_{\varphi_1}}\right)$$

we obtain

$$v = \lambda_1 T v,$$

i.e. v is a solution of

$$\begin{cases} -\Delta v = \lambda_1 v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

By Proposition 2.5, we deduce that $v = \pm\varphi_1$. Hence for k large enough, either

$$u_{n_k} \geqslant \frac{1}{2}\|u_{n_k}\|_{\varphi_1}\varphi_1 \geqslant C\varphi_1 \gg \alpha$$

or

$$u_{n_k} \leqslant -\frac{1}{2}\|u_{n_k}\|_{\varphi_1}\varphi_1 \leqslant -C\varphi_1 \ll \beta,$$

which contradicts the localization $u_{n_k} \in \overline{\mathcal{O}}$ and the claim is proved.

Conclusion. We apply Theorem 3.2 to the problem (11) with

$$r = R := 1 + \max\{K, \|\alpha\|_\infty, \|\beta\|_\infty\}.$$

Let $w \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ be the solution of

$$\begin{cases} -\Delta w = \left(\lambda_1 - \dfrac{1}{R}d\right)w + d + \gamma, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

which exists by Proposition 2.6. Choose $a > 0$ large enough such that $\beta_2 := w + a\varphi_1 \geqslant C\varphi_1 \geqslant \alpha_2$. It is then easy to see that β_2 is an upper solution of (11) and in the same way, for $b > 0$ large enough, $\alpha_1 := -w - b\varphi_1 \leqslant -C\varphi_1 \leqslant \beta_1$ is a lower solution of (11). Hence we have the two pairs of lower and upper solutions required by Theorem 3.2.

Assume α_2 is not a strict lower solution. Then there exists a solution u of (11) with $u \geqslant \alpha_2$ and $u \not\gg \alpha_2$. As further $u(x_0) \geqslant \alpha_2(x_0) > \beta_1(x_0)$, we see that $u \in \overline{\mathcal{O}}$ and, by the Claim, $\|u\|_{\varphi_1} < R$. Hence u is a solution of (1) in $\overline{\mathcal{O}}$. The same argument holds in case β_1 is not a strict upper solution.

It remains to consider the case where α_2 and β_1 are strict. In that case, we deduce from Theorem 3.2 the existence of three solutions of (11), one of them, namely u, being in $\mathcal{O}$. Hence, from the Claim, we have $\|u\|_{\varphi_1} < R$ and u is a solution of (1) which concludes the proof. □

The assumption $\alpha \leqslant C\varphi_1$ is satisfied in case we require a little more regularity on f.

Proposition 3.4. *Let Ω be a bounded domain and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that Assumptions* (H-1) *and* (H-2) *are satisfied.*

Assume that there exist $\alpha \in \mathcal{C}(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ lower solution of (1). *Moreover suppose that there exists $h \in \mathcal{A}$ such that, for all $u \in [\alpha, \|\alpha\|_\infty + 1]$, $|f(x, u(x))| \leqslant h(x)$ a.e. in Ω.*

Then there exists $C > 0$ such that $\alpha \leqslant C\varphi_1$.

Proof. Let $R = \|\alpha\|_\infty$ and consider the function

$$\begin{aligned} \bar{f}(x,u) &= f(x,u) && \text{if } u \leqslant R \\ &= (R + 1 - u) f(x,u) && \text{if } R < u \leqslant R + 1 \\ &= 0 && \text{if } R + 1 < u, \end{aligned}$$

and the modified problem

$$\begin{cases} -\Delta u = \bar{f}(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Observe that $\beta = R + 1$ is an upper solution of (12) with $\alpha \leqslant \beta$. Hence we conclude by Theorem 3.1 the existence of $u \in \mathcal{C}_{\varphi_1}$ with $\alpha \leqslant u \leqslant \beta$ and hence, there exists $C > 0$ such that $\alpha \leqslant u \leqslant C\varphi_1$. □

Remark 3.5. The same type of result holds true for β.

4. Regular domain

If Ω is a regular domain of $\mathbb{R}^N$, the space $\mathcal{A}$ is simply $L^p(\Omega)$ for some $p > N$ and we recover the classical results.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega$ of class $\mathcal{C}^{1,1}$, f be an L^p-Carathéodory function with $p > N$. Assume that there exist α and $\beta \in \mathcal{C}(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ respectively lower and upper solutions of* (1)*.*

(i) *If $\alpha \leqslant \beta$ and there exists $h \in L^p(\Omega)$ such that, for all $u \in [\alpha, \beta]$, $|f(x,u(x))| \leqslant h(x)$ a.e. in Ω, then the problem* (1) *has at least one solution $u \in \mathcal{C}_{\varphi_1} \cap W^{2,p}(\Omega)$ such that*

$$\alpha \leqslant u \leqslant \beta.$$

(ii) *Assume $\alpha \nleqslant \beta$. If moreover, for every $R > 0$, there exists $h_R \in L^p(\Omega)$ such that, for all $u \in \mathbb{R}$ with $|u| \leqslant R$, $|f(x,u)| \leqslant h_R(x)$ a.e. in Ω and there exists $\gamma \in L^p(\Omega)$ such that, for all $(x,u) \in \Omega \times \mathbb{R}$, $|f(x,u) - \lambda_1 u| \leqslant \gamma(x)$, then the problem* (1) *has at least one solution $u \in \mathcal{C}_{\varphi_1} \cap W^{2,p}(\Omega)$ such that $u \in \overline{\mathcal{O}}$ where*

$$\mathcal{O} = \left\{ u \in \mathcal{C}_{\varphi_1} \,\middle|\, \min(u - \alpha) < 0 < \max(u - \beta) \right\}.$$

Remark 4.1. Recall that in this situation, we have $W^{2,p}(\Omega) \subset \mathcal{C}^1(\overline{\Omega})$.

Proof of Theorem 4.1. We apply the previous results with $\mathcal{A} = L^p(\Omega)$. We know by [11, Section 9.6] or [28, Lemmas 3.21 and 3.22], that for every $h \in L^p(\Omega)$, the problem (4) has a unique solution $u \in W^{2,p}(\Omega)$ and that the operator T is continuous from $L^p(\Omega)$ to $W^{2,p}(\Omega)$. Moreover $W^{2,p}(\Omega)$ is compactly imbedded into $\mathcal{C}^1(\overline{\Omega})$.

On the other hand, $\varphi_1 \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ and, by Theorem 2.2, is such that $\partial_\nu \varphi(x_0) < 0$ for all $x_0 \in \partial\Omega$ where $\nu = \nu(x_0)$ is the outward normal at x_0. Hence, we deduce, arguing as in [14, Lemma 3.1] that $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ is continuously imbedded into $\mathcal{C}_{\varphi_1}$ and Assumption (H-1) is satisfied.

Assumption (H-2) can be deduced easily from the Carathéodory condition.

Hence the result can be deduced from Theorems 3.1, 3.3 and Proposition 3.4. □

5. Polygonal domain of $\mathbb{R}^2$

In this section we consider the case where Ω is a polygonal domain of $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, in the following sense.

