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Global weak solutions in a three-dimensional Keller–Segel–Navier–Stokes system involving a tensor-valued sensitivity with saturation

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Abstract

This paper is concerned with the following Keller–Segel–Navier–Stokes system

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\mathcal{S}(x,n,c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + n, & x \in \Omega,\ t > 0, \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{$\star$}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $\kappa \in \mathbb{R}$ and $\mathcal{S}$ denotes a given tensor-valued function fulfilling

$$|\mathcal{S}(x,n,c)| \le \frac{C_{\mathcal{S}}}{(1+n)^{\alpha}}$$

with some $C_{\mathcal{S}} > 0$ and $\alpha > 0$. As the case $\kappa = 0$ has been considered in [25], it is shown in the present paper that the corresponding initial–boundary problem with $\kappa \neq 0$ admits at least one global weak solution if $\alpha \ge \frac{3}{7}$.

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1. Introduction

Chemotaxis is a kind of partially oriented movement of living cells in response to chemical gradients. In order to describe this biological phenomenon in mathematics, Keller and Segel [13] proposed the following system

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\mathcal{S}(n,c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $n = n(x,t)$ and $c = c(x,t)$ stand for the density of cell population and the concentration of chemoattractant, respectively, and $\mathcal{S} = \mathcal{S}(n,c)$ represents the chemotactic sensitivity. The mathematical analysis of system (1.1) and the variants thereof mainly concentrates on the boundedness and blow-up of the solutions (see [1,5,11,15,21,26]). From these works, it can be observed that there are several different versions of chemotactic sensitivities, such as the signal-dependent sensitivity $\mathcal{S}(n,c) = \frac{C_{\mathcal{S}}}{c}$ or $\frac{C_{\mathcal{S}}}{(1+\mu c)^2}$ with $C_{\mathcal{S}} > 0$ and $\mu > 0$ [5,26], which reflects the inhibition of cell movement in the location where the signal concentration is high [10,14], the n-dependent sensitivity $\mathcal{S}(n,c) = n^q$ with $q > 0$ or $n(n+1)^q$ with $q \in \mathbb{R}$ [11,21], which shows the volume-filling effect in the process of chemotaxis [17], as well as the tensor-valued sensitivity $\mathcal{S} = (\mathcal{S}_{ij})_{2\times 2}$ with $\mathcal{S}_{ij} \in C^2(\bar{\Omega} \times [0,\infty) \times [0,\infty))$ for $i, j \in \{1,2\}$ [1,15], which describes the rotational chemotactic migration happening close to the physical boundary of the domain [33,34]. In comparison with the systems with scalar-valued sensitivities, the systems with tensor-valued sensitivity lose some energy structure, which results in considerable difficulties in the mathematical analysis.

If chemotaxis occurs in incompressible fluid, then the original chemotaxis system needs to be coupled with another equation which characterizes the motion of the fluid, and the resulting system can read as

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\mathcal{S}(n,c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), & x \in \Omega,\ t > 0, \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

which was introduced in [23]. In system (1.2), $u = u(x,t)$ and $P = P(x,t)$ denote the velocity of incompressible fluid and the associated pressure, respectively, f, ϕ are given parameter functions and $\kappa \in \mathbb{R}$ measures the strength of nonlinear fluid convection. As to the mathematical analysis of system (1.2), the global solvability of corresponding initial–boundary problem is the most attractive issue during the past years and the related results have been established in [27,30,31,35]. Among these results, one can find that if $\mathcal{S}$ and f satisfy some suitable structural hypothesis the global solvability of system (1.2) can be derived on the basis of certain natural quasi-Lyapunov functional involving the logarithmic entropy $\int_\Omega n \ln n$, for both linear diffusion [30,31] and porous-medium type nonlinear diffusion [35]. Besides that, for the two-dimensional Navier–Stokes version of system (1.2), this energy-based method also plays an important role in

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the consideration of the large time behavior of the solutions [32]. However, if $\mathcal{S}$ is regarded as a tensor-valued sensitivity as mentioned before, the energy-based reasoning seems to fail due to its excessive dependence on the inflexible structural assumptions on $\mathcal{S}$ and f such as $\left(\frac{f}{\mathcal{S}}\right)'' \leq 0$ on $(0,\infty)$ [30,31,35]. Therefore, in order to relax the rigid structural restrictions on $\mathcal{S}$ and f, for two-dimensional system (1.2) with tensor-valued $\mathcal{S}$, the core of the arguments turns to the analysis of the combinational functional $\int_\Omega n^p + \int_\Omega |\nabla c|^2$ [12], and for the three-dimensional chemotaxis–Stokes version of system (1.2) with tensor-valued $\mathcal{S}$, an alternative approach of a priori estimation is developed in [27].

It is noted that the second equation in system (1.2) describes the situation where the chemical signal is only consumed by cells, while if the chemical signal is secreted by cells, then system (1.2) is transformed into

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\mathcal{S}(n,c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.3}$$

which is the coupled system of (1.1) with a (Navier–)Stokes equation. Due to the mathematical difficulties aroused by $-c+n$ in the second equation of system (1.3), results on global solvability of system (1.3) are much less as compared to that of system (1.2). In fact, under the properly strong logistic dampening effect, the three-dimensional Stokes-version of system (1.3) with $\mathcal{S}(n,c)\equiv C_{\mathcal{S}}>0$ possesses a globally bounded classical solution [20], while for (1.3) with $\kappa=1$, the analogous conclusion is derived in the two-dimensional setting [19]; If $\mathcal{S}$ is a signal-dependent function satisfying $\mathcal{S}(n,c)\leq\frac{C_{\mathcal{S}}}{(1+\mu c)^k}$ with $\mu>0$ and $k>1$, then a unique globally bounded classical solution is constructed for two-dimensional system (1.3) and three-dimensional Stokes-version of system (1.3), and a global weak solution is proved to exist for three-dimensional Navier–Stokes version of system (1.3) [16]; If $\mathcal{S}=\mathcal{S}(x,n,c)$ is a tensor-valued sensitivity fulfilling

$$|\mathcal{S}(x,n,c)| \leq \frac{C_{\mathcal{S}}}{(1+n)^\alpha} \quad \text{for some} \quad \alpha>0 \quad \text{and} \quad C_{\mathcal{S}}>0, \tag{1.4}$$

then two-dimensional Stokes-version of system (1.3) admits a unique global classical solution which is bounded [24], and similar results are also valid for the three-dimensional Stokes-version of system (1.3) with $\alpha>\frac{1}{2}$ [25]. Whereas in the case when $u\equiv 0$, there may exist solutions which blow up in finite time either if the chemotactic sensitivity $\mathcal{S}$ is the constant unit matrix [29] or if $\mathcal{S}$ decays at large values of n but too slowly [11,28,2–4]. From the above results, one can find that the tensor-valued chemotactic sensitivity $\mathcal{S}$ fulfilling (1.4) can enforce the global solvability of (1.3) appropriately, which makes it necessary to introduce saturation effects in the chemotactic sensitivity in some cases. However for the three-dimensional Navier–Stokes-version of system (1.3) with tensor-valued $\mathcal{S}$, the results on global solvability are still unknown. Therefore, the purpose of this paper is to prove that three-dimensional Navier–Stokes-version of system (1.3) with tensor-valued $\mathcal{S}$ satisfying (1.4) is indeed globally solvable if α is suitably large, which, to the best of our knowledge, is the first result on global solvability of three-dimensional Keller–Segel–Navier–Stokes system (1.3) with tensor-valued sensitivity.

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Ahead of the presentation of the main result, we give a precise statement of the evolution problem considered in the sequel. This problem is consisted of system (1.3) with boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.5}$$

and the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.6}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$ and the initial data comply with

$$\begin{cases} n_0 \in C^0(\bar{\Omega}) \text{ and } n_0 \geq 0 \text{ as well as } n_0 \not\equiv 0 \text{ in } \bar{\Omega}, \\ c_0 \in W^{1,\infty}(\Omega) \ \text{ and } c_0 \geq 0 \text{ as well as } c_0 \not\equiv 0 \text{ in } \bar{\Omega}, \\ u_0 \in D(A^\beta) \ \text{ for some } \beta \in (\frac{3}{4}, 1) \end{cases} \tag{1.7}$$

with A standing for the Stokes operator in the solenoidal subspace $L^2_\sigma(\Omega) := \{w \in L^2(\Omega) | \nabla \cdot w = 0\}$ of $L^2(\Omega)$. As to the parameter functions $\mathcal{S}$ and ϕ in (1.3), we assume that $\mathcal{S} = (\mathcal{S}_{ij})_{3\times 3}$ with

$$\mathcal{S}_{ij} \in C^2(\bar{\Omega} \times [0,\infty) \times [0,\infty)) \quad \text{for} \quad i, j \in \{1,2,3\} \tag{1.8}$$

fulfills (1.4) and that

$$\phi \in W^{2,\infty}(\Omega). \tag{1.9}$$

Now, we present the main result as follows.

Theorem 1.1. *Let $\kappa \neq 0$ and $\alpha \geq \frac{3}{7}$, and let (1.4)–(1.9) be valid. Then problem (1.3), (1.5) and (1.6) possesses at least one global weak solution (n, c, u, P) in the sense of Definition 5.1 below.*

The crucial step of our approach is to establish the ε-independent estimates of

$$\int_\Omega n_\varepsilon^{2\alpha}(\cdot,t) + \int_\Omega c_\varepsilon^2(\cdot,t) \quad \text{and} \quad \int_\Omega |u_\varepsilon(\cdot,t)|^2 \quad \text{for all} \quad t \geq 0 \tag{1.10}$$

(see Section 3), where $(n_\varepsilon, c_\varepsilon, u_\varepsilon)_{\varepsilon\in(0,1)}$ is the solution of the approximated system of (1.3) (see (2.1) below). Then by a two-step bootstrap argument it is possible to gain the ε-dependent bounds for n_ε and ∇c_ε in $L^p((0,T_{\max,\varepsilon}); L^p(\Omega))$ and $L^q((0,T_{\max,\varepsilon}); L^q(\Omega))$ for arbitrarily large $p > 1$ and $q > 1$, respectively, which underlies the derivation of the global solvability of the approximated system (see Section 4). In the final, based on the ε-independent estimates of (1.10), we can establish some ε-independent bounds for spatio-temporal integrals of the approximated solutions as well as several ε-independent regularity features of their time derivatives, and then follow a standard procedure to construct a global weak solution in the sense of Definition 5.1 (see Section 5).

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2. Preliminaries

Due to $\kappa \neq 0$ and the presence of tensor-valued $\mathcal{S}$ in system (1.3), we need to consider an appropriately regularized problem of (1.3), (1.5) and (1.6) at first. According to the ideas in [15] and [31], the corresponding regularized problem is introduced as follows:

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon), & x \in \Omega,\ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + F_\varepsilon(n_\varepsilon), & x \in \Omega,\ t > 0, \\ u_{\varepsilon t} + \kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = 0,\ u_\varepsilon = 0, & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_0(x),\ \ c_\varepsilon(x,0) = c_0(x),\ \ u_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where for $\varepsilon \in (0, 1)$,

$$\mathcal{S}_\varepsilon(x, n, c) := \rho_\varepsilon(x)\mathcal{S}(x, n, c), \quad (x, n, c) \in \bar{\Omega} \times [0, \infty) \times [0, \infty) \tag{2.2}$$

with $\{\rho_\varepsilon\}_{\varepsilon \in (0,1)} \subset C_0^\infty(\Omega)$ being a family of standard cut-off functions and fulfilling $0 \leq \rho_\varepsilon \leq 1$ in Ω as well as $\rho_\varepsilon \to 1$ as $\varepsilon \to 0$ in Ω, and moreover,

$$F_\varepsilon(s) := \frac{1}{\varepsilon}\ln(1 + \varepsilon s) \quad \text{for all} \quad s \geq 0,$$

and Y_ε denotes the standard Yosida approximation [18] defined as

$$Y_\varepsilon v := (1 + \varepsilon A)^{-1} v \quad \text{for all} \quad v \in L^2_\sigma(\Omega).$$

From the definition of F_ε, we can also obtain that for any $\varepsilon \in (0, 1)$

$$0 \leq F_\varepsilon'(s) = \frac{1}{1 + \varepsilon s} \leq 1 \quad \text{for all} \quad s \geq 0, \tag{2.3}$$

$$0 \leq F_\varepsilon(s) \leq s \quad \text{for all} \quad s \geq 0 \tag{2.4}$$

and

$$F_\varepsilon(s) \to s, \quad F_\varepsilon'(s) \to 1 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for all} \quad s \geq 0. \tag{2.5}$$

The local solvability of (2.1) can be derived by a slight modification of the well-established fixed point arguments in [30], so here we omit the proof.

