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Semi-classical ground states concentrating on the nonlinear potential for a Dirac equation

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ABSTRACT

We study the semi-classical limit of the least energy solutions to the nonlinear Dirac equation

$$-i\varepsilon\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u = P(x)|u|^{p-2}u$$

for $x \in \mathbb{R}^3$. Since the Dirac operator is unbounded from below and above, the associate energy functional is strongly indefinite, and since the problem is considered in the global space $\mathbb{R}^3$, the Palais–Smale condition is not satisfied. New phenomena and mathematical interests arise in the use of the calculus of variations. We prove that the equation has the least energy solutions for all $\varepsilon > 0$ small, and additionally these solutions converge to the least energy solutions of the associate limit problem and concentrate to the maxima of the nonlinear potential $P(x)$ in certain sense as $\varepsilon \to 0$.

1. Introduction and main results

In this paper we are concerned with the semi-classical ground states to the stationary Dirac equation in relativistic quantum mechanics:

$$-i\hbar\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u = P(x)|u|^{p-2}u \tag{1}$$

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with $p \in (2, 3)$. In contrast to the related results in the literature, we are interested in the following new cases: *firstly, existence of solutions when $P(x)$ depends indeed on x but having neither periodicity nor limit at the infinity*; *secondly, concentration on the maxima of the coefficient of nonlinear external field.* Here, $\hbar$ denotes the Plank's constant, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\partial_k = \frac{\partial}{\partial x_k}$, $a > 0$ is a constant, $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 complex matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $P \in C(\mathbb{R}^3, \mathbb{R})$.

The equation or the more general one

$$-i\hbar \sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u + M(x)u = F_u(x, u) \tag{2}$$

arises when one seeks for the standing wave solutions of the nonlinear Dirac equation

$$-i\hbar \partial_t \psi = ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k \psi - mc^2 \beta \psi - V(x)\psi + G_\psi(x, \psi). \tag{3}$$

Assuming that $G(x, e^{i\theta}\psi) = G(x, \psi)$ for all $\theta \in [0, 2\pi]$, a standing wave solution of (3) is a solution of the form $\psi(t, x) = e^{\frac{i\mu t}{\hbar}} u(x)$. It is clear that $\psi(t, x)$ solves (3) if and only if $u(x)$ solves (2) with $a = mc$, $M(x) = V(x)/c + \mu I_4$ and $F(x, u) = G(x, u)/c$.

There are many papers devoted to the study on the existence of solutions of (2) under various hypotheses on the potential and the nonlinearity (see [18] for a review). In [4,5,10,24] the authors studied the problem with $M(x) \equiv \omega \in (-a, a)$ and the nonlinearity (the so-called Soler model)

$$F(u) = \frac{1}{2} H(\tilde{u}u), \quad H \in C^2(\mathbb{R}, \mathbb{R}),\ H(0) = 0,\ \tilde{u}u := (\beta u, u)_{\mathbb{C}^4},$$

and in [19] Finkelstein et al. considered the nonlinearity

$$F(u) = \frac{1}{2} |\tilde{u}u|^2 + b|\tilde{u}\alpha u|^2, \quad \tilde{u}\alpha u := (\beta u, \alpha u)_{\mathbb{C}^4},\ \alpha := \alpha_1\alpha_2\alpha_3,$$

by using shooting methods. Such a kind of nonlinearities was later studied in Esteban and Séré [17] by using firstly a variational method (in fact, [17] also considered certain more general super-linear subcritical $F(u)$ independent of x). If the equation is periodic, that is, $M(x)$ and $F(x, u)$ depend periodically on x, by using a critical point theory Bartsch and Ding [6] established also the existence and multiplicity of solutions of (2) with scalar potentials of the type $M(x) = V(x)\beta$ (see also [33]). If the potential is non-periodic (typically, the Coulomb-type potential), in [15] Ding and Ruf considered some asymptotically quadratic nonlinearities, and in [16] Ding and Wei treated the super-quadratic subcritical nonlinearities with mainly the limits of $M(x)$ and $F(x, u)$ existing as $|x| \to \infty$.

For small $\hbar$, the standing waves are referred to as semi-classical states. To describe the translation from quantum to classical mechanics, the existence of solutions u_ε, $\hbar$ small, possesses an important physical interest. Only very recently, the paper [14] studied the existence of a family of ground states of the problem

$$-i\hbar\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + M(x)u = |u|^{p-2}u$$

for all $\hbar$ small, and showed that the family concentrates around the minima of $M(x)$ as $\hbar \to 0$.

Comparing with [14], in this paper we are interested in the existence and concentration phenomenon around the maxima of the nonlinear external field. Remark that, mathematically, since the Dirac operator is unbounded from above and below, the associated energy functional is strongly indefinite, and since the problem is posed on the whole space $\mathbb{R}^3$, the functional does not satisfy the Palais–Smale condition. In addition, since the solutions depend on the coefficient of nonlinear term, the present argument seems to be more delicate than those of [14].

We now describe our result. For notational convenience, writing $\varepsilon = \hbar$, $w = u$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha \cdot \nabla = \sum_{k=1}^{3}\alpha_k\partial_k$, we reread Eq. (1) as

$$-i\varepsilon\alpha \cdot \nabla w + a\beta w = P(x)|w|^{p-2}w. \tag{4}$$

It is standard that (4) is equivalent to

$$-i\alpha \cdot \nabla u + a\beta u = P_\varepsilon(x)|u|^{p-2}u \tag{5}$$

where $P_\varepsilon(x) = P(\varepsilon x)$ with $u(x) \leftrightarrow w(\varepsilon x)$. Assume P satisfies

(P_0) $\inf P > 0$ and $\limsup_{|x|\to\infty} P(x) < \max P(x)$.

Set $m := \max_{x\in\mathbb{R}^3} P(x)$, and

$$\mathscr{P} := \bigl\{x \in \mathbb{R}^3\colon\ P(x) = m\bigr\}.$$

The limit problem associated with (5) reads as

$$-i\alpha \cdot \nabla u + a\beta u = m|u|^{p-2}u. \tag{6}$$

Denote the energy of a solution $w \neq 0$ of (4) by

$$E(w) := \int_{\mathbb{R}^3}\left(\frac{1}{2}\bigl\langle(-i\varepsilon\alpha\cdot\nabla + a\beta)w, w\bigr\rangle - \frac{1}{p}P(x)|w|^p\right)dx,$$

here and in the sequel $\langle\cdot,\cdot\rangle$ denotes the usual scale product in $\mathbb{C}^4$. Setting

$$\tau_\varepsilon := \inf\bigl\{E(w)\colon\ w \neq 0 \text{ is a solution of } (4)\bigr\},$$

a solution $w_0 \neq 0$ with $E(w_0) = \tau_\varepsilon$ is called a least energy solution. Let $\mathscr{S}_\varepsilon$ denote the set of all least energy solutions of (4).