Definition 5.1. Let Ω be an open subset of $\mathbb{R}^2$. We say that Ω is a *polygonal domain* if it has the following properties:

(i) Ω is bounded, connected and is only on one side of its boundary;
(ii) the boundary of Ω is the union of a finite number of linear segments $\bar{\Gamma}_j$, $j \in \{1, \dots, J\}$; Γ_j being supposed to be open.

Denote by S_j, $j = 1, \dots, J$, the vertices of $\partial\Omega$ enumerated clockwise. Without loss of generality we may assume that $B(S_j, 1) \cap \Omega$ does not contain any other vertex of Ω. For $j \in \{1, 2, \dots, J\}$, let ψ_j be the interior angle of Ω at the vertex S_j, $\lambda_j = \frac{\pi}{\psi_j}$ and (r_j, θ_j) the polar coordinates centered at S_j such that

$$B(S_j, 1) \cap \Omega = \bigl\{(r_j \cos\theta_j, r_j \sin\theta_j) \bigm| 0 < r_j < 1,\ 0 < \theta_j < \psi_j\bigr\}.$$

It is well known [6,13,16,18] that the solution of the Dirichlet problem in Ω is not smooth in general. The singularities of this problem are of the form

$$r_j^{k\lambda_j} \sin(k\lambda_j\theta_j) \quad \text{near } S_j$$

with $k \in \mathbb{N}^*$. From this expression, we see that this last function is in $W^{2,p}(\Omega)$ if and only if $k\lambda_j > 2 - \frac{2}{p}$.

We now introduce our space $\mathcal{A}$.

Definition 5.2. We introduce the Banach space

$$L^p_{\vec{\mu}}(\Omega) := \bigl\{f \in L^p_{\mathrm{loc}}(\Omega):\ w^{\frac{1}{p}} f \in L^p(\Omega)\bigr\},$$

where the weight w is given by

$$w(x) = \begin{cases} r_j^{\mu_j}(x), & \text{on } B(S_j, 1),\ \forall j = 1, \dots, J, \\ 1, & \text{else,} \end{cases}$$

for $\vec{\mu} = (\mu_1, \dots, \mu_J) \in \mathbb{R}^J$.

Its natural norm is

$$\|f\|^p_{L^p_{\vec{\mu}}(\Omega)} = \int_\Omega w(x)\bigl|f(x)\bigr|^p\, dx.$$

Remark 5.1. If $\vec{\mu}_1$ and $\vec{\mu}_2$ are such that, $\forall j = 1, \dots, J$, $\mu_{1j} \leqslant \mu_{2j}$, then we have $L^p_{\vec{\mu}_1}(\Omega) \subset L^p_{\vec{\mu}_2}(\Omega)$. Hence, we try to take the powers as large as possible.

Our result is the next one

Theorem 5.1. *Let Ω be a polygonal domain of $\mathbb{R}^2$, $p > 2$ and $\mathcal{A} = L^p_{\vec{\mu}}(\Omega)$ with, in the previous notations,*

$$\mu_j < 2(p-1) - \lambda_j p, \quad \forall j = 1, \dots, J. \tag{13}$$

Assume that f is an $\mathcal{A}$-Carathéodory function and that there exist α and $\beta \in \mathcal{C}(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$, respectively lower and upper solutions of (1).

(i) *If $\alpha \leqslant \beta$ and there exists $h \in \mathcal{A}$ such that, for all $u \in [\alpha, \beta]$, $|f(x, u(x))| \leqslant h(x)$ a.e. in Ω, then the problem* (1) *has at least one solution $u \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ such that*

$$\alpha \leqslant u \leqslant \beta.$$

(ii) *Assume $\alpha \nleqslant \beta$ and, for some $C > 0$, $\alpha \leqslant C\varphi_1$ and $-C\varphi_1 \leqslant \beta$. If moreover, for every $R > C$, there exists $h_R \in \mathcal{A}$ such that, for all $u \in [\alpha, R\varphi_1] \cup [-R\varphi_1, \beta] \cup [-R\varphi_1, R\varphi_1]$, $|f(x, u(x))| \leqslant h_R(x)$ a.e. in Ω and there exists $\gamma \in \mathcal{A}$ such that, for all $(x, u) \in \Omega \times \mathbb{R}$, $|f(x, u) - \lambda_1 u| \leqslant \gamma(x)$, then the problem* (1) *has at least one solution $u \in \mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ such that $u \in \overline{\mathcal{O}}$ where*

$$\mathcal{O} = \left\{ u \in \mathcal{C}_{\varphi_1} \;\middle|\; \min(u - \alpha) < 0 < \max(u - \beta) \right\}.$$

To prove this result we need the following result on the first eigenfunction.

Lemma 5.2. *Let Ω be a polygonal domain of $\mathbb{R}^2$ and let S be one of its vertices. Denote by ψ the interior angle of Ω at the vertex S and $\lambda = \pi/\psi$. Then there exists $C_1 > 0$ such that*

$$\varphi_1(x) \geqslant C_1 r^\lambda \sin(\lambda\theta), \quad \text{in } B(S, 1) \cap \Omega, \tag{14}$$

where (r, θ) are the polar coordinates centered in S.

Moreover, for all $\gamma < \lambda$, there exists $C_2 > 0$ such that

$$\varphi_1(x) \leqslant C_2 r^\gamma, \quad \text{in } B(S, 1) \cap \Omega.$$

Proof. Let us denote $D := B(S, 1) \cap \Omega$.

By Proposition 2.4, $\varphi_1(r, \theta) > 0$ for $(r, \theta) \in]0, 1] \times]0, \psi[$ and by Theorem 2.2, we know that $\partial_\nu \varphi_1(1, \theta) < 0$ for $\theta \in \{0, \psi\}$. Hence, there exists $C_1 > 0$ such that $\varphi_1(1, \theta) \geqslant C_1 \sin(\lambda\theta)$ for $\theta \in [0, \psi]$. As

$$\begin{cases} -\Delta\varphi_1 = \lambda_1 \varphi_1 \geqslant 0 = -\Delta\big(C_1 r^\lambda \sin(\lambda\theta)\big), & \text{in } D, \\ \varphi_1 \geqslant C_1 r^\lambda \sin(\lambda\theta), & \text{on } \partial D, \end{cases}$$

we deduce from Corollary 2.3 that

$$\varphi_1(x) \geqslant C_1 r^\lambda \sin(\lambda\theta), \quad \text{on } D,$$

which proves the first part of the result.

To prove the second part of the result, we need the following claim.