Lemma 2.1. *Let $\kappa \neq 0$, and let (1.4)–(1.9) be satisfied. Then for each $\varepsilon \in (0, 1)$, there exist a maximal existence time $T_{\max,\varepsilon} \in (0, +\infty)$ and a unique quadruple $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ of functions which solve (2.1) and comply with*

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$$\begin{cases} n_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max,\varepsilon})) \bigcap C^{2,1}(\bar{\Omega} \times (0, T_{\max,\varepsilon})), \\ c_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max,\varepsilon})) \bigcap C^{2,1}(\bar{\Omega} \times (0, T_{\max,\varepsilon})) \bigcap_{q>3} C^0([0, T_{max,\varepsilon}); W^{1,q}(\Omega)), \\ u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max,\varepsilon}); \mathbb{R}^3) \bigcap C^{2,1}(\bar{\Omega} \times (0, T_{\max,\varepsilon}); \mathbb{R}^3), \\ P_\varepsilon \in C^{1,0}(\bar{\Omega} \times (0, T_{\max,\varepsilon})) \end{cases} \tag{2.6}$$

as well as

$$n_\varepsilon \geq 0, \quad c_\varepsilon \geq 0 \quad in \quad \bar{\Omega} \times [0, T_{\max,\varepsilon}). \tag{2.7}$$

In addition, if $T_{\max,\varepsilon} < \infty$, then for $q \in (3, \infty)$ and β chosen as in (1.7),

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} + \|A^\beta u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \to \infty \tag{2.8}$$

as $t \to T_{\max,\varepsilon}$.

Upon a straightforward integration of the first two equations in (2.1) over Ω, we can establish the following basic estimates for n_ε and c_ε.

Lemma 2.2. *For each $\varepsilon \in (0, 1)$, the components n_ε and c_ε of the solution to (2.1) satisfy*

$$\int_\Omega n_\varepsilon(\cdot, t) = \int_\Omega n_0 \quad for\ all \quad t \in (0, T_{\max,\varepsilon}) \tag{2.9}$$

and

$$\int_\Omega c_\varepsilon(\cdot, t) \leq \left\{ \int_\Omega c_0, \int_\Omega n_0 \right\} \quad for\ all \quad t \in (0, T_{\max,\varepsilon}). \tag{2.10}$$

Proof. Integrating the first two equations in (2.1) over Ω separately, we can derive (2.9) and

$$\frac{d}{dt} \int_\Omega c_\varepsilon + \int_\Omega c_\varepsilon = \int_\Omega n_\varepsilon = \int_\Omega n_0 \tag{2.11}$$

for all $t \in (0, T_{\max,\varepsilon})$, which along with an ODE comparison argument entails (2.10). The proof is completed. □

As an auxiliary lemma in this paper, Lemma 2.3 in [20] will be used in our subsequent analysis.

Lemma 2.3. *(See [20].) Let $T > 0$, $\tau \in (0, T)$, $a > 0$ and $b > 0$, and suppose that $y : [0, T) \to [0, \infty)$ is absolutely continuous such that*

$$y'(t) + ay(t) \leq h(t) \quad for\ a.e. \quad t \in (0, T)$$

with some nonnegative function $h \in L^1_{loc}[0, T)$ satisfying

$$\int_t^{t+\tau} h(s)ds \le b \quad \textit{for all} \quad t \in [0, T-\tau).$$

Then,

$$y(t) \le \max\left\{y(0)+b, \frac{b}{a\tau}+2b\right\} \quad \textit{for all} \quad t \in (0, T).$$

3. A priori estimates

Based on Lemma 2.1, we can derive a ε-independent bound for (1.10) following an idea from [25].

Lemma 3.1. *Let $\kappa \neq 0$ and $\alpha \ge \frac{3}{7}$. Then one can find a constant $C > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^{2\alpha}(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}), \tag{3.1}$$

$$\int_t^{t+\tau}\int_\Omega |\nabla n_\varepsilon^\alpha|^2 + \int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^2 \le C \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}-\tau) \tag{3.2}$$

with $\tau := \min\left\{1, \frac{T_{\max,\varepsilon}}{2}\right\}$ *and*

$$\int_0^T\int_\Omega |\nabla n_\varepsilon^\alpha|^2 + \int_0^T\int_\Omega |\nabla c_\varepsilon|^2 + \int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \le C(T+1) \quad \textit{for all} \quad T \in (0, T_{\max,\varepsilon}). \tag{3.3}$$

Proof. As mentioned in the introduction, the derivation of the time evolution of $\int_\Omega n_\varepsilon^{2\alpha} + \int_\Omega c_\varepsilon^2$ is considered as a cornerstone of the arguments in the sequel. This can be achieved by a subtle reasoning as in [25, Lemma 2.3] on the basis of the same construction exhibited by the first two equations in (2.1). Thereupon, we have

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon^{2\alpha} + C_1\int_\Omega c_\varepsilon^2\right\} + C_2\left\{\int_\Omega |\nabla n_\varepsilon^\alpha|^2 + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^2\right\} \le C_3 \tag{3.4}$$

and

$$\|n_\varepsilon(\cdot,t)\|_{L^{2\alpha}(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_4 \tag{3.5}$$

for all $t \in (0, T_{\max,\varepsilon})$, where C_1, C_2, C_3 and C_4 are positive constants. In addition, according to Lemma 3.5 in [31], we can derive from the third equation in (2.1) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\phi$$

for all $t \in (0, T_{\max,\varepsilon})$, which combined with (1.9) and the Hölder inequality yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \le C_5 \|u_\varepsilon\|_{L^6(\Omega)} \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \tag{3.6}$$

with some $C_5 > 0$ for all $t \in (0, T_{\max,\varepsilon})$. Due to $u_\varepsilon = 0$ on $\partial\Omega$, the Poincaré inequality provides $C_6 > 0$ such that

$$\|u_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} \le C_6 \|\nabla u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \tag{3.7}$$

for all $t \in (0, T_{\max,\varepsilon})$, which together with the embedding theorem entails

$$\|u_\varepsilon(\cdot, t)\|_{L^6(\Omega)} \le C_7 \|u_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} \le C_6 C_7 \|\nabla u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \tag{3.8}$$

with certain $C_7 > 0$ for all $t \in (0, T_{\max,\varepsilon})$. In light of (3.7) and (3.8), we apply the Young inequality to (3.6) and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 &\le C_8 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \\ &\le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{C_8^2}{2}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)}\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, i.e.

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \le C_8^2 \|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \tag{3.9}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $C_8 := C_5 C_6 C_7$. By another application of the Poincaré inequality, we have

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + C_9 \int_\Omega |u_\varepsilon|^2 \le C_8^2 \|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \tag{3.10}$$

with some $C_9 > 0$ for all $t \in (0, T_{\max,\varepsilon})$.

In the case when $\frac{3}{7} \le \alpha < \frac{3}{5}$, setting $\tau := \min\left\{1, \frac{T_{\max,\varepsilon}}{2}\right\}$ and integrating (3.4) over $(t, t+\tau)$, we can find $C_{10} > 0$ such that

$$\int_t^{t+\tau}\int_\Omega |\nabla n_\varepsilon^\alpha|^2 \le C_{10} \tag{3.11}$$

for all $t \in (0, T_{\max,\varepsilon} - \tau)$. Apart from that, for any $T \in (0, T_{\max,\varepsilon})$, an integration of (3.4) over $(0, T)$ yields

$$\int_0^T \int_\Omega |\nabla n_\varepsilon^\alpha|^2 + \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \leq C_{11}(T+1) \tag{3.12}$$

with some $C_{11} > 0$. Along with (3.5) and (3.11), we employ the Gagliardo–Nirenberg inequality to gain

$$\begin{aligned} \int_t^{t+\tau} \|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} &= \int_t^{t+\tau} \|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{6}{5\alpha}}(\Omega)} \\ &\leq C_{12} \int_t^{t+\tau} \left(\|\nabla n_\varepsilon^\alpha\|^a_{L^2(\Omega)} \|n_\varepsilon^\alpha\|^{1-a}_{L^2(\Omega)} + \|n_\varepsilon^\alpha\|_{L^2(\Omega)} \right)^{\frac{2}{\alpha}} \\ &\leq C_{13} \int_t^{t+\tau} \left(\|\nabla n_\varepsilon^\alpha\|^{\frac{2a}{\alpha}}_{L^2(\Omega)} + 1 \right) \end{aligned} \tag{3.13}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $C_{12}, C_{13} > 0$ and $a = \frac{3-5\alpha}{2} \in (0,1)$ due to $\alpha \in [\frac{3}{7}, \frac{3}{5})$. Meanwhile, it can be verified from $\alpha \geq \frac{3}{7}$ that

$$\frac{2a}{\alpha} \leq 2.$$

Therefore, we can deduce from (3.12), (3.13) and the Young inequality that

$$\begin{aligned} \int_t^{t+\tau} \|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} &\leq C_{13} \int_t^{t+\tau} \int_\Omega |\nabla n_\varepsilon^\alpha|^2 + 2C_{13} \\ &\leq C_{13}(C_{10}+2) \end{aligned} \tag{3.14}$$

for all $t \in (0, T_{\max,\varepsilon})$. Likewise, by virtue of (3.5) and (3.12), we can also find $C_{14} > 0$ such that

$$\int_0^T \|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \leq C_{14}(T+1) \tag{3.15}$$

for any $T \in (0, T_{\max,\varepsilon})$. Combining (3.10) with (3.14), we can infer from Lemma 2.3 that

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C_{15} \tag{3.16}$$

with certain $C_{15} > 0$ for all $t \in (0, T_{\max,\varepsilon})$, which together with (3.5) gives (3.1). In addition, along with (3.14) and (3.15), integrations of (3.9) over $(t, t+\tau)$ and $(0, T)$ provide $C_{16} > 0$ and $C_{17} > 0$ such that

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$$\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^2 \le C_{16} \tag{3.17}$$

for all $t \in (0, T_{\max,\varepsilon} - \tau)$ and

$$\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \le C_{17}(T+1) \tag{3.18}$$

for any $T \in (0, T_{\max,\varepsilon})$. Thereupon, (3.2) follows from (3.11) as well as (3.17), and (3.3) is valid from (3.12) and (3.18).

Whereas for $\alpha \ge \frac{3}{5}$, we know from (3.5) that there exists $C_{18} > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)} \le C_{18}$$

for all $t \in (0, T_{\max,\varepsilon})$, which combined with (3.10) and an ODE comparison argument entails

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le \max\left\{\|u_0\|_{L^2(\Omega)}, \frac{C_8 C_{18}}{C_9^{\frac{1}{2}}}\right\} \tag{3.19}$$

for all $t \in (0, T_{\max,\varepsilon})$. Thereby, (3.1) is implied by (3.5) and (3.19). Moreover, in view of the boundedness of $\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}$ and (1.7) we can derive (3.2) by integrating (3.4) and (3.9) over $(t, t+\tau)$ for all $t \in (0, T_{\max,\varepsilon} - \tau)$, respectively. Similarly, (3.3) is valid from an integration of (3.4) and (3.9) over $(0, T)$ with respect to t for all $T \in (0, T_{\max,\varepsilon})$, respectively. We complete the proof. □

4. The global solvability of regularized problem (2.1)

The main task of this section is to prove the global solvability of regularized problem (2.1). For this purpose, we need to establish some ε-dependent estimates of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ firstly.

4.1. Some ε-dependent estimates

In this subsection, we try to establish the desired ε-dependent estimates for n_ε and c_ε by a two-step bootstrap argument.