Theorem 1.1. *Assume that $p \in (2,3)$ and (P_0) is satisfied. Then for all $\varepsilon > 0$ small,*

(i) *Eq. (4) has at least one least energy solution $w_\varepsilon \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \geqslant 2$;*

(ii) *$\mathscr{S}_\varepsilon$ is compact in* $H^1(\mathbb{R}^3, \mathbb{C}^4)$;
(iii) *there is a maximum point x_ε of $|w_\varepsilon|$ such that* $\lim_{\varepsilon\to 0}\operatorname{dist}(x_\varepsilon, \mathscr{P}) = 0$, *and, for any sequence of such x_ε, $u_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$ converges uniformly to a least energy solution of* (6);
(iv) $|w_\varepsilon(x)| \leqslant C\exp(-\frac{c}{\varepsilon}|x - x_\varepsilon|)$ *for some $C, c > 0$.*

In fact, we are going to focus on studying the equivalent problem (5), this means that we will prove an equivalent form of the theorem: *for all small $\varepsilon \geqslant 0$*, (i) *Eq.* (5) *has a least energy solution $u_\varepsilon \in \bigcap_{q\geqslant 2} W^{1,q}$*; (ii) *the set of all least energy solutions is compact in H^1*; (iii) *there exists a maximum point x_ε of $|u_\varepsilon|$ such that* $\lim_{\varepsilon\to 0}\operatorname{dist}(\varepsilon x_\varepsilon, \mathscr{P}) = 0$, *and, for any sequence of such x_ε, $v_\varepsilon(x) := u_\varepsilon(x + x_\varepsilon)$ converges in H^1 to a least energy solution of* (6); (iv) $|u_\varepsilon(x)| \leqslant C\exp(-c|x - x_\varepsilon|)$ *for some $C, c > 0$.*

It should be compared with the investigation on Schrödinger equations. There is a large number of literature contributed to the study on the semi-classical states of Schrödinger equations

$$\hbar^2\Delta w - V(x)w + f(w) = 0, \quad w \in H^1(\mathbb{R}^N). \tag{7}$$

It is the first time that Floer and Weinstein [20] construct a single-peak solution of (7) for $N = 1$ and $f(w) = w^3$ which concentrates around any given non-degenerate critical point of V. Oh [26] extended this result in a higher dimension for $f(w) = |w|^{p-2}w$, $2 < p < 2N/(N-2)$, and proved the existence of multi-peak solutions concentrating around non-degenerate critical points of V. The arguments in [20,26] are based on a Lyapunov–Schmidt reduction. Subsequently, variational arguments were applied to such issues and the existence of spike layer solutions in the semi-classical limit has been established under various conditions of $V(x)$. In particular, the existence of a ground state of (7) (with more general nonlinearity) was first proved by Rabinowitz [28]. Later Wang [31] proved that the least energy solution concentrates at a global minimum of $V(x)$ provided $\liminf_{|x|\to\infty} V(x) > \inf V > 0$. See also [2,3,7–9,12,22,23,25] and the references therein. Note that, since the Schrödinger operator $-\Delta + V$ is bounded from below, techniques based on the Mountain-pass theorem are well applied to the investigation.

2. The functional-analytic setting

We start with discussing the functional-analytic framework.

Without loss of generality we may assume that $0 \in \mathscr{P}$ throughout the paper.

In what follows by $|\cdot|_q$ we denote the usual L^q-norm, and $(\cdot,\cdot)_2$ the usual L^2-inner product. Let $H_0 = -i\alpha\cdot\nabla + a\beta$ denote the selfadjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$. It is well known, by a Fourier analysis, that $\sigma(H_0) = \sigma_c(H_0) = \mathbb{R}\setminus(-a, a)$ where $\sigma(\cdot)$ and $\sigma_c(\cdot)$ denote the spectrum and the continuous spectrum. Thus the space L^2 possesses the orthogonal decomposition:

$$L^2 = L^- \oplus L^+, \qquad u = u^- + u^+ \tag{8}$$

so that H_0 is negative definite (resp. positive definite) in L^- (resp. L^+). Let $E := \mathcal{D}(|H_0|^{1/2}) = H^{1/2}$ be equipped with the inner product

$$(u, v) = \Re\big(|H_0|^{1/2}u, |H_0|^{1/2}v\big)_2$$

and the induced norm $\|u\| = (u, u)^{1/2}$, where $|H_0|$ and $|H_0|^{1/2}$ denote respectively the absolute value of H_0 and the square root of $|H_0|$. Since $\sigma(H_\varepsilon) \subset \mathbb{R}\setminus(-a, a)$, one has

$$a|u|_2^2 \leqslant \|u\|^2 \quad \text{for all } u \in E. \tag{9}$$

Note that this norm is equivalent to the usual $H^{1/2}$-norm, hence E embeds continuously into L^q for all $q \in [2,3]$ and compactly into L^q_{loc} for all $q \in [1,3)$. It is clear that E possesses the following decomposition

$$E = E^- \oplus E^+ \quad \text{with } E^\pm = E \cap L^\pm, \tag{10}$$

orthogonal with respect to both $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$ inner products. This decomposition induces also a natural decomposition of L^p, hence there is $\pi_p > 0$ such that

$$\pi_p \left|u^+\right|_p^p \leqslant |u|_p^p \quad \text{for all } u \in E. \tag{11}$$

On E we define the functional

$$\Phi_\varepsilon(u) = \frac{1}{2}\left(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\right) - \Psi_\varepsilon(u)$$

for $u = u^- + u^+$, where

$$\Psi_\varepsilon(u) := \frac{1}{p}\int\limits_{\mathbb{R}^3} P_\varepsilon(x)|u|^p.$$

Defining the form $a(u,v) := \int_{\mathbb{R}^3}\langle H_0 u, v\rangle$ and setting $a(u) = a(u,u)$ one has

$$\begin{aligned}\Phi_\varepsilon(u) &= \frac{1}{2}\Re a(u) - \Psi_\varepsilon(u)\\ &= \frac{1}{2}a(u) - \Psi_\varepsilon(u).\end{aligned}$$

Clearly, $\Phi_\varepsilon \in C^1(E,\mathbb{R})$. In fact, for any $u, v \in E$,

$$\begin{aligned}\frac{d}{ds}\Phi_\varepsilon(u+sv)\Big|_{s=0} &= \Re a(u,v) - \Re\int\limits_{\mathbb{R}^3} P_\varepsilon(x)|u|^{p-2}\langle u,v\rangle\\ &= \left(u^+ - u^-, v\right) - \Re\int\limits_{\mathbb{R}^3} P_\varepsilon(x)|u|^{p-2}\langle u,v\rangle.\end{aligned}$$

A standard argument shows that critical points of Φ_ε are solutions of (5).

Lemma 2.1. *Ψ_ε is weakly sequentially lower semi-continuous and Φ'_ε is weakly sequentially continuous.*

Proof. The lemma follows easily because E embeds continuously into L^q for $q \in [2,3]$ and compactly into L^q_{loc} for $q \in [1,3)$ [13]. $\square$

Set, for $r > 0$, $B_r^+ = \{u \in E^+ \colon \|u\| \leqslant r\}$ and $S_r^+ = \{u \in E^+ \colon \|u\| = r\}$, and for $e \in E^+$

$$E_e := E^- \oplus \mathbb{R}^+ e$$

with $\mathbb{R}^+ = [0,\infty)$.

