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On the Cauchy problem for the integrable modified Camassa–Holm equation with cubic nonlinearity

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ABSTRACT

Considered in this paper is the modified Camassa–Holm equation with cubic nonlinearity, which is integrable and admits the single peaked solitons and multi-peakon solutions. The short-wave limit of this equation is known as the short-pulse equation. The main investigation is the Cauchy problem of the modified Camassa–Holm equation with qualitative properties of its solutions. It is firstly shown that the equation is locally well-posed in a range of the Besov spaces. The blow-up scenario and the lower bound of the maximal time of existence are then determined. A blow-up mechanism for solutions with certain initial profiles is described in detail and nonexistence of the smooth traveling wave solutions is also demonstrated.

1. Introduction

In this paper, we are concerned with the following Cauchy problem of the integrable modified Camassa–Holm equation with cubic nonlinearity,

$$\begin{cases} m_t + (u^2 - u_x^2)m_x + 2u_x m^2 + \gamma u_x = 0, \quad m = u - u_{xx},\ t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{1.1}$$

The equation in (1.1) was introduced by Fuchssteiner [19] and Olver and Rosenau [30] (see also [18]) as a new generalization of integrable system by implementing a simple explicit algorithm based

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on the bi-Hamiltonian representation of the classically integrable system. It also arises from a non-stretching invariant curve flow in the two-dimensional Euclidean geometry [21]. In most cases, these new nonlinear systems are endowed with nonlinear dispersion, and thus support non-smooth soliton-like structures. It was shown in [31] that the equation in (1.1) admits the Lax pair and the Cauchy problem (1.1) may be solved by the inverse scattering transform method. It was also found that the equation in (1.1) is related to the short-pulse equation derived by Schäfer and Wayne [32],

$$v_{xt} = \frac{1}{3}\left(v^3\right)_{xx} + \gamma v, \tag{1.2}$$

which is a model for the propagation of ultra-short light pulses in silica optical fibers [32] and is also an approximation of nonlinear wave packets in dispersive media in the limit of few cycles on the ultra-short pulse scale [6].

Indeed, the short-pulse equation (1.2) is a short-wave limit of the equation in (1.1) by applying the following scaling transformation [21]

$$x \mapsto \epsilon x, \qquad t \mapsto \epsilon^{-1} t, \qquad u \mapsto \epsilon^2 u$$

where

$$u(t,x) = u_0(t,x) + \epsilon u_1(t,x) + \epsilon^2 u_2(t,x) + \cdots$$

is expanded in powers of the small parameter ϵ. Then $v = u_{0,x}(t,x)$ satisfies the short-pulse equation (1.2).

The equation in (1.1) is formally integrable and can be rewritten as the bi-Hamiltonian form [30], that is

$$m_t = -\left(\left(u^2 - u_x^2\right)m\right)_x - \gamma u_x = J\frac{\delta H_0}{\delta m} = K\frac{\delta H_1}{\delta m},$$

where

$$J = -\partial m \partial^{-1} m \partial - \frac{\gamma}{2}\partial \quad \text{and} \quad K = \partial^3 - \partial,$$

corresponding to the Hamiltonian

$$H_0 = \int_{\mathbb{R}} mu\, dx,$$

and the Hamiltonian

$$H_1 = \frac{1}{4}\int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4 + 2\gamma u^2\right) dx.$$

It also admits the Lax pair [31], that is

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(m,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = V(m,u,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

where

$$U(m,\lambda)=\frac{1}{2}\begin{pmatrix}-Q & \lambda m\\ -\lambda m & Q\end{pmatrix},$$

$$Q=Q(\lambda,\gamma)=\sqrt{1+\lambda^2\gamma},$$

and

$$\begin{aligned}&V(m,u,\lambda)\\&=-\frac{1}{2}\begin{pmatrix}\lambda^{-2}Q+\frac{1}{2}Q(u^2-u_x^2) & -\lambda^{-1}(u-Qu_x)-\frac{1}{2}\lambda(u^2-u_x^2)m\\ \lambda^{-1}(u+Qu_x)+\frac{1}{2}\lambda(u^2-u_x^2)m & -\lambda^{-2}Q-\frac{1}{2}Q(u^2-u_x^2)\end{pmatrix}.\end{aligned}$$

The Camassa–Holm (CH) equation [2,20] defined by

$$m_t+um_x+2u_xm+\gamma u_x=0,\quad m=u-u_{xx}$$

has attracted much attention in the last twenty years because of its interesting properties: complete integrability, existence of peaked solitons and multi-peakons [2,3], geometric formulations [5,12,13,26,28] and the presence of breaking waves (i.e. a solution that remains bounded while its slope becomes unbounded in finite time) [7,9–11]. Note that the nonlinearity in the CH equation is quadratic. In contrast to the integrable modified KdV equation with a cubic nonlinearity, it is our interest to find an integrable CH-type equations with a cubic nonlinearity. Indeed, two integrable CH-type equations with cubic nonlinearity have been discovered recently. One is the equation in (1.1) and the second one is the so-called Novikov equation [29]. The integrability, peaked solitons, well-posedness and blow-up phenomena to the Novikov equation have been studied extensively, see the references [22,29,33,34], for example.

The goal of the present paper is to establish qualitative results for the Cauchy problem (1.1).

We first study the local well-posedness for the strong solutions to the Cauchy problem (1.1) (see Theorem 3.1). The proof of the local well-posedness is inspired by the argument of approximate solutions by Danchin [16] in the study of the local well-posedness to the CH equation. However, one problematic issue is that we here deal with a higher-order nonlinearity in the Besov spaces, making the proof of several required nonlinear estimates somewhat delicate. These difficulties are nevertheless overcomed by careful estimates for each iterative approximation of solutions to the Cauchy problem (1.1).

With the local well-posedness obtained in hand, we then present a refined local well-posedness, i.e. local existence in the Besov space $B^s_{2,1}$ with the critical index $s=\frac{5}{2}$ (see Theorem 3.2). Then a precise blow-up scenario (see Theorem 4.2) and a lower bound of the maximal time of existence (see Theorem 4.3) are obtained.

Blow-up in finite time depends on strong nonlinear dispersion usually and makes, of course, the analysis more challenging in our case with higher nonlinearities. It is known that a solution of the Camassa–Holm equation, which can be considered as the transport equation, blows up in finite time when its slope u_x is unbounded from below. This idea is expected to be applied to the modified CH equation in (1.1), since it can be written as a transport equation in terms of m along the flow generated by $u^2-u_x^2$, that is

$$m_t+\left(u^2-u_x^2\right)m_x=-2u_xm^2-\gamma u_x. \tag{1.3}$$

Generally speaking, the transport equation theory ensures that, if the slope

$$\left(u^2-u_x^2\right)_x=2u_xm \tag{1.4}$$

is bounded, the solution will remain regular and, therefore, cannot blow up in finite time. In view of this property, together with the Sobolev embedding theorem, it can be shown that the solution blows up in finite time if and only if the slope in (1.4) is unbounded from below. Thus to prevent the solution from blow-up in finite time, the main issue is that it is impossible to control the bound of $u_x m$ in (1.4) in terms of the H^1-norm of the solution unless a higher, positive conserved quantity involved in H^3-norm of the solution can be found. To overcome this difficulty, we may regard the evolution equation (1.3) in terms of the quantity (1.4) being transported along the flow generated by $u^2 - u_x^2$. Then blow-up result can be established by using the global conservative property of the potential density m along the characteristics. This new idea was used in [21] in the case $\gamma = 0$. Inspired by this method, we are able to improve the blow-up result in [21] by using the conservation quantity

$$I_0 = \int_{\mathbb{R}} u(0,x)\,dx = \int_{\mathbb{R}} u(t,x)\,dx \quad \text{(see Theorem 5.1).}$$

As mentioned above, it is well known that the CH equation has the peakons [2], which are shown to be orbitally stable in the intriguing papers [14,15]. Stability of the periodic peakons of the CH equation can be found in [27]. So it is of interest to identify traveling wave solutions of the equation in (1.1). Indeed, it is found by Gui, Liu, Olver, and Qu in [21] (see also [24]) that the equation in (1.1) with $\gamma = 0$ has single peakons given by

$$u_c(t,x) = \sqrt{\frac{3c}{2}}\, e^{-|x-ct|}, \quad c > 0$$

and multi-peakons. In particular, the two-peakons can be given explicitly by

$$\begin{aligned} u(t,x) = {} & \sqrt{\frac{3}{2}c_1}\, \exp\left\{-\left|x - c_1 t - \frac{3\sqrt{c_1 c_2}}{c_1 - c_2} e^{(c_1-c_2)t}\right|\right\} \\ & + \sqrt{\frac{3}{2}c_2}\, \exp\left\{-\left|x - c_2 t - \frac{3\sqrt{c_1 c_2}}{c_1 - c_2} e^{(c_1-c_2)t}\right|\right\}, \quad 0 < c_1 < c_2. \end{aligned}$$

As a part of the present paper, we are able to show that the equation in (1.1) with $\gamma = 0$ does not have any nontrivial smooth traveling wave solutions.

The rest of the paper is organized as follows. In Section 2, some preliminary properties, which will be used later, are presented. The local well-posedness in the Besov spaces is established in Section 3. In Section 4, a blow-up scenario and a lower bound of the maximal existence time of (1.1) will be derived. A new blow-up mechanism is described and some blow-up data are determined in Section 5. Nonexistence of smooth traveling waves for $\gamma = 0$ is demonstrated in Section 6.

Notation. In the following, for a given Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity. We denote $\mathcal{F}u$ or $\hat{u}$ the Fourier transform of the function u.

2. Preliminaries

For convenience of the reader, we recall some basic facts on the Littlewood–Paley theory for the transport equations. One may check [1,4,16,17] for more details.

Proposition 2.1 *(Littlewood–Paley decomposition). (See [1,4].) Let* $\mathcal{B} \overset{\text{def}}{=} \{\xi \in \mathbb{R},\ |\xi| \leqslant \frac{4}{3}\}$ *and* $\mathcal{C} \overset{\text{def}}{=} \{\xi \in \mathbb{R},\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$. *There exist two radial functions* $\chi \in C_c^\infty(\mathcal{B})$ *and* $\varphi \in C_c^\infty(\mathcal{C})$ *such that*

$$\chi(\xi)+\sum_{q\geqslant 0}\varphi\big(2^{-q}\xi\big)=1,\quad \forall\xi\in\mathbb{R}^d,$$

$$\big|q-q'\big|\geqslant 2\quad\Rightarrow\quad \operatorname{Supp}\varphi\big(2^{-q}\cdot\big)\cap\operatorname{Supp}\varphi\big(2^{-q'}\cdot\big)=\varnothing,$$

$$q\geqslant 1\quad\Rightarrow\quad \operatorname{Supp}\chi(\cdot)\cap\operatorname{Supp}\varphi\big(2^{-q}\cdot\big)=\varnothing,$$

and

$$\frac{1}{3}\leqslant\chi(\xi)^2+\sum_{q\geqslant 0}\varphi\big(2^{-q}\xi\big)^2\leqslant 1,\quad\forall\xi\in\mathbb{R}^d.$$

Furthermore, let $h\overset{\text{def}}{=}\mathcal{F}^{-1}\varphi$ and $\tilde{h}\overset{\text{def}}{=}\mathcal{F}^{-1}\chi$. Then the dyadic operators Δ_q and S_q can be defined as follows

$$\Delta_q f\overset{\text{def}}{=}\varphi\big(2^{-q}D\big)f=2^{qd}\int_{\mathbb{R}^d}h\big(2^q y\big)f(x-y)\,dy\quad\text{for }q\geqslant 0,$$

$$S_q f\overset{\text{def}}{=}\chi\big(2^{-q}D\big)f=\sum_{-1\leqslant k\leqslant q-1}\Delta_k f=2^{qd}\int_{\mathbb{R}^d}\tilde{h}\big(2^q y\big)f(x-y)\,dy,$$

$$\Delta_{-1}f\overset{\text{def}}{=}S_0 f\quad\text{and}\quad\Delta_q f\overset{\text{def}}{=}0\quad\text{for }q\leqslant -2.$$

Definition 2.1 *(Besov space).* (See [1,4].) Let $s\in\mathbb{R}$, $1\leqslant p,r\leqslant\infty$. The inhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ ($B^s_{p,r}$ for short) is defined by

$$B^s_{p,r}\overset{\text{def}}{=}\big\{f\in\mathcal{S}'\big(\mathbb{R}^d\big);\ \|f\|_{B^s_{p,r}}<\infty\big\},$$

where

$$\|f\|_{B^s_{p,r}}\overset{\text{def}}{=}\begin{cases}(\sum_{q\in\mathbb{Z}}2^{qsr}\|\Delta_q f\|^r_{L^p})^{\frac{1}{r}}, & \text{for }r<\infty,\\ \sup_{q\in\mathbb{Z}}2^{qs}\|\Delta_q f\|_{L^p}, & \text{for }r=\infty.\end{cases}$$

If $s=\infty$, $B^\infty_{p,r}\overset{\text{def}}{=}\bigcap_{s\in\mathbb{R}}B^s_{p,r}$.

Proposition 2.2. *(See [1,16,17].) The following properties hold.*

i) *Density: if $p,r<\infty$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $B^s_{p,r}(\mathbb{R}^d)$.*
ii) *Sobolev embeddings: if $p_1\leqslant p_2$ and $r_1\leqslant r_2$, then $B^s_{p_1,r_1}\hookrightarrow B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$. If $s_1<s_2$, $1\leqslant p\leqslant+\infty$ and $1\leqslant r_1,r_2\leqslant+\infty$, then the embedding $B^{s_2}_{p,r_2}\hookrightarrow B^{s_1}_{p,r_1}$ is locally compact.*
iii) *Algebraic properties: for $s>0$, $B^s_{p,r}\cap L^\infty$ is an algebra. Moreover, ($B^s_{p,r}$ is an algebra) $\Leftrightarrow$ ($B^s_{p,r}\hookrightarrow L^\infty$) $\Leftrightarrow$ ($s>\frac{d}{p}$ or ($s\geqslant\frac{d}{p}$ and $r=1$)).*
iv) *Fatou property: if $(u^{(n)})_{n\in\mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$ which tends to u in $\mathcal{S}'$, then $u\in B^s_{p,r}$ and*

$$\|u\|_{B^s_{p,r}}\leqslant\liminf_{n\to\infty}\big\|u^{(n)}\big\|_{B^s_{p,r}}.$$

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v) *Complex interpolation*: *if* $u \in B^s_{p,r} \cap B^{\tilde{s}}_{p,r}$ *and* $\theta \in [0,1]$, $1 \leqslant p, r \leqslant \infty$, *then* $u \in B^{\theta s+(1-\theta)\tilde{s}}_{p,r}$ *and* $\|u\|_{B^{\theta s+(1-\theta)\tilde{s}}_{p,r}} \leqslant \|u\|^{\theta}_{B^s_{p,r}} \|u\|^{1-\theta}_{B^{\tilde{s}}_{p,r}}$.

vi) *Let* $m \in \mathbb{R}$ *and* f *be an* S^m*-multiplier (that is,* $f : \mathbb{R}^d \to \mathbb{R}$ *is smooth and satisfies that for all multi-index* α*, there exists a constant* C_α *such that for any* $\xi \in \mathbb{R}^d$, $|\partial^\alpha f(\xi)| \leqslant C_\alpha (1+|\xi|)^{m-|\alpha|}$.*) Then for all* $s \in \mathbb{R}$ *and* $1 \leqslant p, r \leqslant \infty$*, the operator* $f(D)$ *is continuous from* $B^s_{p,r}$ *to* $B^{s-m}_{p,r}$.