Lemma 4.1. *Let $\kappa \ne 0$ and $\alpha \ge \frac{3}{7}$. Then there exists a constant $C > 0$ depending on $\varepsilon \in (0,1)$ such that*

$$\int_\Omega n_\varepsilon^2(\cdot,t) \le C \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}), \tag{4.1}$$

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le C \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{4.2}$$

and

$$\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2 \le C \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}). \tag{4.3}$$

Proof. Testing the first equation in (2.1) by n_ε, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega n_\varepsilon^2 + \int_\Omega |\nabla n_\varepsilon|^2 = \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon \cdot \nabla n_\varepsilon \tag{4.4}$$

for all $t \in (0, T_{\max,\varepsilon})$. Recalling (1.4) and (2.3), one can see that

$$n_\varepsilon F_\varepsilon'(n_\varepsilon) \le \frac{1}{\varepsilon} \quad \text{and} \quad |\mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \le C_{\mathcal{S}}$$

for all $t \in (0, T_{\max,\varepsilon})$. Thereupon, from the Young inequality, (4.4) can be rewritten as

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega n_\varepsilon^2 + \int_\Omega |\nabla n_\varepsilon|^2 &\le \frac{C_{\mathcal{S}}}{\varepsilon}\int_\Omega |\nabla c_\varepsilon||\nabla n_\varepsilon| \\ &\le \frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 + C_1\int_\Omega |\nabla c_\varepsilon|^2 \end{aligned} \tag{4.5}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $C_1 := \frac{C_{\mathcal{S}}^2}{2\varepsilon^2}$. Similarly, multiplying the second equation in (2.1) by c_ε and using the Young inequality, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^2 &= \int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon \\ &\le \frac{1}{2}\int_\Omega n_\varepsilon^2 + \frac{1}{2}\int_\Omega c_\varepsilon^2 \end{aligned} \tag{4.6}$$

for all $t \in (0, T_{\max,\varepsilon})$. Adding (4.5) with $C_1 \times$ (4.6), we derive

$$\frac{d}{dt}\left(\int_\Omega n_\varepsilon^2 + C_1\int_\Omega c_\varepsilon^2\right) + \left(\int_\Omega |\nabla n_\varepsilon|^2 + C_1\int_\Omega c_\varepsilon^2\right) \le C_1\int_\Omega n_\varepsilon^2 \tag{4.7}$$

for all $t \in (0, T_{\max,\varepsilon})$. Based on (2.9), we deduce from the Gagliardo–Nirenberg inequality and the Young inequality that

$$\begin{aligned}\int_\Omega n_\varepsilon^2(\cdot,t) &= \|n_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}\\ &\le C_2\left(\|\nabla n_\varepsilon(\cdot,t)\|^{\frac{3}{5}}_{L^2(\Omega)}\|n_\varepsilon(\cdot,t)\|^{\frac{2}{5}}_{L^1(\Omega)}+\|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)}\right)^2\\ &\le C_3\left(\|\nabla n_\varepsilon(\cdot,t)\|^{\frac{6}{5}}_{L^2(\Omega)}+1\right)\\ &\le \delta\int_\Omega|\nabla n_\varepsilon(\cdot,t)|^2+C_4\end{aligned}\tag{4.8}$$

with any $\delta>0$ and positive constants C_2, C_3, C_4 for all $t\in(0,T_{\max,\varepsilon})$. Combining (4.7) with (4.8), we choose $\delta=\frac{1}{1+C_1}$ and gain

$$\frac{d}{dt}\left(\int_\Omega n_\varepsilon^2+C_1\int_\Omega c_\varepsilon^2\right)+\left(\int_\Omega n_\varepsilon^2+C_1\int_\Omega c_\varepsilon^2\right)\le C_4(C_1+1)\tag{4.9}$$

for all $t\in(0,T_{\max,\varepsilon})$, which implies (4.1) by an ODE comparison argument. Hence, if we set $\tau:=\min\left\{1,\frac{T_{\max,\varepsilon}}{2}\right\}$, then there exists $C_5>0$ fulfilling

$$\int_t^{t+\tau}\int_\Omega n_\varepsilon^2\le C_5\tag{4.10}$$

for all $t\in(0,T_{\max,\varepsilon}-\tau)$.

In view of (3.1) and $D(1+\varepsilon A)=W^{2,2}(\Omega)\bigcap W^{1,2}_{0,\sigma}(\Omega)\hookrightarrow L^\infty(\Omega)$, one can find $C_6>0$ and $C_7>0$ such that

$$\begin{aligned}\|Y_\varepsilon u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &= \|(1+\varepsilon A)^{-1}u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\\ &\le C_6\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\\ &\le C_7\end{aligned}\tag{4.11}$$

for all $t\in(0,T_{\max,\varepsilon})$. Applying the Helmholz projection $\mathcal{P}$ to the third equation in (2.1) and testing the resulting equation by Au_ε, we deduce from $\|\mathcal{P}w\|_{L^2(\Omega)}\le\|w\|_{L^2(\Omega)}$ for all $w\in L^2(\Omega)$ and the Young inequality that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|A^{\frac{1}{2}}u_\varepsilon|^2+\int_\Omega|Au_\varepsilon|^2 &= \int_\Omega Au_\varepsilon\cdot\mathcal{P}[-(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon+n_\varepsilon\nabla\phi]\\ &\le\frac{1}{2}\int_\Omega|Au_\varepsilon|^2+\int_\Omega|(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon|^2+\int_\Omega|n_\varepsilon\nabla\phi|^2\\ &\le\frac{1}{2}\int_\Omega|Au_\varepsilon|^2+C_7^2\int_\Omega|\nabla u_\varepsilon|^2+C_8\int_\Omega n_\varepsilon^2,\end{aligned}$$

i.e.

$$\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}}u_\varepsilon|^2+\int_\Omega |Au_\varepsilon|^2 \le 2C_7^2\int_\Omega |\nabla u_\varepsilon|^2+2C_8\int_\Omega n_\varepsilon^2 \tag{4.12}$$

for all $t\in(0,T_{\max,\varepsilon})$, where C_8 originates from (1.9). Since the known regularity estimates for the Stokes operator in bounded domain [8, p. 82], the Poincaré inequality and the Sobolev embedding theorem guarantee the existence of $C_9>0$ fulfilling $C_9\int_\Omega |A^{\frac{1}{2}}u_\varepsilon(\cdot,t)|^2 \le \int_\Omega |Au_\varepsilon(\cdot,t)|^2$ for all $t\in(0,T_{\max,\varepsilon})$, we can rewrite (4.12) as

$$\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}}u_\varepsilon|^2+C_9\int_\Omega |A^{\frac{1}{2}}u_\varepsilon|^2 \le 2\left(C_7^2\int_\Omega |\nabla u_\varepsilon|^2+C_8\int_\Omega n_\varepsilon^2\right)$$

for all $t\in(0,T_{\max,\varepsilon})$, where if we let $y(t):=\int_\Omega |A^{\frac{1}{2}}u_\varepsilon(\cdot,t)|^2$ and $h(t):=2(C_7^2\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2+C_8\int_\Omega n_\varepsilon^2(\cdot,t))$ for all $t\in(0,T_{\max,\varepsilon})$, then Lemma 2.3 along with (3.2) and (4.10) provides a $C_{10}>0$ such that

$$\int_\Omega |A^{\frac{1}{2}}u_\varepsilon(\cdot,t)|^2 \le C_{10}$$

for all $t\in(0,T_{\max,\varepsilon})$, which together with $\int_\Omega |A^{\frac{1}{2}}w|^2=\int_\Omega |\nabla w|^2$ for all $w\in D(A)$ implies (4.2).

Now, multiplying the second equation in (2.1) by $-\Delta c_\varepsilon$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2+\int_\Omega |\Delta c_\varepsilon|^2+\int_\Omega |\nabla c_\varepsilon|^2=-\int_\Omega \Delta c_\varepsilon\cdot F_\varepsilon(n_\varepsilon)+\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\Delta c_\varepsilon \tag{4.13}$$

for all $t\in(0,T_{\max,\varepsilon})$, where we derive from (2.4), (4.1) and the Young inequality that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2+\frac{3}{4}\int_\Omega |\Delta c_\varepsilon|^2+\int_\Omega |\nabla c_\varepsilon|^2 &\le \int_\Omega n_\varepsilon^2+\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\Delta c_\varepsilon\\ &\le C_{11}+\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\Delta c_\varepsilon\end{aligned} \tag{4.14}$$

with certain $C_{11}>0$ for all $t\in(0,T_{\max,\varepsilon})$. Applying the Hölder inequality to the second integral on the right-hand side of (4.14), we obtain

$$\begin{aligned}\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\Delta c_\varepsilon &\le \|u_\varepsilon\|_{L^6(\Omega)}\|\nabla c_\varepsilon\|_{L^3(\Omega)}\|\Delta c_\varepsilon\|_{L^2(\Omega)}\\ &\le C_{12}\|\nabla c_\varepsilon\|_{L^3(\Omega)}\|\Delta c_\varepsilon\|_{L^2(\Omega)}\end{aligned} \tag{4.15}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $C_{12} > 0$ stems from (4.2) and the Sobolev embedding theorem. Along with (3.1), the Gagliardo–Nirenberg inequality gives

$$\begin{aligned} \|\nabla c_\varepsilon\|_{L^3(\Omega)} &\le C_{13}\left(\|\Delta c_\varepsilon\|_{L^2(\Omega)}^{\frac{3}{4}}\|c_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{4}} + \|c_\varepsilon\|_{L^2(\Omega)}\right) \\ &\le C_{14}\|\Delta c_\varepsilon\|_{L^2(\Omega)}^{\frac{3}{4}} + C_{14} \end{aligned} \tag{4.16}$$

with $C_{13}, C_{14} > 0$ for all $t \in (0, T_{\max,\varepsilon})$. Combining (4.15) with (4.16), we gain

$$\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon \le C_{12}C_{14}(\|\Delta c_\varepsilon\|_{L^2(\Omega)}^{\frac{7}{4}} + \|\Delta c_\varepsilon\|_{L^2(\Omega)}) \tag{4.17}$$

for all $t \in (0, T_{\max,\varepsilon})$. Thus, by applying the Young inequality to (4.17), we obtain

$$\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon \le \delta \int_\Omega |\Delta c_\varepsilon|^2 + C_{15} \tag{4.18}$$

with some $C_{15} > 0$ and any $\delta > 0$ for all $t \in (0, T_{\max,\varepsilon})$. Choosing $\delta > 0$ suitably large and inserting (4.18) into (4.14), we can find $C_{16} > 0$ and $C_{17} > 0$ such that

$$\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + C_{16}\int_\Omega |\nabla c_\varepsilon|^2 \le C_{17}$$

for all $t \in (0, T_{\max,\varepsilon})$, which combined with an ODE comparison argument entails (4.3). The proof is completed. □

Lemma 4.2. *Let $\kappa \ne 0$ and $\alpha \ge \frac{3}{7}$. Then for $p > 1$ and $q \ge 2$ satisfying*

$$\max\left\{\frac{2}{3}, \frac{2\alpha}{q - \frac{1}{3}}\right\} < \frac{p - \frac{2}{3}}{q - \frac{1}{3}} < 6\alpha, \tag{4.19}$$

one can find a constant $C > 0$ depending on $\varepsilon \in (0, 1)$ such that

$$\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} + \|\nabla c_\varepsilon(\cdot, t)\|_{L^{2q}(\Omega)} \le C \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}). \tag{4.20}$$

Proof. Testing the first equation in (2.1) by pn_ε^{p-1} with $p > 1$, we deduce from (1.4), (2.3) and the Young inequality that

$$\begin{aligned} \frac{d}{dt}\int_\Omega n_\varepsilon^p + p(p-1)\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \le{}& \frac{p(p-1)}{4}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \\ &+ C_S^2 p(p-1)\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \end{aligned}$$

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for all $t \in (0, T_{\max,\varepsilon})$, which implies that

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{3(p-1)}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 \le C_{\mathcal{S}}^2 p(p-1)\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \tag{4.21}$$

for all $t \in (0, T_{\max,\varepsilon})$. By virtue of (2.4), we can derive the same conclusion as Lemma 2.8 in [25] by a slight adaptation of the reasoning, i.e.

