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# Global existence and stabilization in a degenerate chemotaxis-Stokes system with mildly strong diffusion enhancement

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**Abstract**

A class of chemotaxis-Stokes systems generalizing the prototype

$$\begin{cases} n_t + u \cdot \nabla n &= \nabla \cdot \big(n^{m-1}\nabla n\big) - \nabla \cdot \big(n \nabla c\big), \\ c_t + u \cdot \nabla c &= \Delta c - nc, \\ u_t + \nabla P &= \Delta u + n \nabla \phi, \qquad \nabla \cdot u = 0, \end{cases}$$

is considered in bounded convex three-dimensional domains, where $\phi \in W^{2,\infty}(\Omega)$ is given.

The paper develops an analytical approach which consists in a combination of energy-based arguments and maximal Sobolev regularity theory, and which allows for the construction of global bounded weak solutions to an associated initial-boundary value problem under the assumption that

$$m > \frac{9}{8}. \tag{0.1}$$

Moreover, the obtained solutions are shown to approach the spatially homogeneous steady state $(\frac{1}{|\Omega|}\int_\Omega n_0, 0, 0)$ in the large time limit.

This extends previous results which either relied on different and apparently less significant energy-type structures, or on completely alternative approaches, and thereby exclusively achieved comparable results under hypotheses stronger than (0.1).


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## 1. Introduction

We consider the chemotaxis-Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot\Big(D(n)\nabla n\Big) - \nabla\cdot\big(n\nabla c\big), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega,\ t>0,\\ u_t + \nabla P = \Delta u + n\nabla\phi, \qquad \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

which was proposed in [35] and [9] as a model for the spatio-temporal evolution in populations of oxytactically moving bacteria that interact with a surrounding fluid through transport and buoyancy, where $n, c, u$ and $P$ denote the density of cells, the oxygen concentration, the fluid velocity and its associated pressure, respectively, and where the diffusivity $D$ and the gravitational potential $\phi = \phi(x)$ are given smooth parameter functions (cf. also [2] for a recent independent derivation of (1.1) on the basis of fundamental principles from the kinetic theory of active particles). Indeed, as reported in [10] and [35], even in such a simple setting lacking any reinforcement of chemotactic motion by signal production through cells, quite a colorful collective behavior can be observed, including the formation of aggregates and the emergence of large-scale convection patterns.

In modification of the original model from [35] in which $D\equiv 1$, the authors in [9] suggested to adequately account for the finite size of bacteria by assuming that the random movement of cells is nonlinearly enhanced at large densities, leading to the choice

$$D(s) = s^{m-1} \qquad \text{for } s\ge 0 \tag{1.2}$$

with some $m>1$ in the prototypical case of porous medium type diffusion. In comparison to the case $D\equiv 1$, nonlinear diffusion mechanisms of this type may suppress the occurrence of blow-up phenomena, as known to be enforced by chemotactic cross-diffusion e.g. in frameworks such as that addressed by the classical Keller–Segel system ([19], [43]). In fact, in three-dimensional initial value problems for (1.1) with $D\equiv 1$, global smooth and bounded solutions could be shown to exist only under appropriate smallness assumptions on the initial data ([11], [23], [7], [6]), while for arbitrarily large data so far only certain global weak solutions have been constructed, which do become smooth eventually but may develop singularities prior to such ultimate regularization ([42], [47]). Contrary to this, assuming (1.2) to hold, recent analysis has revealed the condition

$$m > \frac{7}{6} \tag{1.3}$$

as sufficient for global existence and boundedness of weak solutions to an associated no-flux-no-flux-Dirichlet initial-boundary value problem for all reasonably regular initial data in three-dimensional bounded convex domains ([45], cf. also [27]). This partially extended a precedent result which asserted global solvability within the larger range $m>\frac{8}{7}$, but only in a class of weak solutions locally bounded in $\overline{\Omega} \times [0, \infty)$ ([33]). For smaller values of $m > 1$, up to now existence results are limited to classes of possibly unbounded solutions ([12]).

In view of lacking complementary results on possibly occurring singularity formation phenomena, the question of identifying an *optimal* condition on $m \geq 1$ ensuring global boundedness in the three-dimensional version of (1.1) remains an open challenge, thus marking a substantial difference to the two-dimensional situation in which global existence and boundedness results are available for several variants of (1.1) already in presence of linear cell diffusion, and even when the fluid flow is governed by the corresponding full nonlinear Navier–Stokes system ([11], [42], [44], [8], [48], see also [22] and [32]).

**Main results.** It is the purpose of this work to demonstrate how an adequate combination of energy-based arguments and maximal Sobolev regularity theory can be used to further advance the analysis of (1.1), with $D$ essentially of the form in (1.2), even in previously unexplored ranges of $m$. In fact, in the first step our approach we will make use of an observation to be stated in Lemma 3.1, according to which the system (1.1) also for $m > 1$ continues to feature an energy-type structure known to be present when $m = 1$ even in an associated chemotaxis-Navier–Stokes system ([46]; cf. also [11] and [42] for precedent partial findings in this direction). By means of a first iterative bootstrap procedure, the correspondingly obtained a priori estimates will be turned into some regularity information on the solution component $n$ (Section 4 and Section 5), which itself can be used as a starting point for a second recursive argument: Namely, investigating how far regularity information of the latter type influences integrability properties of $u$ and $\nabla c$ through maximal Sobolev regularity estimates (Section 6), we will be able to successively improve our knowledge on available integral bounds for all solution components under the mild assumption that in the setup of (1.2) we merely have

$$m > \frac{9}{8} \tag{1.4}$$

(Section 7 and Section 8). The estimates thereby obtained will provide appropriate compactness properties which will firstly allow us to construct global bounded weak solutions to (1.1) via a suitable approximation procedure (Section 9), and which thereafter secondly enable us to assert stabilization toward spatially homogeneous equilibria (Section 10).

In order to formulate our results in these directions, let us specify the setup of our analysis by declaring that throughout the sequel we shall assume $D$ to generalize the choice in (1.2) in that

$$D \in C^{\vartheta}_{loc}([0, \infty)) \cap C^2((0, \infty)) \quad \text{is such that} \quad D(s) \geq k_D s^{m-1} \quad \text{for all } s \geq 0 \tag{1.5}$$

with some $\vartheta \in (0, 1)$, $k_D > 0$ and $m > 1$, and by considering the initial-boundary value problem for (1.1) associated with the requirements that

$$n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x) \quad \text{and} \quad u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{1.6}$$

as well as

$$\Big(D(n)\nabla n - n\nabla c\Big) \cdot \nu = 0, \quad \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \qquad \text{on } \partial\Omega, \tag{1.7}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. As for the initial data herein, we shall suppose for convenience that

$$\begin{cases} n_0 \in C^\omega(\overline{\Omega}) \quad \text{for some } \omega > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega \text{ and } n_0 \not\equiv 0, \quad \text{that} \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{satisfies } c_0 \geq 0 \text{ in } \Omega, \quad \text{and that} \\ u_0 \in D(A^\alpha) \quad \text{for some } \alpha \in (\frac{3}{4}, 1), \end{cases} \tag{1.8}$$

where $A = -\mathcal{P}\Delta$ denotes the Stokes operator in $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0\}$ with its domain given by $D(A) := W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$, and with $\mathcal{P}$ representing the Helmholtz projection on $L^2(\Omega)$ ([29]).

We shall then obtain the following result on global existence and large time behavior, where as in several places below we make use of the abbreviation $\overline{\varphi} := \frac{1}{|\Omega|}\int_\Omega \varphi$ for $\varphi \in L^1(\Omega)$.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary and $\phi \in W^{2,\infty}(\Omega)$, and suppose that $D$ is such that (1.5) holds with some*

$$m > \frac{9}{8}. \tag{1.9}$$

*Then for each $n_0, c_0$ and $u_0$ satisfying (1.8) there exist functions*

$$\begin{cases} n \in L^\infty(\Omega \times (0,\infty)) \cap C^0([0,\infty); (W^{2,2}_0(\Omega))^\star), \\ c \in \bigcap_{p>1} L^\infty((0,\infty); W^{1,p}(\Omega)) \cap C^0(\overline{\Omega} \times [0,\infty)) \cap C^{1,0}(\overline{\Omega} \times (0,\infty)), \\ u \in L^\infty(\Omega \times (0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)) \cap C^0(\overline{\Omega} \times [0,\infty)) \end{cases} \tag{1.10}$$

*such that the triple $(n,c,u)$ forms a global weak solution of (1.1), (1.6), (1.7) in the sense of Definition 9.1 below.*

*Moreover, this solution has the property that for arbitrary $p \geq 1$ we have*

$$\|n(\cdot,t) - \overline{n_0}\|_{L^p(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{1.11}$$

As a by-product, this trivially extends previous results on blow-up suppression in the associated fluid-free chemotaxis system with porous medium-type diffusion and signal consumption, as obtained on letting $u \equiv 0$ in (1.1). Even for the latter, apparently somewhat simpler system, only under the assumption (1.3) global bounded solutions have been known to exist ([36]), with again no example of blow-up available for any choice of $D$ yet.

In order to further put these results in perspective, let us note that alternative modeling approaches suggest to introduce as blow-up inhibiting mechanisms certain saturation effects in the cross-diffusive term in (1.1) at large cell densities (cf. e.g. the survey [20]). Indeed, if in (1.1) the summand $-\nabla \cdot (n\nabla c)$ is replaced by $-\nabla \cdot (nS(n)\nabla c)$ with $S$ suitably generalizing the prototype given by $S(s) = (s+1)^{-\alpha}$ for all $s \geq 0$ and some $\alpha > 0$, then known results assert global existence of bounded solutions to a corresponding initial-boundary value problem when in the context of (1.5) we have $m \geq 1$ and $m + \alpha > \frac{7}{6}$ ([40]), which in the particular case $\alpha = 0$ considered here rediscovers (1.3) and is thereby stronger than (1.9). An interesting open problem, partially addressed in [38] and [39], consists in determining optimal conditions on the interplay between these two mechanisms which indeed prevent explosions. This may be viewed as part of a more comprehensive ambition to understand the destabilizing potential of chemotaxis-fluid interaction that has been in the focus also of studies including further relevant processes such as logistic proliferation and death, or off-diagonal cross-diffusive migration ([28], [37], [4], [6], [5], [3]).

## 2. Approximation by non-degenerate problems

In order to construct solutions of (1.1) through an appropriate approximation, following natural regularization procedures we fix a family $(D_\varepsilon)_{\varepsilon\in(0,1)}$ of functions

$$\begin{aligned} &D_\varepsilon \in C^2([0,\infty)) \quad \text{such that} \quad D_\varepsilon(s) \ge \varepsilon \quad \text{for all } s \ge 0 \text{ and } \varepsilon \in (0,1) \quad \text{and} \\ &D(s) \le D_\varepsilon(s) \le D(s) + 2\varepsilon \quad \text{for all } s \ge 0 \text{ and } \varepsilon \in (0,1), \end{aligned} \tag{2.1}$$

and we moreover regularize the cross-diffusive term in (1.1) by introducing a family $(\chi_\varepsilon)_{\varepsilon\in(0,1)} \subset C_0^\infty([0,\infty))$ fulfilling

$$0 \le \chi_\varepsilon \le 1 \text{ in } [0,\infty), \quad \chi_\varepsilon \equiv 1 \text{ in } [0,\tfrac{1}{\varepsilon}] \quad \text{and} \quad \chi_\varepsilon \equiv 0 \text{ in } [\tfrac{2}{\varepsilon},\infty), \tag{2.2}$$

and by letting

$$F_\varepsilon(s) := \int_0^s \chi_\varepsilon(\sigma)d\sigma, \qquad s \ge 0, \tag{2.3}$$

for $\varepsilon \in (0,1)$. Then $F_\varepsilon \in C^\infty([0,\infty))$ satisfies

$$0 \le F_\varepsilon(s) \le s \quad \text{and} \quad 0 \le F_\varepsilon'(s) \le 1 \qquad \text{for all } s \ge 0 \tag{2.4}$$

as well as

$$F_\varepsilon(s) \nearrow s \quad \text{for all } s \ge 0 \quad \text{and} \quad F_\varepsilon'(s) \nearrow 1 \quad \text{for all } s > 0 \qquad \text{as } \varepsilon \searrow 0. \tag{2.5}$$

These choices in particular guarantee that each of the approximate variants of (1.1), (1.6), (1.7) given by

$$\begin{cases} \partial_t n_\varepsilon + u_\varepsilon \cdot \nabla n_\varepsilon = \nabla \cdot \Big(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\Big) - \nabla \cdot \Big(n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon\Big), & x \in \Omega,\ t > 0, \\ \partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon, & x \in \Omega,\ t > 0, \\ \partial_t u_\varepsilon + \nabla P_\varepsilon = \Delta u_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n_\varepsilon}{\partial\nu} = \frac{\partial c_\varepsilon}{\partial\nu} = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.6}$$

for $\varepsilon \in (0,1)$, possesses globally defined classical solutions:

**Lemma 2.1.** *Assume (1.8), and let $\varepsilon \in (0,1)$. Then there exist functions*

$$\begin{cases} n_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ u_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ P_\varepsilon \in C^{1,0}(\overline{\Omega} \times (0,\infty)), \end{cases}$$

*such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.6) classically in $\Omega \times (0,\infty)$, and such that $n_\varepsilon$ and $c_\varepsilon$ are nonnegative in $\Omega \times (0,\infty)$.*

**Proof.** By means of standard arguments from the local existence theories of taxis-type cross diffusive parabolic systems and the Stokes evolution equation ([1], [29], [25], [42]), it follows that there exist $T_{max,\varepsilon} \in (0,\infty]$ and at least one classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon) \in \Big(C^0(\overline{\Omega} \times [0, T_{max,\varepsilon}); \mathbb{R}^5) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon}); \mathbb{R}^5)\Big) \times C^{1,0}(\overline{\Omega} \times (0, T_{max,\varepsilon}))$ which is such that $n_\varepsilon \ge 0$ and $c_\varepsilon \ge 0$ in $\Omega \times (0, T_{max,\varepsilon})$, that $c_\varepsilon \in C^0([0, T_{max,\varepsilon}); W^{1,p}(\Omega))$ for all $p \ge 1$ and that if $T_{max,\varepsilon} < \infty$ then

$$\limsup_{t \nearrow T_{max,\varepsilon}} \Big( \|n_\varepsilon(\cdot,t)\|_{C^2(\overline{\Omega})} + \|c_\varepsilon(\cdot,t)\|_{C^2(\overline{\Omega})} + \|u_\varepsilon(\cdot,t)\|_{C^2(\overline{\Omega})} \Big) = \infty. \tag{2.7}$$

For each $T > 0$, however, using that for any fixed $\varepsilon \in (0,1)$ the function $F'_\varepsilon$ has its support located in $[0, \frac{2}{\varepsilon}]$ according to (2.3) and (2.2), successive application of well-established $L^p$ estimation techniques and methods from higher order regularity theories for scalar parabolic equations and the Stokes system yields $C_1(\varepsilon, T) > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{C^2(\overline{\Omega})} + \|c_\varepsilon(\cdot,t)\|_{C^2(\overline{\Omega})} + \|u_\varepsilon(\cdot,t)\|_{C^2(\overline{\Omega})} \le C_1(\varepsilon, T) \qquad \text{for all } t \in (\tau_\varepsilon, \widehat{T}_{max,\varepsilon}),$$

where $\tau_\varepsilon := \min\{\frac{1}{2}T, \frac{1}{2}T_{max,\varepsilon}\}$ and $\widehat{T}_{max,\varepsilon} := \min\{T, T_{max,\varepsilon}\}$. This shows that (2.7) cannot hold when $T_{max,\varepsilon}$ is finite, whence we actually must have $T_{max,\varepsilon} = \infty$. $\square$

In order to simplify presentation, throughout the sequel we shall tacitly assume that $(n_0, c_0, u_0)$ satisfies (1.8), and that for $\varepsilon \in (0,1)$, $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ denotes the corresponding solution to (2.6) obtained in Lemma 2.1.