Lemma 2.1. *(See [1,16,17].) Suppose that* $(p,r) \in [1,+\infty]^2$ *and* $s > -\frac{d}{p}$*. Let* v *be a vector field such that* ∇v *belongs to* $L^1([0,T]; B^{s-1}_{p,r})$ *if* $s > 1 + \frac{d}{p}$ *or to* $L^1([0,T]; B^{\frac{d}{p}}_{p,r} \cap L^\infty)$ *otherwise. Suppose also that* $f_0 \in B^s_{p,r}$, $F \in L^1([0,T]; B^s_{p,r})$ *and that* $f \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ *solves the* d*-dimensional linear transport equations*

$$(T) \qquad \begin{cases} \partial_t f + v \cdot \nabla f = F, \\ f|_{t=0} = f_0. \end{cases}$$

Then there exists a constant C *depending only on* s, p *and* d *such that the following statements hold*:

1) *If* $r = 1$ *or* $s \neq 1 + \frac{d}{p}$*, then*

$$\|f\|_{B^s_{p,r}} \leqslant \|f_0\|_{B^s_{p,r}} + \int_0^t \big\|F(\tau)\big\|_{B^s_{p,r}}\, d\tau + C\int_0^t V'(\tau)\big\|f(\tau)\big\|_{B^s_{p,r}}\, d\tau,$$

or

$$\|f\|_{B^s_{p,r}} \leqslant e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\big\|F(\tau)\big\|_{B^s_{p,r}}\, d\tau\right) \tag{2.1}$$

hold, where $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{\frac{d}{p}}_{p,r} \cap L^\infty}\, d\tau$ *if* $s < 1 + \frac{d}{p}$ *and* $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{s-1}_{p,r}}\, d\tau$ *else.*

2) *If* $s \leqslant 1 + \frac{d}{p}$ *and, in addition,* $\nabla f_0 \in L^\infty$, $\nabla f \in L^\infty([0,T] \times \mathbb{R}^d)$ *and* $\nabla F \in L^1([0,T]; L^\infty)$*, then*

$$\begin{aligned} &\big\|f(t)\big\|_{B^s_{p,r}} + \big\|\nabla f(t)\big\|_{L^\infty} \\ &\quad \leqslant e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \|\nabla f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)}\Big(\big\|F(\tau)\big\|_{B^s_{p,r}} + \big\|\nabla F(\tau)\big\|_{L^\infty}\Big)\, d\tau\right) \end{aligned}$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{\frac{d}{p}}_{p,r} \cap L^\infty}\, d\tau$.

3) *If* $f = v$*, then for all* $s > 0$*, the estimate* (2.1) *holds with* $V(t) = \int_0^t \|\partial_x u(\tau)\|_{L^\infty}\, d\tau$.

4) *If* $r < +\infty$*, then* $f \in C([0,T]; B^s_{p,r})$*. If* $r = +\infty$*, then* $f \in C([0,T]; B^{s'}_{p,1})$ *for all* $s' < s$.

Lemma 2.2. *(See [17].) Let* $(p, p_1, r) \in [1,+\infty]^3$*. Assume that* $s > -d\min\{\frac{1}{p_1}, \frac{1}{p'}\}$ *with* $p' \stackrel{\text{def}}{=} (1 - \frac{1}{p})^{-1}$*. Let* $f_0 \in B^s_{p,r}$ *and* $F \in L^1([0,T]; B^s_{p,r})$*. Let* v *be a time dependent vector field such that* $v \in L^\rho([0,T]; B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$, $M > 0$ *and* $\nabla v \in L^1([0,T]; B^{\frac{d}{p_1}}_{p_1,\infty} \cap L^\infty)$ *if* $s < 1 + \frac{d}{p_1}$*, and* $\nabla v \in L^1([0,T]; B^{s-1}_{p_1,r})$ *if* $s > 1 + \frac{d}{p_1}$ *or* $s = 1 + \frac{d}{p_1}$ *and* $r = 1$*. Then the transport equations* (T) *has a unique solution* $f \in L^\infty([0,T]; B^s_{p,r}) \cap$

$(\bigcap_{s'<s} C([0,T]; B^{s'}_{p,1}))$ and the inequalities in Lemma 2.1 hold true. If, moreover, $r<\infty$, then we have $f \in C([0,T]; B^s_{p,r})$.

Lemma 2.3 *(1-D Moser-type estimates). (See [4].) Assume that $1 \leqslant p, r \leqslant +\infty$, the following estimates hold:*

(i) *for $s>0$, $\|fg\|_{B^s_{p,r}} \leqslant C(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|g\|_{B^s_{p,r}}\|f\|_{L^\infty})$;*
(ii) *for $s_1 \leqslant \frac{1}{p}$, $s_2 > \frac{1}{p}$ ($s_2 \geqslant \frac{1}{p}$ if $r=1$) and $s_1+s_2>0$,*

$$\|fg\|_{B^{s_1}_{p,r}} \leqslant C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}},$$

where the constant C is independent of f and g.

Lemma 2.4. *(See [17].) Denote $\bar{\mathbb{N}} = \mathbb{N} \cup \infty$. Let $(v^{(n)})_{n\in\bar{\mathbb{N}}}$ be a sequence of functions belonging to $C([0,T]; B^{\frac{1}{2}}_{2,1})$. Assume that $v^{(n)}$ is the solution to*

$$\begin{cases} \partial_t v^{(n)} + a^{(n)}\partial_x v^{(n)} = f, \\ v^{(n)}\big|_{t=0} = v_0 \end{cases} \tag{2.2}$$

with $v_0 \in B^{\frac{1}{2}}_{2,1}$, $f \in L^1(0,T; B^{\frac{1}{2}}_{2,1})$ and that, for some $\beta \in L^1(0,T)$,

$$\sup_{n\in\mathbb{N}} \big\|\partial_x a^{(n)}(t)\big\|_{B^{\frac{1}{2}}_{2,1}} \leqslant \beta(t).$$

If in addition $a^{(n)}$ tends to a^∞ in $L^1(0,T; B^{\frac{1}{2}}_{2,1})$ then $v^{(n)}$ tends to v^∞ in $C(0,T; B^{\frac{1}{2}}_{2,1})$.

Next we reformulate the Cauchy problem (1.1) in a more convenient form. Note that the equation in (1.1) is equivalent to the following one:

$$u_t - u_{xxt} + 3u^2u_x - 4uu_xu_{xx} + u_x^2u_{xxx} + 2u_xu_{xx}^2 - u^2u_{xxx} - u_x^3 + \gamma u_x = 0.$$

Applying the operator $(1-\partial_x^2)^{-1}$ to both sides of the above equation, we obtain

$$u_t + \big(u^2 - \frac{1}{3}u_x^2\big)u_x + \partial_x\big(1-\partial_x^2\big)^{-1}\left(\frac{2}{3}u^3 + uu_x^2\right) + \big(1-\partial_x^2\big)^{-1}\left(\frac{u_x^3}{3} + \gamma u_x\right) = 0, \tag{2.3}$$

which enables us to define the weak solution of the Cauchy problem (1.1).

3. Local well-posedness

3.1. Local existence

In this section, we shall discuss the local well-posedness of the Cauchy problem (1.1). At first, we introduce the following spaces.

Definition 3.1. For $T>0$, $s\in\mathbb{R}$ and $1 \leqslant p \leqslant +\infty$, we set

$$E^s_{p,r}(T) \stackrel{\text{def}}{=} C\big([0,T]; B^s_{p,r}\big) \cap C^1\big([0,T]; B^{s-1}_{p,r}\big) \quad \text{if } r<+\infty,$$

$$E^s_{p,\infty}(T) \stackrel{\text{def}}{=} L^\infty\big([0,T]; B^s_{p,\infty}\big) \cap \mathrm{Lip}\big([0,T]; B^{s-1}_{p,\infty}\big)$$

and $E^s_{p,r} \stackrel{\text{def}}{=} \bigcap_{T>0} E^s_{p,r}(T)$.

The result of the local well-posedness in the Besov space may now be enunciated.

Theorem 3.1. *Suppose that $1 \leqslant p, r \leqslant +\infty$, $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$ and $u_0 \in B^s_{p,r}$. Then there exists a time $T > 0$ such that the initial-value problem (1.1) has a unique solution $u \in E^s_{p,r}(T)$, and the map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $B^s_{p,r}$ into*

$$C\big([0,T]; B^{s'}_{p,r}\big) \cap C^1\big([0,T]; B^{s'-1}_{p,r}\big)$$

for every $s' < s$ when $r = +\infty$ and $s' = s$ whereas $r < +\infty$.

Remark 3.1. When $p = r = 2$, the Besov space $B^s_{p,r}$ coincides with the Sobolev space H^s. Theorem 3.1 implies that under the condition $u_0 \in H^s$ with $s > 5/2$, we can obtain the local well-posedness for the initial-value problem (1.1).

Remark 3.2. As in Remark 4.1 in [21], the existence time for the initial-value problem (1.1) may be chosen independently of s in the following sense. If

$$u \in C\big([0,T]; H^s\big) \cap C^1\big([0,T]; H^{s-1}\big)$$

is the solution of the initial-value problem (1.1) with initial data $u_0 \in H^r$ for some $r > 5/2$, $r \neq s$, then

$$u \in C\big([0,T]; H^r\big) \cap C^1\big([0,T]; H^{r-1}\big)$$

with the same time T. In particular, if $u_0 \in H^\infty$, then $u \in C([0,T]; H^\infty)$.

Remark 3.3. For a strong solution $m = u - u_{xx}$ in Theorem 3.1, if, in addition, the initial data $u_0 \in L^1$, then the following three functionals are conserved:

$$I_0 = \int_{\mathbb{R}} u(t)\,dx, \qquad I_1 = \int_{\mathbb{R}} \big(u^2 + u_x^2\big)\,dx,$$
$$I_2 = \int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4 + 2\gamma u^2\right)dx. \tag{3.1}$$

Under the assumptions in Theorem 3.1 (especially $p = r = 2$), we introduce the flow generated by $u^2 - u_x^2$:

$$\begin{cases} \dfrac{dq(t,x)}{dt} = \big(u^2 - u_x^2\big)\big(t, q(t,x)\big), & x \in \mathbb{R},\ t \in [0,T). \\ q(0,x) = x, \end{cases} \tag{3.2}$$

If $\gamma = 0$, then it is easy to check that [21]

$$m\big(t, q(t,x)\big)q_x(t,x) = m_0(x), \quad \text{for all } (t,x) \in [0,T) \times \mathbb{R}.$$

Remark 3.4. Note that from the above flow, it follows that [21]

$$q_x(t,x) = \left(2\int_0^t (mu_x)\big(s,q(s,x)\big)\,ds\right) > 0, \quad \text{for all } (t,x)\in[0,T)\times\mathbb{R}.$$

In view of the above conservation law, we deduce that: if $u_0(x)$ has compact support in x in the interval $[a,b]$, then so does $m(t,\cdot)$ in the corresponding interval $[q(t,a),q(t,b)]$. This property of retaining compact support for m is actually similar to the case of the CH equation [8,23]. Moreover, if $m_0=(1-\partial_x^2)u_0$ does not change sign, then $m(t,x)$ will not change sign for any $t\in[0,T)$. On the other hand, the L^∞-norm of any function $v(t,\cdot)\in L^\infty$ is preserved under the family of diffeomorphisms $q(t,\cdot)$, that is,

$$\big\|v(t,\cdot)\big\|_{L^\infty} = \big\|v\big(t,q(t,\cdot)\big)\big\|_{L^\infty}, \quad t\in[0,T).$$

In the following, we denote $C>0$ a generic constant only depending on p,r,s. Uniqueness and continuity with respect to the initial data are an immediate consequence of the following result.

Proposition 3.1. *Let $1\leqslant p,r\leqslant +\infty$ and $s>\max\{2+\frac{1}{p},\ \frac{5}{2}\}$. Let $u^{(1)}$, $u^{(2)}$ be two given solutions of the initial-value problem (1.1) with the initial data $u_0^{(1)},u_0^{(2)}\in B^s_{p,r}$ satisfying $u^{(1)},u^{(2)}\in L^\infty([0,T];B^s_{p,r})\cap C([0,T];\mathcal{S}')$. Then for every $t\in[0,T]$:*

$$\begin{aligned}&\big\|\big(u^{(1)}-u^{(2)}\big)(t)\big\|_{B^{s-1}_{p,r}}\\ &\quad\leqslant \big\|u_0^{(1)}-u_0^{(2)}\big\|_{B^{s-1}_{p,r}}\exp\left\{C\int_0^t\left(\big\|u^{(1)}(\tau)\big\|^2_{B^s_{p,r}}+\big\|u^{(2)}(\tau)\big\|^2_{B^s_{p,r}}+|\gamma|\right)d\tau\right\}.\end{aligned} \tag{3.3}$$

Proof. Denote $u^{(12)}\stackrel{\text{def}}{=}u^{(2)}-u^{(1)}$. It is obvious that

$$u^{(12)}\in L^\infty\big([0,T];B^s_{p,r}\big)\cap C\big([0,T];\mathcal{S}'\big),$$

which along with the equivalent formulation (2.3) of (1.1) implies that $u^{(12)}\in C([0,T];B^{s-1}_{p,r})$ and $u^{(12)}$ solves the transport equation

$$\partial_t u^{(12)} + \left[\big(u^{(1)}\big)^2 - \frac{1}{3}\Big(\big(u_x^{(1)}\big)^2 + u_x^{(1)}u_x^{(2)} + \big(u_x^{(2)}\big)^2\Big)\right]\partial_x u^{(12)} = f\big(u^{(12)},u^{(1)},u^{(2)}\big) \tag{3.4}$$

with

$$\begin{aligned} f\big(u^{(12)},u^{(1)},u^{(2)}\big) = {}& -\big(1-\partial_x^2\big)^{-1}\left(\frac{1}{3}\Big(\big(u_x^{(1)}\big)^2+u_x^{(1)}u_x^{(2)}+\big(u_x^{(2)}\big)^2\Big)u_x^{(12)}+\gamma u_x^{(12)}\right)\\ &-\big(u^{(1)}+u^{(2)}\big)u_x^{(2)}u^{(12)}\\ &-\partial_x\big(1-\partial_x^2\big)^{-1}\left(\frac{2}{3}\Big(\big(u^{(1)}\big)^2+u^{(1)}u^{(2)}+\big(u^{(2)}\big)^2\Big)u^{(12)}\right.\\ &\left.+u^{(1)}\big(u_x^{(1)}+u_x^{(2)}\big)u_x^{(12)}+\big(u_x^{(2)}\big)^2u^{(12)}\right). \end{aligned} \tag{3.5}$$