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2q} + \frac{q-1}{2q}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2 \\
&\le (2q^2+q)\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2q-2} + (4q^2+2q)\int_\Omega |\nabla c_\varepsilon|^{2q}|u_\varepsilon|^2 + C_1
\end{aligned} \tag{4.22}$$

with some $C_1 > 0$ for all $t \in (0, T_{\max,\varepsilon})$. Adding (4.21) and (4.22) together and invoking the Hölder inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^{2q}\right) + \frac{3(p-1)}{p}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{q-1}{2q}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2 \\
&\le C_{\mathcal{S}}^2 p(p-1)\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 + (2q^2+q)\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2q-2} + (4q^2+2q)\int_\Omega |\nabla c_\varepsilon|^{2q}|u_\varepsilon|^2 + C_1 \\
&\le C_{\mathcal{S}}^2 p(p-1)\left(\int_\Omega n_\varepsilon^{3(p-2\alpha)}\right)^{\frac{1}{3}}\left(\int_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} \\
&\quad + (2q^2+q)\left(\int_\Omega n_\varepsilon^3\right)^{\frac{2}{3}}\left(\int_\Omega |\nabla c_\varepsilon|^{6(q-1)}\right)^{\frac{1}{3}} \\
&\quad + (4q^2+2q)\left(\int_\Omega |u_\varepsilon|^6\right)^{\frac{1}{3}}\left(\int_\Omega |\nabla c_\varepsilon|^{3q}\right)^{\frac{2}{3}} + C_1
\end{aligned} \tag{4.23}$$

for all $t \in (0, T_{\max,\varepsilon})$. Then we apply the Gagliardo–Nirenberg inequality to find positive constants C_2, C_3, C_4 and C_5 such that

$$\begin{aligned}
\left(\int_\Omega n_\varepsilon^{3(p-2\alpha)}\right)^{\frac{1}{3}} &= \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{6(p-2\alpha)}{p}}(\Omega)}^{\frac{2(p-2\alpha)}{p}} \\
&\le C_2\left(\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{a_1}\|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}^{1-a_1} + \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}\right)^{\frac{2(p-2\alpha)}{p}}
\end{aligned} \tag{4.24}$$

for all $t \in (0, T_{\max,\varepsilon})$ and

$$\begin{aligned}\left(\int_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} &= \left\| |\nabla c_\varepsilon|^q \right\|^{\frac{2}{q}}_{L^{\frac{3}{q}}(\Omega)} \\ &\le C_3 \left(\left\| \nabla |\nabla c_\varepsilon|^q \right\|^{b_1}_{L^2(\Omega)} \left\| |\nabla c_\varepsilon|^q \right\|^{1-b_1}_{L^{\frac{2}{q}}(\Omega)} + \left\| |\nabla c_\varepsilon|^q \right\|_{L^{\frac{2}{q}}(\Omega)} \right)^{\frac{2}{q}}\end{aligned} \tag{4.25}$$

for all $t \in (0, T_{\max,\varepsilon})$ as well as

$$\begin{aligned}\left(\int_\Omega n_\varepsilon^3\right)^{\frac{2}{3}} &= \| n_\varepsilon^{\frac{p}{2}} \|^{\frac{4}{p}}_{L^{\frac{6}{p}}(\Omega)} \\ &\le C_4 \left(\| \nabla n_\varepsilon^{\frac{p}{2}} \|^{a_2}_{L^2(\Omega)} \| n_\varepsilon^{\frac{p}{2}} \|^{1-a_2}_{L^{\frac{4}{p}}(\Omega)} + \| n_\varepsilon^{\frac{p}{2}} \|_{L^{\frac{4}{p}}(\Omega)} \right)^{\frac{4}{p}}\end{aligned} \tag{4.26}$$

for all $t \in (0, T_{\max,\varepsilon})$ and

$$\begin{aligned}\left(\int_\Omega |\nabla c_\varepsilon|^{6(q-1)}\right)^{\frac{1}{3}} &= \left\| |\nabla c_\varepsilon|^q \right\|^{\frac{2(q-1)}{q}}_{L^{\frac{6(q-1)}{q}}(\Omega)} \\ &\le C_5 \left(\left\| \nabla |\nabla c_\varepsilon|^q \right\|^{b_2}_{L^2(\Omega)} \left\| |\nabla c_\varepsilon|^q \right\|^{1-b_2}_{L^{\frac{2}{q}}(\Omega)} + \left\| |\nabla c_\varepsilon|^q \right\|_{L^{\frac{2}{q}}(\Omega)} \right)^{\frac{2(q-1)}{q}}\end{aligned} \tag{4.27}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $a_1 = \frac{3p - \frac{2p}{p-2\alpha}}{3p-2} \in (0,1)$ due to $p > 2\alpha + \frac{2}{3}$, $b_1 = \frac{q}{3q-1} \in (0,1)$, $a_2 = \frac{p}{3p-2} \in (0,1)$ and $b_2 = \frac{3q - \frac{q}{q-1}}{3q-1} \in (0,1)$. In view of Lemma 4.1, we combine (4.24) with (4.25) and (4.26) with (4.27), respectively, and gain positive constants C_6 as well as C_7 such that for all $t \in (0, T_{\max,\varepsilon})$

$$\begin{aligned}&\left(\int_\Omega n_\varepsilon^{3(p-2\alpha)}\right)^{\frac{1}{3}} \left(\int_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} \\ &\le C_6 \left\{ \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + 1 \right\}^{\frac{a_1(p-2\alpha)}{p}} \cdot \left\{ \int_\Omega \left| \nabla |\nabla c_\varepsilon|^q \right|^2 + 1 \right\}^{\frac{b_1}{q}}\end{aligned} \tag{4.28}$$

and

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$$
\begin{aligned}
&\left(\int_\Omega n_\varepsilon^3\right)^{\frac{2}{3}} \left(\int_\Omega |\nabla c_\varepsilon|^{6(q-1)}\right)^{\frac{1}{3}} \\
&\le C_7 \left\{\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + 1\right\}^{\frac{2a_2}{p}} \cdot \left\{\int_\Omega \left|\nabla |\nabla c_\varepsilon|^q\right|^2 + 1\right\}^{\frac{b_2(q-1)}{q}}.
\end{aligned}
\tag{4.29}
$$

From (4.19), it can be verified that

$$
\frac{a_1(p-2\alpha)}{p} + \frac{b_1}{q} < 1 \qquad \text{and} \qquad \frac{2a_2}{p} + \frac{b_2(q-1)}{q} < 1,
$$

thus an application of the Young inequality provides some $C_8 > 0$ satisfying

$$
\begin{aligned}
&\left(\int_\Omega n_\varepsilon^{3(p-2\alpha)}\right)^{\frac{1}{3}} \left(\int_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} + \left(\int_\Omega n_\varepsilon^3\right)^{\frac{2}{3}} \left(\int_\Omega |\nabla c_\varepsilon|^{6(q-1)}\right)^{\frac{1}{3}} \\
&\le \delta \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \delta \int_\Omega \left|\nabla |\nabla c_\varepsilon|^q\right|^2 + C_8
\end{aligned}
\tag{4.30}
$$

with any $\delta > 0$ for all $t \in (0, T_{\max,\varepsilon})$. Apart from that, in light of (4.2) and (4.3), we deduce from the embedding theorem and the Gagliardo–Nirenberg inequality that

$$
\begin{aligned}
\left(\int_\Omega |u_\varepsilon|^6\right)^{\frac{1}{3}} \left(\int_\Omega |\nabla c_\varepsilon|^{3q}\right)^{\frac{2}{3}} &\le C_9 \left\| |\nabla c_\varepsilon|^q \right\|_{L^3(\Omega)}^2 \\
&\le C_{10} \left(\left\| \nabla |\nabla c_\varepsilon|^q \right\|_{L^2(\Omega)}^{a_3} \left\| |\nabla c_\varepsilon|^q \right\|_{L^{\frac{2}{q}}(\Omega)}^{1-a_3} + \left\| |\nabla c_\varepsilon|^q \right\|_{L^{\frac{2}{q}}(\Omega)} \right)^2 \\
&\le C_{11} \left\| \nabla |\nabla c_\varepsilon|^q \right\|_{L^2(\Omega)}^{2a_3} + C_{11}
\end{aligned}
\tag{4.31}
$$

with positive constants C_9, C_{10} and C_{11} as well as $a_3 = \frac{3q-2}{3q-1} \in (0,1)$ for all $t \in (0, T_{\max,\varepsilon})$, which combined with the Young inequality implies

$$
\left(\int_\Omega |u_\varepsilon|^6\right)^{\frac{1}{3}} \left(\int_\Omega |\nabla c_\varepsilon|^{3q}\right)^{\frac{2}{3}} \le \delta \int_\Omega \left|\nabla |\nabla c_\varepsilon|^q\right|^2 + C_{12}
\tag{4.32}
$$

with some $C_{12} > 0$ and any $\delta > 0$ for all $t \in (0, T_{\max,\varepsilon})$. Additionally, based on (3.1) and (4.3), another application of the Gagliardo–Nirenberg inequality along with the Young inequality gives positive constants C_{13}, C_{14}, C_{15} and C_{16} fulfilling

$$
\begin{aligned}
\int_\Omega n_\varepsilon^p &= \|n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \\
&\le C_{13}\left(\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{a_4}\|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{4\alpha}{p}}(\Omega)}^{1-a_4}+\|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{4\alpha}{p}}(\Omega)}\right)^2 \\
&\le C_{14}\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{2a_4}+C_{14} \\
&\le C_{14}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2+2C_{14}
\end{aligned}
\tag{4.33}
$$

for all $t \in (0, T_{\max,\varepsilon})$ and

$$
\begin{aligned}
\int_\Omega |\nabla c_\varepsilon|^{2q} &= \left\||\nabla c_\varepsilon|^q\right\|_{L^2(\Omega)}^2 \\
&\le C_{15}\left(\left\|\nabla|\nabla c_\varepsilon|^q\right\|_{L^2(\Omega)}^{a_5}\left\||\nabla c_\varepsilon|^q\right\|_{L^{\frac{2}{q}}(\Omega)}^{1-a_5}+\left\||\nabla c_\varepsilon|^q\right\|_{L^{\frac{2}{q}}(\Omega)}\right)^2 \\
&\le C_{16}\left\|\nabla|\nabla c_\varepsilon|^q\right\|_{L^2(\Omega)}^{2a_5}+C_{16} \\
&\le C_{16}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^q\right|^2+2C_{16}
\end{aligned}
\tag{4.34}
$$

for all $t \in (0, T_{\max,\varepsilon})$, where $a_4 = \frac{3p-6\alpha}{3p-2\alpha} \in (0, 1)$ and $a_5 = \frac{3q-3}{3q-1} \in (0, 1)$. Choosing properly large $\delta > 0$, we substitute (4.30), (4.32), (4.33) as well as (4.34) into (4.23) and are able to find $C_{17} > 0$ and $C_{18} > 0$ such that

$$
\frac{d}{dt}\left(\int_\Omega n_\varepsilon^p+\int_\Omega |\nabla c_\varepsilon|^{2q}\right)+C_{17}\left(\int_\Omega n_\varepsilon^p+\int_\Omega |\nabla c_\varepsilon|^{2q}\right)\le C_{18} \tag{4.35}
$$

for all $t \in (0, T_{\max,\varepsilon})$, and thereby (4.35) implies (4.20) by an ODE comparison argument. We complete the proof. □

In light of Lemma 4.2, we can derive the following corollary according to the reasoning of Corollary 2.3 in [25].

Corollary 4.1. *Let $\kappa \neq 0$ and $\alpha \ge \frac{3}{7}$. Then for any $p > 1$ and $q > 1$, there exists some $C > 0$ depending on $\varepsilon \in (0, 1)$ such that*

$$
\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}+\|\nabla c_\varepsilon(\cdot,t)\|_{L^q(\Omega)}\le C \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}). \tag{4.36}
$$

4.2. Global solvability of problem (2.1)

With Corollary 4.1 at hand, we are able to prove the global solvability of problem (2.1).

