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Symmetry analysis, conserved quantities and applications to a dissipative DGH equation

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Abstract

In this paper, we study a weakly dissipative Dullin–Gottwald–Holm equation from the viewpoint of Lie symmetry analysis. We first perform symmetry analysis and the nonlinear self-adjointness of this equation. Due to a mixed derivatives term in the equation, we need to rewrite the corresponding form Lagrangian in symmetric form to construct conservation laws. From the viewpoint, we present a general procedure of how these conserved quantities come about. Based on these conserved quantities, blow-up analysis and global existence of strong solutions are presented. Finally, we show that this equation admits a weak peakon-type solution.

MSC: 35G25; 35B44; 35Q35

Keywords: Lie symmetry analysis; Conserved quantity; Blow-up analysis; Wave-breaking; Weak peakon-type solution

1. Introduction

Recently, Novruzov and the co-author studied the following shallow water equation with dissipation in [26,27]

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + k(u - u_{xx})_x + \lambda(u - u_{xx}) = 0, \tag{1}$$

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where $x \in \mathbb{R}$, $t > 0$ and $k \in \mathbb{R}, \lambda \geq 0$ are dispersion coefficient and dissipative parameter respectively. Some certain conditions on the initial datum are established to guarantee that the solution exists globally or to lead to finite time blow-up of the solution. Also, propagation speed for the equation under consideration is investigated. Thereafter, some blow-up results are presented in [34]. Later, Novruzov studied the blow-up of some related equations with dissipation in [28,29].

The equation (1) is a special case of the Dullin–Gottwald–Holm (DGH) equation

$$u_t - \alpha^2 u_{txx} + k u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \qquad x \in \mathbb{R},\ t > 0,$$

with weakly dissipative term $\lambda(u - u_{xx})$. The DGH equation was derived by Dullin, Gottwald and Holm in 2001 [14], using asymptotic expansions directly in the Hamiltonian for Euler's equation in the shallow water regime. It is completely integrable with a bi-Hamiltonian as well as a Lax pair in [14]. The equation was also found independently as a model for nonlinear waves in cylindrical hyperelastic rods [12,13]. When $k = \lambda = 0$, the equation (1) is reduced to the well-known CH equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0,$$

which is a model describing the unidirectional propagation of surface waves on a shallow layer of water that is at rest at infinity [5,11,22–24]. It also arises in the study of a certain non-Newtonian fluids [4] and finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [12]. The novelty of the CH equation is due to its remarkable properties, such as a bi-Hamiltonian structure, Lax completely integrability, infinitely many conservation laws, peakons, wave breaking, etc. [6–10,25]. Lately, Novruzov and Yazar studied the blow-up problem of the following generalized equation

$$u_t - u_{xxt} + [f(u)]_x - [f(u)]_{xxx} + \left[g(u) + \frac{1}{2} f''(u) u_x^2\right]_x + \lambda(u - u_{xx}) = 0, \tag{2}$$

and improved the earlier related blow-up results to a unified one which is a local-in-space blow-up criterion in [29].

In the present paper, we would like to study (1) from the viewpoint of Lie symmetry analysis. Some conservation laws of (1) will be derived and then be used to construct some exact solutions and prove that certain initial data develop into blow-up or global solutions. In this direction, the pioneer work due to Sophus Lie, who introduced the notion of Lie group in order to study the solutions of ordinary differential equations at the end of the nineteenth century. He showed that the order of an ordinary differential equation can be reduced if it is invariant under one-parameter Lie group of point transformations. The applications of Lie groups to differential systems were mainly established by Lie and Emmy Noether. In 1918, Noether presented the relationship between a mathematics symmetry and conservation law of a physical system. Noether's (first) theorem states that every differentiable symmetry of the action of a physical system has a corresponding conservation law. Application of Noether's theorem allows physicists to gain powerful insights into a general theory in physics, by just analyzing the various transformations that would make the form of the laws involved invariant, such as the laws of conservation of linear momentum, angular momentum or energy and so on.

Although Noether's theorem provides an elegant approach to find conservation laws, it has a strong limitation: it can only be applied to equations with variational structure. As we know, a large number of differential equations without variational structure admit conservation laws, for example, equation (1). Thus, many authors developed some methods which do not rely on the knowledge of Lagrangian functions to obtain conservation laws, such as characteristic method [31], direct method [1,2] and so on. About a decade ago, Ibragimov proved a result in [17] which allows one to construct conservation laws for equations without variational structure. Essentially, Ibragimov's theorem is an extension of Noether's theorem by introducing formal Lagrangian to get rid of the variational limitation. So, we study (1) from the viewpoint of Lie symmetry analysis and construct some conservation laws by Ibragimov's theorem in this paper. First, we carry out Lie symmetry analysis, derive some symmetry reductions and invariant solutions for (1). Then, we study the self-adjointness and conservation laws of equation (1) by the Ibragimov's theorem. Based on these conserved quantities, some properties of solutions to (1) are established. In particular, we show a local-in-space blow-up result (which is the particular case of the Theorem 3 (i) in [29]) by an alternative proof and present the estimates of the life span and blow-up rate by taking advantage of obtained conserved quantities.

For the sake of completeness, we briefly present the notations, definition of nonlinear self-adjointness and Ibragimov's theorem on conservation laws. Consider a s-th order nonlinear equation

$$E(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(s)}) = 0 \tag{3}$$

with n independent variables $x = (x_1, x_2, \cdots, x_n)$ and a dependent variable $u = u(x)$, where $u_{(s)} = \partial^s u$. Let

$$E^*(x, u, v, u_{(1)}, v_{(1)}, \cdots, u_{(s)}, v_{(s)}) := \frac{\delta \mathcal{L}}{\delta u} = 0 \tag{4}$$

be the *adjoint equation* of equation (3), where $\mathcal{L} = vE$ is called *formal Lagrangian*, $v = v(x)$ is a new dependent variable and

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{m=1}^{s} (-1)^m D_{i_1} \cdots D_{i_m} \frac{\partial}{\partial u_{i_1 \cdots i_m}}$$

denotes the Euler–Lagrange operator.

Now let us state the definition of nonlinear self-adjointness for a equation, see [15,16,18–20,32] and references therein.

Definition 1. Equation (3) is said to be *nonlinearly self-adjoint* if the equation obtained from the adjoint equation (4) by the substitution $v = \phi(x, u)$ with a certain function $\phi(x, u) \neq 0$ is identical with the original equation (3). In other words, the following equation holds:

$$E^*|_{v=\phi} = \lambda_0 E + \lambda_1 D_t E + \lambda_2 D_x E + \cdots \tag{5}$$

for some differential functions $\lambda_i = \lambda_i(x, u, u_{(1)}, \cdots)$.