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Thanks to the transport theory in Lemma 2.1, one gets

$$\begin{aligned} \|u^{(12)}(t)\|_{B^{s-1}_{p,r}} \leqslant C\int_0^t \Big\|\big(u^{(1)}\big)^2-\frac{1}{3}\big(\big(u^{(1)}_x\big)^2+u^{(1)}_x u^{(2)}_x+\big(u^{(2)}_x\big)^2\big)\Big\|_{B^{s-1}_{p,r}}\|u^{(12)}(\tau)\|_{B^{s-1}_{p,r}}\,d\tau \\ +\int_0^t \big\|f\big(u^{(12)},u^{(1)},u^{(2)}\big)(\tau)\big\|_{B^{s-1}_{p,r}}\,d\tau+\|u^{(12)}(0)\|_{B^{s-1}_{p,r}}. \end{aligned} \tag{3.6}$$

Applying the product law in the Besov spaces, we have

$$\Big\|\big(u^{(1)}\big)^2-\frac{1}{3}\big(\big(u^{(1)}_x\big)^2+u^{(1)}_x u^{(2)}_x+\big(u^{(2)}_x\big)^2\big)\Big\|_{B^{s-1}_{p,r}} \leqslant C\big(\|u^{(1)}\|^2_{B^s_{p,r}}+\|u^{(2)}\|^2_{B^s_{p,r}}\big).$$

Similarly, one gets

$$\begin{aligned} &\Big\|\big(1-\partial_x^2\big)^{-1}\Big(\frac{1}{3}\big(\big(u^{(1)}_x\big)^2+u^{(1)}_x u^{(2)}_x+\big(u^{(2)}_x\big)^2\big)u^{(12)}_x+\gamma u^{(12)}_x\Big)\Big\|_{B^{s-1}_{p,r}} \\ &\quad\leqslant C\Big\|\frac{1}{3}\big(\big(u^{(1)}_x\big)^2+u^{(1)}_x u^{(2)}_x+\big(u^{(2)}_x\big)^2\big)u^{(12)}_x+\gamma u^{(12)}_x\Big\|_{B^{s-2}_{p,r}} \\ &\quad\leqslant C\big(\|u^{(1)}\|^2_{B^{s-1}_{p,r}}+\|u^{(1)}\|_{B^{s-1}_{p,r}}\|u^{(2)}\|_{B^{s-1}_{p,r}}+\|u^{(2)}\|^2_{B^{s-1}_{p,r}}+|\gamma|\big)\|u^{(12)}\|_{B^{s-1}_{p,r}}, \end{aligned}$$

$$\big\|\big(u^{(1)}+u^{(2)}\big)u^{(2)}_x u^{(12)}\big\|_{B^{s-1}_{p,r}} \leqslant C\big(\|u^{(1)}\|_{B^{s-1}_{p,r}}+\|u^{(2)}\|_{B^{s-1}_{p,r}}\big)\|u^{(2)}\|_{B^s_{p,r}}\|u^{(12)}\|_{B^{s-1}_{p,r}},$$

and

$$\begin{aligned} &\Big\|\partial_x\big(1-\partial_x^2\big)^{-1}\Big(\frac{2}{3}\big(\big(u^{(1)}\big)^2+u^{(1)}u^{(2)}+\big(u^{(2)}\big)^2\big)u^{(12)}+u^{(1)}\big(u^{(1)}_x+u^{(2)}_x\big)u^{(12)}_x+\big(u^{(2)}_x\big)^2u^{(12)}\Big)\Big\|_{B^{s-1}_{p,r}} \\ &\quad\leqslant C\Big\|\frac{2}{3}\big(\big(u^{(1)}\big)^2+u^{(1)}u^{(2)}+\big(u^{(2)}\big)^2\big)u^{(12)}+u^{(1)}\big(u^{(1)}_x+u^{(2)}_x\big)u^{(12)}_x+\big(u^{(2)}_x\big)^2u^{(12)}\Big\|_{B^{s-2}_{p,r}} \\ &\quad\leqslant C\big(\|u^{(1)}\|^2_{B^{s-1}_{p,r}}+\|u^{(2)}\|^2_{B^{s-1}_{p,r}}\big)\|u^{(12)}\|_{B^{s-1}_{p,r}}, \end{aligned}$$

which leads to

$$\big\|f\big(u^{(12)},u^{(1)},u^{(2)}\big)\big\|_{B^{s-1}_{p,r}} \leqslant C\big(\|u^{(1)}\|^2_{B^s_{p,r}}+\|u^{(2)}\|^2_{B^s_{p,r}}+|\gamma|\big)\|u^{(12)}\|_{B^{s-1}_{p,r}}.$$

Hence, one obtains from (3.6) that

$$\begin{aligned} &\|u^{(12)}(t)\|_{B^{s-1}_{p,r}} \\ &\quad\leqslant \|u^{(12)}(0)\|_{B^{s-1}_{p,r}}+C\int_0^t \big(\|u^{(1)}(\tau)\|^2_{B^s_{p,r}}+\|u^{(2)}(\tau)\|^2_{B^s_{p,r}}+|\gamma|\big)\|u^{(12)}(\tau)\|_{B^{s-1}_{p,r}}\,d\tau, \end{aligned}$$

and then applying Gronwall's inequality, we reach (3.3). □

Now let us start the proof of Theorem 3.1, which is motivated by the proof of local existence theorem about the Camassa–Holm equation in [16]. Firstly, we shall use the classical Friedrichs regularization method to construct the approximate solutions to the Cauchy problem (1.1).

Lemma 3.1. *Let u_0, p, r and s be as in the statement of Theorem 3.1. Assume that $u^{(0)} := 0$. There exists a sequence of smooth functions $(u^{(n)})_{n\in\mathbb{N}} \in C(\mathbb{R}^+; B^{\infty}_{p,r})$ solving the following linear transport equation by induction*:

$$(T_n)\quad \begin{cases} \{\partial_t + [(u^{(n)})^2 - (u_x^{(n)})^2]\partial_x\}m^{(n+1)} = -2u_x^{(n)}(m^{(n)})^2 - \gamma u_x^{(n)}, & t>0,\ x\in\mathbb{R},\\ u^{(n+1)}|_{t=0} = u_0^{(n+1)}(x) = S_{n+1}u_0, & x\in\mathbb{R}. \end{cases} \tag{3.7}$$

Moreover, there exists a $T>0$ such that the solutions satisfying the following properties:

(i) $(u^{(n)})_{n\in\mathbb{N}}$ *is uniformly bounded in* $E^s_{p,r}(T)$.
(ii) $(u^{(n)})_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $C([0,T]; B^{s-1}_{p,r})$.

Proof. Since all data $S_{n+1}u_0$ belongs to $B^{\infty}_{p,r}$, Lemma 2.2 enables us to show by induction that for all $n\in\mathbb{N}$, the equation (T_n) has a global solution which belongs to $C(\mathbb{R}; B^{\infty}_{p,r})$. Applying Lemma 2.1 to (T_n), we get for all $n\in\mathbb{N}$:

$$\begin{aligned} &\|m^{(n+1)}(t)\|_{B^{s-2}_{p,r}}\\ &\quad\leqslant e^{C\int_0^t \|[(u^{(n)})^2-(u_x^{(n)})^2](\tau)\|_{B^{s-1}_{p,r}}\,d\tau}\|S_{n+1}u_0\|_{B^s_{p,r}}\\ &\qquad + C\int_0^t e^{C\int_\tau^t \|[(u^{(n)})^2-(u_x^{(n)})^2](\tau')\|_{B^{s-1}_{p,r}}\,d\tau'}\|2u_x^{(n)}(m^{(n)})^2+\gamma u_x^{(n)}(\tau)\|_{B^{s-2}_{p,r}}\,d\tau. \end{aligned}\tag{3.8}$$

Thanks to the product law in Besov spaces, one has

$$\begin{aligned} &\|(u^{(n)})^2-(u_x^{(n)})^2\|_{B^{s-1}_{p,r}} \leqslant C\|u^{(n)}\|^2_{B^s_{p,r}},\\ &\|2u_x^{(n)}(m^{(n)})^2+\gamma u_x^{(n)}\|_{B^{s-2}_{p,r}} \leqslant C\big(\|u^{(n)}\|^3_{B^s_{p,r}}+\|u^{(n)}\|_{B^s_{p,r}}\big), \end{aligned}$$

which along with (3.8) leads to

$$\begin{aligned} \|u^{(n+1)}(t)\|_{B^s_{p,r}} &\leqslant e^{C\int_0^t \|u^{(n)}(\tau)\|^2_{B^s_{p,r}}\,d\tau}\|u_0\|_{B^s_{p,r}}\\ &\quad + \frac{C}{\sqrt{2}}\int_0^t e^{C\int_\tau^t \|u^{(n)}(\tau')\|^2_{B^s_{p,r}}\,d\tau'}\big(\|u^{(n)}(\tau)\|^3_{B^s_{p,r}}+\|u^{(n)}(\tau)\|_{B^s_{p,r}}\big)\,d\tau. \end{aligned}\tag{3.9}$$

Let us choose a $T>0$ such that

$$T \leqslant \min\left\{\frac{1}{8C\|u_0\|^2_{B^s_{p,r}}}, \frac{3(\sqrt{2}-1)}{4C}\right\},$$

and suppose by induction that for all $t \in [0,T]$,

$$\big\|u^{(n)}(t)\big\|_{B^s_{p,r}} \leqslant \frac{\sqrt{2}\|u_0\|_{B^s_{p,r}}}{(1-8C\|u_0\|^2_{B^s_{p,r}}t)^{1/2}}. \tag{3.10}$$

Indeed, one obtains from (3.10) that for any $0 \leqslant \tau \leqslant t$,

$$\begin{aligned} C\int_\tau^t \big\|u^{(n)}(\tau')\big\|^2_{B^s_{p,r}}\,d\tau' &\leqslant C\int_\tau^t \frac{2\|u_0\|^2_{B^s_{p,r}}}{1-8C\|u_0\|^2_{B^s_{p,r}}\tau'}\,d\tau' \\ &= \frac{1}{4}\ln\big(1-8C\|u_0\|^2_{B^s_{p,r}}\tau\big) - \frac{1}{4}\ln\big(1-8C\|u_0\|^2_{B^s_{p,r}}t\big). \end{aligned}$$

And then inserting the above inequality and (3.10) into (3.9) leads to

$$\begin{aligned} \big\|u^{(n+1)}(t)\big\|_{B^s_{p,r}} \leqslant{}& \frac{\|u_0\|_{B^s_{p,r}}}{\sqrt[4]{1-8C\|u_0\|^2_{B^s_{p,r}}t}} + \frac{C}{\sqrt{2}\sqrt[4]{1-8C\|u_0\|^2_{B^s_{p,r}}t}} \\ &\times \int_0^t \left(\frac{2\sqrt{2}\|u_0\|^3_{B^s_{p,r}}}{(1-8C\|u_0\|^2_{B^s_{p,r}}\tau)^{\frac{5}{4}}} + \frac{\sqrt{2}\|u_0\|_{B^s_{p,r}}}{(1-8C\|u_0\|^2_{B^s_{p,r}}\tau)^{\frac{1}{4}}}\right)d\tau \\ \leqslant{}& \frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-8C\|u_0\|^2_{B^s_{p,r}}t}} + \frac{1-\sqrt[4]{(1-8C\|u_0\|^2_{B^s_{p,r}}t)^3}}{6\|u_0\|_{B^s_{p,r}}\sqrt[4]{1-8C\|u_0\|^2_{B^s_{p,r}}t}}, \end{aligned}$$

which implies

$$\big\|u^{(n+1)}(t)\big\|_{B^s_{p,r}} \leqslant \frac{\sqrt{2}\|u_0\|_{B^s_{p,r}}}{(1-8C\|u_0\|^2_{B^s_{p,r}}t)^{1/2}}.$$

Therefore, $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^s_{p,r})$.

On the other hand, using the Moser-type estimates (see Lemma 2.3(ii)), one finds that

$$\begin{aligned} \big\|\big[\big(u^{(n)}\big)^2-\big(u^{(n)}_x\big)^2\big]\partial_x m^{(n+1)}\big\|_{B^{s-3}_{p,r}} &\leqslant C\big\|m^{(n+1)}\big\|_{B^{s-2}_{p,r}}\big(\big\|u^{(n)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n)}_x\big\|^2_{B^{s-1}_{p,r}}\big) \\ &\leqslant C\big\|u^{(n+1)}\big\|_{B^s_{p,r}}\big\|u^{(n)}\big\|^2_{B^s_{p,r}}, \end{aligned}$$

and

$$\big\|u^{(n)}_x\big(m^{(n)}\big)^2\big\|_{B^{s-3}_{p,r}} \leqslant C\big\|m^{(n)}\big\|^2_{B^{s-2}_{p,r}}\big\|u^{(n)}\big\|_{B^s_{p,r}} \leqslant C\big\|u^{(n)}\big\|^3_{B^s_{p,r}}.$$

Hence, using the equation (T_n), we have

$$\partial_t u^{(n+1)} \in C\big([0,T];B^{s-1}_{p,r}\big)$$

uniformly bounded, which yields that the sequence $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$.

