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Pullback attractors for a non-autonomous homogeneous two-phase flow model

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ABSTRACT

This article studies the pullback asymptotic behavior of solutions for a non-autonomous homogeneous two-phase flow model in a two-dimensional domain. We prove the existence of pullback attractors $\mathcal{A}^{\mathbb{V}}$ in $\mathbb{V}$ (the velocity has the H^1-regularity) and $\mathcal{A}^{\mathbb{Y}}$ in $\mathbb{Y}$ (the velocity has the L^2-regularity). Then we verify the regularity of the pullback attractors by proving that $\mathcal{A}^{\mathbb{V}} = \mathcal{A}^{\mathbb{Y}}$, which implies the pullback asymptotic smoothing effect of the model in the sense that the solutions eventually become more regular than the initial data. The method used in this article is similar to the one used in Zhao and Zhou (2007) [42] in the case of the non-autonomous incompressible non-Newtonian fluid in a two-dimensional domain. Let us mention that the nonlinearity involved in the model considered in this article is stronger than the one in the two-dimensional non-Newtonian flow studied in Zhao and Zhou (2007) [42].

1. Introduction

The study of non-autonomous dynamical systems is an important subject and has been paid much attention as evidence by the references cited in [13,12,15,14,16,30,31,36,40,6,27,26]. In [24], the author considers some special classes of non-autonomous dynamical systems and studies the existence and uniqueness of uniform attractors. In [14], the authors present a general approach that is well suited to construct the uniform attractors of some equations arising in mathematical physics, see also [38,1,14]. In this approach, instead of considering a single process associated to the dynamical system, the authors consider a family of processes depending on a parameter (symbol) σ in some Banach space. The approach preserves the leading concept of invariance, which implies the structure of the uniform attractors.

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It is well known that the concept of attractors has been proved an extremely useful tool for studying the long-term behavior of solutions of a wide variety of dynamical systems [38,39,2,25,5,34,8,9,17,26–29]. With the development of non-autonomous and random dynamical systems, a new type of attractor, called pullback attractor was formulated and investigated in [20,26]. As summarized in [37,42], it consists of a parameterized family of nonempty compact subsets of the state space. Usually, non-autonomous dynamical systems can be formulated in terms of a cocycle mapping for the dynamics in the state space. If the cocycle is continuous with respect to a group θ, which itself is continuous, then the non-autonomous dynamical system can be reduced to a semigroup via the skew-product flow. Results on global attractors for autonomous semi-dynamical systems can thus be adapted to such non-autonomous dynamical systems [14,11,24]. Recently, using the concept of measure of non-compactness (see [33]), the authors of [41] (see also [40,36,16,30,31]) derived some necessary and sufficient conditions for the existence of pullback attractors of non-autonomous dynamical systems. The result was later improved in [37], where the authors proposed sufficient condition for the existence of pullback attractors for norm-to-weak continuous cocycles (that is, cocycles mapping convergent into weakly convergent sequences) in a Banach space.

In this article, we study the pullback asymptotic behavior of solutions for a non-autonomous homogeneous two-phase flow model in a two-dimensional domain. We prove the existence of pullback attractors $\mathcal{A}^{\mathbb{V}}$ in $\mathbb{V}$ (the velocity has the H^1-regularity) and $\mathcal{A}^{\mathbb{Y}}$ in $\mathbb{Y}$ (the velocity has the L^2-regularity). The we verify the regularity of the pullback attractors by proving that $\mathcal{A}^{\mathbb{V}} = \mathcal{A}^{\mathbb{Y}}$, which implies the pullback asymptotic smoothing effect of the model in the sense that the solutions eventually become more regular than the initial data. The method used in this article is the same as the one used in [42] in the case of the non-autonomous incompressible non-Newtonian fluid in a two-dimensional domain. The proof follows similar steps as in [42] (see also [32,40,41,6,31,36,7,27,26]). Let us mention that the nonlinearity involved in the model considered in this article is stronger than the one in the two-dimensional non-Newtonian flow studied in [42].

Let us recall that the incompressible Navier–Stokes equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids [22]. For instance, this approach is used in [3] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with the phase-field system [4,22,21,23]. In the isothermal compressible case, the existence of a global weak solution is proved in [19]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity u and the order parameter ϕ. This system can be written as a NS equation coupled with a convective Allen–Cahn equation [22]. The associated initial and boundary value problem was studied in [22], in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor $\mathcal{A}$. They also established the existence of an exponential attractor $\mathcal{E}$. This entails that $\mathcal{A}$ has a finite fractal dimension, which is estimated in [22] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated although some important issues remain unresolved [38]. In the case of binary fluids, the analysis is even more complicate and the mathematical studied is still at it infancy as noted in [22].

The article is divided as follows. In the next section, we recall from [22] the non-autonomous homogeneous two-phase flow and its mathematical setting. We also derive some a priori estimates. In Section 3, we recall from [42] preliminaries on pullback attractors for cocycle. Then, Section 4 studies the existence of pullback attractors in $\mathcal{A}^{\mathbb{V}}$using the result of [37]. In Section 5, we prove the existence of pullback attractors $\mathcal{A}^{\mathbb{Y}}$ in $\mathbb{Y}$ when the external force is normal. Then in Section 6, using a method of [42] we verify the regularity of the pullback attractors by proving that $\mathcal{A}^{\mathbb{V}} = \mathcal{A}^{\mathbb{Y}}$, which implies the pullback asymptotic smoothing effect of the model in the sense that the solutions eventually become more regular than the initial data.

2. A two-phase flow model and its mathematical setting

2.1. Governing equations

In this article, we consider a model of homogeneous incompressible two-phase flow with singularly oscillating forces. More precisely, we assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\mathbb{R}^2$. Then, we consider the system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu_1 \Delta u + (u \cdot \nabla) u + \nabla p - \mathcal{K} \mu \nabla \phi = g, \\ \operatorname{div} u = 0, \\ \dfrac{\partial \phi}{\partial t} + u \cdot \nabla \phi + \mu = 0, \\ \mu = -\nu_2 \Delta \phi + \alpha f(\phi), \end{cases} \tag{2.1}$$

in $\mathcal{M} \times (0, +\infty)$.

In (2.1), the unknown functions are the velocity $u = (u_1, u_2)$ of the fluid, its pressure p and the order (phase) parameter ϕ. The quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_{\mathcal{M}} \left(\frac{\nu_2}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) ds, \tag{2.2}$$

where, e.g., $F(r) = \int_0^r f(\zeta)\, d\zeta$. Here, the constants $\nu_1 > 0$ and $\mathcal{K} > 0$ correspond to the kinematic viscosity of the fluid and the capillarity (stress) coefficient, respectively, $\nu_2, \alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, ν_2 is related with the thickness of the interface separating the two fluids. Hereafter, as in [22] we assume that $\nu_2 \leqslant \alpha$. We endow (2.1) with the boundary condition

$$u = 0, \qquad \frac{\partial \phi}{\partial \eta} = 0 \quad \text{on } \partial \mathcal{M} \times (0, +\infty), \tag{2.3}$$

where $\partial \mathcal{M}$ is the boundary of $\mathcal{M}$ and η is its outward normal.

The initial condition is given by

$$(u, \phi)(0) = (u_0, \phi_0) \quad \text{in } \mathcal{M}. \tag{2.4}$$

2.2. Mathematical setting

We first recall from [22] a weak formulation of (2.1)–(2.4). Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial \mathcal{M}$ (e.g., of class $\mathcal{C}^2$). We also assume that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies

$$\begin{cases} \lim\limits_{|r| \to +\infty} f'(r) > 0, \\ \left| f'(r) \right| \leqslant c_f \left(1 + |r|^m \right), \quad \forall r \in \mathbb{R}, \end{cases} \tag{2.5}$$

where c_f is some positive constant and $m \in [1, +\infty)$ is fixed. It follows from (2.5) that

$$\left| f(r) \right| \leqslant c_f \left(1 + |r|^{m+1} \right), \quad \forall r \in \mathbb{R}. \tag{2.6}$$

If X is a real Hilbert space with inner product $(\cdot,\cdot)_X$, we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. We set

$$\mathcal{V} = \{u \in \mathcal{C}_c^\infty(\mathcal{M}):\ \operatorname{div} u = 0 \text{ in } \mathcal{M}\}.$$

We denote by H and V the closure of $\mathcal{V}$ in $(L^2(\mathcal{M}))^2$ and $(H_0^1(\mathcal{M}))^2$, respectively. The scalar product in H is denoted by $(\cdot,\cdot)_{L^2}$ and the associated norm by $|\cdot|_{L^2}$. Moreover, the space V is endowed with the scalar product

$$((u,v)) = \sum_{i=1}^{2} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \|u\| = ((u,u))^{1/2}.$$

We now define the operator A by

$$Au = \mathcal{P}\Delta u, \quad \forall u \in D(A) = H^2(\mathcal{M}) \cap V,$$

where $\mathcal{P}$ is the Leray–Helmotz projector in $L^2(\mathcal{M})$ onto H. Then, A is a self-adjoint positive unbounded operator in H which is associated with the scalar product defined above. Furthermore, A^{-1} is a compact linear operator on H and $|A\cdot|_{L^2}$ is a norm on $D(A)$ that is equivalent to the H^2-norm.

Note that from (2.5), we can find $\gamma > 0$ such that

$$\lim_{|r|\to+\infty} f'(r) > 2\gamma > 0. \tag{2.7}$$

We define the linear positive unbounded operator A_γ on $L^2(\mathcal{M})$ by:

$$A_\gamma \phi = -\Delta\phi + \gamma\phi, \quad \forall \phi \in D(A_\gamma), \tag{2.8}$$

where

$$D(A_\gamma) = \left\{\rho \in H^2(\mathcal{M});\ \frac{\partial\rho}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}\right\}.$$

Note that A_γ^{-1} is a compact linear operator on $L^2(\mathcal{M})$ and $|A_\gamma\cdot|_{L^2}$ is a norm on $D(A_\gamma)$ that is equivalent to the H^2-norm.

We introduce the bilinear operators B_0, B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0, which are defined from $D(A)\times D(A)$ into H, $D(A)\times D(A_\gamma)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M})\times D(A_\gamma^{3/2})$ into H, respectively. More precisely, we set

$$\begin{aligned}
\big(B_0(u,v),w\big) &= \int_{\mathcal{M}} \big[(u\cdot\nabla)v\big]\cdot w\,dx = b_0(u,v,w), \quad \forall u,v,w \in D(A),\\
\big(B_1(u,\phi),\rho\big) &= \int_{\mathcal{M}} \big[(u\cdot\nabla)\phi\big]\rho\,dx = b_1(u,\phi,\rho), \quad \forall u \in D(A),\ \phi,\rho \in D(A_\gamma),\\
\big(R_0(\mu,\phi),w\big) &= \int_{\mathcal{M}} \mu[\nabla\phi\cdot w]\,dx = b_1(w,\phi,\mu), \quad \forall w \in D(A),\ (\mu,\phi) \in L^2(\mathcal{M})\times D\big(A_\gamma^{3/2}\big).
\end{aligned} \tag{2.9}$$

Note that

$$R_0(\mu,\phi)=\mathcal{P}\mu\nabla\phi.$$

Now we define the Hilbert spaces $\mathbb{Y}$ and $\mathbb{V}$ by

$$\mathbb{Y}=H\times H^1(\mathcal{M}),\qquad \mathbb{V}=V\times D(A_\gamma), \tag{2.10}$$

endowed with the scalar products whose associated norms are

$$\big|(u,\phi)\big|^2_{\mathbb{Y}}=\mathcal{K}^{-1}|u|^2_{L^2}+\nu_2\big(|\nabla\phi|^2_{L^2}+\gamma|\phi|^2_{L^2}\big),\qquad \big\|(u,\phi)\big\|^2_{\mathbb{V}}=\|u\|^2+|A_\gamma\phi|^2_{L^2}. \tag{2.11}$$

We also set

$$f_\gamma(r)=f(r)-\alpha^{-1}\nu_2\gamma r$$

and observe that f_γ still satisfies (2.7) with γ in place of 2γ since $\nu_2\leqslant\alpha$. Also its primitive $F_\gamma(r)=\int_0^r f_\gamma(\zeta)\,d\zeta$ is bounded from below.