Lemma 2.2. *Φ_ε possesses the linking structure*:

1) *There exist $r > 0$ and $\rho > 0$ both independent of ε such that $\Phi_\varepsilon|_{B_r^+}(u) \geqslant 0$ and $\Phi_\varepsilon|_{S_r^+} \geqslant \rho$.*
2) *For any $e \in E^+ \setminus \{0\}$, there exist $R_e > 0$ and $C = C_e > 0$ both independent of ε such that $\Phi_\varepsilon(u) < 0$ for all $u \in E_e \setminus B_R$ and $\max \Phi_\varepsilon(E_e) \leqslant C$.*

Proof. Recall that $a|u|_2^2 \leqslant \|u\|^2$ for all $u \in E$ by (9).

1) follows easily because, for $u \in E^+$,

$$\begin{aligned}\Phi_\varepsilon(u) &= \frac{1}{2}\|u\|^2 - \Psi_\varepsilon(u)\\ &\geqslant \frac{1}{2}\|u\|^2 - \frac{1}{p}m|u|_p^p\end{aligned}$$

and $p > 2$.

For checking 2) take $e \in E^+ \setminus \{0\}$. In virtue of (11), one gets

$$\begin{aligned}\Phi_\varepsilon(u) &= \frac{1}{2}\|se\|^2 - \frac{1}{2}\|v\|^2 - \Psi_\varepsilon(u)\\ &\leqslant \frac{1}{2}s^2\|e\|^2 - \frac{1}{2}\|v\|^2 - \frac{\pi_p s^p}{p}\inf P|e|_p^p,\end{aligned} \tag{12}$$

hence 2) since $p > 2$. □

Define (see [27,30])

$$c_\varepsilon := \inf_{e \in E^+ \setminus \{0\}} \max_{u \in E_e} \Phi_\varepsilon(u).$$

As a consequence of Lemma 2.2 we have

Lemma 2.3. *There is $C > 0$ independent of ε such that $\rho \leqslant c_\varepsilon < C$.*

Proof. By 1) of Lemma 2.2 and the definition of c_ε one has $c_\varepsilon \geqslant \rho$. Take $e \in E^+$ with $\|e\| = 1$. It follows from (12) the following

$$c_\varepsilon \leqslant C \equiv C_e,$$

ending the proof. □

Recall that a sequence $\{u_n\} \subset E$ is said to be a $(PS)_c$, $c \in \mathbb{R}$, sequence for Φ_ε if $\Phi_\varepsilon(u_n) \to c$ and $\Phi_\varepsilon'(u_n) \to 0$, and Φ_ε is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence for Φ_ε has a convergent subsequence. A standard linking argument shows that if Φ_ε satisfies the $(PS)_c$ condition then c_ε is a critical value of Φ_ε (see, e.g., [13,30]).

Following Ackermann [1], for a fixed $u \in E^+$ we introduce $\phi_u : E^- \to \mathbb{R}$ defined by

$$\phi_u(v) = \Phi_\varepsilon(u + v).$$

For any $v, w \in E^-$,

$$\phi_u''(v)[w, w] = -\|w\|^2 - \Psi_\varepsilon''(u + v)[w, w] \leqslant -\|w\|^2.$$

In addition,

$$\phi_u(v) \leqslant \frac{1}{2}\big(\|u\|^2 - \|v\|^2\big).$$

Therefore, there is a unique $h_\varepsilon(u) \in E^-$ such that

$$\phi_u\big(h_\varepsilon(u)\big) = \max_{v\in E^-} \phi_u(v).$$

It is clear that

$$0 = \phi_u'\big(h_\varepsilon(u)\big)v = -\big(h_\varepsilon(u), v\big) - \Psi_\varepsilon'\big(u + h_\varepsilon(u)\big)v$$

for all $v \in E^-$, and

$$v \neq h_\varepsilon(u) \quad \Leftrightarrow \quad \Phi_\varepsilon(u+v) < \Phi_\varepsilon\big(u + h_\varepsilon(u)\big).$$

Observe that for $u \in E^+$ and $v \in E^-$,

$$\begin{aligned}
&\phi_u(v) - \phi_u\big(h_\varepsilon(u)\big)\\
&\quad = \int_0^1 (1-t)\phi_u''\big(h_\varepsilon(u) + t\big(v - h_\varepsilon(u)\big)\big)\big[v - h_\varepsilon(u), v - h_\varepsilon(u)\big]\,dt\\
&\quad = -\int_0^1 (1-t)\left(\big\|v - h_\varepsilon(u)\big\|^2 + (p-1)\int_{\mathbb{R}^3} P_\varepsilon(x)\big|u + h_\varepsilon(u) + t\big(v - h_\varepsilon(u)\big)\big|^{p-2}\big|v - h_\varepsilon(u)\big|^2\right)dt,
\end{aligned}$$

hence,

$$\begin{aligned}
&(p-1)\int_0^1\int_{\mathbb{R}^3} (1-t)P_\varepsilon(x)\big|u + h_\varepsilon(u) + t\big(v - h_\varepsilon(u)\big)\big|^{p-2}\big|v - h_\varepsilon(u)\big|^2\,dx\,dt + \frac{1}{2}\big\|v - h_\varepsilon(u)\big\|^2\\
&\quad \leqslant \Phi_\varepsilon\big(u + h_\varepsilon(u)\big) - \Phi_\varepsilon(u+v).
\end{aligned} \tag{13}$$

Define $I_\varepsilon : E^+ \to \mathbb{R}$ by

$$I_\varepsilon(u) = \Phi_\varepsilon\big(u + h_\varepsilon(u)\big) = \frac{1}{2}\big(\|u\|^2 - \big\|h_\varepsilon(u)\big\|^2\big) - \Psi_\varepsilon\big(u + h_\varepsilon(u)\big).$$

Set

$$\mathscr{N}_\varepsilon := \big\{u \in E^+ \setminus \{0\}:\ I_\varepsilon'(u)u = 0\big\}.$$

Lemma 2.4. *For any $u \in E^+ \setminus \{0\}$, there is a unique $t = t(u) > 0$ such that $t(u)u \in \mathscr{N}_\varepsilon$.*

Proof. See [1,16]. □

Now we define

$$d_\varepsilon := \inf_{u \in \mathscr{N}_\varepsilon} I_\varepsilon(u).$$

Lemma 2.5. *$d_\varepsilon = c_\varepsilon$, hence, there is $C > 0$ independent of ε such that $d_\varepsilon \leqslant C$.*

Proof. Indeed, given $e \in E^+$, if $u = v + se \in E_e$ with $\Phi_\varepsilon(u) = \max_{z \in E_e} \Phi_\varepsilon(z)$ then the restriction $\Phi_\varepsilon|_{E_e}$ of Φ_ε on E_e satisfies $(\Phi_\varepsilon|_{E_e})'(u) = 0$ which implies $v = h_\varepsilon(se)$ and $I'_\varepsilon(se)(se) = \Phi'_\varepsilon(u)(se) = 0$, i.e. $se \in \mathscr{N}_\varepsilon$. Thus $d_\varepsilon \leqslant c_\varepsilon$. On the other hand, if $w \in \mathscr{N}_\varepsilon$ then $(\Phi_\varepsilon|_{E_w})'(w + h_\varepsilon(w)) = 0$ so $c_\varepsilon \leqslant \max_{u \in E_w} \Phi_\varepsilon(u) = I_\varepsilon(w)$. Thus $d_\varepsilon \geqslant c_\varepsilon$. This proves $d_\varepsilon = c_\varepsilon$. Now, this, together with Lemma 2.3, yields immediately the last conclusion of the lemma. $\square$