The following two basic properties thereof are immediate consequences of an integration in the first equation in (2.6), as well as an application of the maximum principle to the second.

**Lemma 2.2.** *We have*

$$\|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \int_\Omega n_0 \qquad \textit{for all } t > 0 \tag{2.8}$$

*as well as*

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t > 0. \tag{2.9}$$

## 3. Directly exploiting the natural quasi-energy structure of (2.6)

Some first regularity properties beyond those from Lemma 2.2 can be obtained by making use of a quasi-energy structure which the approximate problems (2.6) inherit from (1.1) thanks to the particular link between the dependence on $n_\varepsilon$ of the interaction terms $-\nabla\cdot(n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon)$ and $-F_\varepsilon(n_\varepsilon)c_\varepsilon$ therein. Similar energy-like properties have been used in previous studies on related problems ([11], [33], [42]), but only in few cases the fluid velocity has been included ([24], [46], [47]).

**Lemma 3.1.** *There exist $\kappa>0$ and $C>0$ such that*

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega n_\varepsilon\ln n_\varepsilon+\frac12\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\kappa\int_\Omega|u_\varepsilon|^2\Big\}\\
&\qquad+\frac1C\Big\{\int_\Omega n_\varepsilon\ln n_\varepsilon+\frac12\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\kappa\int_\Omega|u_\varepsilon|^2\Big\}\\
&\qquad+\frac1C\Big\{\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2+\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\int_\Omega|\nabla u_\varepsilon|^2\Big\}\le C\qquad \textit{for all } t>0. \quad (3.1)
\end{aligned}$$

**Proof.** The derivation of (3.1) follows a standard reasoning combining ideas from [11], [42] and [46]: By means of straightforward computation using the first two equations in (2.6) (cf. [42, Lemma 3.2] for details), we obtain the identity

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega n_\varepsilon\ln n_\varepsilon+\frac12\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}\Big\}+\int_\Omega\frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon}|\nabla n_\varepsilon|^2+\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2\\
&\quad=-\frac12\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)+\int_\Omega\frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon)\\
&\qquad-\frac12\int_\Omega F_\varepsilon(n_\varepsilon)\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac12\int_{\partial\Omega}\frac{1}{c_\varepsilon}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}\qquad\text{for all } t>0, \quad (3.2)
\end{aligned}$$

where

$$\int_\Omega\frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon}|\nabla n_\varepsilon|^2\ge k_D\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2\qquad\text{for all } t>0 \quad (3.3)$$

by (2.1) and (1.5), and where the two last summands on the right are nonpositive by nonnegativity of $F_\varepsilon$ and due to the fact that $\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}\le 0$ on $\partial\Omega\times(0,\infty)$ thanks to the convexity of $\Omega$ ([26, Lemme 2.I.1]). We next recall from [42, Lemma 3.3] that

$$\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \le C_1 \int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 \qquad \text{for all } t>0$$

with $C_1 := (2+\sqrt{3})^2$, and, after two integrations by parts in (3.2), combine (2.9) with Young's inequality to estimate

$$\begin{aligned}
-\frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon) + \int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon) &= \frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon) - \int_\Omega \frac{1}{c_\varepsilon}\nabla c_\varepsilon\cdot(D^2 c_\varepsilon\cdot\nabla u_\varepsilon) \\
&\quad - \int_\Omega \frac{1}{c_\varepsilon}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\
&= -\int_\Omega \frac{1}{c_\varepsilon}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\
&\le \frac{1}{2C_1}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C_2\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } t>0
\end{aligned} \tag{3.4}$$

with $C_2 := \frac{1}{2}\|c_0\|_{L^\infty(\Omega)}$. Now testing the third equation in (2.6) by $u_\varepsilon$, thanks to the continuity of the embeddings $W_0^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and $W^{1,2}(\Omega) \hookrightarrow L^{\frac{12}{5m}}(\Omega)$ we independently see using the Gagliardo–Nirenberg inequality, Young's inequality and (2.8) that there exist positive constants $C_3, C_4, C_5$ and $C_6$ such that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 &= \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\phi \\
&\le \|\nabla\phi\|_{L^\infty(\Omega)}\|u_\varepsilon\|_{L^6(\Omega)}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \\
&\le C_3\|\nabla u_\varepsilon\|_{L^2(\Omega)}\|n_\varepsilon^{\frac{m}{2}}\|_{L^{\frac{12}{5m}}(\Omega)}^{\frac{2}{m}} \\
&\le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{C_3^2}{2}\|n_\varepsilon^{\frac{m}{2}}\|_{L^{\frac{12}{5m}}(\Omega)}^{\frac{4}{m}} \\
&\le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + C_4\cdot\Big\{\|\nabla n_\varepsilon^{\frac{m}{2}}\|_{L^2(\Omega)}^{\frac{2}{3m-1}}\|n_\varepsilon^{\frac{m}{2}}\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{10m-4}{3m^2-m}} + \|n_\varepsilon^{\frac{m}{2}}\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{4}{m}}\Big\} \\
&\le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + C_5\|\nabla n_\varepsilon^{\frac{m}{2}}\|_{L^2(\Omega)}^{\frac{2}{3m-1}} + C_5 \\
&\le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{k_D}{4(C_2+1)}\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + C_6 \qquad \text{for all } t>0.
\end{aligned}$$

In combination with (3.2), (3.3) and (3.4), this shows that

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + (C_2+1)\int_\Omega |u_\varepsilon|^2\right\} + \frac{k_D}{2}\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + \frac{1}{2C_1}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega |\nabla u_\varepsilon|^2 \le 2(C_2+1)C_6 \qquad \text{for all } t>0.$$

Since finally from the Gagliardo–Nirenberg inequality along with Young's inequality and (2.9) we readily obtain $C_7>0$ such that

$$\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + (C_2+1)\int_\Omega |u_\varepsilon|^2 \le C_7\cdot\left\{\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega |\nabla u_\varepsilon|^2 + 1\right\}$$

for all $t>0$, this readily establishes (3.1) upon evident choices of $\kappa$ and $C$. □

In the sequel we shall make use of the latter exclusively through the following direct consequences.

**Lemma 3.2.** *There exists $C>0$ such that for all $\varepsilon\in(0,1)$,*

$$\int_t^{t+1}\int_\Omega |\nabla n_\varepsilon^{\frac{m}{2}}|^2 \le C \qquad \textit{for all } t\ge 0 \tag{3.5}$$

*and*

$$\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon|^4 \le C \qquad \textit{for all } t\ge 0 \tag{3.6}$$

*as well as*

$$\int_t^{t+1}\int_\Omega |\nabla u_\varepsilon|^2 \le C \qquad \textit{for all } t\ge 0. \tag{3.7}$$

**Proof.** All inequalities immediately result from an integration of (3.1) because of (2.9) and the fact that $\int_\Omega n_\varepsilon \ln n_\varepsilon \ge -\frac{|\Omega|}{e}$ for all $t\ge 0$. □

## 4. Preparing an inductive argument

We next address the question how far an informational background such as the one provided by Lemma 3.2 and Lemma 2.2 can be exploited so as to derive further regularity features of solutions to (2.6). More precisely, we shall be concerned with the problem of finding appropriate conditions on $m$ and the numbers $p_\star \ge 1$ and $p^\star > p_\star$ such that bounds of the form

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le C \quad \text{and} \quad \int_t^{t+1}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le C \qquad \text{for all } t \ge 0, \tag{4.1}$$

assumed to be present for $p = p_\star$, can be shown to imply the same estimates for the corresponding quantities for $p = p^\star$.

Our first result in this direction actually requires a bound for $n_\varepsilon$ in the single space $L^\infty((0,\infty); L^{p_\star}(\Omega))$ only, but additionally relies on a space-time regularity property of $\nabla c_\varepsilon$ in asserting the following.

**Lemma 4.1.** *Let $m > 1$, $p_\star \ge 1$, $p > 1$ and $q \ge 2$ be such that*

$$p \le \frac{2(q-1)}{3} p_\star + (2q-1)(m-1). \tag{4.2}$$

*Then for all $K > 0$ there exists $C = C(p_\star, p, q, K) > 0$ such that if for some $\varepsilon \in (0,1)$ we have*

$$\int_\Omega n_\varepsilon^{p_\star}(\cdot,t) \le K \qquad \text{for all } t \ge 0 \tag{4.3}$$

*and*

$$\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon|^{2q} \le K \qquad \text{for all } t \ge 0, \tag{4.4}$$

*then*

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le C \qquad \text{for all } t \ge 0 \tag{4.5}$$

*and*

$$\int_t^{t+1}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le C \qquad \text{for all } t \ge 0. \tag{4.6}$$

**Proof.** In view of (2.8) and Lemma 3.2, since $\frac{2(q-1)}{3}p_\star + (2q-1)(m-1) \ge (2q-1)(m-1) \ge 3(m-1)$ we may assume without loss of generality that $p > m-1$ and $p \ge p_\star$. We then test the first equation in (2.6) by $n_\varepsilon^{p-1}$ and use Young's inequality along with (2.1), (1.5) and (2.4) to see that for all $t > 0$,

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + (p-1)k_D\int_\Omega n_\varepsilon^{p+m-3}|\nabla n_\varepsilon|^2 &\le \frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + (p-1)\int_\Omega n_\varepsilon^{p-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 \\
&= (p-1)\int_\Omega n_\varepsilon^{p-1}F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\
&\le \frac{(p-1)k_D}{2}\int_\Omega n_\varepsilon^{p+m-3}|\nabla n_\varepsilon|^2 \\
&\quad + \frac{p-1}{2k_D}\int_\Omega n_\varepsilon^{p-m+1}|\nabla c_\varepsilon|^2
\end{aligned}$$

so that

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p + C_1\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le \frac{p(p-1)}{2k_D}\int_\Omega n_\varepsilon^{p-m+1}|\nabla c_\varepsilon|^2 \qquad \text{for all } t>0 \tag{4.7}$$

with $C_1 := \frac{2p(p-1)k_D}{(p+m-1)^2}$. Now in order to further estimate the right-hand side herein, we invoke the Hölder inequality to obtain

$$\int_\Omega n_\varepsilon^{p-m+1}|\nabla c_\varepsilon|^2 \le \left\{\int_\Omega n_\varepsilon^{(p-m+1)q'}\right\}^{\frac{1}{q'}}\cdot\left\{\int_\Omega |\nabla c_\varepsilon|^{2q}\right\}^{\frac{1}{q}} \qquad \text{for all } t>0 \tag{4.8}$$

with $q' := \frac{q}{q-1}$, where we firstly note that in the case when $(p-m+1)q' \le p_\star$, (4.3) together with the Hölder inequality yield $C_2 > 0$ such that

$$\left\{\int_\Omega n_\varepsilon^{(p-m+1)q'}\right\}^{\frac{1}{q'}} \le C_2 \qquad \text{for all } t>0. \tag{4.9}$$

If, conversely, $(p-m+1)q' > p_\star$ then due to our assumption $q \ge 2$ we have

$$\frac{2(p-m+1)q'}{p+m-1} \le 2q' \le 4 < 6$$

and thus $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2(p-m+1)q'}{p+m-1}}(\Omega) \hookrightarrow L^{\frac{2p_\star}{p+m-1}}(\Omega)$, whence in particular the number

$$a := \frac{3(p+m-1)[(p-m+1)q'-p_\star]}{(p-m+1)[3(p+m-1)-p_\star]q'}$$

satisfies $a \in [0,1]$, and accordingly the Gagliardo–Nirenberg inequality provides $C_3 > 0$ such that

$$\begin{aligned}\Big\{\int_\Omega n_\varepsilon^{(p-m+1)q'}\Big\}^{\frac{1}{q'}} &= \Big\| n_\varepsilon^{\frac{p+m-1}{2}} \Big\|_{L^{\frac{2(p-m+1)q'}{p+m-1}}(\Omega)}^{\frac{2(p-m+1)}{p+m-1}} \\ &\le C_3 \Big\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \Big\|_{L^2(\Omega)}^{\frac{6[(p-m+1)q'-p_\star]}{[3(p+m-1)-p_\star]q'}} \Big\| n_\varepsilon^{\frac{p+m-1}{2}} \Big\|_{L^{\frac{2p_\star}{p+m-1}}(\Omega)}^{\frac{2(p-m+1)}{p+m-1}(1-a)} \\ &\quad + C_3 \Big\| n_\varepsilon^{\frac{p+m-1}{2}} \Big\|_{L^{\frac{2p_\star}{p+m-1}}(\Omega)}^{\frac{2(p-m+1)}{p+m-1}}\end{aligned}$$

for all $t > 0$. As

$$\Big\| n_\varepsilon^{\frac{p+m-1}{2}} \Big\|_{L^{\frac{2p_\star}{p+m-1}}(\Omega)}^{\frac{2p_\star}{p+m-1}} = \int_\Omega n_\varepsilon^{p_\star} \le K \qquad \text{for all } t > 0$$

by (4.3), together with (4.9), (4.8) and Young's inequality this shows that regardless of the sign of $(p-m+1)q' - p_\star$ we can find $C_4 > 0$ and $C_5 > 0$ fulfilling

$$\begin{aligned}\frac{p(p-1)}{2k_D}\int_\Omega n_\varepsilon^{p-m+1}|\nabla c_\varepsilon|^2 &\le C_4 \cdot \Big\{ \Big\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \Big\|_{L^2(\Omega)}^{\frac{6[(p-m+1)q'-p_\star]}{[3(p+m-1)-p_\star]q'}} + 1 \Big\} \cdot \|\nabla c_\varepsilon\|_{L^{2q}(\Omega)}^2 \\ &\le 2^{-q'} C_1 \cdot \Big\{ \Big\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \Big\|_{L^2(\Omega)}^{\frac{6[(p-m+1)q'-p_\star]}{[3(p+m-1)-p_\star]q'}} + 1 \Big\}^{q'} + C_5 \|\nabla c_\varepsilon\|_{L^{2q}(\Omega)}^{2q} \\ &\le \frac{C_1}{2} \cdot \Big\{ \Big\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \Big\|_{L^2(\Omega)}^{\frac{6[(p-m+1)q'-p_\star]}{3(p+m-1)-p_\star}} + 1 \Big\} + C_5 \|\nabla c_\varepsilon\|_{L^{2q}(\Omega)}^{2q} \qquad (4.10)\end{aligned}$$

for all $t > 0$, the latter inequality being valid because $(\xi+\eta)^{q'} \le 2^{q'-1}(\xi^{q'} + \eta^{q'})$ for all $\xi \ge 0$ and $\eta \ge 0$.