We recall the conservation theorem given by Ibragimov in [17].

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Theorem *(Ibragimov [17]). Any Lie point, Lie–Bäcklund and non-local symmetry generated by*

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} \tag{6}$$

of equation (3) provides a conservation law $D_i(C^i) = 0$ for the system comprising equation (3) and its adjoint equation (4). The conserved vector $\mathbf{C} = (C^i)$ is given by

$$\begin{aligned} C^i = {} & \xi^i \mathcal{L} + W\Big[\tfrac{\partial \mathcal{L}}{\partial u_i} - D_j\Big(\tfrac{\partial \mathcal{L}}{\partial u_{ij}}\Big) + D_k D_j\Big(\tfrac{\partial \mathcal{L}}{\partial u_{ijk}}\Big) - D_l D_k D_j\Big(\tfrac{\partial \mathcal{L}}{\partial u_{ijkl}}\Big) + \cdots\Big] \\ & + D_j(W)\Big[\tfrac{\partial \mathcal{L}}{\partial u_{ij}} - D_k\Big(\tfrac{\partial \mathcal{L}}{\partial u_{ijk}}\Big) + \cdots\Big] \\ & + D_k D_j(W)\Big[\tfrac{\partial \mathcal{L}}{\partial u_{ijk}} - D_l\Big(\tfrac{\partial \mathcal{L}}{\partial u_{ijkl}}\Big) + \cdots\Big] + \cdots, \end{aligned} \tag{7}$$

where $W = \eta - \xi^j u_j$ is the Lie characteristic function and $\mathcal{L} = vE$ is the formal Lagrangian.

Now, let us state our results. First, by classical Lie symmetry analysis [30] we know that (1) admits Lie point symmetry operators.

Proposition 1. *Equation (1) admits the following Lie point symmetry operators:*

$$X_1 = \partial_t, \quad X_2 = \partial_x \quad \text{and} \quad X_3 = e^{\lambda t}(\partial_t + k\partial_x - \lambda u \partial_u).$$

Moreover, the optimal system of one-dimensional subalgebras of the Lie algebra spanned by X_1, X_2, X_3 of (1) given by

$$X_1, \quad X_2, \quad X_1 + aX_2 \quad \text{and} \quad X_2 + bX_3,$$

where a, b are nonzero constants.

From above proposition, we have the following assertion.

Theorem 2. *If $u = f(t, x)$ solves the equation (1), then so do*

$$\begin{aligned} & u_1 = f(t - \varepsilon, x), \quad u_2 = f(t, x - \varepsilon), \\ & u_3 = \frac{1}{1 + \varepsilon\lambda e^{\lambda t}} f\Big(t - \frac{1}{\lambda}\ln(1 + \varepsilon\lambda e^{\lambda t}), x - \frac{k}{\lambda}\ln(1 + \varepsilon\lambda e^{\lambda t})\Big). \end{aligned}$$

Where ε is a constant.

This result shows us that one can gain some new exact solutions of (1) from a seed solution $f(t, x)$ by the above expressions.

In order to construct some conservation laws for (1) by Ibragimov's theorem, we need to investigate the self-adjointness of this equation.

Proposition 3. *Equation (1) is nonlinear self-adjoint with substitution functions:*

$$\begin{aligned} &v_1 = e^{\lambda t}, \quad v_2 = e^{2\lambda t} u \quad \text{and} \\ &v_3 = e^{3\lambda t}\left(u_x^2 + 2uu_{xx} + 2\lambda u_x + 2ku_{xx} + 2u_{xt} - 3u^2\right). \end{aligned} \tag{8}$$

Note that (1) is a nonlinear equation with higher-order mixed derivatives term u_{xxt}. To apply Ibragimov's theorem to (1), we need to rewrite the corresponding form Lagrangian $\mathcal{L} = vE$ of (1) in symmetric form:

$$\begin{aligned} \mathcal{L} = v\Big[&u_t - \frac{1}{3}(u_{xxt} + u_{xtx} + u_{txx}) + 3uu_x - 2u_x u_{xx} - uu_{xxx} \\ &+ k(u - u_{xx})_x + \lambda(u - u_{xx})\Big], \end{aligned}$$

which allows us to successfully apply Ibragimov's theorem to construct some conservation laws for equation (1).

Theorem 4. *Equation (1) possesses conservation laws $D_t C^t + D_x C^x = 0$ with following local components:*

$$C_1^t = e^{\lambda t}(u - u_{xx}), \quad C_1^x = \frac{1}{2} e^{\lambda t}[3u^2 + 2k(u - u_{xx}) - 2uu_{xx} - u_x^2]; \tag{9}$$

$$\begin{cases} C_2^t = e^{2\lambda t} u(u - u_{xx}), \\ C_2^x = e^{2\lambda t}(2u^3 - 2u^2 u_{xx} + ku^2 - 2kuu_{xx} + ku_x^2 + u_x u_t - uu_{xt}) \end{cases} \tag{10}$$

and

$$\begin{cases} C_3^t = 3e^{3\lambda t} u(u^2 + u_x^2), \\ C_3^x = \dfrac{3}{4} e^{3\lambda t}\Big[4u_{xt}^2 + 4u_{xt}(2uu_{xx} + u_x^2 - 3u^2 + 2\lambda u_x + 2ku_{xx}) - 4u_t^2 \\ \qquad - 8u_t(uu_x + \lambda u + ku_x) + 4u_{xx}(u + k)(u_x^2 - 3u^2 + 2\lambda u_x) \\ \qquad + 4u_{xx}^2(u + k)^2 + u_x^4 - 2(3u^2 + 2ku + 2k^2 - 2\lambda^2)u_x^2 \\ \qquad + 4\lambda u_x^3 - 4\lambda uu_x(2u + 3u + 2k) + u^2(9u^2 + 4ku - 4\lambda^2)\Big]. \end{cases} \tag{11}$$

Immediately, from the above theorem, we derive the following conserved quantities for equation (1).

Corollary 5. *Let $\lambda > 0$, u be a solution of (1) and $y = u - u_{xx}$, then the quantities*

$$e^{\lambda t} \int_{\mathbb{R}} u dx = e^{\lambda t} \int_{\mathbb{R}} y dx, \quad e^{2\lambda t} \int_{\mathbb{R}} (u^2 + u_x^2) dx, \quad e^{3\lambda t} \int_{\mathbb{R}} u(u^2 + u_x^2) dx \tag{12}$$

are conserved.