In order to clarify the assumptions on the external force g, we introduce the following notations. Given a Banach space X, we denote by $L^2_{loc}(\mathbb{R};X)$ the metrizable space of functions $\psi(s)$, $s\in\mathbb{R}$ with values in X that are locally 2-power integrable in the Bochner sense [15,31]. It is equipped with the local 2-power mean convergence topology. We will also denote by $L^2_b(\mathbb{R};X)$ the subspace of $L^2_{loc}(\mathbb{R};X)$ of translation bounded functions; i.e., for $\psi(s)\in L^2_b(\mathbb{R};X)$, we have

$$\|\psi\|^2_{L^2_b}\equiv\|\psi\|^2_{L^2_b(\mathbb{R};X)}=\sup_{t\in\mathbb{R}}\int_t^{t+1}\big\|\psi(s)\big\|^2_X\,ds<\infty. \tag{2.12}$$

If $g\in L^2_{loc}(\mathbb{R};X)$, we set

$$\mathcal{H}(g)=\big\{g(\cdot+s),\ s\in\mathbb{R}\big\}. \tag{2.13}$$

Note that for $g_0\in L^2_b(\mathbb{R};X)$, we have

$$\|g\|_{L^2_b(\mathbb{R};X)}\leqslant\|g_0\|_{L^2_b(\mathbb{R};X)},\qquad\forall g\in\mathcal{H}(g_0). \tag{2.14}$$

Throughout this article, we will denote by c a generic positive constant depending on the domain $\mathcal{M}$. Using the notations above, we rewrite (2.1)–(2.3) as (see [22] for the details)

$$\begin{cases}\dfrac{du}{dt}+\nu_1Au+B_0(u,u)-\mathcal{K}R_0(\nu_2A_\gamma\phi,\phi)=g, & \text{a.e. in } \mathcal{M}\times(0,+\infty),\\ \mu=\nu_2A_\gamma\phi+\alpha f_\gamma(\phi), & \text{a.e. in } \mathcal{M}\times(0,+\infty),\\ \dfrac{d\phi}{dt}+\mu+B_1(u,\phi)=0, & \text{a.e. in } \mathcal{M}\times(0,+\infty).\end{cases} \tag{2.15}$$

Remark 2.1. In the weak formulation (2.15), the term $\mu\nabla\phi$ is replaced by $\nu_2A_\gamma\nabla\phi$. This is justified since $f'_\gamma(\phi)\nabla\phi$ is the gradient $F_\gamma(\phi)$ and can be incorporated into the pressure gradient, see [22] for details.

Definition 2.1. Suppose that $(u_0, \phi_0) \in \mathbb{Y}$, $g \in L^2(0, T; V^*)$ and $T > 0$. A pair (u, ϕ) is called a weak solution to (2.15), (2.4) on $[0, T]$ if it satisfies (2.15), (2.4) in a weak sense on $[0, T]$ and

$$(u, \phi) \in \mathcal{C}\big([0, T]; \mathbb{Y}\big) \cap L^2\big([0, T]; \mathbb{V}\big),$$
$$\frac{du}{dt} \in L^{4/3}\big([0, T]; V^*\big), \qquad \frac{d\phi}{dt}, \mu \in L^2\big([0, T]; L^2(\mathcal{M})\big). \tag{2.16}$$

If $(u_0, \phi_0) \in \mathbb{V}$, a weak solution (u, ϕ) is called a strong solution on the time interval $[0, T]$ if in addition to (2.16), it satisfies

$$u \in \mathcal{C}\big([0, T]; V\big) \cap L^2\big(0, T; D(A)\big), \qquad \phi \in \mathcal{C}\big([0, T]; D(A_\gamma)\big) \cap L^2\big(0, T; D\big(A_\gamma^{3/2}\big)\big). \tag{2.17}$$

The weak formulation of (2.15), (2.4) was proposed and studied in [22,21], and the existence and uniqueness of solution was proved when the external force is time independent and non-oscillating. In this article, we study the pullback asymptotic behavior of solutions (2.15), (2.4). We prove the existence of pullback attractors $\mathcal{A}^{\mathbb{V}}$ in $\mathbb{V}$ and $\mathcal{A}^{\mathbb{Y}}$ in $\mathbb{Y}$. Then we verify the regularity of the pullback attractors by proving that $\mathcal{A}^{\mathbb{V}} = \mathcal{A}^{\mathbb{Y}}$, which implies the pullback asymptotic smoothing effect of (2.15), (2.4) in the sense that the solutions eventually become more regular than the initial data.

2.3. Some a priori estimates

In this part, we first derive some a priori estimates on the solution to (2.15), (2.4). We then use these estimates to construct bounded absorbing sets in $\mathbb{V}$ and $\mathbb{Y}$. As pointed in [22], if (u, ϕ) is a smooth solution to (2.1), by taking the scalar product in H of $(2.1)_1$ with u, then taking the scalar product in $L^2(\mathcal{M})$ of $(2.1)_3$ with μ, we derive that

$$\frac{d}{dt}\left[\frac{1}{2}\mathcal{K}^{-1}|u|_{L^2}^2 + \mathcal{F}(\phi)\right] - \mathcal{K}^{-1}(u, g)_{L^2} + \nu_1\mathcal{K}^{-1}\|u\|^2 + |\mu|_{L^2}^2 = 0. \tag{2.18}$$

We will need the following lemma, whose proof is given in [15] (see Lemma 2.1 in [15]).

Lemma 2.1. *Let a real function $z(t)$, $t \geqslant 0$, be uniformly continuous and satisfy the inequality*

$$\frac{dz}{dt} + \lambda z(t) \leqslant f(t), \quad \forall t \geqslant 0, \tag{2.19}$$

where $\lambda > 0$, $f(t) \geqslant 0$ for all $t \geqslant 0$ and $f \in L^1_{loc}(\mathbb{R}+)$. Suppose also that

$$\int_t^{t+1} f(s)\,ds \leqslant M, \quad \forall t \geqslant 0. \tag{2.20}$$

Then,

$$z(t) \leqslant z(0)e^{-\lambda t} + M\big(1 + \lambda^{-1}\big), \quad \forall t \geqslant 0. \tag{2.21}$$

Proof. See [15]. □

Proposition 2.2. *Let $g_0 \in L^2_b(\mathbb{R}; V^*)$ and $g \in \mathcal{H}(g_0)$. For $(u_0, \phi_0) \in \mathbb{Y}$ and $f \in \mathcal{C}^1(\mathbb{R})$. The system* (2.15), (2.4) *has a unique weak solution $(u, \phi)(t)$ that satisfies*

$$(u,\phi) \in \mathcal{C}\big([0,T];\mathbb{Y}\big) \cap L^2\big([0,T];\mathbb{V}\big),$$
$$\frac{du}{dt} \in L^{4/3}\big([0,T];V^*\big), \qquad \frac{d\phi}{dt} \in L^2\big([0,T];L^2(\mathcal{M})\big). \tag{2.22}$$

Moreover, the following estimate holds:

$$\big|(u,\phi)(t)\big|^2_{\mathbb{Y}} \leqslant Q\left(\big|(u,\phi)(\tau)\big|^2_{\mathbb{Y}}\right)e^{-\rho(t-\tau)} + c\big(\|g_0\|^2_{L^2_b(\mathbb{R},V^*)} + c_1\big), \quad \forall t \geqslant \tau \geqslant 0,$$

$$\big|(u,\phi)(t)\big|^2_{\mathbb{Y}} + \int_\tau^t \left(\frac{\nu_1}{\mathcal{K}}\big\|u(s)\big\|^2 + \big|\mu(s)\big|^2_{L^2} + \big|F_\gamma\big(\phi(s)\big)\big|_{L^1}\right) ds$$
$$\leqslant Q\left(\big|(u,\phi)(\tau)\big|^2_{\mathbb{Y}}\right) + \int_\tau^t \big(\|g_0\|^2_{L^2_b(\mathbb{R},V^*)} + c_1\big)\, ds, \quad \forall t \geqslant \tau \geqslant 0,$$

$$\int_\tau^t |A_\gamma \phi|^2_{L^2} \leqslant Q_1\big(t-\tau, \big|(u,\phi)(\tau)\big|^2_{\mathbb{Y}}, \|g_0\|_{L^2_b(\mathbb{R},V^*)}, c_1\big), \quad \forall t \geqslant \tau \geqslant 0, \tag{2.23}$$

where Q nonnegative function given below, Q_1 is a monotone non-decreasing function and c_1 is given by (2.30)*.*

Proof. The existence and uniqueness of weak solutions as well as (2.22) is given in [22]. To derive (2.23), we proceed as in [22] (see Proposition 3.1 and Lemma 3.3 in [22]). Let us take the scalar product in $L^2(\mathcal{M})$ of $(2.15)_3$ with 2ϕ. Adding the resulting equation to (2.18), it follows that (see [22] for the details)

$$\frac{dE}{dt} + \kappa E(t) = \wedge_1(t), \tag{2.24}$$

where

$$E(t) = \big|(u,\phi)(t)\big|^2_{\mathbb{Y}} + 2\alpha\big(F_\gamma\big(\phi(t)\big), 1\big)_{L^2} + \big|\phi(t)\big|^2_{L^2} + C_e, \tag{2.25}$$

and

$$\begin{aligned} \wedge_1(t) = {} & -2\nu_1\mathcal{K}^{-1}\|u\|^2 + \kappa\mathcal{K}^{-1}|u|^2_{L^2} - 2|\mu|^2_{L^2} - (2-\kappa)\nu_2\big(|\nabla\phi|^2_{L^2} + \gamma\big|\phi(t)\big|^2_{L^2}\big) \\ & + 2\alpha\big[\kappa\big(F_\gamma(\phi) - f_\gamma(\phi)\phi, 1\big)_{L^2} - (1-\kappa)\big(f_\gamma(\phi)\phi, 1\big)_{L^2}\big] \\ & + 2\mathcal{K}^{-1}(u,g)_{L^2} + \kappa\big|\phi(t)\big|^2_{L^2} + 2\kappa\alpha C_{F_\gamma}|\mathcal{M}|, \end{aligned} \tag{2.26}$$

$C_e = 2\alpha C_{F_\gamma}|\mathcal{M}| > 0$, $|\mathcal{M}| > 0$ being the Lebesgue measure of $\mathcal{M}$, and $C_{F_\gamma} > 0$ is a constant large enough to ensure that $E(t)$ is nonnegative.

Note that F_γ is bounded from below by a constant independent of ν_2 and α. With this choice of C_e, we can find $C_f > 0$ such that (see [22] for details)

$$\big|\big(u(t),\phi(t)\big)\big|_{\mathbb{Y}}^2 \leqslant E(t) \leqslant C_f\big(1+\big|\big(u(t),\phi(t)\big)\big|_{\mathbb{Y}}^2+\big|\phi(t)\big|_{L^{2m+2}}^{2m+2}\big). \tag{2.27}$$

From (2.5), we have

$$\begin{aligned} c_*\big|f_\gamma(y)\big|\big(1+|y|\big) &\leqslant 2f_\gamma(y)y+c_f\big(1+\alpha^{-1}\nu_2\big),\\ F_\gamma(y)-f_\gamma(y)y &\leqslant c'_f\big(1+\alpha^{-1}\nu_2\big)|y|^2+c''_f, \end{aligned} \tag{2.28}$$

for any $y\in\mathbb{R}$, where c_f, c_*, c'_f and c''_f are positive, sufficiently large constants that depend only on f.

From [22], we also note that

$$\begin{aligned} \wedge_1(t) \leqslant{}& -\mathcal{K}^{-1}\big(\nu_1-\kappa C_m|\mathcal{M}|\big)\big\|u(t)\big\|^2-2\big|\mu(t)\big|_{L^2}^2-(2-\kappa)\nu_2\big|\nabla\phi(t)\big|_{L^2}^2\\ &-\big(2-\kappa\big(1+2c'_f(\alpha+\nu_2)\big)(\nu_2\gamma)^{-1}\big)\nu_2\gamma\big|\phi(t)\big|_{L^2}^2\\ &-c_*\alpha(1-\kappa)\big(\big|f_\gamma\big(\phi(t)\big)\big|,1+\big|\phi(t)\big|\big)_{L^2}+(\nu_1\mathcal{K})^{-1}\|g\|_{V^*}^2+c_1, \end{aligned} \tag{2.29}$$

where C_m depends on the shape of $\mathcal{M}$, but not its size and c_1 is given by

$$c_1=2\kappa\alpha C_{F_\gamma}|\mathcal{M}|+2\alpha c''_f|\mathcal{M}|+c_f(\alpha+\nu_2)(1-\kappa)|\mathcal{M}|. \tag{2.30}$$

Let us choose $\kappa\in(0,1)$ as

$$\kappa=\min\big\{\nu_1\big(2C_m|\mathcal{M}|\big)^{-1},\big(\big(1+2c'_f(\alpha+\nu_2)\big)(\nu_2\gamma)^{-1}\big)^{-1}\big\}. \tag{2.31}$$

From now on, c_i will denote a positive constant independent on the initial data and on time.

It follows from (2.25)–(2.31) that

$$\begin{aligned} &\frac{dE}{dt}+\kappa E(t)+c_2\bigg(\frac{\nu_1}{\mathcal{K}}\big\|u(t)\big\|^2+\nu_2\big|\nabla\phi(t)\big|_{L^2}^2+\nu_2\gamma\big|\phi(t)\big|_{L^2}^2\bigg)+2\big|\mu(t)\big|_{L^2}^2\\ &\quad+c_3\big(\big|f_\gamma\big(\phi(t)\big)\big|,1+\big|\phi(t)\big|\big)_{L^2}\leqslant(\nu_1\mathcal{K})^{-1}\|g\|_{V^*}^2+c_1, \end{aligned} \tag{2.32}$$

which gives

$$\frac{dE}{dt}+\kappa E(t)\leqslant(\nu_1\mathcal{K})^{-1}\|g\|_{V^*}^2+c_1. \tag{2.33}$$

Applying Lemma 2.1 with

$$z(t)=E(t+\tau),\qquad f(t)=\frac{1}{\nu_1\mathcal{K}}\|g\|_{V^*}^2+c_1,\qquad \lambda=\kappa,\qquad M=(\nu_1\mathcal{K})^{-1}\|g\|_{L_b^2(\mathbb{R};V^*)}^2+c_1,$$

we obtain

$$E(t+\tau)\leqslant E(\tau)e^{-\kappa t}+\big(1+\kappa^{-1}\big)\big((\nu_1\mathcal{K})^{-1}\|g\|_{L_b^2(\mathbb{R};V^*)}^2+c_1\big), \tag{2.34}$$

which proves $(2.23)_1$ with $Q(|(u,\phi)(\tau)|_{\mathbb{Y}}^2)=E(\tau)$.