Now our assumption (4.2) enters by ensuring that

$$\begin{aligned}\frac{6[(p-m+1)q'-p_\star]}{3(p+m-1)-p_\star} - 2 &= \frac{6(q'-1)p - 4p_\star - 6(m-1)(q'+1)}{3(p+m-1)-p_\star} \\ &= \frac{6}{[3(p+m-1)-p_\star](q-1)} \\ &\quad \cdot \Big\{ p - \frac{2(q-1)}{3}p_\star - (2q-1)(m-1) \Big\} \\ &\le 0,\end{aligned}$$

whence another application of Young's inequality yields

$$\frac{C_1}{2} \cdot \Big\{ \Big\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \Big\|_{L^2(\Omega)}^{\frac{6[(p-m+1)q'-p_\star]}{3(p+m-1)-p_\star}} + 1 \Big\} \le \frac{C_1}{2}\int_\Omega \Big|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\Big|^2 + C_1 \qquad \text{for all } t > 0.$$

Together with (4.10), this shows that (4.7) implies that

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{C_1}{2}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le C_5 \int_\Omega |\nabla c_\varepsilon|^{2q} + C_1 \qquad \text{for all } t>0, \tag{4.11}$$

where a linear absorptive term can be generated again by interpolation in a straightforward manner: As according to our restriction $p > p_\star$ we know that $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2p}{p+m-1}}(\Omega) \hookrightarrow L^{\frac{2p_\star}{p+m-1}}(\Omega)$, the number $b := \frac{3(p+m-1)(p-p_\star)}{[3(p+m-1)-p_\star]p}$ satisfies $b \in [0,1]$ and from the Gagliardo–Nirenberg inequality, (4.3) and Young's inequality we obtain $C_6 > 0$ and $C_7 > 0$ such that

$$\begin{aligned}
\int_\Omega n_\varepsilon^p &= \left\| n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2p}{p+m-1}} \\
&\le C_6 \left\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^2(\Omega)}^{\frac{6(p-p_\star)}{3(p+m-1)-p_\star}} \left\| n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^{\frac{2p_\star}{p+m-1}}(\Omega)}^{\frac{2p}{p+m-1}(1-b)} + C_6 \left\| n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^{\frac{2p_\star}{p+m-1}}(\Omega)}^{\frac{2p}{p+m-1}} \\
&\le C_7 \left\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^2(\Omega)}^{\frac{6(p-p_\star)}{3(p+m-1)-p_\star}} + C_7 \\
&\le C_7 \int_\Omega \left| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right|^2 + 2C_7 \qquad \text{for all } t>0,
\end{aligned}$$

because $\frac{6(p-p_\star)}{3(p+m-1)-p_\star} \le \frac{6(p-p_\star)}{3p-p_\star} \le 2$ by nonnegativity of $m-1$ and $p_\star$. Therefore, (4.11) shows that if we let $y(t) := \int_\Omega n_\varepsilon^p(\cdot,t)$, $t \ge 0$, and $h(t) := C_5 \int_\Omega |\nabla c_\varepsilon(\cdot,t)|^{2q} + \frac{3}{2}C_1$, $t>0$, than

$$y'(t) + \frac{C_1}{4C_7} y(t) + \frac{C_1}{4}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le h(t) \qquad \text{for all } t>0, \tag{4.12}$$

where in view of our assumption (4.4) we have

$$\int_t^{t+1} h(s)ds \le C_8 := C_5 K + \frac{3}{2}C_1 \qquad \text{for all } t \ge 0. \tag{4.13}$$

In view of an elementary lemma on decay in linear first-order ODEs with suitably decaying inhomogeneities (see e.g. [30, Lemma 3.4]), (4.12) thus firstly implies that with some $C_9 > 0$ we have $y(t) \le C_9$ for all $t > 0$, whereupon (4.12) and (4.13) secondly entail that

$$\frac{C_1}{4}\int_t^{t+1}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 ds \le y(t) + \int_t^{t+1} h(s)ds \le C_8 + C_9 \qquad \text{for all } t \ge 0,$$

so that indeed both (4.5) and (4.6) hold with some conveniently large $C = C(K) > 0$. $\square$

## 5. Uniform $L^p$ bounds on $n_\varepsilon$ for $p < 9(m-1)$ by a first iteration

In a first series of applications of Lemma 4.1, with regard to the regularity assumptions on $\nabla c_\varepsilon$ we shall exclusively rely on the corresponding estimate provided by Lemma 3.2 and intend to repeatedly increase the integrability parameter in (4.5) and (4.6), thus keeping the number $q := 2$ in Lemma 4.1 fixed while successively choosing larger values of $p_\star$ and $p$. We shall see that this indeed leads to improved information whenever $m > \frac{10}{9}$, and thereby we partially re-discover a similar observation that was already made in [33], with an important difference consisting in the fact that unlike in the latter reference, here the achieved bounds are global in time.

**Lemma 5.1.** *Let $m > \frac{10}{9}$. Then for all $p \in [1, 9(m-1))$ there exists $C(p) > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le C(p) \qquad \textit{for all } t \ge 0 \tag{5.1}$$

*and*

$$\int_t^{t+1} \int_\Omega \left| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right|^2 \le C(p) \qquad \textit{for all } t \ge 0. \tag{5.2}$$

**Proof.** We define $(p_k)_{k\in\mathbb{N}_0} \subset \mathbb{R}$ by letting $p_0 := 1$ and

$$p_{k+1} := \frac{2}{3} p_k + 3(m-1) \qquad \text{for } k \ge 0. \tag{5.3}$$

It can the readily be verified that due to our assumption $m > \frac{10}{9}$ the sequence $(p_k)_{k\in\mathbb{N}_0}$ is strictly increasing with $p_k \nearrow 9(m-1)$ as $k \to \infty$, so that by means of an interpolation argument it is clear that we only need to prove (5.1) and (5.2) for $p = p_k$ and each $k \in \mathbb{N}_0$. To this end, we note that the case $k = 0$ can be covered by combining Lemma 3.2 with (2.8), so that in view of an inductive reasoning we are left with the verification of the property that whenever $k \in \mathbb{N}_0$ is such that

$$\int_\Omega n_\varepsilon^{p_k}(\cdot,t) \le C_1(k) \quad \text{and} \quad \int_t^{t+1} \int_\Omega \left| \nabla n_\varepsilon^{\frac{p_k+m-1}{2}} \right|^2 \le C_1(k) \qquad \text{for all } t \ge 0 \text{ and each } \varepsilon \in (0,1) \tag{5.4}$$

with some $C_1(k) > 0$, we can find $C_2(k) > 0$ satisfying

$$\int_\Omega n_\varepsilon^{p_{k+1}}(\cdot,t) \le C_2(k) \quad \text{and} \quad \int_t^{t+1} \int_\Omega \left| \nabla n_\varepsilon^{\frac{p_{k+1}+m-1}{2}} \right|^2 \le C_2(k) \qquad \text{for all } t \ge 0 \text{ and any } \varepsilon \in (0,1). \tag{5.5}$$

To achieve this, we observe that according to the first inequality in (5.4) and (3.6), the requirements (4.3) and (4.5) from Lemma 4.1 are fulfilled for $p_\star := p_k$ and $q := 2$. In light of (5.3), both inequalities in (5.5) therefore result from an application of Lemma 4.1 to $p := p_{k+1}$. $\square$

## 6. Improving estimates for $\nabla c_\varepsilon$ via maximal Sobolev regularity

We next plan to apply Lemma 4.1 by using the outcome of Lemma 5.1 as a starting point with respect to the regularity assumptions on $n_\varepsilon$, but with regard to the hypothesis (4.4) no longer going back to Lemma 3.2 but rather using suitably improved integrability information on $\nabla c_\varepsilon$. Within a range of $m$ which is smaller than that in Lemma 5.1 but yet larger than the interval $(\frac{9}{8}, \infty)$ we shall finally focus on, such further properties can indeed be gained under the assumptions provided by the result of Lemma 5.1 by means of the key Lemma 6.3 below which in turn relies on the following statement on time-independent bounds for $u_\varepsilon$ in appropriate Lebesgue spaces.

**Lemma 6.1.** *Let $m > \frac{215}{192}$. Then there exists $\delta_1(m) > 0$ such that for all $p > 1$ fulfilling $p > 9(m-1) - \delta_1(m)$ and $K > 0$ one can find $C(p, K) > 0$ with the property that if for some $\varepsilon \in (0, 1)$ we have*

$$\int_\Omega n_\varepsilon^p(\cdot, t) \le K \qquad \text{for all } t \ge 0, \tag{6.1}$$

*then*

$$\int_\Omega |u_\varepsilon(\cdot, t)|^{\frac{2(5p+3m-3)}{3}} \le C(p, K) \qquad \text{for all } t \ge 0. \tag{6.2}$$

**Proof.** We let

$$\rho(p) := 20p^2 - (33 - 12m)p - 18(m-1), \qquad p \in \mathbb{R}.$$

Then our assumption $m > \frac{215}{192}$ precisely warrants that

$$\begin{aligned}\rho(9(m-1)) &= 1620(m-1)^2 - 9 \cdot (33 - 12m)(m-1) - 18(m-1)\\ &= 9(m-1)(192m - 215) > 0,\end{aligned}$$

while since $\frac{215}{192} > \frac{131}{124}$ we moreover have

$$\begin{aligned}\rho'(p) &= 40p - 33 + 12m \ge 360(m-1) - 33 + 12m\\ &= 3 \cdot (124m - 131) > 0 \qquad \text{for all } p > 9(m-1).\end{aligned}$$

We can therefore pick $\delta_1 = \delta_1(m) > 0$ such that

$$\rho(p) > 0 \qquad \text{for all } p > 9(m-1) - \delta_1(m),$$

and given $p > 1$ such that $p > 9(m-1) - \delta_1(m)$ we thus obtain that $q := \frac{2(5p+3m-3)}{3}$ satisfies $q > 1$ and

$$3(3-2p) \cdot \Big(q - \frac{3p}{3-2p}\Big) = -\rho(p) < 0$$

and hence

$$\frac{3}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big) < 1. \tag{6.3}$$

Now assuming (6.1) for some $\varepsilon \in (0,1)$ and $K > 0$, on the basis of a variation-of-constants representation of $u_\varepsilon$ we can estimate

$$\|u_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le \|e^{-tA}u_0\|_{L^q(\Omega)} + \int_0^t \Big\|e^{-(t-s)A}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big\|_{L^q(\Omega)} ds, \qquad t > 0, \tag{6.4}$$

and recall known regularization properties of the Dirichlet Stokes semigroup $(e^{-tA})_{t\ge 0}$ ([16, p. 201]) to find $C_1 > 0$, $C_2 > 0$ and $\lambda > 0$ such that

$$\|e^{-tA}u_0\|_{L^q(\Omega)} \le C_1\|u_0\|_{L^q(\Omega)} \qquad \text{for all } t > 0 \tag{6.5}$$

and

$$\begin{aligned}
&\int_0^t \Big\|e^{-(t-s)A}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big\|_{L^q(\Omega)} ds \\
&\quad \le C_2 \int_0^t \Big(1 + (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\Big) e^{-\lambda(t-s)} \Big\|\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big\|_{L^p(\Omega)} ds
\end{aligned} \tag{6.6}$$

for all $t > 0$. Here by boundedness of $\nabla\phi$ on $\Omega$ and the continuity of the Helmholtz projection when acting as an operator in $L^p(\Omega;\mathbb{R}^3)$ ([14]), we see that with some $C_3 > 0$ we have

$$\Big\|\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big\|_{L^p(\Omega)} \le C_3\|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \le C_3 K^{\frac{1}{p}} \qquad \text{for all } s > 0$$

according to (6.1). Therefore, (6.6) entails that

$$\begin{aligned}
\int_0^t \Big\|e^{-(t-s)A}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big\|_{L^q(\Omega)} ds &\le C_2C_3K^{\frac{1}{p}} \int_0^t \Big(1 + (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\Big) e^{-\lambda(t-s)} ds \\
&\le C_2C_3C_4K^{\frac{1}{p}} \qquad \text{for all } t > 0
\end{aligned}$$

with $C_4 := \int_0^\infty (1 + \sigma^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}) e^{-\lambda\sigma} d\sigma$ being finite thanks to (6.3). When combined with (6.5) and (6.4), in view of our choice of $q$ this establishes (6.2). □

As a second preliminary for Lemma 6.3, let us note how a pair of hypotheses in the flavor of (4.1) influences space-time integrability of $n_\varepsilon$ by means of straightforward interpolation.

