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Bounded solutions for a forced bounded oscillator without friction

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Abstract

Under the validity of a Landesman–Lazer type condition, we prove the existence of solutions bounded on the real line, together with their first derivatives, for some second order nonlinear differential equation of the form $\ddot{u} + g(u) = p(t)$, where the reaction term g is bounded. The proof is variational, and relies on a dual version of the Nehari method for the existence of oscillating solutions to superlinear equations.

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1. Introduction

This paper concerns the existence of solutions, bounded on the real line together with their first derivative, for the differential equation

$$\ddot{u} + g(u) = p(t), \tag{1}$$

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where $g \in \mathcal{C}^2(\mathbb{R})$ is bounded, increasing, and has exactly one inflection point, and $p \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ admits asymptotic average $A(p) \in \mathbb{R}$, that is

$$\lim_{T \to +\infty} \frac{1}{T} \int_t^{t+T} p(s)\, ds = A(p),$$

uniformly in $t \in \mathbb{R}$. Such an equation describes the forced motions of an oscillator exhibiting saturation effects. As a model problem, the reader may think to the equation

$$\ddot{u} + \arctan u = p(t),$$

even though we do not require any symmetry assumption on the reaction term g. Under the above assumptions, the main result we prove is the following theorem.

Theorem 1.1. *Eq. (1) admits a bounded solution if and only if*

$$g(-\infty) < A(p) < g(+\infty). \tag{2}$$

In such a case, Eq. (1) admits a countable set of bounded solutions, having arbitrarily large L^∞-norm.

The motivation for our investigation relies on the papers [1,6], which in turn have been inspired by some classical results of Landesman–Lazer type holding in the periodic framework. Such studies concern the equation

$$\ddot{u} + c\dot{u} + g(u) = p(t), \tag{3}$$

where $c \in \mathbb{R}$ and the continuous function g, not necessarily monotone, admits limits at $\pm\infty$, with the property that

$$g(-\infty) < g(s) < g(+\infty)$$

for every s. Also the cases $g(\pm\infty) = \pm\infty$ can be considered, requiring g to be sublinear at infinity if $c = 0$. When p is T-periodic, it is nowadays well known that Eq. (1) admits a periodic solution if and only if the Landesman–Lazer condition

$$g(-\infty) < \frac{1}{T} \int_0^T p(s)\, ds < g(+\infty)$$

is satisfied, regardless of the constant c; this result was first proved by Lazer, using the Schauder fixed point theorem, see [4]. When p is merely bounded, one would like to find analogous conditions for the search of bounded solutions. This problem was first studied by Ahmad [1], under the assumption that p has asymptotic average, in the sense explained above; by means of techniques of the qualitative theory of dissipative equations, the existence of a bounded solution is characterized, whenever $c \neq 0$, by (2). The case in which p is an arbitrary continuous function was solved by Ortega [6], who assumes $c \neq 0$ and provides a sharp necessary and sufficient condition: (3) has a bounded solution if and only if p can be written as $p^* + p^{**}$, where p^* has bounded primitive and p^{**} assumes values strictly contained between $g(-\infty)$ and $g(+\infty)$. This result relies on the Krasnoselskii method of guiding functions, and was generalized by Ortega and Tineo [7] to equations of higher order, using the notions of lower and upper averages of p; again, the condition $c \neq 0$ sticks as a crucial assumption. Later, by means of the method of lower and upper solutions, Mawhin and Ward [5] achieved some results in the case $c = 0$, but in the complementary situation in which $g(-\infty) \geqslant g(+\infty)$. Up to our knowledge, this last is the unique extension of the Landesman–Lazer theory to second order equations without friction, and the question in the case $g(-\infty) < g(+\infty)$ is still open. Under this perspective, in this paper we go back to the setting originally considered by Ahmad, and we prove that its aforementioned result holds also in the case $c = 0$, at least for the particular class of g that we consider.

The proof of our result is variational: we use a dual Nehari method which was first introduced in [8] to obtain bounded solutions in the case of a sublinear reaction (i.e. $g(s) = s^{1/3}$). The method consists of two steps.

Firstly, we consider the boundary value problem

$$\begin{cases} \ddot{u} + g(u) = p(t), & t \in (a,b), \\ u(a) = 0 = u(b), & \\ u(t) > 0, & t \in (a,b), \end{cases} \tag{4}$$

searching for solutions as minimizers of the action functional

$$J_{(a,b)}(u) := \int_a^b \left[\frac{1}{2}\dot{u}^2(t) - G\big(u(t)\big) + p(t)u(t) \right] dt$$

in the weakly closed set $\{u \in H_0^1(a,b)\colon\ u \geqslant 0\}$. In Section 3 we obtain some general properties of the nonnegative minimizers of $J_{(a,b)}$ in any interval (a,b); in Section 4 we prove that, when $b-a$ is sufficiently large, the minimizer $u_+(\cdot;a,b)$ is unique and solves problem (4). The proof of these results is substantially different from the corresponding one in the sublinear case [8]: indeed in the present situation the nonlinearity g and the forcing term p have the same order of growth (they are both bounded), while, as far as $b-a$ is sufficiently large, the forced sub-linear problem can be considered as a small perturbation of the unforced one. This fact introduces a lot of complications, which we can overcome thanks to a careful analysis of the balance between g and p, via measure theory tools, and of the asymptotic properties of the functional $J_{(a,b)}$ as $b-a \to +\infty$. Of course, analogous results can be obtained for negative minimizers $u_-(\cdot;a,b)$. To proceed, it is necessary to prove that $u_\pm(\cdot;a,b)$ is non-degenerate and that $J_{(a,b)}(u_\pm(\cdot;a,b))$ is differentiable as a function of (a,b). This is the object of Sections 5, 6, and it is the only part which requires $g \in \mathcal{C}^2$. We believe that this assumption can be weakened by a suitable approximating procedure, but we prefer to avoid further technicalities at this point.

Once the existence of one-signed solutions is established, in Section 7 we juxtapose positive and negative minimizers with alternate signs to obtain oscillating solutions. Indeed, let us fix $k \geqslant 1$, a bounded interval $[A,B]$ sufficiently large, and let us consider the class of partitions

$$\mathcal{B}_k := \left\{ (t_1,\ldots,t_k) \in \mathbb{R}^k \;\middle|\; \begin{array}{l} A =: t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t_{k+1} := B, \\ t_{i+1} - t_i \text{ is sufficiently large for any } i \end{array} \right\}.$$

For each partition $P = (t_1, \dots, t_k)$ of $\mathcal{B}_k$ there is a function u_P obtained by juxtaposing $u_\pm(\cdot; t_i, t_{i+1})$ with alternate signs $+$ and $-$. In general, this function is not a solution of Eq. (1), because the derivatives $\dot u_P(t_i^\pm)$ may not coincide. We prove that these corner points disappear for the partition maximizing the quantity

$$\psi(P) = \sum_{i=0}^{k} J_{(t_i,t_{i+1})}\big(u_\pm(\cdot; t_i, t_{i+1})\big).$$

This argument provides a solution of (1) having k zeros in $[A, B]$, together with some estimates which depend only on the ratio $(B-A)/k$. Therefore, taking $A \to -\infty$, $B \to +\infty$ and $k \to +\infty$ in an appropriate way, one can pass to the limit and obtain the desired bounded solution. In doing this, one must again modify the corresponding arguments in the sub-linear case, indeed they do not allow to treat the non-symmetric case $g(+\infty) - A(p) \neq A(p) - g(-\infty)$.

Incidentally, assuming p to be T-periodic, a simple variation of the argument above allows to obtain the well-known existence of infinitely many subharmonic solutions [3], i.e. solutions which have minimal period nT, $n \in \mathbb{N}$, with sharp nodal characterization (see Theorem 7.7 at the end of the paper).

To conclude, we remark that also the case of infinite limits $g(\pm\infty)$ can be treated by variational methods. On one hand, as already mentioned, infinitely many bounded solutions for Eq. (1) were obtained in [8] when $g(s) = |s|^{q-1}s$, $0 < q < 1$, and $p \in L^\infty(\mathbb{R})$. On the other hand, the original Nehari method, together with a limiting procedure, allows to obtain an analogous result also when g is superlinear at infinity, as done in [9,10].

2. Preliminaries

It is not difficult to check that if Eq. (1) admits a bounded solution with bounded derivative, then necessarily condition (2) is satisfied. Indeed, by integrating Eq. (1) in $(t, t+T)$, we obtain

$$\frac{\dot u(t+T) - \dot u(t)}{T} = \frac{1}{T}\int_t^{t+T} \big(p(s) - g\big(u(s)\big)\big)\,ds.$$

Since $\dot u$ is bounded, passing to the limit as $T \to +\infty$ we deduce that the left hand side tends to 0, so that

$$\begin{aligned} 0 &= \lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} \big(p(s) - g\big(u(s)\big)\big)\,ds \\ &= A(p) - \lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} g\big(u(s)\big)\,ds. \end{aligned} \tag{5}$$

Now, the boundedness of u and the monotonicity of g imply also that for every $s \in \mathbb{R}$

$$g(-\infty) < g\big(-\|u\|_\infty\big) \leqslant g\big(u(s)\big) \leqslant g\big(\|u\|_\infty\big) < g(+\infty), \tag{6}$$

and a comparison between (5) and (6) gives the desired result (in fact, from this point of view, it is sufficient that $g(-\infty) < g(s) < g(+\infty)$ for every s).

We observe that, by means of suitable translations, it is not restrictive to assume that

$$\begin{aligned} &g(0)=0,\ \ g \in \mathcal{C}^2(\mathbb{R}) \text{ is bounded, strictly increasing in } \mathbb{R}, \\ &\text{strictly concave in } (0,+\infty) \text{ and strictly convex in } (-\infty,0). \end{aligned} \tag{h1}$$

We denote as G the primitive of g vanishing in 0, and

$$\lim_{s\to\pm\infty} g(s) = g_\pm,$$

so that

$$\lim_{s\to\pm\infty} \frac{G(s)}{s} = g_\pm \quad \text{and} \quad g_- < \frac{G(s)}{s} < g_+ \quad \forall s \in \mathbb{R}.$$

As far as the function p is concerned, as we already mentioned, we assume that $p \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is such that for every $\varepsilon > 0$ there exists $\bar{T} > 0$ such that if $T > \bar{T}$ then

$$\sup_{t\in\mathbb{R}} \left| \frac{1}{T} \int_t^{t+T} p(s)\,ds - A(p) \right| < \varepsilon,$$

in such a way that

$$p \text{ is bounded and continuous in } \mathbb{R}, \text{ and has asymptotic average } g_- < A(p) < g_+. \tag{h2}$$

Note that we do not make any assumption on the L^∞ norm of p.

In view of the previous considerations and notations, we can rephrase Theorem 1.1 as follows.

Theorem 2.1. *Under assumptions (h1)–(h2), there exists a sequence (u_m) of solutions of (1) defined in $\mathbb{R}$, with $u_m, \dot{u}_m \in L^\infty(\mathbb{R})$ and $\|u_m\|_\infty \to \infty$ as $m \to \infty$. Moreover, each u_m has infinitely many zeros in $\mathbb{R}$.*

3. Existence and basic properties of nonnegative minimizers

In this section we deal with the boundary value problem (4):

$$\begin{cases} \ddot{u}(t) + g\big(u(t)\big) = p(t), & t \in (a,b), \\ u(a) = 0 = u(b), & \\ u(t) > 0, & t \in (a,b). \end{cases}$$

We seek solutions as minimizers of the related action functional

$$J_{(a,b)}(u) := \int_a^b \left[\frac{1}{2}\dot{u}^2(t) - G\big(u(t)\big) + p(t)u(t) \right] dt$$

in the H^1-weakly closed set

$$H_0^1(a,b)^+ := \{u \in H_0^1(a,b)\colon\ u \geqslant 0\}.$$

We introduce the value

$$\varphi^+(a,b) := \inf_{u \in H_0^1(a,b)^+} J_{(a,b)}(u).$$

Remark 1. Of course, even though in the following we focus on positive solutions, negative ones can be treated similarly as well, seeking solutions to the boundary value problem

$$\begin{cases} \ddot{u}(t) + g(u(t)) = p(t), & t \in (a,b), \\ u(a) = 0 = u(b), & \\ u(t) < 0, & t \in (a,b), \end{cases}$$

associated to the candidate critical value

$$\varphi^-(a,b) := \inf_{u \in H_0^1(a,b)^-} J_{(a,b)}(u),$$

where $H_0^1(a,b)^- := \{u \in H_0^1(a,b)\colon\ u \leqslant 0\}$. Indeed, the two problems are related by the change of variable $v = -u$, $\bar{g}(s) = -g(-s)$ and $\bar{p} = -p$, and $\bar{g}$, $\bar{p}$ satisfy (h1)–(h2) if and only if g, p do. In particular, when dealing with negative solutions, in all the explicit constants we will find, the quantity $g_\pm$ should be replaced by $-g_\mp$, and $A(p)$ by $-A(p)$.