The first two conserved quantities obtained in Corollary 5 can be obtained in a simple standard way. In fact, multiplying the original equation by functions $e^{\lambda t}$ and $e^{2\lambda t}u$ and integrating the resultant equations by parts, we easily get these conserved quantities. However, for the third conserved quantity, it seems not easy to arrive at it by this way. In general, in this way one may guess which quantity would be conserved and which function could be choose to yield this quantity. But it is not always easy to do so. In Theorem 4, we obtain some conservation laws of (1) by just analyzing the Lie point symmetries and self-adjointness. Here, we present a general procedure of how these conserved quantities come about from the viewpoint of Lie symmetry analysis. This method can be apply to other nonlinear differential equations.

Conserved quantities are very useful to show some properties of solutions for nonlinear equations. As one of applications of the obtained conserved quantities, we can show some blow-up and existence results for Equation (1). Our blow-up result is stated as follows.

Theorem 6. *Let $\lambda \geq 0$, $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Assume that u is the solution of (1) in $C([0,T^*), H^s(\mathbb{R})) \cap C^1([0,T^*), H^{s-1}(\mathbb{R}))$ arising from u_0. If there exists a point $x_0 \in \mathbb{R}$ such that*

$$u_{0,x}(x_0) < -|u_0(x_0)| - 2\lambda, \tag{13}$$

then the solution u blows up in finite time, the maximal time T^* is estimated by

$$0 < T^* \leq \begin{cases} \frac{1}{\lambda}\log\left(\sqrt{u_{0,x}^2(x_0) - u_0^2(x_0)} \Big/ \left(\sqrt{u_{0,x}^2(x_0) - u_0^2(x_0)} - 2\lambda\right)\right), & \text{if } \lambda > 0; \\ 2 \Big/ \sqrt{u_{0,x}^2(x_0) - u_0^2(x_0)}\,, & \text{if } \lambda = 0. \end{cases}$$

Moreover, for some $x(t) \in \mathbb{R}$, the blow-up rate is

$$u_x(t, x(t)) \sim -\frac{2}{T^* - t} \quad \text{as } t \to T^*. \tag{14}$$

The condition (13) is local in space, which was improved to a class of nonlinear dispersive wave equations with dissipation (2) in [29] recently. Here, we give an alternative proof and present the estimates of the life span and blow-up rate by using the conserved quantities obtained in Corollary 5.

The other application of conservation laws is a global existence result, if $y_0 = (1 - \partial_x^2)u_0$ satisfies some additional information on the sign.

Theorem 7. *Let $\lambda \geq 0$, $u_0 \in H^3$ and $y_0 = (1 - \partial_x^2)u_0$. Then the solution of the problem (1) remains regular globally in time provided that one of the following conditions occurs:*

(i) y_0 doesn't change sign on $\mathbb{R}$; or

(ii) there exists $x_0 \in \mathbb{R}$ such that $y_0 \leq 0$ as $x \in (-\infty, 0)$ and $y_0 \geq 0$ as $x \in [x_0, \infty)$.

In fact, (i) can be viewed as a special case of (ii) in Theorem 7, if one taking $x_0 = -\infty$ or $x_0 = \infty$.

Finally, using the conserved quantities obtained in Corollary 5, we construct some exact solutions of (1), which differ to the unique solution obtained below (see Lemma 9 in Section 3) since they do not belong to the space $C([0,T); H^s) \cap C^1([0,T); H^{s-1})$ $(s > 3/2)$. Thus, our result implies that the uniqueness in Lemma 9 may not be valid if we don't restrict the initial value u_0 in H^s $(s > 3/2)$. Moreover, we also find a peakon-type solution to (1).

We say a function $u(t,x)$ is a weak solution of (1) if $u(t,x)$ satisfies Equation (1) (or (40) in Section 4) only in the sense of distribution.

Theorem 8. *Equation (1) admits a weak peakon-type solution with the form*

$$u(t,x) = ce^{-\lambda t - |x - kt + ce^{-\lambda t}/\lambda + c_0|}, \tag{15}$$

where c is a nonzero constant and c_0 is an arbitrary constant.

The peakon-type solution (15) is similar to the invariant solution (19) (in Section 2) in terms of expressions. So, invariant solutions of a equation maybe give a hint for study on some special solutions.

The remainder of this paper is organized as follows. In Section 2, we perform Lie symmetry analysis for (1) and derive symmetry reductions and invariant solutions. In Section 3, we discuss the nonlinear self-adjointness of equation (1) and establish some conserved quantities based on this concept. In Section 4, we show some applications of the conservation laws obtained above. More precisely, we present some results on blow-up and global existence of solutions to equation (1). In the final section, we show that (1) admits some special exact solutions.

2. Symmetry analysis

In this section, we carry out Lie symmetry analysis for the equation (1) and derive symmetry reductions and invariant solutions. The symmetries of equation (1) can be obtained by the classical Lie symmetry analysis [30].

Proof of Proposition 1. Consider the vector field

$$X = \xi^t \partial_t + \xi^x \partial_x + \eta \partial_u, \tag{16}$$

which has the third-order prolongation, from (1),

$$X^{(3)} = X + \eta_t^{(1)} \partial_{u_t} + \eta_x^{(1)} \partial_{u_x} + \eta_{xx}^{(2)} \partial_{u_{xx}} + \eta_{xxt}^{(3)} \partial_{u_{xxt}} + \eta_{xxx}^{(3)} \partial_{u_{xxx}},$$

where the functions $\eta_t^{(1)}$, $\eta_x^{(1)}$, $\eta_{xx}^{(2)}$, $\eta_{xxt}^{(3)}$ and $\eta_{xxx}^{(3)}$ can be expressed via the components ξ^t, ξ^x and η of the vector field. The invariant condition is

$$X^{(3)} E = \tau(t,x,u) E.$$

Solving this equation leads to

$$\xi^t = C_1 + C_2 e^{\lambda t}, \quad \xi^x = C_3 + C_2 k e^{\lambda t}, \quad \eta = -C_2 \lambda u e^{\lambda t}.$$

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Table 1
Commutator table.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	λX_3
X_2	0	0	0
X_3	$-\lambda X_3$	0	0

Table 2
Adjoint representation.