**Lemma 6.2.** *Let $m > 1$. Then for all $p \ge 1$ and any $K > 0$ there exists $C(p, K) > 0$ such that if for some $\varepsilon \in (0, 1)$ we have*

$$\int_\Omega n_\varepsilon^p(\cdot, t) \le K \qquad \textit{for all } t \ge 0 \tag{6.7}$$

*and*

$$\int_t^{t+1} \int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le K \qquad \textit{for all } t \ge 0, \tag{6.8}$$

*then*

$$\int_t^{t+1} \int_\Omega n_\varepsilon^{\frac{5p+3m-3}{3}} \le C(p, K) \qquad \textit{for all } t \ge 0. \tag{6.9}$$

**Proof.** Using that $p > 0$ and $m \ge 1$ imply that

$$\frac{2p}{p+m-1} \le \frac{2(5p+3m-3)}{3(p+m-1)} \le 6$$

and hence $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2(5p+3m-3)}{3(p+m-1)}}(\Omega) \hookrightarrow L^{\frac{2p}{p+m-1}}(\Omega)$, from the Gagliardo–Nirenberg inequality we obtain $C_1 > 0$ such that

$$\begin{aligned}
\int_\Omega n_\varepsilon^{\frac{5p+3m-3}{3}} &= \left\| n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^{\frac{2(5p+3m-3)}{3(p+m-1)}}(\Omega)}^{\frac{2(5p+3m-3)}{3(p+m-1)}} \\
&\le C_1 \left\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^2(\Omega)}^2 \left\| n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{4p}{3(p+m-1)}} + C_1 \left\| n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2(5p+3m-3)}{3(p+m-1)}} \\
&\qquad \text{for all } t > 0.
\end{aligned}$$

Noting that $\|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2p}{p+m-1}} \le K$ for all $t > 0$ by (6.7), on integrating in time we thus infer that

$$\begin{aligned}
\int_t^{t+1} \int_\Omega n_\varepsilon^{\frac{5p+3m-3}{3}} &\le C_1 K^{\frac{2}{3}} \int_t^{t+1} \int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 + C_1 K^{\frac{5p+3m-3}{3p}} \\
&\le C_1 K^{\frac{5}{3}} + C_1 K^{\frac{5p+3m-3}{3p}}
\end{aligned}$$

for all $t \ge 0$. $\square$

We can now proceed to the main result of this section which, on the basis of a maximal regularity property of scalar parabolic equations, asserts that bounds of the flavor in (4.1) entail an estimate for $\nabla c_\varepsilon$ in a spatio-temporal $L^{2q}$ space with some positive $q$ which indeed satisfies $q > 2$ if $p \ge 1$ is suitably large.

**Lemma 6.3.** *Let $m > \frac{215}{192}$, and let $\delta_1(m) > 0$ be as in Lemma 6.1. Then for all $p > 9(m-1) - \delta_1(m)$ and each $K > 0$ one can find $C(p,K) > 0$ with the property that if for some $\varepsilon \in (0,1)$,*

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le K \qquad \text{for all } \ge 0 \tag{6.10}$$

*and*

$$\int_t^{t+1} \int_\Omega \left| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right|^2 \le K \qquad \text{for all } t \ge 0, \tag{6.11}$$

*then*

$$\int_t^{t+1} \int_\Omega |\nabla c_\varepsilon|^{\frac{2(5p+3m-3)}{3}} \le C(p,K) \qquad \text{for all } t \ge 0. \tag{6.12}$$

**Proof.** We abbreviate $q := \frac{5p+3m-3}{3}$ and apply a standard result on maximal Sobolev regularity in scalar parabolic equations ([17]) to find $C_1 > 0$, as all subsequently appearing constants $C_2, C_3, \ldots$ possibly depending on $p$, with the property that whenever $t_\star \in \mathbb{R}$, $z \in C^{2,1}(\overline{\Omega} \times [t_\star, t_\star + 2])$ and $f \in C^0(\overline{\Omega} \times [t_\star, t_\star + 2])$ are such that

$$\begin{cases} z_t = \Delta z + f(x,t), & x \in \Omega,\ t \in (t_\star, t_\star + 2), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (t_\star, t_\star + 2), \\ z(x, t_\star) = 0, & x \in \Omega, \end{cases}$$

then

$$\int_{t_\star}^{t_\star+2} \|z(\cdot,t)\|_{W^{2,q}(\Omega)}^q dt \le C_1 \int_{t_\star}^{t_\star+2} \|f(\cdot,t)\|_{L^q(\Omega)}^q dt. \tag{6.13}$$

Furthermore, let us fix $C_2 > 0$ and $C_3 > 0$ such that in accordance with a well-known regularization feature of the Neumann heat semigroup ([41]) and the Gagliardo–Nirenberg inequality we have

$$\|\nabla e^{t\Delta}\varphi\|_{L^{2q}(\Omega)} \le C_2 \|\varphi\|_{W^{1,2q}(\Omega)} \qquad \text{for all } \varphi \in W^{1,2q}(\Omega) \text{ and any } t > 0 \tag{6.14}$$

as well as

$$\|\nabla\varphi\|_{L^{2q}(\Omega)}^{2q} \le C_3\|\varphi\|_{W^{2,q}(\Omega)}^{q}\|\varphi\|_{L^\infty(\Omega)}^{q} \qquad \text{for all } \varphi\in W^{2,q}(\Omega), \tag{6.15}$$

where in establishing the latter we note that $W^{2,q}(\Omega)\hookrightarrow W^{1,2q}(\Omega)\hookrightarrow L^\infty(\Omega)$ due to the fact that $q\ge\frac{5}{3}>\frac{3}{2}$.

As a final preparation, let us observe that according to Lemma 6.2 and Lemma 6.1, our assumptions (6.10) and (6.11) ensure that we can choose $C_4(K)>0$ and $C_5(K)>0$ such that

$$\int_t^{t+2}\|n_\varepsilon(\cdot,s)\|_{L^q(\Omega)}^q ds \le C_4(K) \qquad \text{for all } t\ge 0 \tag{6.16}$$

and

$$\|u_\varepsilon(\cdot,t)\|_{L^{2q}(\Omega)} \le C_5(K) \qquad \text{for all } t\ge 0, \tag{6.17}$$

the latter conclusion relying on our hypothesis on $p$.

In order to make appropriate use of these preliminaries in the present context, we pick a nondecreasing $\zeta_0\in C^\infty(\mathbb{R})$ such that $\zeta_0\equiv 0$ in $(-\infty,-1]$ and $\zeta_0\equiv 1$ in $[1,\infty)$, and for fixed $t_0\ge 0$ we let $\zeta(t)\equiv\zeta^{(t_0)}(t):=\zeta_0(t-t_0)$, $t\ge 0$, and

$$z(\cdot,t):=\zeta(t)\cdot\Big\{c_\varepsilon(\cdot,t)-e^{t\Delta}c_0\Big\}, \qquad t\ge(t_0-1)_+.$$

Then by (2.6) and the identity $\partial_t e^{t\Delta}c_0=\Delta e^{t\Delta}c_0$,

$$\begin{aligned} z_t &= \zeta(t)\cdot\Big\{\Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\Big\} - \zeta(t)\Delta e^{t\Delta}c_0 + \zeta'(t)\cdot\Big\{c_\varepsilon - e^{t\Delta}c_0\Big\} \\ &= \Delta z - \zeta(t)F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon\cdot\nabla z \\ &\quad -\zeta(t)u_\varepsilon\cdot\nabla e^{t\Delta}c_0 + \zeta'(t)c_\varepsilon - \zeta'(t)e^{t\Delta}c_0 \qquad \text{in } \Omega\times((t_0-1)_+,\infty), \end{aligned} \tag{6.18}$$

and clearly

$$\frac{\partial z}{\partial\nu}=0 \qquad \text{on } \partial\Omega\times((t_0-1)_+,\infty). \tag{6.19}$$

Moreover, at the respective initial time we have

$$z\Big(\cdot,(t_0-1)_+\Big)\equiv 0 \qquad \text{in } \Omega, \tag{6.20}$$

because if $t_0\ge 1$ then $\zeta(t_0-1)=0$ and hence

$$z\Big(\cdot,(t_0-1)_+\Big)=z(\cdot,t_0-1)=\zeta(t_0-1)\cdot\Big\{c_\varepsilon(\cdot,t_0-1)-e^{(t_0-1)\Delta}c_0\Big\}=0 \qquad \text{in } \Omega,$$

whereas if $t_0\in[0,1)$ then

$$z\Big(\cdot,(t_0-1)_+\Big)=z(\cdot,0)=\zeta(0)\cdot\Big\{c_\varepsilon(\cdot,0)-c_0\Big\}=0 \qquad \text{in } \Omega$$

by (2.6).

As a consequence of (6.18)–(6.20), we may now invoke (6.13) which along with (6.15) and (2.9) shows that abbreviating $t_\star := (t_0-1)_+$ and noting that $(\xi_1+\xi_2+\xi_3+\xi_4+\xi_5)^q \le 5^q(\xi_1^q+\xi_2^q+\xi_3^q+\xi_4^q+\xi_5^q)$ for all nonnegative $\xi_1,\xi_2,\xi_3,\xi_4$ and $\xi_5$, we have

$$\begin{aligned}
\int_{t_\star}^{t_\star+2} \|\nabla z(\cdot,t)\|_{L^{2q}(\Omega)}^{2q}dt &\le C_3\|c_0\|_{L^\infty(\Omega)}^q \int_{t_\star}^{t_\star+2} \|z(\cdot,t)\|_{W^{2,q}(\Omega)}^q dt \\
&\le 5^q C_1C_3\|c_0\|_{L^\infty(\Omega)}^q \int_{t_\star}^{t_\star+2} \Big\{ \|\zeta(t)F_\varepsilon(n_\varepsilon)c_\varepsilon\|_{L^q(\Omega)}^q + \|u_\varepsilon\cdot\nabla z\|_{L^q(\Omega)}^q \\
&\qquad + \|\zeta(t)u_\varepsilon\cdot\nabla e^{t\Delta}c_0\|_{L^q(\Omega)}^q + \|\zeta'(t)c_\varepsilon\|_{L^q(\Omega)}^q \\
&\qquad + \|\zeta'(t)e^{t\Delta}c_0\|_{L^q(\Omega)}^q \Big\} dt. \qquad (6.21)
\end{aligned}$$

Here we use that by (2.4) we have $0\le F_\varepsilon(s)\le s$ for all $s\ge 0$ and that $0\le\zeta(t)\le 1$ for all $t\in\mathbb{R}$ to see, again by means of (2.9), that

$$\begin{aligned}
\int_{t_\star}^{t_\star+2} \|\zeta(t)F_\varepsilon(n_\varepsilon)c_\varepsilon\|_{L^q(\Omega)}^q dt &\le \|c_0\|_{L^\infty(\Omega)}^q \int_{t_\star}^{t_\star+2} \|n_\varepsilon(\cdot,t)\|_{L^q(\Omega)}^q dt \\
&\le C_4(K)\|c_0\|_{L^\infty(\Omega)}^q \qquad (6.22)
\end{aligned}$$

according to (6.16), while the Cauchy–Schwarz inequality together with (6.17) and (6.14) shows that

$$\begin{aligned}
\int_{t_\star}^{t_\star+2} \|\zeta(t)u_\varepsilon\cdot\nabla e^{t\Delta}c_0\|_{L^q(\Omega)}^q dt &\le \int_{t_\star}^{t_\star+2} \|u_\varepsilon(\cdot,t)\|_{L^{2q}(\Omega)}^q \|\nabla e^{t\Delta}c_0\|_{L^{2q}(\Omega)}^q dt \\
&\le 2C_2^qC_5^q(K)\|c_0\|_{W^{1,2q}(\Omega)}^q. \qquad (6.23)
\end{aligned}$$

Next, by (2.9) and the contractivity of the semigroup $(e^{t\Delta})_{t\ge 0}$ on $L^q(\Omega)$, writing $C_6 := \|\zeta_0'\|_{L^\infty(\mathbb{R})}$ we obtain

$$\int_{t_\star}^{t_\star+2} \|\zeta'(t)c_\varepsilon\|_{L^q(\Omega)}^q dt \le 2C_6^q|\Omega|\cdot\|c_0\|_{L^\infty(\Omega)}^q \qquad (6.24)$$

and

$$\int_{t_\star}^{t_\star+2} \|\zeta'(t)e^{t\Delta}c_0\|_{L^q(\Omega)}^q dt \le 2C_6^q\|c_0\|_{L^q(\Omega)}^q, \qquad (6.25)$$

so that it remains to estimate the corresponding integral associated with the second summand in brackets on the right of (6.21). For this purpose, after employing the Cauchy–Schwarz inequality we additionally make use of Young's inequality to see, again by means of (6.17), that

$$\begin{aligned}
5^q C_1 C_3 \|c_0\|^q_{L^\infty(\Omega)} \int_{t_\star}^{t_\star+2} \|u_\varepsilon \cdot \nabla z\|^q_{L^q(\Omega)} dt &\le 5^q C_1 C_3 \|c_0\|^q_{L^\infty(\Omega)} \int_{t_\star}^{t_\star+2} \|u_\varepsilon\|^q_{L^{2q}(\Omega)} \|\nabla z\|^q_{L^{2q}(\Omega)} dt \\
&\le \frac{1}{2} \int_{t_\star}^{t_\star+2} \|\nabla z\|^{2q}_{L^{2q}(\Omega)} dt \\
&\quad + \frac{25^q C_1^2 C_3^2 \|c_0\|^{2q}_{L^\infty(\Omega)}}{2} \int_{t_\star}^{t_\star+2} \|u_\varepsilon\|^{2q}_{L^{2q}(\Omega)} dt \\
&\le \frac{1}{2} \int_{t_\star}^{t_\star+2} \|\nabla z\|^{2q}_{L^{2q}(\Omega)} dt + 25^q C_1^2 C_3^2 C_5^{2q}(K) \|c_0\|^{2q}_{L^\infty(\Omega)}.
\end{aligned}$$

In conjunction with (6.22)–(6.25), this shows that (6.21) leads to the inequality

$$\begin{aligned}
\frac{1}{2} \int_{t_\star}^{t_\star+2} \|\nabla z(\cdot,t)\|^{2q}_{L^{2q}(\Omega)} dt \le C_7(K) &:= 25^q C_1^2 C_3^2 C_5^{2q}(K) \|c_0\|^{2q}_{L^\infty(\Omega)} \\
&+ 5^q C_1 C_3 \|c_0\|^q_{L^\infty(\Omega)} \cdot \Big\{ C_4(K) \|c_0\|^q_{L^\infty(\Omega)} + 2C_2^q C_5^q(K) \|c_0\|^q_{W^{1,2q}(\Omega)} \\
&\qquad + 2C_6^q \|c_0\|^q_{L^\infty(\Omega)} + 2C_6^q \|c_0\|^q_{L^q(\Omega)} \Big\},
\end{aligned}$$

so that since $(t_0, t_0+1) \subset \Big((t_0-1)_+, (t_0-1)_+ + 2\Big)$ and thus $\zeta \equiv 1$ in $(t_0, t_0+1)$, in particular we infer that

$$\int_{t_0}^{t_0+1} \Big\| \nabla c_\varepsilon(\cdot,t) - \nabla e^{t\Delta} c_0 \Big\|^{2q}_{L^{2q}(\Omega)} dt \le 2C_7(K) \qquad \text{for all } t_0 \ge 0.$$

Once more recalling (6.14) and using that $(\xi+\eta)^{2q} \le 2^{2q-1}(\xi^{2q} + \eta^{2q})$ for all $\xi \ge 0$ and $\eta \ge 0$, we therefore obtain that

$$\begin{aligned}
\int_{t_0}^{t_0+1} \|\nabla c_\varepsilon(\cdot,t)\|^{2q}_{L^{2q}(\Omega)} dt &\le 2^{2q-1} \int_{t_0}^{t_0+1} \Big\| \nabla c_\varepsilon(\cdot,t) - \nabla e^{t\Delta} c_0 \Big\|^{2q}_{L^{2q}(\Omega)} dt \\
&\quad + 2^{2q-1} \int_{t_0}^{t_0+1} \|\nabla e^{t\Delta} c_0\|^{2q}_{L^{2q}(\Omega)} dt \\
&\le 2^{2q} C_7(K) + 2^{2q-1} C_2^{2q} \|c_0\|^{2q}_{W^{1,2q}(\Omega)} \qquad \text{for all } t \ge 0,
\end{aligned}$$

which in view of our definition of $q$ precisely yields (6.12). □

## 7. Arbitrary $L^p$ bounds for $n_\varepsilon$ by a second iteration

Now in light of Lemma 6.3, our general regularity statement from Lemma 4.1 can readily be developed to the following basis for a second iterative reasoning.