Next we are going to show that

$$\big(u^{(n)}\big)_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } C\big([0,T];B^{s-1}_{p,r}\big).$$

In fact, according to (3.7), we obtain that, for all $n,\ell\in\mathbb{N}$,

$$\begin{aligned}&\big\{\partial_t+\big[\big(u^{(n+\ell)}\big)^2-\big(u_x^{(n+\ell)}\big)^2\big]\partial_x\big\}\big(m^{(n+\ell+1)}-m^{(n+1)}\big)\\&\quad=g\big(u^{(n+\ell)},u^{(n)},m^{(n+\ell)},m^{(n)},m^{(n+1)}\big),\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}&g\big(u^{(n+\ell)},u^{(n)},m^{(n+\ell)},m^{(n)},m^{(n+1)}\big)\\&\quad=\big[\big(u^{(n)}-u^{(n+\ell)}\big)\big(u^{(n)}+u^{(n+\ell)}\big)-\big(u_x^{(n)}-u_x^{(n+\ell)}\big)\big(u_x^{(n)}+u_x^{(n+\ell)}\big)\big]\partial_x m^{(n+1)}\\&\qquad-2u_x^{(n+\ell)}\big(m^{(n+\ell)}-m^{(n)}\big)\big(m^{(n)}+m^{(n+\ell)}\big)+2\big(u_x^{(n)}-u_x^{(n+\ell)}\big)\big(m^{(n)}\big)^2+\gamma\big(u_x^{(n)}-u_x^{(n+\ell)}\big).\end{aligned}$$

We rewrite (3.11) as the equation in terms of $(u^{(n+\ell+1)}-u^{(n+1)})$,

$$\begin{aligned}&\big\{\partial_t+\big[\big(u^{(n+\ell)}\big)^2-\big(u_x^{(n+\ell)}\big)^2\big]\partial_x\big\}\big(1-\partial_x^2\big)\big(u^{(n+\ell+1)}-u^{(n+1)}\big)\\&\quad=g\big(u^{(n+\ell)},u^{(n)},m^{(n+\ell)},m^{(n)},m^{(n+1)}\big),\end{aligned}$$

which is equivalent to

$$\big(1-\partial_x^2\big)\big\{\big(\partial_t+\big[\big(u^{(n+\ell)}\big)^2-\big(u_x^{(n+\ell)}\big)^2\big]\partial_x\big)\big(u^{(n+\ell+1)}-u^{(n+1)}\big)\big\}=h^{(n,\ell)}\tag{3.12}$$

with

$$\begin{aligned}h^{(n,\ell)}&=2\partial_x\big[\big(u^{(n+\ell)}\big)^2-\big(u_x^{(n+\ell)}\big)^2\big]\partial_x^2\big(u^{(n+\ell+1)}-u^{(n+1)}\big)\\&\quad+\partial_x^2\big[\big(u^{(n+\ell)}\big)^2-\big(u_x^{(n+\ell)}\big)^2\big]\partial_x\big(u^{(n+\ell+1)}-u^{(n+1)}\big)\\&\quad+g\big(u^{(n+\ell)},u^{(n)},m^{(n+\ell)},m^{(n)},m^{(n+1)}\big).\end{aligned}$$

Applying the operator $(1-\partial_x^2)^{-1}$ to (3.12) gives rise to

$$\big\{\partial_t+\big[\big(u^{(n+\ell)}\big)^2-\big(u_x^{(n+\ell)}\big)^2\big]\partial_x\big\}\big(u^{(n+\ell+1)}-u^{(n+1)}\big)=\big(1-\partial_x^2\big)^{-1}h^{(n,\ell)}.\tag{3.13}$$

Thanks to Lemma 2.1 again, then for every $t\in[0,T]$, we obtain

$$\begin{aligned}&e^{-C\int_0^t\|[(u^{(n+\ell)})^2-(u_x^{(n+\ell)})^2](\tau)\|_{B^{s-1}_{p,r}}\,d\tau}\big\|\big(u^{(n+\ell+1)}-u^{(n+1)}\big)(t)\big\|_{B^{s-1}_{p,r}}\\&\quad\leqslant\big\|u_0^{(n+\ell+1)}-u_0^{(n+1)}\big\|_{B^{s-1}_{p,r}}\\&\qquad+C\int_0^t e^{-C\int_0^\tau\|[(u^{(n+\ell)})^2-(u_x^{(n+\ell)})^2](\tau')\|_{B^{s-1}_{p,r}}\,d\tau'}\big\|h^{(n,\ell)}\big\|_{B^{s-3}_{p,r}}\,d\tau.\end{aligned}\tag{3.14}$$

In the case of $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, one can deduce from the product law in Besov spaces that

$$\begin{aligned}
&\big\|\big[\big(u^{(n)} - u^{(n+\ell)}\big)\big(u^{(n)} + u^{(n+\ell)}\big) - \big(u_x^{(n)} - u_x^{(n+\ell)}\big)\big(u_x^{(n)} + u_x^{(n+\ell)}\big)\big]\partial_x m^{(n+1)}\big\|_{B^{s-3}_{p,r}} \\
&\quad \leqslant C\big\|m^{(n+1)}\big\|_{B^{s-2}_{p,r}}\big(\big\|u^{(n+\ell)} - u^{(n)}\big\|_{B^{s-1}_{p,r}}\big\|u^{(n+\ell)} + u^{(n)}\big\|_{B^{s-1}_{p,r}} \\
&\qquad + \big\|u_x^{(n+\ell)} - u_x^{(n)}\big\|_{B^{s-2}_{p,r}}\big\|u_x^{(n+\ell)} + u_x^{(n)}\big\|_{B^{s-2}_{p,r}}\big) \\
&\quad \leqslant C\big\|u^{(n+\ell)} - u^{(n)}\big\|_{B^{s-1}_{p,r}}\big(\big\|u^{(n)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n+1)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n+\ell)}\big\|^2_{B^s_{p,r}}\big),
\end{aligned}$$

$$\begin{aligned}
&\big\|u_x^{(n+l)}\big(m^{(n+\ell)} - m^{(n)}\big)\big(m^{(n)} + m^{(n+\ell)}\big)\big\|_{B^{s-3}_{p,r}} \\
&\quad \leqslant C\big\|u^{(n+\ell)}\big\|_{B^s_{p,r}}\big\|m^{(n+\ell)} - m^{(n)}\big\|_{B^{s-3}_{p,r}}\big\|m^{(n+\ell)} + m^{(n)}\big\|_{B^{s-2}_{p,r}} \\
&\quad \leqslant C\big\|u^{(n+\ell)} - u^{(n)}\big\|_{B^{s-1}_{p,r}}\big(\big\|u^{(n)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n+\ell)}\big\|^2_{B^s_{p,r}}\big),
\end{aligned}$$

and

$$\begin{aligned}
\big\|\big(u_x^{(n)} - u_x^{(n+\ell)}\big)\big(m^{(n)}\big)^2\big\|_{B^{s-3}_{p,r}} &\leqslant C\big\|u^{(n+\ell)} - u^{(n)}\big\|_{B^{s-2}_{p,r}}\big\|m^{(n)}\big\|^2_{B^{s-2}_{p,r}} \\
&\leqslant C\big\|u^{(n+\ell)} - u^{(n)}\big\|_{B^{s-1}_{p,r}}\big\|u^{(n)}\big\|^2_{B^s_{p,r}}.
\end{aligned}$$

From this, one finds that

$$\begin{aligned}
&\big\|g\big(u^{(n+\ell)}, u^{(n)}, m^{(n+\ell)}, m^{(n)}, m^{(n+1)}\big)\big\|_{B^{s-3}_{p,r}} \\
&\quad \leqslant C\big\|u^{(n+\ell)} - u^{(n)}\big\|_{B^{s-1}_{p,r}}\big(\big\|u^{(n)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n+1)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n+\ell)}\big\|^2_{B^s_{p,r}} + |\gamma|\big).
\end{aligned}$$

Similarly, we may check that

$$\begin{aligned}
&\big\|2\partial_x\big[\big(u^{(n+\ell)}\big)^2 - \big(u_x^{(n+\ell)}\big)^2\big]\partial_x^2\big(u^{(n+\ell+1)} - u^{(n+1)}\big)\big\|_{B^{s-3}_{p,r}} \\
&\quad \leqslant C\big\|u^{(n+\ell+1)} - u^{(n+1)}\big\|_{B^{s-1}_{p,r}}\big\|u^{(n+\ell)}\big\|^2_{B^s_{p,r}}
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\partial_x^2\big[\big(u^{(n+\ell)}\big)^2 - \big(u_x^{(n+\ell)}\big)^2\big]\partial_x\big(u^{(n+\ell+1)} - u^{(n+1)}\big)\big\|_{B^{s-3}_{p,r}} \\
&\quad \leqslant C\big\|u^{(n+\ell+1)} - u^{(n+1)}\big\|_{B^{s-1}_{p,r}}\big\|u^{(n+\ell)}\big\|^2_{B^s_{p,r}}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\big\|h^{(n,\ell)}\big\|_{B^{s-3}_{p,r}} &\leqslant C\big\|u^{(n+\ell+1)} - u^{(n+1)}\big\|_{B^{s-1}_{p,r}}\big\|u^{(n+\ell)}\big\|^2_{B^s_{p,r}} \\
&\quad + C\big\|u^{(n+\ell)} - u^{(n)}\big\|_{B^{s-1}_{p,r}}\big(\big\|u^{(n+\ell)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n+1)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n)}\big\|^2_{B^s_{p,r}} + |\gamma|\big).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&e^{-C\int_0^t \|[(u^{(n+\ell)})^2-(u_x^{(n+\ell)})^2](\tau)\|_{B^{s-1}_{p,r}}\,d\tau}\big\|\big(u^{(n+\ell+1)}-u^{(n+1)}\big)(t)\big\|_{B^{s-1}_{p,r}}\\
&\quad\leqslant \big\|u_0^{(n+\ell+1)}-u_0^{(n+1)}\big\|_{B^{s-1}_{p,r}}\\
&\qquad + C\int_0^t e^{-C\int_0^\tau \|[(u^{(n+\ell)})^2-(u_x^{(n+\ell)})^2](\tau')\|_{B^{s-1}_{p,r}}\,d\tau'}\big\|\big(u^{(n+\ell)}-u^{(n)}\big)(\tau)\big\|_{B^{s-1}_{p,r}}\\
&\qquad \times\big(\big\|u^{(n)}(\tau)\big\|^2_{B^s_{p,r}}+\big\|u^{(n+\ell)}(\tau)\big\|^2_{B^s_{p,r}}+\big\|u^{(n+1)}(\tau)\big\|^2_{B^s_{p,r}}+|\gamma|\big)\,d\tau\\
&\qquad + C\int_0^t e^{-C\int_0^\tau \|[(u^{(n+\ell)})^2-(u_x^{(n+\ell)})^2](\tau')\|_{B^{s-1}_{p,r}}\,d\tau'}\big\|\big(u^{(n+\ell+1)}-u^{(n+1)}\big)(\tau)\big\|_{B^{s-1}_{p,r}}\\
&\qquad \times\big\|u^{(n+\ell)}(\tau)\big\|^2_{B^s_{p,r}}\,d\tau.
\end{aligned}\tag{3.15}$$

Since $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$ and

$$u_0^{(n+\ell+1)}-u_0^{(n+1)}=S_{n+\ell+1}u_0-S_{n+1}u_0=\sum_{q=n+1}^{n+\ell}\Delta_q u_0,$$

then there exists a constant C_T independent of n and ℓ such that for all $t\in[0,T]$,

$$\big\|\big(u^{(n+\ell+1)}-u^{(n+1)}\big)(t)\big\|_{B^{s-1}_{p,r}}\leqslant C_T\left(2^{-n}+\int_0^t\big\|\big(u^{(n+\ell)}-u^{(n)}\big)(\tau)\big\|_{B^{s-1}_{p,r}}\,d\tau\right).$$

Arguing by induction with respect to the index n, one can easily prove that

$$\big\|u^{(n+\ell+1)}-u^{(n+1)}\big\|_{L^\infty_T(B^{s-1}_{p,r})}\leqslant\frac{(TC_T)^{n+1}}{(n+1)!}\big\|u^{(\ell)}\big\|_{L^\infty_T(B^s_{p,r})}+C_T\sum_{k=0}^n 2^{-(n-k)}\frac{(TC_T)^k}{k!}.$$

Similarly $\|u^{(\ell)}\|_{L^\infty_T(B^s_{p,r})}$ can be bounded independently of ℓ, we conclude that there exist some new constant C'_T independent of n and ℓ such that

$$\big\|u^{(n+\ell+1)}-u^{(n+1)}\big\|_{L^\infty_T(B^{s-1}_{p,r})}\leqslant 2^{-n}C'_T.$$

Hence $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$. □

Proof of Theorem 3.1. Thanks to Lemma 3.1, we obtain that $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$, so it converges to some function $u\in C([0,T];B^{s-1}_{p,r})$. We now have to check that u belongs to $E^s_{p,r}(T)$ and solves the Cauchy problem (1.1). Since $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,T];B^s_{p,r})$ according to Lemma 3.1, the Fatou property for the Besov spaces (Proposition 2.2iv)) guarantees that u also belongs to $L^\infty([0,T];B^s_{p,r})$.

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On the other hand, as $(u^{(n)})_{n\in\mathbb{N}}$ converges to u in $C([0,T]; B^{s-1}_{p,r})$, an interpolation argument ensures that the convergence holds in $C([0,T]; B^{s'}_{p,r})$, for any $s'<s$. It is then easy to pass to the limit in the equation (T_n) and to conclude that u is indeed a solution to the Cauchy problem (1.1). Thanks to the fact that u belongs to $L^\infty([0,T]; B^s_{p,r})$, the right-hand side of the equation

$$\partial_t m + (u^2 - u_x^2)\partial_x m = -2u_x m^2 - \gamma u_x$$

belongs to $L^\infty([0,T]; B^{s-2}_{p,r})$. In particular, for the case $r<\infty$, Lemma 2.2 implies that $u \in C([0,T]; B^{s'}_{p,r})$ for any $s'<s$. Finally, using the equation again, we see that $\partial_t u \in C([0,T]; B^{s-1}_{p,r})$ if $r<\infty$, and in $L^\infty([0,T]; B^{s-1}_{p,r})$ otherwise. Moreover, a standard use of a sequence of viscosity approximate solutions $(u_\epsilon)_{\epsilon>0}$ for the Cauchy problem (1.1) which converges uniformly in

$$C\big([0,T]; B^s_{p,r}\big) \cap C^1\big([0,T]; B^{s-1}_{p,r}\big)$$

leads to the continuity of the solution u in $E^s_{p,r}(T)$. $\square$

3.2. Critical case

Attention is now restricted to the critical case in the local well-posedness.

Theorem 3.2. *Suppose that the initial data $u_0(x) \in B^{\frac{5}{2}}_{2,1}$. Then there exists a maximal $T = T(u_0) > 0$ and a unique solution $u(t,x)$ to the Cauchy problem (1.1) such that*

$$u = u(\cdot, u_0) \in C\big([0,T]; B^{\frac{5}{2}}_{2,1}\big) \cap C^1\big([0,T]; B^{\frac{3}{2}}_{2,1}\big).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$u_0 \mapsto u(\cdot, u_0) : B^{\frac{5}{2}}_{2,1} \mapsto C\big([0,T]; B^{\frac{5}{2}}_{2,1}\big) \cap C^1\big([0,T]; B^{\frac{3}{2}}_{2,1}\big)$$

is continuous.

Remark 3.5. Note that the equation in (1.1) with regard to m is a transport form, that is,

$$m_t + (u^2 - u_x^2)m_x = -2u_x m^2 - \gamma u_x.$$

Roughly speaking, in order to propagate the regularity of the solution m to the Cauchy problem (1.1) in terms of its initial data m_0, the "coefficient" $u^2 - u_x^2$ of m_x needs to satisfy the Lipschitz condition. Toward this purpose, it suffices to guarantee u belonging to $W^{2,\infty}$, the space of bounded functions with bounded first and second derivatives, which satisfies the embedding properties $B^{\frac{5}{2}}_{2,1} \hookrightarrow B^2_{\infty,1} \hookrightarrow W^{2,\infty} \hookrightarrow B^2_{\infty,\infty}$. From this, we call $s = \frac{5}{2}$ the critical regularity index in terms of u for the well-posedness of the initial value problem (1.1) in the following sense:

$$H^s \hookrightarrow B^{\frac{5}{2}}_{2,1} \hookrightarrow H^{\frac{5}{2}} \hookrightarrow B^{\frac{5}{2}}_{2,\infty} \hookrightarrow H^{s'} \quad \text{for all } s' < \frac{5}{2} < s.$$

Remark 3.6. Similar to the result of the Camassa–Holm equation presented by Danchin in [16], using the estimates in the proofs of Theorem 3.1–3.2, we may demonstrate the well-posedness of Eq. (1.1) with the initial data u_0 belonging to the critical space $B^{\frac{5}{2}}_{2,\infty} \cap W^{2,\infty}$. We leave the details to the readers.

