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Initial–boundary layer associated with the nonlinear Darcy–Brinkman system

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Abstract

We study the interaction of initial layer and boundary layer in the nonlinear Darcy–Brinkman system in the vanishing Darcy number limit. In particular, we show the existence of a function of corner layer type (so-called initial–boundary layer) in the solution of the nonlinear Darcy–Brinkman system. An approximate solution is constructed by the method of multiple scale expansion in space and in time. We establish the optimal convergence rates in various Sobolev norms via energy method.

Keywords: Initial layer; Boundary layer; Initial–boundary layer; Vanishing Darcy number limit; Darcy–Brinkman system; Multiple scale expansion

1. Introduction

In this article we investigate a singular perturbation problem in fluid dynamics, which is governed by the following incompressible nonlinear Darcy–Brinkman system in a 2D periodic channel $\Omega = [0, 1] \times [0, 1]$

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$$\begin{cases} \epsilon\left(\dfrac{\partial \mathbf{v}^\epsilon}{\partial t} + \left(\mathbf{v}^\epsilon \cdot \nabla\right)\mathbf{v}^\epsilon\right) + \mathbf{v}^\epsilon - \epsilon\Delta\mathbf{v}^\epsilon + \nabla p^\epsilon = \mathbf{F}, \\ \operatorname{div}\mathbf{v}^\epsilon = 0, \\ \mathbf{v}^\epsilon\big|_{z=0,1} = 0, \quad \mathbf{v}^\epsilon \text{ periodic in } x\text{-direction}, \\ \mathbf{v}^\epsilon\big|_{t=0} = \mathbf{v}_0, \end{cases} \tag{1.1}$$

where we use (x, z)-coordinates, suppressing the y variable, so that the z variable is always in the direction normal to the boundary. Here ϵ is a dimensionless parameter which is small in our problem. $\mathbf{v}^\epsilon = (v_1^\epsilon, v_2^\epsilon)$ is the velocity field, p^ϵ is the pressure, and $\mathbf{F}$ is the given external forcing which can be time dependent. We also assume the zeroth order compatibility condition $\mathbf{v}_0|_{z=0,1} = 0$.

The Darcy–Brinkman equation (1.1) can be viewed as an appropriately non-dimensionalized version of the following volume-averaged Navier–Stokes equation (Eq. (2.13) in [29])

$$\rho_l \frac{\partial}{\partial t}\mathbf{u} + \rho_l \mathbf{u} \cdot \nabla\left(\frac{\mathbf{u}}{\chi}\right) = -\chi\nabla[p_l]^l + \eta\Delta\mathbf{u} + \chi\rho_l\mathbf{g} - \frac{\eta\chi}{\Pi(\chi)}\mathbf{u}, \tag{1.2}$$

where ρ_l is the density of the fluid, $\mathbf{u}$ the Darcy velocity, χ the porosity (liquid volume fraction), $[p_l]^l$ the average value of the liquid pressure, η the viscosity, and $\Pi(\chi)$ the permeability defined in Darcy's law. For a given velocity scale V and a given length scale L, if one chooses Darcy's pressure scale $P = \frac{\eta L V}{\Pi_0}$ (Π_0 is the scale of permeability), one can non-dimensionalize (1.2) to obtain (Eq. (2.16) in [29])

$$Da\,Re\left[\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u} \cdot \nabla\left(\frac{\mathbf{u}}{\chi}\right)\right] = -\chi\left[\nabla[p_l]^l + \frac{\rho_l g \Pi_0}{\eta V}\mathbf{e}_z + \frac{\Pi_0}{\Pi(\chi)}\mathbf{u}\right] + Da\Delta\mathbf{u}, \tag{1.3}$$

where $Da = \frac{\Pi_0}{L^2}$ is the Darcy number, $Re = \frac{\rho_l V L}{\eta}$ the Reynolds number, and $\mathbf{e}_z$ a unit vertical vector. In the formal limit of $Da \to 0$ (small permeability), the Darcy–Brinkman equation reduces to the classical Darcy equation

$$\mathbf{u} = -\frac{\Pi(\chi)}{\Pi_0}\left(\nabla[p_l]^l + \frac{\rho_l g \Pi_0}{\eta V}\mathbf{e}_z\right). \tag{1.4}$$

If the variation of the porosity χ is not large, one can simply take χ as a constant. Introducing the seepage velocity $\mathbf{v} = \frac{\mathbf{u}}{\chi}$ and utilizing the Carman–Kozeny permeability function [5]

$$\Pi(\chi) = \Pi_0 \frac{\chi^3}{(1-\chi)^2},$$

the Darcy–Brinkman system (1.3) becomes

$$\frac{\chi^2}{(1-\chi)^2} Da\,Re\left[\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{v}\right] = -\nabla p + \mathbf{F} - \mathbf{v} + \frac{\chi^2}{(1-\chi)^2} Da\Delta\mathbf{v}, \tag{1.5}$$

where $p = \frac{\chi^2}{(1-\chi)^2}[p_l]^l$ and $\mathbf{F} = -\frac{\chi^2}{(1-\chi)^2}\frac{\rho_l g \Pi_0}{\eta V}\mathbf{e}_z$. For the convenience of mathematical analysis, we denote $\epsilon = \frac{\chi^2}{(1-\chi)^2}Da$, set $Re = 1$, and assume more generic forcing $\mathbf{F}$. This leads to the Darcy–Brinkman model (1.1). The case of general Re can be treated in exactly the same fashion.

We note that in many applications ϵ is a small parameter due to either small permeability or small porosity. Formally taking $\epsilon = 0$ in system (1.1), we arrive at the following Darcy equation

$$\begin{cases} \mathbf{v}^0 + \nabla p^0 = \mathbf{F}, \\ \operatorname{div} \mathbf{v}^0 = 0, \\ \mathbf{v}^0 \cdot \mathbf{n}\big|_{z=0,1} = 0, \quad \mathbf{v}^0 \text{ periodic in } x\text{-direction}. \end{cases} \tag{1.6}$$

Note that there is no initial condition for problem (1.6), and the time dependence of $\mathbf{v}^0$ is through the external forcing $\mathbf{F}$. Moreover, one can only impose no penetration boundary condition for the Darcy equation, whereas the velocity field of Eq. (1.1) must satisfy both the no-slip, and no-penetration boundary conditions. We observe that $\mathbf{v}^0$ can also be viewed as the Helmholtz projection of $\mathbf{F}$, cf. [53] for details.

Our aim in this article is to study the convergence of the nonlinear Darcy–Brinkman system (1.1) to the Darcy equation (1.6) in the vanishing Darcy number limit ($\epsilon \to 0$). This is a singular perturbation problem involving both an initial layer (multiple time scales) and a boundary layer (and hence multiple spatial scales). On the one hand, this is similar to the classical boundary layer problem for incompressible viscous fluids at small viscosity that we recall [50,44,55,60,61,27]. Indeed, following the original work of Prandtl [46], we can derive a Prandtl type equation for this model which indicates the existence of a boundary layer in the velocity field of a width proportional to $\sqrt{\epsilon}$ and with no boundary layer in the pressure field (to the leading order). On the other hand, the problem involves an initial layer as well. In this connection, a similar problem has been studied by the second author in the context of Rayleigh–Bénard convection [65–67], see also [45]. As a consequence, the interaction of the boundary layer and initial layer introduces another singular structure of corner layer type (initial–boundary layer), which is new in the present literature to the best of our knowledge.

There is an abundant literature on boundary layer associated with incompressible flows and the related question of vanishing viscosity (see for instance [2,7,48,49,10,41,15,23,47,20,21,8,28,74,32,3,4,33,30,71,1,72,22,58,59,54,56,57,64,24,9,25,19,26,19,26,39,6,16,34,64,40,37,36,12,13,69,63] among many others). We will refrain from surveying the literature here, but emphasize that the boundary layer problem associated with the Navier–Stokes equation is still open and that there is a need to develop tools and methods to tackle it.

The definitions of all of our function spaces reflect the fact that we are working in a domain that is periodic in the horizontal direction (periodic channel). Thus, for instance, $H^m = H^m_{per}(\Omega)$, m a nonnegative integer, is the Sobolev space consisting of all functions on Ω whose weak derivatives up to order m are square integrable and whose weak derivatives up to order $m-1$ are periodic in the horizontal direction, with the usual norm. Similarly, $H^1_{0,per}(\Omega)$ is the subspace of functions in $H^1_{per}(\Omega)$ that vanish on $z = 0, 1$. We will use the classical function spaces of fluid mechanics,

$$H = H(\Omega) = \left\{ \mathbf{v} \in \left(L^2_{per}(\Omega)\right)^2 \colon \operatorname{div} \mathbf{v} = 0,\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } z = 0, 1 \right\},$$

$$V = V(\Omega) = \left\{ \mathbf{v} \in \left(H^1_{0,per}(\Omega)\right)^2 \colon \operatorname{div} \mathbf{v} = 0 \right\},$$

where $\mathbf{n}$ denotes the unit outer normal to $\partial\Omega$. We put the L^2-norm on H and the H^1-norm on V. Because of the Poincaré inequality, we can equivalently use $\|u\|_V = \|\nabla u\|_{L^2}$. We follow the convention that $\|\cdot\|$ is the L^2-norm.

For system (1.1), we work with weak solutions. The following proposition can be proved in a similar fashion as the classical theory of Navier–Stokes equation, cf. [52,53].

Proposition 1.1. *Let $\mathbf{F}, \mathbf{v}_0$ be given in $L^2(0,T;V')$ and H respectively. Then there exists a unique weak solution $\mathbf{v}^\epsilon$ of (1.1), and $\mathbf{v}^\epsilon \in C([0,T],H) \cap L^2(0,T;V)$.*

In addition, if we assume $\mathbf{v}_0 \in V$ and $\mathbf{F} \in L^2(0,T;H)$, then $\mathbf{v}^\epsilon$ is in $C([0,T];V) \cap L^2(0,T;H^2(\Omega))$, and $\frac{\partial \mathbf{v}^\epsilon}{\partial t}$ is in $L^2(0,T;H)$.

The well-posedness of Darcy equation (1.6) can be found in [35] and references therein. For later use, we assume regular data $\mathbf{F} \in C^1([0,T];H^m(\Omega))$ with $m \geqslant 5$. It follows that $\mathbf{v}^0 \in C^1([0,T];H^m(\Omega))$.

The main result in this paper is summarized in the following theorem.

Theorem 1.2. *Assume $\mathbf{v}_0 \in V \cap H^m(\Omega)$ and $\mathbf{F} \in C^1([0,T];H^m(\Omega))$ with $m \geqslant 5$. Then there exists an approximate solution $\mathbf{v}^{app}$ defined in (4.1) as the sum of the solution to Darcy equation, an explicit initial layer, an explicit boundary layer and an initial–boundary layer, so that the following optimal convergence rates hold*

$$\left\|\mathbf{v}^\epsilon - \mathbf{v}^{app}\right\|_{L^\infty(0,T,L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}, \tag{1.7a}$$

$$\left\|\mathbf{v}^\epsilon - \mathbf{v}^{app}\right\|_{L^\infty(0,T,H^1(\Omega))} \leqslant C\epsilon^{\frac{1}{4}}, \tag{1.7b}$$

$$\left\|\mathbf{v}^\epsilon - \mathbf{v}^{app}\right\|_{L^\infty(0,T,L^\infty(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}, \tag{1.7c}$$

$$\left\|\nabla\left(p^\epsilon - p^0\right)\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}} \tag{1.7d}$$

where C is a generic constant independent of ϵ.

