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Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion

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Abstract

We show the existence of locally bounded global solutions to the chemotaxis system

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\frac{u}{v}\nabla v) & \text{in } \Omega\times(0,\infty) \\ v_t = \Delta v - uv & \text{in } \Omega\times(0,\infty) \\ \partial_\nu u = \partial_\nu v = 0 & \text{in } \partial\Omega\times(0,\infty) \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 & \text{in } \Omega \end{cases}$$

in smooth bounded domains $\Omega\subset\mathbb{R}^N$, $N\geq 2$, for $D(u)\geq \delta u^{m-1}$ with some $\delta>0$, provided that $m > 1+\frac{N}{4}$.

MSC: 35K55; 35A01; 35K65; 92C17

Keywords: Keller–Segel; Chemotaxis; Nonlinear diffusion; Global existence; Boundedness

1. Introduction

Even simple, small organisms can exhibit comparatively complex and macroscopically apparent collective behaviour. Bacteria of the species *E. coli*, for example, when set in a capillary

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tube featuring a gradient of nutrient concentration form bands that are visible to the naked eye and migrate with constant speed. Following experimental works of Adler (see e.g. [1,2]), in 1971 Keller and Segel [14] introduced a phenomenological model to capture this kind of behaviour, a prototypical version of which is given by

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (\frac{u}{v}\nabla v) & \text{in } \Omega \times (0,\infty) \\ v_t = \Delta v - uv & \text{in } \Omega \times (0,\infty) \end{cases} \tag{1}$$

with $D(u) \equiv 1$. Herein, u represents the density of bacteria and v is used to denote the concentration of the nutrient. In the model in [14], the diffusion coefficient $D(u)$ is supposed to be constant, thus leading to the typical effect of linear diffusion which causes any population to spread with infinite speed of propagation. In order to avoid this (biologically clearly unrealistic) behaviour, it might be desirable to allow for diffusion of porous medium type (i.e. $D(u) = u^{m-1}$), cf. also [3, p. 1665].

Nevertheless, starting with [14], the model with linear diffusion has successfully been employed to find travelling wave solutions (see e.g. the overview in [37] and references cited therein) and also their stability has been investigated [19,27].

In spite of the rich literature concerned with travelling wave solutions (for such solutions to related systems see also [24,25,20], or [11,26]), little is known about existence of solutions for more general initial data (see below).

The difficulty lies in the hazardous combination of the consumptive effect of the second equation on the nutrient concentration with the singular chemotactic sensitivity in the first: While the second equation compels v to shrink, it is the cross-diffusive contribution of the chemotaxis term that seeks to enlarge the solutions to (1). And it is this very term that is furnished with a large coefficient whenever v becomes small.

For a moment leaving aside the logarithmic shape of the sensitivity in $\nabla \cdot (\frac{u}{v}\nabla v) = \nabla \cdot (u\nabla \log v)$, we are led to the system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - uv, \end{cases} \tag{2}$$

which also appears as part of chemotaxis fluid systems intensively studied during the past six years. (The interested reader can consult the introduction of [16].) Even in (2), global existence of classical solutions is not yet known, apart from 2-dimensional settings [41] or under smallness conditions on v_0 [35].

Although the mathematical difficulty in treating the system vastly increases when a logarithmic sensitivity is included, this form is important. Not only is it needed for the emergence of travelling waves [14,13,32], there are also models giving a detailed mechanistic basis [45] and experimental evidence asserting this form [12].

In those Keller–Segel models (cf. [10,9,3]) where v does not stand for a nutrient to be consumed but a signalling substance produced by the bacteria themselves, i.e. the evolution is governed by

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (\frac{u}{v}\nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

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the singularity in the sensitivity function is mitigated by v tending to stay away from 0 thanks to the production term in the second equation. (For this system, global solutions are known to exist if χ is sufficiently small, where the precise condition depends on the dimension as well as on whether classical [17,40,4] or weak solutions [34,40] are considered and on radial symmetry of initial data [4,28]; but for large χ also blow-up may occur in the corresponding parabolic–elliptic system [28].) The proof of boundedness of solutions for $\chi < \sqrt{\frac{2}{N}}$ in [6] even relies on the second equation actually ensuring a positive pointwise lower bound for v.

In (1), we cannot hope for such a convenient bound and thus have to deal with the influence of the actual singularity in the sensitivity function.

Nevertheless, for $D \equiv 1$, in the domains $\mathbb{R}^2$ and $\mathbb{R}^3$ a global existence result was achieved for initial data that are $H^1 \times H^1$-close to $(\overline{u}, 0)$ for some $\overline{u} > 0$ [38]. The proof rests on energy estimates for a hyperbolic system into which (1) can be converted by means of the Hopf–Cole type transformation $q := \frac{\nabla v}{v}$ that had been introduced in [18] for the treatment of an angiogenesis model.

More recently it has become possible to treat general initial data (the only restrictions being positivity and regularity assumptions) for the system in bounded planar domains [43], where it was shown that global generalized solutions to (1) with $D \equiv 1$ exist whose second component v moreover converges to 0 with respect to the norm in any $L^p(\Omega)$ for $p \in [1, \infty)$ and to the weak-$*$ topology of $L^\infty(\Omega)$. If, moreover, the initial mass of bacteria is small, the solution becomes eventually smooth [44] and converges to the homogeneous steady state. In [44] also an explicit smallness condition on u_0 in $L \log L(\Omega)$ and $\nabla \ln v_0$ in $L^2(\Omega)$ has been found that ensures the global existence of classical solutions.

Solutions emanating from large data, however, have not been proven to be bounded and might blow up and cease to exist as classical solutions after a finite time, continuing only as generalized solutions in the sense of [43]. In higher-dimensional domains, even the existence of such solutions is unknown. Only in a radially symmetric setting "renormalized solutions" have been constructed [42].

In the present article, we aim to find solutions to (1) that are locally bounded and hence do not blow up in finite time. For this, we will rely on stronger growth of D, i.e. on the nonlinear diffusion we want to include. More precisely, we assume that with some $m \geq 1$, which will be subject to further conditions, and $\delta > 0$

$$D \in \mathcal{C}_{\delta,m} := \left\{ d \in C^1([0,\infty));\, d(s) \geq \delta s^{m-1} \text{ for all } s \in [0,\infty) \right\}.$$

In a first step we will additionally require strict positivity of D, i.e.

$$D \in \mathcal{C}^+_{\delta,m} := \left\{ d \in C^1([0,\infty));\, d(s) \geq \delta s^{m-1} \text{ for all } s \in [0,\infty) \text{ and } d(0) > 0 \right\}$$

and prove global existence of classical solutions to (1):

Theorem 1.1. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Then for every $\delta > 0$ and $m \geq 1$ satisfying*

$$m > 1 + \frac{N}{4}, \tag{3}$$

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every $D \in \mathcal{C}^+_{\delta,m}$ *and every pair* (u_0, v_0) *of initial data fulfilling*

$$u_0 \in C^\alpha(\overline{\Omega}) \text{ for some } \alpha \in (0,1), \qquad v_0 \in W^{1,\infty}(\Omega), \qquad u_0 \geq 0, \quad v_0 > 0 \quad \text{in } \overline{\Omega} \tag{4}$$

the initial boundary value problem

$$u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot \left(\frac{u}{v}\nabla v\right) \qquad \text{in } \Omega \times (0, T_{max}) \tag{5a}$$

$$v_t = \Delta v - uv \qquad \text{in } \Omega \times (0, T_{max}) \tag{5b}$$

$$\partial_\nu u = 0 \qquad \text{in } \partial\Omega \times (0, T_{max}) \tag{5c}$$

$$\partial_\nu v = 0 \qquad \text{in } \partial\Omega \times (0, T_{max}) \tag{5d}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \Omega \tag{5e}$$

$$v(\cdot, 0) = v_0 \qquad \text{in } \Omega \tag{5f}$$

has a classical solution $(u, v) \in (C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})))^2$ *which is global (i.e.* $T_{max} = \infty$*).*

Afterwards dropping the strict positivity assumption on D, we will use an approximation procedure and finally prove the existence of global weak solutions that are locally bounded:

Theorem 1.2. *Let* $N \geq 2$ *and* $\Omega \subset \mathbb{R}^N$ *be a bounded smooth domain. Then for every* $\delta > 0$ *and* $m > 1 + \frac{N}{4}$*, every initial data*

$$u_0 \in L^{\max\{1,m-1\}}(\Omega), \qquad v_0 \in W^{1,\infty}(\Omega), \qquad u_0 \geq 0, \qquad v_0 > 0 \tag{6}$$

and every $D \in \mathcal{C}_{\delta,m}$*, (5) has a global locally bounded weak solution* (u, v) *(in the sense of Definition 4.1), which in particular satisfies*

$$\|u\|_{L^\infty(\Omega\times(0,T))} < \infty \qquad \text{for every } T \in (0, \infty).$$

We will devote Section 2 to the proof of local existence of solutions and an extensibility criterion. In the proof of boundedness that follows, we will sometimes use the system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (u\nabla w) \\ w_t = \Delta w - |\nabla w|^2 + u \end{cases} \tag{7}$$

obtained from the transformation $w = -\log(\frac{v}{\|v_0\|_{L^\infty(\Omega)}})$, which has also been used in [43,44]. We note that while the first equation seems more accessible in (7) due to the lack of any singularity, it is (5), where the second equation is more amenable to the derivation of estimates on ∇v.

The first stepping stone for the proof will be a spatio-temporal L^2-bound for ∇w (Lemma 3.2), already giving some boundedness information for $\int_0^t \int_\Omega |\nabla u^{m-1}|$ and $\int_\Omega u^{m-1}(\cdot, t)$ for $t > 0$, which we can use to obtain bounds on $\int_0^t \int_\Omega |\nabla u^{m-1}|$ (Lemma 3.3) and thereby on $\int_0^t \|u\|_p^r$ for certain p, r and $t > 0$ (Lemma 3.5). One consequence of such bounds is a spatio-temporal

L^q-bound on ∇v (see Lemma 3.7), derived with the help of maximal Sobolev regularity properties of the heat equation (cf. Lemma 3.6). Another is the (local-in-time) boundedness of w (Lemma 3.9). This is important, as it will enable us to transfer bounds from ∇v to ∇w (Lemma 3.10).

Bounds on $\int_0^t\int_\Omega |\nabla w|^q$ now in turn will translate into control over $\int_\Omega u^p$ for some p (Lemma 3.12). If p is sufficiently large, this entails $L^\infty(\Omega\times(0,T))$-boundedness of $|\nabla v|$ and $|\nabla w|$ and thus finally of u (Lemma 3.11 and Lemma 2.1 v)). Thereby, the solution is not only locally bounded, but moreover exists globally, according to the extensibility criterion (15).

In Section 4 we rely on bounds already derived in the previous section to construct locally bounded weak solutions to (1) with functions D causing possibly degenerate diffusion.

Notation. Throughout the article we fix $N\in\mathbb{N}$, $N\geq 2$, and $\Omega\subset\mathbb{R}^N$ as a bounded, smooth domain. When dealing with the solution to a differential equation, we will use T_{max} to denote its maximal time of existence; in the case of (5) such T_{max} is provided by Lemma 2.4. By $\hookrightarrow$ and $\overset{cpt}{\hookrightarrow}$ we refer to continuous and compact embeddings of Banach spaces, respectively. We will sometimes write $D(u)$ for the concatenation $D\circ u$ of functions. The number $\lambda_1>0$ will always be the first positive eigenvalue of the Neumann Laplacian in Ω.

2. Local existence

We begin the proof by ensuring local existence of classical solutions in the non-degenerate case. As a first step let us, for easier reference, collect some basic results on existence of and estimates for solutions of certain parabolic PDEs.

Lemma 2.1.

i) *For any $T>0$, $q>N$ and $r>N$ and every $M>0$ there are $C_i>0$ and $\gamma>0$ such that for all nonnegative functions $v_0\in W^{1,q}(\Omega)$ and $u\in L^\infty((0,T);L^r(\Omega))$ satisfying $\|v_0\|_{W^{1,q}(\Omega)}\leq M$ and $\|u\|_{L^\infty((0,T);L^r(\Omega))}\leq M$ for the solution $v\in V_2=\{v\in L^\infty((0,T);L^2(\Omega));\nabla v\in L^2(\Omega\times(0,T))\}$ of*

$$v_t=\Delta v-uv,\quad \partial_\nu v\big|_{\partial\Omega}=0,\quad v(\cdot,0)=v_0 \tag{8}$$

one has $\|v\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])}<C_i$. If, moreover, $u\in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times(0,T])$ for some $\alpha\in(0,\gamma)$, then $v\in C^0(\overline{\Omega}\times[0,T])\cap C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times(0,T])$. If $u\in L^\infty(\Omega\times(0,T))$, then $v\in C^1(\overline{\Omega}\times(0,T])$.

ii) *For any $r\in(N,\infty]$ there is $C_{ii}=C_{ii}(r)>0$ such that for any $T>0$ and any $q\in[2,\infty]$ for all nonnegative functions $v_0\in W^{1,q}(\Omega)$ and $u\in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times(0,T))$ for some $\alpha\in(0,1)$, the solution $v\in C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])$ (for some $\gamma\in(0,\alpha)$) of (8) satisfies*