Lemma 3.1. *The value $\varphi^+(a,b)$ is a real number and it is achieved by $u_{(a,b)} \in H_0^1(a,b)^+$.*

Proof. It is not difficult to check that $J_{(a,b)}$ is weakly lower semi-continuous and coercive, so that the direct method of the calculus of variations applies. □

In what follows we are going to show, that, if (a,b) is sufficiently large, a minimizer $u_{(a,b)}$ is an actual solution of (4); this is not obvious, because in principle $u_{(a,b)}$ could vanish somewhere. Having in mind to let (a,b) vary and wishing to catch the behavior of the minimizers $u_{(a,b)}$ under variations of the domain, it is convenient to introduce suitable scaling to work on a common time-interval. To be precise, for every $u \in H_0^1(a,b)^+$ we can define

$$\widehat{u}(t) := \frac{1}{(b-a)^2} u\big(a + t(b-a)\big) \quad \Leftrightarrow \quad u(t) = (b-a)^2 \widehat{u}\left(\frac{t-a}{b-a}\right), \tag{7}$$

and $\widehat{p}_{(a,b)}(t) := p(a + t(b-a))$. Of course, $\widehat{u} \in H_0^1(0,1)^+$ and

$$\begin{aligned} J_{(a,b)}(u) &= (b-a)^3 \int_0^1 \left[\frac{1}{2}\dot{\widehat{u}}^2(t) - \frac{1}{(b-a)^2} G\big((b-a)^2 \widehat{u}(t)\big) + \widehat{p}_{(a,b)}(t)\widehat{u}(t)\right] dt \\ &=: (b-a)^3 \widehat{J}_{(a,b)}(\widehat{u}). \end{aligned} \tag{8}$$

This reveals that the minimization of $J_{(a,b)}$ in $H_0^1(a,b)^+$ is equivalent to that of $\widehat{J}_{(a,b)}$ in $H_0^1(0,1)^+$; in particular, the function $\widehat{u}_{(a,b)}$ defined by (7) with $u = u_{(a,b)}$ is a minimizer of $\widehat{J}_{(a,b)}$ in $H_0^1(0,1)^+$.

The Euler–Lagrange equation associated to the functional $\widehat{J}_{(a,b)}$ yields the research of solutions to

$$\begin{cases} \ddot{w}(t) + g\big((b-a)^2 w(t)\big) = \widehat{p}_{(a,b)}(t), & \text{in } (0,1), \\ w(0) = 0 = w(1), \\ w(t) > 0, & \text{in } (0,1). \end{cases} \tag{9}$$

Our aim is to show that if $b-a$ is sufficiently large, then a minimizer $\widehat{u}_{(a,b)}$ is an actual solution of (9). We start showing that where it is positive it solves Eq. (1), and it is of class $\mathcal{C}^1$ in the whole $(0,1)$.

Lemma 3.2. *Let $(c,d) \subset (0,1)$ be such that*

$$\widehat{u}_{(a,b)} > 0 \quad \text{in } (c,d).$$

Then $\widehat{u}_{(a,b)}$ is a classical solution of the first equation in (9) in (c,d). Moreover, if $c > 0$ then $\dot{\widehat{u}}_{(a,b)}(c^+) = 0$, and if $d < 1$ then $\dot{\widehat{u}}_{(a,b)}(d^-) = 0$.

Proof. The fact that $\widehat{u}_{(a,b)}$ is a (classical) solution in (c,d) follows from the extremality of $\widehat{u}_{(a,b)}$ with respect to variations with compact support in (c,d). Concerning the second part of the statement, the proof is the same as that of Lemma 2.3 in [8]. □

In the following lemma we prove that the family of the minimizers $\{\widehat{u}_{(a,b)}\}$ is uniformly bounded and equi-Lipschitz-continuous.

Lemma 3.3. *For every $(a,b) \subset \mathbb{R}$ and any $\widehat{u}_{(a,b)}$, it holds*

$$\big|\widehat{u}_{(a,b)}(t)\big| \leqslant \big(\|g\|_\infty + \|p\|_\infty\big) \quad \forall t \in (0,1),$$
$$\big|\dot{\widehat{u}}_{(a,b)}(t)\big| \leqslant \big(\|g\|_\infty + \|p\|_\infty\big) \quad \forall t \in (0,1).$$

Proof. Let $(c,d) \subset [0,1]$ be such that $\widehat{u}_{(a,b)} > 0$ in (c,d), vanishing at c and d. From Lemma 3.2 it follows that

$$\big|\ddot{\widehat{u}}_{(a,b)}(t)\big| \leqslant \big|g\big((b-a)^2 \widehat{u}_{(a,b)}(t)\big)\big| + \big|p(t)\big| \leqslant \|g\|_\infty + \|p\|_\infty \quad \forall t \in (c,d).$$

Since $\widehat{u}_{(a,b)}(c) = 0 = \widehat{u}_{(a,b)}(d)$ and $\widehat{u}_{(a,b)} \in \mathcal{C}^1(0,1)$, there exists $\tau \in (c,d)$ such that $\dot{\widehat{u}}_{(a,b)}(\tau) = 0$. Hence

$$\big|\dot{\widehat{u}}_{(a,b)}(t)\big| \leqslant \big|\dot{\widehat{u}}_{(a,b)}(\tau)\big| + \|g\|_\infty + \|p\|_\infty = \|g\|_\infty + \|p\|_\infty \quad \forall t \in (c,d).$$

Since this relation holds in each interval (c,d) as before, one can easily conclude by recalling that, being $u \in H^1$, it holds

$$\int_{\{u(t)=0\}} |\dot{u}(t)|\,dt = 0. \qquad \square$$

Let

$$s(t) = \sum_{k=0}^{n-1} y_k \chi_{[t_k,t_{k+1})}(t)$$

denote a simple function. We define the quantity

$$\delta(s) := \inf\{t_{k+1} - t_k \colon\ k = 0,\dots,n-1\}. \tag{10}$$

Given any measurable and bounded function $u \in \mathcal{M}(0,1)$, it is well known that for every $\varepsilon > 0$ there is a simple function s_u such that $\|u - s_u\|_\infty < \varepsilon$. In general the quantity $\delta(s_u)$ depends on u and ε. The following lemma says that if we consider the family of the minimizers $\{\widehat{u}_{(a,b)}\}$, given $\varepsilon > 0$ it is possible to find a family of approximating simple functions $\{s_{(a,b)}\}$ such that $\delta(s_{(a,b)})$ is bounded below uniformly with respect to (a,b).

Lemma 3.4. *For every $\varepsilon > 0$, let $m \in \mathbb{N}$ be such that $m > (\|g\|_\infty + \|p\|_\infty)/\varepsilon$. Then for every $(a,b) \subset \mathbb{R}$*

$$s_{(a,b)}(t) := \sum_{k=0}^{m-1} \widehat{u}_{(a,b)}\left(\frac{k}{m}\right)\chi_{[\frac{k}{m},\frac{k+1}{m})}(t)$$

is such that

$$\|\widehat{u}_{(a,b)} - s_{(a,b)}\|_\infty < \varepsilon \quad \text{and} \quad \delta(s_{(a,b)}) = \bar{\delta} := \frac{1}{m}.$$

In particular, m can be chosen only depending on ε and $\|p\|_\infty$, and not on p.

Proof. For every $t \in (0,1)$ there exists $k \in \{0,\dots,m-1\}$ such that $t \in [k/m,(k+1)/m)$, so that by Lemma 3.3

$$\left|\widehat{u}_{(a,b)}(t) - s_{(a,b)}(t)\right| = \left|\int_{\frac{k}{m}}^{t} \dot{\widehat{u}}_{(a,b)}(\tau)\,d\tau\right| \leqslant \frac{1}{m}\left(\|g\|_\infty + \|p\|_\infty\right) \quad \forall t \in (0,1). \qquad \square$$

4. The boundary value problem for large intervals

Here and in the next section we consider the minimizer $u_{(a,b)}$ as function of a, b and p. For this reason, we write

- $u(\cdot; a, b; p)$ and $\widehat{u}(\cdot; a, b; p)$ instead of $u_{(a,b)}$ and $\widehat{u}_{(a,b)}$ respectively,
- $J_{(a,b),p}$ and $\widehat{J}_{(a,b),p}$ instead of $J_{(a,b)}$ and $\widehat{J}_{(a,b)}$ respectively,
- $\varphi^+(a, b; p)$ instead of $\varphi^+(a, b)$,

to emphasize the dependence we are considering. As we have already mentioned, we can introduce an auxiliary problem which carries the asymptotic behavior of (9) for $b - a \to +\infty$. Let us consider

$$\begin{cases} \ddot{w}(t) = -\big(g_+ - A(p)\big) =: -k, & \text{in } (0, 1), \\ w(0) = 0 = w(1), \end{cases} \tag{11}$$

with $k > 0$ thanks to (h2). Of course, this problem has the unique solution

$$w_k(t) = \frac{k}{2} t(1 - t). \tag{12}$$

The related action functional is

$$J_k^\infty(w) := \int_0^1 \left[\frac{1}{2}\dot{w}^2(t) - kw(t)\right] dt, \tag{13}$$

which has the unique minimizer w_k in $H_0^1(0, 1)^+$ (the uniqueness follows from the strict convexity of J_k^∞). A direct computation gives

$$J_k^\infty(w_k) = -\frac{k^2}{24}.$$

Having in mind to compare minimizers related to different forcing terms, for any p satisfying (h2) it is convenient to introduce a subset $\mathcal{P}$ of $L^\infty(\mathbb{R})$ such that the mentioned threshold can be chosen independently of $q \in \mathcal{P}$. To this aim, first of all we recall the following result.

Lemma 4.1. *(See [6, Lemma 2.2].) Let p satisfy (h2). For every $\varepsilon > 0$ there exists a decomposition $p = p_{1,\varepsilon} + \dot{p}_{2,\varepsilon}$, where $\|p_{1,\varepsilon} - A(p)\|_\infty < \varepsilon/2$ and $p_{2,\varepsilon} \in L^\infty(\mathbb{R})$.*

This means that if p has asymptotic average it can be written as a sum between a term $p_{1,\varepsilon}$ which is arbitrarily close to the average $A(p)$, plus a term $\dot{p}_{2,\varepsilon}$ which has bounded primitive.

Given $p \in L^\infty(\mathbb{R})$, we compute $\|p\|_\infty$ and $A(p)$, and for any $0 < \varepsilon < 1$ we consider a decomposition as in Lemma 4.1; we introduce

$$M_1 := \|p\|_\infty + 1 \quad \text{and} \quad M_\varepsilon := \|p_{2,\varepsilon}\|_\infty + 1.$$

We define

$$\mathcal{P} := \left\{ q \in L^\infty(\mathbb{R}) \;\middle|\; \begin{array}{l} \|q\|_\infty < M_1,\ q \text{ has asymptotic average},\ A(q) = A(p), \\ \text{and for any } \varepsilon \in (0,1) \text{ there exists a decomposition} \\ q = q_{1,\varepsilon} + \dot{q}_{2,\varepsilon} \text{ as in Lemma 4.1, with } \|q_{2,\varepsilon}\|_\infty < M_\varepsilon \end{array} \right\}. \tag{14}$$

Remark 2. Note that given any p satisfying assumption (h2) we can define the set $\mathcal{P}$, which definition depends on p. Clearly, $p \in \mathcal{P}$ and the constant function $A(p)$ belongs to $\mathcal{P}$. Moreover, if q is of type

$$q(t) = A(p) + \dot{q}_2(t) \quad \text{or} \quad q(t) = p(t) + \dot{q}_2(t),$$

with $\|q_2\|_\infty, \|\dot{q}_2\|_\infty < 1$, then $q \in \mathcal{P}$.