**Lemma 7.1.** *Let $m > \frac{215}{192}$ and $p_\star > 9(m-1) - \delta_1(m)$ with $\delta_1(m) > 0$ taken from Lemma 6.1. Then for all $p > 1$ fulfilling*

$$p \le \frac{10p_\star^2 + (36m-42)p_\star + (m-1)(18m-27)}{9}, \tag{7.1}$$

*and any choice of $K > 0$ one can pick $C(p,K) > 0$ such that if for some $\varepsilon \in (0,1)$ we have*

$$\int_\Omega n_\varepsilon^{p_\star}(\cdot,t) \le K \qquad \text{for all } t \ge 0 \tag{7.2}$$

*and*

$$\int_t^{t+1}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p_\star+m-1}{2}}\right|^2 \le K \qquad \text{for all } t \ge 0, \tag{7.3}$$

*then*

$$\int_\Omega n_\varepsilon^{p}(\cdot,t) \le C(p,K) \qquad \text{for all } t \ge 0 \tag{7.4}$$

*and*

$$\int_t^{t+1}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le C(p,K) \qquad \text{for all } t \ge 0. \tag{7.5}$$

**Proof.** Since $p_\star > 9(m-1) - \delta_1(m)$, we may invoke Lemma 6.3 to see that writing $q := \frac{5p_\star+3m-3}{3}$ we can find $C_1(K) > 0$ such that

$$\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon|^{2q} \le C_1(K) \qquad \text{for all } t \ge 0. \tag{7.6}$$

Now observing that in our situation the right-hand side of (4.2) can be rewritten according to

$$\begin{aligned}\frac{2(q-1)}{3}p_\star + (2q-1)(m-1) &= \frac{2\cdot\frac{5p_\star+3m-6}{3}}{3}\cdot p_\star + \frac{10p_\star+6m-9}{3}\cdot(m-1)\\ &= \frac{10p_\star^2 + (36m-42)p_\star + (m-1)(18m-27)}{9},\end{aligned}$$

given any $p > 1$ fulfilling (7.1) we may apply Lemma 4.1 to infer that due to (7.2) and (7.6) both inequalities in (7.4) and (7.5) hold if we fix $C(p, K) > 0$ suitably large. $\square$

With regard to the question how far the above lemma through its condition (7.1) indeed allows for an improvement in knowledge, let us briefly prove the following elementary observations which highlight the role of the restriction $m > \frac{9}{8}$ made in Theorem 1.1.

**Lemma 7.2.** *For $m > 1$, let*

$$\psi(p) := \frac{10p^2 + (36m - 42)p + (m-1)(18m - 27)}{9}, \qquad p \in \mathbb{R}. \tag{7.7}$$

*Then*

$$\psi\Big(9(m-1)\Big) > 9(m-1) \quad \textit{if and only if} \quad m > \frac{9}{8}, \tag{7.8}$$

*and there exist $\delta_2(m) > 0$ and $\Gamma > 1$ such that*

$$\psi(p) \ge \Gamma p \qquad \textit{for all } p > 9(m-1) - \delta_2(m). \tag{7.9}$$

**Proof.** Computing

$$\begin{aligned}
\frac{\psi(9(m-1)) - 9(m-1)}{m-1} &= \frac{810(m-1)^2 + 9(36m-42)(m-1) + (m-1)(18m-27)}{9(m-1)} - 9 \\
&= 16(8m - 9) \\
&> 0,
\end{aligned}$$

we directly obtain (7.8). To verify (7.9), we let

$$\widetilde{\psi}(p) := \frac{\psi(p)}{p} \qquad \text{for } p > 0,$$

so that since (7.8) asserts that $C_1 := \widetilde{\psi}(9(m-1)) - 1$ is positive, by continuity we can pick $\delta_2 = \delta_2(m) > 0$ such that $9(m-1) - \delta_2 > 0$ and

$$\widetilde{\psi}(p) \ge \Gamma := 1 + \frac{C_1}{2} \qquad \text{for all } p \in \Big(9(m-1) - \delta_2, 9(m-1)\Big]. \tag{7.10}$$

As

$$\widetilde{\psi}'(p) = \frac{10}{9} - \frac{(m-1)(2m-3)}{p^2} \qquad \text{for all } p > 0, \tag{7.11}$$

it thus immediately follows that if $m \le \frac{3}{2}$ then $\widetilde{\psi}' \ge \frac{10}{9} > 0$ throughout $(0, \infty)$. If $m > \frac{3}{2}$, then for $p \ge 9(m-1)$ we can use (7.11) to estimate

$$\widetilde{\psi}'(p) \ge \frac{10}{9} - \frac{(m-1)(2m-3)}{81(m-1)^2} = \frac{88m-87}{81(m-1)} > 0,$$

because $m > 1$. In both cases, we thus obtain that $\widetilde{\psi}' > 0$ on $[9(m-1), \infty)$ and hence $\widetilde{\psi} \ge \Gamma$ on $(9(m-1) - \delta_2, \infty)$ by (7.10). □

We are thereby prepared for our second recursive argument, with its outcome being as follows.

**Lemma 7.3.** *Let $m > \frac{9}{8}$. Then for all $p > 1$ there exists $C(p) > 0$ such that*

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le C(p) \qquad \text{for all } t \ge 0 \tag{7.12}$$

*and*

$$\int_t^{t+1}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le C(p) \qquad \text{for all } t \ge 0. \tag{7.13}$$

**Proof.** As $m > \frac{9}{8} > \frac{215}{192}$, taking $\delta_1(m) > 0$ and $\delta_2(m) > 0$ as given by Lemma 6.1 and Lemma 7.2, respectively, we may pick $p_0 \in (1, 9(m-1))$ such that

$$p_0 > 9(m-1) - \min\{\delta_1(m), \delta_2(m)\}, \tag{7.14}$$

and thereupon recursively define

$$p_k := \psi(p_{k-1}), \qquad k \in \mathbb{N} = \{1, 2, 3, ...\}, \tag{7.15}$$

with $\psi : \mathbb{R} \to \mathbb{R}$ taken from Lemma 7.2. Then since $p_0 > 9(m-1) - \delta_2(m)$ by (7.14), according to (7.8) an inductive argument shows that

$$p_k \ge \Gamma^k p_0 \qquad \text{for all } k \in \mathbb{N} \tag{7.16}$$

with $\Gamma > 1$ as provided by Lemma 7.2, whence in particular $p_k \to \infty$ as $k \to \infty$. Now due to the boundedness of $\Omega$, in order to verify the lemma it is sufficient to show that for all $k \ge 0$ there exists $C_1(k) > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\int_\Omega n_\varepsilon^{p_k}(\cdot,t) \le C_1(k) \quad \text{and} \quad \int_t^{t+1}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p_k+m-1}{2}}\right|^2 \le C_1(k) \qquad \text{for all } t \ge 0, \tag{7.17}$$

which will again result from an iterative reasoning: Namely, for $k = 0$ the claimed inequality is a direct consequence of Lemma 5.1, because $m > \frac{9}{8} > \frac{10}{9}$ and $p_0 \in (1, 9(m-1))$. If (7.17) holds for some $k_0 \ge 0$ and some $C_1(k_0) > 0$, however, then since (7.16) and (7.14) warrant that $p_k \ge p_0 > 9(m-1) - \delta_1(m)$, and again since $m > \frac{215}{192}$, Lemma 7.1 provides $C_2 > 0$ such that

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le C_2 \quad \text{and} \quad \int_t^{t+1}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le C_2 \qquad \text{for all } t \ge 0$$

with

$$p := \frac{10p_{k_0}^2 + (36m-42)p_{k_0} + (m-1)(18m-27)}{9}.$$

As thus $p = \psi(p_{k_0}) = p_{k_0+1}$ by (7.15), this asserts (7.17) also for $k = k_0 + 1$ and thereby completes the proof. □

## 8. Further regularity properties

With Lemma 7.3 at hand, further regularity properties can now be obtained by essentially straightforward arguments: We firstly recall Lemma 6.1 and a standard regularization feature of the heat semigroup to obtain the following.

**Lemma 8.1.** *Assume that $m > \frac{9}{8}$, and let $p > 1$. Then there exists $C(p) > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^p \le C(p) \qquad \textit{for all } t \ge 0 \tag{8.1}$$

*and*

$$\int_\Omega |u_\varepsilon(\cdot,t)|^p \le C(p) \qquad \textit{for all } t \ge 0. \tag{8.2}$$

**Proof.** In view of Lemma 7.3, (8.2) is an evident consequence of Lemma 6.1. Thereafter, (8.1) can be derived from (8.2) and again Lemma 7.3 by well-known results on gradient regularity in semilinear heat equations ([21]). □

By means of a Moser iteration, the latter together with Lemma 7.3 entails an $\varepsilon$-independent $L^\infty$ bound for $n_\varepsilon$.

**Lemma 8.2.** *If $m > \frac{9}{8}$, then there exists $C > 0$ such that for arbitrary $\varepsilon \in (0,1)$ we have*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \ge 0. \tag{8.3}$$

**Proof.** In view of Lemma 8.1 and Lemma 7.3 when applied to suitably large $p > 1$, this directly follows from a Moser-type iterative procedure (see [31, Lemma A.1] for a version precisely covering the present case). □

Again by means of maximal Sobolev regularity properties combined with an appropriate embedding result, the estimates collected above imply Hölder bounds for $c_\varepsilon$, $u_\varepsilon$ and $\nabla c_\varepsilon$. This will be achieved in Lemma 8.4 on the basis of the following lemma in which any influence of the respective initial data is faded out.

**Lemma 8.3.** *Let $m > \frac{9}{8}$. Then there exist $\theta \in (0,1)$ and $C > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\|c_\varepsilon - \widehat{c}\|_{C^{1+\theta,\theta}(\overline{\Omega}\times[t,t+1])} \le C \qquad \textit{for all } t \ge 0 \tag{8.4}$$

*and*

$$\|u_\varepsilon - \widehat{u}\|_{C^{1+\theta,\theta}(\overline{\Omega}\times[t,t+1])} \le C \qquad \textit{for all } t \ge 0, \tag{8.5}$$

*where*

$$\widehat{c}(\cdot,t) := e^{t\Delta}c_0 \quad \textit{and} \quad \widehat{u}(\cdot,t) := e^{-tA}u_0 \qquad \textit{for } t \ge 0. \tag{8.6}$$

**Proof.** Since $\widehat{c}_t = \Delta\widehat{c}$, it follows from (2.6) that

$$\partial_t(c_\varepsilon - \widehat{c}) = \Delta(c_\varepsilon - \widehat{c}) - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon, \qquad x\in\Omega,\ t>0,$$

where given $p > 1$ we may invoke Lemma 8.1, Lemma 8.2 and (2.9) and recall (2.4) to find $C_1 > 0$ fulfilling

$$\int_t^{t+2}\int_\Omega \Big| - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\Big|^p \le C_1 \qquad \text{for all } t\ge 0 \text{ and } \varepsilon\in(0,1).$$

Therefore, by means of maximal Sobolev regularity estimates along with an appropriate time localization in the style of the argument from Lemma 6.3, we infer the existence of $C_2 > 0$ such that

$$\int_t^{t+2} \Big\{ \|c_\varepsilon(\cdot,s) - \widehat{c}(\cdot,s)\|_{W^{2,p}(\Omega)}^p + \|\partial_t(c_\varepsilon(\cdot,s) - \widehat{c}(\cdot,s))\|_{L^p(\Omega)}^p \Big\} ds \le C_2$$
$$\text{for all } t\ge 0 \text{ and } \varepsilon\in(0,1).$$

In view of a known embedding property ([1]), an application thereof to suitably large $p > 1$ establishes (8.4).

Likewise, using that

$$\partial_t(u_\varepsilon - \widehat{u}) = -A(u_\varepsilon - \widehat{u}) + \mathcal{P}[n_\varepsilon\nabla\phi], \qquad x\in\Omega,\ t>0,$$

and that herein for $p > 1$ we can use the boundedness of $\mathcal{P}$ on $L^p(\Omega;\mathbb{R}^3)$ ([14]) together with Lemma 8.2 to find $C_3 > 0$ such that

$$\int_t^{t+2}\int_\Omega \Big|\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big|^p \le C_3 \qquad \text{for all } t\ge 0 \text{ and } \varepsilon\in(0,1),$$

we obtain (8.5) from corresponding maximal Sobolev regularity estimates for the Stokes evolution equation ([17]). $\square$

Indeed, the latter inter alia implies the following Hölder estimates, which with regard to the gradient bound in (8.9) must remain local in time due to possibly lacking appropriate regularity and compatibility properties of $c_0$.