Note that $\mathbf{v}^{app}$ comprises the "inviscid" solution to Darcy equation, an initial layer, a boundary layer and an initial–boundary layer. The convergence rate estimates in Theorem 1.2 demonstrate the singular structure of the solution $\mathbf{v}^\epsilon$ in terms of asymptotic expansion: (1.7a) indicates the necessary existence of an initial layer (initial layer is of order 1 in $L^\infty(L^2)$); both (1.7b) and (1.7c) show the presence of an initial layer, a boundary layer and an initial–boundary layer (boundary layer and initial–boundary layer are of order $\epsilon^{-\alpha}$, $\alpha > 0$ in $L^\infty(H^1)$ and of order 1 in $L^\infty(L^\infty)$); (1.7d) verifies that pressure has no singular structure to the leading order. The approach that we take is classical in the sense that we follow a Prandtl type approach, cf. [62,68,18]. Specifically, we derive the Prandtl type effective equations for the correctors (initial layer function, boundary layer function and initial–boundary layer function) that approximate $\mathbf{v}^\epsilon - \mathbf{v}^0$ in the region of initial layer, boundary layer and corner layer, respectively. This alternative approach has the advantage that the matching procedure is conceptually simple: the sum of the Darcy solution and the correctors is a natural candidate for an approximation to the solution of Darcy–Brinkman system. The analysis of the initial–boundary layer problem then consists of the study of the Prandtl type equations, and proof of convergence of the approximate solution to the exact solution of the Darcy–Brinkman system. To show the optimality of the convergence rates, we resort to the order $\sqrt{\epsilon}$ expansion. The key to our success here is a mild nonlinear term in the sense that the convection term $\mathbf{v}^\epsilon \cdot \nabla \mathbf{v}^\epsilon$ has a small coefficient ϵ. Because of this, the Prandtl type equations for the boundary layer and initial–boundary layer (to the leading order) are all linear though the Darcy–Brinkman model (1.1) itself is nonlinear. This is similar to the case of boundary layer for the incompressible Navier–Stokes flows with non-characteristic boundary conditions [61,60] as well as secondly boundary layer associated with the Navier–Stokes equations under Navier type slip boundary conditions [71,69,63,21]. The main difficulty for us is the existence of an initial–boundary layer which necessitates the simultaneous treatment of multiple scales in space and in time.

The paper is organized as follows. An example is given in Section 2 to illustrate the phenomenon inducing the existence of an initial–boundary layer. In Section 3, we give a detailed asymptotic analysis of the singular perturbation problem (1.1), and derive the Prandtl type equations for each layer. In Section 4, the approximate solution is constructed, and the convergence result is proved via energy method. A high order expansion is given in the fifth section. We provide the conclusion at the end.

We would like to point out that though Theorem 1.2 is proved for the case of 2D, it is possible to generalize the results to the 3D case provided the solution to the Darcy equation is sufficiently smooth. Indeed, since the velocity components of the boundary layer (initial–boundary layer) are related only through the divergence-free condition (Eqs. (3.6), (3.16)), explicit formulas for truncated boundary layer and initial–boundary layer can be constructed in 3D easily. A formula for modified boundary layer in 3D is given in [27], see also [70] for construction of background flows with prescribed tangential velocity at the boundary. The construction of modified initial–boundary layer is analogous.

2. An example

The interaction of initial layer and boundary layer can be well illustrated through the following simple example. We consider here a special case of shear flow in the half plane $z > 0$. Assuming in system (1.1) that the data take the form of

$$\mathbf{F} = \left(e^{-z}, 0\right), \qquad \mathbf{v}_0 = \left(v_0(z), 0\right),$$

we seek a solution of the form

$$\mathbf{v}^\epsilon = \left(v_1^\epsilon(t, z), 0\right), \qquad p^\epsilon = 0.$$

Then the system (1.1) reduces to a scalar parabolic equation on the half line $z > 0$

$$\begin{cases} \epsilon \dfrac{\partial v_1^\epsilon}{\partial t} + v_1^\epsilon - \epsilon \dfrac{\partial^2 v_1^\epsilon}{\partial z^2} = e^{-z}, \\ v_1^\epsilon(0, z) = v_0(z), \\ v_1^\epsilon(t, 0) = 0, \\ v_1^\epsilon \to 0, \quad \text{as } z \to \infty. \end{cases} \tag{2.1}$$

The explicit solution to Eq. (2.1) can be found by using the Green's function, see for instance [43]. One can then try to explore the structure of the solution by the method of asymptotic expansion of integrals. Here we take an alternative approach. One first solves the ODE

$$\begin{cases} u^\epsilon - \epsilon \dfrac{\partial^2 u^\epsilon}{\partial z^2} = e^{-z}, \\ u^\epsilon(0) = 0, \\ u^\epsilon \to 0, \quad \text{as } z \to \infty. \end{cases} \tag{2.2}$$

And the solution is found to be

$$u^\epsilon = \frac{1}{1-\epsilon}\left(e^{-z} - e^{-\frac{z}{\sqrt{\epsilon}}}\right).$$

Note that u^ϵ contains a boundary layer type component $e^{-\frac{z}{\sqrt{\epsilon}}}$. Defining $w^\epsilon = v_1^\epsilon - u^\epsilon$, one sees that $w^\epsilon = w_I^\epsilon + w_C^\epsilon$ satisfies

$$\begin{cases} \epsilon \dfrac{\partial w^\epsilon}{\partial t} + w^\epsilon - \epsilon \dfrac{\partial^2 w^\epsilon}{\partial z^2} = 0, \\ w^\epsilon(0, z) = f_R(z) + f_B(z), \\ w^\epsilon(t, 0) = 0, \\ w^\epsilon \to 0, \quad \text{as } z \to \infty, \end{cases} \tag{2.3}$$

with

$$\begin{aligned} f_R(z) &= v_0(z) - \frac{1}{1-\epsilon} e^{-z}, \\ f_B(z) &= \frac{1}{1-\epsilon} e^{-\frac{z}{\sqrt{\epsilon}}}. \end{aligned} \tag{2.4}$$

The regular function $f_R(z)$ contributes to an initial layer type solution $w_I^\epsilon = e^{-\frac{t}{\epsilon}} r(t, z)$ with $r(t, z)$ satisfying the heat equation on the positive half line

$$\begin{cases} \dfrac{\partial r}{\partial t} - \dfrac{\partial^2 r}{\partial z^2} = 0, \\ r(0, z) = f_R(z), \\ r(t, 0) = 0, \\ r \to 0, \quad \text{as } Z \to \infty. \end{cases} \tag{2.5}$$

On the other hand, the boundary layer type initial data f_B develops an initial–boundary layer type solution, since, if one defines $\tau = \frac{t}{\epsilon}$, $Z = \frac{z}{\sqrt{\epsilon}}$, one finds $w_C^\epsilon = \frac{1}{1-\epsilon} w(\tau, Z)$ satisfying the equation

$$\begin{cases} \dfrac{\partial w}{\partial \tau} + w - \dfrac{\partial^2 w}{\partial Z^2} = 0, \\ w(0, Z) = e^{-Z}, \\ w(t, 0) = 0, \\ w \to 0, \quad \text{as } z \to \infty. \end{cases} \tag{2.6}$$

To sum up, one finds

$$v_1^\epsilon = \frac{1}{1-\epsilon}e^{-z} - \frac{1}{1-\epsilon}e^{-\frac{z}{\sqrt{\epsilon}}} + e^{-\frac{t}{\epsilon}}r(t,z) + \frac{1}{1-\epsilon}w\left(\frac{t}{\epsilon}, \frac{z}{\sqrt{\epsilon}}\right),$$

which clearly reveals the existence of a boundary layer, an initial layer, and an initial–boundary layer.

3. Asymptotic analysis

In this section, we will derive the equations satisfied by the initial layer, boundary layer and initial–boundary layer, respectively. The approach we take is of Prandtl type. We will focus on deriving the Prandtl type equations near the boundary $z = 0$; the equations near $z = 1$ are entirely analogous. Motivated by the example in Section 2, we formally assume the solutions of system (1.1) have an asymptotic expansion of the form

$$\begin{aligned} &\mathbf{v}^\epsilon = \mathbf{v}^0(t,x,z) + \mathbf{v}^I(t/\epsilon,x,z) + \mathbf{v}^B(t,x,z/\sqrt{\epsilon}) + \mathbf{v}^C(t/\epsilon,x,z/\sqrt{\epsilon}) + \cdots, \\ &p^\epsilon = p^0(t,x,z) + \cdots, \end{aligned} \tag{3.1}$$

with the superscripts I, B, C denoting the initial layer, boundary layer, and initial–boundary layer, respectively. Here $\mathbf{v}^I$ and $\mathbf{v}^B$ (more precisely v_1^B) take care of the difference in initial and boundary conditions between systems (1.1) and (1.6), and $\mathbf{v}^C$ takes care of the extra boundary condition introduced by $\mathbf{v}^I$ and the extra initial condition introduced by $\mathbf{v}^B$. Introducing the stretched variables $\tau = \frac{t}{\epsilon}$ and $Z = \frac{z}{\sqrt{\epsilon}}$, we impose the matching conditions as follows

$$\begin{aligned} &\mathbf{v}^I \to 0, \quad \text{as } \tau \to \infty, \\ &v_1^B \to 0, \quad \text{as } Z \to \infty, \\ &\mathbf{v}^C \to 0, \quad \text{as } \tau \to \infty, \\ &v_1^C \to 0, \quad \text{as } Z \to \infty. \end{aligned} \tag{3.2}$$

Note we did not impose decaying condition for v_2^B and v_2^C when $Z \to \infty$, since the boundary conditions in normal direction are the same in systems (1.1) and (1.6). We will see later that v_2^B is not of boundary layer type.

Plug the expansion (3.1) into Eq. (1.1) and keep all the $O(1)$ terms (in terms of $\sqrt{\epsilon}$). Outside the initial layer and boundary layer region ($\tau, Z \to \infty$, respectively), one rederives the Darcy equation (1.6) by the matching conditions (3.2) and using the incompressibility condition. Within the initial layer region but outside the boundary layer region ($Z \to \infty$), one deduces the initial layer equation (3.4). Likewise, one has the boundary layer equation (3.6) within the boundary layer region and outside the initial layer region ($\tau \to \infty$). After subtracting the Darcy equation, initial layer equation, and boundary layer equation, one is left with initial–boundary layer equation (3.16). The corresponding initial and boundary conditions are derived in the same way.

Note that the boundary layer and the initial–boundary layer exist both at $z = 0$ and $z = 1$. For constructing an approximate solution which satisfies the same boundary conditions as in (1.1), we have to modify the boundary layer and initial–boundary layer profile. The rest of this section is devoted to the study of initial layer, and construction of modified boundary layer and initial–boundary layer. Throughout the rest of the paper, the following convention will be assumed

$$a(t,x)=v_1^0(t,x,0),\qquad b(t,x)=v_1^0(t,x,1),\qquad c(x)=v_1^0(0,x,0),$$
$$d(x)=v_1^0(0,x,1),\qquad \Omega_\infty=\{(x,Z)\mid x\in[0,1],\ Z\in(0,\infty)\}. \tag{3.3}$$

3.1. Initial layer

The initial layer $\mathbf{v}^I$ satisfies an ODE

$$\begin{cases} \epsilon\dfrac{\partial \mathbf{v}^I}{\partial t}+\mathbf{v}^I=0,\\ \mathbf{v}^I\big|_{t=0}=\mathbf{v}_0(x,z)-\mathbf{v}^0(0,x,z), \end{cases} \tag{3.4}$$

where $\mathbf{v}^0$ is the solution to the Darcy equation (1.6). Its solution is given by $(\mathbf{v}_0(x,z)-\mathbf{v}^0(0,x,z))e^{-\frac{t}{\epsilon}}$. It follows that

$$\|\mathbf{v}^I,\nabla\mathbf{v}^I,\Delta\mathbf{v}^I\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C,$$
$$\|\mathbf{v}^I,\nabla\mathbf{v}^I,\Delta\mathbf{v}^I\|_{L^2(0,T;L^2(\Omega))}\leqslant C\sqrt{\epsilon}, \tag{3.5}$$

provided that $\mathbf{v}_0(x,z)-\mathbf{v}^0(0,x,z)\in H^2(\Omega)$. Note that $\nabla\cdot\mathbf{v}^I=0$ since $\mathbf{v}_0$ is divergence-free by assumption.