We are ready to show that problem (11) is the limit problem of (4) as $b - a \to +\infty$, in the following sense.

Proposition 4.2. *Let p satisfy assumption (h2), and let $\mathcal{P}$ be defined by (14). For every $0 < \varepsilon < (g_+ - A(p))^2/24$ there exists $L_1 > 0$ depending only on ε such that if $b - a \geqslant L_1$ then*

$$-\underline{\alpha} \leqslant \widehat{J}_{(a,b),q}\big(\widehat{u}(\cdot; a, b; q)\big) \leqslant -\overline{\alpha} \quad \forall q \in \mathcal{P},$$

where

$$\underline{\alpha} := \frac{(g_+ - A(p))^2}{24} + \varepsilon \quad \textit{and} \quad \overline{\alpha} := \frac{(g_+ - A(p))^2}{24} - \varepsilon. \tag{15}$$

Remark 3. The upper bound on ε implies that $\widehat{u}(\cdot; a, b; q)$ cannot vanish identically whenever $b - a > L_1$.

To prove Proposition 4.2 we need some intermediate results.

Lemma 4.3. *Let $\mathfrak{F} \subset H_0^1(0,1)^+$ be such that*

$$\|u\|_{L^1(0,1)} \leqslant M \quad \forall u \in \mathfrak{F}.$$

For every $\varepsilon > 0$ there exists $L_2 = L_2(\varepsilon) > 0$ such that, if $b - a > L_2$, then

$$\left| \int_0^1 \left[\frac{1}{(b-a)^2} G\big((b-a)^2 u\big) - g_+ u \right] \right| < \varepsilon,$$

$$\left| \int_0^1 \big[g\big((b-a)^2 u\big) u - g_+ u \big] \right| < \varepsilon,$$

$$\left|\int_0^1 \left[g\big((b-a)^2u\big)u - \frac{1}{(b-a)^2}G\big((b-a)^2u\big)\right]\right| < \varepsilon,$$

for every $u \in \mathfrak{F}$.

Proof. Let $K_1 := 2(1+Mg_+)$ and $\varepsilon > 0$ be fixed. By assumption (h1) we infer the existence of $\bar{s} > 0$ such that

$$s > \bar{s} \quad \Rightarrow \quad \left(1 - \frac{\varepsilon}{K_1}\right)g_+ \leqslant \frac{G(s)}{s} \leqslant g_+.$$

For every (a, b) and for every $u \in \mathfrak{F}$ we can write

$$\int_0^1 \frac{G((b-a)^2u)}{(b-a)^2} = \int_{\{(b-a)^2u \leqslant \bar{s}\}} \frac{G((b-a)^2u)}{(b-a)^2u}u + \int_{\{(b-a)^2u > \bar{s}\}} \frac{G((b-a)^2u)}{(b-a)^2u}u. \tag{16}$$

As far as the first integral on the right hand side is concerned, since $s > 0$ implies $0 \leqslant G(s)/s \leqslant g_+$, it results

$$0 \leqslant \int_{\{(b-a)^2u \leqslant \bar{s}\}} \frac{G((b-a)^2u)}{(b-a)^2u}u \leqslant \int_{\{(b-a)^2u \leqslant \bar{s}\}} g_+u \leqslant \frac{g_+\bar{s}}{(b-a)^2} < \frac{\varepsilon}{K_1}, \tag{17}$$

whenever $b - a > L_2$ sufficiently large, for every $u \in \mathfrak{F}$. Note also that the same choice of L_2 gives

$$b - a > L_2 \quad \Rightarrow \quad 0 \leqslant g_+\left(\int_0^1 u - \int_{\{(b-a)^2u > \bar{s}\}} u\right) < \frac{\varepsilon}{K_1} \quad \forall u \in \mathfrak{F}.$$

Let us consider the second integral on the right hand side of (16). Our choice of $\bar{s}$ and the previous relation imply that, if $b - a > L_2$, then

$$\begin{aligned}
-\left(1 - \frac{\varepsilon}{K_1}\right)\frac{\varepsilon}{K_1} + g_+\left(1 - \frac{\varepsilon}{K_1}\right)\int_0^1 u \leqslant g_+\left(1 - \frac{\varepsilon}{K_1}\right) \int_{\{(b-a)^2u > \bar{s}\}} u \\
\leqslant \int_{\{(b-a)^2u > \bar{s}\}} \frac{G((b-a)^2u)}{(b-a)^2u}u \\
\leqslant g_+ \int_{\{(b-a)^2u > \bar{s}\}} u \leqslant g_+ \int_0^1 u,
\end{aligned}$$

for every $u \in \mathfrak{F}$. Due to the boundedness of the family $\mathfrak{F}$ in $L^1(0,1)$, it results

$$0 \leqslant g_+ \int_0^1 u - \int_{\{(b-a)^2 u > \bar{s}\}} \frac{G((b-a)^2 u)}{(b-a)^2 u} u \leqslant (1 + M g_+) \frac{\varepsilon}{K_1} = \frac{\varepsilon}{2}, \tag{18}$$

for every $u \in \mathfrak{F}$. Collecting together (16), (17) and (18), we obtain the first estimate of the thesis. To prove the second one, we can adapt the same argument because of assumption (h1). The third estimate follows easily. $\square$

Lemma 4.4. *Let $\mathfrak{F} \subset H_0^1(0,1)^+$ be such that*

$$\|u\|_{L^1(0,1)} \leqslant M \quad \forall u \in \mathfrak{F}.$$

For $\varepsilon > 0$, $\delta_1 > 0$ and for every $u \in \mathfrak{F}$, let us assume the existence of a simple function s_u such that

$$\|u - s_u\|_\infty < \varepsilon_1 \quad \text{and} \quad \delta(s_u) \geqslant \delta_1,$$

where $\delta(\cdot)$ is defined as in (10) and $\varepsilon_1 := \varepsilon/(M_1 + M + \|g\|_\infty + 1)$. Then there exists $L_3 > 0$, depending on ε, δ_1 but independent of $q \in \mathcal{P}$, such that, if $b - a > L_3$, then

$$\left| \int_0^1 \big(\widehat{q}_{(a,b)} - A(p)\big) u \right| < \varepsilon,$$

for every $u \in \mathfrak{F}$ and $q \in \mathcal{P}$.

Proof. Let $K_2 := (M_1 + M + \|g\|_\infty + 1)$, and let us assume that $(a,b) = (0,L)$ to ease the notation. It is straightforward to apply the following argument for a general $(a,b) \subset \mathbb{R}$. Let us consider, for $(c,d) \subset [0,1]$,

$$\int_c^d \widehat{q}_L(t)\,dt = \frac{1}{L} \int_{cL}^{dL} q(t)\,dt = \frac{d-c}{L(d-c)} \int_{cL}^{dL} q(t)\,dt.$$

For any $\varepsilon > 0$ sufficiently small, we consider the decomposition $q = q_{1,\varepsilon} + \dot{q}_{2,\varepsilon}$ given by Lemma 4.1. By the definition of $\mathcal{P}$, we know that

$$\sup_{t\in\mathbb{R}} \left| \frac{1}{T} \int_t^{t+T} q(\sigma)\,d\sigma - A(p) \right| \leqslant \sup_{t\in\mathbb{R}} \left(\frac{1}{T} \int_t^{t+T} \big|q_{1,\varepsilon}(\sigma) - A(p)\big|\,d\sigma + \left| \frac{1}{T} \int_t^{t+T} \dot{q}_{2,\varepsilon}(\sigma)\,d\sigma \right| \right)$$
$$< \frac{\varepsilon}{2} + \frac{2}{T}\|q_{2,\varepsilon}\|_\infty < \frac{\varepsilon}{2} + \frac{2}{T} M_\varepsilon < \varepsilon$$

whenever $T > \bar{T}(\varepsilon) := 4M_\varepsilon/\varepsilon$, independently of $q \in \mathcal{P}$. Therefore, if $(d-c)L > \bar{T}(\varepsilon/K_2)$, then

$$\left|\frac{1}{L(d-c)}\int_{Lc}^{Ld} q(t)\,dt - A(p)\right| < \frac{\varepsilon}{K_2} \quad \forall q \in \mathcal{P}.$$

Let us consider the family of simple functions $\{s_u\colon\ u \in \mathfrak{F}\}$. Let us set $L_3 := (1/\delta_1)\bar{T}(\varepsilon/K_2)$; for $s_u = \sum_{k=0}^{n-1} y_k \chi_{[t_k,t_{k+1})}$, we note that if $L > L_3$, then

$$(t_{k+1}-t_k)L \geqslant \delta_1 L_3 = \bar{T}\left(\frac{\varepsilon}{K_2}\right),$$

so that

$$\begin{aligned}\left|\int_0^1 \left(\widehat{q}_L - A(p)\right)s_u\right| &\leqslant \sum_{k=0}^{n-1} |y_k|(t_{k+1}-t_k)\left|\frac{1}{L(t_{k+1}-t_k)}\int_{Lt_k}^{Lt_{k+1}} q(\sigma)\,d\sigma - A(p)\right| \\ &< \frac{\varepsilon}{K_2}\int_0^1 |s_u| < \frac{\varepsilon}{K_2}(M+1),\end{aligned}$$

independently of $u \in \mathfrak{F}$ and on $q \in \mathcal{P}$, where for the last inequality we used the boundedness of $\mathfrak{F}$ in $L^1(0,1)$. Therefore, if $L \geqslant L_3$, then

$$\begin{aligned}\left|\int_0^1 \left(\widehat{q}_L - A(p)\right)u\right| &\leqslant \int_0^1 \left|\widehat{q}_L + A(p)\right||u-s_u| + \left|\int_0^1 \left(\widehat{q}_L - A(p)\right)s_u\right| \\ &< \left(\|q\|_\infty + \|g\|_\infty\right)\|u-s_u\|_\infty + \frac{\varepsilon}{K_2}(M+1) \\ &< \varepsilon,\end{aligned}$$

for every $u \in \mathfrak{F}$ and for every $q \in \mathcal{P}$ (for the reader's convenience, we recall that by definition $M_1 > \|q\|_\infty$ for every $q \in \mathcal{P}$). □

We are in a position to prove Proposition 4.2.

Proof of Proposition 4.2. Let us consider the family

$$\mathfrak{F} := \left\{\widehat{u}(\cdot; a, b; q)\colon\ (a,b) \subset \mathbb{R},\ q \in \mathcal{P}\right\} \cup \{w_{(g_+ - A(p))}\},$$

where we recall that $\widehat{u}(\cdot; a, b; q)$ is the minimizer of $\widehat{J}_{(a,b),q}$ (defined by (8)), and $w_{(g_+ - A(p))}$ has been defined by (12). In light of Lemmas 3.3 and 3.4, the family satisfies the assumptions of Lemmas 4.3 and 4.4.

Let $L_1 := \max\{L_2(\varepsilon/2), L_3(\varepsilon/2)\}$, where L_2 and L_3 have been defined in the quoted statements, and we recall that L_3 is independent of $q \in \mathcal{P}$. By definition, if $b - a > L_1$, then

$$\begin{aligned}\widehat{J}_{(a,b),q}\big(\widehat{u}(\cdot;a,b;q)\big) &> \int_0^1 \left[\frac{1}{2}\dot{\widehat{u}}^2(t;a,b;q) - \big(g_+ - A(p)\big)\widehat{u}(t;a,b;q)\right] dt - \varepsilon \\ &\geqslant \inf_{H_0^1(0,1)^+} J^\infty_{(g_+-A(p))} - \varepsilon = -\frac{(g_+ - A(p))^2}{24} - \varepsilon,\end{aligned}$$

for every $q \in \mathcal{P}$, where we recall that J_k^∞ has been defined as in (13) for any $k \in \mathbb{R}$. Moreover, by minimality,

$$\begin{aligned}\widehat{J}_{(a,b),q}\big(\widehat{u}(\cdot;a,b;q)\big) &\leqslant \widehat{J}_{(a,b),q}\big(w_{(g_+-A(p))}\big) \\ &< \int_0^1 \left[\frac{1}{2}\dot{w}^2_{(g_+-A(p))}(t) - \big(g_+ - A(p)\big)w_{(g_+-A(p))}(t)\right] dt + \varepsilon \\ &= \inf_{H_0^1(0,1)^+} J^\infty_{(g_+-A(p))} + \varepsilon = -\frac{(g_+ - A(p))^2}{24} + \varepsilon,\end{aligned}$$

whenever $b - a > L_1$. $\square$

Now we can come back on the time interval $[a,b]$: due to the explicit relations (7) and (8), we can summarize the previous results in the following statement.