Claim. *For $\gamma < \lambda$, if there exists $C_3 > 0$ such that $\varphi_1 \leqslant C_3 r^{\gamma-2}$ on D, then there exists $C_4 > 0$ such that $\varphi_1 \leqslant C_4 r^{\gamma}$ on D.*

To prove the claim, consider the function

$$u(r,\theta) = 2C_5 r^{\gamma} \frac{\sin \frac{\gamma(\psi-\theta)}{2} \sin \frac{\gamma\theta}{2}}{\gamma^2 \cos \frac{\gamma\psi}{2}},$$

where (r,θ) are the polar coordinates centered in S and $C_5 \geqslant \lambda_1 C_3$ is large enough such that $u(1,\theta) \geqslant \varphi_1(1,\theta)$ for $\theta \in [0,\psi]$. Observe then that

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 \leqslant \lambda_1 C_3 r^{\gamma-2} \leqslant C_5 r^{\gamma-2} = -\Delta u, & \text{in } D, \\ \varphi_1 = 0 = u, & \text{on } \partial D \cap B(0,1), \\ \varphi_1 \leqslant u, & \text{on } \partial D \cap \partial B(0,1). \end{cases}$$

Hence, by Corollary 2.3, $\varphi_1 \leqslant u$ and in particular, there exists $C_4 > 0$ such that $\varphi_1 \leqslant C_4 r^{\gamma}$ on D which proves the Claim.

Now to conclude the proof, observe that, as φ_1 is bounded, using the Claim, we can prove recursively that, for all $\gamma < \lambda$, there exists $C > 0$ with $\varphi_1 \leqslant C r^{\gamma}$ on D. $\square$

Proof of Theorem 5.1. In the course of this proof we will use the following notations: $a \lesssim b$ means the existence of a positive constant C, which is independent of the quantities a and b under consideration such that $a \leqslant Cb$ and $a \sim b$ means $a \lesssim b$ and $b \lesssim a$.

By Remark 5.1, we can suppose without loss of generality that

$$\mu_j > \max\{-2 - \lambda_j p, 2(\lambda_j - 1)(1-p)\}.$$

We apply the results of Section 3 with $\mathcal{A} = L^p_{\mu}(\Omega)$. The verification of Assumption (H-2) is easy via the L^p_{μ}-Carathéodory conditions. Moreover, by Remark 2.4, the cone K is normal and by Remark 2.5 or [13, Theorem 4.4.3.7], $\varphi_1 \in \mathcal{C}_0(\overline{\Omega})$. To verify that $\mathcal{C}_{\varphi_1}$ is continuously imbedded in $L^p_{\mu}(\Omega)$, let $f \in \mathcal{C}_{\varphi_1}$. By Lemma 5.2, for all $\gamma_j < \lambda_j$, we have $|f| \lesssim r_j^{\gamma_j}$ on $C_j(1) = \Omega \cap B(S_j, 1)$. Hence f is in $L^p_{\mu}(\Omega)$ as there exists $\gamma_j < \lambda_j$ such that

$$\int_0^1 r^{\gamma_j p + \mu_j + 1}\, dr < \infty$$

(which holds if $\mu_j > -2 - \lambda_j p$).

Let us concentrate on the verification that $T : L^p_{\mu}(\Omega) \to \mathcal{C}_{\varphi_1}$ is well defined and compact. We introduce the operator

$$T_0 : L^p_{\mu}(\Omega) \to H^1_0(\Omega) : h \to u,$$

where $u \in H_0^1(\Omega)$ is the unique (variational) solution of

$$\begin{cases} -\Delta u = h, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Step 1. The operator T_0 is well defined and continuous from $L^p_{\vec\mu}(\Omega)$ into $H_0^1(\Omega)$. To apply Lax–Milgram theorem, it suffices to show that

$$L : H_0^1(\Omega) \to \mathbb{R} : v \mapsto \int_\Omega h(x)v(x)\,dx$$

is continuous for $h \in L^p_{\vec\mu}(\Omega)$.

Let us concentrate on $\int_{D_j} h(x)v(x)\,dx$ for $D_j = \Omega \cap B(S_j, 1)$.

In case $\mu_j \leqslant 0$, let $q = \frac{p}{p-1}$. Using the continuous imbedding of $H_0^1(D_j)$ into $L^q(D_j)$, we obtain

$$\begin{aligned} \int_{D_j} hv\,dx &= \int_{D_j} r^{\mu_j/p} h r^{-\mu_j/p} v\,dx \\ &\leqslant \left\| r^{\mu_j/p} h \right\|_{L^p(D_j)} \left\| r^{-\mu_j/p} \right\|_{L^\infty(D_j)} \|v\|_{L^q(D_j)} \leqslant C\|v\|_{H_0^1(D_j)}. \end{aligned}$$

If $\mu_j > 0$, observe that $r^{-\mu_j/p} \in L^{q_j}(D_j)$ if $q_j < \frac{2p}{\mu_j}$. Hence, choosing $q_j \in]\frac{p}{p-1}, \frac{2p}{\mu_j}[$ (which is possible as $0 < \mu_j < 2(p-1) - \lambda_j p$) and defining $s_j = \frac{pq_j}{(p-1)q_j - p}$ so that $\frac{1}{p} + \frac{1}{q_j} + \frac{1}{s_j} = 1$ we have, using the continuous imbedding of $H_0^1(D_j)$ into $L^{s_j}(D_j)$

$$\begin{aligned} \int_{D_j} hv\,dx &= \int_{D_j} r^{\mu_j/p} h r^{-\mu_j/p}\, v\,dx \\ &\leqslant \left\| r^{\mu_j/p} h \right\|_{L^p(D_j)} \left\| r^{-\mu_j/p} \right\|_{L^{q_j}(D_j)} \|v\|_{L^{s_j}(D_j)} \leqslant C\|v\|_{H_0^1(D_j)}. \end{aligned}$$

Hence L is continuous and we conclude by Lax–Milgram theorem.

Step 2. The operator T is well defined and compact. Observe first that, by Corollary 2.3, (15) has at most one solution in $\mathcal{C}_{\varphi_1} \cap W^{2,p}_{\mathrm{loc}}(\Omega)$.

Let $(h_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p_{\vec\mu}(\Omega)$, i.e., there exists $R > 0$ such that, for all $n \in \mathbb{N}$,

$$\|h_n\|_{L^p_{\vec\mu}(\Omega)} \leqslant R. \tag{16}$$

Denote by $u_n \in H_0^1(\Omega)$ the unique solution of (15) with datum h_n.

Using regularity results far from the singular points of Ω for the Laplace equation with Dirichlet boundary conditions (see [11,13]), $u_n \in W^{2,p}(\tilde\Omega)$, where $\tilde\Omega$ is a subdomain of $\overline{\Omega}$ with a smooth boundary, its boundary being the same as Ω except in $\bigcup_{j=1}^J B(S_j, \delta)$ for some $\delta > 0$, with the estimate

$$\|u_n\|_{W^{2,p}(\tilde\Omega)} \lesssim \|h_n\|_{L^p_{\vec\mu}(\Omega)}, \quad \forall n \in \mathbb{N}.$$

As $W^{2,p}(\tilde{\Omega})$ is compactly imbedded into $\mathcal{C}^1(\overline{\tilde{\Omega}})$, we deduce that there exists $u \in \mathcal{C}^1(\overline{\tilde{\Omega}})$ such that $u_n \to u$ in $\mathcal{C}^1(\overline{\tilde{\Omega}})$ and hence, arguing as in [14, Lemma 3.1], we obtain

$$\sup_{x\in\overline{\tilde{\Omega}}\setminus\bigcup_{j=1}^{J}B(S_j,\delta)} \frac{|u_n(x)-u(x)|}{\varphi_1(x)} \to 0 \quad \text{as } n\to\infty. \tag{17}$$