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$e^{-\lambda} X_3$
X_2	X_1	X_2	X_3
X_3	$X_1 + \varepsilon\lambda X_3$	X_2	X_3

So, the infinitesimal generators of the equation (1) is spanned by the vector fields:

$$X_1 = \partial_t, \quad X_2 = \partial_x \quad \text{and} \quad X_3 = e^{\lambda t}(\partial_t + k\partial_x - \lambda u \partial_u).$$

The commutation relations of Lie algebra determined by X_1, X_2, X_3 are given in Table 1 as follows.

From the commutator table of this algebra and the Lie series

$$Ad(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2!}[X_i, [X_i, X_j]] - \frac{\varepsilon^3}{3!}[X_i, [X_i, [X_i, X_j]]] + \cdots$$

in [30], we derive the adjoint representation in Table 2 as above. Thanks to the adjoint representation, we infer that the optimal system of one-dimension subalgebras of the Lie algebra spanned by X_1, X_2, X_3 of (1) is given by

$$X_1, \quad X_2, \quad X_1 + aX_2, \quad X_2 + bX_3,$$

where a, b are nonzero constants. The proof of Proposition 1 is completed. □

Proof of Theorem 2. From the vector fields X_1, X_2, X_3 given in Proposition 1, we see that one-parameter symmetry groups $f_i : (t, x, u) \to (\tilde{t}, \tilde{x}, \tilde{u})$ of the infinitesimal generators X_i $(i = 1, 2, 3)$ are given as follows:

$$\begin{aligned}
&f_1 : (t, x, u) \to (t + \varepsilon, x, u),\\
&f_2 : (t, x, u) \to (t, x + \varepsilon, u),\\
&f_3 : (t, x, u) \to \Big(t - \tfrac{1}{\lambda}\ln(1 - \lambda\varepsilon e^{\lambda t}), x - \tfrac{k}{\lambda}\ln(1 - \lambda\varepsilon e^{\lambda t}), (1 - \lambda\varepsilon e^{\lambda t})u\Big),
\end{aligned}$$

where ε is a group parameter such that f_i's make sense, and f_1 is a time translation, f_2 is a space translation and f_3 is a scaling transformation. Consequently, we arrive at the conclusion of Theorem 2. □

Now, we derive symmetry reductions and invariant solutions of (1) by using the optimal system in Proposition 1.

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Case 1. $X_1 + aX_2 = \partial_t + a\partial_x (a \neq 0)$. It's easy to see that the corresponding characteristic equations are given by

$$\frac{dt}{1} = \frac{dx}{a} = \frac{du}{0}.$$

This solution of the above equations yields two invariants of the operator $X_1 + aX_2$:

$$\zeta = x - at, \quad u = f(\zeta).$$

Thus the equation (1) is reduced to the following ODE:

$$a(f''' - f') + 3ff' - 2f'f'' - ff''' + k(f' - f''') + \lambda(f - f'') = 0,$$

whose nontrivial solution yields a group invariant solution of (1)

$$u(t,x) = C_1 e^{x-at} + C_2 e^{-x+at}, \tag{17}$$

where C_1, C_2 are arbitrary constants. It is a traveling wave solution of (1).

Case 2. $X_2 + bX_3 = \partial_x + be^{\lambda t}(\partial_t + k\partial_x - \lambda u\partial_u)(b \neq 0)$. The corresponding characteristic equations are

$$\frac{dt}{be^{\lambda t}} = \frac{dx}{1 + kbe^{\lambda t}} = \frac{du}{-b\lambda e^{\lambda t} u}.$$

Solving these equations we obtain the following invariants of $X_2 + bX_3$:

$$\zeta = x - kt + \frac{1}{b\lambda e^{\lambda t}}, \quad e^{\lambda t} u = f(\zeta). \tag{18}$$

Substitution of (18) into the equation (1) leads to that $f(\zeta)$ satisfies the reduced ODE

$$b(3ff' - ff''' - 2f'f'') + f''' - f' = 0.$$

Hence, the invariant solution of (1), resulting from its invariance under $X_2 + bX_3$, is given by

$$u(t,x) = e^{-\lambda t} f(\zeta) = e^{-\lambda t}(C_1 e^{\zeta} + C_2 e^{-\zeta}), \tag{19}$$

where $\zeta = x - kt + \frac{1}{b\lambda e^{\lambda t}}$ and C_1, C_2 are arbitrary constants.

Similarly, for $X_1 = \partial_t$ and $X_2 = \partial_x$ we repeat above processes and obtain that

$$u = C_1 e^{x} + C_2 e^{-x} \quad \text{and} \quad u = C_1 e^{-\lambda t} \tag{20}$$

are the invariant solutions of (1). Here, we omit the details.

To end this section, we point out that one can gain some new exact solutions to (1) by Theorem 2 with the seed solutions (17), (19) and (20). For example, we take the special solution

$u = C_1 e^{x-at}$ ($C_2 = 0$ in (17)) as a seed solution, by f_3 in Theorem 2 we see that the function

$$C_1(1+\lambda\varepsilon e^{\lambda t})^{\frac{a-k}{\lambda}-1} e^{x-at}$$

is also a solution of (1).

3. Nonlinear self-adjointness and conservation laws for (1)

Conserved quantities of nonlinear equations are very important, since they can help us understand motion of the objects and show some properties of solutions to the considered equations, such as apriori estimates, global existence and stability of solutions. They are useful tools for qualitative analysis. However, it seems not easy to find some useful conserved quantities for nonlinear problems. In this section, we will apply the concepts of self-adjointness and the Ibragimov's theorem on conservation laws to construct some conserved quantities for equation (1), which will be served for qualitative analysis of solutions to (1). We begin with the study of self-adjointness for (1).

Proof of Proposition 3. We rewrite equation (1) as

$$E = u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + k(u - u_{xx})_x + \lambda(u - u_{xx}) = 0. \tag{21}$$

Computing the variational derivative of the formal Lagrangian $\mathcal{L} = vE$, we obtain the adjoint equation of (1) is

$$\begin{aligned} E^* = \frac{\delta \mathcal{L}}{\delta u} &= -v_t + v_{xxt} + 3vu_x - 3(vu)_x + (v_x u_x)_x + uv_{xxx} \\ &\quad -k(v - v_{xx})_x + \lambda(v - v_{xx}) = 0, \end{aligned} \tag{22}$$

where $v = v(t, x)$ is a new dependent variable. Let $\lambda_i = \lambda_i(t, x, u, u_t, u_x, \cdots)$ $(i = 0, 1, 2, \cdots)$ be differential functions and, by the concept of nonlinear self-adjointness, we have

$$E^*|_{v=\psi(t,x,u,u_t,u_x,\cdots)} = \lambda_0 E + \lambda_1 D_t(E) + \lambda_2 D_x(E) + \cdots.$$

The comparison of the coefficients of the derivatives of u in both sides of the above equation yields an algebraic system. Solving this system, we obtain the substitution function v given by (8). Consequently, (1) is nonlinear self-adjoint with the substitution functions in (8). Therefore, we have demonstrated the self-adjointness result. □

Now we construct conservation laws of (1) by the Ibragimov's theorem on conservation laws.