From (2.32), we also have

$$
\begin{aligned}
E(t) &+ c_2 \int_\tau^t \big(\nu_1 \mathcal{K}^{-1} \|u(s)\|^2 + \nu_2 |\nabla\phi(s)|_{L^2}^2 + \nu_2\gamma |\phi(s)|_{L^2}^2\big)\, ds \\
&+ \int_\tau^t \big[2|\mu(s)|_{L^2}^2 + c_3\big(|f_\gamma(\phi(s))|, 1 + |\phi(s)|\big)_{L^2}\big]\, ds \\
&\leqslant E(\tau) + \int_\tau^t \big((\nu_1\mathcal{K})^{-1} \|g(s)\|_{V^*}^2 + c_1\big)\, ds.
\end{aligned}
\tag{2.35}
$$

Note that (2.5) implies that

$$
|F_\gamma(y)| \leqslant |f_\gamma(y)|\big(1+|y|\big) + c_4, \quad \forall y \in \mathbb{R}, \tag{2.36}
$$

for some positive constant c_4.

Therefore, $(2.23)_2$ follows from (2.35) and (2.36).

For $(2.23)_3$, using $(2.23)_1$–$(2.23)_2$ we obtain as in [22] that

$$
\begin{aligned}
\nu_2 \int_\tau^t |A_\gamma\phi|_{L^2}^2\, ds &\leqslant \int_\tau^t |\mu(s)|_{L^2}^2\, ds + \alpha^2 c_f \int_\tau^t \big(1 + |\phi|_{L^{2m+2}}^{2m+2}\big)\, ds + \gamma^2 \int_\tau^t |\phi(s)|_{L^2}^2\, ds \\
&\leqslant Q_1\big(t-\tau, |(u,\phi)(\tau)|_{\mathbb{Y}}, \|g_0\|_{L_b^2(\mathbb{R};V^*)}, c_1\big),
\end{aligned}
\tag{2.37}
$$

and $(2.23)_3$ follows. □

Proposition 2.3. *Let $g_0 \in L_b^2(\mathbb{R}; H)$, then for any $g \in \mathcal{H}(g_0)$ and $(u_0, \phi_0) \in \mathbb{V}$, the system (2.15), (2.4) has a unique strong solution $(u, \phi)(t)$ that satisfies*

$$
\begin{aligned}
&(u,\phi) \in \mathcal{C}\big([0,T];\mathbb{V}\big) \cap L^2\big([0,T]; D(A) \times D\big(A_\gamma^{3/2}\big)\big), \\
&\frac{du}{dt} \in L^{4/3}\big([0,T];H\big), \qquad \frac{d\phi}{dt} \in L^2\big([0,T]; D\big(A_\gamma^{1/2}\big)\big),
\end{aligned}
\tag{2.38}
$$

and the following estimate holds:

$$
(t-\tau)\|(u,\phi)(t)\|_{\mathbb{V}}^2 \leqslant C\big(t-\tau, |(u,\phi)(\tau)|_{\mathbb{Y}}, \|g_0\|_{L_b^2(\mathbb{R};H)}\big), \quad \forall t \geqslant \tau \geqslant 0, \tag{2.39}
$$

where $C = C(R_1, R_2, R_3)$ is a monotone continuous functions of R_1, R_2 and R_3.

Proof. The existence and uniqueness of strong solution as well as (2.38) is proved as in [22]. We only need to prove (2.39). Taking the inner product in H of $(2.15)_1$ with $2Au$, the inner product in $L^2(\mathcal{M})$ of $(2.15)_2$ and $(2.15)_3$ with $2A_\gamma^2\phi$ and adding the resulting equalities gives (see [22] for the details)

$$
\begin{aligned}
&\frac{d\mathcal{Y}}{dt} + 2\nu_1|Au|_{L^2}^2 + 2\nu_2\big|A_\gamma^{3/2}\phi\big|_{L^2}^2 \\
&\quad = 2\mathcal{K}\big(R_0(\nu_2 A_\gamma\phi, \phi), Au\big)_{L^2} - 2\big(B_0(u,u), Au\big)_{L^2} + (g, Au)_{L^2} \\
&\qquad - 2\alpha\big(A_\gamma^{1/2} f_\gamma(\phi), A_\gamma^{3/2}\phi\big)_{L^2} - 2\big(A_\gamma^{1/2} B_1(u,\phi), A_\gamma^{3/2}\phi\big)_{L^2},
\end{aligned}
\tag{2.40}
$$

where

$$\mathcal{Y}(t) = \|u(t)\|^2 + \left|A_\gamma \phi(t)\right|_{L^2}^2.$$

We note that

$$\begin{aligned}
2\mathcal{K}\left|\left(R_0(\nu_2 A_\gamma \phi, \phi), Au\right)_{L^2}\right| &\leqslant 2\mathcal{K}\nu_2\left|R_0(A_\gamma\phi,\phi)\right|_{L^2}|Au|_{L^2} \\
&\leqslant c|A_\gamma\phi|_{L^2}|\nabla\phi|_{L^\infty}|Au|_{L^2} \\
&\leqslant c|A_\gamma\phi|_{L^2}^{4/3}|\phi|_{H^1}^{2/3}|Au|_{L^2}^{4/3} + \frac{\nu_2}{8}\left|A_\gamma^{3/2}\phi\right|_{L^2}^2 \\
&\leqslant \frac{\nu_1}{4}|Au|_{L^2}^2 + \frac{\nu_2}{8}\left|A_\gamma^{3/2}\phi\right|_{L^2}^2 + c|A_\gamma\phi|_{L^2}^4|\phi|_{H^1}^2,
\end{aligned} \tag{2.41}$$

$$\begin{aligned}
2\left|\left(B_0(u,u), Au\right)_{L^2}\right| &\leqslant c\left|B_0(u,u)\right|_{L^2}|Au|_{L^2} \\
&\leqslant c|u|_{L^2}^{1/2}\|u\||Au|_{L^2}^{3/2} \\
&\leqslant \frac{\nu_1}{4}|Au|_{L^2}^2 + c|u|_{L^2}^2\|u\|^4,
\end{aligned} \tag{2.42}$$

$$\begin{aligned}
2\alpha\left|\left(A_\gamma^{1/2} f_\gamma(\phi), A_\gamma^{3/2}\phi\right)_{L^2}\right| &\leqslant c\left|\nabla f_\gamma(\phi)\right|_{L^2}\left|A_\gamma^{3/2}\phi\right|_{L^2} \\
&\leqslant \frac{\nu_2}{8}\left|A_\gamma^{3/2}\phi\right|_{L^2}^2 + c\left|f_\gamma'(\phi)\nabla\phi\right|_{L^2}^2,
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
2\left|\left(A_\gamma^{1/2}B_1(u,\phi), A_\gamma^{3/2}\phi\right)_{L^2}\right| &\leqslant 2\left|A_\gamma^{1/2}B_1(u,\phi)\right|_{L^2}\left|A_\gamma^{3/2}\phi\right|_{L^2} \\
&\leqslant c\|u\|^{1/2}|Au|_{L^2}^{1/2}|\phi|_{H^1}^{1/2}|A_\gamma\phi|_{L^2}^{1/2}\left|A_\gamma^{3/2}\phi\right|_{L^2} \\
&\quad + c|u|_{L^2}^{1/2}\|u\|^{1/2}|A_\gamma\phi|_{L^2}^{1/2}\left|A_\gamma^{3/2}\phi\right|_{L^2}^{3/2} \\
&\leqslant \frac{\nu_2}{8}\left|A_\gamma^{3/2}\phi\right|_{L^2}^2 \\
&\quad + c\left(\|u\||Au|_{L^2}|\phi|_{H^1}|A_\gamma\phi|_{L^2} + |u|_{L^2}^2\|u\|^2|A_\gamma\phi|_{L^2}^2\right) \\
&\leqslant \frac{\nu_2}{8}\left|A_\gamma^{3/2}\phi\right|_{L^2}^2 + \frac{\nu_1}{4}|Au|_{L^2}^2 \\
&\quad + c\left(\|u\|^2|\phi|_{H^1}^2|A_\gamma\phi|_{L^2}^2 + |u|_{L^2}^2\|u\|^2|A_\gamma\phi|_{L^2}^2\right).
\end{aligned} \tag{2.44}$$

It follows from (2.40)–(2.44) that

$$\begin{aligned}
&\frac{d}{dt}\left((t-\tau)\mathcal{Y}(t)\right) + (t-\tau)\frac{\nu_1}{2}|Au|_{L^2}^2 + (t-\tau)\frac{\nu_2}{2}\left|A_\gamma^{3/2}\phi\right|_{L^2}^2 \\
&\quad \leqslant (t-\tau)\mathcal{G}(t)\mathcal{Y}(t) + (t-\tau)\Upsilon(t) + \mathcal{Y}(t),
\end{aligned} \tag{2.45}$$

where

$$\begin{aligned}
\mathcal{G}(t) &= c\left(|A_\gamma\phi|_{L^2}^2|\phi|_{H^1}^2 + |u|_{L^2}^2\|u\|^2\right), \\
\Upsilon(t) &= c\left(\left|f_\gamma'(\phi)\nabla\phi\right|_{L^2}^2 + |g|_{L^2}^2\right).
\end{aligned} \tag{2.46}$$

Note that from (2.23), we have

$$\int_\tau^t \big((s-\tau)\Upsilon(s)+\mathcal{Y}(s)\big)\,ds \leqslant Q\left(t-\tau, \big|(u,\phi)(\tau)\big|_{\mathbb{Y}}^2, \int_\tau^t |g|_{L^2}^2\,ds\right), \tag{2.47}$$

$$\begin{aligned}\int_\tau^t (s-\tau)\mathcal{G}(s)\,ds &\leqslant c\int_\tau^t (s-\tau)\big(\big|A_\gamma^{3/2}\phi\big|_{L^2}^2\big\|\phi(s)\big\|^2+\big|u(s)\big|_{L^2}^2\big\|u(s)\big\|^2\big)\,ds\\ &\leqslant Q\left(t-\tau, \big|(u,\phi)(\tau)\big|_{\mathbb{Y}}^2, \int_\tau^t |g|_{L^2}^2\,ds\right).\end{aligned} \tag{2.48}$$

It follows from (2.45)–(2.48) that

$$(t-\tau)\mathcal{Y}(t) \leqslant Q\left(t-\tau, \big|(u,\phi)(\tau)\big|_{\mathbb{Y}}^2, \int_\tau^t |g|_{L^2}^2\,ds\right), \quad \forall t \geqslant \tau \geqslant 0, \tag{2.49}$$

and (2.39) follows. □

3. Preliminary

Let (E,d) be a complete metric space, (P,ρ) be a metric space which will be called the parameter space. We assume that we are given a mapping $\theta:\mathbb{R}\times P\to P$ such that $\theta_t\equiv\theta(t,\cdot):P\to P$ forms a group, that is, θ satisfies

$$\begin{aligned}\theta_{t+\tau}&=\theta_t\cdot\theta_\tau, \quad \forall t,\tau\in\mathbb{R},\\ \theta_0&=Id.\end{aligned} \tag{3.1}$$

Hereafter, we will denote by $\mathcal{B}(E)$ the set of all bounded subsets of E.

Definition 3.1. A mapping $\Psi:\mathbb{R}_+\times P\times E\to E$ is said to be a cocycle on E with respect to the group θ if the following conditions are satisfied:

$$\begin{aligned}&\Psi(0,p,x)=x, \quad \forall(p,x)\in P\times E;\\ &\Psi(t+\tau,p,x)=\Psi\big(t,\theta_\tau(p),\Psi(\tau,p,x)\big), \quad \forall t,\tau\in\mathbb{R}_+,\ (p,x)\in P\times E.\end{aligned} \tag{3.2}$$

The cocycle is said to be continuous on E if for all $(t,p)\in\mathbb{R}_+\times P$, the mapping $\Psi(t+\tau,p,\cdot):E\to E$ is continuous.

The cocycle is said to be norm-to-weak continuous on E if for each $p\in P$ and $t\geqslant 0$,

$$\|x_n\|_E\to\|x\|_E \text{ implies that } \Psi(t,p,x_n)\rightharpoonup\Psi(t,p,x) \text{ in } E \text{ as } n\to\infty. \tag{3.3}$$

The mapping $\pi:\mathbb{R}_+\times P\times E\to P\times E$ defined by

$$\pi(t,p,x)=\big(\theta_t(p),\Psi(t,p,x)\big), \quad \forall t\in R,\ (p,x)\in P\times E, \tag{3.4}$$

forms a semigroup on $P\times E$ and is called a skew-product flow [42].