Lemma 2.6. *For any $e \in E^+ \setminus \{0\}$, there is $T_e > 0$ independent of $\varepsilon > 0$ such that $t_\varepsilon \leqslant T_e$ for $t_\varepsilon > 0$ satisfying $t_\varepsilon e \in \mathscr{N}_\varepsilon$.*

Proof. Since $I'_\varepsilon(t_\varepsilon e)(t_\varepsilon e) = 0$ one sees that the restriction of Φ_ε satisfies $(\Phi_\varepsilon|_{E_e})'(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)) = 0$. Thus

$$\Phi_\varepsilon\big(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\big) = \max_{w \in E_e} \Phi_\varepsilon(w).$$

This, together with Lemma 2.5 and (12), implies the desired conclusion. $\square$

Let $\mathscr{K}_\varepsilon := \{u \in E\colon\ \Phi'_\varepsilon(u) = 0\}$ be the critical set of Φ_ε. It is easy to see that if $\mathscr{K}_\varepsilon \setminus \{0\} \neq \emptyset$ then

$$d_\varepsilon = \inf\big\{\Phi_\varepsilon(u)\colon\ u \in \mathscr{K}_\varepsilon \setminus \{0\}\big\}$$

(see an argument of [16]). Using the same iterative argument of [17, Proposition 3.2] one obtains easily the following

Lemma 2.7. *If $u \in \mathscr{K}_\varepsilon$ with $|\Phi_\varepsilon(u)| \leqslant C_1$ and $|u|_2 \leqslant C_2$, then, for any $q \in [2, \infty)$, $u \in W^{1,q}(\mathbb{R}^3)$ with $\|u\|_{W^{1,q}} \leqslant \Lambda_q$ where Λ_q depends only on C_1, C_2 and q.*

Let $\mathscr{S}_\varepsilon$ be the set of all least energy solutions of Φ_ε. If $u \in \mathscr{S}_\varepsilon$ then $\Phi_\varepsilon(u) = d_\varepsilon$ and a standard argument shows that $\mathscr{S}_\varepsilon$ is bounded in E, hence, $|u|_2 \leqslant C_2$ for $u \in \mathscr{S}_\varepsilon$, some $C_2 > 0$ independent of ε. Therefore, as a consequence of Lemmas 2.5 and 2.7 we see that, for each $q \in [2, \infty)$, there is $C_q > 0$ independent of ε such that

$$\|u\|_{W^{1,q}} \leqslant C_q \quad \text{for all } u \in \mathscr{S}_\varepsilon. \tag{14}$$

This, together with the Sobolev embedding theorem, implies that there is $C_\infty > 0$ independent of ε with

$$|u|_\infty \leqslant C_\infty \quad \text{for all } u \in \mathscr{S}_\varepsilon. \tag{15}$$

3. The limit problem

We will make use of the limit equation for proving our main result. To this end we discuss in this section the existence and some properties of the least energy solutions of the limit problem.

For any $b > 0$, consider the equation

$$-i\alpha \cdot \nabla u + a\beta u = b|u|^{p-2}u, \quad u \in H^1\big(\mathbb{R}^3, \mathbb{C}^4\big) \tag{16}$$

($p \in (2, 3)$). Its solutions are critical points of the functional

$$\begin{aligned}\Gamma_b(u) &:= \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \frac{1}{p}b\int_{\mathbb{R}^3} |u|^p \\ &= \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \Psi_b(u)\end{aligned}$$

defined for $u = u^- + u^+ \in E = E^- \oplus E^+$ where $\Psi_b = \frac{1}{p}b|u|_p^p$. Denote the critical set, the least energy, and the set of least energy solutions of Γ_b as follows

$$\begin{aligned}\mathscr{L}_b &:= \big\{u \in E\colon\ \Gamma_b'(u) = 0\big\}, \\ \gamma_b &:= \inf\big\{\Gamma_b(u)\colon\ u \in \mathscr{L}_b \setminus \{0\}\big\}, \\ \mathscr{R}_b &:= \big\{u \in \mathscr{L}_v\colon\ \Gamma_b(u) = \gamma_b,\ \big|u(0)\big| = |u|_\infty\big\}.\end{aligned}$$

The following lemma is from [16].

Lemma 3.1. *There hold the following*:

i) $\mathscr{L}_b \neq \emptyset$, $\gamma_b > 0$, *and* $\mathscr{L}_b \subset \bigcap_{q\geqslant 2} W^{1,q}$ *for all* $q \geqslant 2$;
ii) γ_b *is attained, and* $\mathscr{R}_b$ *is compact in* $H^1(\mathbb{R}^3, \mathbb{C}^4)$;
iii) *there exist* $C, c > 0$ *such that*

$$\big|u(x)\big| \leqslant C\exp\big(-c|x|\big) \quad \textit{for all } x \in \mathbb{R}^3,\ u \in \mathscr{R}_b.$$

As before (replacing P_ε with b) we introduce the following notations:

$$\begin{aligned}&\mathscr{J}_b : E^+ \to E^-\colon \quad \Gamma_b\big(u + \mathscr{J}_b(u)\big) = \max_{v\in E^-} \Gamma_b(u + v); \\ &J_b : E^+ \to \mathbb{R}\colon \quad J_b(u) = \Gamma_b\big(u + \mathscr{J}_b(u)\big); \\ &\mathscr{M}_b := \big\{u \in E^+ \setminus \{0\}\colon\ J_b'(u)u = 0\big\}.\end{aligned}$$

Plainly, critical points of J_b and Γ_b are in one-to-one correspondence via the injective map $u \to u + \mathscr{J}_b(u)$ from E^+ into E.

Notice that, similar to (13), we have for $u \in E^+$ and $v \in E^-$

$$\begin{aligned}&(p-1)\int_0^1\int_{\mathbb{R}^3}(1-t)b\big|u + \mathscr{J}_b(u) + t\big(v - \mathscr{J}_b(u)\big)\big|^{p-2}\big|v - \mathscr{J}_b(u)\big|^2\,dx\,dt + \frac{1}{2}\big\|v - \mathscr{J}_b(u)\big\|^2 \\ &\quad\leqslant \Gamma_b\big(u + \mathscr{J}_b(u)\big) - \Gamma_b(u + v).\end{aligned} \tag{17}$$

It is not difficult to check that, for each $u \in E^+ \setminus \{0\}$, there is a unique $t = t(u) > 0$ such that $tu \in \mathscr{M}_b$ (see [1,16]).