Proof of Theorem 4. For a general generator X given by (16), the corresponding Lie characteristic function is $W = \eta - \xi^t u_t - \xi^x u_x$. Form the Ibragimov's theorem, we obtain that the density

$$C^t = \xi^t \mathcal{L} + W\left(v - \frac{1}{3}v_{xx}\right) + \frac{1}{3}v_x D_x W - \frac{1}{3}v D_x^2 W, \tag{23}$$

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and the flux is

$$\begin{aligned} C^x &= \xi^x \mathcal{L} + W\Big[3uv - 2u_{xx}v + kv + D_x(2u_xv + \lambda v) - D_x^2(uv+kv) - \frac{2}{3}v_{xt}\Big] \\ &\quad + \frac{1}{3}v_x D_t W + D_x W\Big[\frac{1}{3}v_t - 2u_xv - \lambda v + D_x(vu+kv)\Big] - \frac{2}{3}vD_{xt}W \\ &\quad - (uv+kv)D_x^2 W. \end{aligned} \tag{24}$$

For the symmetries X_i $(i = 1, 2, 3)$, from the formula (23) and (24) we obtain readily some conservation laws for equation (1).

Let us construct the conserved vector corresponding to the time translation group with the generator $X_1 = \partial_t$. For this operator, we have $W = -u_t$. Therefore, from the formula (23) and (24) we obtain the conserved vector (C^t, C^x) with components

$$\begin{aligned} C^t &= \mathcal{L} - u_tv + \frac{1}{3}u_tv_{xx} - \frac{1}{3}v_xu_{xt} + \frac{1}{3}vu_{xxt} \\ &= \mathcal{L} - v(u_t - u_{xxt}) + \frac{1}{3}D_x(u_tv_x - 2vu_{xt}), \\ C^x &= -u_t\Big(3vu - vu_{xx} + kv + \lambda v_x - uv_{xx} - kv_{xx} - \frac{2}{3}v_{xt}\Big) - \frac{1}{3}u_{tt}v_x \\ &\quad + u_{xt}\Big(vu_x + \lambda v - uv_x - kv_x - \frac{1}{3}v_t\Big) + \frac{2}{3}vu_{xtt} + u_{xxt}(uv+kv), \end{aligned}$$

which can be reduced to

$$\begin{aligned} C^t &= v(3uu_x - 2u_xu_{xx} - uu_{xxx} + ku_x - ku_{xxx} + \lambda u - \lambda u_{xx}), \\ C^x &= u_t(vu_{xx} + uv_{xx} - 3uv - kv - \lambda v_x + kv_{xx} + v_{xt}) \\ &\quad + u_{xt}(vu_x + \lambda v - uv_x - kv_x - v_t) + u_{xxt}(uv+kv), \end{aligned} \tag{25}$$

here we have used the equation (1). For the substitution function $v_1 = e^{\lambda t}$, (25) yields

$$\begin{aligned} C^t &= \lambda e^{\lambda t}(u - u_{xx}) + D_x\Big[e^{\lambda t}\Big(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx} + ku - ku_{xx}\Big)\Big], \\ C^x &= \frac{1}{2}\lambda e^{\lambda t}[3u^2 + 2k(u-u_{xx}) - 2uu_{xx} - u_x^2] \\ &\quad - D_t\Big[e^{\lambda t}\Big(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx} + ku - ku_{xx}\Big)\Big], \end{aligned}$$

which can be reduced to the following simple form

$$\begin{aligned} C^t &= \lambda e^{\lambda t}(u - u_{xx}), \\ C^x &= \frac{1}{2}\lambda e^{\lambda t}[3u^2 + 2k(u-u_{xx}) - 2uu_{xx} - u_x^2]. \end{aligned} \tag{26}$$

Similarly, insetting the substitution functions v_2 and v_3 into (25) and after some tedious and lengthy calculations, we obtain the corresponding conservation laws (10) and (11) respectively.

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Processing as above, for the space and scaling translation operators $X_2 = \partial_x$ and $X_3 = e^{\lambda t}(\partial_t + k\partial_x - \lambda u\partial_u)$, we also derive some conservation laws. However, for these two infinitesimal generators, the corresponding conservation laws are trivial. For example, substituting v_1 into the formula (23) and (24) and taking into account equation (1), we have

$$\begin{aligned} C^t &= \frac{1}{3}e^{\lambda t}(u_{xxx} - 3u_x) = \frac{1}{3}D_x[e^{\lambda t}(u_{xx} - 3u)], \\ C^x &= \frac{1}{3}e^{\lambda t}(3\lambda u - \lambda u_{xx} + 3u_t - u_{xxt}) = -\frac{1}{3}D_t[e^{\lambda t}(u_{xx} - 3u)], \end{aligned}$$

which can be simplified to

$$C^t = 0, \quad C^x = 0,$$

that is, this conservation law is trivial. Similarly, we can check that for X_2, X_3 and the substitutions v_i $(i = 1, 2, 3)$ given in (8), the corresponding conservation laws are trivial. □

Proof of Corollary 5. From Theorem 4, integrating $D_t C^t + D_x C^x = 0$ over $\mathbb{R}$ and applying the conditions for u at infinity, it's easy to find Corollary 5 holds. □

4. Applications: blow-up and global existence of solutions

In this section, we will use the conservation laws obtained in the above section to show some blow-up and global existence results of strong solutions to (1) for $\lambda \geq 0$ and any $k \in \mathbb{R}$, provided the initial data u_0, y_0 and the parameter λ satisfy suitable conditions.

Let us present and recall several useful results that will be used in the sequel. We set $G(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$ and denote $\Lambda := (1 - \partial_x^2)^{1/2}$. For $f \in L^2(\mathbb{R})$, the operator Λ^2 acting on f has the representation $\Lambda^{-2} f(x) = G * f$, where $*$ denotes the spatial convolution. It is convenient to rewrite the Cauchy problem associated with equation (1) in the following weak form:

$$\begin{cases} u_t + (u + k)u_x = -\partial_x G * \left(u^2 + \frac{1}{2}u_x^2\right) - \lambda u, & x \in \mathbb{R},\ t > 0, \\ u_0(x) = u(0, x), & x \in \mathbb{R}. \end{cases} \tag{27}$$

Let's recall the local well-posedness and blowup scenario for the Cauchy problem of equation (1) in $H^s(\mathbb{R})$, $s > \frac{3}{2}$, see [26].