It then remains to look at the behaviour of u_n near the corners. Therefore for any $j = 1,\dots,J$, we fix a cut-off function $\eta_j \in \mathcal{D}(\mathbb{R}^2)$ such that

$$\eta_j \equiv 1 \quad \text{near } C_j(1/2), \qquad \eta_j \equiv 0 \quad \text{on } \Omega\setminus C_j(1),$$

where $C_j(r)$ is the truncated cone $C_j(r) = \Omega \cap B(S_j, r)$. For shortness we write $C = C_j(1)$ and drop the index j. Let us set

$$\tilde{u}_n = \eta u_n.$$

This function satisfies

$$\begin{cases} -\Delta\tilde{u}_n = \tilde{h}_n, & \text{in } C,\\ \tilde{u}_n = 0, & \text{on } \partial C, \end{cases} \tag{18}$$

where $\tilde{h}_n = \eta h_n - 2\nabla\eta\cdot\nabla u_n - u_n\Delta\eta$. Moreover due to the above results (regularity far from the corners), $\tilde{h}_n$ belongs to $L^p_{\tilde{\mu}}(C)$ and there exists $R' > 0$ such that, for all $n\in\mathbb{N}$,

$$\|\tilde{h}_n\|_{L^p_{\tilde{\mu}}(C)} \leqslant R'.$$

Let us introduce the polar coordinates (r,θ) centered in S.

Let $\alpha = \frac{\psi}{2-p}(\mu - 2p + \lambda p + 2)$ and $\gamma = \frac{\alpha}{\psi}$. Observe that $\alpha \in\,]0,\pi[$ as $2(\lambda-1)(1-p) < \mu < 2(p-1) - \lambda p$. Using the change of variables similar to the one used in [23] $\rho = r^\gamma$, $\theta' = \gamma\theta$, the above problem (18) is transformed into

$$\begin{cases} -\Delta U_n = F_n, & \text{in } C_\alpha,\\ U_n = 0, & \text{on } \partial C_\alpha, \end{cases} \tag{19}$$

where $U_n(\rho,\theta') = \tilde{u}_n(\rho^{\frac{1}{\gamma}}, \frac{\theta'}{\gamma})$, $F_n(\rho,\theta') = \frac{1}{\gamma^2}\rho^{\frac{2(1-\gamma)}{\gamma}}\tilde{h}_n(\rho^{\frac{1}{\gamma}}, \frac{\theta'}{\gamma})$ and

$$C_\alpha = \bigl\{(\rho\cos\theta', \rho\sin\theta') \bigm| 0<\rho<1,\ 0<\theta'<\alpha\bigr\}.$$

Moreover U_n belongs to $H^1_0(C_\alpha)$, while F_n belongs to $L^p_{\mu_1}(C_\alpha)$ where $\mu_1 = (1-\frac{\pi}{\alpha})p$ with the estimate

$$\|F_n\|_{L^p_{\mu_1}(C_\alpha)} \lesssim \|\tilde{h}_n\|_{L^p_{\tilde{\mu}}(C)} \lesssim 1, \quad \forall n\in\mathbb{N}.$$

By definition of η, we have also that U_n is a solution of

$$\begin{cases} -\Delta U_n = F_n, & \text{in } \tilde{C}_\alpha,\\ U_n = 0, & \text{on } \partial\tilde{C}_\alpha, \end{cases} \tag{20}$$

where $\tilde{C}_\alpha$ has a smooth boundary except in S and coincide with C_α except in a neighbourhood of $\partial B(S,1)$.

Let us introduce the spaces $V^{k,p}(\tilde{C}_\alpha,\kappa)$ as the closure of

$$\mathcal{C}_S^\infty(\tilde{C}_\alpha)=\bigl\{v\in\mathcal{C}^\infty(\overline{\tilde{C}_\alpha})\bigm| S\notin\operatorname{supp} v\bigr\}$$

with respect to the norm

$$\|u\|_{V^{k,p}(\tilde{C}_\alpha,\kappa)}=\biggl(\sum_{|\gamma|\leqslant k}\int_{\tilde{C}_\alpha}\bigl|D^\gamma u(x)\bigr|^p r^{p(\kappa-k+|\gamma|)}(x)\,dx\biggr)^{1/p}.$$

As, by [17, Remark 9.11], $L^p_{\mu_1}(\tilde{C}_\alpha)=V^{0,p}(\tilde{C}_\alpha,1-\frac{\pi}{\alpha})$, applying [18, Lemma 11.2(ii)] (as in [18, Example 11.3]) we prove that $U_n\in V^{2,2}(\tilde{C}_\alpha,1)$. Hence, by [18, Corollary (iv) of Theorem 10.2] (see also [16, Section 8.4.1])

$$U_n=D_n\rho^{\frac{\pi}{\alpha}}\sin\biggl(\frac{\pi}{\alpha}\theta'\biggr)+W_n\quad\text{in a neighbourhood of } S$$

with $D_n\in\mathbb{R}$ and $W_n\in V^{2,p}(\tilde{C}_\alpha,1-\frac{\pi}{\alpha})$. By [18, Theorem 10.3], we have also

$$\|W_n\|_{V^{2,p}(\tilde{C}_\alpha,1-\frac{\pi}{\alpha})}+|D_n|\lesssim\|F_n\|_{V^{0,p}(\tilde{C}_\alpha,1-\frac{\pi}{\alpha})}\lesssim 1.$$

Now observe that the application

$$V^{2,p}\biggl(\tilde{C}_\alpha,1-\frac{\pi}{\alpha}\biggr)\to W^{2,p}(\tilde{C}_\alpha):u\to u/\rho^{\frac{\pi}{\alpha}-1}$$

is continuous and hence we have also

$$\biggl\|\frac{W_n}{\rho^{\frac{\pi}{\alpha}-1}}\biggr\|_{W^{2,p}(\tilde{C}_\alpha)}\lesssim\|F_n\|_{V^{0,p}(\tilde{C}_\alpha,1-\frac{\pi}{\alpha})}\lesssim 1.$$

By the compact imbedding of $W^{2,p}(\tilde{C}_\alpha)$ in $\mathcal{C}^1(\overline{\tilde{C}}_\alpha)$ (see [1, Theorem 6.2]), we have, passing to a subsequence that $D_{n_k}\to D$ and $\frac{W_{n_k}}{\rho^{\frac{\pi}{\alpha}-1}}\to W$ in $\mathcal{C}^1(\overline{\tilde{C}}_\alpha)$ and hence

$$\sup_{\tilde{C}_\alpha}\biggl|\frac{W_{n_k}(\rho,\theta')-\rho^{\frac{\pi}{\alpha}-1}W(\rho,\theta')}{\rho^{\frac{\pi}{\alpha}}\sin(\frac{\pi}{\alpha}\theta')}\biggr|\to 0,\quad\text{as } k\to\infty. \tag{21}$$

Going back to $\tilde{u}_{n_k}$, we get, for some $\epsilon>0$,

$$\sup_{C\cap B(S,\epsilon)}\biggl|\frac{\tilde{u}_{n_k}(r,\theta)-\tilde{u}(r,\theta)}{r^\lambda\sin(\lambda\theta)}\biggr|\to 0,\quad\text{as } k\to\infty,$$

where $\tilde{u}(r,\theta) = Dr^\lambda \sin(\lambda\theta) + r^{\lambda-\frac{\alpha}{\psi}} W(r^{\frac{\alpha}{\psi}}, \frac{\alpha}{\psi}\theta)$. Introducing (14) in the above convergence yields

$$\sup_{C\cap B(S,\epsilon)} \frac{|u_{n_k}(r,\theta) - \tilde{u}(r,\theta)|}{\varphi_1(r,\theta)} \to 0, \quad \text{as } k \to \infty, \tag{22}$$

if $\epsilon \leqslant \frac{1}{2}$ (as in that case $\tilde{u}_{n_k} = u_{n_k}$ in $C \cap B(S,\epsilon)$).