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Proposition 4.1. *Let $\kappa \neq 0$ and $\alpha \geq \frac{3}{7}$. Then the solution established in Lemma 2.1 is global in time for each $\varepsilon \in (0, 1)$.*

Proof. Applying the Helmholtz projection to the third equation in (2.1), we have

$$u_{\varepsilon t} + Au_\varepsilon = \mathcal{P}[-(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon + n_\varepsilon \nabla\phi]. \tag{4.37}$$

Set $h_\varepsilon(x,t) := \mathcal{P}[-(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon + n_\varepsilon \nabla\phi]$, then from (4.11), (4.2), (4.36) and (1.9) one can find $C_1 > 0$ such that

$$\|h_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C_1 \tag{4.38}$$

for all $t \in (0, T_{\max,\varepsilon})$. It follows from the variation-of-constants formula that

$$u_\varepsilon(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}h_\varepsilon(\cdot,s)ds \tag{4.39}$$

for all $t \in (0, T_{\max,\varepsilon})$. According to the smoothing properties of Stokes semigroup [6], for any $\beta \in (\frac{3}{4}, 1)$ we apply A^β to (4.39) and then obtain

$$\begin{aligned}
\|A^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &\leq \|A^\beta e^{-tA}u_0\|_{L^2(\Omega)} + \int_0^t \|A^\beta e^{-(t-s)A}h_\varepsilon(\cdot,s)\|_{L^2(\Omega)}ds \\
&\leq C_2 t^{-\beta}\|u_0\|_{L^2(\Omega)} + C_2\int_0^t (t-s)^{-\beta}\|h_\varepsilon(\cdot,s)\|_{L^2(\Omega)}ds \\
&\leq C_2 t^{-\beta}\|u_0\|_{L^2(\Omega)} + \frac{C_1C_2T_{\max,\varepsilon}^{1-\beta}}{1-\beta}
\end{aligned} \tag{4.40}$$

for all $t \in (0, T_{\max,\varepsilon})$. If we let $\tau := \min\{1, \frac{T_{\max,\varepsilon}}{3}\}$, then due to $D(A^\beta) \hookrightarrow L^\infty(\Omega)$ [9,7] and (4.40), there exist $C_3 > 0$ and $C_4 > 0$ fulfilling

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C_3\|A^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C_4 \tag{4.41}$$

for all $t \in (\tau, T_{\max,\varepsilon})$. Another application of the variation-of-constants formula to the first equation in (2.1) entails

$$\begin{aligned}
n_\varepsilon(\cdot,t) = e^{(t-\tau)\Delta}n_\varepsilon(\cdot,\tau) - \int_\tau^t e^{(t-s)\Delta}\nabla\cdot\{&n_\varepsilon(\cdot,s)F'_\varepsilon(n_\varepsilon(\cdot,s))\mathcal{S}_\varepsilon(x, n_\varepsilon(\cdot,s), c_\varepsilon(\cdot,s))\nabla c_\varepsilon(\cdot,s) \\
&+ n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s)\}ds
\end{aligned} \tag{4.42}$$

for all $t \in (2\tau, T_{\max,\varepsilon})$. Define $B := -\Delta + 1$ under homogeneous Neumann boundary conditions in $L^r(\Omega)$, and then for fixed $\rho \in (0, \frac{1}{2})$ we choose $r > 3$ sufficiently large fulfilling $r > \frac{3}{2\rho}$, i.e. $2\rho - \frac{3}{r} > 0$, which implies $D(B^\rho) \hookrightarrow L^\infty(\Omega)$ [9]. From (1.4) and (2.3), it is clear that $|\mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \le C_{\mathcal{S}}$ and $sF'_\varepsilon(s) < \frac{1}{\varepsilon}$ for all $s \ge 0$. Thereupon, together with (2.9), (4.36) and (4.41), we deduce from (4.42) that

$$\begin{aligned}
&\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \\
\le\ & \|e^{(t-\tau)\Delta} n_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)} \\
&+ \int_\tau^t \|e^{(t-s)\Delta}\nabla\cdot\{n_\varepsilon(\cdot,s)F'_\varepsilon(n_\varepsilon(\cdot,s))\mathcal{S}_\varepsilon(x,n_\varepsilon(\cdot,s),c_\varepsilon(\cdot,s))\nabla c_\varepsilon(\cdot,s) \\
&+ n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s)\}\|_{L^\infty(\Omega)} ds \\
\le\ & \|e^{(t-\tau)\Delta} n_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)} \\
&+ \int_\tau^t \|B^\rho e^{(t-s)\Delta}\nabla\cdot\{n_\varepsilon(\cdot,s)F'_\varepsilon(n_\varepsilon(\cdot,s))\mathcal{S}_\varepsilon(x,n_\varepsilon(\cdot,s),c_\varepsilon(\cdot,s))\nabla c_\varepsilon(\cdot,s) \\
&+ n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s)\}\|_{L^r(\Omega)} ds \\
=\ & \|e^{(t-\tau)\Delta} n_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)} \\
&+ \int_\tau^t e^{\frac{t-s}{2}} \|B^\rho e^{-\frac{t-s}{2}B} e^{\frac{t-s}{2}\Delta}\nabla\cdot\{n_\varepsilon(\cdot,s)F'_\varepsilon(n_\varepsilon(\cdot,s))\mathcal{S}_\varepsilon(x,n_\varepsilon(\cdot,s),c_\varepsilon(\cdot,s))\nabla c_\varepsilon(\cdot,s) \\
&+ n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s)\}\|_{L^r(\Omega)} ds \\
\le\ & C_5(t-\tau)^{-\frac{3}{2}}\|n_\varepsilon(\cdot,\tau)\|_{L^1(\Omega)} + C_5\int_\tau^t e^{\frac{t-s}{2}}\left(\frac{t-s}{2}\right)^{-\rho-\frac{1}{2}}\{\|\nabla c_\varepsilon(\cdot,s)\|_{L^r(\Omega)} \\
&+ \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\|n_\varepsilon(\cdot,s)\|_{L^r(\Omega)}\}ds \\
\le\ & C_6
\end{aligned} \tag{4.43}$$

with $C_5 > 0$ and $C_6 > 0$ for all $t \in (2\tau, T_{\max,\varepsilon})$. Recalling (4.36), we know that there exists $C_7 > 0$ satisfying

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^{\frac{6q}{q+3}}(\Omega)} \le C_7 \tag{4.44}$$

for any $q > 2$ and all $t \in (0, T_{\max,\varepsilon})$. From (4.44), (3.1) and the Gagliardo–Nirenberg inequality, we can derive

$$\begin{aligned}
\|c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} &\le C_8\left(\|\nabla c_\varepsilon(\cdot,t)\|^a_{L^{\frac{6q}{q+3}}(\Omega)}\|c_\varepsilon(\cdot,t)\|^{1-a}_{L^2(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\right) \\
&\le C_9
\end{aligned} \tag{4.45}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $C_8, C_9 > 0$ and $a = \frac{3q-6}{4q-3} \in (0, 1)$. Combining (4.36) with (4.45), one can find $C_{10} > 0$ fulfilling

$$\|c_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} \le C_{10} \tag{4.46}$$

for any $q > 3$ and all $t \in (0, T_{\max,\varepsilon})$, which along with (4.41) and (4.43) contradicts to the blow-up criterion (2.8). Thus, we infer that $T_{\max,\varepsilon} = \infty$ and prove the lemma. □

5. Proof of the main result

5.1. Some further ε-independent estimates

In this subsection, we devote to establishing some ε-independent bounds for spatio-temporal integrals of n_ε, c_ε and u_ε, as well as deriving certain regularity features of the time derivatives in (2.1), so that it is possible to take limits of n_ε, c_ε and u_ε as $\varepsilon \to 0$ in the next subsection.

Lemma 5.1. *Let $\kappa \ne 0$ and $\alpha \ge \frac{3}{7}$. Then there exists $C > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\int_0^T \int_\Omega n_\varepsilon^\beta \le C(T+1) \quad \text{for all} \quad T \in (0, \infty) \quad \text{with} \quad \beta = \begin{cases} \frac{2(3\alpha+1)}{3}, & \text{if} \quad \frac{3}{7} \le \alpha < \frac{1}{2}, \\ \frac{10\alpha}{3}, & \text{if} \quad \alpha \ge \frac{1}{2}, \end{cases} \tag{5.1}$$

$$\int_0^T \int_\Omega |\nabla n_\varepsilon|^\gamma \le C(T+1) \quad \text{for all} \quad T \in (0, \infty) \quad \text{with} \quad \gamma = \begin{cases} \frac{3\alpha+1}{2}, & \text{if} \quad \frac{3}{7} \le \alpha < \frac{1}{2}, \\ \frac{10\alpha}{2\alpha+3}, & \text{if} \quad \frac{1}{2} \le \alpha < 1, \\ 2, & \text{if} \quad \alpha \ge 1, \end{cases} \tag{5.2}$$

and

$$\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} \le C(T+1) \quad \text{for all} \quad T \in (0, \infty). \tag{5.3}$$

Proof. In the case when $\frac{3}{7} \le \alpha < \frac{1}{2}$, we can derive from (2.9) and the Gagliardo–Nirenberg inequality that

$$\begin{aligned} \int_0^T \int_\Omega n_\varepsilon^{\frac{2(3\alpha+1)}{3}} &= \int_0^T \|n_\varepsilon^\alpha(\cdot, t)\|_{L^{\frac{2(3\alpha+1)}{3\alpha}}(\Omega)}^{\frac{2(3\alpha+1)}{3\alpha}} dt \\ &\le C_1 \int_0^T \left\{ \|\nabla n_\varepsilon^\alpha(\cdot, t)\|_{L^2(\Omega)}^{\frac{3\alpha}{3\alpha+1}} \|n_\varepsilon^\alpha(\cdot, t)\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{1}{3\alpha+1}} + \|n_\varepsilon^\alpha(\cdot, t)\|_{L^{\frac{1}{\alpha}}(\Omega)} \right\}^{\frac{2(3\alpha+1)}{3\alpha}} dt \qquad (5.4) \\ &\le C_2 \int_0^T \left\{ \|\nabla n_\varepsilon^\alpha(\cdot, t)\|_{L^2(\Omega)}^2 + 1 \right\} dt \end{aligned}$$

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with $C_1 > 0$ and $C_2 > 0$. Whereas for $\alpha \geq \frac{1}{2}$, based on (3.1), another application of the Gagliardo–Nirenberg inequality yields

$$\begin{aligned}
\int_0^T \int_\Omega n_\varepsilon^{\frac{10\alpha}{3}} &= \int_0^T \|n_\varepsilon^\alpha(\cdot,t)\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{10}{3}} dt \\
&\leq C_3 \int_0^T \left\{ \|\nabla n_\varepsilon^\alpha(\cdot,t)\|_{L^2(\Omega)}^{\frac{3}{5}} \|n_\varepsilon^\alpha(\cdot,t)\|_{L^2(\Omega)}^{\frac{2}{5}} + \|n_\varepsilon^\alpha(\cdot,t)\|_{L^2(\Omega)} \right\}^{\frac{10}{3}} dt \qquad (5.5)\\
&\leq C_4 \int_0^T \left\{ \|\nabla n_\varepsilon^\alpha(\cdot,t)\|_{L^2(\Omega)}^2 + 1 \right\} dt,
\end{aligned}$$

where C_3 and C_4 are positive constants. Hence, (5.1) is implied by (3.3), (5.4) and (5.5).

When $\frac{3}{7} \leq \alpha < \frac{1}{2}$, one can see that

$$\frac{4}{3\alpha+1} > 1 \qquad \text{and} \qquad \frac{(3\alpha+1)(1-\alpha)}{2} > 0.$$

By virtue of the Hölder inequality with exponents $\frac{4}{3\alpha+1}$ and $\frac{4}{3-3\alpha}$, we deduce that

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla n_\varepsilon|^{\frac{3\alpha+1}{2}} &= \int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^{\frac{3\alpha+1}{2}}}{n_\varepsilon^{\frac{(3\alpha+1)(1-\alpha)}{2}}} \cdot n_\varepsilon^{\frac{(3\alpha+1)(1-\alpha)}{2}} \\
&\leq \left(\int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^{2(1-\alpha)}} \right)^{\frac{3\alpha+1}{4}} \left(\int_0^T \int_\Omega n_\varepsilon^{\frac{2(3\alpha+1)}{3}} \right)^{\frac{3-3\alpha}{4}} \qquad (5.6)\\
&= \left(\frac{1}{\alpha^2} \int_0^T \int_\Omega |\nabla n_\varepsilon^\alpha|^2 \right)^{\frac{3\alpha+1}{4}} \left(\int_0^T \int_\Omega n_\varepsilon^{\frac{2(3\alpha+1)}{3}} \right)^{\frac{3-3\alpha}{4}},
\end{aligned}$$

which along with (3.3) and (5.1) yields (5.2) with $\gamma = \frac{3\alpha+1}{2}$ for $\frac{3}{7} \leq \alpha < \frac{1}{2}$. Apart from that, if $\frac{1}{2} \leq \alpha < 1$, it is clear that

$$\frac{2\alpha+3}{5\alpha} > 1 \qquad \text{and} \qquad \frac{10\alpha(1-\alpha)}{2\alpha+3} > 0.$$

Thereupon, by means of a similar argument as (5.6), we achieve that

$$\int_0^T \int_\Omega |\nabla n_\varepsilon|^{\frac{10\alpha}{2\alpha+3}} = \int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^{\frac{10\alpha}{2\alpha+3}}}{n_\varepsilon^{\frac{10\alpha(1-\alpha)}{2\alpha+3}}} \cdot n_\varepsilon^{\frac{10\alpha(1-\alpha)}{2\alpha+3}}$$

$$\leq \left(\int_0^T\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^{2(1-\alpha)}}\right)^{\frac{5\alpha}{2\alpha+3}} \left(\int_0^T\int_\Omega n_\varepsilon^{\frac{10\alpha}{3}}\right)^{\frac{3-3\alpha}{2\alpha+3}} \tag{5.7}$$

$$= \left(\frac{1}{\alpha^2}\int_0^T\int_\Omega |\nabla n_\varepsilon^\alpha|^2\right)^{\frac{5\alpha}{2\alpha+3}} \left(\int_0^T\int_\Omega n_\varepsilon^{\frac{10\alpha}{3}}\right)^{\frac{3-3\alpha}{2\alpha+3}},$$

and whence for $\frac{1}{2} \leq \alpha < 1$ (5.2) with $\gamma = \frac{10\alpha}{2\alpha+3}$ is valid from (3.3) and (5.1). Whereas for $\alpha \geq 1$, we test the first equation by n_ε and deduce from (2.3), $|n_\varepsilon \mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \leq C_{\mathcal{S}}$ and the Young inequality that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega n_\varepsilon^2 + \int_\Omega |\nabla n_\varepsilon|^2 &= \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon \cdot \nabla n_\varepsilon \\ &\leq C_{\mathcal{S}}\int_\Omega |\nabla c_\varepsilon| \cdot |\nabla n_\varepsilon| \\ &\leq \frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 + \frac{C_{\mathcal{S}}^2}{2}\int_\Omega |\nabla c_\varepsilon|^2,\end{aligned}$$

i.e.

$$\frac{d}{dt}\int_\Omega n_\varepsilon^2 + \int_\Omega |\nabla n_\varepsilon|^2 \leq C_{\mathcal{S}}^2\int_\Omega |\nabla c_\varepsilon|^2 \tag{5.8}$$

for all $t \in (0, \infty)$. Together with (3.3), an integration of (5.8) over $(0, T)$ with respect to t entails (5.2) with $\gamma = 2$ for $\alpha \geq 1$.