Corollary 4.5. *For $0 < \varepsilon < (1 - A(p))^2/24$, let $L_1(\varepsilon)$ be defined as in Proposition 4.2. If $b - a > L_1(\varepsilon)$ then*

$$-\underline{\alpha}(b-a)^3 \leqslant \varphi^+(a,b;q) \leqslant -\overline{\alpha}(b-a)^3,$$

for every $q \in \mathcal{P}$, where $\underline{\alpha}$, $\overline{\alpha}$ are defined as in Eq. (15).

Remark 4. By definition, $L_1 \geqslant L_2, L_3$. Therefore, if $b - a > L_1$, Lemmas 4.3 and 4.4 hold true; in particular, we deduce that for every $0 < \varepsilon < \frac{1}{24}(1 - A(p))^2$, if $b - a > L_1(\varepsilon)$, then

$$\left| \int_a^b \big[g\big(u(t;a,b;q)\big)u(t;a,b;q) - G\big(u(t;a,b;q)\big)\big]\, dt \right| < \varepsilon(b-a)^3$$

for every $q \in \mathcal{P}$.

In the next statement and in the rest the symbol $\|\cdot\|$ denotes the Dirichlet H_0^1 norm on the considered interval, that is,

$$\|u\| = \left(\int_a^b \dot{u}^2(t)\, dt\right)^{1/2} \qquad \forall u \in H_0^1(a,b).$$

Corollary 4.6. *There exist $L_4 > 0$ and a positive constant $C_1 > 0$ such that, if $b - a \geqslant L_4$, then $\|u(\cdot; a, b; q)\| \geqslant C_1(b-a)^{3/2}$ and $\|u(\cdot; a, b; q)\|_\infty \geqslant C_1(b-a)^2$ for every $q \in \mathcal{P}$.*

Proof. Since the function $\lambda \mapsto J_{(a,b),q}(\lambda u(\cdot; a, b; q))$ reaches its minimum at $\lambda = 1$, it results

$$\int_a^b \left[\dot{u}^2(t; a, b; q) - g\big(u(t; a, b; q)\big)u(t; a, b; q) + q(t)u(t; a, b; q)\right] dt = 0.$$

We can solve this identity for the last term and substitute in the expression of $J_{(a,b),q}(u(\cdot; a, b; q))$:

$$J_{(a,b),q}\big(u(\cdot; a, b; q)\big) = -\int_a^b \frac{1}{2}\dot{u}^2(t; a, b; q)\, dt + \int_a^b \left[g\big(u(t; a, b; q)\big)u(t; a, b; q) - G\big(u(t; a, b; q)\big)\right] dt.$$

Given $\varepsilon > 0$ sufficiently small, if $b - a > L_1(\varepsilon)$, defined as in Proposition 4.2, then we have (we refer also to Corollary 4.5 and to Remark 4)

$$J_{(a,b),q}\big(u(\cdot; a, b; q)\big) > -\frac{1}{2}\big\|\dot{u}(\cdot; a, b; q)\big\|^2 - \varepsilon(b-a)^3 \quad \text{and}$$

$$J_{(a,b),q}\big(u(\cdot; a, b; q)\big) \leqslant \left(-\frac{(g_+ - A(p))^2}{24} + \varepsilon\right)(b-a)^3,$$

for every $q \in \mathcal{P}$, from which we deduce

$$\big\|\dot{u}(\cdot; a, b; q)\big\|^2 > \left(\frac{(g_+ - A(p))^2}{12} - 4\varepsilon\right)(b-a)^3 \quad \forall q \in \mathcal{P}.$$

We choose $\bar{\varepsilon} = (g_+ - A(p))^2/96$ and set $L_4 = L_1(\bar{\varepsilon})$. Hence

$$\big\|u(\cdot; a, b; q)\big\| \geqslant \frac{(g_+ - A(p))}{\sqrt{24}}(b-a)^{\frac{3}{2}} \quad \forall q \in \mathcal{P},$$

and

$$\begin{aligned} \frac{(g_+ - A(p))^2}{24}(b-a)^3 &\leqslant \int_a^b \dot{u}^2(t; a, b; q)\, dt \\ &= \int_a^b \left[g\big(u(t; a, b; q)\big)u(t; a, b; q) - q(t)u(t; a, b; q)\right] dt \\ &\leqslant \big(\|g\|_\infty + M_1\big)\big\|u(\cdot; a, b; q)\big\|_\infty (b-a), \end{aligned}$$

which gives the desired result for

$$C_1 := \frac{(g_+ - A(p))^2}{24(\|g\|_\infty + M_1)}. \qquad \square$$

Finally, we can prove that if $b-a$ is sufficiently large, then any minimizer $u(\cdot;a,b;q)$ with $q \in \mathcal{P}$ is an actual solution of the boundary problem (4).

Proposition 4.7 *(Existence). Let p satisfy assumption (h2), and let $\mathcal{P}$ be defined by (14). There exists $\tilde{L} \geqslant L_4$ such that, if $b-a \geqslant \tilde{L}$, then $u(t;a,b;q) > 0$ for every $t \in (a,b)$, $q \in \mathcal{P}$. Hence, $u(\cdot;a,b;q)$ is a solution of (4).*

Proof. For $q \in \mathcal{P}$, let

$$\big\{t \in (a,b)\colon\ u(t;a,b;q) > 0\big\} = \bigcup_{i\in I}(a_i,b_i),$$

where I is a family of indexes and $u(t;a,b;q) > 0$ for $t \in (a_i,b_i)$ (thus the (a_i,b_i) are disjoint intervals). By continuity, there exists $j \in I$ such that in (a_j,b_j) there exists a point τ of global maximum for $u(\cdot;a,b;q)$. By Corollary 4.6, we know that $u(\tau;a,b;q) \geqslant C_1(b-a)^2$ whenever $b-a \geqslant L_4$, for every $q \in \mathcal{P}$. Assume by contradiction that $(a_j,b_j) \neq (a,b)$; say, for instance, $a_j > a$. In order to obtain a contradiction, we consider separately the cases $A(p) > 0$ or $A(p) \leqslant 0$.

The case $A(p) > 0$**.** We choose $0 < \varepsilon < \min\{C_1, 2A(p)/3\}$, where we recall that C_1 has been defined as in Corollary 4.6, and we consider the decomposition of Lemma 4.1 for the forcing term q. By the monotonicity of g, assumption (h1), there exists $s_\varepsilon := g^{-1}(A(p) - 3\varepsilon/2)$. Assuming $b-a$ sufficiently large in such a way that $u(\tau) > s_\varepsilon$ we can introduce

$$\begin{aligned} T &:= \inf\big\{\bar{t} > a_j\colon\ u(t;a,b;q) > s_\varepsilon \text{ for every } t \in (\bar{t},\tau)\big\},\\ a' &:= \inf\big\{\bar{t} \leqslant T\colon\ \dot{u}(t;a,b;q) \geqslant 0 \text{ for every } t \in [\bar{t},T]\big\} \end{aligned}$$

(in particular, if $\dot{u}(T;a,b;q) = 0$ then $a' := T$). Note that, by definition,

$$\begin{cases} 0 \leqslant u(t;a,b;q) \leqslant s_\varepsilon & \text{if } t \in [a',T],\\ u(t;a,b;q) \geqslant s_\varepsilon & \text{if } t \in [T,\tau]. \end{cases} \tag{19}$$

As $u(\cdot;a,b;q) \in \mathcal{C}^1(a,b)$, $a' \geqslant a_j > a$ necessarily implies $\dot{u}(a';a,b;q) = 0$. As a consequence, if we reach a contradiction, we deduce that both $a' = a_j = a$ and $\dot{u}(a;a,b;q) > 0$.

Step 1) *There exists $C_2 > 0$ independent of $q \in \mathcal{P}$ such that $T - a' \leqslant C_2$.*

By the monotonicity of g and (19), we deduce that, for every $t \in (a',T)$,

$$\begin{aligned} \ddot{u}(t;a,b;q) &= -g\big(u(t;a,b;q)\big) + q_{1,\varepsilon}(t) + \dot{q}_{2,\varepsilon}(t)\\ &\geqslant -g(s_\varepsilon) + A(p) - \frac{\varepsilon}{2} + \dot{q}_{2,\varepsilon}(t) = \varepsilon + \dot{q}_{2,\varepsilon}(t). \end{aligned}$$

By integrating twice in (a',t), and using the fact that $\dot{u}(a';a,b;q) = 0$, we obtain

$$s_\varepsilon \geqslant u(T;a,b;q) - u\big(a';a,b;q\big) \geqslant \frac{\varepsilon}{2}\big(T-a'\big)^2 - 2M_\varepsilon\big(T-a'\big),$$

which provides the desired estimate.

Step 2) *There exists $C_3 > 0$ independent of $q \in \mathcal{P}$ such that $\dot{u}(T;a,b;q) \leqslant C_3$.*

As $g(s) \geqslant 0$ for $s \geqslant 0$, we see that, for every $t \in (a', T)$,

$$\begin{aligned}\ddot{u}(t;a,b;q) &= -g\big(u(t;a,b;q)\big) + q_{1,\varepsilon}(t) + \dot{q}_{2,\varepsilon}(t)\\ &\leqslant A(p) + \frac{\varepsilon}{2} + \dot{q}_{2,\varepsilon}(t).\end{aligned}$$

By integrating in (a', T), we deduce that

$$\dot{u}(T;a,b;q) \leqslant \left(A(p) + \frac{\varepsilon}{2}\right)\big(T-a'\big) + 2M_\varepsilon \leqslant C_3,$$

where we used the first step and the fact that $\dot{u}(a';a,b;q) = 0$.

Step 3) *Conclusion of the proof in the case $A(p) > 0$.*

By the monotonicity of g (assumption (h1)) and (19), we deduce that, for every $t \in (T, \tau)$,

$$\begin{aligned}\ddot{u}(t;a,b;q) &= -g\big(u(t;a,b;q)\big) + q_{1,\varepsilon}(t) + \dot{q}_{2,\varepsilon}(t)\\ &\leqslant -g(s_\varepsilon) + A(p) + \frac{\varepsilon}{2} + \dot{q}_{2,\varepsilon}(t) = 2\varepsilon + \dot{q}_{2,\varepsilon}(t).\end{aligned}$$

By integrating twice in (T, t) and evaluating in τ, we deduce

$$\begin{aligned}u(\tau;a,b;q) &\leqslant \varepsilon(b-a)^2 + \big(\dot{u}(T;a,b;q) + 2M_\varepsilon\big)(b-a) + u(T;a,b;q)\\ &\leqslant \varepsilon(b-a)^2 + (C_3 + 2M_\varepsilon)(b-a) + s_\varepsilon,\end{aligned}$$

where we used the result of the previous step and the definition of T. The choice of $\varepsilon < C_1$ gives a contradiction with Corollary 4.6 for $b-a$ sufficiently large (greater than a constant $\tilde{L}$ depending only on $\mathcal{P}$ and not on the particular choice of q).