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Proof of Theorem 3.2. Theorem 3.2 will be divided into the following three lemmas. □

We first present the existence of the solution.

Lemma 3.2. *Assume that $u_0 \in B^{\frac{5}{2}}_{2,1}$. Then there exists a time $T > 0$ such that the Cauchy problem (1.1) has a solution $u \in C([0,T]; B^{\frac{5}{2}}_{2,1}) \cap C^1([0,T]; B^{\frac{3}{2}}_{2,1})$.*

Proof. On account of $u_0 \in B^{\frac{5}{2}}_{2,1}$, the transport theory (see Lemma 2.1) can be applied. Similar to the case $u_0(x) \in B^s_{p,r}$, $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, we can establish this lemma. The proof of the lemma is therefore omitted without details. □

We are now in a position to establish estimates in $L^\infty(0,T; B^{\frac{3}{2}}_{2,1})$ for the difference of two solutions of the Cauchy problem (1.1) belonging to $L^\infty([0,T]; B^{\frac{5}{2}}_{2,1}) \cap C([0,T]; B^{\frac{3}{2}}_{2,1})$. Uniqueness is a corollary of the following result.

Lemma 3.3. *Suppose that u_0 (resp. v_0) $\in B^{\frac{5}{2}}_{2,1}$ such that u (resp. v) $\in L^\infty([0,T]; B^{\frac{5}{2}}_{2,1}) \cap C([0,T]; B^{\frac{3}{2}}_{2,1})$ is a solution to the Cauchy problem (1.1) with initial data u_0 (resp. v_0). Let $w = u - v$ and $w_0 = u_0 - v_0$. Then for every $t \in [0,T]$:*

$$\|w(t)\|_{B^{\frac{3}{2}}_{2,1}} \leqslant \|w(0)\|_{B^{\frac{3}{2}}_{2,1}} \exp\left\{C\int_0^t \left(\|u(\tau)\|^2_{B^{\frac{5}{2}}_{2,1}} + \|v(\tau)\|^2_{B^{\frac{5}{2}}_{2,1}} + |\gamma|\right) d\tau\right\}. \tag{3.16}$$

Proof. Thanks to the formulation (2.3), we see that w solves the linear equation

$$\begin{aligned} &\partial_t w + \left[u^2 - \frac{1}{3}\left(u_x^2 + u_x v_x + v_x^2\right)\right]\partial_x w + \left(1 - \partial_x^2\right)^{-1}\left(\frac{1}{3}\left(u_x^2 + u_x v_x + v_x^2\right)w_x + \gamma w_x\right) \\ &\quad + (u+v)v_x w + \partial_x\left(1 - \partial_x^2\right)^{-1}\left(\frac{2}{3}\left(u^2 + uv + v^2\right)w + u(u_x + v_x)w_x + v_x^2 w\right) = 0. \end{aligned} \tag{3.17}$$

Consider that u_0 (resp. v_0) $\in B^{\frac{5}{2}}_{2,1}$ such that u (resp. v) $\in L^\infty([0,T]; B^{\frac{5}{2}}_{2,1}) \cap C([0,T]; B^{\frac{3}{2}}_{2,1})$, by virtue of the transport theory in Lemma 2.1, the following inequality holds true:

$$\begin{aligned} \|w(t)\|_{B^{\frac{3}{2}}_{2,1}} &\leqslant \|w(0)\|_{B^{\frac{3}{2}}_{2,1}} + C\int_0^t \left\|u^2 - \frac{1}{3}\left(u_x^2 + u_x v_x + v_x^2\right)\right\|_{B^{\frac{3}{2}}_{2,1}}(\tau)\|w(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\, d\tau \\ &\quad + \int_0^t \left\|f(u, v, u_x, v_x, w, w_x)(\tau)\right\|_{B^{\frac{3}{2}}_{2,1}}\, d\tau, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} &f(u, v, u_x, v_x, w, w_x) \\ &\quad = \left(1 - \partial_x^2\right)^{-1}\left(\frac{1}{3}\left(u_x^2 + u_x v_x + v_x^2\right)w_x + \gamma w_x\right) \\ &\qquad + (u+v)v_x w + \partial_x\left(1 - \partial_x^2\right)^{-1}\left(\frac{2}{3}\left(u^2 + uv + v^2\right)w + u(u_x + v_x)w_x + v_x^2 w\right). \end{aligned}$$

Applying the product law in the Besov spaces, we have

$$\left\| u^2 - \frac{1}{3}\big(u_x^2 + u_x v_x + v_x^2\big)\right\|_{B^{\frac{3}{2}}_{2,1}} \leqslant C\big(\|u\|^2_{B^{\frac{5}{2}}_{2,1}} + \|v\|^2_{B^{\frac{5}{2}}_{2,1}}\big).$$

Similarly, one gets

$$\begin{aligned}
&\left\|\big(1-\partial_x^2\big)^{-1}\left(\frac{1}{3}\big(u_x^2 + u_x v_x + v_x^2\big)w_x + \gamma w_x\right)\right\|_{B^{\frac{3}{2}}_{2,1}} \\
&\quad \leqslant C\left\|\frac{1}{3}\big(u_x^2 + u_x v_x + v_x^2\big)w_x + \gamma\, w_x\right\|_{B^{\frac{1}{2}}_{2,1}} \\
&\quad \leqslant C\big(\|u\|^2_{B^{\frac{3}{2}}_{2,1}} + \|u\|_{B^{\frac{3}{2}}_{2,1}}\|v\|_{B^{\frac{3}{2}}_{2,1}} + \|v\|^2_{B^{\frac{3}{2}}_{2,1}} + |\gamma|\big)\|w\|_{B^{\frac{3}{2}}_{2,1}},
\end{aligned}$$

$$\big\|(u+v)v_x w\big\|_{B^{\frac{3}{2}}_{2,1}} \leqslant C\big(\|u\|_{B^{\frac{3}{2}}_{2,1}} + \|v\|_{B^{\frac{3}{2}}_{2,1}}\big)\|v\|_{B^{\frac{5}{2}}_{2,1}}\|w\|_{B^{\frac{3}{2}}_{2,1}},$$

and

$$\begin{aligned}
&\left\|\partial_x\big(1-\partial_x^2\big)^{-1}\left(\frac{2}{3}\big(u^2 + uv + v^2\big)w + u(u_x+v_x)w_x + v_x^2 w\right)\right\|_{B^{\frac{3}{2}}_{2,1}} \\
&\quad \leqslant C\left\|\frac{2}{3}\big(u^2 + uv + v^2\big)w + u(u_x+v_x)w_x + v_x^2 w\right\|_{B^{\frac{1}{2}}_{2,1}} \\
&\quad \leqslant C\big(\|u\|^2_{B^{\frac{1}{2}}_{2,1}} + \|v\|^2_{B^{\frac{1}{2}}_{2,1}} + \|u\|_{B^{\frac{1}{2}}_{2,1}}\big(\|u\|_{B^{\frac{3}{2}}_{2,1}} + \|v\|_{B^{\frac{3}{2}}_{2,1}}\big) + \|v\|^2_{B^{\frac{3}{2}}_{2,1}}\big)\|w\|_{B^{\frac{3}{2}}_{2,1}},
\end{aligned}$$

which leads to

$$\big\|f(u,v,u_x,v_x,w,w_x)\big\|_{B^{\frac{3}{2}}_{2,1}} \leqslant C\big(\|u\|^2_{B^{\frac{5}{2}}_{2,1}} + \|v\|^2_{B^{\frac{5}{2}}_{2,1}} + |\gamma|\big)\|w\|_{B^{\frac{3}{2}}_{2,1}}.$$

Hence, we obtain from (3.18) that

$$\big\|w(t)\big\|_{B^{\frac{3}{2}}_{2,1}} \leqslant \big\|w(0)\big\|_{B^{\frac{3}{2}}_{2,1}} + C\int_0^t \big(\big\|u(\tau)\big\|^2_{B^{\frac{5}{2}}_{2,1}} + \big\|v(\tau)\big\|^2_{B^{\frac{5}{2}}_{2,1}} + |\gamma|\big)\big\|w(\tau)\big\|_{B^{\frac{3}{2}}_{2,1}}\,d\tau.$$

Therefore, due to the Gronwall inequality, we deduce that

$$\big\|w(t)\big\|_{B^{\frac{3}{2}}_{2,1}} \leqslant \big\|w(0)\big\|_{B^{\frac{3}{2}}_{2,1}} \exp\left\{C\int_0^t \big(\big\|u(\tau)\big\|^2_{B^{\frac{5}{2}}_{2,1}} + \big\|v(\tau)\big\|^2_{B^{\frac{5}{2}}_{2,1}} + |\gamma|\big)\,d\tau\right\},$$

which is the desired result. □

At last, we are going to verify the continuity of the solution with regard to initial data in $B^{\frac{5}{2}}_{2,1}$.

Lemma 3.4. *For any $u_0 \in B^{\frac{5}{2}}_{2,1}$, there exists a time $T > 0$ and a neighborhood V of u_0 in $B^{\frac{5}{2}}_{2,1}$ such that for any $v \in V$, which is the solution of the Cauchy problem (1.1) with the initial data v_0, the map*

$$\Phi : v_0 \to v(\cdot, v_0) : V \subset B^{\frac{5}{2}}_{2,1} \to C\big([0,T]; B^{\frac{5}{2}}_{2,1}\big) \cap C^1\big([0,T]; B^{\frac{3}{2}}_{2,1}\big)$$

is continuous.

Motivated by [17], Lemma 3.4 can be established by applying Lemma 3.3 and a continuity result Lemma 2.4 for the linear transport equations.

Proof of Lemma 3.4. We first prove the continuity of the map Φ in $C([0,T]; B^{\frac{3}{2}}_{2,1})$. Let us fix a $u_0 \in B^{\frac{5}{2}}_{2,1}$ and a $\delta > 0$. We claim that there exists a $T > 0$ and $M > 0$ such that for any $\tilde{u}_0 \in B^{\frac{5}{2}}_{2,1}$ with $\|\tilde{u}_0 - u_0\|_{B^{\frac{5}{2}}_{2,1}} \leqslant \delta$, the solution $\tilde{u} = \Phi(\tilde{u}_0)$ of the Cauchy problem (1.1) associated to $\tilde{u}_0$ belongs to $C([0,T]; B^{\frac{5}{2}}_{2,1})$ and satisfies $\|\tilde{u}\|_{L^\infty(0,T;B^{\frac{5}{2}}_{2,1})} \leqslant M$. Indeed, from the proof of the local well-posedness we know that when we fix a $T > 0$ such that

$$T \leqslant \min\left\{\frac{1}{8C\|\tilde{u}_0\|^2_{B^{\frac{5}{2}}_{2,1}}}, \frac{3(\sqrt{2}-1)}{4C}\right\},$$

then from (3.10) similarly we deduce that

$$\big\|\tilde{u}(t)\big\|_{B^{\frac{5}{2}}_{2,1}} \leqslant \frac{\sqrt{2}\|\tilde{u}_0\|_{B^{\frac{5}{2}}_{2,1}}}{(1 - 8C\|\tilde{u}_0\|^2_{B^{\frac{5}{2}}_{2,1}} t)^{1/2}} \quad \text{for all } t \in [0,T]. \tag{3.19}$$

Since $\|\tilde{u}_0 - u_0\|_{B^{\frac{5}{2}}_{2,1}} \leqslant \delta$, we get $\|\tilde{u}_0\|_{B^{\frac{5}{2}}_{2,1}} \leqslant \|u_0\|_{B^{\frac{5}{2}}_{2,1}} + \delta$. One can choose some suitable constant C, such that

$$T = \frac{3}{32C(\|u_0\|_{B^{\frac{5}{2}}_{2,1}} + \delta + 1)^2} \quad \text{and} \quad M = 2\sqrt{2}\big(\|u_0\|_{B^{\frac{5}{2}}_{2,1}} + \delta\big).$$

Now combining the above uniform bounds with Lemma 3.3, we infer that

$$\big\|\Phi(\tilde{u}_0) - \Phi(u_0)\big\|_{L^\infty(0,T;B^{\frac{3}{2}}_{2,1})} \leqslant \delta e^{C(2M^2+|\gamma|)T}.$$

In view of this inequality, we know that Φ is the Hölder continuous from $B^{\frac{5}{2}}_{2,1}$ into $C([0,T]; B^{\frac{3}{2}}_{2,1})$.

Next, we present the continuity of the map Φ in $C([0,T]; B^{\frac{5}{2}}_{2,1})$. Let $u_0^{(\infty)} \in B^{\frac{5}{2}}_{2,1}$ and $(u_0^{(n)})_{n\in\mathbb{N}}$ tend to $u_0^{(\infty)}$ in $B^{\frac{3}{2}}_{2,1}$. Denote by $u^{(n)}$ the solution corresponding to datum $u_0^{(n)}$. According to the above argument, one can find $T, M > 0$ independent of n such that for all $n \in \mathbb{N}$, $u^{(n)}$ is defined on $[0,T]$ and

$$\sup_{n\in\mathbb{N}}\big\|u^{(n)}\big\|_{L^\infty_T(B^{\frac{5}{2}}_{2,1})} \leqslant M. \tag{3.20}$$

Thanks to the first step, proving that $u^{(n)}$ tends to $u^{(\infty)}$ in $C([0,T];B^{\frac{5}{2}}_{2,1})$ amounts to proving that $m^{(n)}=u^{(n)}-u^{(n)}_{xx}$ tends to $m^{(\infty)}=u^{(\infty)}-u^{(\infty)}_{xx}$ in $C([0,T];B^{\frac{1}{2}}_{2,1})$.

