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The rate at which energy decays in a viscously damped hinged Euler–Bernoulli beam

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Abstract

We study the best decay rate of the solutions of a damped Euler–Bernoulli beam equation with a homogeneous Dirichlet boundary conditions. We show that the fastest decay rate is given by the supremum of the real part of the spectrum of the infinitesimal generator of the underlying semigroup, if the damping coefficient is in $L^{\infty}(0,1)$.

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Contents

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1. Introduction and main result

We consider the following damped Euler–Bernoulli equation:

$$\partial_t^2 u(x,t) + \partial_x^4 u(x,t) + 2a(x)\partial_t u(x,t) = 0, \quad 0 < x < 1,\ t > 0, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \qquad \partial_x^2 u(0,t) = \partial_x^2 u(1,t) = 0, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u^0(x), \qquad \partial_t u(x,0) = u^1(x), \quad 0 < x < 1, \tag{1.3}$$

where $a \in L^\infty(0,1)$ is non-negative satisfying the following condition:

$$\exists c > 0 \text{ s.t., } a(x) \geq c, \quad \text{a.e., in a non-empty open subset } I \text{ of } (0,1). \tag{1.4}$$

In order to formulate our results we consider the Hilbert space

$$\left[H^2(0,1) \cap H_0^1(0,1)\right] \times L^2(0,1) =: V \times L^2(0,1),$$

where we denote by $H^s(0,1)$, $s \in \mathbb{R}$, the usual Sobolev spaces. We endow this space with the inner product:

$$\left\langle [f,g],[u,v]\right\rangle := \int_0^1 \left(f^{(2)}(x)\overline{u^{(2)}(x)} + g(x)\overline{v(x)}\right)dx, \quad \text{for all } [f,g],[u,v] \text{ in } V \times L^2(0,1).$$

From now on, we shall represent a pair of functions by $[f,g]$ rather than (f,g) to avoid confusion with classical inner product on $L^2(0,1)$.

We define the energy of a solution u of (1.1)–(1.3), at time t, as

$$E\big(u(t)\big) = \frac{1}{2}\int_0^1 \left(\left|\partial_t u(x,t)\right|^2 + \left|\partial_x^2 u(x,t)\right|^2\right)dx. \tag{1.5}$$

To prove the decay of the energy, we multiply (1.1) by $\partial_t u$ and integrate on [0, 1] by using the boundary condition. We can easily check that every sufficiently smooth solution of (1.1)–(1.3) satisfies

$$\frac{d}{dt}E\big(u(t)\big) = -2\int_0^1 a(x)\big|\partial_t u(x,t)\big|^2\,dx,$$

which in particular implies the energy identity

$$E\big(u(0)\big) - E\big(u(t)\big) = 2\int_0^t\int_0^1 a(x)\big|\partial_s u(x,s)\big|^2\,dx\,ds, \quad \forall t \geq 0.$$

In our case, i.e., when the damping term a is non-negative, and positive on an open subset, it is known that the energy of a solution will decay exponentially in time. More precisely, the following result follows from [14] (see also [3]).

Proposition 1.1. *Let $a \in L^\infty(0,1)$, be non-negative satisfying (1.4).*

For all $(u^0, u^1) \in V \times L^2(0,1)$, the problem (1.1)–(1.3) admits a unique solution u in $C([0,+\infty[; V) \cap C^1([0,+\infty[; L^2(0,1))$. Moreover, there exist constants $C > 0$ and $\omega_0 < 0$ depending only on $a(x)$ such that

$$E\big(u(t)\big) \leq Ce^{2\omega_0 t}E\big(u(0)\big), \quad \forall t > 0. \tag{1.6}$$

The system (1.1)–(1.3) can be written as an abstract evolution equation on $V \times L^2(0,1)$:

$$\begin{cases} \partial_t U = A_a U \\ U(x,0) = U^0(x) := \big(u^0(x), u^1(x)\big), \end{cases} \tag{1.7}$$

where $U = [u, \partial_t u]$ and the operator A_a is given by

$$A_a := \begin{pmatrix} 0 & \mathrm{Id} \\ -\frac{d^4}{dx^4} & -2a(x) \end{pmatrix} \quad \text{from } \mathcal{D} \text{ to } V \times L^2(0,1), \tag{1.8}$$

with $\mathcal{D} = \{[u,v] \in V \times L^2(0,1);\ v \in V,\ u \in H^4(0,1),\ \frac{d^2u}{dx^2}(0) = 0,\ \frac{d^2u}{dx^2}(1) = 0\}$. Here Id denotes the identity on V. Note that the domain of the operator A_a is independent of a, since $a \in L^\infty(0,1)$.

The eigenvalue problem for the non-self-adjoint, quadratic operator pencil generated by (1.1)–(1.3) is obtained by replacing u in (1.1) by

$$u(x,t) = e^{\lambda t}\phi(x).$$

We obtain from (1.7) the standard form

$$(A_a - \lambda \mathrm{Id})\Phi = 0, \quad \Phi = [\phi, \lambda\phi] := \phi[1,\lambda].$$

The condition for the existence of non-trivial solutions is that $\lambda \in \sigma(A_a)$ (the spectrum of A_a). Since $\mathcal{D}$ is compactly embedded in the energy space $V \times L^2(0,1)$ then the spectrum $\sigma(A_a)$ is discrete and the eigenvalues of A_a have a finite algebraic multiplicity. On the other hand, since A_a is a bounded monotone perturbation of a skew-adjoint operator (undamped A_0), it follows from the Hill–Yosida theorem that A_a generates a C_0-semigroup of contractions on the energy space $V \times L^2(0,1)$ (see [26]).

In this work, we give the value of the best decay rate ω_0 (see (1.6)) in terms of the spectral abscissa of the generator A_a. More precisely, let us define:

$$\omega(a) = \inf\big\{\omega;\ \text{there exists } C = C(\omega) > 0 \text{ such that } E\big(u(t)\big) \le C(\omega)e^{2\omega t} E\big(u(0)\big),$$
$$\text{for every solution } u \text{ of (1.1)–(1.3) with initial data in } V \times L^2(0,1)\big\}, \quad (1.9)$$

and

$$\mu(a) = \sup\big\{\mathrm{Re}(\lambda);\ \lambda \in \sigma(A_a)\big\}. \quad (1.10)$$

It follows easily from the above observations that

$$\mu(a) \le \omega(a). \quad (1.11)$$

Our main result establishes the reverse inequality under the assumption that $a(x)$ is in $L^\infty(0,1)$.

Theorem 1.2. *Let $a \in L^\infty(0,1)$, be non-negative satisfying (1.4). Then*

$$\mu(a) = \omega(a). \quad (1.12)$$

Although the literature on the decay estimates of the energy of the wave equation with locally distributed damping is quite impressive (see [3,7,13,17,20,19,22–24,30–33]), little is known on the decay estimate of the energy plate equations with locally distributed damping (see [3,11,14,16,20,29,30]). The determination of optimal decay rates was performed mostly for the damped wave operator in the 1-d case (a vibrating string), see [1,2,4,6,8–10]. For higher dimension, G. Lebeau gives in [18] the explicit (and optimal) value of the best decay rate in terms of the spectral abscissa of the generator of the semigroup and the mean value of the function a along the rays of geometrical optics. It is not our intention to do a complete review on this subject here. We refer the readers to the references in the mentioned above for more information.

As in [8] we will establish the reverse inequality of (1.11) by proving that the system of generalized eigenvectors of the operator A_a constitutes a Riesz basis in the energy space $V \times L^2(0,1)$, and that all eigenvalues of A_a with sufficiently large modulus are algebraically simple. To do this, we require precise knowledge of the spectrum of the non-self-adjoint operator A_a.

In [8], Cox and Zuazua adopt the shooting method based on an ansatz of Horn. This approach consists in constructing an explicit approximation of the characteristic equation of the underlying system. Under the assumption that the damping is of bounded variation (i.e., $a \in BV(0,1)$), they obtained high frequency asymptotic expansions of the spectrum. The shooting method can be used only for one-dimensional boundary value problems.