The case $A(p) \leqslant 0$. We choose $0 < \varepsilon < C_1$, where we recall that C_1 has been defined as in Corollary 4.6, and consider the decomposition of Lemma 4.1 for the forcing term q. For every $t \in (a_j, b_j)$ we have

$$\ddot{u}(t;a,b;q) = -g\big(u(t;a,b;q)\big) + q_{1,\varepsilon}(t) + \dot{q}_{2,\varepsilon}(t) \leqslant \frac{\varepsilon}{2} + \dot{q}_{2,\varepsilon}(t),$$

where we used the fact that $g(s) \geqslant 0$ for $s \geqslant 0$. By integrating twice in (a_j, t) with $t \in (a_j, b_j)$, and evaluating in τ, we obtain

$$u(\tau;a,b;q) \leqslant \varepsilon(b-a)^2 + 2M_\varepsilon(b-a).$$

Having chosen $\varepsilon < C_1$, this immediately contradicts Corollary 4.6 for $b-a$ sufficiently large. □

For the results of the next sections it is important to prove the uniqueness of the minimizer of the functional $J_{(a,b),q}$ with $q \in \mathcal{P}$. In light of the previous and the next statements, this uniqueness is guaranteed provided $b - a > \tilde{L}$. In the following proposition the forcing term p is fixed; therefore, we will use the simplified notation of the previous section.

Proposition 4.8 *(Uniqueness). Let u and v be functions in $\mathcal{C}^2(a,b) \cap H_0^1(a,b)$ such that $u > 0$ and $v > 0$ in (a,b). Assume that*

$$J_{(a,b)}(u) = J_{(a,b)}(v) = \varphi^+(a,b).$$

Then $u \equiv v$ in $[a,b]$.

Proof. Let us consider the function

$$\Phi(\lambda) := J_{(a,b)}\big((1-\lambda)u + \lambda v\big).$$

We note that $\Phi \in \mathcal{C}^1(\mathbb{R})$ and

$$\Phi'(\lambda) = dJ_{(a,b)}\big((1-\lambda)u + \lambda v\big)[v-u].$$

As $\Phi(0) = \Phi(1)$, there exists $\bar{\lambda} \in (0,1)$ such that $\Phi'(\bar{\lambda}) = 0$, that is,

$$\int_a^b \big[(1-\bar{\lambda})\dot{u} + \bar{\lambda}\dot{v}\big](\dot{v} - \dot{u}) - g\big((1-\bar{\lambda})u + \bar{\lambda}v\big)(v-u) + p(v-u) = 0. \tag{20}$$

Also, by minimality we know that $\Phi'(0) = \Phi'(1) = 0$, that is

$$\int_a^b \dot{u}(\dot{v} - \dot{u}) - g(u)(v-u) + p(v-u) = 0, \tag{21}$$

$$\int_a^b \dot{v}(\dot{v} - \dot{u}) - g(v)(v-u) + p(v-u) = 0. \tag{22}$$

If we consider (20) and subtract $(1-\bar{\lambda})$ times (21) and $\bar{\lambda}$ times (22), we obtain

$$\int_a^b \big[(1-\bar{\lambda})g(u) + \bar{\lambda}g(v) - g\big((1-\bar{\lambda})u + \lambda v\big)\big](v-u) = 0. \tag{23}$$

We claim that

$$\text{either} \quad u \equiv v \quad \text{or} \quad \text{the function } v - u \text{ changes sign in } (a,b). \tag{24}$$

Indeed, assume $u \not\equiv v$ and, w.l.o.g., $v \geqslant u$ in (a,b). The set $A := \{t \in (a,b)\colon\ v(t) > u(t)\}$ is not empty and has positive measure. Hence, by (23) and the strict concavity of g in $(0,+\infty)$, assumption (h1), we deduce that

$$0 = \int_a^b \big[(1-\bar{\lambda})g(u) + \bar{\lambda}g(v) - g\big((1-\bar{\lambda})u + \lambda v\big)\big](v-u)$$
$$= \int_A \big[(1-\bar{\lambda})g(u) + \bar{\lambda}g(v) - g\big((1-\bar{\lambda})u + \lambda v\big)\big](v-u) < 0,$$

a contradiction. This proves the claim (24), so that it remains to show that $v-u$ cannot change sign in (a,b). By contradiction again, assume that $v-u$ changes sign in (a,b), so that in particular there exists τ in (a,b) such that $u(\tau) = v(\tau)$. Say, for instance, that

$$\int_a^\tau \left(\frac{1}{2}\dot{u}^2 - G(u) + pu\right) \leqslant \int_a^\tau \left(\frac{1}{2}\dot{v}^2 - G(v) + pv\right);$$

necessarily it results

$$\int_\tau^b \left(\frac{1}{2}\dot{u}^2 - G(u) + pu\right) \geqslant \int_\tau^b \left(\frac{1}{2}\dot{v}^2 - G(v) + pv\right).$$

Let

$$\tilde{u}(t) := \begin{cases} u(t) & \text{if } t \in (a,\tau), \\ v(t) & \text{if } t \in [\tau,b). \end{cases}$$

By definition $\tilde{u} \in H_0^1(a,b)^+$, $\tilde{u} > 0$ in (a,b) and $J_{(a,b)}(\tilde{u}) \leqslant J_{(a,b)}(u) = \varphi^+(a,b)$, that is, $\tilde{u}$ is a minimizer of $J_{(a,b)}$ in $H_0^1(a,b)^+$ which is strictly positive in (a,b); hence, it solves the boundary value problem (4) and has to be of class $\mathcal{C}^2(a,b)$. This implies that $\dot{u}(\tau) = \dot{v}(\tau)$, and recalling that $u(\tau) = v(\tau)$, we can apply the uniqueness theorem for the initial value problems, proving that $u \equiv v$ in (a,b). $\square$

Let $p \in \mathcal{P}$, and let $\mathcal{P}$ be defined by (14). Collecting together the results of Propositions 4.7 and 4.8, we can conclude that there exists $\tilde{L} > 0$ such that for every $(a,b) \subset \mathbb{R}$ with $b-a \geqslant \tilde{L}$ and for every $q \in \mathcal{P}$ there exists a unique minimizer $u(\cdot;a,b;q)$ of the functional $J_{(a,b),q}$ in $H_0^1(a,b)^+$, which is strictly positive in (a,b) and hence solves problem (4) with forcing term q. It is then possible to define a map which associates to each triple (a,b,q), with $b-a \geqslant \tilde{L}$ and $q \in \mathcal{P}$, the unique minimizer $u(\cdot;a,b;q)$. It is not difficult to adapt the proof of Lemma 3.4 in [8], proving that this map is continuous.

Lemma 4.9. *Let p satisfy (h2), and let $\mathcal{P}$ be defined by (14). Let A and B be fixed and let*

$$\mathcal{I} := \big\{(t,a,b) \in \mathbb{R}^3\colon\ b-a > \tilde{L},\ A < a \leqslant t \leqslant b < B\big\},$$

where $\tilde{L}$ has been defined as in Proposition 4.7. Let us consider the metric space $\mathcal{P}$ endowed with the distance $d(q_1, q_2) = \|q_1 - q_2\|_{L^2(A,B)}$. The map

$$(t, a, b, q) \in \overline{\mathcal{I}} \times \mathcal{P} \mapsto \big(u(t; a, b; q), \dot{u}(t; a, b; q)\big) \in \mathbb{R}^2$$

is continuous.

5. Non-degeneracy of positive minimizers

Assume that u solves (4) in (a, b); we can consider the variational equation

$$\begin{cases} \ddot{\psi}(t) + g'\big(u(t)\big)\psi(t) = 0, & t \in (a, b), \\ \psi(a) = 0 = \psi(b). \end{cases} \tag{25}$$

Definition 1. We say that u is *non-degenerate* as solution of (4) if problem (25) has only the trivial solution $\psi \equiv 0$ in $H^2(a, b) \cap H_0^1(a, b)$.

The main result of this section is the following.

Proposition 5.1. *Let p satisfy (h2), $\mathcal{P}$ be defined by (14), and $\tilde{L}$ be defined as in Proposition 4.7, and let us assume that $b - a \geqslant \tilde{L}$. The function $u(\cdot; a, b; p)$ is non-degenerate as solution of the boundary value problem (4).*

For the proof, we will use some known results in singularity theory, which we recall here and for which we refer to Section 3.2 of the book by Ambrosetti and Prodi [2].

Definition 2. Let $\Phi : \Omega \subset E \to F$ be of class $\mathcal{C}^2(\Omega)$, where Ω is open, E and F are Banach spaces and $u_0 \in \Omega$. We say that u_0 is *singular* if $d\Phi(u_0)$ is not invertible. It is *ordinary singular* if it is singular and

(i) $\mathrm{Ker}(d\Phi(u_0))$ is one-dimensional:

$$\mathrm{Ker}\big(d\Phi(u_0)\big) = \mathbb{R}\psi_0 \quad \text{for some } \psi_0 \in E \setminus \{0\};$$

$\mathrm{Range}(d\Phi(u_0))$ is closed and has codimension 1:

$$\mathrm{Range}\big(d\Phi(u_0)\big) = \big\{q \in F\colon \langle \gamma_0, q\rangle = 0\big\} \quad \text{with } \gamma_0 \in F^* \setminus \{0\}.$$

(ii) $\langle \gamma_0, d^2\Phi(u_0)[\psi_0, \psi_0]\rangle \neq 0$.

Theorem 5.2 *(Ambrosetti–Prodi). Let u_0 be an ordinary singular point for Φ, and, say,*

$$\big\langle \gamma_0, d^2\Phi(u_0)[\psi_0, \psi_0]\big\rangle > 0;$$

let $q_0 = \Phi(u_0)$, and let $q \in F$ be such that $\langle \gamma_0, q\rangle > 0$. Then there exist a neighborhood U of u_0 in E and a positive number ε^* such that the equation

$$\Phi(u) = q_0 + \varepsilon q, \quad u \in U,$$

has exactly two solutions for $0 < \varepsilon < \varepsilon^*$ *and no solution for* $-\varepsilon^* < \varepsilon < 0$.

We are ready to show that $u(\cdot; a, b; p)$ is non-degenerate.

Proof of Proposition 5.1. Let

$$X := H^2(a, b) \cap H_0^1(a, b), \qquad \|u\|_X := \|\ddot{u}\|_2, \qquad Y := L^2(a, b).$$

We introduce the map $\mathcal{F} : X \to Y$ defined by

$$\mathcal{F}(u) = -\ddot{u} - g(u).$$

Under assumption (h1), it is immediate to see that $\mathcal{F} \in \mathcal{C}^2(X, Y)$ and

$$d\mathcal{F}(u)\psi = -\ddot{\psi} - g'(u)\psi, \qquad d^2\mathcal{F}(u)[\psi_1, \psi_2] = -g''(u)\psi_1\psi_2.$$

By the Fredholm alternative, $u(\cdot; a, b; p)$ is degenerate as solution of (4) if and only if it is singular for $\mathcal{F}$. So, let us assume by contradiction that $u(\cdot; a, b; p)$ is degenerate as solution of (4).

Step 1) $u(\cdot; a, b; p)$ *is ordinary singular for* $\mathcal{F}$.

It is possible to argue as in the first part of the proof of Proposition 4.1 in [8].

Step 2) *Conclusion of the proof.*

By definition, $\mathcal{F}(u(t; a, b; p)) = p$. We can choose $q \in Y$ such that

- $\int_a^b q\psi_0 > 0$;
- $p + \varepsilon q \in \mathcal{P}$ for every $|\varepsilon|$ sufficiently small.

Indeed, let $\phi \in \mathcal{C}_c^\infty(a, b) \setminus \{0\}$ be negative. Taking $q = \ddot{\phi}$, we obtain

$$\int_a^b \ddot{\phi}\psi_0 = \int_a^b \phi\ddot{\psi}_0 = -\int_a^b g'\big(u(t; a, b; q)\big)\phi\psi_0 > 0,$$

because $-g' < 0$ in $\mathbb{R}$ and $\psi_0 > 0$ in (a, b). Also, it is easy to check that the function $p + \varepsilon q \in \mathcal{P}$ whenever $|\varepsilon|$ is sufficiently small (see Remark 2). So, by definition, $\mathcal{F}(u(\cdot; a, b; p + \varepsilon q)) = p + \varepsilon q$, and by Lemma 4.9 it results $u(\cdot; a, b; p + \varepsilon q) \to u(\cdot; a, b; p)$ in X as $\varepsilon \to 0^-$. On the other hand, by Theorem 5.2 there exists a neighborhood U of $u(\cdot; a, b; p)$ in X such that the equation $\mathcal{F}(u) = p + \varepsilon q$ has no solution in U for $\varepsilon < 0$ sufficiently small, a contradiction. □

Remark 5. The second part of the proof of Proposition 5.1 is inspired by the second part of the proof of Proposition 4.1 in [8], but it contains a subtle difference: in the present situation, we have to consider variations of type $p + \varepsilon q$ which belong to $\mathcal{P}$, otherwise we cannot ensure that the corresponding minimizer $u(\cdot; a, b; p + \varepsilon q)$ solves (4) with forcing term $p + \varepsilon q$, see Proposition 4.7.