Clearly, J_b has the Mountain-pass structure. Set

$$b_1 := \inf\{J_b(u):\ u \in \mathscr{M}_b\},$$
$$b_2 := \inf_{\gamma \in \Omega_b} \max_{t \in [0,1]} J_b(\gamma(t)),$$
$$b_3 := \inf_{\gamma \in \tilde{\Omega}_b} \max_{t \in [0,1]} J_b(\gamma(t)),$$

where

$$\Omega_b := \{\gamma \in C([0,1], E^+):\ \gamma(0) = 0,\ J_b(\gamma(1)) < 0\}$$

and

$$\tilde{\Omega}_b := \{\gamma \in C([0,1], E^+):\ \gamma(0) = 0,\ \gamma(1) = u_0\}$$

for any arbitrarily fixed $u_0 \in E^+$ with $J_b(u_0) < 0$. Then

$$\gamma_b := b_1 = b_2 = b_3$$

(see [16, Lemma 3.8]).

Lemma 3.2. *Let $u \in \mathscr{M}_b$ be such that $J_b(u) = \gamma_b$, and set $E_u = E^- \oplus \mathbb{R}u$. Then*

$$\max_{w \in E_u} \Gamma_b(w) = J_b(u).$$

Proof. Clearly, since $u + \mathscr{J}_b(u) \in E_u$,

$$J_b(u) = \Gamma_b(u + \mathscr{J}_b(u)) \leqslant \max_{w \in E_u} \Gamma_b(w).$$

On the other hand, for any $w = v + su \in E_u$,

$$\begin{aligned}\Gamma_b(w) &= \frac{1}{2}\|su\|^2 - \frac{1}{2}\|v\|^2 - \Psi_b(v + su)\\ &\leqslant \Gamma_b(su + \mathscr{J}_b(su)) = J_b(su).\end{aligned}$$

Thus, since $u \in \mathscr{M}_b$,

$$\max_{w \in E_u} \Gamma_b(w) \leqslant \max_{s \geqslant 0} J_b(su) = J_b(u),$$

giving the conclusion. $\square$

Lemma 3.3. *If $b_1 < b_2$ then $\gamma_{b_1} > \gamma_{b_2}$.*

Proof. Let $u \in \mathscr{L}_{b_1}$ with $\Gamma_{b_1}(u) = \gamma_{b_1}$ and set $e = u^+$. Then

$$\gamma_{b_1} = \Gamma_{b_1}(u) = \max_{w \in E_e} \Gamma_{b_1}(w).$$

Let $u_1 \in E_e$ be such that $\Gamma_{b_2}(u_1) = \max_{w \in E_e} \Gamma_{b_2}(w)$. One has

$$\begin{aligned} \gamma_{b_1} = \Gamma_{b_1}(u) \geqslant \Gamma_{b_1}(u_1) &= \Gamma_{b_2}(u_1) + \frac{1}{p}(b_2 - b_1)|u_1|_p^p \\ &\geqslant \gamma_{b_2} + \frac{1}{p}(b_2 - b_1)|u_1|_p^p \end{aligned}$$

as claimed. □

Lemma 3.4. *$d_\varepsilon \geqslant \gamma_m$ for all $\varepsilon > 0$.*

Proof. Arguing indirectly, assume that $d_\varepsilon < \gamma_m$ for some $\varepsilon > 0$. By definition and Lemma 2.5 we can choose an $e \in E^+ \setminus \{0\}$ such that $\max_{u \in E_e} \Phi_\varepsilon(u) < \gamma_m$. By definition again one has $\gamma_m \leqslant \max_{u \in E_e} \Gamma_m(u)$. Since $P_\varepsilon(x) \leqslant m$, $\Phi_\varepsilon(u) \geqslant \Gamma_m(u)$ for all $u \in E$, and we get

$$\gamma_m > \max_{u \in E_e} \Phi_\varepsilon(u) \geqslant \max_{u \in E_e} \Gamma_m(u) \geqslant \gamma_m,$$

a contradiction. □

4. Proof of the main result

We are now in a position to give the proof of the main result, that is, Theorem 1.1. The key for the proof is that $d_\varepsilon \to \gamma_m$ as $\varepsilon \to 0$ (Lemma 4.1). With this we argue by contradiction to show the existence of semi-classical solutions (Lemma 4.2). The compactness of $\mathscr{S}_\varepsilon$ is easy to check (Lemma 4.3). In order to show the concentration it is sufficient to verify that, for any sequence $\varepsilon_j \to 0$ with $u_j \in \mathscr{S}_{\varepsilon_j}$, there is a subsequence which converges, up to a shift of x-variable, to a least energy solution of the limit problem, and such a subsequence is uniformly small at the infinity with the help of the sub-solution estimate. Lastly, by a Kato's inequality we complete the proof.

Lemma 4.1. *$d_\varepsilon \to \gamma_m$ as $\varepsilon \to 0$.*

Proof. Set $W^0(x) = m - P(x)$ and $W_\varepsilon^0(x) = W^0(\varepsilon x)$. Then

$$\Phi_\varepsilon(v) = \Gamma_m(v) + \frac{1}{p} \int_{\mathbb{R}^3} W_\varepsilon^0(x)|v|^p. \tag{18}$$

In virtue of Lemma 3.1 let $u = u^- + u^+ \in \mathscr{R}_m$, a least energy solution of (16) with $b = m$, and set $e = u^+$. It is clear that $e \in \mathscr{M}_m$, $\mathscr{J}_m(e) = u^-$ and $J_m(e) = \gamma_m$. There is a unique $t_\varepsilon > 0$ such that $t_\varepsilon e \in \mathscr{N}_\varepsilon$. One has

$$d_\varepsilon \leqslant I_\varepsilon(t_\varepsilon e). \tag{19}$$

By Lemma 2.6, $\{t_\varepsilon\}$ is bounded, hence, without loss of generality we can assume $t_\varepsilon \to t_0$ as $\varepsilon \to 0$. Observe that, by (13) and (17),

$$\begin{aligned}
&\frac{1}{2}\left\| \mathscr{J}_m(t_\varepsilon e) - h_\varepsilon(t_\varepsilon e)\right\|^2 + (I) \\
&\quad \leqslant \Phi_\varepsilon\big(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\big) - \Phi_\varepsilon\big(t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\big) \\
&\quad = \Gamma_m\big(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\big) + \frac{1}{p}\int_{\mathbb{R}^3} W_\varepsilon^0(x)\left|t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\right|^p \\
&\qquad - \Gamma_m\big(t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\big) - \frac{1}{p}\int_{\mathbb{R}^3} W_\varepsilon^0(x)\left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p \\
&\quad = -\Big(\Gamma_m\big(t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\big) - \Gamma_m\big(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\big)\Big) \\
&\qquad + \frac{1}{p}\int_{\mathbb{R}^3} W_\varepsilon^0(x)\big(\left|t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\right|^p - \left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p\big),
\end{aligned}$$

hence,

$$\begin{aligned}
&\left\|h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\right\|^2 + (I) + (II) \\
&\quad \leqslant \frac{1}{p}\int_{\mathbb{R}^3} W_\varepsilon^0(x)\big(\left|t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\right|^p - \left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p\big)
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
(I) := (p-1)\int_{\mathbb{R}^3}\int_0^1 &(1-s)P_\varepsilon(x)\Big(\left|t_\varepsilon e + h_\varepsilon(t_\varepsilon e) + s\big(\mathscr{J}_m(t_\varepsilon e) - h_\varepsilon(t_\varepsilon e)\big)\right|^{p-2} \\
&\cdot \left|\mathscr{J}_m(t_\varepsilon e) - h_\varepsilon(t_\varepsilon e)\right|^2\Big)\,ds\,dx,
\end{aligned}$$

$$\begin{aligned}
(II) := (p-1)\int_{\mathbb{R}^3}\int_0^1 &(1-s)m\Big(\left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e) + s\big(h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\big)\right|^{p-2} \\
&\cdot \left|h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\right|^2\Big)\,ds\,dx.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\left|t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\right|^p - \left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p \\
&\quad = \left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^{p-2}\big\langle t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e), h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\big\rangle \\
&\qquad + (p-1)\int_0^1 (1-s)\Big(\left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e) + s\big(h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\big)\right|^{p-2} \\
&\qquad \cdot \left|h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\right|^2\Big)\,ds.
\end{aligned}$$