Finally, in view of (3.1) and (3.3), (5.3) can be established by the reasoning of (3.34) in [31] and thereby we complete the proof. □

Lemma 5.2. *Let $\kappa \neq 0$ and $\alpha \geq \frac{3}{7}$. Then one can find a positive constant C independent of $\varepsilon \in (0, 1)$ satisfying*

$$\int_0^T \|n_{\varepsilon t}(\cdot, t)\|_{(W^{1,\frac{10(3\alpha+1)}{21\alpha-8}}(\Omega))^*}^{\frac{10(3\alpha+1)}{9(\alpha+2)}} dt \leq C(T+1) \quad \textit{for all} \quad T \in (0, \infty) \quad \textit{if} \quad \frac{3}{7} \leq \alpha < \frac{1}{2}, \tag{5.9}$$

$$\int_0^T \|n_{\varepsilon t}(\cdot, t)\|_{(W^{1,\frac{10\alpha}{7\alpha-3}}(\Omega))^*}^{\frac{10\alpha}{3\alpha+3}} dt \leq C(T+1) \quad \textit{for all} \quad T \in (0, \infty) \quad \textit{if} \quad \frac{1}{2} \leq \alpha < 1, \tag{5.10}$$

$$\int_0^T \|n_{\varepsilon t}(\cdot, t)\|_{(W^{1,\frac{5}{2}}(\Omega))^*}^{\frac{5}{3}} dt \leq C(T+1) \quad \textit{for all} \quad T \in (0, \infty) \quad \textit{if} \quad \alpha \geq 1, \tag{5.11}$$

$$\int_0^T \|c_{\varepsilon t}(\cdot,t)\|_{(W^{1,5}(\Omega))^*}^{\frac{5}{4}} dt \le C(T+1) \quad \textit{for all} \quad T \in (0,\infty) \tag{5.12}$$

and

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|_{(W_{0,\sigma}^{1,5}(\Omega))^*}^{\frac{5}{4}} dt \le C(T+1) \quad \textit{for all} \quad T \in (0,\infty). \tag{5.13}$$

Proof. If $\frac{3}{7} \le \alpha < \frac{1}{2}$, an elementary calculation ensures that

$$\frac{10(3\alpha+1)(1-\alpha)}{13-6\alpha} < \frac{2(3\alpha+1)}{3} \qquad \text{and} \qquad \frac{10(3\alpha+1)}{9(\alpha+2)} < \frac{3\alpha+1}{2}. \tag{5.14}$$

For $t \in (0,\infty)$, testing the first equation in (2.1) by any fixed $\psi \in C^\infty(\bar{\Omega})$, we deduce from the Hölder inequality that

$$\begin{aligned} \left|\int_\Omega n_{\varepsilon t}(\cdot,t)\psi\right| &= \left|-\int_\Omega \nabla n_\varepsilon \cdot \nabla\psi + \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon \cdot \nabla\psi + \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\psi\right| \\ &\le \Big\{\|\nabla n_\varepsilon\|_{L^{\frac{10(3\alpha+1)}{9(\alpha+2)}}(\Omega)} + \|n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\|_{L^{\frac{10(3\alpha+1)}{9(\alpha+2)}}(\Omega)} \\ &\quad + \|n_\varepsilon u_\varepsilon\|_{L^{\frac{10(3\alpha+1)}{9(\alpha+2)}}(\Omega)}\Big\} \cdot \|\psi\|_{W^{1,\frac{10(3\alpha+1)}{21\alpha-8}}(\Omega)}. \end{aligned} \tag{5.15}$$

Along with (1.4) and (2.3), (5.15) further implies that

$$\begin{aligned} \int_0^T \|n_{\varepsilon t}(\cdot,t)\|_{(W^{1,\frac{10(3\alpha+1)}{21\alpha-8}}(\Omega))^*}^{\frac{10(3\alpha+1)}{9(\alpha+2)}} dt &\le \int_0^T \Big\{\|\nabla n_\varepsilon\|_{L^{\frac{10(3\alpha+1)}{9(\alpha+2)}}(\Omega)} \\ &\quad + \|n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\|_{L^{\frac{10(3\alpha+1)}{9(\alpha+2)}}(\Omega)} \\ &\quad + \|n_\varepsilon u_\varepsilon\|_{L^{\frac{10(3\alpha+1)}{9(\alpha+2)}}(\Omega)}\Big\}^{\frac{10(3\alpha+1)}{9(\alpha+2)}} dt \\ &\le C_1 \int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{10(3\alpha+1)}{9(\alpha+2)}} + C_1\int_0^T\int_\Omega |n_\varepsilon^{1-\alpha}\nabla c_\varepsilon|^{\frac{10(3\alpha+1)}{9(\alpha+2)}} \\ &\quad + C_1\int_0^T\int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{10(3\alpha+1)}{9(\alpha+2)}} \end{aligned} \tag{5.16}$$

with some $C_1 > 0$ independent of $T \in (0,\infty)$. Due to (5.14) and (5.2), we use the Young inequality to find $C_2 > 0$ such that

$$\begin{aligned}\int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{10(3\alpha+1)}{9(\alpha+2)}} &\le \int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{3\alpha+1}{2}} + |\Omega| T \\ &\le C_2(T+1)\end{aligned} \tag{5.17}$$

for all $T \in (0, \infty)$. Again employing the Hölder inequality, we derive from (5.14), (5.1), (3.3) and (5.3) that

$$\begin{aligned}\int_0^T\int_\Omega |n_\varepsilon^{1-\alpha}\nabla c_\varepsilon|^{\frac{10(3\alpha+1)}{9(\alpha+2)}} &= \int_0^T\int_\Omega n_\varepsilon^{\frac{10(3\alpha+1)(1-\alpha)}{9(\alpha+2)}} |\nabla c_\varepsilon|^{\frac{10(3\alpha+1)}{9(\alpha+2)}} \\ &\le \left(\int_0^T\int_\Omega n_\varepsilon^{\frac{10(3\alpha+1)(1-\alpha)}{13-6\alpha}}\right)^{\frac{13-6\alpha}{9(\alpha+2)}} \left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^2\right)^{\frac{5(3\alpha+1)}{9(\alpha+2)}} \\ &\le (|\Omega| T)^{\frac{9\alpha-2}{9(\alpha+2)}} \left(\int_0^T\int_\Omega n_\varepsilon^{\frac{2(3\alpha+1)}{3}}\right)^{\frac{5(1-\alpha)}{3(\alpha+2)}} \left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^2\right)^{\frac{5(3\alpha+1)}{9(\alpha+2)}} \\ &\le C_3(T+1)\end{aligned} \tag{5.18}$$

for all $T \in (0, \infty)$ and

$$\begin{aligned}\int_0^T\int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{10(3\alpha+1)}{9(\alpha+2)}} &\le \left(\int_0^T\int_\Omega n_\varepsilon^{\frac{2(3\alpha+1)}{3}}\right)^{\frac{5}{3(\alpha+2)}} \left(\int_0^T\int_\Omega u_\varepsilon^{\frac{10}{3}}\right)^{\frac{3\alpha+1}{3(\alpha+2)}} \\ &\le C_4(T+1)\end{aligned} \tag{5.19}$$

for all $T \in (0, \infty)$, where C_3, C_4 are positive constants. Thus, (5.9) is valid from (5.16), (5.17), (5.18) and (5.19).

In the case when $\frac{1}{2} \le \alpha < 1$, it is not difficult to verify that

$$\frac{10\alpha(1-\alpha)}{3-2\alpha} < \frac{10\alpha}{3} \quad \text{and} \quad \frac{10\alpha}{3\alpha+3} < \frac{10\alpha}{2\alpha+3}. \tag{5.20}$$

Along the reasoning of (5.16), we can find $C_5 > 0$ such that

$$\begin{aligned}\int_0^T \|n_{\varepsilon t}(\cdot,t)\|^{\frac{10\alpha}{3\alpha+3}}_{(W^{1,\frac{10\alpha}{7\alpha-3}}(\Omega))^*} dt &\le C_5\int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{10\alpha}{3\alpha+3}} + C_5\int_0^T\int_\Omega |n_\varepsilon^{1-\alpha}\nabla c_\varepsilon|^{\frac{10\alpha}{3\alpha+3}} \\ &\quad + C_5\int_0^T\int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{10\alpha}{3\alpha+3}}.\end{aligned} \tag{5.21}$$

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In light of (5.20) and (5.2), an application of the Young inequality provides $C_6 > 0$ such that

$$\begin{aligned}\int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{10\alpha}{3\alpha+3}} &\le \int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{10\alpha}{2\alpha+3}} + |\Omega|T \\ &\le C_6(T+1)\end{aligned} \tag{5.22}$$

for all $T \in (0,\infty)$. Based on (5.20), (5.1), (3.3) and (5.3), we deduce from the Hölder inequality that

$$\begin{aligned}\int_0^T\int_\Omega |n_\varepsilon^{1-\alpha}\nabla c_\varepsilon|^{\frac{10\alpha}{3\alpha+3}} &= \int_0^T\int_\Omega n_\varepsilon^{\frac{10\alpha(1-\alpha)}{3\alpha+3}} |\nabla c_\varepsilon|^{\frac{10\alpha}{3\alpha+3}} \\ &\le \left(\int_0^T\int_\Omega n_\varepsilon^{\frac{10\alpha(1-\alpha)}{3-2\alpha}}\right)^{\frac{3-2\alpha}{3\alpha+3}} \left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^2\right)^{\frac{5\alpha}{3\alpha+3}} \\ &\le (|\Omega|T)^{\frac{\alpha}{3\alpha+3}} \left(\int_0^T\int_\Omega n_\varepsilon^{\frac{10\alpha}{3}}\right)^{\frac{1-\alpha}{1+\alpha}} \left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^2\right)^{\frac{5\alpha}{3\alpha+3}} \\ &\le C_7(T+1)\end{aligned} \tag{5.23}$$

for all $T \in (0,\infty)$ and

$$\begin{aligned}\int_0^T\int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{10\alpha}{3\alpha+3}} &\le \left(\int_0^T\int_\Omega n_\varepsilon^{\frac{10\alpha}{3}}\right)^{\frac{1}{1+\alpha}} \left(\int_0^T\int_\Omega u_\varepsilon^{\frac{10}{3}}\right)^{\frac{\alpha}{1+\alpha}} \\ &\le C_8(T+1)\end{aligned} \tag{5.24}$$

with $C_7, C_8 > 0$ independent of $T \in (0,\infty)$. Substituting (5.22), (5.23) and (5.24) into (5.21), we achieve (5.10).