Lemma 9. *[26] Suppose that $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then there exists a maximal $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u \in C([0, T); H^s) \cap C^1([0, T); H^{s-1})$ of (1) with $u(0, x) = u_0$. Moreover, the solution u depends continuously on the initial value u_0 and the maximal time of existence T is independent of s.*

Lemma 10 *([26]). Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ and T be the maximal existence time of the solution u to (1) with the initial data u_0. Then the corresponding solution u blows up in finite time if and only if*

$$\lim_{t \to T} \liminf_{x \in \mathbb{R}} u_x(x, t) = -\infty. \tag{28}$$

The following result for ODE theory will be useful.

Lemma 11 *([33]). Let $y \in C^1(\mathbb{R})$, $a > 0$ and $b > 0$. If $y'(t) \geq ay^2(t) - b$ with $y(0) > \sqrt{\frac{b}{a}}$. Then $y(t) \to +\infty$ as $t \to t^* \leq \frac{1}{2\sqrt{ab}} \log\Big(\frac{y(0)+\sqrt{\frac{b}{a}}}{y(0)-\sqrt{\frac{b}{a}}}\Big)$.*

Suppose $u(x,t)$ solves (1). We introduce the standard particle trajectory $q(t;x)$ which satisfies the following system

$$\begin{cases} q_t(t;x) = u(t, q(t;x)) + k, & x \in \mathbb{R}, \ 0 < t < T, \\ q(0;x) = x, & x \in \mathbb{R}, \end{cases} \tag{29}$$

where T is the lifespan of the solution u. Then the map $q(t;\cdot)$ is an increasing diffeomophism of the line with

$$q_x(t;x) = \exp\Big(\int_0^t u_q(s, q(s;x))ds\Big) > 0, \quad \text{for } (t,x) \in [0,T) \times \mathbb{R}. \tag{30}$$

Moreover, $q_x(0;x) = 1$.

4.1. A local-in-space blow-up criterion for equation (1)

In this subsection, to show how to use the conservation laws in Corollary 5, we derive wave breaking result for the strong solutions to equation (1). An alternative proof of blow-up criterion for (1) is given, which differs from the one in [29]. Moreover, we present the estimates of blow-up time and rate.

Proof of Theorem 6. In order to discuss the singularities, we differentiate equation (27) with respect to x and use the identity $\partial_x^2 G * f = G * f - f$ to get

$$u_{tx} + (u+k)u_{xx} = u^2 - \frac{1}{2}u_x^2 - G * \Big(u^2 + \frac{1}{2}u_x^2\Big) - \lambda u_x. \tag{31}$$

Recalling $q(t;x_0)$ given in (29) and introducing the following two functions

$$M(t) = (u - u_x)(t, q(t;x_0)), \qquad N(t) = (u + u_x)(t, q(t;x_0)),$$

we study the dynamics of $M(t)$ and $N(t)$ along the characteristics. In view of (29), equations (27) and (31), we have

$$\begin{aligned} M' &= \big(u_t + u_x(u+k) - [u_{xt} + u_{xx}(u+k)]\big)(t, q(t;x_0)) \\ &= \Big[-u^2 + \frac{1}{2}u_x^2 + (G - \partial_x G) * \Big(u^2 + \frac{1}{2}u_x^2\Big) + \lambda u_x - \lambda u\Big](t, q(t;x_0)) \\ &\geq -\frac{1}{2}MN - \lambda M. \end{aligned} \tag{32}$$

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Here we have used the estimate $(G-\partial_x G)*(u^2+\frac{1}{2}u_x^2)\geq \frac{1}{2}u^2$ (cf. Lemma 3.1 in [3]). Similarly, for the derivative of function $N(t)$ we can derive that

$$\begin{aligned} N' &= \Big[u^2-\frac{1}{2}u_x^2-(G+\partial_x G)*\Big(u^2+\frac{1}{2}u_x^2\Big)-\lambda(u+u_x)\Big](t,q(t;x_0)) \\ &\leq \frac{1}{2}MN-\lambda N. \end{aligned} \tag{33}$$

From (13), we see that the initial data satisfies

$$\begin{aligned} &u_0(x_0)+u_{0,x}(x_0)<-2\lambda\leq 0, \quad u_{0,x}(x_0)-u_0(x_0)<-2\lambda\leq 0, \\ &u_0^2(x_0)-u_{0,x}^2(x_0)<-4\lambda^2. \end{aligned}$$

Also, along the characteristics emanating from x_0, it follows

$$\begin{aligned} &M(0)>2\lambda\geq 0, && N(0)<-2\lambda\leq 0, \\ &M'(0)\geq -\frac{1}{2}M(0)(N(0)+2\lambda)>0, && N'(0)\leq \frac{1}{2}N(0)(M(0)-2\lambda)<0. \end{aligned} \tag{34}$$

Then, over the time of existence we have

$$M'(t)>0, \qquad N'(t)<0. \tag{35}$$

Otherwise, if (35) is not true, then there exists $t_0\in[0,T)$ such that

$$t_0=\min\{t\in[0,T);\ M'(t)=0 \text{ or } N'(t)=0\}.$$

Then from (34) we see that $t_0>0$. The definition of t_0 combining with (32) and (33) implies that

$$0=M'(t_0)\geq -\frac{1}{2}M(t_0)\big[N(t_0)+2\lambda\big] \quad \text{or} \quad 0=N'(t_0)\leq \frac{1}{2}N(t_0)\big[M(t_0)-2\lambda\big]. \tag{36}$$

However, noting that $M(t)$ is increasing and $N(t)$ is decreasing in the interval $[0,t_0]$, due to (34), we find that

$$M(t_0)\geq M(0)>2\lambda, \quad N(t_0)\leq N(0)<-2\lambda.$$

Hence,

$$-M(t_0)\big[N(t_0)+2\lambda\big]>0 \quad \text{and} \quad N(t_0)\big[M(t_0)-2\lambda\big]<0,$$

which contradict with (36). So, we infer that $M'(t)>0$ and $N'(t)<0$ for all $t\in[0,T)$.