Note that $m^{(n)}$ solves the following linear transport equation:

$$\begin{cases}\{\partial_t+[(u^{(n)})^2-(u^{(n)}_x)^2]\partial_x\}m^{(n)}=f^{(n)},\\ m^{(n)}\big|_{t=0}=m^{(n)}_0(x)=u^{(n)}_0-u^{(n)}_{0xx},\end{cases}$$

where $f^{(n)}=-2u^{(n)}_x[m^{(n)}]^2-\gamma u^{(n)}_x$. Following the Kato theory [25], we decompose $m^{(n)}$ into $m^{(n)}=z^{(n)}+w^{(n)}$ with

$$\begin{cases}\{\partial_t+[(u^{(n)})^2-(u^{(n)}_x)^2]\partial_x\}z^{(n)}=f^{(n)}-f^{(\infty)},\\ z^{(n)}\big|_{t=0}=m^{(n)}_0(x)-m^{(\infty)}_0(x).\end{cases}$$

and

$$\begin{cases}\{\partial_t+[(u^{(n)})^2-(u^{(n)}_x)^2]\partial_x\}w^{(n)}=f^{(\infty)},\\ w^{(n)}\big|_{t=0}=m^{(\infty)}_0(x).\end{cases}\tag{3.21}$$

Using Lemma 2.2 and the product law in the Besov spaces, one may check that

$$\begin{aligned}\|z^{(n)}(t)\|_{B^{\frac{1}{2}}_{2,1}}&\leqslant\exp\left\{C\int_0^t\big\|[(u^{(n)})^2-(u^{(n)}_x)^2](\tau)\big\|_{B^{\frac{3}{2}}_{2,1}}\,d\tau\right\}\\&\quad\times\left(\|m^{(n)}_0-m^{(\infty)}_0\|_{B^{\frac{1}{2}}_{2,1}}+C\int_0^t\big\|(f^{(n)}-f^{(\infty)})(\tau)\big\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau\right)\\&\leqslant\exp\left\{C\int_0^t\|u^{(n)}(\tau)\|^2_{B^{\frac{5}{2}}_{2,1}}\,d\tau\right\}\Bigg[\|m^{(n)}_0-m^{(\infty)}_0\|_{B^{\frac{1}{2}}_{2,1}}\\&\quad+C\int_0^t\big(\|u^{(n)}\|^2_{B^{\frac{5}{2}}_{2,1}}+\|u^{(\infty)}\|^2_{B^{\frac{5}{2}}_{2,1}}+|\gamma|\big)\\&\quad\times\big(\big\|(m^{(n)}-m^{(\infty)})(\tau)\big\|_{B^{\frac{1}{2}}_{2,1}}+\big\|(u^{(n)}-u^{(\infty)})(\tau)\big\|_{B^{\frac{3}{2}}_{2,1}}\big)\,d\tau\Bigg].\end{aligned}\tag{3.22}$$

On the other hand, since the sequence $(u^{(n)})_{n\in\bar{\mathbb{N}}}$ is uniformly bounded in $C([0,T];B^{\frac{5}{2}}_{2,1})$ and tends to $u^{(\infty)}$ in $C([0,T];B^{\frac{3}{2}}_{2,1})$, applying Lemma 2.4 to (3.21) implies that $w^{(n)}$ tends to $m^{(\infty)}$ in $C([0,T];B^{\frac{1}{2}}_{2,1})$.

Let $\varepsilon>0$. Thanks to the above result of convergence with estimates (3.20) and (3.22), we deduce that for large enough $n\in\mathbb{N}$,

$$\begin{aligned}
&\left\|\left(m^{(n)}-m^{(\infty)}\right)(\tau)\right\|_{B^{\frac{1}{2}}_{2,1}} \\
&\quad\leqslant \varepsilon + C\left(M^2+|\gamma|\right)e^{C(M^2+|\gamma|)t} \\
&\qquad\times\left\{\left\|m_0^{(n)}-m_0^{(\infty)}\right\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \left(\left\|\left(m^{(n)}-m^{(\infty)}\right)(\tau)\right\|_{B^{\frac{1}{2}}_{2,1}} + \left\|\left(u^{(n)}-u^{(\infty)}\right)(\tau)\right\|_{B^{\frac{3}{2}}_{2,1}}\right) d\tau\right\}.
\end{aligned}$$

As $u^{(n)}$ tends to $u^{(\infty)}$ in $C([0,T];B^{\frac{3}{2}}_{2,1})$, the last term in the above integral is less than ε for large n. Hence, thanks to Gronwall's inequality, we get

$$\left\|\left(m^{(n)}-m^{(\infty)}\right)(\tau)\right\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1})} \leqslant C_{M,T,|\gamma|}\left(\varepsilon + \left\|m_0^{(n)}-m_0^{(\infty)}\right\|_{B^{\frac{1}{2}}_{2,1}}\right)$$

for some constant $C_{M,T,|\gamma|}$ depending only on M, $|\gamma|$ and T, which completes the continuity of the map Φ in $C([0,T];B^{\frac{5}{2}}_{2,1})$.

Finally, applying ∂_t to the equation in (1.1) and using the same argument to the resulting equation in terms of $\partial_t u$, we may verify the continuity of the map Φ in $C^1([0,T];B^{\frac{3}{2}}_{2,1})$. This completes the proof of Lemma 3.4. □

4. Blow-up scenario and a lower bound of the maximal existence time

In [21], the authors derived a new wave-breaking mechanism for solutions to the equation in (1.1) with certain initial profiles. It means that the maximal time of existence of solutions to the equation in (1.1) has some definite upper bounds under given initial conditions. In this section, we will give a lower bound of the existence time for this equation. We first present the following theorem.

Theorem 4.1. *(See [21].) Let $m_0=(1-\partial_x^2)u_0\in H^s(\mathbb{R})$ with $s>\frac{1}{2}$. Let m be the corresponding solution to (1.1). Assume $T^*_{m_0}>0$ is the maximum time of existence. Then*

$$T^*_{u_0}<\infty \quad\Rightarrow\quad \int_0^{T^*_{m_0}} \left\|(mu_x)(\tau)\right\|_{L^\infty} d\tau = \infty. \tag{4.1}$$

Remark 4.1. There is a little difference between this theorem and the original theorem (Theorem 4.2 in [21]). We recall the result in [21] as follows:

$$T^*_{u_0}<\infty \quad\Rightarrow\quad \int_0^{T^*_{m_0}} \left\|m(\tau)\right\|^2_{L^\infty} d\tau = \infty, \tag{4.2}$$

which, together with the maximum principle to the transport equation (1.1) in terms of m applied, implies (4.1). In fact, applying the maximum principle to the transport equation (1.1), we immediately get

$$\left\|m(t)\right\|_{L^\infty} \leqslant \|m_0\|_{L^\infty} + C\int_0^t \left(\left\|(mu_x)(\tau)\right\|_{L^\infty} + |\gamma|\right)\left\|m(\tau)\right\|_{L^\infty} d\tau, \tag{4.3}$$

where we used the estimate $\|u_x\|_{L^\infty} \leqslant C\|m\|_{L^\infty}$ and $\|(u^2-u_x^2)_x\|_{L^\infty} = 2\|mu_x\|_{L^\infty}$. Then, Gronwall's inequality applied to (4.3) yields

$$\big\|m(t)\big\|_{L^\infty} \leqslant \|m_0\|_{L^\infty} \exp\left\{C\int_0^t \left(\big\|(mu_x)(\tau)\big\|_{L^\infty} + |\gamma|\right) d\tau\right\}, \tag{4.4}$$

which along with (4.2) gives rise to (4.1).

In the following, attention is now turned to blow-up issue. We first recall a blow-up scenario in [21].

Theorem 4.2. *(See [21].) Let $u_0 \in H^s$, $s > 5/2$, and $u(t,x)$ be the solution of the Cauchy problem (1.1) with life-span T. Then T is finite if and only if*

$$\liminf_{t\uparrow T}\Big(\inf_{x\in\mathbb{R}}\big(mu_x(t,x)\big)\Big) = -\infty.$$

We now deduce a lower bound depending only on $\|u_0\|_{W^{2,\infty}}$ for the maximal time of existence of the solution to (1.1).

Theorem 4.3. *Assume that $u_0 \in H^s$ with $s > \frac{5}{2}$. Let $T^* > 0$ be the maximum time of existence of the solution u to (1.1) with the initial data u_0. If $\gamma \neq 0$, then T^* satisfies*

$$T^* \geqslant \frac{1}{20|\gamma|}\ln\left(1 + \frac{2|\gamma|}{(2\|u_0\|_{L^\infty} + \|\partial_x u_0\|_{L^\infty} + 2\|\partial_x^2 u_0\|_{L^\infty})^2}\right).$$

Otherwise, if $\gamma = 0$, then

$$T^* \geqslant \frac{1}{2(3\|u_0\|_{L^\infty} + 3\|\partial_x u_0\|_{L^\infty} + \|\partial_x^2 u_0\|_{L^\infty})^2}.$$

Proof. Note that the equation in (1.1) is equivalent to the following equation

$$u_t + \left(u^2 - \frac{1}{3}u_x^2\right)u_x + \partial_x G * \left(\frac{2}{3}u^3 + uu_x^2\right) + G * \left(\frac{1}{3}u_x^3 + \gamma u_x\right) = 0, \tag{4.5}$$

where $u = G * m = (1-\partial_x^2)^{-1}m$ and $G(x) = \frac{1}{2}e^{-|x|}$. Multiplying the above equation by u^{2n-1} and integrating the results in x-variable, in view of Hölder's inequality, we obtain

$$\int_{\mathbb{R}} u^{2n-1}u_t\,dx = \frac{1}{2n}\frac{d}{dt}\|u\|_{L^{2n}}^{2n} = \|u\|_{L^{2n}}^{2n-1}\frac{d}{dt}\|u\|_{L^{2n}},$$

$$\left|\int_{\mathbb{R}} u^{2n-1}u_x^3\,dx\right| \leqslant \|u_x\|_{L^\infty}^2\|u\|_{L^{2n}}^{2n-1}\|u_x\|_{L^{2n}},$$

and

$$\left|\int_{\mathbb{R}} u^{2n-1} G * \left(\partial_x\left(\frac{2}{3}u^3 + uu_x^2\right) + \frac{1}{3}u_x^3 + \gamma u_x\right) dx\right|$$
$$\leqslant \|u\|_{L^{2n}}^{2n-1}\left(\frac{1}{3}\|u_x\|_{L^\infty}^2\|u_x\|_{L^{2n}} + \frac{2}{3}\|u\|_{L^\infty}^2\|u\|_{L^{2n}} + \|u_x\|_{L^\infty}^2\|u\|_{L^{2n}} + |\gamma|\|u_x\|_{L^{2n}}\right).$$

Integrating over $[0,t]$, it follows that

$$\|u(t)\|_{L^{2n}} \leqslant \|u(0)\|_{L^{2n}} + \int_0^t \|u(\tau)\|_{L^{2n}}\left(\|u_x(\tau)\|_{L^\infty}^2 + \frac{2}{3}\|u(\tau)\|_{L^\infty}^2\right) d\tau$$
$$+ \frac{4}{3}\int_0^t \|u_x(\tau)\|_{L^{2n}}\left(\|u_x(\tau)\|_{L^\infty}^2 + |\gamma|\right) d\tau.$$

Letting n tend to infinity in the above inequality, we have

$$\|u(t)\|_{L^\infty} \leqslant \|u(0)\|_{L^\infty} + \int_0^t \left(\|u(\tau)\|_{L^\infty}\|u_x(\tau)\|_{L^\infty}^2 + \frac{2}{3}\|u(\tau)\|_{L^\infty}^3 + |\gamma|\|u_x(\tau)\|_{L^\infty}\right) d\tau$$
$$+ \frac{4}{3}\int_0^t \|u_x(\tau)\|_{L^\infty}^3 d\tau. \tag{4.6}$$

Differentiating (4.5) with respect to x, in view of $(1-\partial_x^2)G * f = f$, we obtain

$$u_{tx} + u^2 u_{xx} + uu_x^2 - \frac{2}{3}u^3 - u_x^2 u_{xx} + G * \left(\frac{2}{3}u^3 + uu_x^2\right) + \partial_x G * \left(\frac{1}{3}u_x^3 + \gamma u_x\right) = 0. \tag{4.7}$$

Multiplying the above equation by u_x^{2n-1} and integrating the results in x over $\mathbb{R}$, still in view of Hölder's inequality, we have

$$\int_{\mathbb{R}} u_x^{2n-1} u_{xt}\, dx = \|u_x\|_{L^{2n}}^{2n-1}\frac{d}{dt}\|u_x\|_{L^{2n}},$$

$$\left|\int_{\mathbb{R}} u_x^{2n-1} u^2 u_{xx}\, dx\right| = \left|\frac{1}{2n}\int_{\mathbb{R}} u^2\left(u_x^{2n}\right)_x dx\right| = \left|-\frac{1}{2n}\int_{\mathbb{R}} 2uu_x^{2n+1}\, dx\right|$$
$$\leqslant \frac{2}{2n}\|u_x\|_{L^\infty}^2\|u_x\|_{L^{2n}}^{2n-1}\|u\|_{L^{2n}},$$

$$\left|\int_{\mathbb{R}} uu_x^{2n+1}\, dx\right| \leqslant \|u_x\|_{L^\infty}^2\left|\int_{\mathbb{R}} u_x^{2n-1} u\, dx\right| \leqslant \|u_x\|_{L^\infty}^2\|u_x\|_{L^{2n}}^{2n-1}\|u\|_{L^{2n}},$$

$$\left|\int_{\mathbb{R}} u^3 u_x^{2n-1}\, dx\right| \leqslant \|u\|_{L^\infty}^2\left|\int_{\mathbb{R}} u_x^{2n-1} u\, dx\right| \leqslant \|u\|_{L^\infty}^2\|u_x\|_{L^{2n}}^{2n-1}\|u\|_{L^{2n}},$$

and

$$\left|\int_{\mathbb{R}} u_x^{2n-1} G * \left(\frac{2}{3}u^3 + uu_x^2 + \partial_x\left(\frac{1}{3}u_x^3 + \gamma u_x\right)\right) dx\right|$$
$$\leqslant \|u_x\|_{L^{2n}}^{2n-1}\left(\frac{1}{3}\|u_x\|_{L^\infty}^2\|u_x\|_{L^{2n}} + \frac{2}{3}\|u\|_{L^\infty}^2\|u\|_{L^{2n}} + \|u_x\|_{L^\infty}^2\|u\|_{L^{2n}} + |\gamma|\|u_x\|_{L^{2n}}\right).$$

Integrating over $[0,t]$ and letting n tend to infinity, it follows that

$$\begin{aligned}\left\|u_x(t)\right\|_{L^\infty} \leqslant \left\|u_x(0)\right\|_{L^\infty} &+ \int_0^t \left(3\left\|u(\tau)\right\|_{L^\infty}\left\|u_x(\tau)\right\|_{L^\infty}^2 + \frac{4}{3}\left\|u(\tau)\right\|_{L^\infty}^3 + |\gamma|\|u_x\|_{L^\infty}\right) d\tau \\ &+ \frac{1}{3}\int_0^t \left\|u_x(\tau)\right\|_{L^\infty}^3 d\tau. \qquad (4.8)\end{aligned}$$

Differentiating (4.7) with respect to x, in view of $(1-\partial_x^2)G * f = f$, we obtain

$$\begin{aligned}&u_{txx} + u^2u_{xxx} + 4uu_xu_{xx} - 2u^2u_x + \frac{2}{3}u_x^3 - 2u_xu_{xx}^2 - u_x^2u_{xxx} - \gamma u_x \\ &\quad + \partial_x G * \left(\frac{2}{3}u^3 + uu_x^2\right) + G * \left(\frac{1}{3}u_x^3 + \gamma u_x\right) = 0. \qquad (4.9)\end{aligned}$$