The conclusion follows from (17) and (22) as δ can be chosen smaller than ϵ by a correct choose of $\tilde{\Omega}$. $\square$

Remark 5.2. In the regular case, the condition (13) becomes $\mu_j < p-2$, so that, we can choose $\mu_j \geqslant 0$. Hence $L^p(\Omega) \subset L^p_{\mu}(\Omega)$ and this result improves Theorem 4.1.

Remark 5.3. The condition (13) is almost necessary. In fact consider the truncated cone C with angle ψ. Then if $\mu > 2(p-1) - \lambda p$ and $\gamma \in]\frac{2p-2-\mu}{p}, \lambda[$, the function $u(r,\theta) = r^\gamma \varphi(r)\sin(\lambda\theta)$ with φ a regular function such that $\varphi(r) = 1$ for $r < 1/4$ and $\varphi(r) = 0$ for $r > 1/2$ is a solution of

$$\begin{cases} -\Delta u = h, & \text{in } C, \\ u = 0, & \text{on } \partial C, \end{cases}$$

with $h \in L^p_\mu(C)$ and $u \notin \mathcal{C}_{\varphi_1}$.

6. Domain of $\mathbb{R}^N$, $N \geqslant 3$, with a conical point

Definition 6.1. Let Ω be an open subset of $\mathbb{R}^N$, $N \geqslant 3$. We say that Ω has a *conical point* if it has the following properties:

(i) Ω is bounded, connected and is only on one side of its boundary $\partial\Omega$;
(ii) the boundary $\partial\Omega$ is Lipschitz;
(iii) the boundary $\partial\Omega$ is smooth except at one point 0 (called a conical point), where there exists a neighborhood U of 0 such that $U \cap \Omega$ coincides with $U \cap K$, when K is an infinite cone centered at 0.

Without loss of generality we may assume that the conical point 0 is at the origin of the cartesian coordinates and that $B(0,1) \subset U$. Now we denote by G the intersection between K and the unit sphere. Denote furthermore by (r,θ) the spherical coordinates centered at 0.

As before, the solution of the Dirichlet problem in Ω is not smooth in general (see e.g. [6,13, 16,18]). The singularities are here of the form

$$r^{\lambda'} \psi_{\lambda'}(\theta),$$

where $\psi_{\lambda'}$ is the eigenfunction of the positive Laplace–Beltrami operator δ_G with Dirichlet boundary conditions of eigenvalue ν':

$$\begin{cases} \delta_G \psi_{\lambda'} = \nu' \psi_{\lambda'}, & \text{in } G, \\ \psi_{\lambda'} = 0, & \text{on } \partial G, \end{cases}$$

the singular exponent λ' being related to the eigenvalue ν' by the relation

$$\lambda' = 1 - \frac{N}{2} + \sqrt{\nu' + \left(1 - \frac{N}{2}\right)^2}.$$

Denote by λ the smallest singular exponent (corresponding to the smallest eigenvalue ν of δ_G). As in the previous section, we are now able to introduce our space $\mathcal{A}$.

Definition 6.2. Let Ω be a domain with one conical point in 0. We introduce the Banach space

$$L^p_\mu(\Omega) := \{f \in L^p_{\mathrm{loc}}(\Omega):\ r^{\mu/p} f \in L^p(\Omega)\}.$$

Its natural norm is

$$\|f\|^p_{L^p_\mu(\Omega)} = \int_\Omega r^\mu |f(x)|^p\,dx.$$

In this setting, we can prove the

Theorem 6.1. *Let Ω be a domain with one conical point in* 0*, $p > N$ such that $1 - N/p \neq \lambda' - \lambda$ for all singular exponent λ' and $\mathcal{A} = L^p_\mu(\Omega)$ with $\mu = (1 - \lambda)p$. Then the statements of Theorem* 5.1 *remain valid.*

As before we need the following result on the first eigenfunction.

Lemma 6.2. *Let Ω be a domain with one conical point* 0*. Then there exists $C_1 > 0$ such that*

$$\varphi_1(x) \geqslant C_1 r^\lambda \psi_\lambda(\theta), \quad \text{in } B(0,1) \cap \Omega. \tag{23}$$

Moreover, for all $\gamma < \lambda$, there exists $C_2 > 0$ such that

$$\varphi_1(x) \leqslant C_2 r^\gamma, \quad \text{in } B(0,1) \cap \Omega.$$

Proof. The proof is quite similar to the one of Lemma 5.2. Let us give it for the sake of completeness. Let us denote $D := B(0,1) \cap \Omega$.

By Proposition 2.4, we know that $\varphi_1(r,\theta) > 0$ for $(r,\theta) \in]0,1] \times G$ and by Theorem 2.2, $\partial_\nu \varphi_1(1,\theta) < 0$ for $\theta \in \partial G$. Hence, there exists $C_1 > 0$ such that $\varphi_1(1,\theta) \geqslant C_1 \psi_\lambda(\theta)$ for $\theta \in \bar{G}$. As

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 \geqslant 0 = -\Delta\big(C_1 r^\lambda \psi_\lambda(\theta)\big), & \text{in } D,\\ \varphi_1 \geqslant C_1 r^\lambda \psi_\lambda(\theta), & \text{on } \partial D,\end{cases}$$

the first part of the result directly follows from Corollary 2.3.

To prove the second part of the result, we need the following claim.

Claim. *For $\gamma < \lambda$, if there exists $C_3 > 0$ such that $\varphi_1 \leqslant C_3 r^{\gamma-2}$ on D, then there exists $C_4 > 0$ such that $\varphi_1 \leqslant C_4 r^\gamma$ on D.*

To prove the Claim, consider the solution U of

$$\begin{cases} -\Delta U = r^{\gamma-2}, & \text{in } K,\\ U = 0, & \text{on } \partial K, \end{cases}$$

where K is given in the definition of a conical point. Such a solution exists and is of the form

$$U(r,\theta) = r^{\gamma}\chi(\theta),$$

for some smooth function χ because γ is not a singular exponent.