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In this paper, we follow the main idea in [8]. However, for high frequency we will use a perturbation method based on some resolvent estimate for the operator A_a. The advantage of this approach is that it works in any dimension and in a very general setting (see [28] and also [15]). On the other hand, we only need that $a \in L^\infty(0,1)$.

In the following, we give an outline of the proof. First, for $a = 0$, the operator A_0 is skew-adjoint with compact resolvent in $V \times L^2(0,1)$. From general operator theory, all its eigenvalues lie on the imaginary axis and the geometric and algebraic multiplicity of each eigenvalue are the same. Moreover, there is a sequence of eigenvectors of A_0 which forms a Riesz (orthonormal, actually) basis for $V \times L^2(0,1)$.

For $a = a_0 = \text{const} > 0$, we compute explicitly all the eigenvalues and eigenvectors of A_{a_0}. In particular, we prove that the algebraic multiplicity is usually one except for ($a_0 = k_0^2\pi^2$, for some $k_0 \in \mathbb{N}$) there is one eigenvalue of multiplicity two. On the other hand, using the explicit expression of the eigenfunctions of A_0 and A_{a_0} we prove that the systems of eigenvectors of A_0 and A_{a_0} are quadratically close in $V \times L^2(0,1)$. Thus, it follows from [27, Appendix D, Theorem 3] that the system of eigenvectors of A_{a_0} constitutes a Riesz basis. Consequently, by a standard argument (see Theorem 2.5), we identify the optimal energy decay rate with the supremum of the real part of A_{a_0}.

In Section 3, we treat the general case, i.e., $a \in L^\infty(0,1)$ and $a(x)$ is non-negative satisfying (1.4). First, we introduce the characteristic determinant of A_a. Recalling that the characteristic determinant $\Delta_{2,4}(1,\lambda)$ of A_a is an entire function whose zeros are the eigenvalues of A_a, with the order of these zeros determining the algebraic multiplicities. By analyzing the function $\Delta_{2,4}(1,\lambda)$, we give in Proposition 3.1 rough preliminary bounds on the spectrum of A_a. Moreover, since A_a is a bounded perturbation of skew-adjoint operator with compact resolvent it follows from [12, Chapter 5, Theorem 10.1] that the generalized eigenvectors of A_a are complete in $V \times L^2(0,1)$. To prove Theorem 1.2 we also need to study the asymptotic behavior of the high frequency of A_a, more precisely, the behavior of the corresponding algebraic multiplicity. In fact, since the distance between two consecutive eigenvalues tends to infinity at infinity, as well as the fact that the damping is bounded, we give some resolvent estimates of the operators A_a and A_0 and then we show that all eigenvalues of A_a with sufficiently large modulus are algebraically simple (see Subsection 3.2). Eventually, we complete the proof of Theorem 1.2 as in the constant case, see Subsection 3.4.

2. Undamped and constant damping operator

2.1. Spectral analysis

Here are some elementary properties of the skew-adjoint (undamped) operator A_0:

$$A_0 := \begin{pmatrix} 0 & \mathrm{Id} \\ -\frac{d^4}{dx^4} & 0 \end{pmatrix} : \mathcal{D} \subset V \times L^2(0,1) \longrightarrow V \times L^2(0,1),$$

where $\mathcal{D} = \{[u,v] \in V \times L^2(0,1);\ v \in V,\ u \in H^4(0,1),\ \frac{d^2u}{dx^2}(0) = 0,\ \frac{d^2u}{dx^2}(1) = 0\}$, with $V = H^2(0,1) \cap H_0^1(0,1)$.

Lemma 2.1. *The eigenvalues and the corresponding eigenvectors of A_0 are given by:*

$$A_0 V_{\pm k} = \pm i k^2 \pi^2 V_{\pm k}, \quad \textit{for all } k \in \mathbb{N}^*,$$
$$\textit{where} \quad V_{\pm k} = \frac{1}{k^2\pi^2} \sin(k\pi x)\big[1, \pm i k^2 \pi^2\big]. \tag{2.1}$$

Moreover, the family $\mathcal{B}_0 := (V_{\pm k})_{k \in \mathbb{N}^*}$ is an orthonormal basis of the energy space $V \times L^2(0,1)$.

Now, we focus on the spectrum of A_{a_0} when the damping $a(x)$ is a positive constant denoted by a_0. Let $W = [u, v] \in \mathcal{D}$ be an eigenvector of A_{a_0} associated to the eigenvalue λ. Then

$$v = \lambda u \quad \text{and} \quad -u^{(4)} - 2a_0 \lambda u = \lambda^2 u \tag{2.2}$$
$$\text{with} \quad u(0) = u(1) = 0 \quad \text{and} \quad u^{(2)}(0) = u^{(2)}(1) = 0. \tag{2.3}$$

It follows that the eigenvalue λ of A_{a_0} satisfies:

$$\lambda^2 + 2a_0 \lambda = -k^4 \pi^4, \quad \text{for all integer } k \geq 1. \tag{2.4}$$

In the rest of the subsection, we characterize the algebraic multiplicity of the eigenvalues of A_{a_0}:

Lemma 2.2. *The algebraic multiplicity of the eigenvalue λ_k, $k \in \mathbb{Z}^*$, of A_{a_0} is its order as a zero of Eq. (2.4). In particular, the algebraic multiplicity of λ_k, $k \in \mathbb{Z}^*$, of A_{a_0} is at most 2.*

Proof. We have two situations:

(i) If $a_0 \in \,]0, +\infty[\, \setminus \{k^2\pi^2; k \in \mathbb{Z}^*\}$ then there exists $k_0 \in \mathbb{Z}$ such that $k_0^2 \pi^2 < a_0 < (k_0+1)^2 \pi^2$. In this case the eigenvalues of A_{a_0} are

$$\lambda_{\pm k} = \begin{cases} -a_0 \pm \sqrt{a_0^2 - k^4\pi^4} & \text{for } k = 1, 2, \cdots, k_0 \\ -a_0 \pm i\sqrt{k^4\pi^4 - a_0^2} & \text{for } k > k_0, \end{cases} \tag{2.5}$$

with the corresponding eigenvector

$$W_{\pm k} = \sin(k\pi x)[1, \lambda_{\pm k}], \quad \forall k \geq 1. \tag{2.6}$$

Now, we want to show that the algebraic multiplicity of the eigenvalues λ_p, $p \in \mathbb{Z}^*$ is exactly 1. If the algebraic multiplicity of λ_p is to exceed one then one must be able to solve $(A_{a_0} - \lambda_p) W_{p,1} = W_p$. With $W_{p,1} = [u_{p,1}, v_{p,1}]$, this requires $v_{p,1} = \lambda_p u_{p,1} + \sin(p\pi x)$ and $-u_{p,1}^{(4)} + p^4 \pi^4 u_{p,1} = 2(a_0 + \lambda_p) \sin(p\pi x)$ with $u_{p,1}(0) = u_{p,1}(1) = u_{p,1}^{(2)}(0) = u_{p,1}^{(2)}(1) = 0$. Since $\lambda_p \neq -a_0$ then the previous equation has no solution. Therefore the eigenvalues of A_{a_0} are simple.