3.2. Boundary layer

Recalling the stretched variable $Z=\frac{z}{\sqrt{\epsilon}}$, one finds that the leading order boundary layer $\mathbf{v}^{B,0}$ defined near $z=0$ satisfies the following Prandtl type equation

$$\begin{cases} v_1^{B,0}-\partial_{ZZ}v_1^{B,0}=0,\qquad Z\in(0,\infty),\\ \dfrac{\partial v_1^{B,0}}{\partial x}+\dfrac{1}{\sqrt{\epsilon}}\dfrac{\partial v_2^{B,0}}{\partial Z}=0,\qquad Z\in(0,\infty),\ x\in(0,1),\\ v_1^{B,0}|_{Z=0}=-a(t,x),\qquad v_2^{B,0}|_{Z=0}=0,\\ v_1^{B,0}\to 0,\qquad \text{as } Z\to\infty. \end{cases} \tag{3.6}$$

The boundary layer near $z=1$ satisfies a similar equation with the stretched variable $Z=\frac{1-z}{\sqrt{\epsilon}}$ and $a(t,x)$ replaced by $b(t,x)$. The solution to Eq. (3.6) is given as

$$v_1^{B,0}=-a(t,x)e^{-\frac{z}{\sqrt{\epsilon}}},\qquad v_2^{B,0}=\sqrt{\epsilon}\frac{\partial a}{\partial x}\bigl(1-e^{-\frac{z}{\sqrt{\epsilon}}}\bigr). \tag{3.7}$$

Clearly $v_1^{B,0}$ and $v_2^{B,0}$ do not vanish at $z=1$. Likewise, the boundary layer functions at $z=1$ are not zero at the boundary $z=0$. A truncation is needed to ensure the overall boundary layer profile satisfies the respective boundary conditions exactly. To maintain the divergence-free condition, we take the truncation at the stream function level. Such an approach of truncation has been extensively used in the study of boundary layer problems for incompressible flow, cf. [55,60,61,27] for instance.

Introduce a cut-off function $\rho \in C^\infty[0,1]$ such that

$$\begin{cases} \rho = 1, & z \in \left[0, \frac{1}{4}\right], \\ \rho = 0, & z \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{3.8}$$

The truncated stream function is thus defined as, in view of the solution formula (3.7),

$$\psi^{B,0} = \sqrt{\epsilon} a(t,x)\left(1 - e^{-z/\sqrt{\epsilon}}\right)\rho(z).$$

Then the modified boundary layer profile at $z = 0$ can be defined as follows

$$\tilde{\mathbf{v}}^{B,0} = \nabla^\perp \psi^{B,0} = \left(-\frac{\partial \psi^{B,0}}{\partial z}, \frac{\partial \psi^{B,0}}{\partial x}\right).$$

Thus

$$\begin{aligned} \tilde{v}_1^{B,0} &= -ae^{-Z}\rho - \sqrt{\epsilon} a\left(1 - e^{-Z}\right)\rho', \\ \tilde{v}_2^{B,0} &= \sqrt{\epsilon}\frac{\partial a}{\partial x}\left(1 - e^{-Z}\right)\rho. \end{aligned} \tag{3.9}$$

The modified boundary layer profile $\tilde{\mathbf{v}}^{B,1}$ at $z = 1$ is constructed in a similar fashion. Letting

$$\tilde{\mathbf{v}}^B = \tilde{\mathbf{v}}^{B,0} + \tilde{\mathbf{v}}^{B,1}, \tag{3.10}$$

we see $\tilde{\mathbf{v}}^B$ is equal to the exact boundary layer profile $\tilde{\mathbf{v}}^{B,0}$ and $\tilde{\mathbf{v}}^{B,1}$ within a fixed width $\frac{1}{4}$ of the respective boundaries. Moreover, $\nabla \cdot \tilde{\mathbf{v}}^B = 0$ by construction, and $\operatorname{supp} \tilde{\mathbf{v}}^{B,0} \cap \operatorname{supp} \tilde{\mathbf{v}}^{B,1} = \emptyset$.

By using (3.9), we infer that $\tilde{\mathbf{v}}^B$ satisfies the following modified Prandtl type equation (in original variables)

$$\begin{cases} \tilde{\mathbf{v}}^B - \epsilon \Delta \tilde{\mathbf{v}}^B = \mathbf{f}^B, \\ \nabla \cdot \tilde{\mathbf{v}}^B = 0, \\ \tilde{\mathbf{v}}^B\big|_{z=0,1} = -\mathbf{v}^0\big|_{z=0,1}, \end{cases} \tag{3.11}$$

where $\mathbf{f}^B = \mathbf{f}^{B,0} + \mathbf{f}^{B,1}$ with $\mathbf{f}^{B,0} = (f_1^{B,0}, f_2^{B,0})$ defined as follows

$$f_1^{B,0} = \epsilon^{\frac{3}{2}} \Delta\left(a\rho'\right)\left(1 - e^{-\frac{z}{\sqrt{\epsilon}}}\right) + \epsilon\left(\frac{\partial^2 a}{\partial x^2}\rho + 3a\rho''\right)e^{-\frac{z}{\sqrt{\epsilon}}} - 2\sqrt{\epsilon} a e^{-\frac{z}{\sqrt{\epsilon}}}\rho' - \sqrt{\epsilon} a\rho', \tag{3.12a}$$

$$f_2^{B,0} = -\epsilon^{\frac{3}{2}} \Delta\left(\frac{\partial a}{\partial x}\rho\right)\left(1 - e^{-\frac{z}{\sqrt{\epsilon}}}\right) - 2\epsilon\frac{\partial a}{\partial x} e^{-\frac{z}{\sqrt{\epsilon}}}\rho' + \sqrt{\epsilon}\frac{\partial a}{\partial x}\rho. \tag{3.12b}$$

$\mathbf{f}^{B,1}$ has similar terms.

Remark 1. It is clear from (3.12a) that the truncation introduces extra error terms of order $\sqrt{\epsilon}$ which are not of boundary layer type (the functions do not decay when $Z \to \infty$).

A direct calculation based on (3.9) gives

$$\begin{aligned}
&\|\tilde{v}_1^{B,0}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{4}},\\
&\|\partial_z\tilde{v}_1^{B,0}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{-\frac{1}{4}},\\
&\|\tilde{v}_2^{B,0}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}},\\
&\|\partial_z\tilde{v}_2^{B,0}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{4}}
\end{aligned} \tag{3.13}$$

with C a generic constant independent of ϵ. Moreover, one has

$$\|\partial_x^j \mathbf{f}^B\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}, \quad j = 0, 1, \tag{3.14}$$

which follows easily from (3.12), provided that

$$\partial_x^3 a = \partial_x^3 v_1^0(t, x, 0) \in L^\infty\big(0, T; L^2(\partial\Omega)\big).$$

For the spatial uniform estimate, we recall here the following version of anisotropic Sobolev embedding (Corollary 7.3 from [27], see also [55,73]).

Lemma 3.1. *There exists a constant C such that for any $u \in H^1_{0,per}(\Omega)$*

$$\|u\|_{L^\infty(\Omega)} \leqslant C\big(\|u\|_{L^2}^{\frac{1}{2}}\|\partial_z u\|_{L^2}^{\frac{1}{2}} + \|\partial_x u\|_{L^2}^{\frac{1}{2}}\|\partial_z u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}^{\frac{1}{2}}\|\partial_x\partial_z u\|_{L^2}^{\frac{1}{2}}\big),$$

where one or both sides of the inequality could be infinite.

Note that taking derivative with respect to x or t does not change the estimate (3.13) as long as a is smooth in x and t. Combining this observation, (3.13) and Lemma 3.1 we deduce

Proposition 3.2. *Assume $\mathbf{v}_0 \in V \cap H^m(\Omega)$ and $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ with $m \geqslant 4$. The following estimates hold*

$$\begin{aligned}
&\|\partial_t^j\tilde{v}_1^B\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{4}}, \quad j = 0, 1,\\
&\|\partial_t^j\tilde{v}_2^B\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}, \quad j = 0, 1,\\
&\|\nabla\tilde{v}_1^B\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{-\frac{1}{4}},\\
&\|\nabla\tilde{v}_2^B\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{4}},\\
&\|\tilde{v}_1^B\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant C,\\
&\|\tilde{v}_2^B\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant C\epsilon^{\frac{1}{4}}.
\end{aligned} \tag{3.15}$$

Remark 2. It is clear from the explicit solution formula (3.7) that the estimates above are all optimal.

3.3. Initial–boundary layer

We recall the definition of the stretched variables $\tau = \frac{t}{\epsilon}$, $Z = \frac{z}{\sqrt{\epsilon}}$. The initial–boundary layer $\mathbf{v}^{C,0}$ at the corner $t=0$, $z=0$ satisfies the following Prandtl type equation

$$\begin{cases} \dfrac{\partial v_1^{C,0}}{\partial \tau} + v_1^{C,0} - \partial_{ZZ} v_1^{C,0} = 0, \quad \tau > 0,\ Z > 0, \\ \dfrac{\partial v_1^{C,0}}{\partial x} + \dfrac{1}{\sqrt{\epsilon}} \dfrac{\partial v_2^{C,0}}{\partial Z} = 0, \\ v_1^{C,0}\big|_{\tau=0} = -v_1^{B,0}\big|_{t=0} = c(x)e^{-Z}, \\ v_1^{C,0}\big|_{Z=0} = -v_1^{I}\big|_{z=0} = c(x)e^{-\tau}, \quad v_2^{C,0}\big|_{Z=0} = 0, \\ v_1^{C,0} \to 0, \quad \text{as } Z \to \infty, \end{cases} \tag{3.16}$$

with $c(x) = v_1^0(0, x, 0)$ defined in (3.3) as the initial–boundary value (at $z = 0$) of the first/tangential component of the Darcy velocity. In the derivation one has applied the compatibility condition $\mathbf{v}_0|_{z=0} = 0$.

Remark 3. Though the initial condition of Darcy–Brinkman system satisfy the compatibility condition $\mathbf{v}_0|_{z=0} = 0$, $c(x)$ in (3.16) may not be zero as the Darcy velocity $\mathbf{v}^0$ can be viewed as the Helmholtz projection of $\mathbf{F}$ into the divergence-free space H.

We recall that Ω_∞ is given in Eq. (3.3). For system (3.16), one has the following a priori estimates.

Lemma 3.3.

$$\begin{aligned} &\left|v_1^{C,0}\right| \leqslant \left|c(x)\right| e^{-\frac{\tau}{2} - \frac{Z}{\sqrt{2}}}, \\ &\left\|v_1^{C,0}\right\|_{L^\infty(0,\infty; L^2(\Omega_\infty))} \leqslant C, \\ &\left\|\partial_Z v_1^{C,0}\right\|_{L^\infty(0,\infty; L^2(\Omega_\infty))} \leqslant C, \\ &\left\|v_1^{C,0}\right\|_{L^2(0,\infty; L^2(\Omega_\infty))} \leqslant C, \\ &\left\|\partial_Z v_1^{C,0}\right\|_{L^2(0,\infty; L^2(\Omega_\infty))} \leqslant C, \\ &\left\|Z\partial_Z v_1^{C,0}\right\|_{L^\infty(0,\infty; L^2(\Omega_\infty))} \leqslant C. \end{aligned}$$

Proof. We first establish the pointwise estimate

$$\left|v_1^{C,0}\right| \leqslant \left|c(x)\right| e^{-\frac{\tau}{2} - \frac{Z}{\sqrt{2}}}.$$

For that, we introduce an auxiliary function $k(\tau, x, Z) = |c(x)|e^{-\frac{\tau}{2}-\frac{Z}{\sqrt{2}}}$. It is clear that k satisfies the equation

$$\frac{\partial k}{\partial \tau} + k - \partial_{ZZ} k = 0,$$

but with

$$k|_{\tau=0} \geqslant v_1^{C,0}\big|_{\tau=0}, \qquad k|_{Z=0} \geqslant v_1^{C,0}\big|_{Z=0}.$$

The comparison principle for parabolic equations (p. 219 in [31]) gives $v_1^{C,0} \leqslant k$. The same argument applies to $-v_1^{C,0}$, which concludes the proof.