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)}\leq C_{ii}\,\|\nabla v_0\|_{L^q(\Omega)}+C_{ii}\,\|v_0\|_{L^\infty(\Omega)}\,\|u\|_{L^\infty((0,T);L^r(\Omega))}\,.$$

iii) *For every $T>0$, $\delta_0>0$, $M>0$ and $K>0$ there is $C_{iii}>0$ such that for every $u_0\in L^\infty(\Omega)$ satisfying $0\leq u_0\leq M$ and every $g\in\left(C^0((\overline{\Omega}\times(0,T))\right)^N$ fulfilling $g\cdot\nu=0$ on $\partial\Omega$ and $\|g\|_{L^\infty(\Omega)}\leq K$, and for all $A\in L^\infty(\Omega\times(0,T))$ with $A>\delta_0$ in $\Omega\times(0,T)$, the unique weak solution of*

$$u_t = \nabla\cdot(A\nabla u - g) \text{ in } \Omega\times(0,T), \quad \partial_\nu u\big|_{\partial\Omega} = 0 \text{ in } (0,T), \quad u(\cdot,0) = u_0 \text{ in } \Omega, \tag{9}$$

satisfies

$$\|u\|_{L^\infty(\Omega\times(0,T))} \le C_{iii} \quad \text{and} \quad \|\nabla u\|_{L^2(\Omega\times(0,T))} \le C_{iii}. \tag{10}$$

iv) *For any $T>0$, for any $D_0 > \delta_0 > 0$, $M>0$, $K>0$ and $\alpha\in(0,1)$ there are $C_{iv}>0$ and $\gamma\in(0,1)$ such that for every $A\in L^\infty(\Omega\times(0,T))$ fulfilling $\delta_0 < A < D_0$ a.e. in $\Omega\times(0,T)$, and for all $g\in\left(C^0(\overline{\Omega}\times(0,T))\right)^N$ with $g\cdot\nu = 0$ on $\partial\Omega$ and $\|g\|_{L^\infty(\Omega\times(0,T))}\le M$ and all $u_0\in C^\alpha(\overline{\Omega})$ with $\|u_0\|_{C^\alpha(\overline{\Omega})}\le M$, any solution u of (9) that obeys the estimate $\|u\|_{L^\infty(\Omega\times(0,T))}\le K$ satisfies*

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])} \le C_{iv}. \tag{11}$$

Moreover, if $g\in C^{\beta,\frac{\beta}{2}}(\overline{\Omega}\times(0,T])$ for some $\beta>0$, then $u\in C^{2,1}(\overline{\Omega}\times(0,T])$.

v) *For every $m\ge1$, $\delta>0$, $K>0$, $p_0\ge1$, $q_1>n+2$ and $T\in(0,\infty]$ there is $C_v>0$ such that for every $D\in C^1(\overline{\Omega}\times[0,T)\times[0,\infty))$ which obeys $D\ge0$, $D(x,t,s)\ge\delta s^{m-1}$ for all $(x,t,s)\in\Omega\times(0,T)\times(0,\infty)$ and every $f\in C^0((0,T);C^0(\overline{\Omega})\cap C^1(\Omega))$, with $f\cdot\nu\le0$ on $\partial\Omega\times(0,T)$, satisfying $\|f\|_{L^\infty((0,T);L^{q_1}(\Omega))}\le K$, for every nonnegative function $u\in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$ that satisfies $\|u\|_{L^\infty((0,T);L^{p_0}(\Omega))}\le K$ and $u_t\le\nabla\cdot(D(x,t,u)\nabla u)+\nabla\cdot f(x,t)$ in $\Omega\times(0,T)$ and $\partial_\nu u\big|_{\partial\Omega}\le0$ on $(0,T)$, we have $\|u(\cdot,t)\|_{L^\infty(\Omega)}\le C_v$ for every $t\in(0,T)$.*

Proof. i) According to [15, III.5.1], (8) has a unique weak solution $v\in V_2$ in $\Omega\times(0,T)$. The first part of the statement thus immediately results from [31, Thm. 1.3 and Remarks 1.3, 1.4], whereas the second is a consequence of a uniqueness statement [15, III.5.1] combined with the existence assertion for classical solutions in [15, IV.5.3] (applied to $\zeta_\epsilon(t)v(x,t)$ for some cutoff function $\zeta_\epsilon\in C_0^\infty([0,\infty))$, $\zeta_\epsilon\big|_{(0,\frac{\epsilon}{2})}\equiv0$, $\zeta_\epsilon\big|_{(\epsilon,\infty)}\equiv1$ for arbitrary $\epsilon>0$). The third part – actually, even Hölder-continuity of ∇v – is provided by [22, Thm. 1.1].

ii) Existence of a solution ensured as in the proof of i), we may rely on [30, Cor. 4.3.3] to represent v as mild solution via the variation of constants formula, and invoking [39, Lemma 1.3 iii)] and [39, Lemma 1.3 ii)], we gain $c_1>0$ and $c_2>0$, respectively, such that with $\rho:=\max\{q,r\}$ and by Hölder's inequality

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le{}& c_1\|\nabla v_0\|_{L^q(\Omega)} \\ &+\int_0^t c_2|\Omega|^{\frac1q-\frac1\rho}\left(1+(t-s)^{-\frac12-\frac N2(\frac1r-\frac1\rho)}\right)\|uv(\cdot,s)\|_{L^r(\Omega)}e^{-\lambda_1(t-s)}\mathrm{d}s \\ \le{}& c_1\|\nabla v_0\|_{L^q(\Omega)} + c_3\|v_0\|_{L^\infty(\Omega)}\|u\|_{L^\infty((0,T);L^r(\Omega))},\end{aligned}$$

where we have used that $\sigma^{-\frac12-\frac N2(\frac1r-\frac1\rho)}\le1+\sigma^{-\frac12-\frac N{2r}}$ for all $\sigma>0$ and all $\rho\in[1,\infty]$ and set $c_3:=\int_0^\infty c_2\left(2+\sigma^{-\frac12-\frac N{2r}}\right)e^{-\lambda_1\sigma}\mathrm{d}\sigma$, which is finite because of $r>N$, and where we have taken into account that by comparison arguments $0\le v(\cdot,t)\le\|v_0\|_{L^\infty(\Omega)}$ in Ω for all $t\in(0,T)$.

iii) This is a combination of [23, Thm. 6.38] (estimate for ∇u), [23, Thm. 6.39] (existence and uniqueness) and [23, Thm. 6.40] (uniform boundedness).

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iv) The same theorems as in the proof of i) apply.
v) This is (part of) Lemma A.1 in [36]. □

Lemma 2.2. *For every positive function $D \in C^0([0,\infty))$, for every $L > 0$ and $\alpha \in (0,1)$ there is $T > 0$ such that for every $u_0 \in C^\alpha(\overline{\Omega})$ satisfying $\|u_0\|_{C^\alpha(\overline{\Omega})} \le L$ and every $v_0 \in W^{1,\infty}(\Omega)$ which satisfies $v_0 > \frac{1}{L}$ in $\overline{\Omega}$ and fulfils $\|v_0\|_{W^{1,\infty}(\Omega)} \le L$ there is a pair of functions $(u,v) \in C^0(\overline{\Omega}\times[0,T]) \cap C^{2,1}(\overline{\Omega}\times(0,T])$ solving (5) in $\Omega\times(0,T)$.*

Proof. We let $R := L+1 \ge \|u_0\|_{L^\infty(\Omega)} + 1$. For the choice of $r := \infty$ we obtain $c_1 := C_{ii} > 0$ with properties as described in Lemma 2.1 ii), and thereupon invoking Lemma 2.1 iii) for parameters $T = 1$, $\delta_0 = \inf_{s>0} D(s)$, $M = R$, $K = c_1 R(1+R)L^2 e^R$, we are given $c_2 := C_{iii} > 0$ as in (10). An application of Lemma 2.1 iv) for $T = 1$, $\delta_0 = \inf_{s>0} D(s)$, $D_0 = \sup_{0<s<R} D(s)$, $M = \max\left\{c_1 R(1+R)L^2 e^R, L\right\}$ and $K = c_2$ provides us with $c_3 := C_{iv} > 0$ and $\gamma \in (0,1)$ as in (11). With these, we choose $T\in(0,1)$ such that $\|u_0\|_{L^\infty(\Omega)} + c_3 T^{\frac{\gamma}{2}} < R$ and introduce

$$S := \left\{\widehat{u} \in C^0(\overline{\Omega}\times[0,T]);\, 0 \le \widehat{u} \le R,\, u(\cdot,0) = u_0,\, \|\widehat{u}\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])} \le c_3\right\} \subset C^0(\overline{\Omega}\times[0,T]). \tag{12}$$

For any $\widehat{u} \in S$ we define $\widehat{u}(t) := u(T)$ for $t \in (T,1]$ and note that the solution v of

$$v_t = \Delta v - \widehat{u} v \quad \text{in } \Omega\times(0,1), \quad \partial_\nu v\big|_{\partial\Omega} = 0, \quad v(\cdot,0) = v_0 \text{ in } \Omega, \tag{13}$$

satisfies

$$\|v_0\|_{L^\infty(\Omega)} \ge v(\cdot,t) \ge (\inf v_0)\, e^{-R} \ge \frac{1}{L} e^{-R} \tag{14}$$

in Ω for all $t \in [0,1]$ and, by definition of c_1, $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le c_1(1+R)\|v_0\|_{W^{1,\infty}(\Omega)} \le c_1(1+R)L$.

We let u be the solution of

$$u_t = \nabla\cdot\left(D(\widehat{u})\nabla u - \frac{\widehat{u}}{v}\nabla v\right) \text{ in } \Omega\times(0,1), \quad \partial_\nu u\big|_{\partial\Omega} = 0, \quad u(\cdot,0) = u_0 \text{ in } \Omega.$$

Then by definition of c_2 and c_3 (with $g = \frac{\widehat{u}}{v}\nabla v$ and $A = D\circ\widehat{u}$ in (9)), $\|u\|_{L^\infty(\Omega\times(0,1))} \le c_2$ and $\|u\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,1])} \le c_3$. Hence if we define $\Phi(\widehat{u}) := u\big|_{\Omega\times(0,T)}$, we have $\|\Phi(\widehat{u})(t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + c_3 t^{\frac{\gamma}{2}} \le R$ for every $t \in (0,T)$ and every $\widehat{u} \in S$, and thus Φ is a function mapping S into itself, where S is a closed convex set in $C^0(\overline{\Omega}\times[0,T])$. Moreover, $\Phi\colon S \to S$ is continuous: We let $\overline{u} \in S$ and $\widehat{u}^k \in S$ for all $k \in \mathbb{N}$ such that $\widehat{u}^k \to \overline{u}$ in $C^0(\overline{\Omega}\times[0,T])$. Then, with respect to $\|\cdot\|_{L^\infty(\Omega\times(0,T))}$ and with respect to the weak-*-topology of $L^\infty((0,T);W^{1,\infty}(\Omega))$, the solutions v^k of (13) with $\widehat{u}$ replaced by $\widehat{u}^k$ converge to $\overline{v}$ solving (13) with $\overline{u}$ instead of $\widehat{u}$: Assuming on the contrary that there were a sequence $(k_l)_{l\in\mathbb{N}}$ such that for each subsequence $(k_{l_m})_{m\in\mathbb{N}}$ thereof the sequence $(v^{k_{l_m}})_{m\in\mathbb{N}}$ did not converge in the indicated topologies, from the uniform bounds on $\left\|v^{k_l}\right\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])}$ and on $\left\|\nabla v^{k_l}\right\|_{L^\infty((0,T);L^\infty(\Omega))}$ asserted by Lemma 2.1 i) and Lemma 2.1 ii), respectively, we could conclude the existence of some subsequence $(v^{k_{l_n}})_{n\in\mathbb{N}}$ being uniformly convergent in $\Omega \times [0,T]$ and weakly-*-convergent in $L^\infty((0,T);W^{1,\infty}(\Omega))$. By passing to the limit in the weak formulation in the equations of the form (13) satisfied by v^{k_l}, the limit can easily be seen to coincide with the unique weak solution $\overline{v}$ of (13) with $\overline{u}$ replacing $\widehat{u}$, contradicting the choice of $(v^{k_l})_{l\in\mathbb{N}}$. We observe that hence and by (14), $\frac{u^k}{v^k}\nabla v^k \overset{*}{\rightharpoonup} \frac{\overline{u}}{\overline{v}}\nabla\overline{v}$ in $L^\infty((0,T);L^\infty(\Omega))$. Similarly taking into account bounds on $\left\|\Phi(\widehat{u}^k)\right\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])}$ and $\left\|\nabla\Phi(\widehat{u}^k)\right\|_{L^2(\Omega\times(0,T))}$ as obtained from Lemma 2.1 iv) and 2.1 iii) and again employing the weak formulation of the equations defining $\Phi(\widehat{u}^k)$ and uniqueness of the solution $\Phi(\overline{u})=u$ of $u_t=\nabla\cdot(D(\overline{u})\nabla u-\frac{\overline{u}}{\overline{v}}\nabla\overline{v})$, $\partial_\nu u\big|_{\partial\Omega}=0$, $u(\cdot,0)=u_0$, we finally see that $\Phi(\widehat{u}^k)\to\Phi(\overline{u})$ in $C^0(\overline{\Omega}\times[0,T])$.

We note that $S\subset C^0(\overline{\Omega}\times[0,T])$ is a closed bounded convex set and $\Phi(S)$ is relatively compact in $C^0(\overline{\Omega}\times[0,T])$, owing to the uniform Hölder bound c_3 and Arzelà–Ascoli's theorem, so that we can apply Schauder's fixed point theorem to find $u\in S$ such that $\Phi(u)=u$. Due to the regularity assertions in Lemma 2.1 iv) even $u\in C^{2,1}(\overline{\Omega}\times(0,T])$; also the corresponding solution v of the second equation belongs to this space by 2.1 i). □

Lemma 2.3. *Let $T>0$. If on $\Omega\times(0,T)$ there is a solution (u,v) to (5) such that*

$$\sup_{t\in(0,T)}\|u(\cdot,t)\|_{L^\infty(\Omega)}<\infty,$$

then there is $\widetilde{T}>T$ such that there is a solution to (5) in $\Omega\times(0,\widetilde{T})$ which on $\Omega\times(0,T)$ coincides with (u,v).

Proof. Successive application of comparison arguments in (5b) and of Lemma 2.1 parts ii), iv) and i) show the existence of $\alpha>0$ and $M>0$ such that

$$\inf_{\Omega\times(0,T)} v>\frac{1}{M},\quad \|v\|_{L^\infty((0,T);W^{1,\infty}(\Omega))}\le M,\quad \|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[0,T])}\le M,\quad \|v\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[0,T])}\le M.$$

Due to the uniform continuity of u and v,

$$\widetilde{u}_0(x):=\lim_{t\nearrow T}u(x,t),\qquad \widetilde{v}_0(x):=\lim_{t\nearrow T}v(x,t),\qquad x\in\overline{\Omega},$$

are well-defined and satisfy $M\ge\widetilde{v}_0\ge\frac{1}{M}$ in $\overline{\Omega}$ as well as $\|\widetilde{u}_0\|_{C^\alpha(\overline{\Omega})}\le M$.