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(ii) Assume that there exist $k_0 \in \mathbb{Z}^*$ such that $a_0 = k_0^2\pi^2$. As in [8], we call such a_0 *defective*. The spectrum is given by (2.5).
For all $k \in \mathbb{N}^* \setminus \{k_0\}$, $\lambda_{\pm k}$ is simple with the corresponding eigenvector given by (2.6). The only difference to the previous case is that $\lambda_{k_0} = \lambda_{-k_0} = -a_0$. It remains to prove that the algebraic multiplicity of $-a_0$ is two as eigenvalue of A_{a_0}. As in the previous case $W_{k_0} = \sin(k_0\pi x)[1, -a_0]$ is the eigenvector associated to the eigenvalue $-a_0$. The generalized eigenvector $W_{k_0,1}$ via $(A_{a_0} + a_0)W_{k_0,1} = W_{k_0}$ and $\langle W_{k_0}|W_{k_0,1}\rangle = 0$ is given by $W_{k_0,1} = \frac{1}{2}\sin(k_0\pi x)[\frac{1}{a_0}, 1]$. Then the algebraic multiplicity of $-a_0$ is at least two. From now on, we denote $W_{k_0,1}$ by W_{-k_0}.
Assume that the algebraic multiplicity of $-a_0$ exceeded two, one must then be able to solve $(A_{a_0} + a_0)W_{k_0,2} = W_{-k_0}$. With $W_{k_0,2} = [u_{k_0,2}, v_{k_0,2}]$, we find $v_{k_0,2} = -a_0u_{k_0,2} + \sin(k_0\pi x)$ and

$$-u^{(4)}_{k_0,2} + k_0^4\pi^4 u_{k_0,2} = \sin(k_0\pi x), \qquad u_{k_0,2}(0) = u_{k_0,2}(1) = u^{(2)}_{k_0,2}(0) = u^{(2)}_{k_0,2}(1) = 0.$$

Since this equation does not possess a solution, the algebraic multiplicity of $-a_0$ may not exceed two. □

2.2. *Generalized eigenvectors*

In this subsection, we show that the family of the generalized eigenvectors associated to the constant damping is a Riesz basis. Here and for the rest of the paper we will use the following notation.

Notation 2.3. We set $\mathcal{B}_{a_0} := (\widetilde{W}_p := \frac{W_p}{\|W_p\|_{V\times L^2(0,1)}})_{p\in\mathbb{Z}^*}$, where

1. when $a_0 \in]0, +\infty[\setminus \{k^2\pi^2; k \in \mathbb{Z}^*\}$, W_p is an eigenvector given by (2.6) for all $p \in \mathbb{Z}^*$,
2. when a_0 is defective, i.e., $a_0 = p_0^2\pi^2$ for some p_0, then for all $p \in \mathbb{Z}^* \setminus \{-p_0\}$, W_p is an eigenvector given by (2.6) and $W_{-p_0} = \frac{1}{2}\sin(p_0\pi x)[\frac{1}{a_0}, 1]$ is a generalized eigenvector of A_{a_0} associated to the eigenvalue $-a_0 = -p_0^2\pi^2$.

We have the following result:

Proposition 2.4. *The family $\mathcal{B}_{a_0}$ is a Riesz basis.*

Proof. The sequence $(W_k)_{k\in\mathbb{Z}^*}$ admits a biorthogonal family $(W_k^*)_{k\in\mathbb{Z}^*}$ in $V \times L^2(0, 1)$ given by

$$W^*_{\pm k} = \sin(k\pi x)[1, -\overline{\lambda_{\pm k}}], \quad k = 1, 2, \cdots, \tag{2.7}$$

if $a_0 \in]0, +\infty[\setminus \{k^2\pi^2; k \in \mathbb{Z}^*\}$. For $a_0 = p_0^2\pi^2$ for some $p_0 \in \mathbb{Z}^*$, we define, for all $k \in \mathbb{Z}^* \setminus \{-p_0\}$, W_k^* as above and $W^*_{-p_0}$ as a generalized eigenvector via $(A_a^* + a_0\mathrm{Id})W^*_{-p_0} = W^*_{p_0}$ and $\langle W^*_{-p_0}|W^*_{p_0}\rangle = 0$. That is, $W^*_{-p_0} = \frac{1}{2}\sin(p_0\pi x)[\frac{1}{a_0}, -1]$.

Note that $W^*_{\pm k}$ are the generalized eigenvectors of the adjoint of A_{a_0},

$$A^*_{a_0} := \begin{pmatrix} 0 & -\mathrm{Id} \\ \frac{d^4}{dx^4} & -2a_0 \end{pmatrix} \quad \text{from } \mathcal{D}(A^*_{a_0}) \text{ to } V \times L^2(0,1), \tag{2.8}$$

with $\mathcal{D}(A^*_{a_0}) = \mathcal{D}$ and remark that the eigenvalues of $A^*_{a_0}$ are precisely those of A_{a_0} (including multiplicities), see (2.5).

If $a_0 \in]0, +\infty[\setminus \{k^2\pi^2; k \in \mathbb{Z}^*\}$, we see that $\langle W_n | W^*_m\rangle = -\lambda_n(a_0 + \lambda_m)\delta_{n,m}$ and hence $(W_k)_{k\in\mathbb{Z}^*}$ is a linearly independent set. Here $\delta_{n,m}$ is the Kronecker's delta, i.e. $\delta_{n,m} = 1$ if $n = m$ and 0 if $n \neq m$.

If $a_0 = p_0^2\pi^2$ for some $p_0 \in \mathbb{Z}^*$, we have

$$\langle W_n | W^*_m\rangle = \begin{cases} -\lambda_n(a_0 + \lambda_n)\delta_{n,m} & \text{if } n \neq \pm p_0,\ m \in \mathbb{Z}^* \\ \frac{a_0}{2}\delta_{n,m} & \text{if } n = \pm p_0,\ m \in \mathbb{Z}^*. \end{cases}$$

Hence, even in the defective case, $(W_k)_{k\in\mathbb{Z}^*}$ is a linearly independent set.

For large k the eigenvalue λ_k is simple and nonreal. The corresponding normalized eigenvector is given by $\widetilde{W}_k = \frac{1}{k^2\pi^2} W_k = \frac{\sin(k\pi x)}{k^2\pi^2}[1, \lambda_k]$. Combining this with (2.1), we obtain

$$\|V_k - \widetilde{W}_k\|^2_{V\times L^2(0,1)} = \frac{1}{2}\left| i - \frac{\lambda_k}{k^2\pi^2}\right|^2 = \mathcal{O}\left(\frac{1}{k^4}\right). \tag{2.9}$$

Then $\sum_{k\in\mathbb{Z}^*} \|V_k - \widetilde{W}_k\|^2_{V\times L^2(0,1)} < \infty$, i.e. $(\widetilde{W}_k)_{k\in\mathbb{Z}^*}$ is quadratically close to $(V_k)_{k\in\mathbb{Z}^*}$.

According to Theorem 3 in [27, Appendix D], a linearly independent set that is quadratically close to an orthonormal basis is in fact equivalent to that basis in the sense there exists a linear isomorphism Φ_{a_0} of $V \times L^2(0,1)$ under which $\widetilde{W}_{\pm k} = \Phi_{a_0} V_{\pm k}$. Thus we have proved Proposition 2.4. □

2.3. *Proof of Theorem 1.2 in the constant damping case*

We are in position to prove the main result in the case of a constant damping:

Theorem 2.5. *If $a(x) = a_0$ is a positive constant then $\mu(a_0) = \omega(a_0)$.*

Proof. As $(\widetilde{W}_k)_{k\in\mathbb{Z}^*}$ is a Riesz basis, we may expand the initial data as

$$[u^0, v^0] = \sum_{k\in\mathbb{Z}^*} c_k \widetilde{W}_k.$$

Then the solution of (1.1)–(1.3) is given by

$$[u, \partial_t u] = \begin{cases} \sum_{k\in\mathbb{Z}^*} c_k \exp(\lambda_k t)\widetilde{W}_k & \text{if } a_0 \text{ is not defective} \\ t\exp(\lambda_{p_0} t)c_{-p_0}\widetilde{W}_{p_0} + \sum_{k\in\mathbb{Z}^*} c_k \exp(\lambda_k t)\widetilde{W}_k & \\ \qquad\qquad \text{if } a_0 = p_0^2\pi^2 \text{ for some } p_0. \end{cases} \tag{2.10}$$

The spectral abscissa $\mu(a_0)$ is equal to $-a_0 + \mathrm{Re}(\sqrt{a_0^2 - \pi^4})$ and this allows us to bound the energy $E(u(t))$ by:

(i) If a_0 is not defective,

$$\begin{aligned}
E\big(u(t)\big) &= \big\|[u, \partial_t u]\big\|^2_{V\times L^2(0,1)} = \Big\|\Phi_{a_0} \sum_{k\in\mathbb{Z}^*} \exp(\lambda_k t) c_k \Phi_{a_0}^{-1}(\widetilde{W}_k)\Big\|^2_{V\times L^2(0,1)} \\
&= \Big\|\Phi_{a_0} \sum_{k\in\mathbb{Z}^*} \exp(\lambda_k t) c_k V_k\Big\|^2_{V\times L^2(0,1)} \le \|\Phi_{a_0}\|^2 \sup_{k\in\mathbb{Z}^*} \big|\exp(\lambda_k t)\big|^2 \sum_{k\in\mathbb{Z}^*} |c_k|^2 \\
&\le \|\Phi_{a_0}\|^2 \exp\big(2\mu(a)t\big) \sum_{k\in\mathbb{Z}^*} |c_k|^2 = \|\Phi_{a_0}\|^2 \exp\big(2\mu(a)t\big) \Big\|\sum_{k\in\mathbb{Z}^*} c_k V_k\Big\|^2_{V\times L^2(0,1)} \\
&= \|\Phi_{a_0}\|^2 \exp\big(2\mu(a)t\big) \Big\|\Phi_{a_0}^{-1} \sum_{k\in\mathbb{Z}^*} c_k \widetilde{W}_k\Big\|^2_{V\times L^2(0,1)} \\
&\le \|\Phi_{a_0}\|^2 \big\|\Phi_{a_0}^{-1}\big\|^2 \exp\big(2\mu(a_0)t\big) E\big(u(0)\big).
\end{aligned} \tag{2.11}$$

(ii) Now assume that a_0 is defective. Repeating the previous argument and using the last equality in the right hand side of (2.10) leads to

$$E\big(u(t)\big) \le \|\Phi_{a_0}\|^2 \big\|\Phi_{a_0}^{-1}\big\|^2 \big(1 + t^2\big) \exp\big(2\mu(a_0)t\big) E\big(u(0)\big).$$

Summing (i) and (ii), we deduce

$$\omega(a_0) \le \mu(a_0),$$

which together with (1.11) yields Theorem 2.5. □

2.4. *Another characterization of the spectrum*

In this subsection the damping $a(x)$ is not necessarily constant. As Lemma 2.2 will not survive to a non-constant damping we characterized differently the eigenvalues of A_a and their algebraic multiplicities. In fact, let $u_2(x, \lambda)$, $u_4(x, \lambda)$ be two solutions of

$$u^{(4)}(x, \lambda) + \lambda\big(2a(x) + \lambda\big) u(x, \lambda) = 0, \quad \forall x \in [0, 1], \tag{2.12}$$

subject to the corresponding initial conditions:

$$u_2(0, \lambda) = 0, \qquad u_2^{(1)}(0, \lambda) = 1, \qquad u_2^{(2)}(0, \lambda) = 0 \quad \text{and} \quad u_2^{(3)}(0, \lambda) = 0, \tag{2.13}$$

$$u_4(0, \lambda) = u_4^{(1)}(0, \lambda) = u_4^{(2)}(0, \lambda) = 0 \quad \text{and} \quad u_4^{(3)}(0, \lambda) = 1. \tag{2.14}$$

Lemma 2.6. *A complex number λ is an eigenvalue of A_a if and only if*

$$\Delta_{2,4}(1, \lambda) := \det \begin{pmatrix} u_2(1, \lambda) & u_4(1, \lambda) \\ u_2^{(2)}(1, \lambda) & u_4^{(2)}(1, \lambda) \end{pmatrix} = 0, \tag{2.15}$$

and the algebraic multiplicity of the eigenvalue λ *of* A_a *is its order as a zero of the equation* $\Delta_{2,4}(1,\lambda)=0$.

Proof. Let $u_1(x,\lambda), u_3(x,\lambda)$ be two solutions of (2.12) subject to the corresponding initial conditions:

$$u_1(0,\lambda)=1, \qquad u_1^{(1)}(0,\lambda)=u_1^{(2)}(0,\lambda)=u_1^{(3)}(0,\lambda)=0,$$

$$u_3(0,\lambda)=u_3^{(1)}(0,\lambda)=0, \qquad u_3^{(2)}(0,\lambda)=1 \quad \text{and} \quad u_3^{(3)}(0,\lambda)=0.$$

Since (u_1,u_2,u_3,u_4) constitutes a basis of solutions of the differential equation (2.12), then all solutions of (2.12) are given by:

$$u(x,\lambda)=u(0)u_1(x,\lambda)+u^{(1)}(0)u_2(x,\lambda)+u^{(2)}(0)u_3(x,\lambda)+u^{(3)}(0)u_4(x,\lambda), \quad \forall x\in[0,1]. \tag{2.16}$$

Let $\lambda\in\mathbb{C}$. λ is an eigenvalue of A_a if and only if there exists a non-trivial solution $W=[u,v]\in\mathcal{D}$ satisfying Eqs. (2.2)–(2.3), i.e. $v(x)=\lambda u(x)$ and

$$u(x)=u^{(1)}(0)u_2(x,\lambda)+u^{(3)}(0)u_4(x,\lambda) \quad \text{for all } x\in[0,1], \tag{2.17}$$

with $(u^{(1)}(0),u^{(3)}(0))\neq(0,0)$ satisfying the following system:

$$\begin{cases} u_2(1,\lambda)u^{(1)}(0)+u_4(1,\lambda)u^{(3)}(0)=u(1)=0 \\ u_2^{(2)}(1,\lambda)u^{(1)}(0)+u_4^{(2)}(1,\lambda)u^{(3)}(0)=u^{(2)}(1)=0. \end{cases}$$

Then, the previous system has non-trivial solution $(u^{(1)}(0),u^{(3)}(0))$ if and only if (2.15) holds. In this case the associated eigenvector $W=[u,v]=u(x,\lambda)[1,\lambda]\in\mathcal{D}$ with $u(x,\lambda)$ is given by:

(i) if $(u_2(1,\lambda),u_4(1,\lambda))\neq(0,0)$,

$$u(x,\lambda)=u_4(1,\lambda)u_2(x,\lambda)-u_2(1,\lambda)u_4(x,\lambda), \tag{2.18}$$

(ii) if $(u_2(1,\lambda),u_4(1,\lambda))=(0,0)$ then $(u_2^{(2)}(1,\lambda),u_4^{(2)}(1,\lambda))\neq(0,0)$ and

$$u(x,\lambda)=u_4^{(2)}(1,\lambda)u_2(x,\lambda)-u_2^{(2)}(1,\lambda)u_4(x,\lambda). \tag{2.19}$$

The matrix $\begin{pmatrix} u_2(1,\lambda) & u_4(1,\lambda) \\ u_2^{(2)}(1,\lambda) & u_4^{(2)}(1,\lambda)\end{pmatrix}$ is not trivial for all $\lambda\in\mathbb{C}$. Otherwise, $u_2(\cdot,\lambda)$ and $u_4(\cdot,\lambda)$ are solutions of problem (2.12) with initial conditions:

$$u(0,\lambda)=u(1,\lambda)=0 \quad \text{and} \quad u^{(2)}(0,\lambda)=u^{(2)}(1,\lambda)=0.$$

Then $u_2(\cdot,\lambda)=u_4(\cdot,\lambda)$, contradiction (since $u_2^{(1)}(0,\lambda)=1$ and $u_4^{(1)}(0,\lambda)=0$).