For the rest of the inequalities, we apply the standard energy estimate to the equation satisfied by $\phi(\tau, x, Z) := v_1^{C,0} - c(x)e^{-\tau-Z}$

$$\begin{cases} \dfrac{\partial \phi}{\partial \tau} + \phi - \partial_{ZZ}\phi = l(\tau, x, Z), \quad \tau > 0,\ Z > 0, \\ \phi|_{\tau=0} = 0, \\ \phi|_{Z=0} = 0; \quad \phi \to 0, \quad \text{as } Z \to \infty, \end{cases}$$

where $l(\tau, x, Z) := c(x)e^{-\tau-Z}$. While the approach is classical, we reproduce the last estimate here for completeness. Multiply the above equation by $Z^2\phi$ and integrate the resulting equation over Ω_∞. We then obtain through integration by parts that

$$\frac{1}{2}\frac{d}{d\tau}\int_{\Omega_\infty} Z^2\phi^2\,dZ + \int_{\Omega_\infty} Z^2\phi^2\,dZ + \int_{\Omega_\infty} \partial_Z\big(Z^2\phi\big)\partial_Z\phi\,dZ \\ = \int_{\Omega_\infty} Z^2\phi l\,dZ.$$

This can be reduced to

$$\frac{d}{d\tau}\int_{\Omega_\infty} Z^2\phi^2\,dZ + \int_{\Omega_\infty} Z^2\phi^2\,dZ + \int_{\Omega_\infty} Z^2(\partial_Z\phi)^2\,dZ \\ \leqslant C\int_{\Omega_\infty} Z^2 l^2\,dZ + C\int_{\Omega_\infty} \phi^2\,dZ.$$

It follows from this and $\|v_1^{C,0}\|_{L^2(0,\infty;L^2(\Omega_\infty))} \leqslant C$ that

$$\|Z\phi\|_{L^\infty(0,\infty;L^2(\Omega_\infty))} + \|Z\phi\|_{L^2(0,\infty;L^2(\Omega_\infty))} + \|Z\partial_Z\phi\|_{L^2(0,\infty;L^2(\Omega_\infty))} \leqslant C.$$

Next, one multiplies the equation by $Z^2\frac{\partial\phi}{\partial\tau}$ and integrates it over Ω_∞. The same manipulation leads to

$$\int_{\Omega_\infty} Z^2(\partial_\tau\phi)^2\,dZ + \frac{d}{d\tau}\biggl(\int_{\Omega_\infty} Z^2\phi^2\,dZ + \int_{\Omega_\infty} Z^2(\partial_Z\phi)^2\biggr)dZ \leqslant C\int_{\Omega_\infty} Z^2 l^2\,dZ + C\int_{\Omega_\infty}(\partial_Z\phi)^2\,dZ.$$

An integration in time yields

$$\|Z\partial_Z\phi\|_{L^\infty(0,\infty;L^2(\Omega_\infty))} \leqslant C,$$

in which the estimate $\|\partial_Z v_1^{C,0}\|_{L^2(0,\infty;L^2(\Omega_\infty))} \leqslant C$ has been used. The estimate for $Z\partial_Z v_1^{C,0}$ then follows from an application of the triangle inequality. □

Remark 4. It is clear that $\partial_x v_1^{C,0}$ satisfies the same type of inequalities as $v_1^{C,0}$.

With Lemma 3.3, the following proposition can be established by using the change of variable and the divergence-free condition.

Proposition 3.4. *Assume* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ *with* $m \geqslant 4$. *One has the following estimates*

$$\begin{aligned}
&\|v_1^{C,0}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{4}},\\
&\|v_2^{C,0}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}},\\
&\|v_1^{C,0}\|_{L^2(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{3}{4}},\\
&\|v_2^{C,0}\|_{L^2(0,T;L^2(\Omega))} \leqslant C\epsilon,\\
&\|\partial_z v_1^{C,0}\|_{L^2(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{4}},\\
&\|\partial_z v_1^{C,0}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{-\frac{1}{4}},\\
&\|\partial_z v_2^{C,0}\|_{L^2(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{3}{4}},\\
&\|\partial_z v_2^{C,0}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{4}}.
\end{aligned}$$

We proceed to construct a modified initial–boundary layer profile by truncating the stream function of system (3.16). Finding the stream function for $(v_1^{C,0}, v_2^{C,0})$ is an inverse problem. Since by definition

$$v_1^{C,0} = -\frac{\partial\psi^{C,0}}{\partial z} = -\frac{1}{\sqrt{\epsilon}}\frac{\partial\psi^{C,0}}{\partial Z},$$

we infer from Eq. (3.16) that $\psi^{C,0}$ should satisfy, assuming $\psi^{C,0}|_{Z=0} = 0$ which is consistent with $\frac{\partial\psi^{C,0}}{\partial x}|_{Z=0} = 0$

$$\begin{cases} \dfrac{\partial \psi^{C,0}}{\partial \tau} + \psi^{C,0} - \partial_{ZZ}\psi^{C,0} = f(\tau, x), \\ \psi^{C,0}\big|_{\tau=0} = -\sqrt{\epsilon}c(x)\big(1 - e^{-Z}\big), \\ \dfrac{\partial \psi^{C,0}}{\partial Z}\bigg|_{Z=0} = -\sqrt{\epsilon}c(x)e^{-\tau}, \\ \dfrac{\partial \psi^{C,0}}{\partial Z} \to 0, \quad \text{as } Z \to \infty, \\ \psi^{C,0}\big|_{Z=0} = 0, \end{cases} \tag{3.17}$$

with an integral function $f(\tau, x)$ to be determined so that the overdetermined system is solvable. To find $\psi^{C,0}$, we first solve the following equation

$$\begin{cases} \dfrac{\partial \psi_1}{\partial \tau} + \psi_1 - \partial_{ZZ}\psi_1 = 0, \\ \psi_1|_{\tau=0} = -\sqrt{\epsilon}c(x)\big(1 - e^{-Z}\big), \\ \dfrac{\partial \psi_1}{\partial Z}\bigg|_{Z=0} = -\sqrt{\epsilon}c(x)e^{-\tau}, \\ \dfrac{\partial \psi_1}{\partial Z} \to 0, \quad \text{as } Z \to +\infty. \end{cases} \tag{3.18}$$

Its solution can be found by using the Green's function approach, cf. [43,11]

$$\psi_1 = -\sqrt{\epsilon}c(x)e^{-\tau}\left\{ \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} \big(1 - e^{-Z_0}\big)\big(e^{-\frac{(Z-Z_0)^2}{4\tau}} + e^{-\frac{(Z+Z_0)^2}{4\tau}}\big)\, dZ_0 \right.$$
$$\left. - 2\int_0^{\tau} \frac{1}{\sqrt{4\pi s}} e^{-\frac{Z^2}{4s}}\, ds \right\}. \tag{3.19}$$

Next, we take $\psi_2 = -\psi_1|_{Z=0}$ so that

$$\psi_2 = 2\sqrt{\epsilon}c(x)e^{-\tau}\left\{ \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} \big(1 - e^{-Z_0}\big) e^{-\frac{Z_0^2}{4\tau}}\, dZ_0 - \int_0^{\tau} \frac{1}{\sqrt{4\pi s}}\, ds \right\}$$
$$= 2\sqrt{\epsilon}c(x)e^{-\tau}\left\{ \frac{1}{2} - \frac{e^{\tau}}{\sqrt{\pi}} \int_{\sqrt{\tau}}^{+\infty} e^{-z^2}\, dz - \int_0^{\tau} \frac{1}{\sqrt{4\pi s}}\, ds \right\}$$
$$\big(\text{by a change of variable } z = (Z_0 + 2\tau)/\sqrt{4\tau}\big).$$

Now defining $\psi^{C,0} = \psi_1 + \psi_2$, we see that $\psi^{C,0}$ satisfies the system (3.17) with

$$f(\tau, x) = -\frac{2\sqrt{\epsilon}c(x)}{\sqrt{\pi}} \int_{\sqrt{\tau}}^{+\infty} e^{-z^2}\, dz = -\frac{2\sqrt{\epsilon}c(x)e^{-\tau}}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\frac{z^2}{4\tau} - z}\, dz. \tag{3.20}$$

Pursuing the same line of thought as in the truncation of boundary layer, we define the truncated stream function as

$$\tilde{\psi}^{C,0} = \psi^{C,0}\rho(z),$$

so that the modified initial–boundary layer is given by

$$\tilde{\mathbf{v}}^{C,0} = \left(-\frac{\partial \tilde{\psi}^{C,0}}{\partial z}, \frac{\partial \tilde{\psi}^{C,0}}{\partial x}\right).$$

Thus

$$\begin{aligned} \tilde{v}_1^{C,0} &= v_1^{C,0}\rho(z) - \psi^{C,0}\rho'(z), \\ \tilde{v}_2^{C,0} &= v_2^{C,0}\rho(z). \end{aligned} \tag{3.21}$$

The corresponding $\tilde{\mathbf{v}}^{C,1}$ at the boundary $z = 1$ is defined similarly. Note that $\tilde{\mathbf{v}}^{C,0}(0, x, Z) = -\tilde{\mathbf{v}}^{B,0}(0, x, Z)$.

Taking

$$\tilde{\mathbf{v}}^C = \tilde{\mathbf{v}}^{C,0} + \tilde{\mathbf{v}}^{C,1}, \tag{3.22}$$

one finds that $\tilde{\mathbf{v}}^C$ satisfies the following system

$$\begin{cases} \epsilon \dfrac{\partial \tilde{\mathbf{v}}^C}{\partial t} + \tilde{\mathbf{v}}^C - \epsilon \Delta \tilde{\mathbf{v}}^C = \mathbf{f}^C, \quad t \in (0, T),\ (x, z) \in \Omega, \\ \nabla \cdot \tilde{\mathbf{v}}^C = 0, \\ \tilde{\mathbf{v}}^C\big|_{t=0} = -\tilde{\mathbf{v}}^B\big|_{t=0}, \\ \tilde{\mathbf{v}}^C\big|_{z=0,1} = -\mathbf{v}^I\big|_{z=0,1}. \end{cases} \tag{3.23}$$

Here $\mathbf{f}^C = \mathbf{f}_1^{C,0} + \mathbf{f}_2^{C,0}$, and $\mathbf{f}_1^{C,0} = (f_1^{C,0}, f_2^{C,0})$ with

$$\begin{aligned} f_1^{C,0} = {} & -f\rho' - 2\epsilon \frac{\partial v_1^{C,0}}{\partial z}\rho' - 3\epsilon v_1^{C,0}\rho'' \\ & + \epsilon \psi^{C,0}\rho''' - \epsilon \frac{\partial^2 v_1^{C,0}}{\partial x^2}\rho + \epsilon \frac{\partial v_2^{C,0}}{\partial x}\rho', \end{aligned} \tag{3.24a}$$

$$f_2^{C,0} = \sqrt{\epsilon}\left(2\frac{\partial v_1^{C,0}}{\partial x}\rho' - v_2^{C,0}\rho'' - \frac{\partial^2 v_2^{C,0}}{\partial x^2}\rho\right). \tag{3.24b}$$

In deriving $f_1^{C,0}$ one has utilized Eq. (3.17).