Noting that $W_\varepsilon^0(x) \leqslant m$, substituting this into (20) yields

$$\begin{aligned}
&\left\|h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\right\|^2 + (I) + \left(1 - \frac{1}{p}\right)(II) \\
&\quad\leqslant \frac{1}{p}\int_{\mathbb{R}^3} W_\varepsilon^0(x)\left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^{p-1}\left|h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\right| \\
&\quad\leqslant \left(\int_{\mathbb{R}^3} \left(W_\varepsilon^0(x)\right)^{p/(p-1)}\left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p\right)^{p/(p-1)}\left|h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\right|_p.
\end{aligned} \tag{21}$$

Since $t_\varepsilon \to t_0$, by the exponential decay of e, we have

$$\limsup_{R\to\infty} \int_{|x|\geqslant R} \left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p = 0$$

which implies that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(W_\varepsilon^0(x)\right)^{p/(p-1)}\left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p \\
&\quad= \left(\int_{|x|\leqslant R} + \int_{|x|>R}\right)\left(W_\varepsilon^0(x)\right)^{p/(p-1)}\left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p \\
&\quad\leqslant \int_{|x|\leqslant R} \left(W_\varepsilon^0(x)\right)^{p/(p-1)}\left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p \\
&\qquad + m^{p/(p-1)}\int_{|x|>R} \left|t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e)\right|^p \\
&\quad= o(1)
\end{aligned}$$

as $\varepsilon \to 0$. Thus by (21) we see that $\|h_\varepsilon(t_\varepsilon e) - \mathscr{J}_m(t_\varepsilon e)\|^2 \to 0$, that is, $h_\varepsilon(t_\varepsilon e) \to \mathscr{J}_m(t_0 e)$. Consequently,

$$\int_{\mathbb{R}^3} W_\varepsilon^0(x)\left|t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\right|^p \to 0$$

as $\varepsilon \to 0$. This, jointly with (18), implies

$$\begin{aligned}
\Phi_\varepsilon\big(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\big) &= \Gamma_m\big(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)\big) + o(1) \\
&= \Gamma_m\big(t_0 e + \mathscr{J}_m(t_0 e)\big) + o(1),
\end{aligned}$$

that is,

$$I_\varepsilon(t_\varepsilon e) = J_m(t_0 e) + o(1)$$

as $\varepsilon \to 0$. Recall that by Lemma 3.2

$$J_m(t_0 e) \leqslant \max_{v\in E_e} \Gamma_m(v) = J_m(e) = \gamma_m.$$

Now, using Lemma 3.4 and (19) we obtain

$$\gamma_m \leqslant \lim_{\varepsilon\to 0} d_\varepsilon \leqslant \lim_{\varepsilon\to 0} I_\varepsilon(t_\varepsilon e) = J_m(t_0 e) \leqslant \gamma_m,$$

hence, $d_\varepsilon \to \gamma_m$ as claimed. □

Lemma 4.2. *d_ε is attained for all small $\varepsilon > 0$.*

Proof. Given $\varepsilon > 0$, let $u_n \in \mathcal{N}_\varepsilon$ be a minimization sequence: $I_\varepsilon(u_n) \to d_\varepsilon$. By the Ekeland variational principle we can assume that u_n is, in addition, a $(PS)_{d_\varepsilon}$ sequence for I_ε on $\mathcal{N}_\varepsilon$. A standard argument shows that u_n is in fact a $(PS)_{d_\varepsilon}$ sequence for I_ε on E^+ (see, e.g., [27,32]). Then $w_n = u_n + \mathscr{I}_\varepsilon(u_n)$ is a $(PS)_{d_\varepsilon}$ sequence for Φ_ε on E. It is easy to see that w_n is bounded. We can assume without loss of generality that $w_n \rightharpoonup w_\varepsilon = z_\varepsilon^+ + z_\varepsilon^- \in \mathscr{K}_\varepsilon$ in E. If $w_\varepsilon \neq 0$ then clearly $\Phi_\varepsilon(w_\varepsilon) = d_\varepsilon$. So we are going to check that $w_\varepsilon \neq 0$ for all $\varepsilon > 0$ small.

For this end, take $\limsup_{|x|\to\infty} P(x) < b < m$ and define

$$P^b(x) = \min\{b, P(x)\}.$$

Consider the functional

$$\Phi_\varepsilon^b(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \frac{1}{p}\int_{\mathbb{R}^3} P_\varepsilon^b(x)|u|^p$$

and as before define correspondingly $h_\varepsilon^b : E^+ \to E^-$, $I_\varepsilon^b : E^+ \to \mathbb{R}$, $\mathcal{N}_\varepsilon^b$, d_ε^b and so on. As done in the proof of Lemma 4.1 before,

$$\gamma_b \leqslant d_\varepsilon^b \to \gamma_b \tag{22}$$

as $\varepsilon \to 0$.

Assume by contradiction that there is a sequence $\varepsilon_j \to 0$ with $w_{\varepsilon_j} = 0$. Then $w_n = u_n + h_{\varepsilon_j}(u_n) \rightharpoonup 0$ in E, $u_n \to 0$ in L^q_{loc} for $q \in (1,3)$, and $w_n(x) \to 0$ a.e. in $x \in \mathbb{R}^3$. Let $t_n > 0$ be such that $t_n u_n \in \mathcal{N}_{\varepsilon_j}^b$. We see as before that $\{t_n\}$ is bounded and one may assume $t_n \to t_0$ as $n \to \infty$. By (P_0), the set $A_\varepsilon := \{x \in \mathbb{R}^3{:}\ P_\varepsilon(x) > b\}$ is bounded. Remark that $h_{\varepsilon_j}^b(t_n u_n) \rightharpoonup 0$ in E and $h_{\varepsilon_j}^b(t_n u_n) \to 0$ in L^q_{loc} as $n \to \infty$ (cf. [1]). Additionally, $\Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^b(t_n u_n)) \leqslant I_{\varepsilon_j}(u_n)$ by virtue of Lemma 3.2. We obtain