Picking a sequence $(t_k)_{k\in\mathbb{N}}\nearrow T$ and referring to $\|v\|_{L^\infty((0,T);W^{1,\infty}(\Omega))}\le M$, we may conclude the existence of a subsequence $(t_{k_l})_{l\in\mathbb{N}}$ such that $\nabla v(\cdot,t_{k_l})\overset{*}{\rightharpoonup}\nabla\widetilde{v}_0$ in $L^\infty(\Omega)$ as $l\to\infty$, and thus infer $\|\widetilde{v}_0\|_{W^{1,\infty}(\Omega)}\le M$. According to Lemma 2.2, we can find $\tau>0$ and $(\widetilde{u},\widetilde{v})\in\big(C^0(\overline{\Omega}\times[0,\tau])\cap C^{2,1}(\overline{\Omega}\times(0,\tau])\big)^2$ solving

$$\widetilde{u}_t=\nabla\cdot\left(D(\widetilde{u})\nabla\widetilde{u}-\frac{\widetilde{u}}{\widetilde{v}}\nabla\widetilde{v}\right),\qquad \widetilde{v}_t=\Delta\widetilde{v}-\widetilde{u}\widetilde{v}\qquad \text{in }\Omega\times(0,\tau),$$
$$\partial_\nu\widetilde{u}\big|_{\partial\Omega}=\partial_\nu\widetilde{v}\big|_{\partial\Omega}=0,\ \ \widetilde{u}(\cdot,0)=\widetilde{u}_0,\ \ \widetilde{v}(\cdot,0)=\widetilde{v}_0.$$

Letting

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$$(\overline{u},\overline{v})(\cdot,t):=\begin{cases}(u,v)(\cdot,t), & t<T,\\ (\widetilde{u},\widetilde{v})(\cdot,t-T), & t\in[T,T+\tau),\end{cases}$$

we obtain a weak solution of (5) in $\Omega\times(0,T+\tau)$, which by Lemma 2.1 parts iv) and i) is classical. □

Lemma 2.4. *Let $\alpha\in(0,1)$, $m\ge 1$, $\delta>0$. For every $D\in\mathcal{C}_{\delta,m}$, $u_0\in C^\alpha(\overline{\Omega})$, $v_0\in W^{1,\infty}(\Omega)$, $u_0\ge 0$, $v_0>0$ in $\overline{\Omega}$, there is $T_{max}>0$ and $(u,v)\in(C^0(\overline{\Omega}\times[0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max})))^2$ such that (u,v) solves (5) and*

$$T_{max}=\infty\quad or\quad \limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty. \tag{15}$$

Moreover, $u\ge 0$ and $0<v\le\|v_0\|_{L^\infty(\Omega)}$ throughout $\Omega\times(0,T_{max})$.

Proof. We let $u_0\in C^\alpha(\overline{\Omega})$ and $v_0\in W^{1,\infty}(\Omega)$, define

$$\mathcal{S}=\left\{(t,u,v);\ t\in(0,\infty),u,v\in C^0(\overline{\Omega}\times[0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max}))\ \ (u,v)\text{ solves (5)}\right\}$$

and introduce the order relation $\preceq$ given by

$$(t_1,u_1,v_1)\preceq(t_2,u_2,v_2):\Longleftrightarrow\ t_1\le t_2, u_2|_{(0,t_1)}=u_1,\ v_2|_{(0,t_1)}=v_1.$$

Every totally ordered set $M_I=\{(t_i,u_i,v_i);\ i\in I\}$ with arbitrary index set I has an upper bound $(\sup_{i\in I}t_i,u,v)$, where $u(\tau)=u_i(\tau)$ if $\tau\in(0,t_i)$ and v is defined analogously. (This yields well-defined functions, since $u_{i_1}(\tau)=u_{i_2}(\tau)$ if $\tau\in(0,t_{i_1})\cap(0,t_{i_2})$, because M_I is totally ordered.) Moreover, $\mathcal{S}$ is not empty, according to Lemma 2.2. By Zorn's lemma there is some maximal element $(T_{max},u,v)\in\mathcal{S}$. Assume that $\limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}<\infty$. Then Lemma 2.3 immediately yields $\widetilde{T}>T$ such that $(\widetilde{T},\widetilde{u},\widetilde{v})\succeq(T_{max},u,v)$, contradicting the maximality of (T_{max},u,v). □

3. The nondegenerate case. Proof of Theorem 1.1

This section is devoted to the derivation of estimates for the solutions, so as to finally obtain their global existence by means of the extensibility criterion (15).

For some manipulations in (5a) it would be more convenient to deal with a nonsingular chemotaxis term of the form $\nabla\cdot(u\nabla w)$ instead of $\nabla\cdot(\frac{u}{v}\nabla v)$. For this purpose, we employ the following transformation and, given a solution $(u,v)\in(C^0(\overline{\Omega}\times[0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max})))^2$ of (5) for initial data (u_0,v_0) as in (4), let

$$w:=-\log\frac{v}{\|v_0\|_{L^\infty(\Omega)}}\qquad\text{in }\Omega\times[0,T_{max}). \tag{16}$$

Then $w\ge 0$ in $\Omega\times(0,T_{max})$ and $(u,w)\in(C^0(\overline{\Omega}\times[0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max})))^2$ solves

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$$u_t = \nabla\cdot(D(u)\nabla u + u\nabla w) \qquad \text{in } \Omega\times(0,T_{max}) \tag{17a}$$
$$w_t = \Delta w - |\nabla w|^2 + u \qquad \text{in } \Omega\times(0,T_{max}) \tag{17b}$$
$$\partial_\nu u = 0 = \partial_\nu w \qquad \text{in } \partial\Omega\times(0,T_{max}) \tag{17c}$$
$$u(\cdot,0) = u_0, \quad w(\cdot,0) = w_0 := -\log\frac{v_0}{\|v_0\|_{L^\infty(\Omega)}} \qquad \text{in } \Omega, \tag{17d}$$

where (u_0, w_0) satisfy

$$u_0 \geq 0, \quad w_0 \geq 0, \quad w_0 \in W^{1,\infty}(\Omega), \quad u_0 \in C^\alpha(\overline{\Omega}) \text{ for some } \alpha\in(0,1). \tag{18}$$

Evidently, given $\|v_0\|_{L^\infty(\Omega)}$ every solution (u, w) to (17) yields a solution to (5) via $v := \|v_0\|_{L^\infty(\Omega)}\, e^{-w}$.

We will proceed in several steps, the first being the following simple observation that the bacterial mass is conserved throughout evolution:

Lemma 3.1. *Let $T > 0$, $\delta > 0$, $m \in \mathbb{R}$, $D \in \mathcal{C}_{\delta,m}$ and let $u \in C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T))$ solve (5c), (5a) with some $v \in C^{2,1}(\overline{\Omega}\times(0,T))$ or (17a), (17c) with some $w \in C^{2,1}(\overline{\Omega}\times(0,T))$. Then, with $u_0 := u(\cdot,0)$,*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \text{for every } t\in(0,T).$$

Proof. This is an immediate consequence of (5a) or (17a), obtained upon integration over Ω. □

In (17), a spatio-temporal L^2-bound for ∇w can be inferred rather directly:

Lemma 3.2. *Let $m \in \mathbb{R}$, $\delta > 0$, $T > 0$ and $(u, w) \in (C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T)))^2$ be a solution to (17) for any $D \in \mathcal{C}_{\delta,m}$. Then, with $u_0 := u(\cdot,0)$, $w_0 := w(\cdot,0)$,*

$$\int_0^t\int_\Omega |\nabla w|^2 \leq \int_\Omega w_0 + t\int_\Omega u_0 \qquad \text{for every } t\in(0,T).$$

Proof. In order to see this, it is sufficient to integrate the second equation of (17) and take into account Lemma 3.1. □

This bound can be transformed into a first information on derivatives of u:

Lemma 3.3. *Let $m > 1$, $\delta > 0$. For any $K > 0$ there is $C > 0$ such that for any $D \in \mathcal{C}_{\delta,m}$ any solution (u, w) of (17) emanating from initial data (u_0, w_0) as in (18) with $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \leq K$, $\|w_0\|_{L^1(\Omega)} \leq K$ obeys the estimates*

$$\int_0^t\int_\Omega u^{2m-4}|\nabla u|^2 \leq C(1+t) \qquad \text{for all } t\in(0,T_{max})$$

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and

$$\int_0^t \int_\Omega D(u)u^{m-3}|\nabla u|^2 \le C(1+t) \qquad \textit{for all } t \in (0, T_{max})$$

as well as

$$\int_\Omega u^{m-1}(\cdot, t) \le C(1+t) \qquad \textit{for all } t \in (0, T_{max}). \tag{19}$$

Proof. Due to (17a), on $(0, T_{max})$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^{m-1} &= (m-1) \int_\Omega u^{m-2} \nabla \cdot (D(u)\nabla u + u\nabla w) \\ &= (m-1)(2-m) \int_\Omega D(u)u^{m-3}|\nabla u|^2 + (m-1)(2-m) \int_\Omega u^{m-2}\nabla u \cdot \nabla w. \end{aligned} \tag{20}$$

We note that by Young's inequality

$$\left| \int_\Omega u^{m-2}\nabla u \cdot \nabla w \right| \le \frac{\delta}{3} \int_\Omega u^{2m-4}|\nabla u|^2 + \frac{3}{4\delta} \int_\Omega |\nabla w|^2 \quad \text{on } (0, T_{max}). \tag{21}$$

The sign of $(m-1)(2-m)$ in (20) depends on the size of m and we therefore distinguish the following cases:

If $m \in (1, 2)$, (20) together with (21) and Lemma 3.2 yields

$$\begin{aligned} &\frac{1}{3} \int_0^t \int_\Omega D(u)u^{m-3}|\nabla u|^2 + \frac{\delta}{3} \int_0^t \int_\Omega u^{2m-4}|\nabla u|^2 \\ &\le \frac{1}{(m-1)(2-m)} \int_\Omega u^{m-1}(\cdot, t) - \frac{1}{(m-1)(2-m)} \int_\Omega u_0 + \frac{3}{4\delta} \int_0^t \int_\Omega |\nabla w|^2 \\ &\le \frac{1}{(m-1)(2-m)} |\Omega|^{\frac{m-2}{m-1}} \left(\int_\Omega u_0 \right)^{\frac{1}{m-1}} + \frac{3}{4\delta} \left(\int_\Omega w_0 + t \int_\Omega u_0 \right) \quad \text{for any } t \in (0, T_{max}). \end{aligned} \tag{22}$$

If $m > 2$, (20) and (21) can be combined to give

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$$\frac{1}{(m-1)(m-2)}\int_\Omega u^{m-1}(\cdot,t)-\frac{1}{(m-1)(m-2)}\int_\Omega u_0^{m-1}+\frac{1}{3}\int_0^t\int_\Omega D(u)u^{m-3}|\nabla u|^2$$
$$+\frac{\delta}{3}\int_0^t\int_\Omega u^{2m-4}|\nabla u|^2\le\frac{3}{4\delta}\int_0^t\int_\Omega|\nabla w|^2\quad\text{for any } t\in(0,T_{max}),$$

which allows for a similarly obvious definition of C as (22). This inequality also entails (19) for $m>2$, the only case that does not immediately result from Lemma 3.1.

If $m=2$, apparently the consideration of $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{m-1}=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u=0$ does not help in achieving an estimate for $\int_0^t\int_\Omega u^{2m-4}|\nabla u|^2=\int_0^t\int_\Omega|\nabla u|^2$. From the analogously obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\log u+\frac{1}{3}\int_\Omega\frac{D(u)}{u}|\nabla u|^2+\frac{2\delta}{3}\int_\Omega|\nabla u|^2\le\frac{\delta}{3}\int_\Omega|\nabla u|^2+\frac{3}{4\delta}\int_\Omega|\nabla w|^2\text{ on }(0,T_{max}),$$

however, we can derive the same form of estimates as in the other cases. □

For convenience let us recall those special cases of the Gagliardo–Nirenberg inequality we are going to use in the following:

Lemma 3.4.
i) *Let $0<q\le p\le\frac{2N}{N-2}$ (or $0<q\le p<\infty$ if $N=2$) and let $s>0$ and $\gamma>0$. Then there is $c>0$ such that*

$$\|u\|^\gamma_{L^p(\Omega)}\le c\,\|\nabla u\|^{a\gamma}_{L^2(\Omega)}\,\|u\|^{(1-a)\gamma}_{L^q(\Omega)}+c\,\|u\|^\gamma_{L^s(\Omega)}\qquad\textit{for all } u\in W^{1,2}(\Omega)\cap L^q(\Omega)\cap L^s(\Omega),$$

where

$$a=\frac{\frac{1}{q}-\frac{1}{p}}{\frac{1}{q}+\frac{1}{N}-\frac{1}{2}}.$$

ii) *Let $p,q\in(1,\infty)$ be such that $p(q-N)=q(2p-N)a$ for some $a\in[\frac{1}{2},1)$. Then there is $c>0$ such that*

$$\|\nabla v\|^q_{L^q(\Omega)}\le c\,\|\Delta v\|^{qa}_{L^p(\Omega)}\,\|v\|^{q(1-a)}_{L^\infty(\Omega)}+c\,\|v\|^q_{L^\infty(\Omega)}$$

for all $v\in W^{2,p}(\Omega)\cap W^{1,q}(\Omega)\cap L^\infty(\Omega)$ with $\partial_\nu v=0$ on $\partial\Omega$.