For $\tilde{\mathbf{v}}^C$ and $\mathbf{f}^C$, one has the following estimates.

Lemma 3.5. *The assumption is the same as the one in Proposition 3.4. Then the following inequalities hold*

$$\begin{aligned}
&\|\tilde{v}_1^C\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{4}},\\
&\|\tilde{v}_2^C\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}},\\
&\|\nabla\tilde{v}_1^C\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{-\frac{1}{4}},\\
&\|\nabla\tilde{v}_2^C\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{4}},\\
&\|\tilde{v}_1^C\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant C,\\
&\|\tilde{v}_2^C\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant C\epsilon^{\frac{1}{4}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\|\partial_x^j \mathbf{f}^C\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}, \quad j=0,1,\\
&\|\partial_x^j \mathbf{f}^C\|_{L^2(0,T;L^2(\Omega))} \leqslant C\epsilon, \quad j=0,1.
\end{aligned}$$

Proof. Since

$$\psi^{C,0} = -\sqrt{\epsilon}\int_0^Z v_1^{C,0}\,dZ_0,$$

Lemma 3.3 implies that $\psi^{C,0}$ satisfies the same estimate as $v_2^{C,0}$. Thus $\tilde{\mathbf{v}}^{C,0}$ has the same estimate as $\mathbf{v}^{C,0}$ in Proposition 3.4. The estimates of $\tilde{\mathbf{v}}^C$ then follow from Proposition 3.4 and Lemma 3.1.

For the estimate of $\mathbf{f}^C$, one only needs to control the term $f\rho'$ in (3.24). It follows from (3.20)

$$\big|f(\tau,x)\big| \leqslant \frac{2\sqrt{\epsilon}|c(x)|e^{-\tau}}{\sqrt{4\pi\tau}}\int_0^{+\infty} e^{-\frac{z^2}{4\tau}}\,dz = \sqrt{\epsilon}\big|c(x)\big|e^{-\tau},$$

which yields

$$\|f\rho'\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}},$$

and by the change of variable $\tau = \frac{t}{\epsilon}$

$$\|f\rho'\|_{L^2(0,T;L^2(\Omega))} \leqslant C\epsilon.$$

We thus proved the lemma. □

Remark 5. We point out that the estimates in Lemma 3.5 are also optimal. In fact, since the initial–boundary layer equation (3.16) is a linear parabolic equation, the solution formula (3.21) can be given explicitly in terms of the Green's function, cf. [43]. (Compare to (3.18) and (3.19).)

4. Approximate solution and error estimate

Based on the asymptotic analysis above, we define the approximate solution as follows

$$\begin{aligned} &\mathbf{v}^{app} = \mathbf{v}^0 + \mathbf{v}^I + \tilde{\mathbf{v}}^B + \tilde{\mathbf{v}}^C, \\ &p^{app} = p^0. \end{aligned} \tag{4.1}$$

Plugging the approximate solution $\mathbf{v}^{app}$ and p^{app} into the system (1.1), one finds they should satisfy

$$\begin{cases} \epsilon\left(\dfrac{\partial \mathbf{v}^{app}}{\partial t} + (\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{app}\right) + \mathbf{v}^{app} - \epsilon\Delta\mathbf{v}^{app} + \nabla p^{app} = \mathbf{F} + \mathbf{F}^{err}, \\ \nabla\cdot\mathbf{v}^{app} = 0, \\ \mathbf{v}^{app}\big|_{z=0,1} = 0, \\ \mathbf{v}^{app}\big|_{t=0} = \mathbf{v}_0, \end{cases} \tag{4.2}$$

with the extra body forcing $\mathbf{F}^{err}$ defined as

$$\mathbf{F}^{err} = \mathbf{f}^B + \mathbf{f}^C + \epsilon\left(\frac{\partial \mathbf{v}^0}{\partial t} + \frac{\partial \tilde{\mathbf{v}}^B}{\partial t}\right) + \epsilon(\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{app} - \epsilon(\Delta\mathbf{v}^0 + \Delta\mathbf{v}^I). \tag{4.3}$$

In view of the estimates (3.5) and (3.14), Proposition 3.2 and Lemma 3.5, one can easily deduce the following estimate for the extra body forcing $\mathbf{F}^{err}$.

Lemma 4.1. *Under the assumption of* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ *with* $m \geqslant 4$, *the following estimate holds*

$$\left\|\partial_x^j \mathbf{F}^{err}\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}, \quad j = 0, 1. \tag{4.4}$$

Define the error functions

$$\mathbf{v}^{err} = \mathbf{v}^\epsilon - \mathbf{v}^{app}, \qquad p^{err} = p^\epsilon - p^{app}.$$

Combining systems (1.1) and (4.2), one can see that $\mathbf{v}^{err}$, p^{err} satisfy the following equations

$$\begin{aligned} &\epsilon\left(\frac{\partial \mathbf{v}^{err}}{\partial t} + (\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err} + (\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err} + (\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{app}\right) + \mathbf{v}^{err} \\ &\quad - \epsilon\Delta\mathbf{v}^{err} + \nabla p^{err} = -\mathbf{F}^{err}, \\ &\nabla\cdot\mathbf{v}^{err} = 0, \\ &\mathbf{v}^{err}\big|_{z=0,1} = 0, \\ &\mathbf{v}^{err}\big|_{t=0} = 0. \end{aligned} \tag{4.5}$$

Now we are in a position to state our main theorem.

Theorem 4.2. *Assume* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ *with* $m \geqslant 4$. *The following convergence rates hold*

$$\left\|\mathbf{v}^\epsilon - \mathbf{v}^{app}\right\|_{L^\infty(0,T,L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}, \tag{4.6a}$$

$$\left\|\mathbf{v}^\epsilon - \mathbf{v}^{app}\right\|_{L^\infty(0,T,H^1(\Omega))} \leqslant C, \tag{4.6b}$$

$$\left\|\mathbf{v}^\epsilon - \mathbf{v}^{app}\right\|_{L^\infty(0,T,L^\infty(\Omega))} \leqslant C\epsilon^{\frac{1}{4}}, \tag{4.6c}$$

$$\left\|\nabla\left(p^\epsilon - p^0\right)\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}. \tag{4.6d}$$

Since $\|\mathbf{v}^I + \tilde{\mathbf{v}}^B + \tilde{\mathbf{v}}^C\|_{L^2(0,T,L^2(\Omega))}$ is of order $O(\epsilon^{\frac{1}{4}})$ (cf. Remarks 2 and 5), estimate (4.6a) implies the following optimal vanishing viscosity limit result

Corollary 4.3. *The assumption is that in Theorem 4.2. One has*

$$C_1\epsilon^{\frac{1}{4}} \leqslant \left\|\mathbf{v}^\epsilon - \mathbf{v}^0\right\|_{L^2(0,T,L^2(\Omega))} \leqslant C_2\epsilon^{\frac{1}{4}}, \tag{4.7}$$

with constants $C_1 < C_2$.

For the estimate of the nonlinear terms in Eq. (4.5), we need the following classical results, see [53,17] for the detailed proof.

Lemma 4.4. *Let* $b(\mathbf{u},\mathbf{v},\mathbf{w})$ *be the trilinear form on* $V \times V \times V$ *defined by*

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx.$$

Then b has the following properties

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \tag{4.8a}$$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}), \tag{4.8b}$$

$$\left|b(\mathbf{u},\mathbf{v},\mathbf{w})\right| \leqslant C\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{v}\|_{L^2}\|\mathbf{w}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{w}\|_{L^2}^{\frac{1}{2}}, \tag{4.8c}$$

$$\left|b(\mathbf{u},\mathbf{v},\mathbf{w})\right| \leqslant C\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\Delta\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\mathbf{w}\|_{L^2}^{\frac{1}{2}}, \quad \textit{provided}$$
$$(\mathbf{u},\mathbf{v},\mathbf{w}) \in V \times \left(V \cap H^2(\Omega)\right) \times H. \tag{4.8d}$$

Proof of Theorem 4.2. Because of our construction, $\mathbf{v}^{err}$ satisfies homogeneous initial and boundary conditions in the z direction, and periodic in the x direction. Therefore, when performing integration by parts the boundary terms vanish, which helps to simplify our calculation. We divide the proof into four steps.

4.1. $L^\infty(L^2)$ estimate

Multiplying Eq. (4.5) by $\mathbf{v}^{err}$ on both sides and integrating over the domain gives

$$\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\epsilon\|\nabla\mathbf{v}^{err}\|^2_{L^2(\Omega)}\\
&\quad=-\epsilon\big(\big(\mathbf{v}^{err}\cdot\nabla\big)\mathbf{v}^{app},\mathbf{v}^{err}\big)-\big(F^\epsilon,\mathbf{v}^{err}\big)\\
&\quad=\epsilon\big(\big(\mathbf{v}^{err}\cdot\nabla\big)\mathbf{v}^{err},\mathbf{v}^{app}\big)-\big(F^\epsilon,\mathbf{v}^{err}\big)\\
&\quad\leqslant\epsilon\|\mathbf{v}^{app}\|_{L^\infty}\|\mathbf{v}^{err}\|_{L^2(\Omega)}\|\nabla\mathbf{v}^{err}\|_{L^2(\Omega)}+\|\mathbf{v}^{err}\|_{L^2(\Omega)}\|\mathbf{F}^\epsilon\|_{L^2(\Omega)}\\
&\quad\leqslant\frac{\epsilon}{2}\|\mathbf{v}^{app}\|^2_{L^\infty}\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\frac{\epsilon}{2}\|\nabla\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\frac{1}{2}\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}+C\|\mathbf{F}^\epsilon\|^2_{L^2(\Omega)},
\end{aligned}$$

which leads to, by Lemma 4.1

$$\epsilon\frac{d}{dt}\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\epsilon\|\nabla\mathbf{v}^{err}\|^2_{L^2(\Omega)}\leqslant C\epsilon\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}+C\epsilon,\tag{4.9}$$

where the uniform estimate $\|\mathbf{v}^{app}\|_{L^\infty}$ follows from Sobolev inequality, Proposition 3.2, and Lemma 3.5.

Applying first Gronwall's inequality to (4.9) yields

$$\epsilon\|\mathbf{v}^{err}(t,\cdot)\|^2_{L^2(\Omega)}+\|\mathbf{v}^{err}\|^2_{L^2(0,t;L^2(\Omega))}+\epsilon\|\nabla\mathbf{v}^{err}\|^2_{L^2(0,t;L^2(\Omega))}\leqslant C\epsilon,\tag{4.10}$$

which gives

$$\|\mathbf{v}^{err}\|_{L^\infty(L^2)}\leqslant C,\tag{4.11a}$$

$$\|\mathbf{v}^{err}\|_{L^2(L^2)}\leqslant C\epsilon^{\frac{1}{2}},\tag{4.11b}$$

$$\|\mathbf{v}^{err}\|_{L^2(H^1)}\leqslant C.\tag{4.11c}$$

By using the estimate (4.11a), inequality (4.9) can be simplified as

$$\epsilon\frac{d}{dt}\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}\leqslant C\epsilon.\tag{4.12}$$

Multiplying the inequality above by an integration factor $e^{\frac{t}{\epsilon}}$ and then integrating the result, we obtain

$$\|\mathbf{v}^{err}\|_{L^\infty(L^2)}\leqslant C\epsilon^{\frac{1}{2}},\tag{4.13}$$

where the constant C is independent of ϵ.

We remark that the argument from (4.9) to (4.13) can be simplified in the sense that a simple integration in time can be used in (4.9), if one assumes that the coefficient of $\|\mathbf{v}^{err}\|^2_{L^2(\Omega)}$ in (4.9) satisfies $C\epsilon\leqslant\frac{1}{2}$. This amounts to assuming $\epsilon\leqslant\frac{C}{\|\mathbf{v}^{app}\|_{L^\infty}}$ in (4.9). We have deliberately avoided this assumption here by taking a detour in the proof.