$$\begin{aligned} d_{\varepsilon_j}^b &\leqslant I_{\varepsilon_j}^b(t_n u_n) = \Phi_{\varepsilon_j}^b\left(t_n u_n + h_{\varepsilon_j}^b(t_n u_n)\right) \\ &= \Phi_{\varepsilon_j}\left(t_n u_n + h_{\varepsilon_j}^b(t_n u_n)\right) + \frac{1}{p}\int_{\mathbb{R}^3}\left(P_{\varepsilon_j} - P_{\varepsilon_j}^b\right)\left|t_n u_n + h_{\varepsilon_j}^b(t_n u_n)\right|^p \\ &\leqslant I_{\varepsilon_j}(u_n) + \frac{1}{2}\int_{A_{\varepsilon_j}}\left(P_{\varepsilon_j} - P_{\varepsilon_j}^b\right)\left|t_n u_n + h_{\varepsilon_j}^b(t_n u_n)\right|^p \\ &= d_{\varepsilon_j} + o(1) \end{aligned}$$

as $n \to \infty$, hence, $d^b_{\varepsilon_j} \leqslant d_{\varepsilon_j}$. By (22), letting $j \to \infty$ yields

$$\gamma_b \leqslant \gamma_m,$$

contradicting with $\gamma_m < \gamma_b$ (see Lemma 3.3). □

Lemma 4.3. *$\mathscr{S}_\varepsilon$ is compact for all small $\varepsilon > 0$.*

Proof. Assume by contradiction that, for some $\varepsilon_j \to 0$, $\mathscr{S}_{\varepsilon_j}$ is not compact in E. Let $u^j_n \in \mathscr{S}_{\varepsilon_j}$ with $u^j_n \rightharpoonup 0$ as $n \to \infty$. As done in proving the above Lemma 4.2, one gets a contradiction. □

For the later use, letting

$$D = -i \sum_{k=1}^{3} \alpha_k \partial_k,$$

we write (5) as

$$Du = -a\beta u + P_\varepsilon(x)|u|^{p-2}u.$$

By Lemma 2.7, $u \in \bigcap_{q \geqslant 2} W^{1,q}$ for any $u \in \mathscr{K}_\varepsilon$. Acting the operator D on the two sides of the above representation and noting that $D^2 = -\Delta$ we get

$$-\Delta u = -a^2 u + r_\varepsilon\big(x, |u|\big)u$$

where

$$r_\varepsilon\big(x, |u|\big) = |u|^{p-2}\Bigg(D\big(P_\varepsilon(x)\big) - i(p-2)P_\varepsilon(x)\sum_{k=1}^{3}\alpha_k \frac{\Re\langle \partial_k u, u\rangle}{|u|^2} + P_\varepsilon(x)\big(-a\beta + P_\varepsilon(x)|u|^{p-2}\big)\Bigg).$$

Letting

$$\operatorname{sgn} u = \begin{cases} \frac{\bar u}{|u|} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0, \end{cases}$$

by the Kato's inequality [11], there holds

$$\Delta|u| \geqslant \Re\big[\Delta u(\operatorname{sgn} u)\big].$$

Observe that

$$\Re\Bigg[\Bigg(D\big(P_\varepsilon(x)\big) - i(p-2)P_\varepsilon(x)\sum_{k=1}^{3}\alpha_k \frac{\Re\langle \partial_k u, u\rangle}{|u|^2}\Bigg)u\frac{\bar u}{|u|}\Bigg] = 0$$

and

$$\Re\left[\beta u \cdot \frac{\bar{u}}{|u|}\right] = 0.$$

Hence

$$\Delta|u| \geqslant \left(a^2 - \left(P_\varepsilon(x)|u|^{p-2}\right)^2\right)|u|. \tag{23}$$

Since $u \in W^{1,q}$ for any $q \geqslant 2$, by the sub-solution estimate [21,29] one has

$$\left|u(x)\right| \leqslant C_0 \int_{B_1(x)} \left|u(y)\right| dy \tag{24}$$

with C_0 independent of x and $u \in \mathscr{K}_\varepsilon$, $\varepsilon > 0$.

Lemma 4.4. *There is a maximum point x_ε of $|u_\varepsilon|$ such that* $\operatorname{dist}(y_\varepsilon, \mathscr{P}) \to 0$ *where $y_\varepsilon = \varepsilon x_\varepsilon$, and, for any such x_ε, $v_\varepsilon(x) := u_\varepsilon(x + x_\varepsilon)$ converges in E to a least energy solution of* (6)*, as $\varepsilon \to 0$.*

Proof. Let $\varepsilon_j \to 0$, $u_j \in \mathscr{S}_j$ where $\mathscr{S}_j = \mathscr{S}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. A concentration argument shows that there exist a sequence $\{x_j\} \subset \mathbb{R}^3$ and constants $R > 0, \delta > 0$ such that

$$\liminf_{j\to\infty} \int_{B(x_j,R)} |u_j|^2 \geqslant \delta.$$

Set

$$v_j(x) = u_j(x + x_j).$$

Then v_j solves, denoting $\hat{P}_{\varepsilon_j}(x) = P(\varepsilon_j(x + x_j))$,

$$-i\alpha \cdot \nabla v_j + a\beta v_j = \hat{P}_{\varepsilon_j}(x)|v_j|^{p-2} v_j \tag{25}$$

with energy

$$\begin{aligned}
\mathcal{E}(v_j) &:= \frac{1}{2}\left(\left\|v_j^+\right\|^2 - \left\|v_j^-\right\|^2\right) - \frac{1}{p}\int_{\mathbb{R}^3} \hat{P}_{\varepsilon_j}(x)|v_j|^p \\
&= \Phi_{\varepsilon_j}(u_j) = \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^3} \hat{P}_{\varepsilon_j}(x)|v_j|^p \\
&= d_{\varepsilon_j}.
\end{aligned} \tag{26}$$

Additionally, $v_j \rightharpoonup v$ in E and $v_j \to v$ in L^q_{loc} for $q \in [1,3)$.

We claim that $\{\varepsilon_j x_j\}$ is bounded in $\mathbb{R}^3$. Assume by contradiction that $\varepsilon_j|x_j| \to \infty$. Without loss of generality assume $P(\varepsilon_j x_j) \to P_\infty$. Clearly, $m > P_\infty$ by (P_0). Since for any $\varphi \in C_0^\infty$,

$$0 = \lim_{j\to\infty} \int_{\mathbb{R}^3} \langle H_0 v_j - \hat{P}_{\varepsilon_j}|v_j|^{p-2}v_j, \varphi\rangle$$
$$= \int_{\mathbb{R}^3} \langle H_0 v - P_\infty |v|^{p-2}v, \varphi\rangle,$$

v solves

$$-i\alpha\cdot\nabla v + a\beta v = P_\infty |v|^{p-2}v. \tag{27}$$

Therefore, the energy

$$\mathcal{E}(v) := \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) - \frac{1}{p}\int_{\mathbb{R}^3} P_\infty |v|^p \geqslant \gamma_{P_\infty}.$$

Remark that since $m > P_\infty$ one has $\gamma_m < \gamma_{P_\infty}$ by Lemma 3.3. Moreover, by the weakly lower semi-continuity of L^p-norm,

$$\lim_{j\to\infty}\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^3} \hat{P}_{\varepsilon_j}|v_j|^p \geqslant \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^3} P_\infty |v|^p = \mathcal{E}(v).$$

Consequently, using (26) one gets

$$\gamma_m < \gamma_{P_\infty} \leqslant \mathcal{E}(v) \leqslant \lim_{j\to\infty} d_{\varepsilon_j} = \gamma_m,$$

a contradiction.