Proof. The Gagliardo–Nirenberg inequality can be found in [29, p. 125], [5, Thm. 10.1] or in [21, Lemma 2.3] (where also the case of $p,q<1$ in i) is covered); replacing D^2v by Δv in the standard formulation of ii) is possible by, e.g., [5, Thm. 19.1]. □

Aided by the Gagliardo–Nirenberg inequality, in the next step, as consequence of the estimates from Lemma 3.3 we shall acquire the bound (23), which will be featured as condition in Lemma 3.7 and Lemma 3.8, and can be seen as an important ingredient of the proof of Theorem 1.1.

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Lemma 3.5. *Let $K > 0$, $T \in (0, \infty)$, $\delta > 0$ and $m > 1$. If either*

i) $m \le 2$, $r > 1$, $p \ge 1$ *satisfy* $p \le \frac{2N}{N-2}(m-1)$ *and* $r(1-\frac{1}{p}) \le 2m - 3 + \frac{2}{N}$ *or*
ii) $m \ge 2$, $r > 1$ *and* $p \in [m-1, \frac{2N}{N-2}(m-1)]$ *are such that* $(\frac{1}{m-1} - \frac{1}{p})r \le 1 + \frac{2}{N}$,

then there is $C > 0$ such that whenever $(u, w) \in (C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T)))^2$ solves (17) for some $D \in \mathcal{C}_{\delta,m}$ and for initial data (u_0, w_0) with (18) and $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \le K$, $\|w_0\|_{L^1(\Omega)} \le K$, we have

$$\int_0^t \|u\|_{L^p(\Omega)}^r < C(1 + t^{r+1}) \quad \textit{for all } t \in (0, T). \tag{23}$$

Proof. i) Due to $m > 1$, the inequality $p \le \frac{2N}{N-2}(m-1)$ is equivalent to $\frac{p}{m-1} \le \frac{2N}{N-2}$, and $p \ge 1$ ensures $\frac{p}{m-1} \ge \frac{1}{m-1}$. Thus, the Gagliardo–Nirenberg inequality (Lemma 3.4 i)) yields $c_1 > 0$ such that with

$$a := \frac{m - 1 - \frac{m-1}{p}}{m - 1 + \frac{1}{N} - \frac{1}{2}},$$

and hence $\frac{r}{m-1}a \le 2$, for all $t \in (0, T_{max})$ we obtain

$$\begin{aligned}
\int_0^t \|u\|_{L^p(\Omega)}^r &= \int_0^t \left\| u^{m-1} \right\|_{L^{\frac{p}{m-1}}(\Omega)}^{\frac{r}{m-1}} \\
&\le c_1 \int_0^t \left\| \nabla u^{m-1} \right\|_{L^2(\Omega)}^{\frac{r}{m-1}a} \left\| u^{m-1} \right\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{r}{m-1}(1-a)} + c_1 \int_0^t \left\| u^{m-1} \right\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{r}{m-1}} \\
&\le c_1 \left(\int_\Omega u_0 \right)^{\frac{r}{m-1}(1-a)} \int_0^t \left(1 + \left\| \nabla u^{m-1} \right\|_{L^2(\Omega)}^2 \right) + c_1 \int_0^t \left(\int_\Omega u_0 \right)^{\frac{r}{m-1}},
\end{aligned}$$

where we have used Lemma 3.1, and can conclude the proof with applications of Lemma 3.3 and Young's inequality.

ii) From Lemma 3.3 we obtain $c_2 > 0$ such that

$$\int_\Omega u^{m-1}(\cdot, t) \le c_2(1 + t) \quad \text{and} \quad \int_0^t \int_\Omega |\nabla u^{m-1}|^2 \le c_2(1 + t) \qquad \text{for all } t \in (0, T_{max}).$$

The fact that $p \in [m-1, \frac{2N}{N-2}(m-1)]$ entails both $\frac{p}{m-1} \ge 1$ and $\frac{p}{m-1} \le \frac{2N}{N-2}$. Therefore, with

$$a := \frac{1 - \frac{m-1}{p}}{1 + \frac{1}{N} - \frac{1}{2}},$$

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Lemma 3.4 i) produces $c_3 > 0$ such that for all $t \in (0, T_{max})$

$$\begin{aligned}\int_0^t \|u\|_{L^p(\Omega)}^r &= \int_0^t \left\|u^{m-1}\right\|_{L^{\frac{p}{m-1}}(\Omega)}^{\frac{r}{m-1}} \\ &\le c_3 \int_0^t \left\|\nabla u^{m-1}\right\|_{L^2(\Omega)}^{\frac{r}{m-1}a} \left\|u^{m-1}\right\|_{L^1(\Omega)}^{\frac{r(1-a)}{m-1}} + c_3 \int_0^t \left\|u^{m-1}\right\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{r}{m-1}} \\ &\le c_2 \left(c_3(1+t)\right)^{\frac{r(1-a)}{m-1}} \int_0^t \left(\left\|\nabla u^{m-1}\right\|_{L^2(\Omega)}^2 + 1\right) + c_3 \|u_0\|_{L^1(\Omega)}^r t \le c_4 + c_5 t^{r+1},\end{aligned}$$

where we have used that $\frac{ra}{m-1} \le 2$ and, aided by Lemma 3.1 and the trivial inequality $r\frac{1-a}{m-1} \le r$, chosen suitable positive constants c_4 and c_5. □

As preparation for exploiting (23) in the second equation of (5), we recall

Lemma 3.6. *Let $p, q \in (1, \infty)$. Then for every $T > 0$ there exists $C > 0$ such that for every $z \in L^q((0, T); L^p(\Omega))$ the unique solution of*

$$v_t = \Delta v - z \quad \text{in } \Omega \times (0, T), \qquad \partial_\nu v\big|_{\partial\Omega} = 0, \qquad v(\cdot, 0) = 0$$

satisfies

$$\int_0^T \|\Delta v\|_{L^p(\Omega)}^q \le C \int_0^T \|z\|_{L^p(\Omega)}^q .$$

Proof. We obtain this lemma as straightforward consequence of well-known maximal regularity assertions, cf. [7,8]. □

Lemma 3.6 empowers us to develop (23) into useful knowledge about the gradient of v:

Lemma 3.7. *Let $p \ge \frac{N}{2}$, $r \ge p$, $(2 - \frac{N}{p})r > 1 - N$, and*

$$\begin{cases} q \in (1, N + (2 - \frac{N}{p})r], & \text{if } p \ge N \\ q \in (1, N + (2 - \frac{N}{p})r] \cap (1, \frac{Np}{N-p}), & \text{if } \frac{N}{2} < p < N. \end{cases}$$

Then for every $K > 0$ and $T > 0$ there is $C > 0$ such that for every $v_0 \in W^{1,\infty}(\Omega)$ with $\|v_0\|_{W^{1,\infty}(\Omega)} \le K$, and every $u \in L^r((0, T); L^p(\Omega))$ for which

$$\int_0^T \|u\|_{L^p(\Omega)}^r < K \tag{24}$$

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is satisfied, the solution v of (5b) fulfils

$$\int_0^T \int_\Omega |\nabla v|^q < C. \tag{25}$$

Proof. In order to prepare the application of Lemma 3.6, we decompose $v(\cdot,t) = \widetilde{v}(\cdot,t) + e^{t\Delta}v_0$ in $\Omega \times (0,T)$, where $\widetilde{v}$ solves

$$\widetilde{v}_t = \Delta \widetilde{v} - uv, \quad \widetilde{v}(\cdot,0) = 0, \quad \partial_\nu \widetilde{v}\big|_{\partial\Omega} = 0.$$

By nonnegativity of v_0 and uv, we clearly have $0 \le \widetilde{v} \le v \le K$ in $\Omega \times (0,T)$.

We let $q \le N + (2 - \frac{N}{p})r$ and without loss of generality assume $q \ge 2p$ (which is possible since $2p = N + 2p - N \le N + (2p - N)\frac{r}{p} = N + (2 - \frac{N}{p})r$ and also $2p < \frac{Np}{N-p}$ if $p \in (\frac{N}{2}, N)$). We note that $q \le N + (2 - \frac{N}{p})r$ implies that $r \ge \frac{q-N}{2-\frac{N}{p}} = \frac{(q-N)p}{2p-N}$ and hence with

$$a := \frac{p(q-N)}{q(2p-N)}$$

we have $aq \le r$. Moreover, $q \ge 2p$ ensures that $pq - Np \ge pq - \frac{Nq}{2} = \frac{1}{2}q(2p - N)$ and thus $a \ge \frac{1}{2}$, and, furthermore, $(p - N)q > -Np$, which is obvious for $p > N$ and holds by assumption on q if $p < N$, entails $2pq - Nq > pq - Np$ and hence $a < 1$. Accordingly, from [39, Lemma 1.3 iii)] and the Gagliardo–Nirenberg inequality (Lemma 3.4 ii)) we obtain $c_1 > 0$, $c_2 > 0$, respectively, such that we have

$$\begin{aligned}\int_0^T \int_\Omega |\nabla v|^q &\le 2^q \int_0^T \int_\Omega |\nabla e^{t\Delta} v_0|^q + 2^q \int_0^T \int_\Omega |\nabla \widetilde{v}|^q \\ &\le c_1 T \, \|\nabla v_0\|^q_{L^q(\Omega)} + 2^q \int_0^T \|\nabla \widetilde{v}\|^q_{L^q(\Omega)} \\ &\le c_1 T |\Omega|^{\frac{1}{q}} \, \|\nabla v_0\|_{L^\infty(\Omega)} + c_2 \int_0^T \|\Delta \widetilde{v}\|^{aq}_{L^p(\Omega)} \, \|\widetilde{v}\|^{(1-a)q}_{L^\infty(\Omega)} + c_2 \int_0^T \|\widetilde{v}\|^q_{L^\infty(\Omega)}.\end{aligned}$$

Since $aq < r$, due to Young's inequality and boundedness of $\widetilde{v}$ this estimate can be turned into

$$\int_0^T \int_\Omega |\nabla v|^q \le c_3 + c_4 \int_0^T \|\Delta \widetilde{v}\|^r_{L^p(\Omega)},$$

for some $c_3 > 0$, $c_4 > 0$, where we may invoke the maximal Sobolev result of Lemma 3.6 for $z = uv$ and hence $\int_0^T \|z\|^r_{L^p(\Omega)} \le K^r \int_0^T \|u\|^r_{L^p(\Omega)}$ to conclude (25) from (24). □

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Another consequence of (23) is (local-in-time) boundedness of w:

Lemma 3.8. *Assume that $r \in (1,\infty)$, $p \in [1,\infty)$ are such that $\frac{Nr}{2p(r-1)} < 1$. Then for every $K > 0$ there is $C > 0$ such that whenever, for some $T > 0$, $w \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ solves (17b), (17c), (17d) for some w_0 as in (18) and some $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ such that $\|w_0\|_{L^\infty(\Omega)} \le K$, $\frac{1}{|\Omega|}\int_\Omega u(\cdot,t) \le K$ on $(0,T)$ and moreover*

$$\int_0^T \|u\|_{L^p(\Omega)}^r < K,$$

then

$$w(x,t) \le C(1+t) \qquad \text{for all } (x,t) \in \Omega \times (0,T).$$

Proof. By nonpositivity of $-|\nabla w|^2$, we have that $0 \le w \le \widetilde{w}$, where $\widetilde{w}$ solves

$$\widetilde{w}_t = \Delta\widetilde{w} + u, \quad \partial_\nu \widetilde{w}\big|_{\partial\Omega} = 0, \quad \widetilde{w}(\cdot,0) = w_0.$$

For this function we can estimate

$$\|\widetilde{w}(\cdot,t)\|_{L^\infty(\Omega)} \le \|w_0\|_{L^\infty(\Omega)} + \int_0^t \left\| e^{(t-s)\Delta}\left(u(\cdot,s) - \overline{u}\right)\right\|_{L^\infty(\Omega)} ds + \overline{u}\cdot t \quad \text{for all } t \in (0,T), \tag{26}$$

where $\overline{u} = \frac{1}{|\Omega|}\int_\Omega u(\cdot,t) \le K$. For assessing the integral in (26) we invoke [39, Lemma 1.3 i)] to obtain $c_1 > 0$ such that

$$\begin{aligned}
&\int_0^t \left\| e^{(t-s)\Delta}\left(u(\cdot,s) - \overline{u}\right)\right\|_{L^\infty(\Omega)} ds \\
&\le c_1 \int_0^t \left(1 + (t-s)^{-\frac{N}{2p}}\right) e^{-\lambda_1(t-s)} \|u(\cdot,s) - \overline{u}\|_{L^p(\Omega)} \\
&\le c_1 \int_0^t \left(1 + (t-s)^{-\frac{N}{2p}}\right) e^{-\lambda_1(t-s)} \|u(\cdot,s)\|_{L^p(\Omega)} ds + c_1 K|\Omega|^{\frac{1}{p}} \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right) e^{-\lambda_1\sigma} d\sigma \\
&\le c_1 \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right)^{\frac{r}{r-1}} e^{-\lambda_1\sigma\frac{r}{r-1}} d\sigma + c_1 \int_0^t \|u(\cdot,s)\|_{L^p(\Omega)}^r + c_1 K|\Omega|^{\frac{1}{p}} \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right) e^{-\lambda_1\sigma} d\sigma
\end{aligned} \tag{27}$$

for all $t \in (0,T)$. Collecting the constants in (26) and (27), we see that for all $(x,t) \in \Omega \times (0,T)$

$$w(x,t) \le \|\widetilde{w}(\cdot,t)\|_{L^\infty(\Omega)} \le C(1+t),$$

where

$$C := K + c_1 K + c_1 K |\Omega|^{\frac{1}{p}} \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right) e^{-\lambda_1 \sigma} d\sigma + k_1 \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right)^{\frac{r}{r-1}} e^{-\lambda_1 \sigma \frac{r}{r-1}} d\sigma,$$

which is finite due to $\frac{Nr}{2p(r-1)} < 1$ (and its consequence $\frac{N}{2p} < 1$). $\square$

If we can find parameters that allow for an application of Lemma 3.5 and Lemma 3.8 at the same time, we can conclude boundedness of w. This is the goal we pursue in the following lemma:

Lemma 3.9. *Let*

$$m > 1 + \frac{N}{4} \tag{28}$$

and $\delta > 0$. *Then for all* $T \in (0, \infty)$ *there is* $C > 0$ *such that for every* $D \in \mathcal{C}_{\delta,m}$ *and every* (u_0, w_0) *as in* (18) *with* $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \le K$, $\|w_0\|_{L^\infty(\Omega)} \le K$, *any solution* $(u, w) \in (C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T)))^2$ *of* (17) *satisfies*

$$w(x, t) \le C \qquad \text{for all } x \in \Omega \text{ and all } t \in (0, T).$$

Proof. Let us first consider the case $m \in (2 - \frac{1}{N}, 2]$ (that is of interest only if $N < 4$, because m is supposed to satisfy $m > 1 + \frac{N}{4}$) and observe that by (28), we have

$$m > \begin{Bmatrix} \frac{3}{2}, & \text{if } N = 2 \\ \frac{7}{4}, & \text{if } N = 3 \end{Bmatrix} = \frac{5}{4} - \frac{1}{2N} + \frac{N}{8} + \sqrt{\left(\frac{5}{4} - \frac{1}{2N} + \frac{N}{8}\right)^2 - \frac{5}{4} + \frac{1}{N} - \frac{3}{8}N}.$$

Therefore, we see that

$$4m^2 - 10m + \frac{4}{N}m - Nm + 5 - \frac{4}{N} + \frac{3}{2}N > 0$$

and hence

$$\begin{aligned}(N-2)\left(m - \frac{3}{2}\right) = Nm - \frac{3}{2}N - 2m + 3 < 4m^2 - 8m + \frac{4}{N}m - 4m + 8 - \frac{4}{N} \\ = 2(m-1)\left(2m - 4 + \frac{2}{N}\right),\end{aligned}$$

so that

$$\frac{N(m - \frac{3}{2})}{2m - 4 + \frac{2}{N}} < \frac{2N}{N-2}(m-1).$$

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Since moreover $\frac{2N}{N-2}(m-1) > 1$, it is possible to choose $p \ge 1$ such that $p \in \left(\frac{N(m-\frac{3}{2})}{2m-4+\frac{2}{N}}, \frac{2N}{N-2}(m-1)\right)$. With this choice of p we let

$$r := \frac{2m-3+\frac{2}{N}}{1-\frac{1}{p}}$$

and note that $2m-3+\frac{2}{N} > 4-\frac{2}{N}-3+\frac{2}{N} = 1 > 1-\frac{1}{p}$ entails $r > 1$. Hence Lemma 3.5 i) is applicable. Moreover,

$$\begin{aligned}\frac{2p}{N}\left(1-\frac{1}{r}\right) &= \frac{2p}{N}\left(1-\frac{1-\frac{1}{p}}{2m-3+\frac{2}{N}}\right)\\ &= \frac{2}{N}\cdot\frac{p(2m-3+\frac{2}{N})-p+1}{2m-3+\frac{2}{N}} > \frac{2}{N}\cdot\frac{N(m-\frac{3}{2})+1}{2(m-\frac{3}{2}+\frac{1}{N})} = 1\end{aligned}$$

and we can additionally invoke Lemma 3.8 so as to obtain the desired boundedness of w on $\Omega \times (0,T)$.

If $m \ge 2$ (and $m > 1+\frac{N}{4}$), we note that

$$\frac{N^2(m-1)}{2N(m-1)+4(m-1)-2N} < \frac{N^2(m-1)}{2N\frac{N}{4}+4\frac{N}{4}-2N} = \frac{N^2(m-1)}{\frac{N^2}{2}-N} = \frac{2N}{N-2}(m-1).$$

Since $m \ge 2$,

$$\frac{1+\frac{2}{N}}{m-1} \le 1+\frac{2}{N} < 1+\frac{4}{N}+\frac{4}{N^2},$$

and hence

$$\frac{1}{m-1}-1-\frac{2}{N} < \frac{2}{N}+\frac{4}{N^2}-\frac{2}{N(m-1)} = \frac{2N(m-1)+4(m-1)-2N}{N^2(m-1)}.$$

Therefore we can pick $p \in \left(\frac{N^2(m-1)}{2N(m-1)+4(m-1)-2N}, \frac{2N}{N-2}(m-1)\right)$ such that $\frac{1}{p} > \frac{1}{m-1}-1-\frac{2}{N}$ and $p > m-1$, and we let $r := \frac{1+\frac{2}{N}}{\frac{1}{m-1}-\frac{1}{p}}$. Then $r > 1$ and, apparently, $(\frac{1}{m-1}-\frac{1}{p})r \le 1+\frac{2}{N}$, warranting applicability of Lemma 3.5. Moreover, $p > \frac{N^2(m-1)}{2N(m-1)+4(m-1)-2N}$ entails $\frac{1}{p} < \frac{2}{N}+\left(\frac{2}{N}\right)^2-\frac{2}{N(m-1)}$ and thus $\frac{N}{2p}(1+\frac{2}{N}) = \frac{N}{2p}+\frac{1}{p} < 1+\frac{2}{N}-\frac{1}{m-1}+\frac{1}{p}$ and hence, finally,

$$\frac{N}{2p} < 1-\frac{\frac{1}{m-1}-\frac{1}{p}}{1+\frac{2}{N}} = 1-\frac{1}{r},$$

which permits us to employ Lemma 3.8 and conclude. □

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Lemma 3.10. *For every $K>0$ and every $q\in(0,\infty]$ there is $C>0$ such that for all $T>0$ and all $v\in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$*

$$\|v_0\|_{L^\infty(\Omega)}\geq\frac{1}{K},\quad w\leq K\quad \text{in } \Omega\times(0,T),\quad \text{and}\quad \|\nabla v\|_{L^q(\Omega\times(0,T))}\leq K$$

implies

$$\|\nabla w\|_{L^q(\Omega\times(0,T))}\leq C$$

Proof. Since $w\leq K$, we have $v=\|v_0\|_{L^\infty(\Omega)}e^{-w}\geq\|v_0\|_{L^\infty(\Omega)}e^{-K}$, and immediately obtain $\frac{1}{v}\leq\|v_0\|^{-1}_{L^\infty(\Omega)}e^K\leq Ke^K$ in $\Omega\times(0,T)$. Thus

$$\|\nabla w\|_{L^q(\Omega\times(0,t))}\leq\left\|\frac{1}{v}\nabla v\right\|_{L^q(\Omega\times(0,t))}\leq Ke^K\|\nabla v\|_{L^q(\Omega\times(0,t))}\leq K^2e^K=:C.\quad\square$$

Lemma 3.11. *Let $\delta>0$, $m\geq1$, $q>2$ and $p>1$. Then for every $K>0$ and $T>0$ there is $C>0$ such that the following holds: If $q\geq N$ and*

$$m\leq2,\qquad p\geq m-\frac{2}{q},\qquad p\leq(q-1)(m-1)+\frac{q-2}{N},\tag{29}$$

$$\text{or}\qquad m\geq2,\qquad p\geq2\left(1-\frac{1}{q}\right)(m-1),\qquad p\leq(m-1)\left(\frac{q}{2}+\frac{(q-2)(N+2)}{2N}\right),\tag{30}$$

then for every function $w\in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$ with

$$\int_0^t\int_\Omega|\nabla w|^q\leq K \text{ for all } t\in(0,T),$$

any solution $u\in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$ of (17a), (17c), (17d) with $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)}\leq K$ and some $D\in\mathcal{C}_{\delta,m}$ fulfils

$$\int_\Omega u^p(\cdot,t)\leq C\qquad \text{for all } t\in(0,T).$$

Proof. Either of (29) and (30) implies $p\geq m-1$. Moreover,

$$\frac{N-2}{2N}\leq\frac{q-2}{2q}\cdot\frac{p+m-1}{p-m+1}.\tag{31}$$

Let us first consider the case $m\leq2$. Then $p\geq m-\frac{2}{q}$ implies $p-m+1\geq\frac{q-2}{q}$ and hence

$$\frac{2}{m+p-1}\leq\frac{2q}{q-2}\cdot\frac{p-m+1}{p+m-1}.\tag{32}$$

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We now let

$$a := \frac{\frac{m+p-1}{2} - \frac{q-2}{2q} \cdot \frac{p+m-1}{p-m+1}}{\frac{m+p-1}{2} + \frac{1}{N} - \frac{1}{2}}$$

and observe that

$$a \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} \le 2, \tag{33}$$

because $p \le (q-1)(m-1) + \frac{q-2}{N}$ implies that $\frac{q}{q-2}p - p = (\frac{q}{q-2} - 1)p = \frac{2}{q-2}p \le \frac{2(q-1)}{q-2}(m-1) + \frac{2}{N} = (1 + \frac{q}{q-2})(m-1) + \frac{2}{N} = m - 1 + \frac{q}{q-2}(m-1) + \frac{2}{N}$, that is, $\frac{q}{q-2}(p-m+1) \le m + p - 1 + \frac{2}{N}$ and hence $\frac{q}{q-2}(p-m+1) - 1 \le (m+p-1) + \frac{2}{N} - 1$, which leads to

$$\begin{aligned} a \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} &= \frac{\frac{m+p-1}{2} - \frac{q-2}{2q}\frac{p+m-1}{p-m+1}}{\frac{m+p-1}{2} + \frac{1}{N} - \frac{1}{2}} \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} \\ &= \frac{\frac{p-m+1}{2} \cdot \frac{2q}{q-2} - 1}{\frac{1}{2}((m+p-1) + \frac{2}{N} - 1)} \le 2. \end{aligned}$$

From Lemma 3.1 we obtain $c_1 > 0$ such that

$$\left\| u^{\frac{m+p-1}{2}}(\cdot, t) \right\|_{L^{\frac{2}{m+p-1}}(\Omega)} = c_1 \qquad \text{for all } t \in (0, T).$$

Due to (31) and (32) we can apply the Gagliardo–Nirenberg inequality in the form of Lemma 3.4 i) to obtain $c_2 > 0$ such that

$$\begin{aligned} \int_\Omega u^{(p+1-m)\frac{q}{q-2}} &= \int_\Omega u^{\frac{m+p-1}{2}(\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1})} \\ &= \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}}(\Omega)}^{\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}} \\ &\le c_2 \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{a\cdot\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}} \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{(1-a)\cdot\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}} + c_2 \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}} \\ &= c_2 c_1^{(1-a)\cdot\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}} \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{a\cdot\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}} + c_2 c_1^{\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}} \end{aligned} \tag{34}$$

on $(0, T)$.

In obtaining such an estimate for $m \ge 2$ we could use the same argument. It is, however, possible to obtain better conditions by relying on Lemma 3.3 instead of Lemma 3.1. Apart from that, the reasoning is analogous: We have $p \ge 2\big(1 - \frac{1}{q}\big)(m-1)$, which implies $qp \ge (q - 2 + q)(m-1)$, thus $q(p-m+1) \ge (m-1)(q-2)$ and hence

$$\frac{2(m-1)}{m+p-1} \le \frac{2q}{q-2} \cdot \frac{p-m+1}{p+m-1} \tag{35}$$

and let

$$b := \frac{\frac{m+p-1}{2(m-1)} - \frac{q-2}{2q} \cdot \frac{p+m-1}{p-m+1}}{\frac{m+p-1}{2(m-1)} + \frac{1}{N} - \frac{1}{2}},$$

noting that

$$b \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} \le 2, \tag{36}$$

because $p \le (m-1)(\frac{q}{2} + \frac{(q-2)(N+2)}{2N})$ implies that $(\frac{q}{q-2} - 1)(\frac{p}{m-1}) = \frac{2}{q-2}\frac{p}{m-1} \le \frac{N+2}{N} + \frac{q}{q-2}$ and hence $\frac{q(p-m+1)}{(m-1)(q-2)} = \frac{q}{q-2}(\frac{p}{m-1} - 1) \le \frac{p}{m-1} + \frac{N+2}{N} = \frac{m+p-1}{m-1} + \frac{2}{N}$, which shows that $\frac{p-m+1}{2(m-1)} \cdot \frac{2q}{q-2} - 1 \le \frac{m+p-1}{m-1} + \frac{2}{N} - 1$ and therefore also

$$\begin{aligned} b \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} &= \frac{\frac{m+p-1}{2(m-1)} - \frac{q-2}{2q} \cdot \frac{p+m-1}{p-m+1}}{\frac{m+p-1}{2(m-1)} + \frac{1}{N} - \frac{1}{2}} \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} \\ &= \frac{\frac{q}{q-2} \cdot \frac{p-m+1}{2(m-1)} - 1}{\frac{m+p-1}{2(m-1)} + \frac{1}{N} - \frac{1}{2}} \le 2. \end{aligned}$$

Lemma 3.3 yields $c_3 > 0$ such that

$$\left\| u^{\frac{m+p-1}{2}}(\cdot, t) \right\|_{L^{\frac{2(m-1)}{m+p-1}}(\Omega)} \le c_3 \qquad \text{for all } t \in (0, T)$$

and hence (31) and (35) enable us to invoke the Gagliardo–Nirenberg inequality and obtain $c_4 > 0$ such that on $(0, T)$

$$\begin{aligned} \int_\Omega u^{(p+1-m)\frac{q}{q-2}} &= \int_\Omega u^{\frac{m+p-1}{2}(\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1})} \\ &= \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}}(\Omega)}^{\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} \\ &\le c_4 \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{a \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2(m-1)}{m+p-1}}(\Omega)}^{(1-a) \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} + c_4 \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} \\ &\le c_4 c_3^{(1-a) \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{a \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} + c_4 c_3^{\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}}. \end{aligned} \tag{37}$$

From either (34) and (33) or (37) and (36) (and possibly Young's inequality) we hence find that with some $c_5 > 0$ we have

$$\int_\Omega u^{(p+1-m)\frac{q}{q-2}} \le c_5 \left\| u^{\frac{m+p-3}{2}} \nabla u \right\|^2_{L^2(\Omega)} + c_5 \qquad \text{on } (0,T). \tag{38}$$

In

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + (p-1)\delta \int_\Omega u^{p+m-3}|\nabla u|^2 \le (p-1)\left|\int_\Omega u^{p-1}\nabla u\cdot\nabla w\right| \qquad \text{on } (0,T)$$

we can apply Young's inequality to see that on $(0,T)$

$$(p-1)\left|\int_\Omega u^{p-1}\nabla u\cdot\nabla w\right| \le \frac{(p-1)\delta}{4}\int_\Omega u^{p+m-3}|\nabla u|^2 + \frac{p-1}{\delta}\int_\Omega u^{p-m+1}|\nabla w|^2.$$

A further application of Young's inequality allows us to separate u and $|\nabla w|$ in the last integral according to

$$\frac{p-1}{\delta}\int_\Omega u^{p-m+1}|\nabla w|^2 \le \frac{c_5(p-1)}{\delta^3}\int_\Omega |\nabla w|^q + \frac{(p-1)\delta}{4c_5}\int_\Omega u^{(p+1-m)\frac{q}{q-2}}, \qquad \text{on } (0,T).$$

Therefore, due to (38), in total,

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{(p-1)\delta}{2}\int_\Omega u^{p+m-3}|\nabla u|^2 \le \frac{(p-1)\delta}{4} + \frac{c_5(p-1)}{\delta^3}\int_\Omega |\nabla w|^q \qquad \text{on } (0,T).$$

Integration with respect to time produces the lemma. □

We are particularly interested in applying the previous lemma for some $p > N$, because for such p, a bound on $\int_\Omega u^p$ on some interval $[0,T]$ already ensures uniform boundedness of ∇v (and hence ∇w) on $\overline{\Omega}\times[0,T]$.