As an easy consequence of the Fredholm alternative, we obtain also the following corollary.

Corollary 5.3. *Let p satisfy (h2), let $\mathcal{P}$ be defined by (14), let $\tilde{L}$ be defined as in Proposition 4.7, and assume that $b-a \geqslant \tilde{L}$. The boundary value problem*

$$\begin{cases} \ddot{\psi}(t) + g'\big(u(t;a,b;q)\big)\psi(t) = 0, & t \in (a,b), \\ \psi(a) = \psi_a, \quad \psi(b) = \psi_b, \end{cases}$$

has a unique solution for every $q \in \mathcal{P}$.

6. Differentiability of $\varphi^+(a,b)$

In this section we will show that $\varphi^+(a,b) = J_{(a,b),p}(u(\cdot;a,b;p))$ is differentiable as function of a and b.

Lemma 6.1. *Let p satisfy (h2), and let $\mathcal{P}$ be defined by (14). Let A and B be fixed and let*

$$\mathcal{I} := \big\{(t,a,b) \in \mathbb{R}^3 \colon b-a > \tilde{L},\ A < a \leqslant t \leqslant b < B\big\},$$

where $\tilde{L}$ has been defined as in Proposition 4.7. If $q \in \mathcal{P}$ is of class $\mathcal{C}^1$, then the map

$$(t,a,b) \in \mathcal{I} \mapsto \big(u(t;a,b;q), \dot{u}(t;a,b;q)\big) \in \mathbb{R}^2$$

is of class $\mathcal{C}^1$, too. More precisely,

$$\begin{aligned} \frac{\partial u}{\partial a}(t;a,b;q) &= \xi_1(t), & \frac{\partial \dot{u}}{\partial a}(t;a,b;q) &= \dot{\xi}_1(t), \\ \frac{\partial u}{\partial b}(t;a,b;q) &= \xi_2(t), & \frac{\partial \dot{u}}{\partial b}(t;a,b;q) &= \dot{\xi}_2(t), \end{aligned}$$

where ξ_1 and ξ_2 are the solutions (unique by Corollary 5.3) of

$$\ddot{\xi}(t) + g'\big(u(t;a,b;q)\big)\xi(t) = 0$$

with the boundary conditions

$$\begin{cases} \xi_1(a) = -\dot{u}\big(a^+;a,b;q\big), \\ \xi_1(b) = 0, \end{cases} \quad \text{or} \quad \begin{cases} \xi_2(a) = 0, \\ \xi_2(b) = -\dot{u}\big(b^-;a,b;q\big), \end{cases}$$

respectively.

Proof. In light of the results of the previous sections, it is not difficult to adapt the proof of Lemma 5.1 in [8]. □

Proposition 6.2. *For every p satisfying (h2), the function $\varphi^+(a,b) = \varphi^+(a,b;p)$ is of class $\mathcal{C}^1$ with respect to a and b in $\{b-a > \tilde{L}\}$, with derivatives*

$$\frac{\partial \varphi^+}{\partial a}(a,b) = \frac{1}{2}\dot{u}^2\big(a^+;a,b;p\big) \quad \text{and} \quad \frac{\partial \varphi^+}{\partial b}(a,b) = -\frac{1}{2}\dot{u}^2\big(b^-;a,b;p\big).$$

Proof. If $p \in \mathcal{C}^1(\mathbb{R})$ then we can apply Lemma 6.1, obtaining that $\varphi^+(a,b) = J_{(a,b),p}(u(\cdot;a,b;p))$ is differentiable. In such a case, the expressions of its derivatives follow by direct computation. In the general case, we claim that

$$\text{there exists } (q_n) \subset \mathcal{P} \cap \mathcal{C}^1(\mathbb{R}) \quad \text{such that} \quad q_n \to p \quad \text{in } L^2(A,B). \tag{26}$$

This is not straightforward, since $\mathcal{P}$ is defined as in (14). Let $\varepsilon_n \to 0$ as $n \to \infty$, and let us consider the decomposition

$$p = p_{1,\varepsilon_n} + \dot{p}_{2,\varepsilon_n}$$

given by Lemma 4.1. For any fixed n, we consider

$$q_{n,m} = A(p) + \frac{d}{dt}(\rho_m * p_{2,\varepsilon_n}) = A(p) + \rho_m * \dot{p}_{2,\varepsilon_n},$$

where (ρ_m) is a family of mollifiers, $*$ denotes the usual product of convolution, and the last identity follows from the fact that $p_{2,\varepsilon} \in \mathcal{C}^1(\mathbb{R})$. It is not difficult to check that $q_{n,m} \in \mathcal{P} \cap \mathcal{C}^1(\mathbb{R})$ for any m,n, and that for any n there exists m_n sufficiently large such that

$$\|q_{n,m_n} - p\|_{L^2(A,B)} < \varepsilon_n.$$

Hence, the sequence (q_{n,m_n}) has the desired properties, and claim (26) follows.

We introduce $\varphi_n(a,b) := \varphi^+(a,b;q_n)$ and $\varphi(a,b) := \varphi^+(a,b;p)$, and observe that, thanks to the previous step, each φ_n is of class $\mathcal{C}^1(\mathbb{R})$. Let $\Delta := \{(a,b)\colon\ b-a > \tilde{L},\ A < a < b < B\}$. We claim that

$$\varphi_n \to \varphi \quad \text{uniformly for } (a,b) \in \overline{\Delta}. \tag{27}$$

If not,

$$\sup_{(a,b)\in\overline{\Delta}} \big|\varphi_n(a,b) - \varphi(a,b)\big| = \sup_{(a,b)\in\overline{\Delta}} \big|\varphi^+(a,b;q_n) - \varphi^+(a,b;p)\big| = c_n \geqslant \bar{c} > 0.$$

By Lemma 4.9 and the continuity of $J_{(a,b),p}(u)$ as function of (u,a,b,p), the function φ^+ is continuous in the three variables, so that by compactness for every n the supremum is achieved by $(a_n,b_n) \in \overline{\Delta}$. Therefore, if (27) does not hold, then

$$\big|\varphi^+(a_n,b_n;q_n) - \varphi^+(a_n,b_n;p)\big| \geqslant \bar{c}$$

for any n. Since, up to subsequences, both a_n and b_n converge, this contradicts the continuity of φ^+.

With a similar argument we see also that $\dot{u}(\tau; a, b; q_n) \to \dot{u}(\tau; a, b; p)$ for $\tau = a, b$, uniformly in $\overline{\Delta}$, so that

$$\frac{\partial \varphi_n}{\partial a}(a, b) \to \frac{1}{2}\dot{u}^2\big(a^+; a, b; p\big) \quad \text{and} \quad \frac{\partial \varphi_n}{\partial b}(a, b) \to -\frac{1}{2}\dot{u}^2\big(b^-; a, b; p\big),$$

uniformly in $\overline{\Delta}$. The convergence of (φ_n) and of the sequences of the derivatives reveals that φ is of class $\mathcal{C}^1$ in Δ, and the thesis follows. □

7. Sign-changing solutions

In this section we complete the proof of Theorem 2.1. Firstly, we prove the existence of sign-changing solutions of (1) in bounded (sufficiently large) intervals; then, by an exhaustion procedure, we pass to the whole real line. To do this, we juxtapose positive and negative solutions on adjacent intervals, the latter existing and satisfying analogous properties of the former ones, as enlightened in Remark 1. To distinguish between positive and negative solutions, and since the forcing term p is now fixed, we change our notations accordingly, denoting such solutions as $u_\pm(\cdot; a, b)$. Resuming, we have the following result.

Proposition 7.1. *For every $\varepsilon > 0$ there exists $L > 0$ such that, if $b - a \geqslant L$, then the value $\varphi^\pm(a, b)$ is achieved by a unique $u_\pm(\cdot; a, b) \in H_0^1(a, b)$, which is strictly positive/negative and solves Eq. (1) in (a, b). Moreover,*

$$\begin{aligned}
&\big\|u_\pm(\cdot; a, b)\big\| \leqslant \big(\|g\|_\infty + \|p\|_\infty\big)(b - a)^{\frac{3}{2}}, \\
&-\underline{\alpha}(b - a)^3 \leqslant \varphi^+(a, b) \leqslant -\overline{\alpha}(b - a)^3, \\
&-\underline{\beta}(b - a)^3 \leqslant \varphi^-(a, b) \leqslant -\overline{\beta}(b - a)^3,
\end{aligned}$$

where $\underline{\alpha}, \overline{\alpha}$ have been defined as in (15) and

$$\underline{\beta} := \frac{(-g_- + A(p))^2}{24} + \varepsilon \quad \textit{and} \quad \overline{\beta} := \frac{(-g_- + A(p))^2}{24} - \varepsilon.$$

Proof. The proposition directly follows from Proposition 4.7, Lemma 3.3, Corollary 4.5 and Remark 1. □

By assumption (h2), there are two possibilities:

$$\text{either} \quad g_+ - A(p) = -g_- + A(p) \quad \text{or} \quad g_+ - A(p) \neq -g_- + A(p).$$

In the former case, we observe that for a given ε it results $\underline{\alpha} = \underline{\beta}$ and $\overline{\alpha} = \overline{\beta}$. Otherwise, it is possible to choose ε sufficiently small in such a way that

$$\text{either} \quad \underline{\alpha} < \overline{\beta} \quad \text{or} \quad \underline{\beta} < \overline{\alpha}.$$

To fix the ideas, in the following we consider the case

$$\overline{\beta} < \underline{\beta} < \overline{\alpha} < \underline{\alpha}. \tag{28}$$

The reader can easily adapt the arguments below in order to cover also the other situations (actually, if $g_+ - A(p) = -g_- + A(p)$, the problem is considerably simplified).

Firstly, we start by choosing $\varepsilon > 0$ sufficiently small in Proposition 7.1 in such a way that

$$\frac{\underline{\beta}}{(1+\sqrt{\underline{\beta}/\underline{\alpha}})^2} < \overline{\beta}; \tag{29}$$

by definition, one can easily check that this choice is possible.

Remark 6. Let $\nu := \underline{\beta}/\underline{\alpha}$. It is useful to observe that Eq. (29) implies that

$$\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 + \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 - \overline{\beta} < 0,$$
$$\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 + \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 - \overline{\alpha} < 0.$$

First of all, by (28) we immediately see that the second of these relations is automatically satisfied provided the first one holds. And for the first one it is sufficient to note that

$$\begin{aligned}\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 + \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 &= \underline{\alpha}\left(\frac{1}{1+\sqrt{\nu}}\right)^3\left[(\sqrt{\nu})^3 + \nu\right] \\ &= \frac{\underline{\alpha}\nu}{(1+\sqrt{\nu})^2} = \frac{\underline{\beta}}{(1+\sqrt{\underline{\beta}/\underline{\alpha}})^2}.\end{aligned}$$

Let $(A, B) \subset \mathbb{R}$ and $k \in \mathbb{N}$ be such that $(k+1)L \leqslant B - A$; hence, it is possible to divide the interval (A, B) in $k+1$ sub-intervals, in such a way that each of them is larger than L. We define the set of admissible partitions of (A, B) in $(k+1)$ sub-intervals as

$$\mathcal{B}_k := \left\{(t_1, \dots, t_k) \in \mathbb{R}^k\colon\ A =: t_0 \leqslant t_1 \leqslant \dots \leqslant t_k \leqslant t_{k+1} := B,\ t_{i+1} - t_i \geqslant L\right\};$$

also, we introduce the function $\psi : \mathcal{B}_k \to \mathbb{R}$ defined by

$$\psi(t_1, \dots, t_k) := \sum_{i=0}^{k} \varphi^{\sigma(i)}(t_i, t_{i+1}), \quad \text{where } \sigma(i) = \begin{cases} + & \text{if } i \text{ is even,} \\ - & \text{if } i \text{ is odd.} \end{cases} \tag{30}$$

We consider the maximization problem

$$c_k(A, B) := \sup\left\{\psi(t_1, \dots, t_k)\colon\ (t_1, \dots, t_k) \in \mathcal{B}_k\right\}. \tag{31}$$

Remark 7. It is possible to consider also the maximization problem for the function having opposite $\sigma(i)$. The situation is essentially the same.