Multiplying the above equation by u_{xx}^{2n-1} and integrating the results in x-variable, still in view of Hölder's inequality, we have

$$\int_{\mathbb{R}} u_{xx}^{2n-1}u_{xxt}\,dx = \|u_{xx}\|_{L^{2n}}^{2n-1}\frac{d}{dt}\|u_{xx}\|_{L^{2n}},$$

$$\begin{aligned}\left|\int_{\mathbb{R}} u_{xx}^{2n-1}u^2u_{xxx}\,dx\right| &= \left|\frac{1}{2n}\int_{\mathbb{R}} u^2\left(u_{xx}^{2n}\right)_x dx\right| = \left|-\frac{1}{2n}\int_{\mathbb{R}} 2uu_xu_{xx}^{2n}\,dx\right| \\ &\leqslant \frac{2}{2n}\|u\|_{L^\infty}\|u_x\|_{L^\infty}\|u_{xx}\|_{L^{2n}}^{2n},\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}} u_{xx}^{2n-1}u_x^2u_{xxx}\,dx\right| &= \left|\frac{1}{2n}\int_{\mathbb{R}} u_x^2\left(u_{xx}^{2n}\right)_x dx\right| = \left|-\frac{1}{2n}\int_{\mathbb{R}} 2u_xu_{xx}^{2n+1}\,dx\right| \\ &\leqslant \frac{2}{2n}\|u_x\|_{L^\infty}\|u_{xx}\|_{L^\infty}\|u_{xx}\|_{L^{2n}}^{2n}.\end{aligned}$$

Similarly, one gets

$$\left|\int_{\mathbb{R}} uu_xu_{xx}^{2n}\,dx\right| \leqslant \|u\|_{L^\infty}\|u_x\|_{L^\infty}\|u_{xx}\|_{L^{2n}}^{2n}, \qquad \left|\int_{\mathbb{R}} u_x^3u_{xx}^{2n-1}\,dx\right| \leqslant \|u_x\|_{L^\infty}^2\|u_{xx}\|_{L^{2n}}^{2n-1}\|u_x\|_{L^{2n}},$$

$$\left|\int_{\mathbb{R}} u^2u_xu_{xx}^{2n-1}\,dx\right| \leqslant \|u\|_{L^\infty}^2\|u_{xx}\|_{L^{2n}}^{2n-1}\|u_x\|_{L^{2n}}, \qquad \left|\int_{\mathbb{R}} u_xu_{xx}^{2n+1}\,dx\right| \leqslant \|u_x\|_{L^\infty}\|u_{xx}\|_{L^\infty}\|u_{xx}\|_{L^{2n}}^{2n},$$

and

$$\left|\int_{\mathbb{R}} u_{xx}^{2n-1} G * \left(\partial_x\left(\frac{2}{3}u^3 + uu_x^2\right) + \frac{1}{3}u_x^3 + \gamma u_x\right) dx\right|$$
$$\leqslant \|u_{xx}\|_{L^{2n}}^{2n-1}\left(\frac{1}{3}\|u_x\|_{L^\infty}^2\|u_x\|_{L^{2n}} + \frac{2}{3}\|u\|_{L^\infty}^2\|u\|_{L^{2n}} + \|u_x\|_{L^\infty}^2\|u\|_{L^{2n}} + |\gamma|\|u_x\|_{L^{2n}}\right).$$

Integrating the resulting inequality over $[0,t]$ and letting n tend to infinity, it follows that

$$\begin{aligned}\|u_{xx}(t)\|_{L^\infty} \leqslant{}& \|u_{xx}(0)\|_{L^\infty} \\ &+ \int_0^t \|u(\tau)\|_{L^\infty}\left(\frac{2}{3}\|u(\tau)\|_{L^\infty}^2 + \frac{7}{2}\|u_x(\tau)\|_{L^\infty}^2 + \frac{5}{2}\|u_{xx}(\tau)\|_{L^\infty}^2\right) d\tau \\ &+ \int_0^t \|u_x(\tau)\|_{L^\infty}\left(2\|u(\tau)\|_{L^\infty}^2 + \|u_x(\tau)\|_{L^\infty}^2 + 3\|u_{xx}(\tau)\|_{L^\infty}^2 + 2|\gamma|\right) d\tau. \qquad (4.10)\end{aligned}$$

If $\gamma \neq 0$, let

$$h(t) = 2\|u(t)\|_{L^\infty} + \|u_x(t)\|_{L^\infty} + 2\|u_{xx}(t)\|_{L^\infty}.$$

Combining (4.6), (4.8) and (4.10), we deduce that

$$\|m(t)\|_{L^\infty} \leqslant h(t) \leqslant h(0) + 5\int_0^t \left[\left(h(\tau)\right)^3 + 2|\gamma| h(\tau)\right] d\tau. \qquad (4.11)$$

Define

$$T = \frac{1}{20|\gamma|}\ln\left(1 + \frac{2|\gamma|}{(2\|u_0\|_{L^\infty} + \|\partial_x u_0\|_{L^\infty} + 2\|\partial_x^2 u_0\|_{L^\infty})^2}\right).$$

By (4.11), then for all $t \leqslant \min\{T, T^*\}$, one can easily get

$$\|m(t)\|_{L^\infty} \leqslant \frac{\sqrt{2|\gamma|}h(0)}{\sqrt{(h^2(0) + 2|\gamma|)e^{-20|\gamma|t} - h^2(0)}}.$$

By virtue of Theorem 3.1, it follows that $T^* \geqslant T$.

If $\gamma = 0$, let

$$h(t) = 3\|u(t)\|_{L^\infty} + 3\|u_x(t)\|_{L^\infty} + \|u_{xx}(t)\|_{L^\infty}.$$

Combining (4.6), (4.8) and (4.10), we obtain that

$$\|m(t)\|_{L^\infty} \leqslant h(t) \leqslant h(0) + \int_0^t \left(h(\tau)\right)^3 d\tau. \qquad (4.12)$$

And let

$$T=\frac{1}{2(3\|u_0\|_{L^\infty}+3\|\partial_x u_0\|_{L^\infty}+\|\partial_x^2 u_0\|_{L^\infty})^2}.$$

Similarly, we obtain that for all $t\leqslant\min\{T,T^*\}$,

$$\left\|m(t)\right\|_{L^\infty}\leqslant h(t)\leqslant\frac{h(0)}{\sqrt{1-8h^2(0)t}}. \tag{4.13}$$

This completes the proof of Theorem 4.2. □

5. Blow-up data for $\gamma=0$

In this section, we will provide sufficient conditions for the blow-up data to the initial-value problem (1.1) with $\gamma=0$. The blow-up result is now established in the following.

Theorem 5.1. *Let $\gamma=0$. Suppose $u_0\in H^s\cap L^1$ with $s>5/2$. Let $T>0$ be the maximal time of existence of the corresponding solution $m(t,x)$ to (1.1) with the initial data $m_0(x)=(1-\partial_x^2)u_0$. Assume $m_0(x)\geqslant 0$ for all $x\in\mathbb{R}$ and $m_0(x_0)>0$ at some point $x_0\in\mathbb{R}$.*

i) *If*

$$\partial_x u_0(x_0)<-\|u_0\|_{H^1}\sqrt{\frac{I_0}{m_0(x_0)}}\quad \text{with } I_0:=\int_{\mathbb{R}}u_0(x)\,dx, \tag{5.1}$$

then the solution $m(t,x)$ blows up at a time

$$T_0\leqslant t^*:=\frac{-\partial_x u_0(x_0)}{I_0\|u_0\|_{H^1}^2}-\sqrt{\left(\frac{\partial_x u_0(x_0)}{I_0\|u_0\|_{H^1}^2}\right)^2-\frac{1}{I_0\|u_0\|_{H^1}^2 m_0(x_0)}}.$$

Moreover when $T_0=t^*$, the following estimate of the blow-up rate holds

$$\liminf_{t\to T_0^-}\Big((T_0-t)\inf_{x\in\mathbb{R}}(mu_x)(t,x)\Big)\leqslant-\frac{1}{2}. \tag{5.2}$$

ii) *If*

$$0>\partial_x u_0(x_0)>-I_0\quad \text{and}$$

$$\frac{1}{m_0(x_0)}-\frac{\partial_x u_0(x_0)}{\sqrt{2}I_0\|u_0\|_{H^1}}<\frac{1}{\sqrt{2}\|u_0\|_{H^1}}\ln\left(\frac{I_0}{I_0+\partial_x u_0(x_0)}\right), \tag{5.3}$$

then the solution $m(t,x)$ blows up at a time

$$T_0\leqslant t^{**}:=\frac{1}{\sqrt{2}I_0\|u_0\|_{H^1}}\ln\left(\frac{I_0}{I_0+\partial_x u_0(x_0)}\right).$$

iii) *If* $\partial_x u_0(x_0) \leqslant -I_0$*, then the solution* $m(t,x)$ *blows up at a time* $T_0 \leqslant t_1$*, where* t_1 *uniquely solves the equation*

$$\frac{\sqrt{2}(I_0+\partial_x u_0(x_0))}{4I_0\|u_0\|_{H^1}}\left(e^{\sqrt{2}I_0\|u_0\|_{H^1}t}-1\right)-I_0t+\frac{1}{m_0(x_0)}=0.$$

Proof. Denoting $M = mu_x$, we first recall from Proposition 5.1 in [21] that for all $(t,x) \in [0,T) \times \mathbb{R}$,

$$M_t+\left(u^2-u_x^2\right)M_x=-2M^2-2m\left(1-\partial_x^2\right)^{-1}\left(u_x^2m\right)-2m\partial_x\left(1-\partial_x^2\right)^{-1}(uu_xm),$$

which along with (3.2) implies

$$\frac{d}{dt}M\big(t,q(t,x)\big)=\left(-2M^2-2m\left(1-\partial_x^2\right)^{-1}\left(u_x^2m\right)-2m\partial_x\left(1-\partial_x^2\right)^{-1}(uu_xm)\right)\big(t,q(t,x)\big). \tag{5.4}$$

Since $m_0(x) \geqslant 0$ for all $x \in \mathbb{R}$, Remark 3.4 implies that

$$m(t,x) \geqslant 0, \tag{5.5}$$

for all $t \in [0,T)$, $x \in \mathbb{R}$, and hence

$$\left(m\left(1-\partial_x^2\right)^{-1}\left(u_x^2m\right)\right)(t,x) \geqslant 0.$$

On the other hand, for $G = \frac{1}{2}e^{-|x|}$, we have

$$\begin{aligned}\partial_x\left(1-\partial_x^2\right)^{-1}(uu_xm)(t,x)&=\partial_xG*(uu_xm)(t,x)\\&=-\frac{1}{2}\int_{-\infty}^{+\infty}\operatorname{sign}(x-y)e^{-|x-y|}(uu_xm)(t,y)\,dy,\end{aligned}$$

which implies

$$\begin{aligned}-2m\partial_x\left(1-\partial_x^2\right)^{-1}(uu_xm)&=m\int_{-\infty}^{+\infty}\operatorname{sign}(x-y)e^{-|x-y|}(uu_xm)(y)\,dy\\&\leqslant m\int_{-\infty}^{+\infty}e^{-|x-y|}\big(u|u_x|m\big)(y)\,dy\\&=2m\left(1-\partial_x^2\right)^{-1}\big(u|u_x|m\big).\end{aligned} \tag{5.6}$$

Therefore, we find from (5.4) and (5.6) that

$$\frac{d}{dt}M\big(t,q(t,x)\big)\leqslant\left(-2M^2+2m\left(1-\partial_x^2\right)^{-1}\big((u-|u_x|)|u_x|m\big)\right)\big(t,q(t,x)\big). \tag{5.7}$$

Notice that

$$u(t,x) = (G * m)(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} m(t,y)\,dy,$$

we have

$$u(t,x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{y} m(t,y)\,dy + \frac{e^{x}}{2}\int_{x}^{+\infty} e^{-y} m(t,y)\,dy,$$

$$u_x(t,x) = -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{y} m(t,y)\,dy + \frac{e^{x}}{2}\int_{x}^{+\infty} e^{-y} m(t,y)\,dy,$$

which along with (5.5) and Remark 3.3 leads to

$$0 \leqslant \int_{x}^{+\infty} e^{x-y} m(t,y)\,dy = u(t,x) + u_x(t,x) \leqslant \int_{-\infty}^{+\infty} m(t,y)\,dy = I_0,$$

$$0 \leqslant \int_{-\infty}^{x} e^{y-x} m(t,y)\,dy = u(t,x) - u_x(t,x) \leqslant \int_{-\infty}^{+\infty} m(t,y)\,dy = I_0.$$

From this, we obtain for all $(t,x) \in [0,T) \times \mathbb{R}$,

$$\big|u_x(t,x)\big| \leqslant u(t,x), \qquad u(t,x) - \big|u_x(t,x)\big| \leqslant I_0 \quad \text{and} \quad u(t,x) \leqslant I_0 + u_x(t,x). \tag{5.8}$$

Therefore, in view of the Sobolev inequality $\|u\|_{L^\infty(\mathbb{R})} \leqslant \frac{1}{\sqrt{2}}\|u\|_{H^1(\mathbb{R})}$ and (5.8), it follows from (5.7) that

$$\begin{aligned}
\frac{d}{dt}M\big(t,q(t,x)\big) &\leqslant -2M^2\big(t,q(t,x)\big) + 2I_0\|u\|_{L^\infty} m\big(t,q(t,x)\big)\big[\big(1-\partial_x^2\big)^{-1} m\big]\big(t,q(t,x)\big) \\
&= -2M^2\big(t,q(t,x)\big) + 2I_0\|u\|_{L^\infty}(mu)\big(t,q(t,x)\big) \\
&\leqslant -2M^2\big(t,q(t,x)\big) + 2I_0\|u\|_{L^\infty}^2 m\big(t,q(t,x)\big) \\
&\leqslant -2M^2\big(t,q(t,x)\big) + I_0\|u_0\|_{H^1}^2 m\big(t,q(t,x)\big)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}M\big(t,q(t,x)\big) &\leqslant -2M^2\big(t,q(t,x)\big) + 2I_0\|u\|_{L^\infty}(mu)\big(t,q(t,x)\big) \\
&\leqslant -2M^2\big(t,q(t,x)\big) + 2I_0\|u\|_{L^\infty}\big[m(I_0+u_x)\big]\big(t,q(t,x)\big) \\
&\leqslant -2M^2\big(t,q(t,x)\big) + \sqrt{2}I_0\|u_0\|_{H^1} M\big(t,q(t,x)\big) + \sqrt{2}I_0^2\|u_0\|_{H^1} m\big(t,q(t,x)\big),
\end{aligned}$$

which, in particular, implies

$$\frac{d}{dt}M\big(t,q(t,x_0)\big) \leqslant -2M^2\big(t,q(t,x_0)\big) + I_0\|u_0\|_{H^1}^2 m\big(t,q(t,x_0)\big) \tag{5.9}$$

and

$$\frac{d}{dt}M\big(t,q(t,x_0)\big) \leqslant -2M^2\big(t,q(t,x_0)\big) + \sqrt{2}I_0\|u_0\|_{H^1}M\big(t,q(t,x_0)\big) \\ + \sqrt{2}I_0^2\|u_0\|_{H^1}m\big(t,q(t,x_0)\big). \tag{5.10}$$