Lemma 3.12. *Let $\delta > 0$. Assume that either*

i) $2-\frac{1}{N} < m \le 2$, $N \ge 2$, $q > N$ *and* $q > 1 + \frac{N^2+1}{Nm-N+1}$, *or*
ii) $m \ge 2$, $N \ge 2$, $q > N$ *and* $q > \frac{2N^2+2m^2+2m-4}{(m-1)(N+m+2)}$.

Then there is $p > N$ and for every $K > 0$ and $T \in (0,\infty)$ there is $C > 0$ such that whenever $u \in C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T))$ solves (17a), (17c), (17d), with some $D \in \mathcal{C}_{\delta,m}$, some u_0 as in (18) and such that $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \le K$, and some $w \in C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T))$ satisfying

$$\int_0^T\int_\Omega |\nabla w|^q \le K,$$

then

$$\int_\Omega u^p(\cdot,t) \le C \qquad \text{for all } t \in (0,T).$$

Proof. i) For $\widetilde{q} = 2$ we have $m - \frac{2}{\widetilde{q}} = m - 1 = (\widetilde{q}-1)(m-1) + \frac{\widetilde{q}-2}{N}$ and because $m > 2 - \frac{1}{N}$, for every $\widetilde{q} \ge 2$ we have

$$\frac{d}{d\widetilde{q}}\left(m - \frac{2}{\widetilde{q}}\right) = \frac{2}{\widetilde{q}^2} \le 1 = 2 - \frac{1}{N} - m + m - 1 + \frac{1}{N} < m - 1 + \frac{1}{N}$$
$$= \frac{d}{d\widetilde{q}}\left((\widetilde{q}-1)(m-1) + \frac{\widetilde{q}-2}{N}\right).$$

Therefore $m - \frac{2}{q} < (q-1)(m-1) + \frac{q-2}{N}$. Furthermore $q > 1 + \frac{N^2+1}{mN-N+1} = \frac{m-1+\frac{2}{N}+N}{m-1+\frac{1}{N}}$ implies that

$$(q-1)(m-1) + \frac{q-2}{N} = q\left(m - 1 + \frac{1}{N}\right) + 1 - m - \frac{2}{N} > N.$$

Hence it is possible to find $p > N$ such that $p > m - \frac{2}{q}$ and $p < (q-1)(m-1) + \frac{q-2}{N}$ and an application of Lemma 3.11 proves the statement.

ii) Since $x + \frac{1}{x} \ge 2$ for all $x > 0$, and since $q \ge 2$, we have

$$2 - \frac{2}{q} \le \frac{q}{2} + \frac{(q-2)(m+2)}{2N}$$

and hence $2(1-\frac{1}{q})(m-1) \le (m-1)(\frac{q}{2} + \frac{(q-2)(m+2)}{2N})$. The fact that $q > \frac{2N^2+2m^2+2m-4}{(m-1)(N+m+2)} = \frac{1}{(m-1)(N+m+2)}(2N^2 + (2m+4)(m-1)) = (\frac{2N^2}{m-1} + 2m + 4)\frac{1}{N+m+2}$ shows that $q(N+m+2) > 2m + 4 + \frac{2N^2}{m-1}$ and hence $N < \frac{m-1}{2N}(q(N+m+2) - 2m - 4) = \frac{m-1}{2N}(Nq + (q-2)(m+2)) = (m-1)(\frac{q}{2} + \frac{(q-2)(m+2)}{2N})$. Therefore we can choose $p > N$ such that

$$p < (m-1)\left(\frac{q}{2} + \frac{(q-2)(m+2)}{2N}\right) \qquad \text{and} \qquad p > 2\left(1 - \frac{1}{q}\right)(m-1)$$

and apply Lemma 3.11 for this choice of p to obtain the assertion. □

The previous lemma requires a bound on some $\int_0^T \int_\Omega |\nabla w|^q$. Fortunately, this is exactly what we have prepared in Lemma 3.5, Lemma 3.7, Lemma 3.9, and Lemma 3.10.

Lemma 3.13. *Let $m > 1 + \frac{N}{4}$ and $\delta > 0$. Then there is $p > N$ and for every $K > 0$ and $T > 0$ there is $C > 0$ such that every solution $(u,w) \in (C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T)))^2$ of (17) with initial data (u_0,w_0) as in (18) and with $\|u_0\|_{L^1(\Omega)} \le K$, $\|w_0\|_{W^{1,\infty}(\Omega)} \le K$ and any $D \in \mathcal{C}^+_{\delta,m}$ satisfies*

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$$\int u^p(\cdot,t) \le C \qquad \textit{for every } t \in (0,T).$$

Proof. By the choice of m, from Lemma 3.9 we know that we can find $C > 0$ such that for any u_0, w_0 and D as above, any solution $(u,w) \in (C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T)))^2$ of (17) satisfies $0 \le w \le C$ in $\Omega \times (0,T)$. Lemma 3.10 therefore warrants that the desired conclusion results from a combination of Lemma 3.5 and Lemma 3.7 with Lemma 3.12 – provided that there are parameters p, q, r that simultaneously satisfy all conditions posed by these lemmata. This is what we ensure in the remainder of the proof:

Case $N = 2$, $m \in (\frac{3}{2}, 2]$: We let $r = 4(m-1)$, $p = 2$, and $q = 4m-2$. Then $r(1-\frac{1}{p}) = 4(m-1)(1-\frac{1}{2}) = 2(m-1) \le 2m-2 = 2m-3+\frac{2}{2}$, which enables us to invoke Lemma 3.5 i). Moreover, $m > \frac{3}{2}$ implies $4m-4 > 2$ and thus $r > p$, and we have $q = 4m-2 = 2+4m-4 \le 2+(2-\frac{2}{2})4(m-1) = N+(2-\frac{N}{p})r$. Therefore, Lemma 3.7 becomes applicable. Thanks to $q = 4m-2 \ge 4\cdot\frac{3}{2}-2 = 4 > 2 = N$ and thanks to $m \ge \frac{3}{2}$, hence $q = 4m-2 > 4\cdot 32-2 > \frac{7}{2} = 1+\frac{5}{2\cdot\frac{3}{2}-1} > 1+\frac{5}{2m-1} = 1+\frac{N^2+1}{Nm-N+1}$ holds true, facilitating the use of Lemma 3.12 i).

Case $N = 3$, $m \in (\frac{7}{4}, 2]$: Here we let $r = p = 2m-\frac{4}{3}$. Then $p \ge 1$, $r \ge 1$, $p = 2m-\frac{4}{3} < 6m-6 = \frac{2N}{N-2}(m-1)$ and $r(1-\frac{1}{p}) = r-1 = 2m-\frac{7}{3} = 2m-3+\frac{2}{N}$, so that Lemma 3.5 i) can be used. Since $m > \frac{7}{4} = \frac{21}{12} > \frac{19}{12}$, we have that $12m^2-19m-\frac{8}{3} = 12(m-\frac{19}{12})m-\frac{8}{3} \ge 12(\frac{7}{4}-\frac{19}{12})\frac{7}{4}-\frac{8}{3} = \frac{7}{2}-\frac{8}{3} = \frac{21-16}{6} > 0$ and thus $3m+8 < 12m^2-16m+\frac{16}{3} = (4m-\frac{8}{3})(3m-2)$, i.e. $2p > \frac{3m+8}{3m-2}$. Furthermore, $p = 2m-\frac{4}{3} \le 4-\frac{4}{3} = \frac{8}{3} < 3$ and $p = 2m-\frac{4}{3} \ge \frac{7}{2}-\frac{4}{3} = \frac{21-8}{6} = \frac{13}{6} > \frac{3}{2}$, so that consequently, also $\frac{3p}{3-p} > 2p$ holds. We choose $q \in (\frac{3m+8}{3m-2}, 2p)$, thereby ensuring the applicability of Lemma 3.7. Since finally $q > \frac{3m+8}{3m-2} = \frac{m+\frac{8}{3}}{m-\frac{2}{3}} = 1+\frac{\frac{10}{3}}{m-\frac{2}{3}} = 1+\frac{3+\frac{1}{3}}{m-1+\frac{1}{3}}$ and $q > \frac{3m+8}{3m-2} = 1+\frac{10}{3m-2} \ge 1+\frac{10}{6-2} = \frac{7}{2} > 3 \ge 2$ we may also draw on Lemma 3.12 i).

Case $N \ge 2$, $m \ge 2$, $m \ge 1+\frac{N}{4}$: Let $r := p := 2\frac{N+1}{N}(m-1)$. Then obviously $p = r > 1$. Moreover, $p \le \frac{2N}{N-2}(m-1)$ (because $\frac{2N}{N-2} > \frac{2+2N}{N}$ is equivalent to $2N^2 > 2N^2+2N-4N-4$ and hence to $0 > -2N-4$) and

$$\left(\frac{1}{m-1}-\frac{1}{p}\right)r = \frac{p}{m-1}-1 = 2\frac{N+1}{N}-1 = \frac{N+2}{N} \le 1+\frac{2}{N},$$

so that the conditions of Lemma 3.5 ii) are satisfied. We furthermore let $q := 2p = \frac{4(N+1)}{N}(m-1)$ and note that $p > \frac{N}{2}$, since $2\frac{N+1}{N}(m-1) > 2\cdot\frac{N+1}{N}\frac{N}{4} = \frac{N+1}{2} > \frac{N}{2}$, and that $q \le 2p = 2r+N-N\frac{r}{p}$, that moreover either $p \ge N$ or $p < N$ and $q = 2p < \frac{Np}{N-p}$, because $p > \frac{N}{2}$, and therefore Lemma 3.7 is applicable. In order to see that these choices also make the use of Lemma 3.12 ii) viable, we first investigate the polynomial

$$P_N(m) := (2N+2)m^3+(2N^2+N)m^2+(-4N^2-11N-6)m-N^3+2N^2+8N+4. \quad (39)$$

It is extremal whenever $P_N'(m) = (6N+6)m^2+(4N^2+2N)m+(-4N^2-11N-6) = 0$, which is the case for exactly two real numbers that lie in $(-\infty,2)$, because for $m \ge 2$ we have $P_N'(m) \ge (24N+24)+(8N^2+4N)+(-4N^2-11N-6) > 0$. We claim that $P_N(m) > 0$ for any $m > \max\{2, 1+\frac{N}{4}\}$ and for this compute $P_N(\max\{2, 1+\frac{N}{4}\})$:

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$$\begin{aligned}
P_N(2) &= 16N+16+8N^2+4N-8N^2-22N-12-N^3+2N^2+8N+4\\
&= -N^3+2N^2+6N+8\\
&= \begin{cases} -8+8+12+8>0, & N=2,\\ -27+18+18+8>0, & N=3,\\ -64+32+24+8=0, & N=4,\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&P_N\left(1+\frac{N}{4}\right)\\
&\quad= \frac{1}{4^3}\Big((2N+2)(N+4)^3+4(2N^2+N)(N+4)^2+16(-4N^2-11N-6)(N+4)\\
&\qquad\quad -64N^3+128N^2+512N+256\Big)\\
&\quad= \frac{2}{4^3}N^2(5N+3)(N-4),
\end{aligned}$$

which is nonnegative for $N \ge 4$. Since P_N is nonnegative in $\max\left\{2, 1+\frac{N}{4}\right\}$ and strictly increasing on $(2,\infty)$, we conclude that $P_N(m) > 0$ for any $m > \max\left\{2, 1+\frac{N}{4}\right\}$. Positivity of $P_N(m)$ is equivalent to

$$2(N+1)(m-1)^2(N+m+2) > N^3+m^2N+mN-2N$$

and hence

$$q = \frac{4(N+1)}{N}(m-1) > \frac{2N^2+2m^2+2m-4}{(m-1)(N+m+2)}.$$

Furthermore by the fact that $p > \frac{N}{2}$, we also have $q > N$, and can invoke Lemma 3.12 ii). □