**Lemma 8.4.** *Let $m > \frac{9}{8}$. Then there exists $\theta \in (0,1)$ with the property that one can find $C > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\|c_\varepsilon\|_{C^\theta(\overline{\Omega}\times[t,t+1])} \le C \qquad \text{for all } t \ge 0 \tag{8.7}$$

*and*

$$\|u_\varepsilon\|_{C^\theta(\overline{\Omega}\times[t,t+1])} \le C \qquad \text{for all } t \ge 0, \tag{8.8}$$

*and that for all $\tau > 0$ it is possible to choose $C(\tau) > 0$ fulfilling*

$$\|\nabla c_\varepsilon\|_{C^\theta(\overline{\Omega}\times[t,t+1])} \le C(\tau) \qquad \text{for all } t \ge \tau \tag{8.9}$$

*whenever $\varepsilon \in (0,1)$.*

**Proof.** We take $\widehat{c}$ and $\widehat{u}$ from (8.6) and note that since $c_0 \in W^{1,\infty}(\Omega) \hookrightarrow \bigcap_{\theta\in(0,1)} C^\theta(\overline{\Omega})$ and $u_0 \in D(A^\alpha) \hookrightarrow \bigcap_{\theta\in(0,2\alpha-\frac{3}{2})} C^\theta(\overline{\Omega})$ ([15], [18]), known smoothing properties of the heat equation and the Stokes evolution system ensure that there exist $\theta_1 \in (0,1)$, $\theta_2 \in (0,1)$, $C_1 > 0$ and $C_2 > 0$ such that

$$\|\widehat{c}\|_{C^{\theta_1}(\overline{\Omega}\times[t,t+1])} \le C_1 \qquad \text{for all } t \ge 0$$

and

$$\|\widehat{u}\|_{C^{\theta_2}(\overline{\Omega}\times[t,t+1])} \le C_2 \qquad \text{for all } t \ge 0,$$

and that for all $\tau > 0$ we can find $C_3(\tau) > 0$ such that

$$\|\nabla\widehat{c}\|_{C^1(\overline{\Omega}\times[t,t+1])} \le C_3(\tau) \qquad \text{for all } t \ge \tau.$$

Therefore, (8.7)–(8.9) result from Lemma 8.3. □

For strongly degenerate cell diffusion present when e.g. $D(s) = s^{m-1}$, $s \ge 0$, with large values of $m$, we do not know whether $n_\varepsilon$ enjoys equicontinuity properties in the classical pointwise sense, which may indeed suffer from a possible dominance of the transport terms in the first equation of (2.6) at small densities. In order to nevertheless provide some compactness and equicontinuity properties of this solution component, let us finally derive two statements on time regularity of $n_\varepsilon$ in a straightforward manner.

**Lemma 8.5.** *Suppose that $m > \frac{9}{8}$, and let $T > 0$. Then there exists $C(T) > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\int_0^T \left\| \partial_t n_\varepsilon^m(\cdot,t) \right\|_{(W_0^{1,\infty}(\Omega))^\star} dt \le C(T) \tag{8.10}$$

*and*

$$\|n_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,2}(\Omega))^\star} \le C(T) \qquad \textit{for all } t \in (0,T). \tag{8.11}$$

**Proof.** We fix $t \in (0,T)$ and $\zeta \in C_0^\infty(\Omega)$ such that $\|\zeta\|_{W^{1,\infty}(\Omega)} \le 1$, and then obtain from the first equation in (2.6) by straightforward manipulations that writing $C_1 := \sup_{\varepsilon\in(0,1)} \|n_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}$ and $C_2 := \|D\|_{L^\infty((0,c_1))} + 2$, according to (1.5) we have

$$\begin{aligned}
&\left| \frac{1}{m} \int_\Omega \partial_t n_\varepsilon^m(\cdot,t)\zeta \right| \\
&\quad = \left| \int_\Omega n_\varepsilon^{m-1} \nabla\cdot \Big\{ D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon - n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon - n_\varepsilon u_\varepsilon \Big\} \zeta \right| \\
&\quad = \bigg| -(m-1)\int_\Omega n_\varepsilon^{m-2} D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2\zeta - \int_\Omega n_\varepsilon^{m-1} D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla\zeta \\
&\qquad\quad +(m-1)\int_\Omega n_\varepsilon^{m-1} F_\varepsilon'(n_\varepsilon)(\nabla n_\varepsilon\cdot\nabla c_\varepsilon)\zeta + \int_\Omega n_\varepsilon^m F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon\cdot\nabla\zeta + \frac{1}{m}\int_\Omega n_\varepsilon^m u_\varepsilon\cdot\nabla\zeta \bigg| \\
&\quad \le (m-1)C_2 \int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + (m-1)C_2\int_\Omega n_\varepsilon^{m-1}|\nabla n_\varepsilon| \\
&\quad +(m-1)\int_\Omega n_\varepsilon^{m-1}|\nabla n_\varepsilon|\cdot|\nabla c_\varepsilon| + \int_\Omega n_\varepsilon^m|\nabla c_\varepsilon| + \frac{1}{m}\int_\Omega n_\varepsilon^m|u_\varepsilon| \\
&\quad \le (m-1)C_2\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + \frac{m-1}{2}C_2\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + \frac{m-1}{2}C_2\int_\Omega n_\varepsilon^m \\
&\quad +\frac{m-1}{2}\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + \frac{m-1}{2}\int_\Omega n_\varepsilon^m|\nabla c_\varepsilon|^2 \\
&\quad +\int_\Omega n_\varepsilon^m|\nabla c_\varepsilon| + \frac{1}{m}\int_\Omega n_\varepsilon^m|u_\varepsilon| \\
&\quad \le (m-1)\Big(\frac{3C_2}{2}+\frac{1}{2}\Big)\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + \frac{(m-1)C_1^m C_2|\Omega|}{2} + \frac{(m-1)C_1^m}{2}\int_\Omega|\nabla c_\varepsilon|^2 \\
&\quad +C_1^m\int_\Omega|\nabla c_\varepsilon| + \frac{C_1^m}{m}\int_\Omega|u_\varepsilon| \qquad \text{for all } \varepsilon\in(0,1).
\end{aligned}$$

In view of the estimates provided by Lemma 3.2 and Lemma 8.1, (8.10) therefore readily results upon integration.

The inequality in (8.11) can similarly be derived from Lemma 8.1 and Lemma 8.2. □

## 9. Existence of a global bounded weak solution

In the sequel, we shall refer to the following natural concept of weak solvability in (1.1), (1.6), (1.7):

**Definition 9.1.** Let

$$\begin{aligned} & n \in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \\ & c \in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \cap L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \qquad \text{and} \\ & u \in L^1_{loc}([0,\infty); W^{1,1}(\Omega;\mathbb{R}^3)), \end{aligned} \tag{9.1}$$

be such that $n \ge 0$ and $c \ge 0$ in $\Omega \times (0,T)$ and

$$D_0(n), \ n|\nabla c| \text{ and } n|u| \ \text{ belong to } L^1_{loc}(\overline{\Omega} \times [0,\infty)), \tag{9.2}$$

where $D_0(s) := \int_0^s D(\sigma)d\sigma$ for $s \ge 0$. Then $(n,c,u)$ will be called a *global weak solution* of (1.1), (1.6), (1.7) if $\nabla \cdot u = 0$ in the distributional sense, if

$$-\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega D_0(n)\Delta\varphi + \int_0^\infty \int_\Omega n\nabla c \cdot \nabla\varphi + \int_0^\infty \int_\Omega nu \cdot \nabla\varphi \tag{9.3}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$, if

$$-\int_0^\infty \int_\Omega c\varphi_t - \int_\Omega c_0 \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi - \int_0^\infty \int_\Omega nc\varphi + \int_0^\infty \int_\Omega cu \cdot \nabla\varphi \tag{9.4}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, and if moreover

$$-\int_0^\infty \int_\Omega u \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega n\nabla\phi \cdot \varphi \tag{9.5}$$

for all $\varphi \in C_0^\infty(\Omega \times [0,\infty);\mathbb{R}^3)$ such that $\nabla \cdot \varphi \equiv 0$ in $\Omega \times (0,\infty)$.

In this context, a series of standard extraction procedures on the basis of our estimates collected above indeed yields global solvability.

**Lemma 9.1.** *Let $m > \frac{9}{8}$. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$, a null set $N \subset (0,\infty)$ and a triple $(n,c,u)$ of functions $n : \Omega \times (0,\infty) \to [0,\infty)$, $c : \Omega \times (0,\infty) \to [0,\infty)$ and $u : \Omega \times (0,\infty) \to \mathbb{R}^3$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and*

$$n_\varepsilon(\cdot,t) \to n(\cdot,t) \qquad \text{a.e. in } \Omega \text{ for all } t \in (0,\infty) \setminus N, \tag{9.6}$$

$$n_\varepsilon \overset{\star}{\rightharpoonup} n \qquad \text{in } L^\infty(\Omega \times (0,\infty)), \tag{9.7}$$

$$n_\varepsilon \to n \qquad \text{in } C^0_{loc}([0,\infty); (W_0^{2,2}(\Omega))^\star), \tag{9.8}$$

$$c_\varepsilon \to c \qquad \text{in } C^0_{loc}(\overline{\Omega} \times [0,\infty)), \tag{9.9}$$

$$c_\varepsilon \overset{\star}{\rightharpoonup} c \qquad \text{in } L^\infty((0,\infty); W^{1,p}(\Omega)) \qquad \text{for all } p \in (1,\infty), \tag{9.10}$$

$$\nabla c_\varepsilon \to \nabla c \qquad \text{in } C^0_{loc}(\overline{\Omega} \times [0,\infty)), \tag{9.11}$$

$$u_\varepsilon \to u \qquad \text{in } C^0_{loc}(\overline{\Omega} \times [0,\infty)), \tag{9.12}$$

$$u_\varepsilon \overset{\star}{\rightharpoonup} u \qquad \text{in } L^\infty(\Omega \times (0,\infty)) \qquad \text{and} \tag{9.13}$$

$$\nabla u_\varepsilon \to \nabla u \qquad \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \tag{9.14}$$

*as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, $(n,c,u)$ forms a global weak solution of (1.1), (1.6), (1.7) in the sense of Definition 9.1, and we have*

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \qquad \text{for all } t \in (0,\infty) \setminus N. \tag{9.15}$$

**Proof.** Since Lemma 3.2, Lemma 8.2 and Lemma 8.5 guarantee that $(n_\varepsilon^m)_{\varepsilon\in(0,1)}$ is bounded in $L^2_{loc}([0,\infty; W^{1,2}(\Omega))$ and that $(\partial_t n_\varepsilon^m)_{\varepsilon\in(0,1)}$ is bounded in $L^2_{loc}([0,\infty); (W_0^{3,2}(\Omega))^\star)$ due to the continuity of the embedding $W_0^{3,2}(\Omega) \hookrightarrow W_0^{1,\infty}(\Omega)$, an Aubin–Lions lemma ([34]) yields $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and that $n_\varepsilon^m \to n^m$ holds a.e. in $\Omega \times (0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$ with some nonnegative function $n$ defined on $\Omega \times (0,\infty)$, whence using the Fubini–Tonelli theorem we readily obtain (9.6). In view of Lemma 8.2, Lemma 3.2 and (8.11), on further extraction we may also achieve (9.7) and (9.8), whereas the bounds provided by Lemma 3.2, Lemma 8.1 and Lemma 8.4 ensure that we can moreover easily achieve (9.9)–(9.14) upon two applications of the Arzelà–Ascoli theorem.

The regularity properties in (9.1) and (9.2) as well as the claimed solenoidality of $u$ are evident from (9.6)–(9.14), while the verification of (9.3), (9.4) and (9.5) is thereafter straightforward. □

## 10. Large time behavior

### *10.1. Basic decay information*

Next addressing the large time asymptotics of our solutions, as in several previous studies on qualitative behavior in related chemotaxis-fluid systems with signal absorption ([44], [24], [47], [45]) we shall rely on the following elementary information indicating a certain decay of the quantities $nc$ and $\nabla c$. Here and throughout the sequel, without further mentioning we shall assume that $m > \frac{9}{8}$ and that $(n,c,u)$ denotes the global weak solution constructed in Lemma 9.1.

**Lemma 10.1.** *There exist* $\varepsilon_\star \in (0,1)$ *and* $C > 0$ *such that*

$$\int_0^\infty \int_\Omega n_\varepsilon c_\varepsilon \le C \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star) \tag{10.1}$$

*and*

$$\int_0^\infty \int_\Omega |\nabla c_\varepsilon|^2 \le C \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star). \tag{10.2}$$

**Proof.** Using Lemma 8.2, we can fix $C_1 > 0$ such that $n_\varepsilon \le C_1$ in $\Omega \times (0,\infty)$ for all $\varepsilon \in (0,1)$, and let $\varepsilon_\star \in (0,1)$ be small enough such that $\frac{1}{\varepsilon_\star} \ge C_1$. Then (2.3) implies that $F_\varepsilon(n_\varepsilon) \equiv n_\varepsilon$ throughout $\Omega \times (0,\infty)$ whenever $\varepsilon \in (0,\varepsilon_\star)$, whence integrating the second equation in (2.6) we obtain

$$\int_\Omega c_\varepsilon(\cdot,t) + \int_0^t \int_\Omega n_\varepsilon c_\varepsilon = \int_\Omega c_0 \qquad \text{for all } \varepsilon \in (0,\varepsilon_\star) \text{ and each } t > 0,$$

from which (10.1) follows. Moreover, testing the same equation by $c_\varepsilon$ and recalling (2.4) yields

$$\frac{1}{2}\int_\Omega c_\varepsilon^2(\cdot,t) + \int_0^t \int_\Omega |\nabla c_\varepsilon|^2 = \frac{1}{2}\int_\Omega c_0^2 - \int_0^t \int_\Omega F_\varepsilon(n_\varepsilon) c_\varepsilon^2 \le \frac{1}{2}\int_\Omega c_0^2 \qquad \text{for all } \varepsilon \in (0,1) \text{ and } t > 0$$

and thereby verifies (10.2). □

### *10.2. Decay of c*

A first application of Lemma 10.1 shows that thanks to the uniform Hölder estimates from Lemma 8.4 the second solution component indeed decays in the sense claimed in Theorem 1.1.