Similarly, one can see from the equation in (1.1) that

$$\frac{d}{dt}m\big(t,q(t,x_0)\big) = -2mM\big(t,q(t,x_0)\big). \tag{5.11}$$

Denote that

$$\overline{M}(t) := 2M\big(t,q(t,x_0)\big) \quad \text{and} \quad \overline{m}(t) := 2m\big(t,q(t,x_0)\big).$$

We first reformulate (5.9) and (5.11) as

$$\frac{d}{dt}\overline{M}(t) \leqslant -\overline{M}(t)^2 + I_0\|u_0\|_{H^1}^2\overline{m}(t)$$

and

$$\frac{d}{dt}\overline{m}(t) = -\overline{m}(t)\overline{M}(t). \tag{5.12}$$

Combining this with (5.5), we deduce that

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{\overline{m}(t)^2}\frac{d}{dt}\overline{m}(t)\right) &= \frac{d}{dt}\left(-\frac{1}{\overline{m}(t)}\overline{M}(t)\right) \\
&= \frac{1}{\overline{m}(t)^2}\left(-\overline{m}(t)\frac{d}{dt}\overline{M}(t) + \overline{M}(t)\frac{d}{dt}\overline{m}(t)\right) \\
&\geqslant \frac{1}{\overline{m}(t)^2}\big(\overline{m}(t)\big(\overline{M}(t)^2 - I_0\|u_0\|_{H^1}^2\overline{m}(t)\big) - \overline{m}(t)\overline{M}(t)^2\big) \\
&= -I_0\|u_0\|_{H^1}^2.
\end{aligned}$$

Integrating from 0 to t leads to

$$\frac{1}{\overline{m}(t)^2}\frac{d}{dt}\overline{m}(t) \geqslant C_0 - I_0\|u_0\|_{H^1}^2 t, \tag{5.13}$$

with

$$C_0 := -\frac{\overline{M}(0)}{\overline{m}(0)} = -(\partial_x u_0)(x_0).$$

Combining this with (5.12) yields

$$\overline{M}(t) = -\frac{1}{\overline{m}(t)}\frac{d}{dt}\overline{m}(t) \leqslant -\overline{m}(t)\big(C_0 - I_0\|u_0\|_{H^1}^2 t\big). \tag{5.14}$$

Integrating (5.13) again on $[0,t]$ implies

$$\frac{1}{\overline{m}(t)}-\frac{1}{\overline{m}(0)}\leqslant\frac{1}{2}I_0\|u_0\|_{H^1}^2t^2-C_0t,$$

and hence

$$\begin{aligned}\frac{1}{\overline{m}(t)}&\leqslant\frac{1}{2}I_0\|u_0\|_{H^1}^2\left(t^2-\frac{2C_0}{I_0\|u_0\|_{H^1}^2}t+\frac{2}{I_0\|u_0\|_{H^1}^2\overline{m}(0)}\right)\\&=\frac{1}{2}I_0\|u_0\|_{H^1}^2\left(t^2-\frac{2C_0}{I_0\|u_0\|_{H^1}^2}t+\frac{1}{I_0\|u_0\|_{H^1}^2m_0(x_0)}\right).\end{aligned}$$

The quadratic equation

$$t^2-\frac{2C_0}{I_0\|u_0\|_{H^1}^2}t+\frac{1}{I_0\|u_0\|_{H^1}^2m_0(x_0)}=0$$

has two roots:

$$t^*:=\frac{C_0}{I_0\|u_0\|_{H^1}^2}-\sqrt{\left(\frac{C_0}{I_0\|u_0\|_{H^1}^2}\right)^2-\frac{1}{I_0\|u_0\|_{H^1}^2m_0(x_0)}},\quad\text{and}$$

$$t_*:=\frac{C_0}{I_0\|u_0\|_{H^1}^2}+\sqrt{\left(\frac{C_0}{I_0\|u_0\|_{H^1}^2}\right)^2-\frac{1}{I_0\|u_0\|_{H^1}^2m_0(x_0)}}.$$

It thus transpires from assumption (5.1) that

$$\left(\frac{C_0}{I_0\|u_0\|_{H^1}^2}\right)^2>\frac{1}{I_0\|u_0\|_{H^1}^2m_0(x_0)},\quad\text{hence } 0<t^*<\frac{C_0}{I_0\|u_0\|_{H^1}^2}<t_*.$$

Thus,

$$0\leqslant\frac{1}{\overline{m}(t)}\leqslant\frac{I_0\|u_0\|_{H^1}^2}{2}\left(t-t^*\right)(t-t_*).\tag{5.15}$$

It is then adduced from (5.15) that there is a time $T_0\in(0,t^*]$ such that

$$m(t)\to+\infty,\quad\text{as } t\to T_0\leqslant t^*,$$

which, by (5.14), implies that

$$M(t)\to-\infty,\quad\text{as } t\to T_0\leqslant t^*.$$

Therefore,

$$\inf_{x\in\mathbb{R}}M(t,x)\leqslant M(t)\to-\infty,\quad\text{as } t\to T_0\leqslant t^*,$$

which, in view of Theorem 4.2, implies that the solution $m(t,x)$ blows up at the time T_0.

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Having established wave breaking results for (1.1) as above, attention is now given to blow-up rate for the solution. In fact, owing to (5.14) and (5.15), we derive that for all $0 < t < T_0$,

$$\begin{aligned}(T_0 - t)\inf_{x\in\mathbb{R}} \overline{M}(t,x) &\leqslant (T_0 - t)\overline{M}(t) \leqslant (T_0 - t)\overline{m}(t)\big((\partial_x u_0)(x_0) + I_0\|u_0\|_{H^1}^2 t\big)\\ &\leqslant (T_0 - t)\frac{2}{I_0\|u_0\|_{H^1}^2(t-t^*)(t-t_*)}\big(I_0\|u_0\|_{H^1}^2 t + (\partial_x u_0)(x_0)\big)\\ &\leqslant \frac{2(T_0 - t)}{(t-t^*)(t-t_*)}\left(t + \frac{(\partial_x u_0)(x_0)}{I_0\|u_0\|_{H^1}^2}\right),\end{aligned}$$

which leads to (5.2) when $T_0 = t^*$. Therefore, we end the proof of Theorem 5.1(i).

On the other hand, we reformulate (5.10) as

$$\frac{d}{dt}\overline{M}(t) \leqslant -\overline{M}(t)^2 + \sqrt{2}I_0\|u_0\|_{H^1}\overline{M}(t) + \sqrt{2}I_0^2\|u_0\|_{H^1}\overline{m}(t).$$

Combining this with (5.12) and (5.5), we deduce that

$$\begin{aligned}\frac{d}{dt}\left(-\frac{\overline{M}(t)}{\overline{m}(t)}\right) &= \frac{d}{dt}\left(\frac{1}{\overline{m}(t)^2}\frac{d}{dt}\overline{m}(t)\right) = \frac{1}{\overline{m}(t)^2}\left(-\overline{m}(t)\frac{d}{dt}\overline{M}(t) + \overline{M}(t)\frac{d}{dt}\overline{m}(t)\right)\\ &\geqslant \frac{1}{\overline{m}(t)^2}\big[\overline{m}(t)\big(\overline{M}(t)^2 - \sqrt{2}I_0\|u_0\|_{H^1}\overline{M}(t) - \sqrt{2}I_0^2\|u_0\|_{H^1}\overline{m}(t)\big) - \overline{m}(t)\overline{M}(t)^2\big]\\ &= -\frac{\overline{M}(t)}{\overline{m}(t)}\sqrt{2}I_0\|u_0\|_{H^1} - \sqrt{2}I_0^2\|u_0\|_{H^1},\end{aligned}$$

which gives rise to

$$\frac{d}{dt}\left(-\frac{\overline{M}(t)}{\overline{m}(t)}e^{-C_1 t}\right) \geqslant -C_2 e^{-C_1 t} \tag{5.16}$$

with

$$C_1 := \sqrt{2}I_0\|u_0\|_{H^1}, \qquad C_2 := \sqrt{2}I_0^2\|u_0\|_{H^1}.$$

Integrating (5.16) from 0 to t leads to

$$-\frac{\overline{M}(t)}{\overline{m}(t)}e^{-C_1 t} \geqslant \frac{C_2}{C_1}e^{-C_1 t} + \frac{C_0 C_1 - C_2}{C_1} \tag{5.17}$$

with

$$C_0 = -\frac{\overline{M}(0)}{\overline{m}(0)} = -(\partial_x u_0)(x_0),$$

which implies

$$\overline{M}(t) \leqslant -\overline{m}(t)\left(\frac{C_2}{C_1} + \frac{C_0 C_1 - C_2}{C_1}e^{C_1 t}\right) \tag{5.18}$$

and

$$-\frac{d}{dt}\left(\frac{1}{\overline{m}(t)}\right) \geqslant \frac{C_2}{C_1} + \frac{C_0C_1 - C_2}{C_1}e^{C_1 t}. \tag{5.19}$$

Integrating (5.19) again on $[0, t]$ implies

$$\frac{1}{\overline{m}(t)} - \frac{1}{\overline{m}(0)} \leqslant \frac{C_2 - C_0C_1}{C_1^2}\left(e^{C_1 t} - 1\right) - \frac{C_2}{C_1}t,$$

and hence

$$0 \leqslant \frac{1}{\overline{m}(t)} \leqslant \frac{C_2 - C_0C_1}{C_1^2}\left(e^{C_1 t} - 1\right) - \frac{C_2}{C_1}t + \frac{1}{\overline{m}(0)} =: f(t).$$

Notice that $f(0) = \frac{1}{\overline{m}(0)} > 0$ and the assumption (5.3) implies the equation $\frac{d}{dt}f(t) = 0$ has only one root

$$t^{**} := \frac{1}{C_1}\ln\left(\frac{C_2}{C_2 - C_0C_1}\right)$$

and then

$$f\left(t^{**}\right) = \frac{C_0}{C_1} + \frac{1}{m_0(x_0)} - \frac{C_2}{C_1^2}\ln\left(\frac{C_2}{C_2 - C_0C_1}\right) < 0.$$

From this, we may find a time $0 < T_0 \leqslant t^{**}$ such that

$$m(t) \to +\infty, \quad \text{as } t \to T_0 \leqslant t^{**},$$

which, by (5.18), implies that

$$M(t) \to -\infty, \quad \text{as } t \to T_0 \leqslant t^{**}.$$

Therefore,

$$\inf_{x\in\mathbb{R}} M(t,x) \leqslant M(t) \to -\infty, \quad \text{as } t \to T_0 \leqslant t^{**},$$

which, in view of Theorem 4.2, implies that the solution $m(t,x)$ blows up at the time T_0. This ends the proof of Theorem 5.1(ii). Similarly, we may prove Theorem 5.1(iii). This completes the proof of Theorem 5.1. □

Remark 5.1. (1) Compared to the blow-up result of Theorem 5.2 in [21] where $\partial_x u_0(x_0) < -\sqrt{\frac{\sqrt{2}\|u_0\|_{H^1}^3}{m_0(x_0)}}$, Theorem 5.1(i) looks better at least in some sense as the following example: taking the initial data $u_0(x) = e^{-x^2}$, then

$$I_0 = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}, \qquad \|u_0\|_{H^1}^2 = \int_{-\infty}^{\infty} e^{-2x^2}\,dx + 4\int_{-\infty}^{\infty} x^2 e^{-2x^2}\,dx = \sqrt{2\pi},$$

which implies

$$\|u_0\|_{H^1}\sqrt{\frac{I_0}{m_0(x_0)}} < \sqrt{\frac{\sqrt{2}\|u_0\|_{H^1}^3}{m_0(x_0)}}.$$

(2) In view of Theorem 5.1(iii), if $\partial_x u_0(x_0) \leqslant -I_0$, then the function

$$F(t) := \frac{\sqrt{2}(I_0 + \partial_x u_0(x_0))}{4I_0\|u_0\|_{H^1}}\left(e^{\sqrt{2}I_0\|u_0\|_{H^1}t} - 1\right) - I_0 t + \frac{1}{m_0(x_0)} = 0$$

has only one root on $[0, +\infty)$. In fact, we need only consider the case $\partial_x u_0(x_0) < -I_0$, that is, $I_0 + \partial_x u_0(x_0) < 0$. Notice that $F(0) = \frac{1}{m_0(x_0)} > 0$, $F(+\infty) < 0$, and $\frac{d}{dt}F(t) < 0$ for all $t \in [0, +\infty)$. We then deduce from the Intermediate Value Theorem that $F(t) = 0$ has only one root on $[0, +\infty)$.

6. Nonexistence of smooth traveling waves for $\gamma = 0$

In this section, we prove that the equation in (1.1) does not have nontrivial smooth traveling waves.

Theorem 6.1. *There is no nontrivial smooth traveling wave solution $u(t,x) = \phi(x - ct)$, $c \in \mathbb{R}$ of the Cauchy problem (1.1) with $\gamma = 0$ in $C([0,\infty); H^3(\mathbb{R})) \cap C^1([0,\infty); H^2(\mathbb{R}))$.*

Proof. We use a contradiction argument. Assume that $\phi \in H^3$ is a strong solution of the Cauchy problem (1.1). Then we have

$$c\left(\phi - \phi''\right)' = \left(\left(\phi^2 - \phi_x^2\right)\left(\phi - \phi''\right)\right)' \quad \text{in } L^2(\mathbb{R}).$$

Since $\phi \in H^3(\mathbb{R}) \subset C_0^2(\mathbb{R})$, we find that

$$c\left(\phi - \phi''\right) = \left(\phi^2 - \phi_x^2\right)\left(\phi - \phi''\right) \quad \text{in } H^1(\mathbb{R}). \tag{6.1}$$

Note that $\phi \not\equiv 0$ and $\phi, \phi', \phi'' \to 0$ as $|x| \to \infty$, it implies that $\phi - \phi'' \neq 0$. Otherwise, $\phi = c_1 e^x + c_2 e^{-x}$, which gives $\phi \equiv 0$, $x \in \mathbb{R}$ since $\phi \to 0$ as $|x| \to \infty$. It then follows from (6.1) that

$$\phi^2 - \phi'^2 = c. \tag{6.2}$$

Let $|x| \to \infty$. Then $\phi, \phi' \to 0$. It yields from (6.2) that $c = 0$. Hence we deduce from (6.2) that

$$\phi^2 - \phi'^2 = 0,$$

which implies that either $\phi = c_1 e^x$ or $\phi = c_2 e^{-x}$. This then leads to a contradiction that $\phi \equiv 0$, since $\phi \to 0$, as $|x| \to \infty$. This completes the proof of the theorem. □

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