Taking account of (34), we know that

$$M(t)\geq M(0)>2\lambda\geq 0 \quad \text{and} \quad N(t)\leq N(0)<-2\lambda\leq 0. \tag{37}$$

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These also imply that $MN < -4\lambda^2$. Thanks to the fact $MN < 0$, we consider the quantity $H(t) = \sqrt{-M(t)N(t)}$. The inequalities in (32) and (33) along with (37) imply that

$$M'(t) \geq -\frac{1}{2}M(N+2\lambda) > 0, \quad N'(t) \leq \frac{1}{2}N(M-2\lambda) < 0$$

for $t \in [0, T)$. So we have $M'N + MN' < \frac{1}{2}(M - N - 4\lambda)MN < 0$. A simple calculation yields

$$H'(t) = -\frac{M'N+MN'}{2\sqrt{-MN}} \geq \frac{\lambda MN}{\sqrt{-MN}} + \frac{(N-M)MN}{4\sqrt{-MN}} \geq -\lambda H(t) + \frac{1}{2}H^2(t). \tag{38}$$

If $\lambda = 0$, (38) is reduced to $H'(t) \geq \frac{1}{2}H^2$. Solving this inequality, we find that

$$H(t) \geq \left(\frac{1}{H(0)} - \frac{1}{2}t\right)^{-1},$$

which tells us that there exists $T^* > 0$ such that $H(t)$ goes to ∞ as $t \to T^*$, where T^* can be estimated as following

$$T^* \leq \frac{2}{H(0)} = \frac{2}{\sqrt{u_{0,x}^2(x_0) - u_0^2(x_0)}}.$$

If $\lambda \neq 0$, then the inequality (38) becomes

$$(H-\lambda)' \geq \frac{1}{2}(H-\lambda)^2 - \frac{1}{2}\lambda^2.$$

Applying Lemma 11, we see that there exists $T^* > 0$ such that $H(t) - \lambda \to \infty$, that is, $H(t) \to \infty$ as $t \to T^*$ and T^* is controlled by

$$0 \leq T^* \leq \frac{1}{\lambda}\log\left(\sqrt{u_{0,x}^2(x_0) - u_0^2(x_0)} \Big/ \left(\sqrt{u_{0,x}^2(x_0) - u_0^2(x_0)} - 2\lambda\right)\right).$$

Consequently, the fact $H(t) \leq -u_x(t, q(t; x_0))$ shows us the finite time wave breaking $u_x(t, q(t; x_0)) \to -\infty$ as $t \to T^*$.

Now, let us give more insight into the blowup mechanism for the wave breaking solutions to equation (1). From (32), in view of Hölder inequality and Corollary 5, we see that

$$\begin{aligned} M' &= -\frac{1}{2}MN - \lambda M + \Big[-\frac{1}{2}u^2 + (G - \partial_x G) * \Big(u^2 + \frac{1}{2}u_x^2\Big)\Big](t, q(t; x_0)) \\ &\leq -\frac{1}{2}MN - \lambda M + \|G - \partial_x G\|_{L^\infty} \Big\|u^2 + \frac{1}{2}u_x^2\Big\|_{L^1} \\ &\leq -\frac{1}{2}MN - \lambda M + \|u_0\|_{H^1}. \end{aligned}$$

This, together with the estimate in (32), infers the estimates for M:

$$-\frac{1}{2}MN - \lambda M \leq M' \leq -\frac{1}{2}MN - \lambda M + \|u_0\|_{H^1}^2.$$

Similarly, for the function N, from (33) we have

$$\frac{1}{2}MN - \lambda N - \|u_0\|_{H^1}^2 \leq N' \leq \frac{1}{2}MN - \lambda N.$$

Consequently, we conclude from the above inequalities that

$$1 + \lambda\Big(\frac{1}{N} - \frac{1}{M}\Big) \leq \frac{N' - M'}{MN} \leq 1 + \lambda\Big(\frac{1}{N} - \frac{1}{M}\Big) - \frac{2\|u_0\|_{H^1}^2}{MN}. \tag{39}$$

Observing that $u_x(t, q(t; x_0)) \to -\infty$ as $t \to T^*$ and u is bounded by $\|u_0\|_{H^1}$, we see $\frac{1}{M} \to 0$ and $\frac{1}{N} \to 0$ as $t \to T^*$. For small $\varepsilon > 0$, we can find $t_0 \in (0, T^*)$ such that t_0 is close enough to T^* in a such way that

$$\Big|\frac{\lambda}{2}\Big(\frac{1}{N} - \frac{1}{M}\Big)\Big| \leq \varepsilon \quad \text{and} \quad \Big|\frac{\lambda}{2}\Big(\frac{1}{N} - \frac{1}{M}\Big) - \frac{\|u_0\|_{H^1}^2}{MN}\Big| \leq \varepsilon$$

on (t_0, T^*). So, on such interval, one can conclude from (39) that

$$\frac{1}{2} - \varepsilon \leq \frac{d}{dt}\Big(\frac{1}{u_x(t, q(t; x_0))}\Big) \leq \frac{1}{2} + \varepsilon.$$

Integrating the above inequalities on (t, T_1^*) with $t \in (t_0, T_1^*)$, we get the blowup rate (14) with $x(t) = q(t, x_0)$. □

4.2. Global existence of the strong solution

In this subsection, we use the obtained conserved quantities in Corollary 5 to show a global existence of the strong solution to (1).

Proof of Theorem 7. With $y := u - u_{xx}$, equation (1) takes the form of a quasi-linear evolution equation of hyperbolic type

$$y_t + (u + k)y_x + 2yu_x + \lambda y = 0. \tag{40}$$

In view of equation (29), one gets easily that

$$\frac{d}{dt}[y(t, q(t, x))q_x(t, x)^2] = -\lambda y(t, q(t, x))q_x^2,$$

which means that

$$e^{\lambda t} y(t, q(t; x))q_x(t; x)^2 = e^{\lambda t} y(t, q(t; x))q_x(t; x)^2|_{t=0} = y_0(x). \tag{41}$$

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When case (i) occurs, since y_0 doesn't change sign on $\mathbb{R}$, without loss of generality, we can assume that $y_0 \geq 0$. Therefore, the identity (41) ensures that $y \geq 0$ for $t \in [0, T)$, $x \in \mathbb{R}$. By the facts that $u = G * y$ and G is positive, we see that u is nonnegative for $t \in [0, T)$, $x \in \mathbb{R}$. On the other hand, note that

$$\begin{aligned} u(t,x) &= \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi} y(t,\xi)d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi} y(t,\xi)d\xi, \\ u_x(t,x) &= -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi} y(t,\xi)d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi} y(t,\xi)d\xi, \end{aligned} \tag{42}$$

which imply that $u + u_x \geq 0$ and $u - u_x \geq 0$. Therefore, the Sobolev's inequality and Corollary 5 tell us that

$$|u_x(t,x)| \leq |u(t,x)| \leq \|u\|_{L^\infty} \leq \frac{\sqrt{2}}{2}\|u\|_{H^1} \leq \frac{\sqrt{2}}{2}e^{-\lambda t}\|u_0\|_{H^1}.$$

Hence, we achieve the desired result from Lemma 10.