**Lemma 10.2.** *We have*

$$c(\cdot,t) \to 0 \quad \text{in } W^{1,\infty}(\Omega) \qquad \text{as } t \to \infty. \tag{10.3}$$

**Proof.** Following a variant of an approach pursued in [44], we first use (9.15) and the Poincaré inequality to see that for all $\varepsilon \in (0,1)$,

$$\begin{aligned} \overline{n_0} \cdot \int_\Omega c_\varepsilon &= \int_\Omega n_\varepsilon \overline{c_\varepsilon} \\ &= \int_\Omega n_\varepsilon c_\varepsilon - \int_\Omega n_\varepsilon (c_\varepsilon - \overline{c_\varepsilon}) \end{aligned}$$

$$\le \int_\Omega n_\varepsilon c_\varepsilon + \sqrt{|\Omega|} C_1 \Big\{ \int_\Omega (c_\varepsilon - \overline{c_\varepsilon})^2 \Big\}^{\frac{1}{2}}$$

$$\le \int_\Omega n_\varepsilon c_\varepsilon + C_2 \Big\{ \int_\Omega |\nabla c_\varepsilon|^2 \Big\}^{\frac{1}{2}} \qquad \text{for all } t > 0$$

with $C_1 := \sup_{\varepsilon \in (0,1)} \|n_\varepsilon\|_{L^\infty(\Omega \times (0,\infty))} < \infty$ by Lemma 8.2, and with some $C_2 > 0$. Thus, by (2.9),

$$\overline{n_0}^2 \cdot \Big\{ \int_\Omega c_\varepsilon \Big\}^2 \le 2 \Big\{ \int_\Omega n_\varepsilon c_\varepsilon \Big\}^2 + 2C_2^2 \int_\Omega |\nabla c_\varepsilon|^2$$

$$\le 2C_1 \|c_0\|_{L^\infty(\Omega)} \int_\Omega n_\varepsilon c_\varepsilon + 2C_2^2 \int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t > 0,$$

so that according to Lemma 10.1 we infer that with some $\varepsilon_\star \in (0,1)$ and $C_3 > 0$ we have

$$\int_0^\infty \|c_\varepsilon(\cdot,t)\|_{L^1(\Omega)}^2 dt \le C_3 \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star)$$

and hence

$$\int_0^\infty \|c(\cdot,t)\|_{L^1(\Omega)}^2 dt \le C_3$$

thanks to Lemma 9.1 and Fatou's lemma. Since the spatio-temporal Hölder continuity property expressed by (8.7) warrants that $0 \le t \mapsto \|c(\cdot,t)\|_{L^1(\Omega)}$ is uniformly continuous, through a standard argument this entails that necessarily

$$c(\cdot,t) \to 0 \quad \text{in } L^1(\Omega) \qquad \text{as } t \to \infty. \tag{10.4}$$

Since Lemma 8.4 moreover guarantees that with some $\theta \in (0,1)$ and $C_4 > 0$ we have

$$\|c(\cdot,t)\|_{C^{1+\theta}(\overline{\Omega})} \le C_4 \qquad \text{for all } t > 1, \tag{10.5}$$

a straightforward reasoning based on interpolation and the compactness of the first among the continuous embeddings $C^{1+\theta}(\overline{\Omega}) \hookrightarrow W^{1,\infty}(\Omega) \hookrightarrow L^1(\Omega)$ shows that (10.4) and (10.5) entail (10.3): In fact, given $\eta > 0$ we may employ an Ehrling-type lemma to pick $C_5 > 0$ fulfilling

$$\|\varphi\|_{W^{1,\infty}(\Omega)} \le \frac{\eta}{2C_4} \|\varphi\|_{C^{1+\theta}(\overline{\Omega})} + C_5 \|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in C^{1+\theta}(\overline{\Omega}), \tag{10.6}$$

and then use (10.4) to choose $t_0 > 1$ satisfying

$$\|c(\cdot,t)\|_{L^1(\Omega)} \le \frac{\eta}{2C_5} \qquad \text{for all } t > t_0.$$

Then by (10.6) and (10.5),

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le \frac{\eta}{2C_4} \cdot C_4 + C_5 \cdot \frac{\eta}{2C_5} = \eta \qquad \text{for all } t > t_0,$$

as desired. □

*10.3. Stabilization of n*

Next concerned with the large time behavior of $n$, in order to circumvent obstacles stemming from possibly strong degeneracies of diffusion when $m$ is large, we rely on another quasi-energy structure in deriving the following result which can be viewed as asserting a certain short-time conservation of smallness of the quantity $\int_\Omega (n_\varepsilon - \overline{n_0})^2$, and which, remarkably, beyond the above properties and in particular (10.2) does not explicitly require the presence of any diffusion mechanism in the first equation in (2.6).

**Lemma 10.3.** *There exists $C > 0$ such that for each $\varepsilon \in (0,1)$ and any choice of $t_\star \ge 0$ we have*

$$\int_\Omega \Big(n_\varepsilon(\cdot,t) - \overline{n_0}\Big)^2 \le C \cdot \bigg\{ \int_\Omega \Big(n_\varepsilon(\cdot,t_\star) - \overline{n_0}\Big)^2 + \int_\Omega |\nabla c_\varepsilon(\cdot,t_\star)|^2 + \sup_{s\in(t_\star,t_\star+1)} \|c_\varepsilon(\cdot,s)\|^2_{L^2(\Omega)} \bigg\}$$
$$\textit{for all } t \in (t_\star, t_\star+1). \qquad (10.7)$$

**Proof.** We start by multiplying the first equation in (2.6) by $n_\varepsilon - \overline{n_0}$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (n_\varepsilon - \overline{n_0})^2 &= -\int_\Omega D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 + \int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon \\ &\le \int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon \qquad \text{for all } t > 0. \end{aligned} \qquad (10.8)$$

Here in order to appropriately estimate the right-hand side, we introduce

$$G_\varepsilon(s) := \int_0^s \sigma F'_\varepsilon(\sigma)d\sigma, \qquad s \ge 0,$$

and once more integrate by parts to rewrite

$$\begin{aligned} \int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon &= \int_\Omega \nabla G_\varepsilon(n_\varepsilon)\cdot\nabla c_\varepsilon \\ &= -\int_\Omega G_\varepsilon(n_\varepsilon)\Delta c_\varepsilon \\ &= -\int_\Omega \Big(G_\varepsilon(n_\varepsilon) - G_\varepsilon(\overline{n_0})\Big)\cdot \Delta c_\varepsilon \qquad \text{for all } t > 0, \end{aligned} \qquad (10.9)$$

because $\int_\Omega \Delta c_\varepsilon(\cdot,t) = 0$ for all $t > 0$. Now since we know from Lemma 8.2 that with some $C_1 > 0$ we have

$$n_\varepsilon \le C_1 \quad \text{in } \Omega \times (0,\infty) \qquad \text{for all } \varepsilon \in (0,1), \tag{10.10}$$

and since $0 \le G_\varepsilon'(s) \le s$ thanks to (2.4), by the mean value theorem we can estimate

$$\begin{aligned}
\Big| G_\varepsilon(n_\varepsilon(x,t)) - G_\varepsilon(\overline{n_0}) \Big| &\le \|G_\varepsilon'\|_{L^\infty((0,C_1))} |n_\varepsilon(x,t) - \overline{n_0}| \\
&\le C_1 |n_\varepsilon(x,t) - \overline{n_0}| \qquad \text{for all } x \in \Omega, t > 0 \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

By means of Young's inequality, (10.9) therefore implies that

$$\begin{aligned}
\int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon) \nabla n_\varepsilon \cdot \nabla c_\varepsilon &\le C_1 \int_\Omega |n_\varepsilon - \overline{n_0}| \cdot |\Delta c_\varepsilon| \\
&\le \frac{1}{2} \int_\Omega (n_\varepsilon - \overline{n_0})^2 + \frac{C_1^2}{2} \int_\Omega |\Delta c_\varepsilon|^2 \qquad \text{for all } t > 0,
\end{aligned}$$

and that in view of (10.8) we thus have

$$\frac{d}{dt} \int_\Omega (n_\varepsilon - \overline{n_0})^2 \le \int_\Omega (n_\varepsilon - \overline{n_0})^2 + C_1^2 \int_\Omega |\Delta c_\varepsilon|^2 \qquad \text{for all } t > 0. \tag{10.11}$$

Here an adequate compensation of the rightmost integral can be achieved by using the second equation in (2.6), which when tested against $-\Delta c_\varepsilon$ yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 &= \int_\Omega F_\varepsilon(n_\varepsilon) c_\varepsilon \Delta c_\varepsilon + \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon) \Delta c_\varepsilon \\
&\le \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 c_\varepsilon^2 \\
&\quad + \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^2 \\
&\le \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 + C_1^2 \int_\Omega c_\varepsilon^2 + C_2^2 \int_\Omega |\nabla c_\varepsilon|^2 \\
&\quad \text{for all } t > 0,
\end{aligned} \tag{10.12}$$

where in accordance with Lemma 8.4 we have chosen $C_2 > 0$ large enough fulfilling $|u_\varepsilon| \le C_2$ in $\Omega \times (0,\infty)$ for all $\varepsilon \in (0,1)$.

In combination, (10.11) and (10.12) now show that

$$\frac{d}{dt}\left\{\int_\Omega (n_\varepsilon-\overline{n_0})^2 + C_1^2\int_\Omega |\nabla c_\varepsilon|^2\right\} \le \int_\Omega (n_\varepsilon-\overline{n_0})^2 + 2C_1^4\int_\Omega c_\varepsilon^2 + 2C_1^2C_2^2\int_\Omega |\nabla c_\varepsilon|^2$$

for all $t>0$,

implying that $y(t):=\int_\Omega (n_\varepsilon(\cdot,t)-\overline{n_0})^2 + C_1^2\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2$, $t\ge 0$, satisfies

$$y'(t)\le C_3 y(t) + C_4\int_\Omega c_\varepsilon^2 \qquad \text{for all } t>0$$

with $C_3:=\max\{1,2C_2^2\}$ and $C_4:=2C_1^4$. By an ODE comparison, this entails that

$$\begin{aligned} y(t) &\le e^{C_3(t-t_\star)}y(t_\star) + C_4\int_{t_\star}^t e^{C_3(t-s)}\cdot\left\{\int_\Omega c_\varepsilon^2(\cdot,s)\right\}ds \\ &\le e^{C_3}y(t_\star) + \frac{C_4e^{C_3}}{C_3}\cdot \sup_{s\in(t_\star,t_\star+1)}\int_\Omega c_\varepsilon^2(\cdot,s) \qquad \text{for all } t\in(t_\star,t_\star+1) \end{aligned}$$

and thereby establishes (10.7). $\square$

By means of another $L^p$ testing procedure applied to the first equation in (2.6), again relying on the estimate (10.2) from Lemma 10.1, the latter implies stabilization of $n$ toward its average, at least when yet considered in $L^2(\Omega)$ and outside a null set of times.

**Lemma 10.4.** *Let $N\subset(0,\infty)$ be as provided by Lemma 9.1. Then*

$$n(\cdot,t)\to\overline{n_0} \quad in\ L^2(\Omega) \qquad as\ (0,\infty)\setminus N\ni t\to\infty. \tag{10.13}$$

**Proof.** We first invoke Lemma 10.3 to find $C_1>0$ such that for any $t_\star\ge 0$ and $\varepsilon\in(0,1)$ we have

$$\int_\Omega \Big(n_\varepsilon(\cdot,t)-\overline{n_0}\Big)^2 \le C_1\cdot\left\{\int_\Omega\Big(n_\varepsilon(\cdot,t_\star)-\overline{n_0}\Big)^2 + \int_\Omega |\nabla c_\varepsilon(\cdot,t_\star)|^2 + \sup_{s\in(t_\star,t_\star+1)}\|c_\varepsilon(\cdot,s)\|_{L^2(\Omega)}^2\right\}$$

$$\text{for all } t\in(t_\star,t_\star+1).$$

Here since $c_\varepsilon\to c$ in $C^0_{loc}(\overline{\Omega}\times[0,\infty))$ and $\nabla c_\varepsilon\to\nabla c$ in $C^0_{loc}(\overline{\Omega}\times[1,\infty))$ as $\varepsilon=\varepsilon_j\searrow 0$ according to Lemma 9.1, and since $(n_\varepsilon-\overline{n_0})_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(\Omega\times(0,\infty))$ by Lemma 8.2, on the basis of (9.6) and the dominated convergence theorem we may let $\varepsilon=\varepsilon_j\searrow 0$ to obtain that

$$\int_\Omega \Big(n(\cdot,t)-\overline{n_0}\Big)^2 \le C_1\cdot\Big\{\int_\Omega \Big(n(\cdot,t_\star)-\overline{n_0}\Big)^2 + \int_\Omega |\nabla c(\cdot,t_\star)|^2 + \sup_{s\in(t_\star,t_\star+1)} \|c(\cdot,s)\|_{L^2(\Omega)}^2\Big\}$$

$$\text{for all } t_\star\in(1,\infty)\setminus N \text{ and any } t\in(t_\star,t_\star+1)\setminus N. \qquad (10.14)$$

In order to prepare an appropriate control of the right-hand side herein, we fix some $\gamma>1$ satisfying $\gamma\ge m-1$, and noting that then $2\gamma-m$ is positive we use $n_\varepsilon^{2\gamma-m}$ as a test function in the first equation from (2.6) to see that for all $\varepsilon\in(0,1)$,

$$\begin{aligned}\frac{1}{2\gamma-m+1}\frac{d}{dt}\int_\Omega n_\varepsilon^{2\gamma-m+1} &+ (2\gamma-m)\int_\Omega n_\varepsilon^{2\gamma-m-1}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2\\ &= (2\gamma-m)\int_\Omega n_\varepsilon^{2\gamma-m}F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon \qquad \text{for all } t>0,\end{aligned}$$

which in light of (2.1), (1.5), (2.4) and Young's inequality implies that

$$\begin{aligned}\frac{1}{2\gamma-m+1}&\int_\Omega n_\varepsilon^{2\gamma-m+1}(\cdot,t) + \frac{(2\gamma-m)k_D}{2}\int_0^t\int_\Omega n_\varepsilon^{2\gamma-2}|\nabla n_\varepsilon|^2\\ &\le \frac{1}{2\gamma-m+1}\int_\Omega n_0^{2\gamma-m+1} - \frac{(2\gamma-m)k_D}{2}\int_0^t\int_\Omega n_\varepsilon^{2\gamma-2}|\nabla n_\varepsilon|^2\\ &\quad +(2\gamma-m)\int_0^t\int_\Omega n_\varepsilon^{2\gamma-m}|\nabla n_\varepsilon|\cdot|\nabla c_\varepsilon|\\ &\le \frac{1}{2\gamma-m+1}\int_\Omega n_0^{2\gamma-m+1} + \frac{2\gamma-m}{2k_D}\int_0^t\int_\Omega n_\varepsilon^{2\gamma-2m+2}|\nabla c_\varepsilon|^2\\ &\le \frac{1}{2\gamma-m+1}\int_\Omega n_0^{2\gamma-m+1} + \frac{2\gamma-m}{2k_D}\|n_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}^{2\gamma-2m+2}\int_0^\infty\int_\Omega|\nabla c_\varepsilon|^2\\ &\quad \text{for all } t>0,\end{aligned}$$

because $2\gamma-2m+2\ge 0$. Due to the boundedness properties asserted by Lemma 8.2 and Lemma 10.1, we therefore conclude that there exist $\varepsilon_\star\in(0,1)$ and $C_2>0$ such that

$$\int_0^\infty\int_\Omega \Big|\nabla n_\varepsilon^\gamma\Big|^2 \le C_2 \qquad \text{for all } \varepsilon\in(0,\varepsilon_\star),$$

and that hence according to the Poincaré inequality we can find $C_3>0$ fulfilling

$$\int_0^\infty \left\| n_\varepsilon^\gamma(\cdot,t) - \mu_\varepsilon^\gamma(t) \right\|_{L^2(\Omega)}^2 dt \le C_3 \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star),$$

where we have set $\mu_\varepsilon(t) := \left\{ \frac{1}{|\Omega|} \int_\Omega n_\varepsilon^\gamma(\cdot,t) \right\}^{\frac{1}{\gamma}}$ for $t > 0$ and $\varepsilon \in (0,1)$. Again using (9.6) along with the dominated convergence theorem, from this we readily infer on invoking Fatou's lemma on the time interval $(0,\infty)$ that

$$\int_0^\infty \left\| n^\gamma(\cdot,t) - \mu^\gamma(t) \right\|_{L^2(\Omega)}^2 dt \le C_3 \tag{10.15}$$

is valid with $\mu(t) := \left\{ \frac{1}{|\Omega|} \int_\Omega n^\gamma(\cdot,t) \right\}^{\frac{1}{\gamma}}$, $t \in (0,\infty) \setminus N$, the latter satisfying $\mu(t) \ge \overline{n_0}$ for all $t \in (0,\infty) \setminus N$ due to the fact that by (9.15) and the Hölder inequality we can estimate

$$\overline{n_0}|\Omega| = \int_\Omega n(\cdot,t) \le \left\{ \int_\Omega n^\gamma(\cdot,t) \right\}^{\frac{1}{\gamma}} \cdot |\Omega|^{1-\frac{1}{\gamma}} \qquad \text{for } t \in (0,\infty) \setminus N.$$