Remark 3.1. It is clear that if Ψ is continuous on E, then it is norm-to-weak continuous on E.

Definition 3.2. Let Ψ be a cocycle on E with respect to a group θ. A set $\mathcal{B}_0 \subset E$ is called a uniformly (with respect to $p \in P$) absorbing set for Ψ if for any $\mathcal{O} \in \mathcal{B}(E)$, there exists $T_0 = T_0(\mathcal{O}) \in \mathbb{R}_+$ such that

$$\Psi(t, p, \mathcal{O}) \subset \mathcal{B}_0, \quad \forall t \geqslant T_0,\ p \in P. \tag{3.5}$$

Definition 3.3. Let Ψ be a cocycle on E with respect to a group θ. Given $\mathcal{O} \in \mathcal{B}(E)$ and $p \in P$, we define the pullback ω-limit set $\omega_p(\mathcal{O})$ by

$$\omega_p(\mathcal{O}) = \bigcap_{s\in\mathbb{R}_+} \overline{\bigcup_{t\geqslant s} \Psi\big(t, \theta_{-t}(p), \mathcal{O}\big)}. \tag{3.6}$$

Definition 3.4. A family $\mathcal{A} = \{A_p\}_{p\in P}$ of nonempty compact sets on E is called a pullback attractor of the cocycle ϕ if it is Ψ-invariant, that is,

$$\Psi(t, p, A_p) = A_{\theta_t(p)}, \quad \forall t \in \mathbb{R}_+,\ p \in P, \tag{3.7}$$

and pullback attracting, that is

$$\lim_{t\to\infty} d_E\big(\Psi\big(t, \theta_{-t}(p), \mathcal{O}\big), A_p\big) = 0, \quad \forall \mathcal{O} \in \mathcal{B}(E),\ p \in P, \tag{3.8}$$

where d_E denotes the metric on E.

Theorem 3.1. *Let Ψ be a continuous cocycle on E with respect to a group θ of continuous mappings on P and let $\pi = (\theta_t, \Psi)$ be the corresponding skew-product flow on $P \times E$. In addition, we assume that there exists a nonempty compact subset $\mathcal{B}_0$ of E such that for every $\mathcal{O} \in \mathcal{B}(E)$, there exists $T(\mathcal{O}) \in \mathbb{R}_+$, which is independent of $p \in P$, such that*

$$\Psi(t, p, \mathcal{O}) \subset \mathcal{B}_0, \quad \forall t > T(\mathcal{O}). \tag{3.9}$$

Then,

(1) *there exists a unique pullback attractor $\mathcal{A} = \{A_p\}_{p\in P}$ for the cocycle ϕ on E, where*

$$A_p = \bigcap_{\tau\in\mathbb{R}_+} \overline{\bigcup_{t>\tau} \Psi\big(t, \theta_{-t}(p), \mathcal{B}_0\big)}. \tag{3.10}$$

Furthermore,

(2) *there exists a global attractor $\hat{A}$ for the autonomous semi-dynamical system Ψ defined on $P \times E$, where*

$$\hat{A} = \bigcap_{\tau\in\mathbb{R}_+} \overline{\bigcup_{t>\tau} \pi(t, P \times \mathcal{B}_0)}. \tag{3.11}$$

Assertions (1) *and* (2) *are equivalent and*

$$\hat{A} = \bigcup_{p\in P} \{p\} \times A_p. \tag{3.12}$$

Proof. The proof of (3.10) is given in [18,35], while (3.11) is proved in [11]. Finally, (3.13) is proved in [10]. □

Let $\mathcal{O} \in \mathcal{B}(E)$. Its Kuratowski measure of non-compactness $\varrho(B)$ is defined by

$$\varrho(\mathcal{O}) = \inf\{\delta\colon \mathcal{O} \text{ admits a finite cover by sets of diameter } \leqslant \delta\}. \tag{3.13}$$

We recall the following properties of ϱ (see Lemma 2.1 in [37]).

Lemma 3.2. *Let $\mathcal{O}, B_1, B_2 \in \mathcal{B}(E)$, then*

(1) $\varrho(\mathcal{O}) = 0 \Leftrightarrow \bar{\mathcal{O}}$ *is compact*;
(2) $\varrho(B_1 + B_2) \leqslant \varrho(B_1) + \varrho(B_2)$;
(3) $\varrho(B_1) \leqslant \varrho(B_2)$ *if* $B_1 \subset B_2$;
(4) $\varrho(B_1 \cup B_2) \leqslant \max\{\varrho(B_1), \varrho(B_2)\}$;
(5) $\varrho(\mathcal{O}) = \varrho(\bar{\mathcal{O}})$.

Definition 3.5. A cocycle Ψ on E is said to be pullback ω-limit compact if for any $\mathcal{O} \in \mathcal{B}(E)$, any $p \in P$, and any $\epsilon > 0$, there exists $t_0 = t_0(\mathcal{O}, p, \epsilon) \in \mathbb{R}_+$ such that

$$\varrho\left(\bigcup_{t \geqslant t_0} \Psi\big(t, \theta_{-t}(p), \mathcal{O}\big)\right) \leqslant \epsilon. \tag{3.14}$$

Definition 3.6. A cocycle Ψ on E is said to satisfy the pullback condition (PC) if for any $p \in P$, $\mathcal{O} \in \mathcal{B}(E)$ and $\epsilon > 0$, there exist $t_0 = t_0(\mathcal{O}, p, \epsilon)$ and a finite dimensional space E_1 of E such that

$$(1) \quad \mathcal{P}_0\left(\bigcup_{t \geqslant t_0} \Psi\big(t, \theta_{-t}(p), \mathcal{O}\big)\right) \text{ is bounded,} \tag{3.15}$$

where $\mathcal{P}_0 : E \to E_1$ is a bounded projector;

$$(2) \quad \sup_{u \in \mathcal{O}} \big\| (I - \mathcal{P}_0)\Psi\big(t, \theta_{-t}(p), u\big)\big\|_E \leqslant \epsilon, \quad \forall t \geqslant t_0. \tag{3.16}$$

Lemma 3.3. *Let E be a Banach space and Ψ a cocycle on E. If Ψ is norm-to-weak continuous on E and has a uniformly absorbing set $\mathcal{B}_0$, then Ψ possesses a pullback attractor $\mathcal{A} = \{\mathcal{A}_p\}_{p \in P}$ satisfying $\mathcal{A}_p = \omega_p(\mathcal{B}_0)$ if and only if Ψ is pullback ω-limit compact.*

Proof. See [42]. □

Lemma 3.4. *Let E be a Banach space and Ψ a cocycle on E. If Ψ satisfies the pullback condition (PC), then Ψ is pullback ω-limit compact. Moreover, if E is uniformly convex, then the converse is true.*

Proof. See [42]. □

4. Existence of a pullback attractor in $\mathbb{V}$

In this section, we suppose that $g_0 \in L^2_b(\mathbb{R}; H)$ and $g \in \mathcal{H}(g_0)$. We will prove the existence of a pullback attractor $\mathcal{A}^{\mathbb{V}} = \{\mathcal{A}^{\mathbb{V}}_g\}_{g \in \mathcal{H}(g_0)}$ in $\mathbb{V}$. Hereafter, we define the group $\{\theta_t\}_{t \in \mathbb{R}}$ acting on $\mathcal{H}(g_0)$ by

$$\theta_t g(\cdot) = g(\cdot + t), \quad \forall t \in \mathbb{R},\ \forall g \in \mathcal{H}(g_0). \tag{4.1}$$

From the existence and uniqueness result proved in [22] and recalled in Proposition 2.2, we define a continuous cocycle $\Psi(t, g, (u_0, \phi_0))$ on $\mathbb{Y}$ by

$$\Psi\big(t, g, (u_0, \phi_0)\big) = (u, \phi)(t), \quad \forall\big(t, g, (u_0, \phi_0)\big) \in \mathbb{R}_+ \times \mathcal{H}(g_0) \times \mathbb{Y}, \tag{4.2}$$

where $(u, \phi)(t)$ is the solution of (2.15), (2.4) with the data $(u_0, \phi_0) \in \mathbb{Y}$ and the external force function $g \in \mathcal{H}(g_0)$.

From Proposition 2.3, we also define a continuous cocycle $\Psi(t, g, (u_0, \phi_0))$ on $\mathbb{V}$ by

$$\Psi\big(t, g, (u_0, \phi_0)\big) = (u, \phi)(t), \quad \forall\big(t, g, (u_0, \phi_0)\big) \in \mathbb{R}_+ \times \mathcal{H}(g_0) \times \mathbb{V}, \tag{4.3}$$

where $(u, \phi)(t)$ is the solution of (2.15), (2.4) with the data $(u_0, \phi_0) \in \mathbb{V}$ and the external force function $g \in \mathcal{H}(g_0)$.

Lemma 4.1. *Let $g_0 \in L^2_b(\mathbb{R}; H)$, then the cocycle $\Psi(t, g, (u_0, \phi_0))$ define by (4.3) possesses a bounded uniformly absorbing set $\mathcal{B}_0^{\mathbb{V}} \subset \mathbb{V}$ and it is norm-to-weak continuous on $\mathbb{V}$.*

Proof. From $(2.23)_1$ and (2.27), we see that for any $\mathcal{O} \in \mathcal{B}(\mathbb{Y})$, there exists $t_0 = t_0(\mathcal{O}) \in \mathbb{R}_+$ such that

$$\Psi(t, g, \mathcal{O}) \subset \mathcal{B}_0 = \big\{(u, \phi) \in \mathbb{Y},\ \big|(u, \phi)\big|^2_{\mathbb{Y}} \leqslant \rho_0^2\big\}, \quad \forall t \geqslant t_0,\ \forall g \in \mathcal{H}(g_0), \tag{4.4}$$

where

$$\rho_0^2 = c\big(c_1 + \|g_0\|^2_{L^2_b(\mathbb{R}; V^*)}\big), \tag{4.5}$$

and c_1 is given by (2.30).

Now let us set

$$\mathcal{B}_0^{\mathbb{V}} = \bigcup_{g \in \mathcal{H}(g_0)} \bigcup_{t \geqslant t_0(\mathcal{B}_0)} \Psi(t + 1, g, \mathcal{B}_0). \tag{4.6}$$

From (2.39)and (4.5), it is clear that $\mathcal{B}_0^{\mathbb{V}}$ is bounded in $\mathbb{V}$, more precisely, we have

$$\big\|(u, \phi)\big\|^2_{\mathbb{V}} \leqslant Q_1\big(1, \rho_0, \|g_0\|_{L^2_b(\mathbb{R}; H)}\big) \equiv \rho_1^2, \quad \forall (u, \phi) \in \mathcal{B}_0^{\mathbb{V}}. \tag{4.7}$$

It is also clear that $\mathcal{B}_0^{\mathbb{V}}$ is a bounded uniformly absorbing set for the cocycle $\Psi(t, g, (u_0, \phi_0))$ defined on $\mathbb{V}$ by (4.3). From Proposition 2.3, the cocycle is also continuous on $\mathbb{V}$, which implies that it is norm-to-weak continuous on $\mathbb{V}$. $\square$

Lemma 4.2. *Let $g_0 \in L^2_b(\mathbb{R}; H)$, then the cocycle $\Psi(t, g, (u_0, \phi_0))$ defined by (4.3) satisfies the pullback condition (PC) on $\mathbb{V}$.*

Proof. Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \lambda_n \leqslant \cdots$, $\lambda_n \to +\infty$ as $n \to +\infty$ and a family of elements $e_n = (w_n, \rho_n) \subset D(A) \times D(A_\gamma^{3/2})$, which forms a basis of $\mathbb{V}$ and is orthonormal in $\mathbb{Y}$ such that

$$(A, A_\gamma) e_n = \lambda_n e_n, \quad \forall n \in \mathbb{N}. \tag{4.8}$$

Let

$$\mathbb{V}_n = \mathrm{span}\{e_1, e_2, \dots, e_n\}, \tag{4.9}$$

where $n \in \mathbb{N}$ will be specified later. Then $\mathbb{V}_n$ is a finite dimensional subspace of $\mathbb{V}$. Denote by $\mathcal{P}_n$ the orthogonal projector from $\mathbb{V}$ into $\mathbb{V}_n$. Then, we have $\|\mathcal{P}_n\| \leqslant 1$ for all $n \in \mathbb{N}$. From Lemma 4.1, for any $\mathcal{O} \in \mathcal{B}(\mathbb{V})$, there exists $t_0 = t_0(\mathcal{O}) > 0$ such that

$$\Psi(t, g, \mathcal{O}) \subset \mathcal{B}_0^{\mathbb{V}}, \quad \forall t \geqslant t_0,\ \forall g \in \mathcal{H}(g_0).$$