Remark 3.14. The condition $m > 1+\frac{N}{4}$ in Lemma 3.13 is first and foremost employed to guarantee boundedness of w, that is, boundedness of v from below by a positive constant. Therefore it seems reasonable to ask whether it would be possible to soften the assumption on m if we already knew that w be bounded. It turns out that the condition $m > 2-\frac{1}{N}$ of Lemma 3.12 is as strict as $m > 1+\frac{N}{4} = \frac{3}{2}$ if $N=2$, whereas for $N=3$ we see that for any value of p choosing $r = \frac{2m-3+\frac{2}{3}}{1-\frac{1}{p}}$ is optimal (cf. Lemma 3.5 i)). Then $r \ge p$ (required by Lemma 3.7) entails $p \le 2m-\frac{4}{3}$. Other conditions on p are either obviously satisfied with this choice of $p = 2m-\frac{4}{3}$ (namely $p < 6m-6$) or are essentially largeness conditions on p. Thus the choice of p, r in the second case in the proof of Lemma 3.13 was optimal and we can follow the calculations there, which leaves us with two more necessary conditions: $p > \frac{3}{2}$ leading to $m > \frac{17}{12}$, and positivity of $12m^2-19m-\frac{8}{3}$, requiring $m > \frac{19}{24}+\frac{1}{8}\sqrt{\frac{163}{3}}$, which therefore remains as condition on m if one already supposes boundedness of w. For $N=4$, $\frac{7}{4} \le m \le 2$, similarly choosing $p = r = 2m-\frac{3}{2}$

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admits application of Lemma 3.5 i), whereas for invoking Lemma 3.7 and Lemma 3.12 we need some q between $1+\frac{N^2+1}{Nm-N+1}=\frac{4m+14}{4m-3}$ and $2p=4m-3$, which exists if $4m-3>\frac{4m+14}{4m-3}$, i.e. $m>\frac{1}{8}(7+\sqrt{69})$. The conditions $r\ge p\ge\frac{N}{2}$ of Lemma 3.7 and $r(1-\frac{1}{p})\le 2m-3+\frac{2}{N}$ (Lemma 3.5i)) imply $\frac{N}{2}-1\le 2m-3+\frac{2}{N}$ and hence $m\ge\frac{N}{4}-\frac{1}{N}+1$ so that for $N\ge 5$ any choice of $m\le 2$ is impossible and for these dimensions we may restrict our attention to Lemma 3.7 and the second parts of Lemmata 3.5 and 3.12. The assumptions of Lemma 3.5 ii) combined with the condition $r\ge p$ of 3.7 imply that $p\le 2\frac{N+1}{N}(m-1)$. Therefore it is necessary that $2\frac{N+1}{N}(m-1)>\frac{N}{2}$, i.e. $m>\frac{(N+2)^2}{4(N+1)}$ – apart from this condition we are led to follow the case "$N\ge 2$, $m\ge 2$" of the proof of Lemma 3.13. In conclusion: If boundedness of w were known a priori, the present proof would be applicable if

$$m>\begin{cases}\frac{3}{2}, & N=2,\\ \frac{19}{24}+\frac{1}{8}\sqrt{\frac{163}{3}}, & N=3,\\ \frac{1}{8}(7+\sqrt{69}), & N=4,\\ \max\left\{\frac{(N+2)^2}{4(N+1)}, \text{largest root of } P_N \text{ from (39)}\right\}, & N\ge 5.\end{cases} \qquad \square$$

Having completed the necessary preparations, we can now turn to the proof of existence of a global solution. In order to lay the groundwork for compactness arguments in Section 4, at the same time we derive a batch of estimates for the solutions.

Lemma 3.15. *Let $\delta>0$, $m>1+\frac{N}{4}$.*

i) *For any (u_0,v_0) as in (4) and any $D\in\mathcal{C}^+_{\delta,m}$ there is a global classical solution $(u,v)\in(C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^2$ to (5).*
ii) *Moreover, for every $T>0$, $K>0$ there is $C_T>0$ such that for every $D\in\mathcal{C}_{\delta,m}$ and (u_0,v_0) as in (4) with $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)}\le K$, $\frac{1}{K}\le\|v_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{W^{1,\infty}(\Omega)}\le K$, every solution $(u,v)\in(C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T)))^2$ to (5) satisfies*

$$\|u\|_{L^\infty(\Omega\times(0,T))}\le C_T \tag{40}$$

$$\|v\|_{L^\infty((0,T);W^{1,\infty}(\Omega))}\le C_T \tag{41}$$

$$\left\|\frac{1}{v}\nabla v\right\|_{L^\infty(\Omega\times(0,T))}\le C_T \tag{42}$$

$$\|D(u)\nabla u\|_{L^2(\Omega\times(0,T))}\le C_T \tag{43}$$

$$\left\|\nabla u^{m-1}\right\|_{L^2(\Omega\times(0,T))}\le C_T, \tag{44}$$

$$\int_0^T\int_\Omega D(u)u^{m-3}|\nabla u|^2\le C_T, \tag{45}$$

$$\|v_t\|_{L^2((0,T);(W_0^{1,1}(\Omega))^*)}\le C_T, \tag{46}$$

$$\|u_t\|_{L^1((0,T);(W_0^{1,N+1}(\Omega))^*)}\le C_T\left(1+\sup_{s\in[0,C_T]}D(s)\right). \tag{47}$$

Proof. According to Lemma 2.4, corresponding to (u_0, v_0) and D as in the hypothesis of the present lemma, there is a local solution $(u, v) \in (C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})))^2$. We now let $T \in (0, T_{max}] \cap (0, \infty)$ and $K > 0$. By $\mathcal{ID}_K$ let us abbreviate the set of initial data

$$\mathcal{ID}_K := \Big\{(u_0, v_0) \in C^\alpha(\overline{\Omega}) \times W^{1,\infty}(\Omega) \text{ for some } \alpha \in (0,1);\ \|u_0\|_{L^\infty(\Omega)} \le K, \|v_0\|_{W^{1,\infty}(\Omega)} \le K\Big\}.$$

Lemma 3.13 provides us with $p > N$ and $c_1 > 0$ such that for every $D \in \mathcal{C}_{\delta,m}$ and every $(u_0, v_0) \in \mathcal{ID}_K$, every classical solution $(u, v) \in (C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T)))^2$ of (5) satisfies

$$\|u\|_{L^\infty((0,T);L^p(\Omega))} \le c_1$$

and hence

$$\|\nabla v\|_{L^\infty(\Omega\times(0,T))} \le c_2 \quad \text{and} \quad \|w\|_{L^\infty(\Omega\times(0,T))} \le c_3$$

as well as

$$\|\nabla w\|_{L^\infty(\Omega\times(0,T))} \le c_4$$

with some c_2, c_3 and c_4 obtained from Lemma 2.1 ii), Lemma 3.9 and Lemma 3.10, respectively, and with w being defined as in (16). This asserts (41) and (42). An application of Lemma 3.11 for sufficiently large values of q and p then ascertains the existence of $c_5 > 0$ such that for all $D \in \mathcal{C}_{\delta,m}$ and all $(u_0, v_0) \in \mathcal{ID}_K$ any classical solution (u, v) of (5) satisfies

$$\|u\nabla w\|_{L^\infty((0,T);L^{N+3}(\Omega))} \le c_5,$$

again with w as in (16). Additionally taking into account Lemma 3.1, we can apply Lemma 2.1 v) with $f := u\nabla w$ so as to obtain $c_6 > 0$ such that for all $D \in \mathcal{C}_{\delta,m}$ and all $(u_0, v_0) \in \mathcal{ID}_K$ every classical solution (u, v) of (5) satisfies

$$\|u\|_{L^\infty(\Omega\times(0,T))} \le c_6,$$

which shows (40) and – in light of the extensibility criterion in (15) – also proves i). Given $D \in \mathcal{C}_{\delta,m}$ we let $\bar{D}(s) := \int_0^s D(\sigma)d\sigma$ and $\bar{\bar{D}}(s) := \int_0^s \bar{D}(\sigma)d\sigma$ for $s > 0$. Then for every $D \in \mathcal{C}_{\delta,m}$ and $(u_0, v_0) \in \mathcal{ID}_K$, any classical solution (u, v) of (5) obeys $u_t = \Delta \bar{D}(u) + \nabla \cdot (u\nabla w)$ with w as in (16), and testing this equation by $\bar{D}(u)$ we obtain

$$\int_0^T \int_\Omega (\bar{\bar{D}}(u))_t = -\int_0^T \int_\Omega |\nabla \bar{D}(u)|^2 - \int_0^T \int_\Omega u\nabla w \cdot \nabla \bar{D}(u),$$

which, by Young's inequality, turns into

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$$\int_\Omega \bar{\bar{D}}(u(\cdot,T)) + \frac{1}{2}\int_0^T\int_\Omega |D(u)\nabla u|^2 \le \int_\Omega \bar{\bar{D}}(u_0) + \frac{1}{2}\int_0^T\int_\Omega u^2|\nabla w|^2 \le |\Omega|\bar{\bar{D}}(K) + \frac{1}{2}|\Omega|Tc_6^2c_4^2,$$

due to nonnegativity of D proving (43). The existence of $c_7>0$, $c_8>0$ such that for any $D\in\mathcal{C}_{\delta,m}$ and any $(u_0,v_0)\in\mathcal{ID}_K$ any solution of (5) satisfies

$$\left\|\nabla u^{m-1}\right\|_{L^2(\Omega\times(0,T))} \le c_7, \qquad \int_0^T\int_\Omega D(u)u^{m-3}|\nabla u|^2 \le c_8$$

immediately results from Lemma 3.3, so that (44) and (45) have been shown. For every $\phi\in C_0^\infty(\Omega)$ we have that any solution (u,v) of (5) for any $D\in\mathcal{C}_{\delta,m}$, $(u_0,v_0)\in\mathcal{ID}_K$ satisfies

$$\left|\int_\Omega v_t\phi\right| = \left|-\int_\Omega \nabla\phi\cdot\nabla v - \int_\Omega uv\phi\right| \le c_2\|\nabla\phi\|_{L^1(\Omega)} + Kc_6\|\phi\|_{L^1(\Omega)}$$

and we can conclude (46). We let $c_9>0$ be such that $\|\varphi\|_{L^\infty(\Omega)}\le c_9$ for every $\varphi\in W_0^{1,N+1}(\Omega)$ with $\|\varphi\|_{W_0^{1,N+1}(\Omega)}\le 1$ and $c_{10}>0$, $c_{11}>0$ such that $\|\varphi\|_{L^2(\Omega\times(0,T))}\le c_{10}$, $\|\varphi\|_{L^1(\Omega\times(0,T))}\le c_{11}$ for every $\varphi\in L^\infty((0,T);L^{N+1}(\Omega))$ with $\|\varphi\|_{L^\infty((0,T);W_0^{1,N+1}(\Omega))}\le 1$. We denote $X := L^1((0,T);(W_0^{1,N+1}(\Omega))^*)$ and thus have $X^* = L^\infty((0,T);W_0^{1,N+1}(\Omega))$. Taking $\phi\in X^*$ with $\|\phi\|_{X^*}\le 1$, for any solution (u,v) of (5) for $D\in\mathcal{C}_{\delta,m}$ and $(u_0,v_0)\in\mathcal{ID}_K$ we have

$$\begin{aligned}
\frac{1}{m-1}\left|\int_0^T\int_\Omega (u^{m-1})_t\phi\right| &= \left|\int_0^T\int_\Omega u^{m-2}u_t\phi\right| \\
&\le \left|\int_0^T\int_\Omega u^{m-2}\phi\nabla\cdot(D(u)\nabla u)\right| + \left|\int_0^T\int_\Omega u^{m-2}\phi\nabla\cdot\left(\frac{u}{v}\nabla v\right)\right| \\
&\le |m-2|\left|\int_0^T\int_\Omega u^{m-3}\phi D(u)|\nabla u|^2\right| + \left|\int_0^T\int_\Omega u^{m-2}D(u)\nabla u\cdot\nabla\phi\right| \\
&\quad + |m-2|\left|\int_0^T\int_\Omega \frac{u^{m-2}\phi}{v}\nabla v\cdot\nabla u\right| + \left|\int_0^T\int_\Omega \frac{u^{m-1}}{v}\nabla v\cdot\nabla\phi\right| =: I_1+I_2+I_3+I_4,
\end{aligned}$$

where we can estimate $I_1\le|m-2|c_8c_9$,

$$I_2 \le \frac{1}{2}\int_0^T\int_\Omega u^{m-3}D(u)|\nabla u|^2 + \frac{1}{2}\int_0^T\int_\Omega u^{m-1}D(u)|\nabla\phi|^2 \le \frac{c_8}{2} + \frac{1}{2}c_6^{m-1}c_{10}^2\sup_{s\in[0,c_6]}D(s),$$

moreover

$$I_3 \le c_9c_4|m-2| \left\| u^{m-2}\nabla u \right\|_{L^1(\Omega\times(0,T))} \le c_9c_4|m-2|\sqrt{|\Omega|T} \left\| u^{m-2}\nabla u \right\|_{L^2(\Omega\times(0,T))}$$
$$= \frac{c_9c_4|m-2|\sqrt{|\Omega|T}}{m-1} \left\| \nabla u^{m-1} \right\|_{L^2(\Omega\times(0,T))} \le \frac{c_9c_4c_7|m-2|\sqrt{|\Omega|T}}{m-1}$$

and $I_4 \le c_6^{m-1}c_4c_{11}$, so that finally

$$\left\| (u^{m-1})_t \right\|_{L^1((0,T);(W_0^{1,N+1}(\Omega))^*)} \le c_{12} + c_{13} \sup_{s\in[0,c_6]} D(s),$$

where $c_{12} := c_8c_9|m-2| + \frac{c_8}{2} + \frac{c_9c_4c_7|m-2|\sqrt{|\Omega|T}}{m-1} + c_6^{m-1}c_4c_{11}$ and $c_{13} := \frac{1}{2}c_6^{m-1}c_{10}^2$, holds for any solution (u,v) of (5) for any $(u_0,v_0) \in \mathcal{ID}_K$ and any $D \in \mathcal{C}_{\delta,m}$. $\square$

Proof of Theorem 1.1. Lemma 3.15 i) together with (40) contains Theorem 1.1. $\square$

4. Weak solutions in the degenerate case. Proof of Theorem 1.2

If the diffusion becomes degenerate at points where $u = 0$, we can no longer hope for classical solutions. Therefore we introduce the following definition of weak solutions that are – in line with our goal of finding solutions that do not blow up in finite time – locally bounded.