4.2. $L^\infty(H^1)$ estimate

We follow the same line of proof as in [61,38]. First, we try to control the tangential derivative $\partial_x \mathbf{v}^{err}$. Applying the operator ∂_x to both sides of Eq. (4.5), one obtains

$$\begin{aligned}&\epsilon\left(\frac{\partial(\partial_x \mathbf{v}^{err})}{\partial t} + (\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{err} + (\mathbf{v}^{app}\cdot\nabla)\partial_x\mathbf{v}^{err} + (\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{app}\right)\\&\quad + \partial_x\mathbf{v}^{err} - \epsilon\Delta\partial_x\mathbf{v}^{err} + \nabla\partial_x p^{err}\\&= -\partial_x\mathbf{F}^{err} - \epsilon\left(\partial_x\mathbf{v}^{err}\cdot\nabla\mathbf{v}^{err} + \partial_x\mathbf{v}^{app}\cdot\mathbf{v}^{err} + \partial_x\mathbf{v}^{err}\cdot\nabla\mathbf{v}^{app}\right).\end{aligned}\tag{4.14}$$

Upon multiplying by $\partial_x \mathbf{v}^{err}$ on both sides of Eq. (4.14), we control the nonlinear terms by using Lemma 4.4 as follows

$$\left|b\left(\mathbf{v}^{err}, \partial_x\mathbf{v}^{app}, \partial_x\mathbf{v}^{err}\right)\right| \leqslant C\left\|\partial_x\mathbf{v}^{app}\right\|_{L^\infty}^2\left\|\mathbf{v}^{err}\right\|_{L^2}^2 + \frac{\|\nabla\partial_x\mathbf{v}^{err}\|_{L^2}^2}{5},\tag{4.15}$$

$$\begin{aligned}\left|b\left(\partial_x\mathbf{v}^{err}, \mathbf{v}^{err}, \partial_x\mathbf{v}^{err}\right)\right| &\leqslant C\left\|\partial_x\mathbf{v}^{err}\right\|_{L^2}\left\|\nabla\partial_x\mathbf{v}^{err}\right\|_{L^2}\left\|\nabla\mathbf{v}^{err}\right\|_{L^2}\\&\leqslant C\left\|\nabla\mathbf{v}^{err}\right\|_{L^2}^2\left\|\partial_x\mathbf{v}^{err}\right\|_{L^2}^2 + \frac{1}{5}\left\|\nabla\partial_x\mathbf{v}^{err}\right\|_{L^2}^2,\end{aligned}\tag{4.16}$$

$$\begin{aligned}\left|b\left(\partial_x\mathbf{v}^{app}, \mathbf{v}^{err}, \partial_x\mathbf{v}^{err}\right)\right| &= \left|b\left(\partial_x\mathbf{v}^{app}, \partial_x\mathbf{v}^{err}, \mathbf{v}^{err}\right)\right|\\&\leqslant \left\|\partial_x\mathbf{v}^{app}\right\|_{L^\infty}\left\|\nabla\partial_x\mathbf{v}^{err}\right\|_{L^2}\left\|\mathbf{v}^{err}\right\|_{L^2}\\&\leqslant C\left\|\partial_x\mathbf{v}^{app}\right\|_{L^\infty}^2\left\|\mathbf{v}^{err}\right\|_{L^2}^2 + \frac{1}{5}\left\|\nabla\partial_x\mathbf{v}^{err}\right\|_{L^2}^2,\end{aligned}\tag{4.17}$$

$$\begin{aligned}\left|b\left(\partial_x\mathbf{v}^{err}, \mathbf{v}^{app}, \partial_x\mathbf{v}^{err}\right)\right| &= \left|b\left(\partial_x\mathbf{v}^{err}, \partial_x\mathbf{v}^{err}, \mathbf{v}^{app}\right)\right|\\&\leqslant \left\|\mathbf{v}^{app}\right\|_{L^\infty}\left\|\partial_x\mathbf{v}^{err}\right\|_{L^2}\left\|\nabla\partial_x\mathbf{v}^{err}\right\|_{L^2}\\&\leqslant C\left\|\partial_x\mathbf{v}^{app}\right\|_{L^\infty}^2\left\|\partial_x\mathbf{v}^{err}\right\|_{L^2}^2 + \frac{1}{5}\left\|\nabla\partial_x\mathbf{v}^{err}\right\|_{L^2}^2.\end{aligned}\tag{4.18}$$

Collecting inequalities (4.15)–(4.18) and using integration by parts, one obtains

$$\begin{aligned}&\frac{\epsilon}{2}\frac{d}{dt}\left\|\partial_x\mathbf{v}^{err}\right\|_{L^2}^2 + \left\|\partial_x\mathbf{v}^{err}\right\|_{L^2}^2 + \frac{\epsilon}{5}\left\|\nabla\partial_x\mathbf{v}^{err}\right\|_{L^2}^2\\&\quad\leqslant C\left\|\partial_x\mathbf{F}^{\epsilon}\right\|_{L^2}^2 + C\epsilon\left\|\partial_x\mathbf{v}^{app}\right\|_{L^\infty}^2\left\|\mathbf{v}^{err}\right\|_{L^2}^2 + C\epsilon\left(\left\|\nabla\mathbf{v}^{err}\right\|_{L^2}^2 + \left\|\partial_x\mathbf{v}^{app}\right\|_{L^\infty}^2\right)\left\|\partial_x\mathbf{v}^{err}\right\|_{L^2}^2\\&\quad\leqslant C\epsilon + C\epsilon\left\|\partial_x\mathbf{v}^{err}\right\|_{L^2}^2,\end{aligned}\tag{4.19}$$

where one has used the estimate (4.11).

The same approach as deriving the $L^\infty(L^2)$ estimate applied to (4.19) leads to

$$\left\|\partial_x\mathbf{v}^{err}\right\|_{L^\infty(L^2)} \leqslant C\epsilon^{\frac{1}{2}},\tag{4.20a}$$

$$\left\|\partial_x\mathbf{v}^{err}\right\|_{L^2(L^2)} \leqslant C\epsilon^{\frac{1}{2}},\tag{4.20b}$$

$$\left\|\partial_x\mathbf{v}^{err}\right\|_{L^2(H^1)} \leqslant C.\tag{4.20c}$$

In a similar fashion, one can show

$$\left\|\partial_{xx}\mathbf{v}^{err}\right\|_{L^\infty(L^2)} \leqslant C\epsilon^{\frac{1}{2}}. \tag{4.21}$$

Indeed, by an induction argument one can show that $\partial_x^j \mathbf{v}^{err}$ satisfies the same estimate as $\mathbf{v}^{err}$ for any integer $j > 0$, provided enough regularity imposed on the data. This is within the expectation since the boundary layer effects only in the normal direction.

Now by using the divergence-free condition

$$\frac{\partial v_1^{err}}{\partial x} + \frac{\partial v_2^{err}}{\partial z} = 0,$$

it follows from (4.20a), (4.21) and the remark above that

$$\left\|\partial_z v_2^{err}\right\|_{L^\infty(L^2)} \leqslant C\epsilon^{\frac{1}{2}}, \tag{4.22}$$

$$\left\|\partial_{xz} v_2^{err}\right\|_{L^\infty(L^2)} \leqslant C\epsilon^{\frac{1}{2}}. \tag{4.23}$$

In view of the anisotropic Sobolev embedding Lemma 3.1, one concludes the uniform estimate for v_2^{err}

$$\left\|v_2^{err}\right\|_{L^\infty(L^\infty)} \leqslant C\epsilon^{\frac{1}{2}}. \tag{4.24}$$

With the help of the uniform estimate (4.24), we can now derive the $L^\infty(H^1)$ estimate. For that, we multiply Eq. (4.5) by $-\Delta\mathbf{v}^{err}$ and integrate over the domain Ω. We examine the trilinear term one by one. First of all, by using Lemma 4.4 one gets

$$\begin{aligned}
&\left|b\left(\mathbf{v}^{err}, \mathbf{v}^{err}, \Delta\mathbf{v}^{err}\right)\right| \\
&\quad\leqslant C\left\|\mathbf{v}^{err}\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\mathbf{v}^{err}\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\mathbf{v}^{err}\right\|_{L^2}^{\frac{1}{2}}\left\|\Delta\mathbf{v}^{err}\right\|_{L^2}^{\frac{1}{2}}\left\|\Delta\mathbf{v}^{err}\right\|_{L^2} \\
&\quad= C\left\|\mathbf{v}^{err}\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\mathbf{v}^{err}\right\|_{L^2}\left\|\Delta\mathbf{v}^{err}\right\|_{L^2}^{\frac{3}{2}} \\
&\quad\leqslant C\left\|\mathbf{v}^{err}\right\|_{L^2}^{2}\left\|\nabla\mathbf{v}^{err}\right\|_{L^2}^{4} + \frac{1}{6}\left\|\Delta\mathbf{v}^{err}\right\|_{L^2}^{2} \\
&\quad\leqslant C\epsilon\left\|\nabla\mathbf{v}^{err}\right\|_{L^2}^{4} + \frac{1}{6}\left\|\Delta\mathbf{v}^{err}\right\|_{L^2}^{2}.
\end{aligned} \tag{4.25}$$

In deriving the inequality, we have employed the Young inequality cf. [14]. Next,

$$\left|b\left(\mathbf{v}^{app}, \mathbf{v}^{err}, \Delta\mathbf{v}^{err}\right)\right| \leqslant C\left\|\mathbf{v}^{app}\right\|_{L^\infty}^{2}\left\|\nabla\mathbf{v}^{err}\right\|_{L^2}^{2} + \frac{1}{6}\left\|\Delta\mathbf{v}^{err}\right\|_{L^2}^{2}. \tag{4.26}$$

Last,

$$b\left(\mathbf{v}^{err}, \mathbf{v}^{app}, \Delta\mathbf{v}^{err}\right) := I_1 + I_2, \tag{4.27}$$

where

$$I_1 = \int_\Omega v_1^{err} \partial_x \mathbf{v}^{app} \cdot \Delta \mathbf{v}^{err} \, dx$$
$$\leqslant C \| \partial_x \mathbf{v}^{app} \|_{L^\infty}^2 \| v_1^{err} \|_{L^2}^2 + \frac{1}{6} \| \Delta \mathbf{v}^{err} \|_{L^2}^2$$
$$\leqslant C\epsilon + \frac{1}{6} \| \Delta \mathbf{v}^{err} \|_{L^2}^2, \tag{4.28}$$

and, by Proposition 3.2, Lemma 3.5, and the uniform estimate (4.24)

$$I_2 = \int_\Omega v_2^{err} \partial_z \mathbf{v}^{app} \cdot \Delta \mathbf{v}^{err} \, dx$$
$$\leqslant C \| v_2^{err} \|_{L^\infty}^2 \| \partial_z \mathbf{v}^{app} \|_{L^2}^2 + \frac{1}{6} \| \Delta \mathbf{v}^{err} \|_{L^2}^2$$
$$\leqslant C\epsilon^{\frac{1}{2}} + \frac{1}{6} \| \Delta \mathbf{v}^{err} \|_{L^2}^2. \tag{4.29}$$

We therefore have

$$\frac{\epsilon}{2} \frac{d}{dt} \| \nabla \mathbf{v}^{err} \|_{L^2}^2 + \| \nabla \mathbf{v}^{err} \|_{L^2}^2 + \frac{\epsilon}{6} \| \Delta \mathbf{v}^{err} \|_{L^2}^2$$
$$\leqslant C + \epsilon \big(\epsilon \| \nabla \mathbf{v}^{err} \|_{L^2}^2 + 1 \big) \| \nabla \mathbf{v}^{err} \|_{L^2}^2 + C\epsilon^2 + C\epsilon^{\frac{3}{2}}. \tag{4.30}$$