Without loss of generality, let $(u, \phi) = \Psi(t, \theta_{-s-t_0}(g), (u_0, g_0)) \in D(A) \times D(A_\gamma^{3/2}) \subset \mathbb{V}$ satisfy

$$\begin{cases} \dfrac{du}{dt} + \nu_1 Au + B_0(u,u) - \mathcal{K}R_0(\nu_2 A_\gamma \phi, \phi) = \theta_{-s-t_0}(g) = g(t-s-t_0), \\ \mu = \nu_2 A_\gamma \phi + \alpha f_\gamma(\phi), \\ \dfrac{d\phi}{dt} + \mu + B_1(u,\phi) = 0, \end{cases} \tag{4.10}$$

where $(u_0, \phi_0) \in \mathcal{B}_0^V$, $t \geqslant t_0$, $s \in \mathbb{R}$. We decompose (u, ϕ) as

$$(u,\phi) = \mathcal{P}_n(u,\phi) + (I - \mathcal{P}_n)(u,\phi) = (u_1, \phi_1) + (u_2, \phi_2). \tag{4.11}$$

Taking the inner product in H of $(4.10)_1$ with $2Au_2$, the inner product in $L^2(\mathcal{M})$ of $(4.10)_2$ and $(4.10)_3$ with $2A_\gamma^2\phi_2$, we derive that

$$\begin{aligned} &\frac{d\mathcal{Y}}{dt} + 2\nu_1 |Au_2|_{L^2}^2 + 2\nu_2 \bigl|A_\gamma^{3/2}\phi_2\bigr|_{L^2}^2 \\ &\quad = 2\mathcal{K}\bigl(R_0(\nu_2 A_\gamma \phi, \phi), Au_2\bigr)_{L^2} - 2\bigl(B_0(u,u), Au\bigr)_{L^2} + \bigl(g(t-s-t_0), Au_2\bigr)_{L^2} \\ &\qquad - 2\alpha\bigl(A_\gamma^{1/2} f_\gamma(\phi), A_\gamma^{3/2}\phi_2\bigr)_{L^2} - 2\bigl(A_\gamma^{1/2} B_1(u,\phi), A_\gamma^{3/2}\phi_2\bigr)_{L^2}, \end{aligned} \tag{4.12}$$

where

$$\mathcal{Y}(t) = \bigl\|u_2(t)\bigr\|^2 + \bigl|A_\gamma \phi_2(t)\bigr|_{L^2}^2.$$

We recall that (see [38])

$$|w|_{L^\infty} \leqslant c\|w\| \left(1 + \log \frac{|Aw|_{L^2}^2}{\lambda_1 \|w\|^2}\right)^{1/2}, \tag{4.13}$$

and

$$|w|_{L^\infty} \leqslant c|w|_{L^2}^{1/2} |Aw|_{L^2}^{1/2}, \tag{4.14}$$

for all $w \in D(A)$. We derive from (4.18) the following:

$$|u_1|_{L^\infty} \leqslant c\|u_1\|\left(1+\log\frac{|Au_1|_{L^2}^2}{\lambda_1\|u_1\|^2}\right)^{1/2} \leqslant c\|u_1\|D^{1/2}, \tag{4.15}$$

$$|\nabla\phi_1|_{L^\infty} \leqslant c|A_\gamma\phi_1|_{L^2}\left(1+\log\frac{|A_\gamma^{3/2}\phi_1|_{L^2}^2}{\lambda_1|A_\gamma\phi_1|_{L^2}^2}\right)^{1/2} \leqslant c|A_\gamma\phi_1|_{L^2}D^{1/2}, \tag{4.16}$$

where

$$D = \left(1+\log\frac{\lambda_{n+1}}{\lambda_1}\right). \tag{4.17}$$

We note that

$$|Au_1|_{L^2}^2 \leqslant \lambda_n\|u_1\|^2, \qquad \left|A_\gamma^{3/2}\phi_1\right|_{L^2}^2 \leqslant \lambda_n|A_\gamma\phi_1|_{L^2}^2. \tag{4.18}$$

Using (4.15)–(4.18), we derive

$$\begin{aligned} &2\mathcal{K}\left|\left(R_0(\nu_2A_\gamma\phi,\phi), Au_2\right)_{L^2}\right| \\ &\quad = 2\mathcal{K}\left|b_1(Au_2, \phi_1+\phi_2, A_\gamma\phi_1+A_\gamma\phi_2)\right| \leqslant I_1+I_2+I_3+I_4, \end{aligned} \tag{4.19}$$

where

$$I_1 = 2\mathcal{K}\left|b_1(Au_2,\phi_1,A_\gamma\phi_1)\right| \leqslant c|Au_2|_{L^2}|\nabla\phi_1|_{L^\infty}|A_\gamma\phi_1|_{L^2} \leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + c\rho_0^2\rho_1^2D, \tag{4.20}$$

$$I_2 = 2\mathcal{K}\left|b_1(Au_2,\phi_1,A_\gamma\phi_2)\right| \leqslant c|Au_2|_{L^2}|\nabla\phi_1|_{L^\infty}|A_\gamma\phi_2|_{L^2} \leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + c\rho_0^2\rho_1^2D, \tag{4.21}$$

$$\begin{aligned} I_3 &= 2\mathcal{K}\left|b_1(Au_2,\phi_2,A_\gamma\phi_1)\right| \leqslant c|Au_2|_{L^2}|\nabla\phi_2|_{L^\infty}|A_\gamma\phi_1|_{L^2} \\ &\leqslant c|Au_2|_{L^2}\|\phi_2\|^{1/2}\left|A_\gamma^{3/2}\phi_2\right|_{L^2}^{1/2}|A_\gamma\phi_1|_{L^2} \leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + \frac{\nu_2}{10}\left|A_\gamma^{3/2}\phi_2\right|_{L^2}^2 + c\rho_0^2\rho_1^4, \end{aligned} \tag{4.22}$$

$$\begin{aligned} I_4 &= 2\mathcal{K}\left|b_1(Au_2,\phi_2,A_\gamma\phi_2)\right| \leqslant c|Au_2|_{L^2}|\nabla\phi_2|_{L^\infty}|A_\gamma\phi_2|_{L^2} \\ &\leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + \frac{\nu_2}{10}\left|A_\gamma^{3/2}\phi_2\right|_{L^2}^2 + c\rho_0^2\rho_1^4. \end{aligned} \tag{4.23}$$

We also have

$$2\left|\left(B_0(u,u), Au_2\right)_{L^2}\right| = 2\left|\left(B_0(u_1+u_2, u_1+u_2), Au_2\right)_{L^2}\right| \leqslant J_1+J_2+J_3+J_4, \tag{4.24}$$

where

$$\begin{aligned} J_1 &= 2\left|b_0(u_1,u_1,Au_2)\right| \leqslant |u_1|_{L^\infty}\|u_1\||Au_2|_{L^2} \\ &\leqslant c\|u_1\|^2D^{1/2}|Au_2|_{L^2} \leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + c\rho_1^4D, \end{aligned} \tag{4.25}$$

$$J_2 = 2\left|b_0(u_1,u_2,Au_2)\right| \leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + c\rho_1^4D, \tag{4.26}$$

$$\begin{aligned} J_3 &= 2\left|b_0(u_2,u_1,Au_2)\right| \leqslant |u_2|_{L^\infty}\|u_1\||Au_2|_{L^2} \\ &\leqslant c|u_2|^2|Au_2|_{L^2}^{3/2}\|u_1\| \leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + c\rho_0^2\rho_1^4, \end{aligned} \tag{4.27}$$

$$\begin{aligned}J_4 = 2\left|b_0(u_2, u_2, Au_2)\right| &\leqslant |u_2|_{L^\infty}\|u_2\||Au_2|_{L^2} \\ &\leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + c\rho_0^2\rho_1^4. \end{aligned}\tag{4.28}$$

Note that

$$\begin{aligned}2\alpha\left|\left(A_\gamma^{1/2} f_\gamma(\phi), A_\gamma^{3/2}\phi_2\right)_{L^2}\right| &\leqslant c\left|\nabla f_\gamma(\phi)\right|_{L^2}\left|A_\gamma^{3/2}\phi\right|_{L^2} \\ &\leqslant \frac{\nu_2}{10}\left|A_\gamma^{3/2}\phi_2\right|_{L^2}^2 + c\left|f_\gamma'(\phi)\nabla\phi\right|_{L^2}^2 \\ &\leqslant \frac{\nu_2}{10}\left|A_\gamma^{3/2}\phi_2\right|_{L^2}^2 + c\rho_2^2. \end{aligned}\tag{4.29}$$

Note that from (2.5) and (2.10), we have

$$\left|f_\gamma'(\phi)\nabla\phi\right|_{L^2}^2 \leqslant \rho_2^2, \tag{4.30}$$

where $\rho_2^2 = \rho_2^2(\rho_1)$. We also have

$$\begin{aligned}2\left|\left(A_\gamma^{1/2} B_1(u, \phi), A_\gamma{}^{3/2}\phi_2\right)_{L^2}\right| &= 2\left|\left(A_\gamma^{1/2} B_1(u_1 + u_2, \phi_1 + \phi_2), A_\gamma{}^{3/2}\phi_2\right)_{L^2}\right| \\ &\leqslant K_1 + K_2 + K_3 + K_4, \end{aligned}\tag{4.31}$$

where

$$\begin{aligned}K_1 &= 2\left|\left(A_\gamma^{1/2} B_1(u_1, \phi_1), A_\gamma{}^{3/2}\phi_2\right)_{L^2}\right| \\ &\leqslant c\left(|\nabla u_1|_{L^2}|\nabla\phi_1|_{L^\infty} + |u_1|_{L^\infty}|A_\gamma\phi_1|_{L^2}\right)\left|A_\gamma{}^{3/2}\phi_2\right|_{L^2} \\ &\leqslant c\|u_1\||A_\gamma\phi_1|_{L^2} D^{1/2}|A_\gamma\phi_1|_{L^2} \\ &\leqslant \frac{\nu_2}{10}|A_\gamma\phi_2|_{L^2}^2 + c\rho_1^4 D, \end{aligned}\tag{4.32}$$

$$K_2 = 2\left|\left(A_\gamma^{1/2} B_1(u_1, \phi_2), A_\gamma{}^{3/2}\phi_2\right)_{L^2}\right| \leqslant \frac{\nu_2}{10}|A_\gamma\phi_2|_{L^2}^2 + c\rho_1^4 D, \tag{4.33}$$

$$\begin{aligned}K_3 &= 2\left|\left(A_\gamma^{1/2} B_1(u_2, \phi_1), A_\gamma{}^{3/2}\phi_2\right)_{L^2}\right| \\ &\leqslant c\left(|\nabla u_2|_{L^2}|\nabla\phi_1|_{L^\infty} + |u_2|_{L^\infty}|A_\gamma\phi_1|_{L^2}\right)\left|A_\gamma{}^{3/2}\phi_2\right|_{L^2} \\ &\leqslant c\left(\|u_2\||A_\gamma\phi_1|_{L^2} D^{1/2} + |u_2|_{L^2}^{1/2}|Au_2|_{L^2}^{1/2}|A_\gamma\phi_1|_{L^2}\right)\left|A_\gamma{}^{3/2}\phi_2\right|_{L^2} \\ &\leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + \frac{\nu_2}{10}\left|A_\gamma{}^{3/2}\phi_2\right|_{L^2}^2 + c\rho_1^4 D, \end{aligned}\tag{4.34}$$

$$\begin{aligned}K_4 &= 2\left|\left(A_\gamma^{1/2} B_1(u_2, \phi_2), A_\gamma{}^{3/2}\phi_2\right)_{L^2}\right| \\ &\leqslant c\left(|\nabla u_2|_{L^2}|\nabla\phi_2|_{L^\infty} + |u_2|_{L^\infty}|A_\gamma\phi_2|_{L^2}\right)\left|A_\gamma{}^{3/2}\phi_2\right|_{L^2} \\ &\leqslant c\|u_2\|\|\phi_2\|^{1/2}\left|A_\gamma{}^{3/2}\phi_2\right|_{L^2}^{3/2} + c|u_2|_{L^2}^{1/2}|Au_2|_{L^2}^{1/2}|A_\gamma\phi_2|_{L^2}\left|A_\gamma{}^{3/2}\phi_2\right|_{L^2} \\ &\leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + \frac{\nu_2}{10}|A_\gamma\phi_2|_{L^2}^2 + c\rho_0^2 + c\rho_1^6. \end{aligned}\tag{4.35}$$

We also have

$$\left|\left(g(t-s-t_0), Au_2\right)_{L^2}\right| \leqslant \frac{\nu_1}{10}|Au_2|_{L^2}^2 + c\left|g(t-s-t_0)\right|_{L^2}^2. \tag{4.36}$$