Now consider $u = C_5 U$, with C_5 large enough such that $C_5 \geqslant \lambda_1 C_3$ and $u(1,\theta) \geqslant \varphi_1(1,\theta)$ for $\theta \in \bar{G}$ (always possible because $\partial_\nu U(1,\theta) < 0$ for $\theta \in \partial G$). Now we observe that

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 \leqslant \lambda_1 C_3 r^{\gamma-2} \leqslant C_5 r^{\gamma-2} = -\Delta u, & \text{in } D,\\ \varphi_1 = 0 = u, & \text{on } \partial D \cap B(0,1),\\ \varphi_1 \leqslant u, & \text{on } \partial D \cap \partial B(0,1). \end{cases}$$

Hence, by Corollary 2.3, $\varphi_1 \leqslant u$, which implies the existence of $C_4 > 0$ such that $\varphi_1 \leqslant C_4 r^{\gamma}$ on D which proves the Claim.

As in Lemma 5.2, we conclude by recurrence since φ_1 is bounded. □

Proof of Theorem 6.1. We apply the results of Section 3 with $\mathcal{A} = L^p_\mu(\Omega)$. The main difficulty is the verification that $T : \mathcal{A} \to \mathcal{C}_{\varphi_1}$ is well defined and compact. The rest of the proof follows as in Theorem 5.1.

Step 1. The operator T_0 is well defined and continuous from $L^p_\mu(\Omega)$ into $H^1_0(\Omega)$. As in 2d, we are reduced to show that $\int_D h(x)v(x)\,dx$ is well defined for $h \in L^p_\mu(\Omega)$ and $v \in H^1_0(\Omega)$, where $D = \Omega \cap B(0,1)$.

In case $\mu \leqslant 0$, let $q = \frac{p}{p-1}$. As the condition $p > N$ implies that $q \leqslant \frac{2N}{N-2}$, using the continuous imbedding of $H^1(D)$ into $L^q(D)$, we conclude as in 2d.

If $\mu > 0$, we here remark that $r^{-\mu/p} \in L^q(D)$ if $q < \frac{Np}{\mu}$. Hence defining $s = \frac{pq}{(p-1)q-p}$ so that $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$ we have, using the continuous imbedding of $H^1(D)$ into $L^s(D)$ for $s \leqslant \frac{2N}{N-2}$

$$\int_D hv\,dx \leqslant \|r^{\mu/p}h\|_{L^p(D)} \|r^{-\mu/p}\|_{L^q(D)} \|v\|_{L^s(D)} \leqslant C\|v\|_{H^1(D)}.$$

Note that the conditions $1 < s \leqslant \frac{2N}{N-2}$ are satisfied if and only if

$$\frac{2Np}{Np+2p-2N} \leqslant q \quad \text{and} \quad \frac{p}{p-1} < q.$$

Since q must be chosen such that $q < \frac{Np}{\mu}$, these two conditions are valid if

$$\frac{2Np}{Np+2p-2N} < \frac{Np}{\mu} \quad \text{and} \quad \frac{p}{p-1} < \frac{Np}{\mu},$$

or equivalently

$$\mu < \frac{Np + 2p - 2N}{2} \quad \text{and} \quad \mu < N(p-1).$$

As $\mu = (1-\lambda)p < p$, these last conditions hold since $p > N$.

Step 2. The operator T is well defined and compact. Let $(h_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p_\mu(\Omega)$, i.e., there exists $R > 0$ such that, for all $n \in \mathbb{N}$,

$$\|h_n\|_{L^p_\mu(\Omega)} \leqslant R. \tag{24}$$

Denote by $u_n \in H^1_0(\Omega)$ the unique solution of (15) with datum h_n.

Let us introduce the spaces $V^{k,p}(\Omega,\mu)$ as the closure of

$$\mathcal{C}^\infty_S(\Omega) = \bigl\{v \in \mathcal{C}^\infty(\overline{\Omega}) \bigm| 0 \notin \operatorname{supp} v\bigr\}$$

with respect to the norm

$$\|u\|_{V^{k,p}(\Omega,\mu)} = \biggl(\sum_{|\gamma|\leqslant k} \int_\Omega \bigl|D^\gamma u(x)\bigr|^p r^{p(\mu-k+|\gamma|)}(x)\,dx\biggr)^{1/p}.$$

As, by [17, Remark 9.11], $L^p_\mu(\Omega) = V^{0,p}(\Omega, 1-\lambda)$, applying [18, Lemma 11.2(ii)] (as in [18, Example 11.3]) we deduce that $u_n \in V^{2,2}(\Omega,1)$. Hence, by [18, Corollary (iv) of Theorem 10.2] (see also [16])

$$u_n = \eta \sum_{1-\frac{N}{2}<\lambda'<\lambda+1-\frac{N}{p}} c_{\lambda',n} r^{\lambda'} \psi_{\lambda'}(\theta) + w_n,$$

where η is a cut-off function equal to 1 near 0 and with a sufficiently small support, $c_{\lambda',n} \in \mathbb{R}$ and $w_n \in V^{2,p}(\Omega, 1-\lambda)$. By [18, Theorem 10.3], we have also

$$\|w_n\|_{V^{2,p}(\Omega,1-\lambda)} + \sum_{1-\frac{N}{2}<\lambda'<\lambda+1-\frac{N}{p}} |c_{\lambda',n}| \lesssim \|h_n\|_{V^{0,p}(\Omega,1-\lambda)} \lesssim 1.$$

Now observe that the application

$$V^{2,p}(\Omega, 1-\lambda) \to W^{2,p}(\Omega): u \to u/r^{\lambda-1}$$

is continuous and hence we have also

$$\biggl\|\frac{w_n}{r^{\lambda-1}}\biggr\|_{W^{2,p}(\Omega)} \lesssim \|h_n\|_{V^{0,p}(\Omega,1-\lambda)} \lesssim 1.$$

By the compact imbedding of $W^{2,p}(\Omega)$ in $\mathcal{C}^1(\overline{\Omega})$ (see [1, Theorem 6.2]), we have, passing to a subsequence that $c_{\lambda',n_k} \to c_{\lambda'}$ and $\frac{w_{n_k}}{r^{\lambda-1}} \to w$ in $\mathcal{C}^1(\overline{\Omega})$. As w_{n_k} is equal to zero on $\partial\Omega \cap B(0,1)$, by Lemma 6.3 below we deduce that

$$\sup_{x\in D}\left|\frac{\frac{w_{n_k}}{r^{\lambda-1}} - w}{r\psi_\lambda(\theta)}\right| \lesssim \left\|\frac{w_{n_k}}{r^{\lambda-1}} - w\right\|_{\mathcal{C}^1(\overline{\Omega})} \to 0 \quad \text{as } k \to \infty. \tag{25}$$

Now setting

$$u := \eta \sum_{1-\frac{N}{2}<\lambda'<\lambda+1-\frac{N}{p}} c_{\lambda'} r^{\lambda'} \psi_{\lambda'}(\theta) + r^{\lambda-1} w,$$

using Lemma 6.2, we see that

$$\begin{aligned}\left|\frac{u_{n_k}(x) - u(x)}{\varphi_1(x)}\right| &\lesssim \sum_{1-\frac{N}{2}<\lambda'<\lambda+1-\frac{N}{p}} |c_{\lambda',n_k} - c_{\lambda'}| r^{\lambda'-\lambda} \frac{|\psi_{\lambda'}(\theta)|}{\psi_\lambda(\theta)} \\ &\quad + \sup_{x\in D}\left|\frac{\frac{w_{n_k}(x)}{r^{\lambda-1}} - w(x)}{r\psi_\lambda(\theta)}\right| + \sup_{x\in\Omega\setminus D}\left|\frac{w_{n_k}(x) - r^{\lambda-1}w(x)}{\varphi_1(x)}\right|.\end{aligned}$$