The zeros of $\lambda\mapsto\Delta_{2,4}(1,\lambda)$ are the eigenvalues of A_a. Moreover, the corresponding algebraic multiplicity is given by its order as a zero of $\Delta_{2,4}(1,\lambda)$ (see [25, Theorem on page 343]). This may be checked easily when $a(x)=a_0$ is constant. In fact,

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$$\Delta_{2,4}(1,\lambda) = \frac{\sinh((-\lambda(2a_0+\lambda))^{\frac{1}{4}})\sin((-\lambda(2a_0+\lambda))^{\frac{1}{4}})}{\sqrt{-\lambda(2a_0+\lambda)}}. \tag{2.20}$$

Then,

$$\frac{\partial \Delta_{2,4}}{\partial \lambda}(1,\lambda_k) = -\frac{(-1)^k \sinh(k\pi)}{2k^5\pi^5}(\lambda_k + a_0), \quad \forall k \in \mathbb{Z}^*.$$

This vanishes only when $\lambda_{k_0} = -a_0$, i.e. when $a_0 = k_0^2\pi^2$ for some k_0. The second derivative at such a root, $\frac{\partial^2 \Delta_{2,4}}{\partial \lambda^2}(1,-a_0) = -\frac{(-1)^{k_0}\sinh(k_0\pi)}{2a_0^{\frac{5}{2}}}$, is however, nonzero. □

3. General L^∞-damping

3.1. Spectral analysis

In this subsection, we assume only that the damping is bounded, i.e. there exist $\alpha, \beta \in [0,+\infty[$ such that

$$0 \le \alpha \le a(x) \le \beta < \infty \quad \text{almost everywhere in } [0,1]. \tag{3.1}$$

Let us introduce the following two solutions w_2, w_4 of $u^{(4)}(x,\lambda) + \lambda(2a(x)+\lambda)u(x,\lambda) = 0$ subject to the corresponding initial conditions:

$$w_2(1,\lambda) = 0, \qquad w_2^{(1)}(1,\lambda) = -1, \qquad w_2^{(2)}(1,\lambda) = 0 \quad \text{and} \quad w_2^{(3)}(1,\lambda) = 0, \tag{3.2}$$

$$w_4(1,\lambda) = w_4^{(1)}(1,\lambda) = w_4^{(2)}(1,\lambda) = 0 \quad \text{and} \quad w_4^{(3)}(1,\lambda) = -1. \tag{3.3}$$

For $\xi \in [0,1]$, we denote by $\wp(f,g,h)(\xi)$ the following determinant:

$$\wp(f,g,h)(\xi) := \begin{vmatrix} f(\xi) & g(\xi) & h(\xi) \\ f^{(1)}(\xi) & g^{(1)}(\xi) & h^{(1)}(\xi) \\ f^{(2)}(\xi) & g^{(2)}(\xi) & h^{(2)}(\xi) \end{vmatrix}, \tag{3.4}$$

where f, g and h are regular functions.

By definition the eigenvalues of A_a are the poles of the resolvent $(A_a - \lambda)^{-1}$.

Solving $(A_a - \lambda)[u,v] = [f,g]$ is equivalent to find the vector $[u,v]$ such that $v = \lambda u + f$ and

$$u^{(4)} + \lambda\big(2a(x)+\lambda\big) = -g - \big(2a(x)+\lambda\big)f.$$

Solving the latter via the Green's operator, $u = -G(\lambda)(g + (2a(x)+\lambda)f)$, we find

$$(A_a - \lambda)^{-1} = \begin{pmatrix} -G(\lambda)(2a(x)+\lambda) & -G(\lambda) \\ \mathrm{Id} - \lambda G(\lambda)(2a(x)+\lambda) & -\lambda G(\lambda) \end{pmatrix}. \tag{3.5}$$

This Green's operator is $[G(\lambda)\varphi](\xi) = \int_0^1 \mathcal{G}(\xi,x;\lambda)\varphi(x)\,dx$, where

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$$\mathcal{G}(\xi, x; \lambda) := \frac{1}{\Delta_{2,4}(1,\lambda)} \begin{cases} \wp(u_2, u_4, w_4)(\xi) w_2(x,\lambda) + \wp(u_2, w_2, u_4)(\xi) w_4(x,\lambda) & \text{if } 0 \le x < \xi \\ \wp(w_2, u_4, w_4)(\xi) u_2(x,\lambda) + \wp(u_2, w_2, w_4)(\xi) u_4(x,\lambda) & \text{if } \xi < x \le 1, \end{cases} \tag{3.6}$$

where $u_2(x,\lambda)$, $u_4(x,\lambda)$, $w_2(x,\lambda)$ and $w_4(x,\lambda)$ solve (2.12) subject to (2.13), (2.14), (3.2) and (3.3) respectively. Here $\wp(\cdot,\cdot,\cdot)(\xi)$ is given by (3.4) and $\Delta_{2,4}(1,\lambda)$ was introduced in (2.15). This kind of representation is used by Birkhoff for more general boundary value problem of ordinary linear differential equations, see [5, p. 377].

Proposition 3.1. *The operator A_a and its spectra satisfy the following properties:*

(1) *The operator A_a possesses a compact inverse and so a discrete spectrum $\sigma(A_a)$ of eigenvalues of finite algebraic multiplicity.*
(2) *The eigenvalues are the roots of $\lambda \mapsto \Delta_{2,4}(1,\lambda)$. If λ_k is such a root then $W(x,\lambda_k) = u(x,\lambda_k)[1,\lambda_k]$, where $u(x,\lambda_k)$ is given by (2.18) or (2.19) at λ_k. It spans the corresponding eigenspace and its algebraic multiplicity is the order to which $\Delta_{2,4}(1,\lambda)$ vanishes.*
(3) *The spectrum of A_a is symmetric about the real axis and is contained in $\mathcal{C} \cup \mathcal{I}$, where $\mathcal{C}$ is a complex strip given by:*

$$\mathcal{C} := \left\{\lambda \in \mathbb{C};\ |\lambda| \ge \pi^2,\ -\beta \le \operatorname{Re}(\lambda) \le -\alpha\right\} \tag{3.7}$$

and $\mathcal{I}$ is the following real interval:

$$\mathcal{I} := \left[-\beta - \left(\beta^2 - \pi^4\right)_+^{\frac{1}{2}}, -\alpha + \left(\beta^2 - \pi^4\right)_+^{\frac{1}{2}}\right]. \tag{3.8}$$

Here $(\beta^2 - \pi^4)_+ = \max(\beta^2 - \pi^4, 0)$.
(4) *The generalized eigenvectors of A_a are complete in $V \times L^2(0,1)$.*

Proof.

(1) Since the domain $\mathcal{D}$ of the operator A_a is compactly embedded in the energy space $V \times L^2(0,1)$ then the spectrum $\sigma(A_a)$ is discrete and the eigenvalues of A_a have a finite algebraic multiplicity. Much relevant information can be obtained directly from the kernel of $G(0)$.
(2) Let λ_k be an eigenvalue of A_a, and let $W(\cdot,\lambda_k)$ be the corresponding eigenvector. We recall that $W(\cdot,\lambda_k) = u(x,\lambda_k)[1,\lambda_k]$, where $u(x,\lambda_k)$ is given by (2.18) or (2.19) at λ_k. As the initial value problem (2.12)–(2.13) (resp. (2.12)–(2.14)) possesses the unique solution $u_2(\cdot,\lambda_k)$ (resp. $u_4(\cdot,\lambda_k)$). Hence, the geometric multiplicity of each eigenvalue is one. Its algebraic multiplicity is its order as a pole of the resolvent, which is equal to its order as a zero of $\lambda \mapsto \Delta_{2,4}(1,\lambda)$. As in [25] (see also Theorem 4.1 in [21]), this follows from (3.5) and (3.6).
(3) Since A_a is a matrix-valued differential operator with real coefficients, it follows that $\overline{W(x,\lambda_k)} = W(x,\overline{\lambda_k}) = u(x,\overline{\lambda}_k)[1,\overline{\lambda}_k]$ is an eigenvector of A_a corresponding to the eigenvalue $\overline{\lambda_k}$.
Multiplying (2.12) by $\overline{u(x,\lambda)} = u(x,-\lambda)$, integrating on $[0,1]$ and using the boundary conditions (2.13), (2.14), (2.15), we check by solving a quadratic equation that

$$\lambda_{\pm k} = \frac{-\int_0^1 a(x)|u(x,\lambda_k)|^2 dx \pm (\int_0^1 a(x)|u(x,\lambda_k)|^2 dx - \|u^{(2)}(\cdot,\lambda_k)\|_{L^2}^2 \|u(\cdot,\lambda_k)\|_{L^2}^2)^{\frac{1}{2}}}{\|u(\cdot,\lambda_k)\|_{L^2}^2}.$$