Lemma 7.2. *The value $c_k(A, B)$ is achieved by a partition $(\bar{t}_1, \dots, \bar{t}_k) \in \mathcal{B}_k$.*

Proof. This follows from the continuity of $\varphi^{\sigma(i)}$ (in fact $\varphi^{\sigma(i)}$ is differentiable, Proposition 6.2), and from the compactness of $\mathcal{B}_k$. $\square$

To each interval $(\bar{t}_i, \bar{t}_{i+1})$ we associate

$$u_i := u_{\sigma(i)}(\cdot; \bar{t}_i, \bar{t}_{i+1}).$$

In this way, it is defined on the whole $[A, B]$ a function

$$u_{(A,B),k}(t) := u_i(t) \quad \text{if } t \in [\bar{t}_i, \bar{t}_{i+1}], \tag{32}$$

which is a solution of (1) in $(A, B) \setminus \{\bar{t}_1, \dots, \bar{t}_k\}$, and has exactly k zeros in (A, B). If we show that it is differentiable in each $\bar{t}_i$, then $u_{(A,B),k}$ will be a solution in the whole (A, B). To prove the smoothness of $u_{(A,B),k}$, we wish to exploit the knowledge of the explicit expression of the derivatives of $\varphi^{\sigma(i)}$, given in Proposition 6.2. Having this in mind, we observe that, if $(\bar{t}_1, \dots, \bar{t}_k)$ is an inner point of $\mathcal{B}_k$, then by maximality it results $\nabla\psi(\bar{t}_1, \dots, \bar{t}_k) = 0$, where the partial derivatives of ψ can be expressed in terms of the partial derivatives of $\varphi^{\sigma(i)}$. Therefore, the next step consists in the proof of the following lemma.

Lemma 7.3. *There exists H, depending only on L and on p, such that for any $(A, B) \subset \mathbb{R}$, $k \in \mathbb{N}$ with*

$$B - A \geqslant H(k + 1),$$

the corresponding maximizing partition $(\bar{t}_1, \dots, \bar{t}_k) \in \mathcal{B}_k$ is an inner point of $\mathcal{B}_k$, that is, $\bar{t}_{i+1} - \bar{t}_i > L$ for every i.

We need two intermediate results. The first one says that the ratio between two adjacent subintervals of a maximizing partition can be controlled by means of a positive constant depending only on L and on p.

Lemma 7.4. *Let $(\bar{t}_1, \dots, \bar{t}_k) \in \mathcal{B}_k$ be a maximizing partition for (31). There exists $\bar{h} \geqslant 1$, depending only on L and on p, such that*

$$\frac{1}{\bar{h}}(\bar{t}_i - \bar{t}_{i-1}) \leqslant \bar{t}_{i+1} - \bar{t}_i \leqslant \bar{h}(\bar{t}_i - \bar{t}_{i+1})$$

for every $i = 1, \dots, k$.

Proof. For an arbitrary i, let $\lambda = \bar{t}_i - \bar{t}_{i-1}$ and $h\lambda = \bar{t}_{i+1} - \bar{t}_i$. We wish to show that h is bounded from below and from above by two positive constants depending only on L and on p. Let $\nu := \underline{\beta}/\underline{\alpha}$, which belongs to $(0, 1)$ by (28). If both λ and $h\lambda$ are smaller than or equal to $L/\sqrt{\nu}$, then $\sqrt{\nu} \leqslant h \leqslant 1/\sqrt{\nu}$. Otherwise, at least one between λ and $h\lambda$ is greater then $L/\sqrt{\nu}$, so that

$$(1 + h)\lambda > \left(1 + \frac{1}{\sqrt{\nu}}\right)L. \tag{33}$$

Firstly, let us consider the case $\sigma(i-1) = +$, that is, $i-1$ is even. Let

$$s := \bar{t}_{i-1} + \frac{\sqrt{\nu}}{1+\sqrt{\nu}}(\bar{t}_{i+1} - \bar{t}_{i-1}) \in (\bar{t}_{i-1}, \bar{t}_{i+1}).$$

We consider the variation of $(\bar{t}_1, \dots, \bar{t}_k)$ obtained replacing $\bar{t}_i$ with s. This is an admissible partition in $\mathcal{B}_k$, as by (33) we have

$$\begin{aligned} s - \bar{t}_{i-1} &= \frac{\sqrt{\nu}}{1+\sqrt{\nu}}(1+h)\lambda > \frac{\sqrt{\nu}}{1+\sqrt{\nu}}\left(1 + \frac{1}{\sqrt{\nu}}\right)L = L, \\ \bar{t}_{i+1} - s &= \frac{1}{1+\sqrt{\nu}}(1+h)\lambda > \frac{1}{1+\sqrt{\nu}}\left(1 + \frac{1}{\sqrt{\nu}}\right)L > L. \end{aligned}$$

The variational characterization of $(\bar{t}_1, \dots, \bar{t}_k)$ implies that

$$\psi(\bar{t}_1, \dots, \bar{t}_{i-1}, s, \bar{t}_{i+1}, \dots, \bar{t}_k) \leqslant \psi(\bar{t}_1, \dots, \bar{t}_{i-1}, \bar{t}_i, \bar{t}_{i+1}, \dots, \bar{t}_k);$$

by definition, this means

$$\varphi^{\sigma(i-1)}(\bar{t}_{i-1}, s) + \varphi^{\sigma(i)}(s, \bar{t}_{i+1}) \leqslant \varphi^{\sigma(i-1)}(\bar{t}_{i-1}, \bar{t}_i) + \varphi^{\sigma(i)}(\bar{t}_i, \bar{t}_{i+1}).$$

Therefore, recalling that we are considering the case $\sigma(i-1) = +$, by Proposition 7.1 we deduce

$$-\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 (1+h)^3\lambda^3 - \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 (1+h)^3\lambda^3 \leqslant -\overline{\alpha}\lambda^3 - \overline{\beta}h^3\lambda^3,$$

that is,

$$\begin{aligned} &\left[\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 + \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 - \overline{\beta}\right]h^3 + 3\left[\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 + \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3\right]\left(h^2 + h\right) \\ &\quad + \left[\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 + \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 - \overline{\alpha}\right] \geqslant 0. \end{aligned}$$

As observed in Remark 6, thanks to the choice of (29), the coefficient of h^3 and the last term are negative, so that this relation cannot be satisfied if h is too small or too large: this implies that necessarily $1/\bar{h}_1 \leqslant h \leqslant \bar{h}_1$ for a positive constant $\bar{h}_1 > 1$, which depends only on L and on p.

In the case $\sigma(i-1) = -$, one can follow the same line of reasoning, replacing the previous definition of s with

$$s := \bar{t}_{i-1} + \frac{1}{1+\sqrt{\nu}}(\bar{t}_{i+1} - \bar{t}_{i-1}) \in (\bar{t}_{i-1}, \bar{t}_{i+1}).$$

Again, the relation

$$\psi(\bar{t}_1, \dots, \bar{t}_{i-1}, s, \bar{t}_{i+1}, \dots, \bar{t}_k) \leqslant \psi(\bar{t}_1, \dots, \bar{t}_{i-1}, \bar{t}_i, \bar{t}_{i+1}, \dots, \bar{t}_k)$$

implies that for the quantity $\bar{h}_1 > 1$ previously introduced it results $1/\bar{h}_1 \leqslant h \leqslant \bar{h}_1$, and the desired result follows choosing $\bar{h} := \max\{1/\sqrt{\nu}, \bar{h}_1\}$. $\square$

Now we can show that, in a maximizing partition, the ratio between the larger sub-interval and the smaller one is bounded by a constant depending only on L and on p.

Lemma 7.5. *Let*

$$\underline{\lambda} := \min_i (\bar{t}_{i+1} - \bar{t}_i) \quad \textit{and} \quad \bar{\lambda} := \max_i (\bar{t}_{i+1} - \bar{t}_i).$$

Then there exists $h^* \geqslant 1$, depending only on L and on p, such that

$$\bar{\lambda} \leqslant h^* \underline{\lambda}.$$

Proof. Let us denote with $i \neq j$, $0 \leqslant i, j \leqslant k$, two indexes such that

$$\bar{\lambda} = \bar{t}_{i+1} - \bar{t}_i \quad \text{and} \quad \underline{\lambda} = \bar{t}_{j+1} - \bar{t}_j.$$

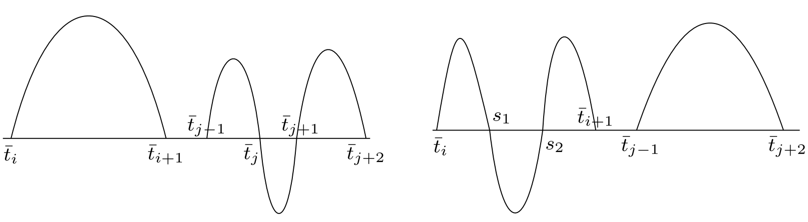
To fix the ideas we consider the case $i < j$. As the previous lemma asserts that the length of any interval is comparable with the one of its neighbors, we can assume without loss of generality i and j to be even, $k \geqslant 5$ and $j - i \geqslant 4$, i.e. $i + 2 \leqslant j - 2$. Let us set again $\nu := \underline{\beta}/\underline{\alpha}$, and let

$$\bar{\sigma} := \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\sqrt[3]{2}}\right)\frac{\sqrt{\nu}}{1 + \sqrt{\nu}}.$$

If $\bar{\lambda} \leqslant \max\{L/\bar{\sigma}, L/(1 - 2\bar{\sigma})\}$, we can choose $h^* = \max\{1/\bar{\sigma}, 1/(1 - 2\bar{\sigma})\}$. Otherwise, we consider a variation of $(\bar{t}_1, \dots, \bar{t}_k)$ introducing two points

$$s_1 := \bar{t}_i + \bar{\sigma}(\bar{t}_{i+1} - \bar{t}_i) \quad \text{and} \quad s_2 := \bar{t}_i + (1 - \bar{\sigma})(\bar{t}_{i+1} - \bar{t}_i)$$

between $\bar{t}_i$ and $\bar{t}_{i+1}$, and eliminating $\bar{t}_j$ and $\bar{t}_{j+1}$ if $j < k$; if $j = k$, we eliminate $\bar{t}_{k-1}$ and $\bar{t}_k$. For the reader's convenience, we explicitly observe that, since $\nu \in (0, 1)$, it results $\bar{t}_i < s_1 < s_2 < \bar{t}_{i+1}$.



In what follows, the notation corresponds to the case $j < k$.