Combining (4.12)–(4.36), we derive that

$$\begin{aligned}
&\frac{d\mathcal{Y}}{dt} + \nu_1|Au_2|_{L^2}^2 + \nu_2\left|A_\gamma^{3/2}\right|_{L^2}^2 \\
&\quad \leqslant c\left(\rho_1^4 D + \rho_1^4\rho_0^2 + \rho_o^2\rho_1^2 D + \rho_1^4 D + \rho_0^2 + \rho_1^6 + \rho_2^2\right) + c\left|g(t-s-t_0)\right|_{L^2}^2,
\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}
\frac{d\mathcal{Y}}{dt} + \nu\lambda_{n+1}\mathcal{Y} &\leqslant c\left(\rho_1^4 D + \rho_1^4\rho_0^2 + \rho_o^2\rho_1^2 D + \rho_1^4 D + \rho_0^2 + \rho_1^6 + \rho_2^2\right) \\
&\quad + c\left|g(t-s-t_0)\right|_{L^2}^2,
\end{aligned} \tag{4.38}$$

where $\nu = \min(\nu_1, \nu_2)$. We note that

$$|Au_2|_{L^2}^2 \geqslant \lambda_{n+1}(Au_2, u_2)_{L^2}, \qquad \left|A_\gamma^{3/2}\phi_2\right|_{L^2}^2 \geqslant \lambda_{n+1}(A_\gamma\phi_2, A_\gamma\phi_2)_{L^2}. \tag{4.39}$$

Let $\tilde{\lambda}_{n+1} = \nu\lambda_{n+1}$ and $\alpha_1 > 0$ such that

$$\left\|(v,\varphi)\right\|_{\mathbb{V}}^2 \leqslant \alpha_1^{-1}\left(\langle Av, v\rangle + \langle A_\gamma\varphi, A_\gamma\varphi\rangle\right), \quad \forall (v,\varphi) \in \mathbb{V}. \tag{4.40}$$

Then the Gronwall Lemma yields

$$\mathcal{Y}(t_0+s) \leqslant \mathcal{Y}(t_0)e^{-\tilde{\lambda}_{n+1}s} + \frac{R_2}{\tilde{\lambda}_{n+1}} + c\int_{t_0}^{s+t_0} e^{-\tilde{\lambda}_{n+1}(s+t_0-t)}\left|g(t-s-t_0)\right|_{L^2}^2\, dt, \tag{4.41}$$

where

$$R_2 = c\left(\rho_1^4 D + \rho_1^4\rho_0^2 + \rho_o^2\rho_1^2 D + \rho_0^2 + \rho_1^6 + \rho_2^2\right). \tag{4.42}$$

Let $n_1 = n_1(\epsilon) \in \mathbb{N}$ such that

$$\frac{R_2}{\tilde{\lambda}_{n+1}} \leqslant \frac{\alpha_1\epsilon}{4}, \quad \forall n \geqslant n_1. \tag{4.43}$$

Let $\eta \in (0,1)$ fixed and $\tau = t-s-t_0$. Then

$$\begin{aligned}
&c\int_{t_0}^{s+t_0} e^{-\tilde{\lambda}_{n+1}(s+t_0-t)}\left|g(t-s-t_0)\right|_{L^2}^2\, dt \\
&\quad = c\int_{-s}^{0} e^{-\tilde{\lambda}_{n+1}\tau}\left|g(\tau)\right|_{L^2}^2\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leqslant c\int_{-\eta}^{0} e^{-\tilde{\lambda}_{n+1}\tau}\big|g(\tau)\big|_{L^2}^2\,d\tau + c\int_{-s}^{-\eta} e^{\tilde{\lambda}_{n+1}\tau}\big|g(\tau)\big|_{L^2}^2\,d\tau \\
&\leqslant c\int_{-\eta}^{0} e^{\tilde{\lambda}_{n+1}\tau}\big|g(\tau)\big|_{L^2}^2\,d\tau + c\int_{-\eta-1}^{-\eta} e^{\tilde{\lambda}_{n+1}\tau}\big|g(\tau)\big|_{L^2}^2\,d\tau \\
&\quad + c\int_{-\eta-2}^{-\eta-1} e^{\tilde{\lambda}_{n+1}\tau}\big|g(\tau)\big|_{L^2}^2\,d\tau + c\int_{-\eta-3}^{-\eta-2} e^{-\tilde{\lambda}_{n+1}\tau}\big|g(\tau)\big|_{L^2}^2\,d\tau + \cdots \\
&\leqslant c\int_{-\eta}^{0} e^{\tau\tilde{\lambda}_{n+1}}\big|g(\tau)\big|_{L^2}^2\,d\tau \\
&\quad + ce^{-\tau\tilde{\lambda}_{n+1}}\big(1+e^{-\tilde{\lambda}_{n+1}}+e^{-2\tilde{\lambda}_{n+1}}+e^{-3\tilde{\lambda}_{n+1}}+\cdots\big)\|g_0\|_{L_b^2(\mathbb{R};H)}^2 \\
&\leqslant c\int_{-\eta}^{0} e^{\tau\tilde{\lambda}_{n+1}}\big|g(\tau)\big|_{L^2}^2\,d\tau + ce^{-\eta\tilde{\lambda}_{n+1}}\|g_0\|_{L_b^2(\mathbb{R};H)}^2\big(1-e^{-\tilde{\lambda}_{n+1}}\big)^{-1}. \qquad (4.44)
\end{aligned}
$$

Since $g \in L_b^2(\mathbb{R}; H)$, we have

$$
\int_{-\eta}^{0} e^{\tau\tilde{\lambda}_{n+1}}\big|g(\tau)\big|_{L^2}^2\,d\tau \leqslant \|g\|_{L_b^2(\mathbb{R};H)}^2 \leqslant \|g_0\|_{L_b^2(\mathbb{R};H)}^2. \qquad (4.45)
$$

By the Lebesgue dominated convergence theorem, we see that for the above $\epsilon > 0$, there exists $n_2 = n_2(\epsilon) \in \mathbb{N}$ such that

$$
\begin{gathered}
c\int_{-\eta}^{0} e^{\tau\tilde{\lambda}_{n+1}}\big|g(\tau)\big|_{L^2}^2\,d\tau \leqslant \frac{\alpha_1\epsilon}{4}, \quad \forall n \geqslant n_2, \\
ce^{-\eta\tilde{\lambda}_{n+1}}\|g\|_{L_b^2(\mathbb{R};H)}^2\big(1-e^{-\tilde{\lambda}_{n+1}}\big)^{-1} \leqslant \frac{\alpha_1\epsilon}{4}, \quad \forall n \geqslant n_2. \qquad (4.46)
\end{gathered}
$$

Now, let $t_1 = t_0 + \frac{1}{\tilde{\lambda}_{n+1}}\ln\big(\frac{4\rho_1^2}{\alpha_1\epsilon}\big) + 1$. Then we have

$$
\mathcal{Y}(t_0)e^{-s\tilde{\lambda}_{n+1}} \leqslant \rho_1^2 e^{-s\tilde{\lambda}_{n+1}} \leqslant \frac{\alpha_1\epsilon}{4}, \quad \forall s \geqslant t_1. \qquad (4.47)
$$

It follows from (4.41)–(4.47) that for $n \geqslant \max\{n_1, n_2\}$ and $s \geqslant t_1$, we have

$$
\begin{aligned}
\big\|(I-\mathcal{P}_n)\Psi\big(s,\theta_{-s}(g),(u_0,\phi_0)\big)\big\|_{\mathbb{V}}^2 &= \big\|(u_2,\phi_2)(s)\big\|_{\mathbb{V}}^2 \leqslant \alpha_1^{-1}\big(\langle Au_2,u_2\rangle + \langle A_\gamma\phi_2, A_\gamma\phi_2\rangle\big) \\
&\leqslant \alpha_1^{-1}\left(\frac{\alpha_1\epsilon}{4}+\frac{\alpha_1\epsilon}{4}+\frac{\alpha_1\epsilon}{4}+\frac{\alpha_1\epsilon}{4}\right) = \epsilon, \qquad (4.48)
\end{aligned}
$$

for all $s \geqslant t_1$, $\forall g \in \mathcal{H}(g_0)$, and the proof is complete. □

Theorem 4.3. *Let $g_0 \in L^2_b(\mathbb{R}; H)$, then the cocycle $\Psi(t, g, (u_0, \phi_0))$ defined by (4.3) possesses a pullback attractor in $\mathbb{V}$*

$$\mathcal{A}^{\mathbb{V}} = \{\mathcal{A}_g^{\mathbb{V}}\}_{g\in\mathcal{H}(g_0)} = \{\omega_g(\mathcal{B}_0^{\mathbb{V}})\}_{g\in\mathcal{H}(g_0)}, \tag{4.49}$$

where $\mathcal{B}_0^{\mathbb{V}}$ is a bounded uniformly absorbing set defined by (4.6) and

$$\omega_g(\mathcal{B}_0^{\mathbb{V}}) = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\Psi(t, \theta_{-t}(g), \mathcal{B}_0^{\mathbb{V}})} \tag{4.50}$$

is the pullback ω-limit set of $\mathcal{B}_0^{\mathbb{V}}$, where the bar denotes the closure in $\mathbb{V}$.

Proof. It follows from Lemmas 3.3, 3.4, 4.1 and 4.2. □

5. Existence of a pullback attractor in $\mathbb{Y}$

In this section, we establish the existence of a pullback attractor $\mathcal{A}^{\mathbb{Y}}$ in $\mathbb{Y}$ with g_0 being normal in $L^2_{log}(\mathbb{R}; V^*)$. We first recall from [42] the following definition.

Definition 5.1. A function $g(t) \in L^2_{log}(\mathbb{R}; V^*)$ is said to be normal if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+\delta}\|g\|^2_{V^*}\,ds \leqslant \epsilon. \tag{5.1}$$

The set of normal functions in $L^2_{log}(\mathbb{R}; V^*)$ will be denoted $L^2_n(\mathbb{R}; V^*)$. Clearly, we have $L^2_n(\mathbb{R}; V^*) \subset L^2_b(\mathbb{R}; V^*)$.

Lemma 5.2. *If $g_0 \in L^2_b(\mathbb{R}; V^*)$, then for any $g \in \mathcal{H}(g_0)$ and $(u_0, \phi_0) \in \mathbb{Y}$, the system (2.15), (2.4) has a unique solution (u, ϕ) that satisfies $(u, \phi)(t)$ that satisfies*

$$\begin{gathered}(u, \phi) \in \mathcal{C}([0, T]; \mathbb{Y}) \cap L^2([0, T]; \mathbb{V}),\\ \frac{du}{dt} \in L^{4/3}([0, T]; V^*), \qquad \frac{d\phi}{dt} \in L^2([0, T]; L^2(\mathcal{M})).\end{gathered} \tag{5.2}$$

Moreover, the following estimate holds:

$$\begin{gathered}|(u, \phi)(t)|^2_{\mathbb{Y}} \leqslant Q(|(u, \phi)(\tau)|^2_{\mathbb{Y}})e^{-\rho(t-\tau)} + c(\|g_0\|^2_{L^2_b(\mathbb{R},V^*)} + c_1), \quad \forall t \geqslant \tau \geqslant 0,\\ |(u, \phi)(t)|^2_{\mathbb{Y}} + \int_\tau^t\left(\frac{\nu_1}{\mathcal{K}}\|u(s)\|^2 + |\mu(s)|^2_{L^2} + |F_\gamma(\phi(s))|_{L^1}\right)ds\\ \leqslant Q(|(u, \phi)(\tau)|^2_{\mathbb{Y}}) + \int_\tau^t(\|g_0\|^2_{L^2_b(\mathbb{R},V^*)} + c_1)\,ds, \quad \forall t \geqslant \tau \geqslant 0,\end{gathered}$$

$$\int_{\tau}^{t} |A_\gamma \phi|_{L^2}^2 \leqslant Q_1\big(t-\tau, \big|(u,\phi)(\tau)\big|_{\mathbb{Y}}^2, \|g_0\|_{L_b^2(\mathbb{R},V^*)}, c_1\big), \quad \forall t \geqslant \tau \geqslant 0, \tag{5.3}$$

where Q nonnegative function given below, Q_1 is a monotone non-decreasing function and c_1 is given by (2.30).

Proof. The proof is similar to that of Proposition 2.2, so we omit it. □

From Lemma 5.2 we define a continuous cocycle $\Psi(t, g, (u_0, \phi_0))$ on $\mathbb{Y}$ by

$$\Psi\big(t, g, (u_0, \phi_0)\big) = (u, \phi)(t), \quad \forall\big(t, g, (u_0, \phi_0)\big) \in \mathbb{R}_+ \times \mathcal{H}_0(g_0) \times \mathbb{Y}, \tag{5.4}$$

where $(u, \phi)(t)$ is the solution of (2.15), (2.4) with the data $(u_0, \phi_0) \in \mathbb{Y}$ and the external force function $g \in \mathcal{H}_0(g_0)$.

Lemma 5.3. *Let $g_0 \in L_b^2(\mathbb{R}; V^*)$, then the cocycle $\Psi(t, g, (u_0, \phi_0))$ define by (5.4) possesses a bounded uniformly absorbing set $\mathcal{B}_0^{\mathbb{Y}} \subset \mathbb{Y}$ and it is norm-to-weak continuous on $\mathbb{Y}$.*

Proof. From $(5.3)_1$, we see that for any $\mathcal{O} \in \mathcal{B}(\mathbb{Y})$, there exists $t_0 = t_0(\mathcal{O}) \in \mathbb{R}_+$ such that

$$\Psi(t, g, \mathcal{O}) \subset \mathcal{B}_0^{\mathbb{Y}} = \big\{(u, \phi) \in \mathbb{Y},\ \big|(u, \phi)\big|_{\mathbb{Y}}^2 \leqslant \rho_3^2\big\}, \quad \forall t \geqslant t_0,\ \forall g \in \mathcal{H}_0(g_0), \tag{5.5}$$

where

$$\rho_3^2 = c\big(c_1 + \|g_0\|_{L_b^2(\mathbb{R};V^*)}^2\big), \tag{5.6}$$

and c_1 is given by (2.30). Finally, from (5.2) and Remark 3.1, we conclude that the cocycle is continuous on $\mathbb{Y}$. □

The following embedding theorem, which can be found in [42] (see Lemma 4.3 of [42]) will be used to prove the pullback ω-limit compactness of the cocycle in $\mathbb{Y}$.