Whereas for $\alpha \ge 1$, it follows that $|n_\varepsilon \mathcal{S}_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \le C_{\mathcal{S}}$ as mentioned before. Along with (2.3) and by a similar argument as the case $\frac{3}{7} \le \alpha < \frac{1}{2}$, we can derive

$$\int_0^T \|n_{\varepsilon t}(\cdot,t)\|^{\frac{5}{3}}_{(W^{1,\frac{5}{2}}(\Omega))^*}\,dt \le C_9\int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{5}{3}} + C_9\int_0^T\int_\Omega |\nabla c_\varepsilon|^{\frac{5}{3}} + C_9\int_0^T\int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{5}{3}} \tag{5.25}$$

with some $C_9 > 0$ independent of $T \in (0,\infty)$, where an application of the Young inequality gives

$$\int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{5}{3}} \le \int_0^T\int_\Omega |\nabla n_\varepsilon|^2 + |\Omega|T \tag{5.26}$$

for all $T \in (0,\infty)$ as well as

$$\int_0^T \int_\Omega |\nabla c_\varepsilon|^{\frac{5}{3}} \leq \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + |\Omega| T \tag{5.27}$$

for all $T \in (0,\infty)$ and a utilization of the Hölder inequality entails

$$\int_0^T \int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{5}{3}} \leq \left(\int_0^T \int_\Omega n_\varepsilon^{\frac{10}{3}}\right)^{\frac{1}{2}} \left(\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}}\right)^{\frac{1}{2}} \tag{5.28}$$

for all $T \in (0,\infty)$. Relying on (5.2), (3.3), (5.1) and (5.3), we can establish (5.11) by inserting (5.26), (5.27) and (5.28) into (5.25).

Next, we test the second equation in (2.1) by ψ and use the Hölder inequality to gain

$$\begin{aligned} \left|\int_\Omega c_{\varepsilon t}(\cdot,t)\psi\right| &= \left|-\int_\Omega \nabla c_\varepsilon \cdot \nabla \psi - \int_\Omega c_\varepsilon \psi + \int_\Omega F_\varepsilon(n_\varepsilon)\psi + \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla \psi\right| \\ &\leq \left\{\|\nabla c_\varepsilon\|_{L^{\frac{5}{4}}(\Omega)} + \|c_\varepsilon\|_{L^{\frac{5}{4}}(\Omega)} + \|F_\varepsilon(n_\varepsilon)\|_{L^{\frac{5}{4}}(\Omega)} + \|c_\varepsilon u_\varepsilon\|_{L^{\frac{5}{4}}(\Omega)}\right\} \cdot \|\psi\|_{W^{1,5}(\Omega)} \end{aligned} \tag{5.29}$$

for all $t \in (0,\infty)$. Since $\frac{5}{4} < \frac{2(3\alpha+1)}{3}$ thanks to $\alpha \geq \frac{3}{7}$, together with (2.4), the Young inequality and the Hölder inequality, (5.29) further implies that

$$\begin{aligned} \int_0^T \|c_{\varepsilon t}(\cdot,t)\|^{\frac{5}{4}}_{(W^{1,5}(\Omega))^*} dt &\leq C_{10} \int_0^T \int_\Omega |\nabla c_\varepsilon|^{\frac{5}{4}} + C_{10} \int_0^T \int_\Omega c_\varepsilon^{\frac{5}{4}} + C_{10} \int_0^T \int_\Omega n_\varepsilon^{\frac{5}{4}} + C_{10} \int_0^T \int_\Omega |c_\varepsilon u_\varepsilon|^{\frac{5}{4}} \\ &\leq C_{10} \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + C_{10} \int_0^T \int_\Omega c_\varepsilon^2 + C_{10} \int_0^T \int_\Omega n_\varepsilon^{\frac{2(3\alpha+1)}{3}} + 3C_{10}|\Omega| T \\ &\quad + C_{10} \left(\int_0^T \int_\Omega c_\varepsilon^2\right)^{\frac{5}{8}} \left(\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}}\right)^{\frac{3}{8}} \end{aligned} \tag{5.30}$$

with certain $C_{10} > 0$ independent of $T \in (0,\infty)$, which combined with (3.3), (3.1), (5.1) and (5.3) yields (5.12).

We have known that $\frac{5}{4} < \frac{2(3\alpha+1)}{3}$, thus according to the reasoning of (3.39) in [31], (5.13) is valid from (3.1), (3.3), (5.1) and (5.3). The proof is completed. □

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5.2. Passing to the limit

Relying on the estimates established in Lemma 3.1, Lemma 5.1 and Lemma 5.2, we can take the limit of the approximated solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ as $\varepsilon \to 0$ and show that the limit is a global weak solution in the sense of the following definition. From now on, for vectors $v \in \mathbb{R}^3$ and $w \in \mathbb{R}^3$ we use $v \otimes w$ to represent the matrix $(b_{ij})_{i,j\in\{1,2,3\}} \in \mathbb{R}^{3\times 3}$ with $b_{ij} := v_i w_j$ for $i, j \in \{1, 2, 3\}$.

Definition 5.1. Let $\kappa \neq 0$ and $\alpha \geq \frac{3}{7}$, and suppose that $\mathcal{S}$ complies with (1.4). Then we call a triple (n, c, u) of functions is a global weak solution of problem (1.3), (1.5) and (1.6), if it fulfills $n \geq 0$, $c \geq 0$ as well as $\nabla \cdot u = 0$ a.e. in $\Omega \times (0, \infty)$, and

$$\begin{cases} n \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \\ c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \\ u \in L^1_{loc}([0,\infty); W^{1,1}_0(\Omega; \mathbb{R}^3)), \end{cases} \tag{5.31}$$

and furthermore,

$$\begin{aligned} &n\mathcal{S}(x,n,c)\nabla c,\ nu \text{ as well as } cu \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0,\infty); \mathbb{R}^3) \\ &\text{and } u \otimes u \in L^1_{loc}(\bar{\Omega} \times [0,\infty); \mathbb{R}^{3\times 3}), \end{aligned} \tag{5.32}$$

additionally, for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0,\infty))$, the components n and c satisfy

$$\begin{aligned} -\int_0^\infty\int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot,0) = &-\int_0^\infty\int_\Omega \nabla n \cdot \nabla\varphi + \int_0^\infty\int_\Omega n\mathcal{S}(x,n,c)\nabla c \cdot \nabla\varphi \\ &+ \int_0^\infty\int_\Omega nu \cdot \nabla\varphi \end{aligned} \tag{5.33}$$

as well as

$$\begin{aligned} -\int_0^\infty\int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) = &-\int_0^\infty\int_\Omega \nabla c \cdot \nabla\varphi - \int_0^\infty\int_\Omega c\varphi + \int_0^\infty\int_\Omega n\varphi \\ &+ \int_0^\infty\int_\Omega cu \cdot \nabla\varphi, \end{aligned} \tag{5.34}$$

while for any $\varphi \in C_0^\infty\left(\Omega \times [0,\infty); \mathbb{R}^3\right)$ fulfilling $\nabla \cdot \varphi \equiv 0$, the component u complies with

$$-\int_0^\infty\int_\Omega u \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla u \cdot \nabla\varphi + \kappa\int_0^\infty\int_\Omega u \otimes u \cdot \nabla\varphi + \int_0^\infty\int_\Omega n\nabla\phi \cdot \varphi. \tag{5.35}$$

Lemma 5.3. *Let $\kappa \neq 0$ and $\alpha \geq \frac{3}{7}$, and let $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ be the solution of (2.1) as constructed in Proposition 4.1. Then one can choose $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ satisfying $\varepsilon_j \searrow 0$ as $j \to \infty$ such that*

$$n_\varepsilon \to n \text{ in } L^\gamma_{loc}\left(\bar{\Omega}\times[0,\infty)\right) \text{ with } \gamma = \begin{cases} \frac{3\alpha+1}{2}, & \text{if } \frac{3}{7} \leq \alpha < \frac{1}{2}, \\ \frac{10\alpha}{2\alpha+3}, & \text{if } \frac{1}{2} \leq \alpha < 1, \\ 2, \quad \text{if} & \alpha \geq 1, \end{cases} \text{ and a.e. in } \Omega\times(0,\infty), \tag{5.36}$$

$$\nabla n_\varepsilon \rightharpoonup \nabla n \quad \text{in} \quad L^\gamma_{loc}\left(\bar{\Omega}\times[0,\infty)\right) \quad \text{with} \quad \gamma = \begin{cases} \frac{3\alpha+1}{2}, & \text{if } \frac{3}{7} \leq \alpha < \frac{1}{2}, \\ \frac{10\alpha}{2\alpha+3}, & \text{if } \frac{1}{2} \leq \alpha < 1, \\ 2, \quad \text{if} & \alpha \geq 1, \end{cases} \tag{5.37}$$

$$c_\varepsilon \to c \quad \text{in} \quad L^2_{loc}\left(\bar{\Omega}\times[0,\infty)\right) \quad \text{and a.e. in} \quad \Omega\times(0,\infty), \tag{5.38}$$

$$\nabla c_\varepsilon \rightharpoonup \nabla c \quad \text{in} \quad L^2_{loc}\left(\bar{\Omega}\times[0,\infty)\right), \tag{5.39}$$

$$\nabla c_\varepsilon \to \nabla c \quad \text{a.e. in} \quad \Omega\times(0,\infty), \tag{5.40}$$

$$u_\varepsilon \to u \quad \text{in} \quad L^2_{loc}\left(\bar{\Omega}\times[0,\infty)\right) \quad \text{and a.e. in} \quad \Omega\times(0,\infty), \tag{5.41}$$

$$u_\varepsilon \rightharpoonup u \quad \text{in} \quad L^{\frac{10}{3}}_{loc}\left(\bar{\Omega}\times[0,\infty)\right) \tag{5.42}$$

and

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in} \quad L^2_{loc}\left(\bar{\Omega}\times[0,\infty)\right) \tag{5.43}$$

hold as $\varepsilon_j \searrow 0$ with some triple (n, c, u) of functions complying with Definition 5.1.

Proof. From Lemma 3.1, Lemma 5.1, Lemma 5.2 and the Aubin–Lions lemma [22], we can derive (5.36), (5.37), (5.38), (5.39), (5.41), (5.42) and (5.43) with some triple (n, c, u) of functions belonging to the indicated spaces by a choice of some sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ with $\varepsilon_j \searrow 0$ as $j \to \infty$. Denote $a_\varepsilon(x,t) := -c_\varepsilon + F_\varepsilon(n_\varepsilon) - (u_\varepsilon \cdot \nabla c_\varepsilon)$. Together with (2.4), we make use of the Young inequality to obtain

$$\int_0^T\int_\Omega c_\varepsilon^{\frac{5}{4}} \leq \int_0^T\int_\Omega c_\varepsilon^2 + |\Omega|T$$

for all $T \in (0,\infty)$ and

$$\int_0^T\int_\Omega |F_\varepsilon(n_\varepsilon)|^{\frac{5}{4}} \leq \int_0^T\int_\Omega n_\varepsilon^{\frac{5}{4}} \leq \int_0^T\int_\Omega n_\varepsilon^{\frac{2(3\alpha+1)}{3}} + |\Omega|T$$

for all $T \in (0,\infty)$ due to $\frac{5}{4} < \frac{2(3\alpha+1)}{3}$. Also, an application of the Hölder inequality entails

$$\int_0^T\int_\Omega |u_\varepsilon\cdot\nabla c_\varepsilon|^{\frac{5}{4}} \le \left(\int_0^T\int_\Omega |u_\varepsilon|^{\frac{10}{3}}\right)^{\frac{3}{8}}\left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^2\right)^{\frac{5}{8}}$$

for all $T\in(0,\infty)$. Therefore, recalling (3.1), (5.1), (5.3) and (3.3), we know that $c_{\varepsilon t}-\Delta c_\varepsilon = a_\varepsilon$ is bounded in $L^{\frac{5}{4}}(\Omega\times(0,T))$ for any $\varepsilon\in(0,1)$, which also implies $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{5}{4}}((0,T);W^{2,\frac{5}{4}}(\Omega))$. Along with (5.12), the Aubin–Lions lemma insures the relative compactness of $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^{\frac{5}{4}}((0,T);W^{1,\frac{5}{4}}(\Omega))$. We can pick an appropriate subsequence which is still written as $(\varepsilon_j)_{j\in\mathbb{N}}$ such that $\nabla c_{\varepsilon_j}\to z_1$ in $L^{\frac{5}{4}}(\Omega\times(0,T))$ for all $T\in(0,\infty)$ and some $z_1\in L^{\frac{5}{4}}(\Omega\times(0,T))$ as $j\to\infty$, and this also implies $\nabla c_{\varepsilon_j}\to z_1$ a.e. in $\Omega\times(0,\infty)$ as $j\to\infty$. By virtue of (5.39), the Egorov theorem shows that $z_1=\nabla c$, and whence (5.40) is valid. In addition, for each fixed $T\in(0,\infty)$, the convergence (5.41) insures the existence of a null set $N_T\subset(0,T)$ such that one can pick a subsequence which we still denote by $(\varepsilon_j)_{j\in\mathbb{N}}$ fulfilling

$$u_\varepsilon(\cdot,t)\to u(\cdot,t) \quad \text{in } L^2(\Omega) \quad \text{for all} \quad t\in(0,T)\backslash N_T \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0. \tag{5.44}$$