As $\bar\lambda > \max\{L/\bar\sigma, L/(1-2\bar\sigma)\}$, the new partition is in $\mathcal{B}_k$: indeed

$$\begin{aligned} s_1 - \bar t_i &= \bar\sigma(\bar t_{i+1} - \bar t_i) = \bar\sigma\bar\lambda > L, \\ s_2 - s_1 &= (1-2\bar\sigma)(\bar t_{i+1} - \bar t_i) = (1-2\bar\sigma)\bar\lambda > L, \\ \bar t_{i+1} - s_2 &= \bar\sigma(\bar t_{i+1} - \bar t_i) = \bar\sigma\bar\lambda > L. \end{aligned} \tag{34}$$

As a consequence, by maximality,

$$\psi(\bar t_1, \dots, \bar t_i, s_1, s_2, \bar t_{i+1}, \dots, \bar t_{j-1}, \bar t_{j+2}, \dots, \bar t_k) \leqslant \psi(\bar t_1, \dots, \bar t_k),$$

that is,

$$\begin{aligned} &\varphi^+(\bar t_i, s_1) + \varphi^-(s_1, s_2) + \varphi^+(s_2, \bar t_{i+1}) + \varphi^+(\bar t_{j-1}, \bar t_{j+2}) \\ &\quad \leqslant \varphi^+(\bar t_i, \bar t_{i+1}) + \varphi^+(\bar t_{j-1}, \bar t_j) + \varphi^-(\bar t_j, \bar t_{j+1}) + \varphi^+(\bar t_{j+1}, \bar t_{j+2}). \end{aligned}$$

We know that $\varphi^+(\bar t_i, \bar t_{i+1}) \leqslant -\bar\alpha\bar\lambda^3$, and the other terms on the right hand side are negative; on the other hand, for the left hand side we can use the expressions (34) and the fact that, by Lemma 7.4, $\bar t_{j+2} - \bar t_{j-1} \leqslant (2\bar h + 1)\underline{\lambda}$. Therefore

$$-2\underline{\alpha}\bar\sigma^3\bar\lambda^3 - \underline{\beta}(1-2\bar\sigma)^3\bar\lambda^3 - \underline{\alpha}(2\bar h + 1)^3\underline{\lambda}^3 \leqslant -\bar\alpha\bar\lambda^3,$$

which gives

$$\left[\bar\alpha - 2\underline{\alpha}\bar\sigma^3 - \underline{\beta}(1-2\bar\sigma)^3\right]\left(\frac{\bar\lambda}{\underline{\lambda}}\right)^3 \leqslant \underline{\alpha}(2\bar h + 1)^3.$$

We claim that

$$\bar\alpha - 2\underline{\alpha}\bar\sigma^3 - \underline{\beta}(1-2\bar\sigma)^3 > 0.$$

As a consequence, the thesis will follow. To show the claim, we note that, by the definition of $\bar\sigma$, it results

$$2\bar\sigma^3 < \left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 \quad \text{and} \quad (1-2\bar\sigma)^3 < \left(\frac{1}{1+\sqrt{\nu}}\right)^3.$$

Thanks to the choice of (29), recalling also Remark 6, we easily deduce

$$\bar\alpha - 2\underline{\alpha}\bar\sigma^3 - \underline{\beta}(1-2\bar\sigma)^3 > \bar\alpha - \underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 - \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 > 0,$$

which completes the proof. □

End of the proof of Lemma 7.3. Let $H = h^*(L+1)$, with h^* introduced in Lemma 7.5. Then any partition of an interval of length $B - A \geqslant H(k+1)$ in $k+1$ sub-intervals has a sub-interval larger than $h^*(L+1)$, and in particular $\bar{\lambda} \geqslant h^*(L+1)$. Applying Lemma 7.5, we immediately deduce $\underline{\lambda} \geqslant L+1$. $\square$

We are ready to prove the existence of sign-changing solutions of (1) in large intervals.

Proposition 7.6. *There exists H, depending only on L and on p, such that if $B - A \geqslant H(k+1)$ and $(\bar{t}_1, \ldots, \bar{t}_k)$ is a maximizing partition for (31), then the function $u_{(A,B),k}$ defined by (32) is a solution of (1).*

Proof. By construction, $u_{(A,B),k}$ solves (1) in $(A, B) \setminus \{\bar{t}_1, \ldots, \bar{t}_k\}$. Moreover, by Lemma 7.3, $(\bar{t}_1, \ldots, \bar{t}_k)$ is a free critical point of the function ψ, so that $\nabla\psi(\bar{t}_1, \ldots, \bar{t}_k) = 0$. In view of Proposition 6.2, this writes

$$-\frac{1}{2}\dot{u}_{i-1}^2(\bar{t}_i^-) + \frac{1}{2}\dot{u}_i^2(\bar{t}_i^+) = 0 \quad i = 1, \ldots, k.$$

But then $u_{(A,B),k}$ is $\mathcal{C}^1$ across each $\bar{t}_i$, and the proposition follows. $\square$

Remark 8. Directly from the construction of $u_{(A,B),k}$, it is possible to obtain some estimates which will be useful in the next proof; we keep here the notation previously introduced. First of all, we note that for every $t \in (A, B)$ there exists i such that $t \in [\bar{t}_i, \bar{t}_{i+1})$. Thanks to Lemma 3.3, we deduce that

$$\begin{aligned} \left|u_{(A,B),k}(t)\right| &= \left|u_i(t)\right| \leqslant C(\bar{t}_{i+1} - \bar{t}_i)^2 \leqslant C\bar{\lambda}^2, \\ \left|\dot{u}_{(A,B),k}(t)\right| &= \left|\dot{u}_i(t)\right| \leqslant C(\bar{t}_{i+1} - \bar{t}_i) \leqslant C\bar{\lambda}, \end{aligned}$$

where C is a positive constant depending only on g and p. As a consequence

$$\|u_{(A,B),k}\|_{L^\infty(A,B)} \leqslant C\bar{\lambda}^2 \quad \text{and} \quad \|\dot{u}_{(A,B),k}\|_{L^\infty(A,B)} \leqslant C\bar{\lambda}.$$

On the other hand, let τ be a point of maximum of $|u_{(A,B),k}|$. There exists $j \in \{0, \ldots, k\}$ such that $\tau \in (\bar{t}_j, \bar{t}_{j+1})$, so that by Corollary 4.6 it results

$$\|u_{(A,B),k}\|_{L^\infty(A,B)} = \left|u_j(\tau)\right| \geqslant C_1(\bar{t}_{j+1} - \bar{t}_j) \geqslant C_1\underline{\lambda},$$

where C_1 is a positive constant depending only on g and p.

It is now possible to complete the proof of the main result.

Proof of Theorem 2.1. For a fixed $L > \bar{L}$, let $\bar{h}$, h^* and H be as in Lemmas 7.4, 7.5 and Proposition 7.6 respectively. Let $\mu \geqslant H$ be fixed (we explicitly remark that h^* is independent of μ). For every $n \in \mathbb{N}$ we have $2n\mu \geqslant 2nH$, so that by Proposition 7.6 there exists $u_{\mu,n} := u_{(-\mu n,\mu n),2n-1}$ which is a solution of (1) in $(-\mu n, \mu n)$ with $2n-1$ zeros, and its zeros correspond to a partition

$$-\mu n =: \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_{2n-1} < \bar{t}_{2n} := \mu n,$$

maximizing for $c_{2n-1}(-\mu n, \mu n)$, defined by (31). At least one of the sub-intervals of the partition has to be smaller than or equal to μ; recalling that $\underline{\lambda} := \min_i(\bar{t}_{i+1} - \bar{t}_i)$ and $\overline{\lambda} = \max_i(\bar{t}_{i+1} - \bar{t}_i)$, it results $\underline{\lambda} \leqslant \mu$; this implies, by means of Lemma 7.5, that $\overline{\lambda} \leqslant h^*\mu$, where h^* does not depend on n or on μ. Analogously, from the fact that at least one of the sub-intervals of the partition has to be larger than or equal to μ, it is possible to deduce that $\underline{\lambda} \geqslant \mu/h^*$.

By using the estimates of Remark 8, it is immediate to obtain

$$C_1\left(\frac{\mu}{h^*}\right)^2 \leqslant \|u_{\mu,n}\|_{L^\infty(-\mu n,\mu n)} \leqslant C\left(h^*\mu\right)^2 \quad \text{and} \quad \|\dot{u}_{\mu,n}\|_{L^\infty(-\mu n,\mu n)} \leqslant C\left(h^*\mu\right).$$

Furthermore, being $u_{\mu,n}$ a solution of (1), it results

$$\|\ddot{u}_{\mu,n}\|_{L^\infty(-\mu n,\mu n)} \leqslant \|g\|_\infty + \|p\|_\infty.$$

The previous estimates reveals that the sequence $(u_{\mu,n})_{n\in\mathbb{N}}$ is uniformly bounded in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$, so that by the Ascoli–Arzelà theorem it converges in $\mathcal{C}^1_{\mathrm{loc}}(\mathbb{R})$, up to a subsequence, to a function u_μ which is a solution of (1) in the whole $\mathbb{R}$, and satisfies

$$C_1\left(\frac{\mu}{h^*}\right)^2 \leqslant \|u_\mu\|_{L^\infty(\mathbb{R})} \leqslant C\left(h^*\mu\right)^2 \quad \text{and} \quad \|\dot{u}_\mu\|_{L^\infty(\mathbb{R})} \leqslant C\left(h^*\mu\right) \tag{35}$$

By construction, u_μ has infinitely many zeros tending to infinity in both the directions; indeed, if this were not true, then $|u_\mu(t)| \geqslant C > 0$ on an interval of length greater than $h^*\mu$, and by the $\mathcal{C}^1_{\mathrm{loc}}$ convergence the same should hold also for $u_{\mu,n}$ when n is sufficiently large, which is not possible.

We have constructed a solution of (1) defined in $\mathbb{R}$, which is bounded together with its first derivative. Now, we can obtain the sequence of bounded solutions $u_m = u_{\mu_m}$ simply repeating the same procedure for a sequence of parameters μ_m such that $\mu_m \to +\infty$ and

$$\mu_m > \sqrt{\frac{C}{C_1}}\left(h^*\right)^2 \mu_{m-1}$$

for every m. Indeed, thanks to Eq. (35), we deduce

$$\|u_{m-1}\|_{L^\infty(\mathbb{R})} \leqslant C\left(h^*\mu_{m-1}\right)^2 < C_1\left(\frac{\mu_m}{h^*}\right)^2 \leqslant \|u_m\|_{L^\infty(\mathbb{R})},$$

so that $u_{m-1} \not\equiv u_m$ and $\|u_m\|_\infty \to +\infty$ as $m \to \infty$. □

To conclude, as we mentioned in the introduction, we turn to the periodic framework. We keep the previous notations, in particular H is defined as in Lemma 7.3. We have the following.

Theorem 7.7. *Let g satisfy (h1), and let p be a continuous T-periodic function such that*

$$g_- < A(p) = \frac{1}{T}\int_0^T p(t)\,dt < g_+.$$

Then, for any $(k,n) \in \mathbb{N}^2$ *with* k *odd and* $nT \geqslant H(k+1)$*, there exists an* nT*-periodic solution of* (1)*, having exactly* k *zeros in each interval of periodicity.*

Remark 9. The nodal characterization of the solutions ensures that, whenever T is the minimal period of p, and n and $(k+1)/2$ are coprime integers, then nT is the minimal period of the corresponding solution. This ensures the existence of an infinite sequence of subharmonic solutions, with diverging minimal period.

Proof of Theorem 7.7. Let

$$\mathcal{A}_k := \left\{ (t_0, t_1, \dots, t_k) \in \mathbb{R}^k \;\middle|\; \begin{array}{l} t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t_{k+1} := t_0 + nT, \\ t_{i+1} - t_i \geqslant L,\ t_0 \in [-T, 2T] \end{array} \right\},$$

and let $\psi : \mathcal{A}_k \to \mathbb{R}$ be defined as in (30) (we point out that now t_0 is not fixed). There exists a maximizer $(\bar{t}_0, \bar{t}_1, \dots, \bar{t}_k)$ for ψ. Since p is T-periodic, we can assume $\bar{t}_0 \in [0, T)$. As a consequence, it results $\nabla \psi(\bar{t}_0, \bar{t}_1, \dots, \bar{t}_k) = 0$. The expression of the partial derivatives of ψ with respect to t_i, $i = 1, \dots, k$, says that the function $u_{(\bar{t}_0, \bar{t}_0 + nT), k}$ (defined as in (32)) is a solution of (1) in $(\bar{t}_0, \bar{t}_0 + nT)$; also, the fact that $\partial_{t_0} \psi(\bar{t}_0, \bar{t}_1, \dots, \bar{t}_k) = 0$ implies that

$$-\frac{1}{2} \dot{u}^2_{(\bar{t}_0, \bar{t}_0 + nT), k}\big(\bar{t}_0^+\big) + \frac{1}{2} \dot{u}^2_{(\bar{t}_0, \bar{t}_0 + nT), k}\big((\bar{t}_0 + nT)^-\big) = 0,$$

that is, $u_{(\bar{t}_0, \bar{t}_0 + nT), k}$ can be extended by nT-periodicity as a (smooth) solution of (1) in the whole $\mathbb{R}$. $\square$

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