For case (ii), in view of (30) and (41), it's easy to get

$$y(t,x)\begin{cases} \leq 0, & x \leq q(t,x_0), \\ \geq 0, & x \geq q(t,x_0). \end{cases}$$

Similar to (42), applying Corollary 5 again we see that, for $x \leq q(t;x_0)$,

$$\begin{aligned} -u_x(t,x) &= -u(t,x) + e^{-x}\int_{-\infty}^{q(t;x_0)} e^{\xi} y(t,\xi)d\xi \\ &\leq -u(t,x) \leq \frac{\sqrt{2}}{2}\|u\|_{H^1} \leq \frac{\sqrt{2}}{2}e^{-\lambda t}\|u_0\|_{H^1} \end{aligned}$$

and, for $x \geq q(t;x_0)$,

$$\begin{aligned} -u_x(t,x) &= u(t,x) - e^{x}\int_{q(t;x_0)}^{\infty} e^{-\xi} y(t,\xi)d\xi \\ &\leq u(t,x) \leq \frac{\sqrt{2}}{2}\|u\|_{H^1} \leq \frac{\sqrt{2}}{2}e^{-\lambda t}\|u_0\|_{H^1}. \end{aligned}$$

Hence, from (40) and the above inequalities, a direct computation yields

$$\begin{aligned} \tfrac{d}{dt}\|y\|_{L^2}^2 &= 2\int_{\mathbb{R}} yy_t dx = -2\int_{\mathbb{R}} y[(u+k)y_x + 2yu_x + \lambda y]dx \\ &= 3\int_{\mathbb{R}} (-u_x)y^2 dx - 2\lambda\int_{\mathbb{R}} y^2 dx \leq \Big(\frac{3\sqrt{2}}{2}e^{-\lambda t}\|u_0\|_{H^1} - 2\lambda\Big)\|y\|_{L^2}^2. \end{aligned}$$

Solving this inequality, we arrive at

$$\|y\|_{L^2}^2 \le \begin{cases} \|y_0\|_{L^2}^2 \, e^{\frac{3\sqrt{2}\|u_0\|_{H^1}}{2} t}, & \text{if } \lambda = 0; \\ \|y_0\|_{L^2}^2 \, e^{\frac{3\sqrt{2}}{2\lambda}\|u_0\|_{H^1}}, & \text{if } \lambda > 0. \end{cases}$$

Note that $y = u - u_{xx}$ and we can represent the function u by $G * y$. We derive from the expression of G that

$$|u_x| \le \|\partial_x G\|_{L^2} \|y\|_{L^2} = \frac{1}{2}\|y\|_{L^2}. \tag{43}$$

Thus, it follows that u_x is bounded in $[0, T]$. Thus, we complete the proof by Lemma 10. □

5. Some special solutions

We construct some special exact solutions in this section.

5.1. Solutions obtained using conservation laws

The method of conservation laws for constructing exact solutions has been proposed in [21]. Here we apply this method to construct solutions of (1) and study the asymptotic behavior for one of obtained solutions which is the strong solution given by Lemma 9.

For the conservation law $D_t C_1^t + D_t C_1^x = 0$, according to the method of conservation laws, some particular solutions are obtained by letting $D_t C_1^t = 0$ and $D_t C_1^x = 0$. Solving the two equations, we obtain the functions

$$u_1 = C_1 e^{-\lambda t} \quad \text{and} \quad u_2 = f(t)e^x + g(t)e^{-x},$$

which are analytical solutions of (1), where $f(t)$ and $g(t)$ are differentiable functions of t. We point out that $u_1 = C_1 e^{-\lambda t}$ is a invariant solution of (1) as shown in Section 2, and $u_2 = f(t)e^x + g(t)e^{-x}$ is a self-similar-type solution. However, these solutions do not like as the unique solution obtained in Lemma 9 which belongs to the space $C([0, T); H^s) \cap C^1([0, T); H^{s-1})$ $(s > 3/2)$. Thus, these analytical solutions imply that the uniqueness in Lemma 9 is not valid if we don't restrict the initial value u_0 in H^s.

5.2. Weak peakon-type solution

In this subsection, we provide a shorter, formal proof of Theorem 8, instead of providing a detailed, rigorous proof.

Let $f \in L^1_{loc}(X)$, where X is an open set of $\mathbb{R}$. Assume that f' exists and is continuous except at a single point $x_0 \in X$ and $f' \in L^1_{loc}(X)$, then the left-and right-handed limits $f(x_0\pm)$ exist and $(T_f)' = T_{f'} + [f(x_0+) - f(x_0-)]\delta_{x_0}$, where T_f is the distribution associated to the function f and δ_{x_0} is the Dirac delta distribution centered at $x = x_0$.

Proof of Theorem 8. Formally, from (15) we have

$$u_x = -ce^{-\lambda t - |x-\tau|} sgn(x - \tau) \quad \text{and} \quad u_{xx} = u - 2ce^{-\lambda t}\delta_\tau$$

where $\tau = kt - ce^{-\lambda t}/\lambda - c_0$, $sgn(\cdot)$ is the sign function and δ_τ is the Dirac delta distribution centered at $x = \tau$. The formal computations yield that $y = u - u_{xx} = 2ce^{-\lambda t}\delta_\tau$ and

$$\begin{aligned} & y_t + (u+k)y_x + 2yu_x + \lambda y \\ = & -2c\lambda e^{-\lambda t}\delta_\tau - 2c(k + ce^{-\lambda t})e^{-\lambda t}\delta'_\tau + (ce^{-\lambda t - |x-\tau|} + k)(2ce^{-\lambda t}\delta'_\tau) \\ & +4ce^{-\lambda t}(-ce^{-\lambda t - |x-\tau|}sgn(x-\tau))\delta_\tau + 2\lambda ce^{-\lambda t}\delta_\tau \\ = & \ 0. \end{aligned}$$

This completes the proof. □

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