By the estimate (25) and the estimate (consequence of [14, Lemma 3.1])

$$\frac{|\psi_{\lambda'}(\theta)|}{\psi_\lambda(\theta)} \lesssim 1,$$

we therefore conclude that u_{n_k} converges to u in $\mathcal{C}_{\varphi_1}$, applying again [14, Lemma 3.1] for the last term of the sum. □

Lemma 6.3. *Using the above notations, there exists a positive constant C such that, for all $w \in \mathcal{C}^1(\overline{\Omega})$ with $w = 0$ on $\partial\Omega \cap B(0,1)$,*

$$\sup_{x\in D}\left|\frac{w}{r\psi_\lambda(\theta)}\right| \leqslant C\|w\|_{\mathcal{C}^1(\overline{\Omega})}.$$

Proof. Denote by

$$S = \{x \in K\colon\ 1/2 < |x| < 1\}.$$

Let $\hat{w}$ be a function in $\mathcal{C}^1(\bar{S})$ such that

$$\hat{w} = 0 \quad \text{on } \partial S \cap \partial K.$$

For any $\hat{x}_0 \in \partial S \cap \partial K$ set $\hat{x} = \hat{x}_0 - t\nu$, for $t > 0$ small enough, ν being the outward normal vector at $\hat{x}_0$. By Taylor's theorem we may write

$$|\hat{w}(\hat{x})| \leqslant \|\nabla\hat{w}\|_{\mathcal{C}(S)} t.$$

Applying the same Taylor's expansion to $\hat{w}_0 = r\psi_\lambda(\theta)$, we have

$$\hat{w}_0(\hat{x}) = -\partial_\nu \hat{w}_0(\tilde{x}_0)t,$$

where $\tilde{x}_0 = \hat{x}_0 - \tilde{t}\nu$ for some $\tilde{t} \in]0, t[$. By Theorem 2.2, we know that

$$\partial_\nu \hat{w}_0(\hat{x}_0) < 0 \quad \forall \hat{x}_0 \in \partial S \cap \partial K,$$

and consequently there exists t_0 small enough such that

$$\inf_{0<t<t_0,\, \hat{x}_0 \in \partial S \cap \partial K} \left|\partial_\nu \hat{w}_0(\hat{x}_0 - t\nu)\right| > 0.$$

Therefore there exists $C > 0$ such that

$$\left|\hat{w}(\hat{x})\right| \leqslant C \|\nabla \hat{w}\|_{\mathcal{C}(S)} \hat{w}_0(\hat{x}), \quad \forall \hat{x} \in \hat{V}, \tag{26}$$

where $\hat{V}$ is a fixed neighborhood of $\partial S \cap \partial K$.

Now coming back to w, for any fixed $s < 1$, we set $\hat{x} = x/s$ and

$$\hat{w}(\hat{x}) = w(x).$$

Applying the estimate (26) to $\hat{w}$ and a scaling argument yield

$$\left|w\big(re^{i\theta}\big)\right| \leqslant Cr \|\nabla w\|_{\mathcal{C}(D)} \psi_\lambda(\theta), \quad \forall \theta \in \hat{W},\ r < 1, \tag{27}$$

where $\hat{W}$ is a fixed neighborhood of ∂G. This estimate proves the result because outside $\hat{W}$ the function ψ_λ is strictly positive. □

Remark 6.1. The results of this section are also valid for $N = 2$. But in that case, our condition on μ is not optimal since $(1 - \lambda)p < 2(p - 1) - \lambda p$. This is the reason of the use of finer arguments in 2d. For p near 2, the two conditions are almost similar since $(1 - \lambda)p$ is close to $2(p - 1) - \lambda p$ for p close to 2.

7. Examples

Until now, we have illustrated Assumption (H-1) in three different situations assuming the existence of appropriate lower and upper solutions and related conditions on the nonlinearity f. Now in order to show the usefulness of our results, we turn to some particular nonlinearities for which the above assumptions are satisfied.

Example 7.1. The easiest choice of lower and upper solutions are constant functions. For example, let Ω be a domain as considered in Section 4, 5 or 6 and consider the problem

$$\begin{cases} -\Delta u = g(x)\big(u \sin u + h(x)\big), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $h \in L^\infty(\Omega)$ and $g \in \mathcal{A}$, $g \geqslant 0$, with $\mathcal{A}$ defined in the corresponding section. Then, for $k > 0$ large enough, $\alpha = \frac{\pi}{2} - 2k\pi$ and $\beta = \frac{3\pi}{2} + 2k\pi$ are lower and upper solutions of the problem with $\alpha \leqslant \beta$ and we can apply the corresponding result to prove the existence of a solution to the above problem.

Example 7.2. A more interesting situation is for example the problem

$$\begin{cases} -\Delta u = g(x) - \operatorname{sgn}(u)|u|^{\xi}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{28}$$

where, as in Section 5, Ω is a polygonal domain in $\mathbb{R}^2$, $g \in L^p_{\vec{\mu}}(\Omega)$ with, in the notations of Section 5,

$$\mu_j < 2(p-1) - \lambda_j p, \quad \forall j = 1, \dots, J,$$

and $\xi > 0$. In that case, there is no hope to apply the result with constant lower and upper solutions if ψ_j is small as the constant functions are in $L^p_{\vec{\mu}}(\Omega)$ with $\vec{\mu}$ satisfying the required restriction only if $\psi_j > \frac{\pi}{2}$.

So let $w \in \mathcal{C}_{\varphi_1}$ be the solution of

$$\begin{cases} -\Delta w = g(x), & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

which exists as follows from the proof of Theorem 5.1. Let $R > 0$ be large enough such that $|w| \leqslant R\varphi_1$ in Ω. Then $\alpha = w - R\varphi_1$ and $\beta = w + R\varphi_1$ are lower and upper solutions of (28) with $\alpha \leqslant 0 \leqslant \beta$. This follows from

$$\begin{aligned} -\Delta\alpha &= g - R\lambda_1\varphi_1 \leqslant g - \operatorname{sgn}(\alpha)|\alpha|^{\xi}, \quad \text{in } \Omega, \\ -\Delta\beta &= g + R\lambda_1\varphi_1 \geqslant g - \operatorname{sgn}(\beta)|\beta|^{\xi}, \quad \text{in } \Omega, \end{aligned}$$

as the boundary conditions are trivially satisfied. Now observe that, for all $u \in [\alpha, \beta]$, i.e. such that $\alpha(x) \leqslant u(x) \leqslant \beta(x)$ in Ω (as $\alpha, \beta \in \mathcal{C}_{\varphi_1}$), we have

$$\left|f(x,u)\right| = \left|g(x) - \operatorname{sgn}(u)|u|^{\xi}\right| \leqslant \left|g(x)\right| + K\varphi_1^{\xi}(x),$$

for some $K > 0$. Hence we can apply Theorem 5.1 to prove the existence of a solution of (28) if $\xi > \frac{\lambda_j - 2}{\lambda_j}$, for all $j = 1, \dots, J$ (this last condition guarantees that $\varphi_1^{\xi} \in L^p_{\vec{\mu}}(\Omega)$).