Hence, if λ_k is a nonreal eigenvalue, we find

$$\lambda_{\pm k} = -\frac{\int_0^1 a(x)|u(x,\lambda_k)|^2\,dx}{\|u(\cdot,\lambda_k)\|_{L^2}^2} \pm i\sqrt{\left(\frac{\|u^{(2)}(\cdot,\lambda_k)\|_{L^2}}{\|u(\cdot,\lambda_k)\|_{L^2}}\right)^2 - \left(\frac{\int_0^1 a(x)|u(x,\lambda_k)|^2\,dx}{\|u(\cdot,\lambda_k)\|_{L^2}^2}\right)^2},$$

which together with (3.1) yields,

$$0 < -\beta \le \mathrm{Re}(\lambda_{\pm k}) = -\frac{\int_0^1 a(x)|u(x,\lambda_k)|^2\,dx}{\|u(\cdot,\lambda_k)\|_{L^2}^2} \le -\alpha,$$

and

$$|\lambda_{\pm k}|^2 = \left(\frac{\|u^{(2)}(\cdot,\lambda_k)\|_{L^2}}{\|u(\cdot,\lambda_k)\|_{L^2}}\right)^2 \ge \pi^4.$$

If λ_k is real we observe that

$$\sqrt{\left(\frac{\int_0^1 a(x)|u(x,\lambda_k)|^2\,dx}{\|u(\cdot,\lambda_k)\|_{L^2}^2}\right)^2 - \left(\frac{\|u^{(2)}(\cdot,\lambda_k)\|_{L^2}}{\|u(\cdot,\lambda_k)\|_{L^2}}\right)^2} \le \left(\beta^2 - \pi^4\right)_+^{\frac{1}{2}}.$$

(4) The operator A_a is a bounded perturbation of a skew-adjoint (undamped) operator with compact resolvent. It follows from Theorem 10.1 in [12, Chapter 5] that the system of generalized eigenvectors is complete in $V \times L^2(0,1)$. □

3.2. Results on high frequencies

In this subsection, we will prove that all eigenvalues of A_a with sufficiently large modulus are algebraically simple and that the system of generalized eigenvectors of the operator A_a constitutes a Riesz basis in the energy space $V \times L^2(0,1)$. For this end, since the distance between two consecutive eigenvalues tends to infinity at infinity, as well as the fact that the damping is bounded, we construct a closed curves $(\Gamma^{(k)})_{|k|>N_0}$ (for some integer N_0 sufficiently large) in the complex plane such that:

(i) For all $n \in \mathbb{N}^*$, $\Gamma^{(\pm n)}$ is centered in $\pm in^2\pi^2$.
(ii) Inside each $\Gamma^{(\pm n)}$ there exits exactly one simple eigenvalue of A_a.
(iii) The operator A_a has exactly $2N_0$ eigenvalues including multiplicity in $\mathbb{C} \setminus (\bigcup_{|k|>N_0} \Gamma^{(k)})$.
(iv) $\sum_{|k|>N_0} \|P^a_{\Gamma^{(k)}} - P^0_{\Gamma^{(k)}}\|^2_{\mathcal{L}(V\times L^2(0,1))} < \infty$, where $P^a_{\Gamma^{(k)}}$ (resp. $P^0_{\Gamma^{(k)}}$) denotes the Riesz projection associated to A_a (resp. A_0) corresponding to $\Gamma^{(k)}$.

The proof of the above statements is based on some resolvent estimates of the operators A_a and A_0. Moreover, since the generalized eigenvectors of A_a are complete it follows from (iv) that the system of generalized eigenvectors of A_a constitutes a Riesz basis in $V \times L^2(0,1)$. Notice that one can deduce (ii) and (iv) from Theorem 4.15a in [15]. For completeness we give a self-contained proof.

Let us introduce some notations. For $n \in \mathbb{N}^*$, we let $\delta_n := |i(n+1)^2\pi^2 - in^2\pi^2| = (2n+1)\pi^2$ be the distance between two consecutive eigenvalues of A_0. We define the four complex numbers:

$$a_n = i\left[(n-1)^2\pi^2 + \frac{1}{2}\delta_{n-1}\right], \qquad b_n = \frac{1}{2}\delta_n + in^2\pi^2,$$
$$c_n = i\left[n^2\pi^2 + \frac{1}{2}\delta_n\right] = a_{n+1} \quad \text{and} \quad d_n = -\frac{1}{2}\delta_n + in^2\pi^2. \tag{3.9}$$

Let $\mathrm{Int}(\Gamma^{(n)})$ denote the rectangle with sides $\gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_3^{(n)}$ and $\gamma_4^{(n)}$ (see Fig. 1), where

$$\gamma_1^{(n)} := \left\{\lambda \in \mathbb{C};\ \mathrm{Im}(\lambda) = \mathrm{Im}(a_n) \text{ and } \left|\mathrm{Re}(\lambda)\right| < \frac{\delta_n}{2}\right\},$$
$$\gamma_2^{(n)} := \left\{\lambda \in \mathbb{C};\ \mathrm{Re}(\lambda) = \frac{\delta_n}{2} \text{ and } \mathrm{Im}(a_n) \le \mathrm{Im}(\lambda) \le \mathrm{Im}(c_n)\right\},$$
$$\gamma_3^{(n)} := \left\{\lambda \in \mathbb{C};\ \mathrm{Im}(\lambda) = \mathrm{Im}(c_n) \text{ and } \mathrm{Re}(\lambda) \text{ goes from } \frac{\delta_n}{2} \text{ to } -\frac{\delta_n}{2}\right\},$$

and

$$\gamma_4^{(n)} := \left\{\lambda \in \mathbb{C};\ \mathrm{Re}(\lambda) = -\frac{\delta_n}{2} \text{ and } \mathrm{Im}(\lambda) \text{ goes from } \mathrm{Im}(c_n) \text{ to } \mathrm{Im}(a_n)\right\}.$$

For $n = 1, 2, \dots$, we set

$$\Gamma^{(n)} = \gamma_1^{(n)} \cup \gamma_2^{(n)} \cup \gamma_3^{(n)} \cup \gamma_4^{(n)}, \qquad \Gamma^{(-n)} := \left\{z \in \mathbb{C};\ \bar{z} \in \Gamma^{(n)}\right\} \tag{3.10}$$

and

$$C^{(n)} = \left\{z \in \mathbb{C};\ \left|\mathrm{Im}(z)\right| < \left(n^2 + n + \frac{1}{2}\right)\pi^2 \text{ and } \left|\mathrm{Re}(z)\right| < n\right\}.$$

Note that by construction $\mathrm{Int}(\Gamma^{(k)}) \cap \mathrm{Int}(\Gamma^{(n)}) = \emptyset$ for all $k, n \in \mathbb{Z}^*$ such that $k \neq n$. Here we denote the interior of $\Gamma^{(k)}$ by $\mathrm{Int}(\Gamma^{(k)})$. Moreover, for all $N \in \mathbb{N}^*$ we have $\mathcal{C} \cup \mathcal{I} \subset C^{(N)} \cup (\bigcup_{|k|\ge N} \mathrm{Int}(\Gamma^{(k)}))$, where $\mathcal{C}$ and $\mathcal{I}$ are given by (3.7) and (3.8) respectively.