Lemma 5.4. *Let E_0, E_1, E be three Banach spaces satisfying $E_1 \subset E \subset E_0$, with the injection of E_1 in E being compact. Assume $p_1 \geqslant 1$ and $p_0 > 1$. We set*

$$W_{p_1,p_0}(0, T; E_1, E_0) = \big\{\psi(t),\ t \in [0, T]\colon \psi(t) \in L^{p_1}\big((0, T); E_1\big),\ \psi'(t) \in L^{p_0}\big((0, T); E_0\big)\big\},$$

equipped with the norm

$$\|\psi\|_{W_{p_1,p_0}(0,T;E_1,E_0)} = \left(\int_0^T \big\|\psi(t)\big\|_{E_1}^{p_1} dt\right)^{1/p_1} + \left(\int_0^T \big\|\psi'(t)\big\|_{E_0}^{p_0} dt\right)^{1/p_0}.$$

Then we have

$$W_{p_1,p_0}(0, T; E_1, E_0) \subset L^{p_1}(0, T; E), \quad \forall T > 0,$$

with compact injection.

Lemma 5.5. *Let $g_0 \in L^2_n(\mathbb{R}; V')$, then the cocycle $\Psi(t, g, (u_0, \phi_0))$ defined by (5.4) is pullback ω-limit compact in $\mathbb{Y}$.*

Proof. For any $\mathcal{O} \in \mathcal{B}(\mathbb{Y})$, let $(u_0, \phi_0) \in \mathcal{O}$ and $(u, \phi)(t) = \Psi(t, \theta_{-s}(g), (u_0, \phi_0))$. Then (u, ϕ) satisfies

$$\begin{cases} \dfrac{du}{dt} + \nu_1 Au + B_0(u, u) - \mathcal{K}R_0(\nu_2 A_\gamma \phi, \phi) = g(t-s), \\ \mu = \nu_2 A_\gamma \phi + \alpha f_\gamma(\phi), \\ \dfrac{d\phi}{dt} + \mu + B_1(u, \phi) = 0. \end{cases} \tag{5.7}$$

Let $E(t) = |(u, \phi)(t)|^2_{\mathbb{Y}} + 2\alpha(F_\gamma(\phi(t)), 1)_{L^2} + |\phi(t)|^2_{L^2} + C_e$. As in (2.24)–(2.32), we derive that

$$\begin{aligned} &\frac{dE}{dt} + \kappa E(t) + c_2\left(\frac{\nu_1}{\mathcal{K}}\|u(t)\|^2 + \nu_2|\nabla\phi(t)|^2_{L^2} + \nu_2\gamma|\phi(t)|^2_{L^2}\right) + 2|\mu(t)|^2_{L^2} \\ &\quad + c_3\big(\big|f_\gamma(\phi(t))\big|, 1 + |\phi(t)|\big)_{L^2} \leqslant (\nu_1\mathcal{K})^{-1}\|g(t-s)\|^2_{V^*} + c_1. \end{aligned} \tag{5.8}$$

Let $\tilde{\rho}_0 = \tilde{\rho}_0(\mathcal{O}) > 0$ be a constant such that (see (2.23) and (2.27))

$$\tilde{E}(u, \phi) = \big|(u, \phi)\big|^2_{\mathbb{Y}} + 2\alpha\big(F_\gamma(\phi), 1\big)_{L^2} + |\phi|^2_{L^2} + C_e \leqslant \tilde{\rho}_0^2, \quad \forall (u, \phi) \in \mathcal{O},\ \forall t \geqslant t_0, \tag{5.9}$$

where C_e is the same constant as in (2.25)–(2.26).

Integrating (5.8), we derive that

$$\begin{aligned} &c\int_{t_1}^{t_2}\left(\frac{\nu_1}{\mathcal{K}}\|u(t)\|^2 + \nu_2|\nabla\phi(t)|^2_{L^2} + \nu_2\gamma|\phi(t)|^2_{L^2}\right)dt \\ &\quad \leqslant c\tilde{\rho}_0^2 + c\int_{t_1}^{t_2}\big((\nu_1\mathcal{K})^{-1}\|g(t-s)\|^2_{V^*} + c_1\big)\,dt. \end{aligned} \tag{5.10}$$

From (5.3), we also have $\forall t_2 > t_1 \geqslant t_0^* + t_0$

$$\int_{t_1}^{t_2}\|B_0(u, u)\|^2_{V^*}\,dt \leqslant c\int_{t_1}^{t_2}|u|^2_{L^2}\|u\|^2\,dt \leqslant \rho_0^2\int_{t_1}^{t_2}\|u\|^2\,dt, \tag{5.11}$$

$$\int_{t_1}^{t_2}\|R_0(A_\gamma\phi, \phi)\|^{4/3}_{V^*}\,dt \leqslant c\int_{t_1}^{t_2}\|\phi\||A_\gamma\phi|^2_{L^2}\,dt \leqslant \rho_0^2\int_{t_1}^{t_2}|A_\gamma\phi|^2_{L^2}\,dt, \tag{5.12}$$

$$\int_{t_1}^{t_2}|B_1(u, \phi)|^2_{L^2}\,dt \leqslant c\int_{t_1}^{t_2}|u|_{L^2}\|u\|\|\phi\||A_\gamma\phi|_{L^2}\,dt \leqslant \rho_0^2\int_{t_1}^{t_2}\|u\||A_\gamma\phi|_{L^2}\,dt, \tag{5.13}$$

$$\int_{t_1}^{t_2}\big|A_\gamma^{1/2}f(\phi)\big|^2_{L^2}\,dt \leqslant c\int_{t_1}^{t_2}\big|f'(\phi)\nabla\phi\big|^2_{L^2}\,dt \leqslant c\rho_2^2. \tag{5.14}$$

It follows from (5.8)–(5.14) that

$$\int_{t_1}^{t_2}\left(\left|\frac{du}{dt}\right|_{V^*}^{4/3}+\left|\frac{d\phi}{dt}\right|_{L^2}^{2}\right)dt\leqslant c\rho_0^2\int_{t_1}^{t_2}\left(\|u\|^2+|A_\gamma\phi|_{L^2}^2\right)dt+c\int_{t_1}^{t_2}\left(\big\|g(t-s)\big\|_{V^*}^2+\rho_2^2\right)dt. \tag{5.15}$$

Since $g_0\in L^2_{log}(\mathbb{R},V^*)$, it follows from (5.10) that the set

$$\begin{aligned}\mathcal{B}_{[t_1,t_2]}=\big\{&(u,\phi)(s)=\Psi\big(s,\theta_{-s}(g),(u_0,\phi_0)\big),\ (u_0,\phi_0)\in\mathcal{B}^{\mathbb{Y}},\\ &g\in\mathcal{H}(g_0),\ t_2>t_1\geqslant t_0^*+t_0,\ s\in[t_1,t_2]\big\}\subset L^2\big((t_1,t_2);\mathbb{Y}\big)\end{aligned}\tag{5.16}$$

is bounded in $L^2(t_1,t_2;\mathbb{V})$. We also note that (5.15) implies that

$$\left\{\left(\frac{du}{dt},\frac{d\phi}{dt}\right),\ (u,\phi)\in\mathcal{B}_{[t_1,t_2]}\right\}\tag{5.17}$$

is bounded in $L^{4/3}(t_1,t_2;V^*)\times L^2(t_1,t_2;L^2(\mathcal{M}))$. Noticing that $\mathbb{V}\subset\mathbb{Y}\subset\mathbb{V}^*$ and the embedding $\mathbb{V}\subset\mathbb{Y}$ is compact, we conclude that $\mathcal{B}_{[t_1,t_2]}$ is pre-compact in $L^2(t_1,t_2;\mathbb{Y})$. If $g_0\in L^2_n(\mathbb{R};V^*)$, then for any $\epsilon>0$, there exists $\delta\in(0,1)$ such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+\delta}\big\|g_0(\tau)\big\|_{V^*}^2\,d\tau\leqslant\frac{\epsilon}{M},\tag{5.18}$$

where $M>0$ will be specified later. Let $t^*=t^*(\mathcal{O})=t_0^*(\mathcal{O})+t_0+1=t_0^*+t_0+1$. It is clear that $\mathcal{B}_{[s-\delta,s]}$ is pre-compact in $L^2(s-\delta,s;\mathbb{Y})$ for each $s\geqslant t^*$. Thus, for any $\epsilon>0$, there exists a finite $\frac{\delta\epsilon}{2M}$-net $\{(u_1,\phi_1),(u_2,\phi_2),\dots,(u_{N^*},\phi_{N^*})\}\subset\mathcal{B}_{[s-\delta,s]}$ such that for any $(u,\phi)\in B_{[s-\delta,s]}$, there exits $k\in\{1,2,\dots,N^*\}$ such that

$$\int_{s-\delta}^{s}\big\|(u,\phi)-(u_k,\phi_k)\big\|_{\mathbb{Y}}^2\,dt\leqslant\frac{\delta\epsilon}{2M},\tag{5.19}$$

which implies that there exists $\tilde{t}\in[s-\delta,s]$ such that

$$\big\|(u,\phi)(\tilde{t})-(u_k,\phi_k)(\tilde{t})\big\|_{\mathbb{Y}}^2\leqslant\frac{\delta\epsilon}{2M}.\tag{5.20}$$

Now if $(u_1,\phi_1)(t)=\Psi(t,\theta_{-s}(g_1),(u_{01},\phi_{01}))$, $(u_2,\phi_2)(t)=\Psi(t,\theta_{-s}(g_2),(u_{02},\phi_{02}))$, with $g_1,g_2\in\mathcal{H}_0(g_0)$, $(u_{01},\phi_{01}),(u_{02},\phi_{02})\in\mathbb{Y}$, we set $(w,\psi)=(u_1,\phi_1)-(u_2,\phi_2)$, $\tilde{g}=g_1-g_2$. Then, (w,ψ) satisfies

$$\begin{cases}\dfrac{dw}{dt}+\nu_1Aw+B_0(w,w)+B_0(w,u_1)+B_0(u_1,w)\\ \qquad-\mathcal{K}R_0(\nu_2A_\gamma\psi,\psi)-\mathcal{K}R_0(\nu_2A_\gamma\psi,\phi_1)-\mathcal{K}R_0(\nu_2A_\gamma\phi_1,\psi)=\tilde{g},\\ \chi=\nu_2A_\gamma\psi+\alpha f_\gamma(\phi_1)-\alpha f_\gamma(\phi_2),\\ \dfrac{d\psi}{dt}+\chi+B_1(w,\psi)+B_1(w,\phi_1)+B_1(u_1,\psi)=0.\end{cases}\tag{5.21}$$

If we set

$$\Phi(t) = \left[\mathcal{K}^{-1}|w|_{L^2}^2 + \nu_2|\nabla\psi|_{L^2}^2 + \gamma\nu_2|\psi|_{L^2}^2\right],$$
$$\Upsilon(t) = c\left(|A_\gamma\phi_1|_{L^2}^2 + \|u_1\|^2 + |u_1|_{L^2}^2\|u_1\|^2 + Q\left(|\phi_1|_{H^1}, |\phi_2|_{H^1}\right)\right). \tag{5.22}$$

Then we derive that

$$\Phi'(t) \leqslant c\left|\tilde{g}(t-s)\right|_{L^2}^2 + \Upsilon(t)\Phi(t), \tag{5.23}$$

which gives

$$\begin{aligned}\Phi(s) &\leqslant \left(\Phi(\tilde{t}) + c\int_{\tilde{t}}^{s}\left|\tilde{g}(t-s)\right|_{L^2}^2 dt\right)\exp\left(\int_{\tilde{t}}^{s}\Upsilon(t)dt\right)\\ &\leqslant \left(\Phi(\tilde{t}) + c\int_{\tilde{t}}^{s}\left|\tilde{g}(t-s)\right|_{L^2}^2 dt\right)\exp\left(Q_1\left(\rho_0, \|g_0\|_{L_b^2(\mathbb{R};V^*)}\right)\right).\end{aligned} \tag{5.24}$$

Note that

$$\begin{aligned}\int_{\tilde{t}}^{s}\Upsilon(t)\,dt &= c\int_{\tilde{t}}^{s}\left(|A_\gamma\phi_1|_{L^2}^2 + \|u_1\|^2 + |u_1|_{L^2}^2\|u_1\|^2 + Q\left(|\phi_1|_{H^1}, |\phi_2|_{H^1}\right)\right)dt\\ &\leqslant Q_1(\rho_1),\end{aligned} \tag{5.25}$$

where Q_1 is a monotone function.