As $|\xi^\gamma - \eta^\gamma| \ge \eta^{\gamma-1} \cdot |\xi - \eta|$ for all $\xi \ge 0$ and $\eta \ge 0$, this implies that

$$\begin{aligned} \left| n^\gamma(\cdot,t) - \mu^\gamma(t) \right|^2 &\ge \mu^{2\gamma-2}(t) \cdot |n(\cdot,t) - \mu(t)|^2 \\ &\ge \overline{n_0}^{2\gamma-2} \cdot |n(\cdot,t) - \mu(t)|^2 \quad \text{a.e. in } \Omega \text{ for all } t \in (0,\infty) \setminus N, \end{aligned}$$

so that from (10.15) we obtain that

$$\int_0^\infty \|n(\cdot,t) - \mu(t)\|_{L^2(\Omega)}^2 dt \le C_4 \tag{10.16}$$

with $C_4 := C_3 \cdot \overline{n_0}^{2-2\gamma} > 0$.

Now to derive the desired conclusion from this and (10.14), given $\eta > 0$ we use (10.16) to find some large $t_0 > 1$ such that

$$\int_{t_0-1}^\infty \|n(\cdot,t) - \mu(t)\|_{L^2(\Omega)}^2 dt < \frac{\eta}{12C_1}, \tag{10.17}$$

and such that in accordance with Lemma 10.2 we moreover have

$$c(x,t) < \frac{\eta}{3C_1} \qquad \text{for all } x \in \Omega \text{ and } t > t_0 - 1 \tag{10.18}$$

and well as

$$\int_\Omega |\nabla c(\cdot,t)|^2 < \frac{\eta}{3C_1} \qquad \text{for all } t > t_0 - 1. \tag{10.19}$$

Then for arbitrary $t > t_0$ we may use (10.17) to pick $t_\star = t_\star(t) \in (t-1,t) \setminus N$ such that

$$\int_\Omega \Big| n(\cdot,t_\star) - \mu(t_\star)\Big|^2 < \frac{\eta}{12C_1}. \tag{10.20}$$

Again by (9.15) and the Cauchy–Schwarz inequality, this firstly entails that

$$\begin{aligned} |\overline{n_0} - \mu(t_\star)| \cdot |\Omega| &= \bigg| \int_\Omega \Big( n(\cdot,t_\star) - \mu(t_\star)\Big)\bigg| \\ &\le \bigg\{ \int_\Omega \Big( n(\cdot,t) - \mu(t)\Big)^2 \bigg\}^{\frac{1}{2}} \cdot |\Omega|^{\frac{1}{2}} \\ &\le \sqrt{\frac{\eta|\Omega|}{12C_1}} \end{aligned}$$

and hence

$$\int_\Omega \Big(\overline{n_0} - \mu(t_\star)\Big)^2 \le \frac{\eta}{12C_1},$$

so that, secondly, from (10.20) we obtain that

$$\begin{aligned} \int_\Omega \Big( n(\cdot,t_\star) - \overline{n_0}\Big)^2 &\le 2\int_\Omega \Big( n(\cdot,t_\star) - \mu(t_\star)\Big)^2 + 2\int_\Omega \Big(\overline{n_0} - \mu(t_\star)\Big)^2 \\ &< \frac{\eta}{3C_1}. \end{aligned}$$

In conjunction with (10.18), (10.19) and (10.14), this means that

$$\int_\Omega \Big( n(\cdot,t) - \overline{n_0}\Big)^2 < C_1 \cdot \bigg\{ \frac{\eta}{3C_1} + \frac{\eta}{3C_1} + \frac{\eta}{3C_1} \bigg\} = \eta,$$

because $t \in (t_\star, t_\star + 1) \setminus N$. Since $\eta > 0$ was arbitrary, this completes the proof. $\square$

By interpolation and approximation, in view of the generalized continuity property of $n$ gained in Lemma 9.1 this readily implies convergence in the style claimed in Theorem 1.1.

**Corollary 10.5.** *For all $p \ge 1$,*

$$n(\cdot,t) \to \overline{n_0} \quad \text{in } L^p(\Omega) \text{ as } t \to \infty. \tag{10.21}$$

**Proof.** By boundedness of $\Omega$, we only need to consider the case $p > 2$, in which due to the Hölder inequality,

$$\|n(\cdot,t)-\overline{n_0}\|_{L^p(\Omega)} \le \|n(\cdot,t)-\overline{n_0}\|_{L^\infty(\Omega)}^{\frac{p-2}{p}} \|n(\cdot,t)-\overline{n_0}\|_{L^2(\Omega)}^{\frac{2}{p}} \le C_1\|n(\cdot,t)-\overline{n_0}\|_{L^2(\Omega)}^{\frac{2}{p}} \quad \text{for all } t>0$$

with $C_1 := \left\{\|n\|_{L^\infty(\Omega\times(0,\infty))} + \overline{n_0}\right\}^{\frac{p-2}{p}}$. Therefore, given $\eta > 0$ we may invoke Lemma 10.4 to fix $t_0 > 0$ such that

$$\|n(\cdot,t)-\overline{n_0}\|_{L^p(\Omega)} \le \eta \qquad \text{for all } t \in (t_0,\infty)\setminus N, \tag{10.22}$$

and for the proof of (10.21) it will be sufficient to make sure that the inequality herein actually remains valid for all $t > t_0$. To verify this, for any such $t$ we can use the density of $(t_0,\infty)\setminus N$ in $(t_0,\infty)$ to find $(t_k)_{k\in\mathbb{N}} \subset (t_0,\infty)\setminus N$ such that $t_k \to t$ as $k\to\infty$. Then (10.22) shows that $\|n(\cdot,t_k)-\overline{n_0}\|_{L^p(\Omega)} \le \eta$ for all $k\in\mathbb{N}$, whence we may extract a subsequence $(t_{k_l})_{l\in\mathbb{N}}$ of $(t_k)_{k\in\mathbb{N}}$ such that $n(\cdot,t_{k_l})-\overline{n_0} \rightharpoonup z$ in $L^p(\Omega)$ as $l\to\infty$. But since this trivially entails that also $n(\cdot,t_{k_l})-\overline{n_0} \rightharpoonup z$ in $(W_0^{2,2}(\Omega))^\star$, from the continuity property implied by (9.8) we infer that $z$ must coincide with $n(\cdot,t)-\overline{n_0}$ and that thus

$$\|n(\cdot,t)-\overline{n_0}\|_{L^p(\Omega)} \le \liminf_{l\to\infty} \|n(\cdot,t_{k_l})-\overline{n_0}\|_{L^p(\Omega)} \le \eta,$$

as claimed. □

*10.4. Decay of u*

Finally, uniform decay of $u$ can be achieved on the basis of the following straightforward application of standard regularity theory in the forced Stokes evolution system.

**Lemma 10.6.** *There exist $\lambda > 0$ and $C > 0$ such that for any choice of $\mu \in \mathbb{R}$ and arbitrary $\varepsilon \in (0,1)$ we have*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C + C\int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \big\|n_\varepsilon(\cdot,s)-\mu\big\|_{L^2(\Omega)} ds \qquad \textit{for all } t>0, \tag{10.23}$$

*where $\alpha \in (\frac{3}{4},1)$ is taken from (1.8).*

**Proof.** As gradients of functions from $W^{1,\infty}(\Omega)$ belong to the kernel of the Helmholtz projection $\mathcal{P}$, for arbitrary $\mu\in\mathbb{R}$ the third equation in (2.6) can be rewritten according to

$$u_{\varepsilon t} + Au_\varepsilon = \mathcal{P}\Big[(n_\varepsilon(\cdot,t)-\mu)\nabla\phi\Big], \qquad x\in\Omega,\ t>0. \tag{10.24}$$

Now since $\alpha > \frac{3}{4}$, from a known embedding result ([15], [18]) we obtain that $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$, so that invoking well-known smoothing properties of the analytic semigroup $(e^{-tA})_{t\ge0}$ ([29], [13]) we infer from (10.24) that with some $C_1 > 0$, $C_2 > 0$ and $\lambda > 0$ we have

$$
\begin{aligned}
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le C_1\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \\
&= C_1\left\| A^\alpha e^{-tA}u_0 + \int_0^t A^\alpha e^{-(t-s)A}\mathcal{P}\Big[(n_\varepsilon(\cdot,s)-\mu)\nabla\phi\Big]ds\right\|_{L^2(\Omega)} \\
&\le C_1\|A^\alpha u_0\|_{L^2(\Omega)} + C_2\int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}\left\|\mathcal{P}\Big[(n_\varepsilon(\cdot,s)-\mu)\nabla\phi\Big]\right\|_{L^2(\Omega)}ds
\end{aligned}
$$

for all $t>0$. Since $\mathcal{P}$ is an orthogonal projector and hence

$$
\left\|\mathcal{P}\Big[(n_\varepsilon(\cdot,s)-\mu)\nabla\phi\Big]\right\|_{L^2(\Omega)} \le \left\|(n_\varepsilon(\cdot,s)-\mu)\nabla\phi\right\|_{L^2(\Omega)} \le \|\nabla\phi\|_{L^\infty(\Omega)}\|n_\varepsilon(\cdot,s)-\mu\|_{L^2(\Omega)} \quad \text{for all } s>0,
$$

in view of our regularity assumption $u_0\in D(A^\alpha)$ we thereby obtain (10.23). $\square$

Here the integral on the right-hand side can be estimated by using the following elementary decay property.

**Lemma 10.7.** *Let $\beta\in(0,1)$, $\lambda>0$ and $h:(0,\infty)\to\mathbb{R}$ be measurable and bounded with $h(t)\to 0$ as $t\to\infty$. Then*

$$
\int_0^t (t-s)^{-\beta}e^{-\lambda(t-s)}h(s)ds \to 0 \qquad \textit{as } t\to\infty. \tag{10.25}
$$

**Proof.** Given $\eta>0$, we pick $t_1>0$ large such that $|h(t)|\le\frac{\eta}{2C_1}$ for all $t>t_1$, where $C_1:=\int_0^\infty\sigma^{-\beta}e^{-\lambda\sigma}d\sigma$ is finite since $\beta<1$. Then writing $t_0:=t_1+\left(\frac{2\|h\|_{L^\infty((0,\infty))}}{\lambda\eta}\right)^{\frac{1}{\beta}}$, for arbitrary $t>t_0$ we can estimate

$$
\begin{aligned}
&\left|\int_0^t (t-s)^{-\beta}e^{-\lambda(t-s)}h(s)ds\right| \\
&\le \int_0^{t_1}(t-s)^{-\beta}e^{-\lambda(t-s)}|h(s)|ds + \int_{t_1}^t (t-s)^{-\beta}e^{-\lambda(t-s)}|h(s)|ds \\
&\le (t-t_1)^{-\beta}\|h\|_{L^\infty((0,\infty))}\int_0^{t_1}e^{-\lambda(t-s)}ds + \frac{\eta}{2C_1}\int_{t_1}^t (t-s)^{-\beta}e^{-\lambda(t-s)}ds \\
&= (t-t_1)^{-\beta}\|h\|_{L^\infty((0,\infty))}\cdot\frac{1}{\lambda}(e^{-\lambda(t-t_1)}-e^{-\lambda t}) + \frac{\eta}{2C_1}\int_0^{t-t_1}\sigma^{-\beta}e^{-\lambda\sigma}d\sigma
\end{aligned}
$$

$$\leq (t_0 - t_1)^{-\beta} \|h\|_{L^\infty((0,\infty))} \cdot \frac{1}{\lambda} + \frac{\eta}{2C_1} \int_0^\infty \sigma^{-\beta} e^{-\lambda\sigma} d\sigma$$

$$= \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

and thereby see that indeed (10.25) is valid. $\square$

In view of the stabilization property from Corollary 10.5, Lemma 10.6 thus entails the desired decay feature of $u$.

**Lemma 10.8.** *We have*

$$u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{10.26}$$

**Proof.** With the null set $N \subset (0, \infty)$ taken from Lemma 9.1, on combining Lemma 8.2 with the dominated convergence theorem we obtain that $n_\varepsilon(\cdot, t) - \overline{n_0} \to n(\cdot, t) - \overline{n_0}$ in $L^2(\Omega)$ as $\varepsilon = \varepsilon_j \searrow 0$. Therefore, using the convergence property (9.12) we infer from Lemma 10.6 that there exist $\lambda > 0$ and $C_1 > 0$ fulfilling

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1 + C_1 \int_0^t (t - s)^{-\alpha} e^{-\lambda(t-s)} \|n(\cdot, s) - \overline{n_0}\|_{L^2(\Omega)} ds \qquad \text{for all } t > 0,$$

where $\alpha \in (\frac{3}{4}, 1)$ is as in (1.8). Since $\|n(\cdot, t) - \overline{n_0}\|_{L^2(\Omega)} \to 0$ as $t \to \infty$ by Corollary 10.5, Lemma 10.7 therefore yields (10.26). $\square$

### *10.5. Proof of Theorem 1.1*

We finally only need to collect our previous findings to arrive at our main result.

**Proof of Theorem 1.1.** The statement on global existence of a weak solution with the regularity features in (1.10) has been asserted by Lemma 9.1. The convergence properties in (1.11) are precisely established by Corollary 10.5, Lemma 10.2 and Lemma 10.8. $\square$

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