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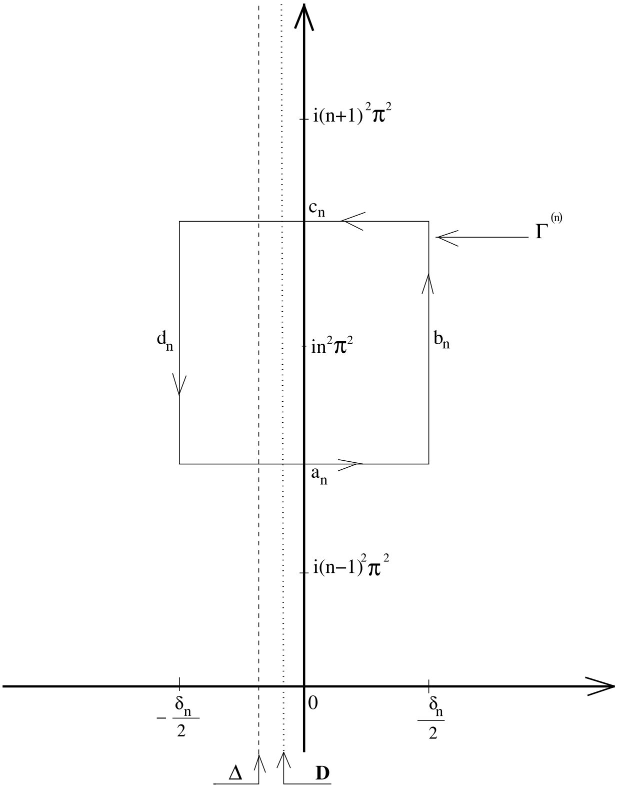

Fig. 1. Here $D=\{z\in\mathbb{C}; \mathrm{Re}(z)=-2\int_0^1 a(s)\,ds\}$, $\Delta=\{z\in\mathbb{C};\ \mathrm{Re}(z)=-2\|a\|_{L^\infty(0,1)}\}$ and a_n, b_n, c_n, d_n given by (3.9). The spectrum of A_a is included in the strip $-2\|a\|_{L^\infty(0,1)} \le \mathrm{Re}(z) \le -\inf a(x) + (\|a\|^2_{L^\infty(0,1)} - \pi^4)_+^{\frac{1}{2}}$.

The principal result of this subsection is the following:

Theorem 3.2. *Let $a(x)$ be in $L^\infty(0,1)$. There exists $N_0\in\mathbb{N}^*$ large enough such that the operator A_a has exactly $2N_0$ eigenvalues, including multiplicity, in C_{N_0} and one simple eigenvalue in $\mathrm{Int}(\Gamma^{(k)})$ for each k with $|k|>N_0$. This exhausts the spectrum of A_a.*

We have divided the proof into a sequence of lemmas.

Lemma 3.3. *Assume that $a\in L^\infty(0,1)$. There exist $C>0$ and $N_0\in\mathbb{N}$ (large enough) such that for $n>N_0$, the following properties hold:*

(i) $\Gamma^{(\pm n)}\cup\partial C^{(n)}\subset\mathbb{C}\setminus(\sigma(A_a)\cup\sigma(A_0))$.

(ii)

$$\left\|(\lambda-A_a)^{-1}-(\lambda-A_0)^{-1}\right\|_{\mathcal{L}(V\times L^2)}\le\frac{C}{n^2},\quad \textit{uniformly on } \lambda\in\Gamma^{(\pm n)}\cup\partial C^{(n)} \tag{3.11}$$

where $\partial C^{(n)}$ is the boundary of the rectangle $C^{(n)}$.

Proof. Since A_0 is skew-adjoint, it follows that

$$\left\|(\lambda - A_0)^{-1}\right\|_{\mathcal{L}(V\times L^2)} \le \frac{1}{\operatorname{dist}(\lambda, \sigma(A_0))}. \tag{3.12}$$

By construction of $\Gamma^{(\pm n)}$ and $C^{(n)}$, we have:

$$\begin{aligned}\operatorname{dist}\big(\Gamma^{(\pm n)}, \sigma(A_0)\big) &= \min\big(\left|b_n - in^2\pi^2\right|, \left|d_n - in^2\pi^2\right|, \left|c_n - i(n+1)^2\pi^2\right|, \left|a_n - i(n-1)^2\pi^2\right|\big)\\ &= \frac{\delta_{n-1}}{2},\end{aligned}$$

and $\operatorname{dist}(\partial C^{(n)}, \sigma(A_0)) \ge n$, which together with (3.12) yields

$$\left\|(\lambda - A_0)^{-1}\right\|_{\mathcal{L}(V\times L^2)} \le \frac{1}{(n-\frac{1}{2})\pi^2}, \quad \text{uniformly on } \lambda \in \Gamma^{(\pm n)}, \tag{3.13}$$

$$\left\|(\lambda - A_0)^{-1}\right\|_{\mathcal{L}(V\times L^2)} \le \frac{1}{n}, \quad \text{uniformly on } \lambda \in \partial C^{(n)}. \tag{3.14}$$

Recalling that $A_0 - A_a =: K_a$ where K_a is the bounded linear operator on $V \times L^2(0,1)$ defined by $K_a = 2a(x)\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}\end{pmatrix}$. From (3.13) and (3.14), we have

$$\left\|K_a(\lambda - A_0)^{-1}\right\|_{\mathcal{L}(V\times L^2)} \le \frac{2\|a\|_\infty}{n}, \quad \text{uniformly on } \lambda \in \Gamma^{(\pm n)} \cup \partial C^{(n)}. \tag{3.15}$$

Choose N_0 such that for $n \ge N_0$:

$$\frac{2\|a\|_\infty}{n} < 1.$$

Now the first statement of the lemma follows from (3.13), (3.14), (3.15) and the following obvious equality:

$$\lambda - A_a = \left[\mathrm{Id} + K_a(\lambda - A_0)^{-1}\right](\lambda - A_0). \tag{3.16}$$

On the other hand (3.16) yields

$$(\lambda - A_a)^{-1} = (\lambda - A_0)^{-1} + (\lambda - A_0)^{-1}\sum_{p\ge 1}\left[-K_a(\lambda - A_0)^{-1}\right]^p,$$

which together with (3.13), (3.14) and (3.15) implies (3.11). □

According to Lemma 3.3, for $n \ge N_0$ the following Riesz projections are well defined:

$$P^a_{\Gamma^{(\pm n)}} := \frac{1}{2\pi i}\int_{\Gamma^{(\pm n)}} (\lambda - A_a)^{-1}\,d\lambda, \qquad P^0_{\Gamma^{(\pm n)}} := \frac{1}{2\pi i}\int_{\Gamma^{(\pm n)}} (\lambda - A_0)^{-1}\,d\lambda,$$

$$P^a_{\partial C^{(n)}} := \frac{1}{2\pi i}\int_{\partial C^{(n)}} (\lambda - A_a)^{-1}\,d\lambda \quad \text{and} \quad P^0_{\partial C^{(n)}} := \frac{1}{2\pi i}\int_{\partial C^{(n)}} (\lambda - A_0)^{-1}\,d\lambda. \tag{3.17}$$

The following result is a simple consequence of (3.11) and the fact that $\operatorname{long}(\partial C^{(n)})$, $\operatorname{long}(\Gamma^{(\pm n)}) = \mathcal{O}(n)$.