Example 7.3. In this third example, we want to consider the so-called Landesman–Lazer conditions at the right of the first eigenvalue. These conditions where introduced by E.M. Landesman and A.C. Lazer [19] in 1970 in the regular case. The first ones to consider such problems at the left of the first eigenvalue via lower and upper solutions seem to be J.L. Kazdan and F.W. Warner [15] in 1975. Extensions to nonlinearities between the two first eigenvalues had been considered mainly by J.-P. Gossez and P. Omari [12] and P. Habets and P. Omari [14] (see also [7]) with the help of non-ordered lower and upper solutions as here.

Consider the problem

$$\begin{cases} -\Delta u = \lambda_1 u + g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{29}$$

where Ω is a domain as in Section 4, 5 or 6 and let $\mathcal{A}$ be defined in the corresponding section. Assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(a) there exist $s_+ \in \mathbb{R}$ and $g_+ \in \mathcal{A}$ such that

$$g(x,u) \geqslant g_+(x) \quad \text{if } u \geqslant s_+\varphi_1(x) \quad \text{and} \quad \int_\Omega g_+(x)\varphi_1(x)\,dx \geqslant 0;$$

(b) there exist $s_- \in \mathbb{R}$ and $g_- \in \mathcal{A}$ such that

$$g(x,u) \leqslant g_-(x) \quad \text{if } u \leqslant s_-\varphi_1(x) \quad \text{and} \quad \int_\Omega g_-(x)\varphi_1(x)\,dx \leqslant 0;$$

(c) there exists $\gamma \in \mathcal{A}$ such that

$$\big|g(x,u)\big| \leqslant \gamma(x), \quad \text{for } (x,u) \in \Omega \times \mathbb{R}.$$

Then (29) has at least one solution.

To prove this result, we need first to prove that, for all $h \in \mathcal{A}$ with $\int_\Omega h(x)\,\varphi_1(x)\,dx = 0$, the problem

$$\begin{cases} -\Delta u = \lambda_1 u + h(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{30}$$

has a solution in $\mathcal{C}_{\varphi_1}$.

Let $X = \{u \in \mathcal{C}_{\varphi_1} \mid \int_\Omega u(x)\varphi_1(x)\,dx = 0\}$ and define $T_1 : X \to X$ by $T_1(v) = u$ where u is the unique solution of

$$\begin{cases} -\Delta u = \lambda_1 v + h, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Observe that this problem has a unique solution in $\mathcal{C}_{\varphi_1}$ as it is proved in the corresponding section but also that

$$0 = \int_\Omega (h + \lambda_1 v)\varphi_1\,dx = -\int_\Omega \Delta u\varphi_1\,dx = \lambda_1 \int_\Omega u\varphi_1\,dx.$$

Hence T_1 is well defined.

Now let us prove that T_1 has a fixed point. Consider the homotopy

$$\begin{cases} -\Delta u = \lambda_1 u + \mu h(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{31}$$

and observe that we have an a priori bound on the solutions of (31) in X for $\mu \in [0,1]$. Otherwise we have a sequence $(u_n)_n \subset X$ of solutions corresponding to the sequence $(\mu_n)_n \subset [0,1]$ such that $\|u_n\|_{\mathcal{C}_{\varphi_1}} \to \infty$. Set $v_n = u_n/\|u_n\|_{\mathcal{C}_{\varphi_1}}$ and observe that v_n satisfies

$$\begin{cases} -\Delta v_n = \lambda_1 v_n + \dfrac{\mu_n}{\|u_n\|_{\mathcal{C}_{\varphi_1}}} h(x), & \text{in } \Omega, \\ v_n = 0, & \text{on } \partial\Omega. \end{cases}$$

As $T : \mathcal{A} \to \mathcal{C}_{\varphi_1}$ is compact, we have that, up to a subsequence, $v_n \to v$ with v solution of

$$\begin{cases} -\Delta v = \lambda_1 v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

such that $\|v\|_{\mathcal{C}_{\varphi_1}} = 1$ and $\int_\Omega v\varphi_1\,dx = 0$. We deduce from Proposition 2.5 that $v = \pm\varphi_1$ which contradicts $\int_\Omega v\varphi_1\,dx = 0$. Hence there exists $R > 0$ such that, for all $\mu \in [0,1]$, every solution $u \in X$ of (31) satisfies $\|u\|_{\mathcal{C}_{\varphi_1}} < R$. As for $\mu = 0$ the only solution of (31) in X is $u = 0$ and the corresponding fixed point operator is linear, we have, by homotopy invariance,

$$\deg\big(I - T_1, B(0,R)\big) \neq 0$$

and there exists $u \in X$ solution of (30).

Using this result, let us show how to construct a lower solution $\alpha \geqslant s_+\varphi_1$. Assume without loss of generality that we choose the normalization of φ_1 in such a way that $\int_\Omega \varphi_1^2(x)\,dx = 1$. Let w be the solution in X of

$$\begin{cases} -\Delta w = \lambda_1 w + g_+ - \Big(\displaystyle\int_\Omega g_+\varphi_1\,dx\Big)\varphi_1, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

and define $\alpha = R\varphi_1 + w$ with R large enough in such a way that $\alpha \geqslant s_+\varphi_1$. Observe that, in Ω,

$$\begin{aligned} -\Delta\alpha &= \lambda_1\alpha + g_+ - \Big(\int_\Omega g_+\varphi_1\,dx\Big)\varphi_1 \\ &\leqslant \lambda_1\alpha + g(x,\alpha) - \Big(\int_\Omega g_+\varphi_1\,dx\Big)\varphi_1 \\ &\leqslant \lambda_1\alpha + g(x,\alpha) \end{aligned}$$

and hence, α is a lower solution. We construct in the same way an upper solution $\beta \leqslant s_-\varphi_1$. Moreover there exists $C > 0$ such that $-C\varphi_1 \leqslant \beta \leqslant \alpha \leqslant C\varphi_1$. By assumption, there exists $\gamma \in \mathcal{A}$ such that $|g(x,u)| \leqslant \gamma(x)$ in $\Omega \times \mathbb{R}$, and, for all $R > C$ and all $u \in [\alpha, R\varphi_1] \cup [-R\varphi_1, \beta] \cup [-R\varphi_1, R\varphi_1]$ we have

$$\big|\lambda_1 u + g(x,u)\big| \leqslant \lambda_1 R\varphi_1(x) + \gamma(x) = h_R(x)$$

with $h_R = \lambda_1 R\varphi_1 + \gamma \in \mathcal{A}$. Hence all the conditions to apply (ii) of Theorem 4.1, 5.1 or 6.1 are satisfied and we have proved the existence of a solution of (29).

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