Now, we try to verify that the triplet (n,c,u) is the desired solution in the sense of Definition 5.1. To begin with, the regularity (5.31) can be inferred from (5.36), (5.37), (5.38), (5.39), (5.41) and (5.43), and moreover, $n\ge 0$, $c\ge 0$ as well as $\nabla\cdot u=0$ a.e. in $\Omega\times(0,\infty)$ are implied by the nonnegativity of n_ε, c_ε and $\nabla\cdot u_\varepsilon=0$ a.e. in $\Omega\times(0,\infty)$, respectively. If $\frac{3}{7}\le\alpha<\frac{1}{2}$, then combining with (1.4) and (2.3), we apply the Hölder inequality to gain

$$\begin{aligned}\int_0^T\int_\Omega |n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon|^{\frac{3\alpha+1}{2}} &\le C_{\mathcal{S}}^{\frac{3\alpha+1}{2}}\int_0^T\int_\Omega |n_\varepsilon^{1-\alpha}\nabla c_\varepsilon|^{\frac{3\alpha+1}{2}}\\ &\le C_{\mathcal{S}}^{\frac{3\alpha+1}{2}}\left(\int_0^T\int_\Omega n_\varepsilon^{\frac{2(3\alpha+1)}{3}}\right)^{\frac{3-3\alpha}{4}}\left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^2\right)^{\frac{3\alpha+1}{4}}\end{aligned} \tag{5.45}$$

for all $T\in(0,\infty)$, and analogously, for $\frac{1}{2}\le\alpha<1$, we also have

$$\begin{aligned}\int_0^T\int_\Omega |n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon|^{\frac{10\alpha}{2\alpha+3}} &\le C_{\mathcal{S}}^{\frac{10\alpha}{2\alpha+3}}\int_0^T\int_\Omega |n_\varepsilon^{1-\alpha}\nabla c_\varepsilon|^{\frac{10\alpha}{2\alpha+3}}\\ &\le C_{\mathcal{S}}^{\frac{10\alpha}{2\alpha+3}}\left(\int_0^T\int_\Omega n_\varepsilon^{\frac{10\alpha}{3}}\right)^{\frac{3-3\alpha}{3+2\alpha}}\left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^2\right)^{\frac{5\alpha}{3+2\alpha}}\end{aligned} \tag{5.46}$$

for all $T\in(0,\infty)$. Whereas for $\alpha\ge 1$, it is clear that for all $T\in(0,\infty)$

$$\int_0^T\int_\Omega |n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon|^2 \le C_{\mathcal{S}}^2\int_0^T\int_\Omega |\nabla c_\varepsilon|^2 \tag{5.47}$$

due to (2.3) and $|n_\varepsilon \mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)| \le C_{\mathcal{S}}$. Thereupon, in view of (5.1) and (3.3), we can infer from (5.45), (5.46) and (5.47) that

$$\begin{aligned} &n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon \rightharpoonup z_2 \quad \text{in} \quad L^\gamma(\Omega\times(0,T)) \\ &\text{with} \quad \gamma=\begin{cases} \frac{3\alpha+1}{2}, & \text{if} \quad \frac{3}{7}\le\alpha<\frac{1}{2}, \\ \frac{10\alpha}{2\alpha+3}, & \text{if} \quad \frac{1}{2}\le\alpha<1, \\ 2, \quad \text{if} \quad \alpha\ge 1, \end{cases} \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0 \end{aligned} \tag{5.48}$$

for each $T\in(0,\infty)$, and on the other hand, it follows from (1.8), (2.2), (2.5), (5.36), (5.38) and (5.40) that

$$n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon \to n\mathcal{S}(x,n,c)\nabla c \quad \text{a.e. in} \quad \Omega\times(0,\infty) \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0. \tag{5.49}$$

Again by the Egorov theorem, we gain $z_2 = n\mathcal{S}(x,n,c)\nabla c$, and hence (5.48) can be rewritten as

$$\begin{aligned} &n_\varepsilon F_\varepsilon'(n_\varepsilon)\mathcal{S}_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon \rightharpoonup n\mathcal{S}(x,n,c)\nabla c \quad \text{in} \quad L^\gamma(\Omega\times(0,T)) \\ &\text{with} \quad \gamma=\begin{cases} \frac{3\alpha+1}{2}, & \text{if} \quad \frac{3}{7}\le\alpha<\frac{1}{2}, \\ \frac{10\alpha}{2\alpha+3}, & \text{if} \quad \frac{1}{2}\le\alpha<1, \\ 2, \quad \text{if} \quad \alpha\ge 1, \end{cases} \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0 \end{aligned} \tag{5.50}$$

for each $T\in(0,\infty)$, which shows the integrability of $n\mathcal{S}(x,n,c)\nabla c$ in (5.32) as well. It is not difficult to verify that

$$\frac{10(3\alpha+1)}{9(\alpha+2)}>1 \quad \text{for} \quad \frac{3}{7}\le\alpha<\frac{1}{2} \qquad \text{and} \qquad \frac{10\alpha}{3\alpha+3}>1 \quad \text{for} \quad \frac{1}{2}\le\alpha<1.$$

Thereupon, recalling (5.19), (5.24) and (5.28), we infer that for each $T\in(0,\infty)$

$$n_\varepsilon u_\varepsilon \rightharpoonup z_3 \quad \text{in} \quad L^\theta(\Omega\times(0,T)) \quad \text{with} \quad \theta=\begin{cases} \frac{10(3\alpha+1)}{9(\alpha+2)}, & \text{if} \quad \frac{3}{7}\le\alpha<\frac{1}{2}, \\ \frac{10\alpha}{3\alpha+3}, & \text{if} \quad \frac{1}{2}\le\alpha<1, \\ \frac{5}{3}, \quad \text{if} \quad \alpha\ge 1, \end{cases} \quad \text{as } \varepsilon=\varepsilon_j\searrow 0, \tag{5.51}$$

and furthermore, (5.36) and (5.41) imply that

$$n_\varepsilon u_\varepsilon \to nu \quad \text{a.e. in} \quad \Omega\times(0,\infty) \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0. \tag{5.52}$$

Along with (5.51) and (5.52), the Egorov theorem guarantees that $z_3 = nu$, whereupon we derive from (5.51) that

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$$n_\varepsilon u_\varepsilon \rightharpoonup nu \quad \text{in} \quad L^\theta(\Omega \times (0,T)) \quad \text{with} \quad \theta = \begin{cases} \frac{10(3\alpha+1)}{9(\alpha+2)}, & \text{if} \quad \frac{3}{7} \le \alpha < \frac{1}{2}, \\ \frac{10\alpha}{3\alpha+3}, & \text{if} \quad \frac{1}{2} \le \alpha < 1, \\ \frac{5}{3}, & \text{if} \quad \alpha \ge 1, \end{cases} \quad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{5.53}$$

for each $T \in (0,\infty)$. As a straightforward consequence of (5.38) and (5.41), it holds that

$$c_\varepsilon u_\varepsilon \to cu \quad \text{in} \quad L^1_{loc}(\bar{\Omega} \times [0,\infty)) \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0. \tag{5.54}$$

Thus, the integrability of nu and cu in (5.32) is verified by (5.53) and (5.54). According to the argument of [18, Theorem V.3.1.1] and along with (5.44), a utilization of $\|Y_\varepsilon w\|_{L^2(\Omega)} \le \|w\|_{L^2(\Omega)}$ for all $w \in L^2_\sigma(\Omega)$ and $Y_\varepsilon w \to w$ in $L^2(\Omega)$ as $\varepsilon \searrow 0$ entails that for each $t \in (0,T)\backslash N_T$

$$\begin{aligned} \|Y_\varepsilon u_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} &\le \|Y_\varepsilon\left(u_\varepsilon(\cdot,t) - u(\cdot,t)\right)\|_{L^2(\Omega)} + \|Y_\varepsilon u(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \\ &\le \|u_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} + \|Y_\varepsilon u(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \\ &\to 0 \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0, \end{aligned}$$

and apart from that, (3.1) and (5.44) allow for a choice of $C_2 > 0$ fulfilling

$$\begin{aligned} \|Y_\varepsilon u_\varepsilon(\cdot,t) - u(\cdot,t)\|^2_{L^2(\Omega)} &\le \left(\|Y_\varepsilon u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + \|u(\cdot,t)\|_{L^2(\Omega)}\right)^2 \\ &\le \left(\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + \|u(\cdot,t)\|_{L^2(\Omega)}\right)^2 \\ &\le C_2 \quad \text{for all} \quad t \in (0,T)\backslash N_T \quad \text{and} \quad \varepsilon \in (0,1), \end{aligned}$$

so that we can infer from the dominated convergence theorem that

$$\int_0^T \|Y_\varepsilon u_\varepsilon(\cdot,t) - u(\cdot,t)\|^2_{L^2(\Omega)} dt \to 0 \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0 \quad \text{for each} \quad T \in (0,\infty),$$

i.e.

$$Y_\varepsilon u_\varepsilon \to u \quad \text{in} \quad L^2_{loc}(\bar{\Omega} \times [0,\infty)) \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0.$$

Hence, combining with (5.41), we arrive at

$$Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \to u \otimes u \quad \text{in} \quad L^1_{loc}(\bar{\Omega} \times [0,\infty)) \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0. \tag{5.55}$$

From (5.55), we know that $u \otimes u$ is integrable, and consequently (5.32) is valid. Finally, for any fixed $T \in (0,\infty)$, we can derive

$$\begin{aligned} \|F_\varepsilon(n_\varepsilon) - n\|_{L^\gamma(\Omega\times(0,T))} &\le \|F_\varepsilon(n_\varepsilon) - F_\varepsilon(n)\|_{L^\gamma(\Omega\times(0,T))} + \|F_\varepsilon(n) - n\|_{L^\gamma(\Omega\times(0,T))} \\ &\le \|F'_\varepsilon\|_{L^\infty(0,\infty)}\|n_\varepsilon - n\|_{L^\gamma(\Omega\times(0,T))} + \|F_\varepsilon(n) - n\|_{L^\gamma(\Omega\times(0,T))} \end{aligned} \tag{5.56}$$

with

$$\gamma = \begin{cases} \frac{3\alpha+1}{2}, & \text{if } \frac{3}{7} \le \alpha < \frac{1}{2}, \\ \frac{10\alpha}{2\alpha+3}, & \text{if } \frac{1}{2} \le \alpha < 1, \\ 2, \quad \text{if} & \alpha \ge 1, \end{cases}$$

where we have known that

$$\|n_\varepsilon - n\|_{L^\gamma(\Omega\times(0,T))} \to 0 \tag{5.57}$$

thanks to (5.36). Besides that, we also deduce from (2.4) that

$$\|F_\varepsilon(n(\cdot,t)) - n(\cdot,t)\|^\gamma_{L^\gamma(\Omega)} \le 2^\gamma \|n(\cdot,t)\|^\gamma_{L^\gamma(\Omega)} \tag{5.58}$$

for each $t \in (0,T)$, which together with (5.36) shows the integrability of $\|F_\varepsilon(n(\cdot,t)) - n(\cdot,t)\|^\gamma_{L^\gamma(\Omega)}$ on $(0,T)$. Thereupon, by virtue of (2.5), we infer from the dominated convergence theorem that

$$\int_0^T \|F_\varepsilon(n(\cdot,t)) - n(\cdot,t)\|^\gamma_{L^\gamma(\Omega)} dt \to 0 \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0 \tag{5.59}$$

for each $T \in (0,\infty)$. From (5.56), (5.57) and (5.59), we can see clearly that

$$F_\varepsilon(n_\varepsilon) \to n \quad \text{in} \quad L^\gamma_{loc}(\bar{\Omega}\times[0,\infty)) \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0 \tag{5.60}$$

with

$$\gamma = \begin{cases} \frac{3\alpha+1}{2}, & \text{if } \frac{3}{7} \le \alpha < \frac{1}{2}, \\ \frac{10\alpha}{2\alpha+3}, & \text{if } \frac{1}{2} \le \alpha < 1, \\ 2, \quad \text{if} & \alpha \ge 1. \end{cases}$$

Relying on (5.36)–(5.39), (5.41), (5.43), (5.50), (5.53)–(5.55) and (5.60), we can take $\varepsilon = \varepsilon_j \searrow 0$ and thus establish (5.33), (5.34) and (5.35) by standard arguments from the corresponding weak formulations in (2.1). The proof is completed. □

Proof of Theorem 1.1. Theorem 1.1 directly follows from Proposition 4.1 and Lemma 5.3. □

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