Lemma 3.4. *There exists $C > 0$ (independent of n) and $N_0 \in \mathbb{N}$ such that for $n \geq N_0$, we have*

$$\left\| P^a_{\Gamma^{(\pm n)}} - P^0_{\Gamma^{(\pm n)}} \right\|_{\mathcal{L}(V \times L^2)} \leq \frac{C}{n} < 1, \tag{3.18}$$

$$\left\| P^a_{\partial C^{(n)}} - P^0_{\partial C^{(n)}} \right\|_{\mathcal{L}(V \times L^2)} \leq \frac{C}{n} < 1. \tag{3.19}$$

End of the proof of Theorem 3.2. First, recalling that if P and Q are two projectors with $\|P - Q\| < 1$, then $\operatorname{rank}(P) = \operatorname{rank}(Q)$ (see Lemma 3.1 in [12]). Thus, in the notation of Lemma 3.3, we have

$$\operatorname{rank}\bigl(P^a_{\partial C^{(n)}}\bigr) = \operatorname{rank}\bigl(P^0_{\partial C^{(n)}}\bigr), \qquad \operatorname{rank}\bigl(P^a_{\Gamma^{(\pm n)}}\bigr) = \operatorname{rank}\bigl(P^0_{\Gamma^{(\pm n)}}\bigr), \quad \text{for } n \geq N_0.$$

Next, we conclude from (3.7) and (3.8) that $\mathcal{C} \cup \mathcal{I} \subset C^{(N_0)} \cup (\bigcup_{|k| \geq N_0} \operatorname{Int}(\Gamma^{(k)}))$, hence that $\sigma(A_a)$ is a subset of $C^{(N_0)} \cup (\bigcup_{|k| \geq N_0} \operatorname{Int}(\Gamma^{(k)}))$. Now Theorem 3.2 follows from the fact that

$$\operatorname{rank}\bigl(P^0_{\partial C^{(N_0)}}\bigr) = 2N_0 \quad \text{and} \quad \operatorname{rank}\bigl(P^0_{\Gamma^{(\pm n)}}\bigr) = 1. \qquad \square$$

Remark 3.5. In the proofs of Lemmas 3.3–3.4, we have used only the fact that the distance between two consecutive eigenvalues of A_0 tends to infinity and the fact that A_0 is a skew-adjoint operator. Similar general results are well-known (see Theorem 4.15a in [15]). Note that this approach cannot be applied to the damped wave equation since the spectral gap in this case is equal to π (do not tends to infinity at infinity).

3.3. Riesz basis

In this subsection we construct a Riesz basis consisting of generalized eigenvectors of A_a. First, since the associated high frequencies are simple, then for all $k \in \mathbb{N}^*$, $k > N_0$ (N_0 given by Theorem 3.2), we define $\varphi_{\pm k} := P^a_{\Gamma^{(\pm k)}} V_{\pm k}$ where $V_{\pm k}$ is the eigenvector of A_0 associated to the eigenvalue $\pm i k^2 \pi^2$ and $P^a_{\Gamma^{(\pm k)}}$ is given by (3.17). We get the following lemma:

Lemma 3.6. *For all $k \in \mathbb{N}^*$, $k > N_0$, the vector $\varphi_{\pm k}$ is an eigenvector of A_a associated to the eigenvalue $\lambda_{\pm k}$. Moreover, there exists $C > 0$ such that*

$$\|\varphi_n - V_n\|_{V \times L^2} \leq \frac{C}{|n|}, \quad \textit{for all } n \in \mathbb{Z}^*,\ |n| > N_0. \tag{3.20}$$

In particular, $\|\varphi_n\|_{V \times L^2} = 1 + \mathcal{O}(\frac{1}{|n|})$ uniformly for $n \in \mathbb{Z}^*$, $|n| > N_0$.

Proof. For all $m \in \mathbb{Z}^*$, $|m| > N_0$, we have $A_a \varphi_m = A_a P^a_{\Gamma(m)} V_m = \lambda_m P^a_{\Gamma(m)} V_m = \lambda_m \varphi_m$. Using Lemma 3.4 and the fact that $P^0_{\Gamma(n)} V_n = V_n$ with $\|V_n\|_{V \times L^2} = 1$, we get:

$$\|\varphi_m - V_m\|_{V \times L^2} = \left\| \left(P^a_{\Gamma(m)} - P^0_{\Gamma(m)}\right) V_m \right\|_{V \times L^2} \le \left\| P^a_{\Gamma(m)} - P^0_{\Gamma(m)} \right\|_{\mathcal{L}(V \times L^2)} \le \frac{C}{|m|},$$

for all $m \in \mathbb{Z}^*$, $|m| > N_0$ (C independent of m). In particular, parallelogram inequality and recalling that $\|V_m\|_{V \times L^2} = 1$ give that $\|\varphi_m\|_{V \times L^2} = 1 + \mathcal{O}(\frac{1}{|m|})$ uniformly for $m \in \mathbb{Z}^*$, $|m| > N_0$. $\square$

Now, we complete the sequence $(\varphi_k)_{|k|>N_0}$ of the eigenvectors associated to the high frequencies of A_a by considering the generalized eigenvectors associated to the low frequencies of A_a. Note that the number of these generalized eigenvectors associated to the low frequencies of A_a is finite, at most $2N_0$ by Theorem 3.2. For $k \in \mathbb{Z}^*$ such that $|k| \le N_0$, we denote by m_k the algebraic multiplicity of λ_k and we associated to it the Jordan chain of generalized eigenvectors, $(W_{k,p})_{p=0}^{m_k-1}$,

$$W_{k,0} = u(x, \lambda_k)[1, \lambda_k], \quad \text{where } u(x, \lambda_k) \text{ is given by (2.18) (or (2.19))}, \tag{3.21}$$

$$A_a W_{k,p} = \lambda_k W_{k,p} + W_{k,p-1}, \qquad \langle W_{k,p}, W_{k,p-1} \rangle = 0, \quad p = 1, \cdots, m_k - 1. \tag{3.22}$$

The vector $W_{k,0}$ is an eigenvector of A_a associated to λ_k and the chain is a basis for the root subspace $\mathcal{E}_k := \{W \in V \times L^2(0,1);\ (A_a - \lambda_k)^{m_k} W = 0\}$.

Now, we take the family of generalized eigenvectors of A_a:

$$\mathcal{B}_a := (W_{k,p})_{|k| \le N_0, 0 \le p \le m_k - 1} \cup (\varphi_n)_{|n|>N_0}.$$

Since $\overline{\mathrm{Vect}(\mathcal{B}_a)} = V \times L^2(0,1)$ (see Proposition 3.1(4)) and the family $\mathcal{B}_a$ is quadratically close to the orthonormal basis $(V_k)_{k \in \mathbb{Z}^*}$ of eigenvectors of the undamped operator (see (3.20)), it now follows from the Fredholm Alternative, see e.g., [27, Appendix D, Theorem 3], the following result:

Theorem 3.7. *The set $\mathcal{B}_a$ is a Riesz basis for the energy space $V \times L^2(0,1)$. Moreover, there exists a linear isomorphism Φ_a of $V \times L^2(0,1)$ such that for all $n \in \mathbb{Z}^*$, $|n| > N_0$, $\Phi_a V_n = \varphi_n$ and $\Phi_a(\mathrm{Vect}(V_n, |n| \le N_0)) = \mathrm{Vect}(W_{k,p}, |k| \le N_0, 0 \le p \le m_k - 1)$.*

3.4. Proof of the main result

For the proof of Theorem 1.2 in the general setting, we follow the same strategy as in the constant damping case. Using Theorem 3.7, we may expand the initial data as

$$[u^0, v^0] = \sum_{|k| \le N_0} \sum_{p=0}^{m_k-1} c_{k,p} W_{k,p} + \sum_{|n|>N_0} c_n \varphi_n.$$

Then the solution of (1.1)–(1.3) is given by

$$[u, \partial_t u] = \sum_{|k|\le N_0} \exp(\lambda_k t) \sum_{p=0}^{m_k-1} c_{k,p} \sum_{l=0}^{p} \frac{t^{p-l}}{(p-l)!} W_{k,l} + \sum_{|n|>N_0} c_n \exp(\lambda_n t)\varphi_n. \tag{3.23}$$

Recalling from Theorem 3.2 that at most $2N_0$ eigenvalues may be of algebraic multiplicity greater than one and that $2N_0$ is the maximum of such multiplicity, and the family $\mathcal{B}_0 := (V_{\pm k})_{k\in\mathbb{N}^*}$ is an orthonormal basis of the energy space $V \times L^2(0, 1)$ (see Lemma 2.1), by the linear isomorphism Φ_a, (as in the constant case), we get

$$E\big(u(t)\big) = \big\|[u, \partial_t u]\big\|^2_{V\times L^2(0,1)} \le \|\Phi_a\|^2 \big\|\Phi_a^{-1}\big\|^2 \big(1 + t^{2N_0}\big) \exp\big(2\mu(a)t\big) E\big(u(0)\big).$$

Then $\omega(a) \le \mu(a)$, and with inequality (1.11) we have established our main result. □

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