Thus $\{\varepsilon_j x_j\}$ is bounded. Hence, we can assume $y_j = \varepsilon_j x_j \to y_0$. Then v solves

$$-i\alpha\cdot\nabla v + a\beta v = P(y_0)|v|^{p-2}v.$$

Since $P(y_0) \leqslant m$, the energy

$$\mathcal{E}(v) := \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) - \frac{1}{p}\int_{\mathbb{R}^3} P(y_0)|v|^p \geqslant \gamma_{P(y_0)} \geqslant \gamma_m.$$

Using (26) again

$$\mathcal{E}(v) = \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^3} P(y_0)|v|^p \leqslant \lim_{j\to\infty} d_{\varepsilon_j} = \gamma_m.$$

This implies $\mathcal{E}(v) = \gamma_m$, hence $P(y_0) = m$, so by Lemma 3.3, $y_0 \in \mathscr{P}$.

The above argument shows also that

$$\lim_{j\to\infty}\int_{\mathbb{R}^3} \hat{P}_{\varepsilon_j}(x)|v_j|^p = \int_{\mathbb{R}^3} P(y_0)|v|^p = \frac{2p\gamma_m}{p-2}$$

which implies $|v_j - v|_p \to 0$ by the Brezis–Lieb lemma, then $|(v_j - v)^{\pm}|_p \to 0$ by (11). For proving that $v_j \to v$ in E, denote $z_j = v_j - v$. Remarking that $z_j^{\pm} \to 0$ in L^p, the scale product of (25) with z_j^+ yields

$$\left(v_j^+, z_j^+\right) = o(1).$$

Similarly, using the exponential decay of v together with the fact that $z_j^{\pm} \to 0$ in L^2_{loc}, it follows from (27) that

$$\left(v^+, z_j^+\right) = o(1).$$

Thus

$$\left\|z_j^+\right\|^2 = o(1).$$

Similarly,

$$\left\|z_j^-\right\|^2 = o(1).$$

We then get $v_j \to v$ in E.

In order to verify that $v_j \to v$ in H^1, observe that by (25) and (27)

$$H_0 z_j = \hat{P}_{\varepsilon_j}(x)\left(|v_j|^{p-2} v_j - |v|^{p-2} v\right) + \left(\hat{P}_{\varepsilon_j}(x) - m\right)|v|^{p-2} v,$$

and

$$\lim_{R\to\infty} \int\limits_{|x|\leqslant R} \left|\left(\hat{P}_{\varepsilon_j}(x) - m\right)|v|^{p-2} v\right|^2 = 0$$

by the exponential decay of v. This, together with the uniform estimate (15), shows that $|H_0 z_j|_2 \to 0$. Therefore $v_j \to v$ in H^1.

By virtue of (24) it is clear that one may assume that $x_j \in \mathbb{R}^3$ is a maximum point of $|u_j|$. Moreover, from the above argument we readily see that, any sequence of such points satisfies $y_j = \varepsilon_j x_j$ converging to some point in $\mathscr{P}$ as $j \to \infty$. $\square$

Lemma 4.5. *Let v_j be given in the proof of the above lemma. Then $|v_j(x)| \to 0$ as $|x| \to \infty$ uniformly in $j \in \mathbb{N}$.*

Proof. Assume by contradiction that the conclusion of the lemma does not hold. Then by (24) there exist $\kappa > 0$ and $x_j \in \mathbb{R}^3$ with $|x_j| \to \infty$ such that $\kappa \leqslant |v_j(x_j)| \leqslant C_0 \int_{B_1(x_j)} |v_j|$. Since $v_j \to v$ in H^1 one gets

$$\begin{aligned} \kappa &\leqslant C_0 \int\limits_{B_1(x_j)} |v_j| \leqslant C_0 \int\limits_{B_1(x_j)} |v_j - v| + C_0 \int\limits_{B_1(x_j)} |v| \\ &\leqslant C' \left(\int\limits_{\mathbb{R}^3} |v_j - v|^2\right)^{1/2} + C_0 \int\limits_{B_1(x_j)} |v| \to 0, \end{aligned}$$

a contradiction. $\square$

Lemma 4.6. *There exists $C > 0$ such that for all $j \in \mathbb{N}$*

$$\left|u_j(x)\right| \leqslant Ce^{-\frac{a}{\sqrt{\tau}}|x-x_j|} \quad \forall j \in \mathbb{N}.$$

Proof. By Lemma 4.5 we may take $0 < \delta$ and $R > 0$ such that $|v_j(x)| \leqslant \delta$ and

$$\left|\Re\left[r_{\varepsilon_j}\big(x, |v_j|\big)v_j \frac{\overline{v_j}}{|v_j|}\right]\right| \leqslant \frac{a^2}{2}|v_j|$$

for all $|x| \geqslant R$, $j \in \mathbb{N}$. This, together with (23), implies

$$\Delta|v_j| \geqslant \frac{a^2}{2}|v_j| \quad \text{for all } |x| \geqslant R,\ j \in \mathbb{N}.$$

Let $\Gamma(y) = \Gamma(y, 0)$ be a fundamental solution to $-\Delta + a^2/2$ (see, e.g., [29]). Using the uniform boundedness, one may choose Γ so that $|v_j(y)| \leqslant \frac{a^2}{2}\Gamma(y)$ holds on $|y| = R$, all $j \in \mathbb{N}$. Let $z_j = |v_j| - \frac{a^2}{2}\Gamma$. Then

$$\begin{aligned}\Delta z_j &= \Delta|v_j| - \frac{a^2}{2}\Delta\Gamma \\ &= \frac{a^2}{2}\left(|v_j| - \frac{a^2}{2}\Gamma\right) = \frac{a^2}{2}z_j.\end{aligned}$$

By the maximum principle we can conclude that $z_j(y) \leqslant 0$ on $|y| \geqslant R$. It is well known that there is $C' > 0$ such that $\Gamma(y) \leqslant C' \exp(-\frac{a}{\sqrt{2}}|y|)$ on $|y| \geqslant 1$. We see that

$$\left|v_j(y)\right| \leqslant C \exp\left(-\frac{a}{\sqrt{2}}|y|\right)$$

for all $y \in \mathbb{R}^3$ and all $j \in \mathbb{N}$, that is,

$$\left|u_j(x)\right| \leqslant C \exp\left(-\frac{a}{\sqrt{2}}|x - x_j|\right)$$

for all $x \in \mathbb{R}^3$ and all $j \in \mathbb{N}$.

The proof is completed. □

Proof of Theorem 1.1. Going back to Eq. (4) with the variable substitution: $x \mapsto x/\varepsilon$, Lemma 4.2, jointly with Lemma 2.7, shows that, for all $\varepsilon > 0$ small, Eq. (4) has at least one least energy solution $w_\varepsilon \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \geqslant 2$, that is, the conclusion (i) of Theorem 1.1. Lemma 4.3 is nothing but the conclusion (ii). And finally, the conclusions (iii) and (iv) follow from Lemma 4.4 and Lemma 4.6 respectively. □

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