Since $\|\nabla \mathbf{v}^{err}\|_{L^2(L^2)} \leqslant C$ by (4.11c), the application of Gronwall's inequality and the method of integration factor implies

$$\| \nabla \mathbf{v}^{err} \|_{L^\infty(L^2)} \leqslant C. \tag{4.31}$$

4.3. $L^\infty(L^\infty)$ estimate

We already derived the uniform estimate for v_2^{err} in (4.24). In view of the anisotropic Sobolev embedding Lemma 3.1, one only needs to control $\|\partial_x \partial_z \mathbf{v}^{err}\|_{L^\infty(L^2)}$ in order to get the uniform estimate for v_1^{err}. Therefore, multiplying Eq. (4.14) by $-\Delta \partial_x \mathbf{v}^{err}$, using the same technique as in proving the $L^\infty(H^1)$ estimate, one has

$$\frac{\epsilon}{2} \frac{d}{dt} \| \nabla \partial_x \mathbf{v}^{err} \|_{L^2}^2 + \| \nabla \partial_x \mathbf{v}^{err} \|_{L^2}^2 + \frac{\epsilon}{6} \| \Delta \partial_x \mathbf{v}^{err} \|_{L^2}^2$$
$$\leqslant C + \epsilon \big(\epsilon \| \nabla \partial_x \mathbf{v}^{err} \|_{L^2}^2 + 1 \big) \| \nabla \partial_x \mathbf{v}^{err} \|_{L^2}^2 + C\epsilon^2 + C\epsilon^2 \| \partial_x \partial_z \mathbf{v}^{app} \|_{L^2}^2. \tag{4.32}$$

By using (4.20c), Proposition 3.2, Lemma 3.5, applying Gronwall's inequality and integration factor, one concludes

$$\| \nabla \partial_x \mathbf{v}^{err} \|_{L^\infty(L^2)} \leqslant C. \tag{4.33}$$

The application of Lemma 3.1 implies

$$\left\| v_1^{err} \right\|_{L^\infty(L^\infty)} \leqslant C\epsilon^{\frac{1}{4}}. \tag{4.34}$$

4.4. *Estimate of pressure*

From the analysis above, one has

$$\left\| \mathbf{v}^{err} \right\|_{L^\infty(L^\infty)} \leqslant C\epsilon^{\frac{1}{4}}, \qquad \left\| \nabla \mathbf{v}^{err} \right\|_{L^\infty(L^2)} \leqslant C.$$

Thus by using Proposition 3.2 and Lemma 3.5, one concludes an estimate for the nonlinear terms

$$\left\| \epsilon\left(\left(\mathbf{v}^{err} \cdot \nabla \right) \mathbf{v}^{err} + \left(\mathbf{v}^{app} \cdot \nabla \right) \mathbf{v}^{err} + \left(\mathbf{v}^{err} \cdot \nabla \right) \mathbf{v}^{app} \right) \right\|_{L^\infty(L^2)} \leqslant C\epsilon.$$

One can then write the error equation (4.5) as a time-dependent Stokes system

$$\epsilon \frac{\partial \mathbf{v}^{err}}{\partial t} - \epsilon \Delta \mathbf{v}^{err} + \nabla p^{err} = \tilde{\mathbf{F}}^{err},$$

where $\|\tilde{\mathbf{F}}^{err}\|_{L^\infty(L^2)} \leqslant C\epsilon^{\frac{1}{2}}$. The regularity theory of Stokes system [53] (applied to $\epsilon \mathbf{v}^{err}$ and p^{err}) implies

$$\left\| \nabla p^{err} \right\|_{L^\infty(L^2)} \leqslant C\epsilon^{\frac{1}{2}}.$$

This completes the proof of Theorem 4.2. □

5. A higher order expansion

It is clear from (3.9) and (3.21) that the truncation introduces extra errors of order $\sqrt{\epsilon}$ into the approximate equation (4.2) (the last terms in (3.12a) and the first term in (3.24a)). To correct these errors, it is natural to look at the expansion up to the order $\sqrt{\epsilon}$. Thus one assumes an ansatz

$$\begin{aligned} \mathbf{v}^\epsilon = {} & \mathbf{v}^0 + \left(\mathbf{v}^I + \tilde{\mathbf{v}}^B + \tilde{\mathbf{v}}^C \right) + \sqrt{\epsilon}\left(\mathbf{u}^0(t, x, z) + \mathbf{u}^I(t/\epsilon, x, z) \right) \\ & + \sqrt{\epsilon}\left(\mathbf{u}^{B,0}(t, x, z/\sqrt{\epsilon}) + \mathbf{u}^{C,0}(t/\epsilon, x, z/\sqrt{\epsilon}) \right) \\ & + \sqrt{\epsilon}\left(\mathbf{u}^{B,1}\left(t, x, (1-z)/\sqrt{\epsilon}\right) + \mathbf{u}^{C,1}\left(t/\epsilon, x, (1-z)/\sqrt{\epsilon}\right) \right) + O(\epsilon), \end{aligned} \tag{5.1}$$

$$p^\epsilon = p^0 + \sqrt{\epsilon} p^1(t, x, z) + O(\epsilon), \tag{5.2}$$

with the matching condition defined similarly as in (3.2). One recalls the definition of $\tilde{\mathbf{v}}^B$ and $\tilde{\mathbf{v}}^C$ in (3.10) and (3.22), respectively.

Remark 6. The ansatz (5.2) for pressure makes sense, since the estimate (4.6d) in Theorem 4.2 suggests there is no boundary layer or initial–boundary layer of leading order for pressure.

Following the same approach as in Section 3, we can derive the Prandtl type equations as follows

- $\mathbf{u}^0$ and p^1

$$\begin{cases} \mathbf{u}^O + \nabla p^1 = \mathbf{f}^O, & (x,z) \in \Omega, \\ \nabla \cdot \mathbf{u}^O = 0, \\ u_2^o\big|_{z=0,1} = 0, \end{cases} \tag{5.3}$$

where $\mathbf{f}^O = (\rho_0' a + \rho_1' b, -\frac{\partial a}{\partial x}\rho_1 - \frac{\partial b}{\partial x}\rho_2)$, ρ_0 and ρ_1 are the cut-off functions at $z=0$ and $z=1$, respectively. Since $\nabla \cdot \mathbf{f}^O = 0$, the pressure p^1 satisfies a Neumann problem

$$\begin{cases} \Delta p^1 = 0, & (x,z) \in \Omega, \\ \left.\dfrac{\partial p^1}{\partial z}\right|_{z=0} = -\dfrac{\partial a}{\partial x}, \\ \left.\dfrac{\partial p^1}{\partial z}\right|_{z=1} = -\dfrac{\partial b}{\partial x}. \end{cases} \tag{5.4}$$

- $\mathbf{u}^I$

$$\begin{cases} \dfrac{\partial \mathbf{u}^I}{\partial \tau} + \mathbf{u}^I = \mathbf{f}^I, \\ \mathbf{u}^I\big|_{\tau=0} = -\mathbf{u}^0(0,x,z), \end{cases} \tag{5.5}$$

where $\mathbf{f}^I = (-(c(x)\rho_0' + d(x)\rho_1')\frac{2}{\sqrt{\pi}}\int_{\sqrt{\tau}}^{\infty} e^{-z^2}\,dz, 0)$.

- $\mathbf{u}^{B,0}$

$$\begin{cases} u_1^{B,0} - \partial_{ZZ} u_1^{B,0} = 0, & Z \in (0,\infty), \\ \dfrac{\partial u_1^{B,0}}{\partial x} + \dfrac{1}{\sqrt{\epsilon}}\dfrac{\partial u_2^{B,0}}{\partial Z} = 0, & Z \in (0,\infty), \\ u_1^{B,0}\big|_{Z=0} = -u_1^O\big|_{z=0}, \quad u_2^{B,0}\big|_{Z=0} = 0, \\ u_1^{B,0} \to 0, \quad \text{as } Z \to \infty. \end{cases} \tag{5.6}$$

- $\mathbf{u}^{C,0}$

$$\begin{cases} \dfrac{\partial u_1^{C,0}}{\partial_\tau} + u_1^{C,0} - \partial_{ZZ} u_1^{C,0} = 0, & Z \in (0,\infty), \\ \dfrac{\partial u_1^{C,0}}{\partial x} + \dfrac{1}{\sqrt{\epsilon}}\dfrac{\partial u_2^{C,0}}{\partial Z} = 0, & Z \in (0,\infty), \\ u_1^{C,0}\big|_{\tau=0} = -u_1^{B,0}\big|_{t=0}, \\ u_1^{C,0}\big|_{Z=0} = -u_1^I\big|_{z=0}, \quad u_2^{B,0}\big|_{Z=0} = 0, \\ u_1^{C,0} \to 0, \quad \text{as } Z \to \infty. \end{cases} \tag{5.7}$$

Remark 7. Note that the boundary layer type function $-2ae^{-\frac{z}{\sqrt{\epsilon}}}\rho'$ from (3.12) is not included in Eq. (5.6). Similarly, the initial–boundary layer type functions $-2\epsilon\frac{\partial v_1^{C,0}}{\partial z}\rho'$ from (3.24a) and $2\sqrt{\epsilon}\frac{\partial v_1^{C,0}}{\partial x}\rho'$ from (3.24b) are not added to Eq. (5.7). In the proof of Lemma 5.2, we will show that these terms can be bounded above by $C\epsilon^{\frac{5}{4}}$ in the $L^\infty(L^2)$ norm.

One observes the Prandtl type equations (5.5)–(5.7) are entirely analogous to the ones studied in Section 3. Therefore the truncated boundary layer profile $\tilde{\mathbf{u}}^B$ and initial–boundary layer profile $\tilde{\mathbf{u}}^C$ can be constructed in the same way as $\tilde{\mathbf{v}}^B$ and $\tilde{\mathbf{v}}^C$. Moreover $\tilde{\mathbf{u}}^B$ and $\tilde{\mathbf{u}}^C$ follow the same estimates as in Proposition 3.2 and Lemma 3.5.