Let us choose M as

$$M = \exp\left(Q_1(\rho_1)\right). \tag{5.26}$$

Then it follows from (5.20)–(5.26) that

$$\begin{aligned}&\left\|(u,\phi)(s) - (u_k,\phi_k)(s)\right\|_{\mathbb{Y}}^2\\ &\quad\leqslant M\left[\left\|(u,\phi)(\tilde{t}) - (u_k,\phi_k)(\tilde{t})\right\|_{\mathbb{Y}}^2 + c\int_{s-\delta}^{s}\left(\left|g_1(t-s)\right|_{L^2}^2 + \left|g_2(t-s)\right|_{L^2}^2\right)dt\right]\\ &\quad\leqslant M\left(\frac{\epsilon}{2M} + \frac{\epsilon}{2M}\right) = \epsilon, \quad \forall s \geqslant t^*.\end{aligned} \tag{5.27}$$

Therefore, for each $g \in \mathcal{H}(g_0)$, $\Psi(s, \theta_{-s}(g), \mathcal{O})$ is pre-compact in $\mathbb{Y}$ for all $s \geqslant t^*$.

Similarly, substituting $\mathcal{B}_0^{\mathbb{Y}}$ for $\mathcal{O}$, we obtain that there exists $t^*(\mathcal{B}_0^{\mathbb{Y}}) > 0$ and $t_0^*(\mathcal{B}_0^{\mathbb{Y}}) + t_0 + 1$ such that $\Psi(s, \theta_{-s}(g), \mathcal{B}_0^{\mathbb{Y}})$ is pre-compact in $\mathbb{Y}$ for all $s \geqslant t^*(\mathcal{B}_0^{\mathbb{Y}})$. By the invariance property and the continuity of the cocycle, we derive that

$$\begin{aligned}\bigcup_{s\geqslant t^*(\mathcal{O})+t^*(\mathcal{B}_0^{\mathbb{Y}})}\Psi\big(s,\theta_{-s}(g),\mathcal{O}\big) &= \bigcup_{s\geqslant t^*(\mathcal{O})}\Psi\big(t^*\big(\mathcal{B}_0^{\mathbb{Y}}\big),\theta_{-t^*(\mathcal{B}_0^{\mathbb{Y}})}(g),\Psi\big(s,\theta_{-s}(g),\mathcal{O}\big)\big)\\ &\subset \Psi\Big(t^*\big(\mathcal{B}_0^{\mathbb{Y}}\big),\theta_{-t^*(\mathcal{B}_0^{\mathbb{Y}})}(g),\bigcup_{s\geqslant t^*(\mathcal{O})}\Psi\big(s,\theta_{-s}(g),\mathcal{O}\big)\Big)\\ &\subset \Psi\big(t^*\big(\mathcal{B}_0^{\mathbb{Y}}\big),\theta_{-t^*(\mathcal{B}_0^{\mathbb{Y}})}(g),\mathcal{B}_0^{\mathbb{Y}}\big).\end{aligned}\tag{5.28}$$

Thus $\bigcup_{s\geqslant t^*(\mathcal{O})+t^*(\mathcal{B}_0^{\mathbb{Y}})}\Psi(s,\theta_{-s}(g),\mathcal{O})$ is pre-compact in $\mathbb{Y}$ and the proof is complete. □

Theorem 5.6. *Let $g_0\in L_n^2(\mathbb{R};V^*)$, then the cocycle $\Psi(s,g,(u_0,\phi_0))$ defined by (5.4) possesses a pullback attractor in $\mathbb{Y}$*

$$\mathcal{A}^{\mathbb{Y}}=\big\{\mathcal{A}_g^{\mathbb{Y}}\big\}=\big\{\omega_g\big(\mathcal{B}_0^{\mathbb{Y}}\big)\big\}_{g\in\mathcal{H}(g_0)},\tag{5.29}$$

where $\mathcal{B}_0^{\mathbb{Y}}$ is a bounded uniformly absorbing set defined by (5.5) and

$$\omega_g\big(\mathcal{B}_0^{\mathbb{Y}}\big)=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\Psi\big(t,\theta_{-t}(g),\mathcal{B}_0^{\mathbb{Y}}\big)}\tag{5.30}$$

is the pullback ω-limit set of $\mathcal{B}_0^{\mathbb{Y}}$.

Proof. It follows from Lemmas 5.3 and 5.5. □

6. Regularity of pullback attractors

In this section we assume that $g_0\in L_n^2(\mathbb{R};V^*)\cap L_b^2(\mathbb{R};H)$ such that Theorems 4.3 and 5.6 simultaneously hold. Our goal is to prove that $\mathcal{A}^{\mathbb{V}}=\mathcal{A}^{\mathbb{Y}}$, which implies the pullback asymptotic smoothing effect of the model in the sense that the solutions eventually become more regular than the initial data. Using the uniform Gronwall Lemma, we first prove that the solutions to (2.15), (2.4) with the initial value in any bounded set of $\mathbb{Y}$ eventually enter a bounded set of $\mathbb{V}$.

Lemma 6.1. *Let $g_0\in L_b^2(\mathbb{R};H)$ and $\mathcal{O}\subset\mathcal{B}(\mathbb{Y})$ be arbitrary. Let $(u,\phi)(t)=\Psi(t,g,(u_0,\phi_0))$ be the corresponding solution of (2.15), (2.4) with $(u_0,\phi_0)\in\mathcal{O}$ and $g\in\mathcal{H}(g_0)$. Then there exist a time $T_0(\mathcal{O})$ and a positive constant K such that*

$$\big\|(u,\phi)(t)\big\|_{\mathbb{V}}=\big\|\Psi\big(t,g,(u_0,\phi_0)\big)\big\|_{\mathbb{V}}\leqslant K,\quad \forall(u_0,\phi_0)\in\mathcal{O},\ \forall g\in\mathcal{H}(g_0),\ \forall t\geqslant T_0(\mathcal{O}).\tag{6.1}$$

Proof. The proof is similar to that of Proposition 2.3. For the reader's convenience, we sketch it here. Taking the inner product in H of $(2.15)_1$ with $2Au$, the inner product in $L^2(\mathcal{M})$ of $(2.15)_2$ and $(2.15)_3$ with $2A_\gamma^2\phi$ and adding the resulting equalities gives

$$\begin{aligned}&\frac{d\mathcal{Y}}{dt}+2\nu_1|Au|_{L^2}^2+2\nu_2\big|A_\gamma^{3/2}\phi\big|_{L^2}^2\\ &\quad=2\mathcal{K}\big(R_0(\nu_2A_\gamma\phi,\phi),Au\big)_{L^2}-2\big(B_0(u,u),Au\big)_{L^2}+(g,Au)_{L^2}\\ &\qquad-2\alpha\big(A_\gamma^{1/2}f_\gamma(\phi),A_\gamma^{3/2}\phi\big)_{L^2}-2\big(A_\gamma^{1/2}B_1(u,\phi),A_\gamma^{3/2}\phi\big)_{L^2},\end{aligned}\tag{6.2}$$

where

$$\mathcal{Y}(t) = \|u(t)\|^2 + |A_\gamma \phi(t)|_{L^2}^2.$$

As in (2.40)–(2.44), we derive that

$$\frac{d\mathcal{Y}}{dt} + c|Au|_{L^2}^2 + c\left|A_\gamma^{3/2}\phi\right|_{L^2}^2 \leqslant \mathcal{G}(t)\mathcal{Y}(t) + \Upsilon(t), \tag{6.3}$$

where

$$\begin{aligned}\mathcal{G}(t) &= c\left(|A_\gamma \phi|_{L^2}^2 |\phi|_{H^1}^2 + |u|_{L^2}^2 \|u\|^2\right),\\ \Upsilon(t) &= c\left(\left|f_\gamma'(\phi)\nabla\phi\right|_{L^2}^2 + |g|_{L^2}^2\right).\end{aligned} \tag{6.4}$$

Note that from (2.23) and (4.4), for $t \geqslant T_0(\mathcal{O}) \equiv t_0 = t_0(\mathcal{O})$ we have

$$\int_t^{t+1} \mathcal{G}(s)\,ds \leqslant a_1, \qquad \int_t^{t+1} \mathcal{Y}(s)\,ds \leqslant a_3, \qquad \int_t^{t+1} \Upsilon(s)\,ds \leqslant a_2, \tag{6.5}$$

where the constants a_i depend on $\mathcal{O}$ and not on $t \geqslant t_0$.

It follows from (6.4)–(6.5) and the uniform Gronwall Lemma that

$$\mathcal{Y}(t) \leqslant (a_3 + a_2)e^{a_1}, \quad \forall t \geqslant T_0(\mathcal{O}) + 1, \tag{6.6}$$

and (6.1) follows. □

Theorem 6.2. *Let $g_0 \in L_n^2(\mathbb{R}; V^*) \cap L_b^2(\mathbb{R}; H)$, then*

$$\mathcal{A}^{\mathbb{Y}} = \{\mathcal{A}_g^{\mathbb{Y}}\} = \{\mathcal{A}_g^{\mathbb{V}}\} = \mathcal{A}^{\mathbb{V}}. \tag{6.7}$$

Proof. The proof is similar to that of [42] in the case of the incompressible non-Newtonian fluid. For the reader's convenience, we sketch it here. It is enough to prove that

$$\mathcal{A}_g^{\mathbb{Y}} = \mathcal{A}_g^{\mathbb{V}}, \quad \forall g \in \mathcal{H}(g_0). \tag{6.8}$$

First we note that $\mathcal{A}_g^{\mathbb{V}}$ is bounded in $\mathbb{V}$ for all $g \in \mathcal{H}(g_0)$. Thus $\mathcal{A}_{\theta_{-t}(g)}^{\mathbb{V}}$ is bounded in $\mathbb{Y}$ for any $t \in \mathbb{R}$. By the Ψ-invariance property and pullback attracting property of the pullback attractor, we have

$$\begin{aligned}\operatorname{dist}_{\mathbb{Y}}\left(\mathcal{A}_g^{\mathbb{V}}, \mathcal{A}_g^{\mathbb{Y}}\right) &= \operatorname{dist}_{\mathbb{Y}}\left(\Psi\left(t, \theta_{-t}(g), \mathcal{A}_{\theta_{-t}(g)}^{\mathbb{V}}\right), \mathcal{A}_g^{\mathbb{Y}}\right) \quad (\forall t \in \mathbb{R}_+)\\ &= \lim_{t\to+\infty} \operatorname{dist}_{\mathbb{Y}}\left(\Psi\left(t, \theta_{-t}(g), \mathcal{A}_{\theta_{-t}(g)}^{\mathbb{V}}\right), \mathcal{A}_g^{\mathbb{Y}}\right)\\ &= 0, \quad \forall g \in \mathcal{H}(g_0),\end{aligned} \tag{6.9}$$

which implies

$$\mathcal{A}_g^{\mathbb{V}} \subset \mathcal{A}_g^{\mathbb{Y}}, \quad \forall g \in \mathcal{H}(g_0). \tag{6.10}$$

From Lemma 6.1, we also note that

$$A^{\mathbb{Y}}_{\theta_{-t}(g)} = \omega_{\theta_{-t}(g)}\big(\mathcal{B}^{\mathbb{Y}}_0\big) = \bigcap_{s\geqslant 0}\overline{\bigcup_{\tau\geqslant s}\Psi\big(\tau,\theta_{-\tau}\theta_{-t}(g),\mathcal{B}^{\mathbb{Y}}_0\big)} \tag{6.11}$$

is bounded in $\mathbb{V}$ for any $t\in\mathbb{R}$. By the Ψ-invariance property and pullback attracting property of the pullback attractor, we have

$$\begin{aligned}
\operatorname{dist}_{\mathbb{Y}}\big(\mathcal{A}^{\mathbb{Y}}_g,\mathcal{A}^{\mathbb{V}}_g\big) &= \operatorname{dist}_{\mathbb{Y}}\big(\Psi\big(t,\theta_{-t}(g),\mathcal{A}^{\mathbb{Y}}_{\theta_{-t}(g)}\big),\mathcal{A}^{\mathbb{V}}_g\big) \quad (\forall t\in\mathbb{R}_+)\\
&\leqslant \operatorname{dist}_{\mathbb{V}}\big(\Psi\big(t,\theta_{-t}(g),\mathcal{A}^{\mathbb{Y}}_{\theta_{-t}(g)}\big),\mathcal{A}^{\mathbb{V}}_g\big) \quad (\forall t\in\mathbb{R}_+)\\
&= \lim_{t\to+\infty}\operatorname{dist}_{\mathbb{V}}\big(\Psi\big(t,\theta_{-t}(g),\mathcal{A}^{\mathbb{Y}}_{\theta_{-t}(g)}\big),\mathcal{A}^{\mathbb{V}}_g\big)\\
&= 0, \quad \forall g\in\mathcal{H}(g_0),
\end{aligned} \tag{6.12}$$

which implies

$$\mathcal{A}^{\mathbb{Y}}_g\subset\mathcal{A}^{\mathbb{Y}}_g, \quad \forall g\in\mathcal{H}(g_0), \tag{6.13}$$

and (6.8) follows from (6.10) and (6.13). The proof is complete. □

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