Definition 4.1. Let $\delta > 0$, $m \ge 1$ and $D \in \mathcal{C}_{\delta,m}$ and define $\bar{D}(s) := \int_0^s D(\sigma)d\sigma$ for $s \in [0,\infty)$. Moreover, let (u_0, v_0) be as in (4). By a *locally bounded global weak solution* to (5) we mean a pair of functions $(u,v)\colon \Omega\times[0,\infty) \to \mathbb{R}^2$ such that

$$u \in L^\infty_{loc}([0,\infty); L^\infty(\Omega))$$
$$\bar{D}(u) \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$$
$$v \in L^\infty_{loc}([0,\infty); W^{1,\infty}(\Omega))$$

and for every $\phi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ we have

$$-\int_0^\infty\int_\Omega u\phi_t - \int_\Omega u_0\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla\bar{D}(u)\cdot\nabla\phi + \int_0^\infty\int_\Omega \frac{u}{v}\nabla v\cdot\nabla\phi \tag{48}$$

and

$$-\int_0^\infty\int_\Omega v\phi - \int_\Omega v_0\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla v\cdot\nabla\phi - \int_0^\infty\int_\Omega uv\phi. \tag{49}$$

Having prepared a lot of bounds on solutions to (5) for $D \in \mathcal{C}^+_{\delta,m}$ that are uniform in $D \in \mathcal{C}^+_{\delta,m}$ (Lemma 3.15), we approximate $D \in \mathcal{C}_{\delta,m}$ and find a limit of the corresponding solutions.

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Proof of Theorem 1.2. Let $D \in \mathcal{C}_{\delta,m}$. For any $\epsilon > 0$ we define $D_\epsilon(s) := D(s+\epsilon)$, $s \in [0,\infty)$, and note that, for any $\epsilon > 0$, $D_\epsilon \in \mathcal{C}^+_{\delta,m}$. We choose $(u_{0,\epsilon}, v_{0,\epsilon}) \in (C^1(\overline{\Omega}))^2$ such that $u_{0,\epsilon} \to u_0$ and $v_{0,\epsilon} \to v_0$ in $L^1(\Omega)$ as $\epsilon \searrow 0$ and that there is $K > 0$ such that for all $\epsilon \in (0,1)$ we have $\|u_{0,\epsilon}\|_{L^{\max\{1,m-1\}}(\Omega)} + \|v_{0,\epsilon}\|_{W^{1,\infty}(\Omega)} \le K$ and $\|v_{0,\epsilon}\|_{L^\infty(\Omega)} > \frac{1}{K}$, and let $(u_\epsilon, v_\epsilon) \in (C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)))^2$ denote a solution to

$$\begin{cases} u_{\epsilon t} = \nabla \cdot (D_\epsilon \nabla u_\epsilon) - \nabla \cdot \left(\frac{u_\epsilon}{v_\epsilon} \nabla v_\epsilon\right) & \text{in } \Omega \times (0,\infty), \\ v_{\epsilon t} = \Delta v_\epsilon - u_\epsilon v_\epsilon & \text{in } \Omega \times (0,\infty), \\ u_\epsilon(\cdot,0) = u_{0,\epsilon}, \quad v_\epsilon(\cdot,0) = v_{0,\epsilon} & \text{in } \Omega, \\ \partial_\nu u_\epsilon\big|_{\partial\Omega} = 0 = \partial_\nu v_\epsilon\big|_{\partial\Omega} & \text{in } (0,\infty), \end{cases} \tag{50}$$

which exists due to 3.15 i).

Let us define $\bar{D}_\epsilon(s) := \int_0^s D_\epsilon(\sigma)d\sigma$, $s \in [0,\infty)$. We claim that for every $n \in \mathbb{N}$ there is a sequence $(\epsilon_{n,k})_{k\in\mathbb{N}}$ such that $\epsilon_{n,k} \to 0$ as $k \to \infty$ for any $n \in \mathbb{N}$, that for $n > 1$ the sequence $(\epsilon_{n,k})_{k\in\mathbb{N}}$ is a subsequence of $(\epsilon_{n-1,k})_{k\in\mathbb{N}}$ and that for any $n \in \mathbb{N}$

$$\begin{cases} u_{\epsilon_{n,k}} & \text{converges a.e. in } \Omega \times (0,n) \text{ and in } L^1(\Omega \times (0,n)) \\ \bar{D}_{\epsilon_{n,k}}(u_{\epsilon_{n,k}}) & \text{converges weakly in } L^2((0,n); W_0^{1,2}(\Omega)) \\ v_{\epsilon_{n,k}} & \text{converges uniformly in } \Omega \times (0,n) \\ \nabla v_{\epsilon_{n,k}} & \text{converges weakly* in } L^\infty(\Omega \times (0,n)) \\ \frac{1}{v_{\epsilon_{n,k}}} \nabla v_{\epsilon_{n,k}} & \text{converges weakly* in } L^\infty(\Omega \times (0,n)) \end{cases} \tag{51}$$

as $k \to \infty$. For $n = 0$ we choose an arbitrary monotone sequence $(\epsilon_{0,k})_{k\in\mathbb{N}} \subset (0,1)$ which converges to 0. Let $n \in \mathbb{N}$ and let us assume that some sequence $(\epsilon_{n-1,k})_{k\in\mathbb{N}}$ with properties as in (51) is given. Then by Lemma 3.15 ii), more precisely, by (40), there is $c_1(n) > 0$ such that

$$\|u_{\epsilon_{n-1,k}}\|_{L^\infty(\Omega\times(0,n))} \le c_1(n) \qquad \text{for all } k \in \mathbb{N}. \tag{52}$$

We abbreviate

$$d_n := \sup_{\epsilon\in(0,1)} \sup_{0\le s\le c_1(n)} D_\epsilon(s) \le \sup_{0\le s\le c_1(n)+1} D(s).$$

Then

$$\bar{D}_{\epsilon_{n-1,k}}(u_{\epsilon_{n-1,k}}(x,t)) \le \int_0^{c_1(n)} d_n = c_1(n)d_n \quad \text{for all } (x,t) \in \Omega \times (0,n)$$

and combining this with (43), we find $c_2(n) > 0$ such that for all $k \in \mathbb{N}$

$$\|\bar{D}_{\epsilon_{n-1,k}}(u_{\epsilon_{n-1,k}})\|_{L^2((0,n);W^{1,2}(\Omega))} \le c_2(n).$$

Hence there is a subsequence $(\epsilon^{(1)}_{n,k})_{k\in\mathbb{N}}$ of $(\epsilon_{n-1,k})_{k\in\mathbb{N}}$ such that $(\bar{D}_{\epsilon^{(1)}_{n,k}}(u_{\epsilon^{(1)}_{n,k}}))_{k\in\mathbb{N}}$ is weakly convergent in $L^2((0,n);W^{1,2}(\Omega))$. Moreover, (44), (40) and (47) show that there is $c_3(n)$ such that for all $k\in\mathbb{N}$

$$\left\| u^{m-1}_{\epsilon^{(1)}_{n,k}} \right\|_{L^2((0,n);W^{1,2}(\Omega))} \le c_3(n), \qquad \left\| \left(u^{m-1}_{\epsilon^{(1)}_{n,k}} \right)_t \right\|_{L^1((0,n);(W_0^{1,N+1})^*)} \le c_3(n).$$

Since $W^{1,2}(\Omega) \overset{cpt}{\hookrightarrow} L^2(\Omega) \hookrightarrow (W_0^{1,N+1}(\Omega))^*$, we can invoke a version of the Aubin–Lions lemma ([33, Cor. 8.4]) to find a subsequence $(\epsilon^{(2)}_{n,k})_{k\in\mathbb{N}}$ of $(\epsilon^{(1)}_{n,k})_{k\in\mathbb{N}}$ such that $(u^{m-1}_{\epsilon^{(2)}_{n,k}})_{k\in\mathbb{N}}$ is convergent in $L^2((0,n);L^2(\Omega))$, and a further subsequence $(\epsilon^{(3)}_{n,k})_{k\in\mathbb{N}}$ of $(\epsilon^{(2)}_{n,k})_{k\in\mathbb{N}}$ such that $(u^{m-1}_{\epsilon^{(3)}_{n,k}})_{k\in\mathbb{N}}$ and thus, by continuity of $[0,\infty)\ni x\mapsto x^{\frac{1}{m-1}}$, also $(u_{\epsilon^{(3)}_{n,k}})_{k\in\mathbb{N}}$ converge a.e. in $\Omega\times(0,n)$ as well as with respect to the norm of $L^1(\Omega\times(0,n))$ due to Lebesgue's dominated convergence theorem and the fact that the constant $c_1(n)$ is integrable over $\Omega\times(0,n)$. Moreover, (41) and (46) ensure the existence of $c_4(n)>0$ such that

$$\left\| v_{\epsilon^{(3)}_{n,k}} \right\|_{L^\infty((0,n);W^{1,\infty}(\Omega))} \le c_4(n), \qquad \left\| \left(v_{\epsilon^{(3)}_{n,k}} \right)_t \right\|_{L^2((0,n);(W_0^{1,1}(\Omega))^*)} \le c_4(n) \qquad \text{for all } k\in\mathbb{N}$$

and again due to $W^{1,\infty}(\Omega) \overset{cpt}{\hookrightarrow} C^0(\overline{\Omega}) \hookrightarrow (W_0^{1,1}(\Omega))^*$ and [33, Cor. 8.4] we find a subsequence $(\epsilon^{(4)}_{n,k})_{k\in\mathbb{N}}$ of $(\epsilon^{(3)}_{n,k})_{k\in\mathbb{N}}$ such that $(v_{\epsilon^{(4)}_{n,k}})_{k\in\mathbb{N}}$ converges uniformly in $\Omega\times(0,n)$. Additionally, (41) produces another subsequence $(\epsilon^{(5)}_{n,k})_{k\in\mathbb{N}}$ of $(\epsilon^{(4)}_{n,k})_{k\in\mathbb{N}}$ such that $(\nabla v_{\epsilon^{(5)}_{n,k}})_{k\in\mathbb{N}}$ converges weakly* in $L^\infty(\Omega\times(0,n))$. Finally, owing to the bound in (42), we can extract a further subsequence $(\epsilon_{n,k})_{k\in\mathbb{N}}$ of $(\epsilon^{(5)}_{n,k})_{k\in\mathbb{N}}$ such that also $\left(\frac{1}{v_{\epsilon_{n,k}}}\nabla v_{\epsilon_{n,k}}\right)_{k\in\mathbb{N}}$ is weakly* convergent in $L^\infty(\Omega\times(0,n))$. We then use the diagonal sequence $(\widetilde{\epsilon}_k)_{k\in\mathbb{N}} := (\epsilon_{k,k})_{k\in\mathbb{N}}$ to find functions $u,v,z\colon \Omega\times[0,\infty)\to\mathbb{R}$ and $\zeta,\xi\colon\Omega\times[0,\infty)\to\mathbb{R}^N$ such that

$$u_{\widetilde{\epsilon}_k} \to u \qquad \text{in } L^1_{loc}([0,\infty),L^1(\Omega)) \text{ and a.e. in } \Omega\times(0,\infty), \tag{53}$$

$$v_{\widetilde{\epsilon}_k} \to v \qquad \text{in } L^\infty_{loc}([0,\infty);C^0(\overline{\Omega})), \tag{54}$$

$$\bar{D}_{\widetilde{\epsilon}_k}(u_{\widetilde{\epsilon}_k}) \rightharpoonup z \qquad \text{in } L^2_{loc}([0,\infty);W^{1,2}(\Omega)), \tag{55}$$

$$\nabla v_{\widetilde{\epsilon}_k} \overset{*}{\rightharpoonup} \zeta \qquad \text{in } L^\infty_{loc}([0,\infty),L^\infty(\Omega)) \text{ and} \tag{56}$$

$$\frac{1}{v_{\widetilde{\epsilon}_k}}\nabla v_{\widetilde{\epsilon}_k} \overset{*}{\rightharpoonup} \xi \qquad \text{in } L^\infty_{loc}([0,\infty),L^\infty(\Omega)) \tag{57}$$

as $k\to\infty$. Since $u_{\widetilde{\epsilon}_k}+\widetilde{\epsilon}_k\to u$ a.e. and $\bar{D}$ is continuous, also $\bar{D}_{\widetilde{\epsilon}_k}(u_{\widetilde{\epsilon}_k}) = \bar{D}(u_{\widetilde{\epsilon}_k}+\widetilde{\epsilon}_k) - \bar{D}(\widetilde{\epsilon}_k) \to \bar{D}(u)-\bar{D}(0)=\bar{D}(u)$ a.e., and hence $z=\bar{D}(u)$. Also, (54) and (56) imply $\zeta=\nabla v$ and the combination of (54) and (57) shows that $\xi=\frac{1}{v}\nabla v$. We let $\phi\in C_0^\infty(\Omega\times[0,\infty))$. Then (50) entails that

$$-\int_0^\infty\int_\Omega u_{\widetilde{\epsilon}_k}\phi_t - \int_\Omega u_{0\widetilde{\epsilon}_k}\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla \bar{D}_{\widetilde{\epsilon}_k}(u_{\widetilde{\epsilon}_k})\cdot\nabla\phi + \int_0^\infty\int_\Omega \frac{u_{\widetilde{\epsilon}_k}}{v_{\widetilde{\epsilon}_k}}\nabla v_{\widetilde{\epsilon}_k}\cdot\nabla\phi$$

and

$$-\int_0^\infty\int_\Omega v_{\widetilde{\epsilon}_k}\phi - \int_\Omega v_{0\widetilde{\epsilon}_k}\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla v_{\widetilde{\epsilon}_k}\cdot\nabla\phi - \int_0^\infty\int_\Omega u_{\widetilde{\epsilon}_k}v_{\widetilde{\epsilon}_k}\phi,$$

so that passing to the limit as $k\to\infty$ in each of these integrals shows that (u,v) satisfies (48) and (49). That $u\in L^\infty_{loc}([0,\infty);L^\infty(\Omega))$ is also entailed by (53) and (52). Hence (u,v) is a locally bounded global weak solution to (5) in the sense of Definition 4.1. □

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