We define the approximate solution as follows

$$\tilde{\mathbf{w}}^{app} = \mathbf{v}^0 + \mathbf{v}^I + \tilde{\mathbf{v}}^B + \tilde{\mathbf{v}}^C + \sqrt{\epsilon}\big(\mathbf{u}^0 + \mathbf{u}^I + \tilde{\mathbf{u}}^B + \tilde{\mathbf{u}}^C\big), \tag{5.8a}$$

$$\tilde{p}^{app} = p^0 + \sqrt{\epsilon}p^1. \tag{5.8b}$$

We see $\tilde{\mathbf{v}}^{app}$ and $\tilde{p}^{app}$ satisfy a similar equation as (4.2) with a different forcing term

$$\begin{cases} \epsilon\left(\dfrac{\partial \tilde{\mathbf{w}}^{app}}{\partial t} + \big(\tilde{\mathbf{w}}^{app}\cdot\nabla\big)\tilde{\mathbf{w}}^{app}\right) + \tilde{\mathbf{w}}^{app} - \epsilon\Delta\tilde{\mathbf{w}}^{app} + \nabla\tilde{p}^{app} = \mathbf{F} + \tilde{\mathbf{F}}^{err}, \\ \nabla\cdot\tilde{\mathbf{w}}^{app} = 0, \\ \tilde{\mathbf{w}}^{app}\big|_{z=0,1} = 0, \\ \tilde{\mathbf{w}}^{app}\big|_{t=0} = \mathbf{v}_0, \end{cases} \tag{5.9}$$

where the forcing term $\tilde{\mathbf{F}}^{err}$ takes the form of

$$\begin{aligned} \tilde{\mathbf{F}}^{err} &= \epsilon\left(\frac{\partial \mathbf{v}^0}{\partial t} + \frac{\partial \tilde{\mathbf{v}}^B}{\partial t}\right) + \epsilon\big(\tilde{\mathbf{w}}^{app}\cdot\nabla\big)\tilde{\mathbf{w}}^{app} - \epsilon\big(\Delta\mathbf{v}^0 + \Delta\mathbf{v}^I\big) \\ &\quad + \epsilon^{\frac{3}{2}}\left(\frac{\partial \mathbf{u}^0}{\partial t} + \frac{\partial \tilde{\mathbf{u}}^B}{\partial t}\right) - \epsilon^{\frac{3}{2}}\big(\Delta\mathbf{u}^0 + \Delta\mathbf{u}^I\big) + \tilde{\mathbf{f}}^B + \tilde{\mathbf{f}}^C. \end{aligned} \tag{5.10}$$

Here $\tilde{\mathbf{f}}^B$ and $\tilde{\mathbf{f}}^C$ have similar terms as $\mathbf{f}^B$ and $\mathbf{f}^C$ except those $O(\sqrt{\epsilon})$ terms. As an illustration, we give the explicit formulation of $\tilde{\mathbf{f}}^B = \tilde{\mathbf{f}}^{B,0} + \tilde{\mathbf{f}}^{B,1}$:

$$\begin{aligned} \tilde{f}_1^{B,0} &= \epsilon^{\frac{3}{2}}\Delta\big(a\rho'\big)\big(1 - e^{-\frac{z}{\sqrt{\epsilon}}}\big) + \epsilon\left(\frac{\partial^2 a}{\partial x^2}\rho + 3a\rho''\right)e^{-\frac{z}{\sqrt{\epsilon}}} \\ &\quad - 2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho' - \epsilon^{\frac{3}{2}}\Delta u_1^O + \epsilon^2\Delta\big(\bar{a}\rho'\big)\big(1 - e^{-\frac{z}{\sqrt{\epsilon}}}\big) \\ &\quad + \epsilon^{\frac{3}{2}}\left(\frac{\partial^2 \bar{a}}{\partial x^2}\rho + 3\bar{a}\rho''\right)e^{-\frac{z}{\sqrt{\epsilon}}} - 2\epsilon\bar{a}e^{-\frac{z}{\sqrt{\epsilon}}}\rho' - \epsilon\bar{a}\rho', \end{aligned} \tag{5.11a}$$

$$\begin{aligned} \tilde{f}_2^{B,0} &= -\epsilon^{\frac{3}{2}}\Delta\left(\frac{\partial a}{\partial x}\rho\right)\big(1 - e^{-\frac{z}{\sqrt{\epsilon}}}\big) - 2\epsilon\frac{\partial a}{\partial x}e^{-\frac{z}{\sqrt{\epsilon}}}\rho' - \epsilon^{\frac{3}{2}}\Delta u_2^O \\ &\quad - \epsilon^2\Delta\left(\frac{\partial \bar{a}}{\partial x}\rho\right)\big(1 - e^{-\frac{z}{\sqrt{\epsilon}}}\big) - 2\epsilon^{\frac{3}{2}}\frac{\partial \bar{a}}{\partial x}e^{-\frac{z}{\sqrt{\epsilon}}}\rho' + \epsilon\frac{\partial \bar{a}}{\partial x}\rho, \end{aligned} \tag{5.11b}$$

with $\bar{a} = u_1^O(t,x,0)$.

For the estimate of $\tilde{\mathbf{F}}^{err}$, we need the following version of Hardy's inequality (Lemma 13.4 of [51]).

Lemma 5.1. *For $p > 1$, if $f \in L^p(R^+)$ and $g(t) = \frac{1}{t}\int_0^t f(s)\,ds$, then $g \in L^p(R^+)$ and $\|g\|_{L^p} \leqslant \frac{p}{p-1}\|f\|_{L^p}$. For $p = \infty$, one replaces $\frac{p}{p-1}$ by 1.*

Now we are ready to prove

Lemma 5.2. *Assume $\mathbf{v}_0 \in V \cap H^m(\Omega)$ and $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ with $m \geqslant 5$. The following estimate holds*

$$\left\|\partial_x^j \tilde{\mathbf{F}}^{err}\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon, \quad j = 0, 1. \tag{5.12}$$

Proof. In view of inequalities (3.5) and (3.14), Proposition 3.2, and Lemma 3.5, we only need to take care of four potentially troublesome terms: $-2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho'$ in $\tilde{\mathbf{f}}^B$ (from Eqs. (3.12)), $-2\epsilon\frac{\partial v_1^{C,0}}{\partial z}\rho'$ and $2\sqrt{\epsilon}\frac{\partial v_1^{C,0}}{\partial x}\rho'$ in $\tilde{\mathbf{f}}^C$ (from Eqs. (3.24a) and (3.24b)), and $\epsilon(\tilde{\mathbf{v}}^{app}\cdot\nabla)\tilde{\mathbf{v}}^{app}$. The rest terms in (3.24b) are all bounded above by $C\epsilon$ at least in $L^\infty(L^2)$, which follows from Proposition 3.4.

First of all, by the definition of the cut-off function, one has

$$\begin{aligned}
\left\|-2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho'\right\|_{L^2}^2 &\leqslant C\epsilon\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\frac{2z}{\sqrt{\epsilon}}}\rho'^2\,dz \\
&\leqslant C\epsilon^2\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{z^2}{\epsilon}e^{-\frac{2z}{\sqrt{\epsilon}}}\rho'^2\,dz \\
&\leqslant C\epsilon^{\frac{5}{2}}\left\|Ze^{-Z}\right\|_{L^2(0,\infty)}^2 \\
&\leqslant C\epsilon^{\frac{5}{2}}.
\end{aligned} \tag{5.13}$$

The same trick applied to $-2\epsilon\frac{\partial v_1^{C,0}}{\partial z}\rho'$ renders

$$\left\|-2\epsilon\frac{\partial v_1^{C,0}}{\partial z}\rho'\right\|_{L^2(\Omega)} \leqslant C\epsilon^{\frac{5}{4}}\left\|Z\frac{\partial v_1^{C,0}}{\partial Z}\right\|_{L^\infty(0,+\infty;L^2(\Omega_\infty))} \leqslant C\epsilon^{\frac{5}{4}}, \tag{5.14}$$

where the last estimate in Lemma 3.3 has been used. Next, note that Lemma 3.3 also implies

$$\left|\frac{\partial v_1^{C,0}}{\partial x}\right| \leqslant \left|c'(x)\right|e^{-\frac{\tau}{2}-\frac{Z}{\sqrt{2}}}.$$

Thus the above argument yields

$$\left\| 2\sqrt{\epsilon}\frac{\partial v_1^{C,0}}{\partial x}\rho' \right\|_{L^2} \leqslant C\epsilon^{\frac{5}{4}}. \tag{5.15}$$

For the estimate of $\epsilon(\tilde{\mathbf{v}}^{app}\cdot\nabla)\tilde{\mathbf{v}}^{app}$, we only need to control terms like $v_2^0\frac{\partial\tilde{v}_1^B}{\partial z}$ or $v_2^I\frac{\partial\tilde{v}_1^C}{\partial z}$. Since $v_2^0|_{z=0,1}=0$, a direct application of Hardy's inequality Lemma 5.1 yields

$$\begin{aligned}
\left\| v_2^0\frac{\partial\tilde{v}_1^B}{\partial z}\right\|_{L^2} &\leqslant \left\| v_2^0\frac{\partial\tilde{v}_1^{B,0}}{\partial z}\right\|_{L^2} + \left\| v_2^0\frac{\partial\tilde{v}_1^{B,1}}{\partial z}\right\|_{L^2} \\
&\leqslant \left\|\partial_z v_2^0\right\|_{L^\infty}\left(\left\| z\frac{\partial\tilde{v}_1^{B,0}}{\partial z}\right\|_{L^2} + \left\|(1-z)\frac{\partial\tilde{v}_1^{B,1}}{\partial z}\right\|_{L^2}\right) \\
&\leqslant C\epsilon^{\frac{1}{4}}\left\| Ze^{-Z}\right\|_{L^2(0,\infty)}.
\end{aligned} \tag{5.16}$$

We thus proved the lemma. □

The approach in proving Theorem 4.2 then leads to

Theorem 5.3. *Assume $\mathbf{v}_0\in V\cap H^m(\Omega)$ and $\mathbf{F}\in C^1([0,T];H^m(\Omega))$ with $m\geqslant 5$. The following convergence rates hold*

$$\left\|\mathbf{v}^\epsilon - \tilde{\mathbf{w}}^{app}\right\|_{L^\infty(0,T,L^2(\Omega))} \leqslant C\epsilon, \tag{5.17a}$$

$$\left\|\mathbf{v}^\epsilon - \tilde{\mathbf{w}}^{app}\right\|_{L^\infty(0,T,H^1(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}, \tag{5.17b}$$

$$\left\|\mathbf{v}^\epsilon - \tilde{\mathbf{w}}^{app}\right\|_{L^\infty(0,T,L^\infty(\Omega))} \leqslant C\epsilon^{\frac{3}{4}}, \tag{5.17c}$$

$$\left\|\nabla\left(p^\epsilon - \tilde{p}^{app}\right)\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\epsilon. \tag{5.17d}$$

Since (compare to Remarks 2 and 5)

$$\left\|\sqrt{\epsilon}\left(\mathbf{u}^0+\mathbf{u}^I+\tilde{\mathbf{u}}^B+\tilde{\mathbf{u}}^C\right)\right\|_{L^\infty(L^2)} \approx O\left(\epsilon^{\frac{1}{2}}\right), \tag{5.18a}$$

$$\left\|\sqrt{\epsilon}\left(\mathbf{u}^0+\mathbf{u}^I+\tilde{\mathbf{u}}^B+\tilde{\mathbf{u}}^C\right)\right\|_{L^\infty(H^1)} \approx O\left(\epsilon^{\frac{1}{4}}\right), \tag{5.18b}$$

$$\left\|\sqrt{\epsilon}\left(\mathbf{u}^0+\mathbf{u}^I+\tilde{\mathbf{u}}^B+\tilde{\mathbf{u}}^C\right)\right\|_{L^\infty(L^\infty)} \approx O\left(\epsilon^{\frac{1}{2}}\right), \tag{5.18c}$$

$$\left\|\sqrt{\epsilon}p^1\right\|_{L^\infty(L^2)} \approx O\left(\epsilon^{\frac{1}{2}}\right), \tag{5.18d}$$

Theorem 5.3 immediately implies

Corollary 5.4. *Suppose that $\mathbf{v}_0\in V\cap H^m(\Omega)$ and $\mathbf{F}\in C^1([0,T];H^m(\Omega))$ with $m\geqslant 5$. Then the convergence rates (4.6a) and (4.6d) in Theorem 4.2 are optimal. Moreover, one has the following* *improved optimal convergence rates*

$$\left\| \mathbf{v}^{\epsilon} - \mathbf{v}^{app} \right\|_{L^{\infty}(0,T,H^1(\Omega))} \leqslant C\epsilon^{\frac{1}{4}}, \tag{5.19a}$$

$$\left\| \mathbf{v}^{\epsilon} - \mathbf{v}^{app} \right\|_{L^{\infty}(0,T,L^{\infty}(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}. \tag{5.19b}$$

6. Conclusion

In this paper, we have provided a detailed rigorous leading order asymptotic analysis of the nonlinear Darcy–Brinkman system in the vanishing Darcy number limit, which involves a boundary layer, an initial layer and their interaction-**initial–boundary layer**. The optimal convergence rates in Sobolev norms are proved rigorously by including the next order expansion. We remark that the analysis of the initial–boundary layer is novel, involving simultaneous two scale expansion in space and in time. The rigorous convergence result derived in this manuscript further validates the applicability of the Darcy model for flows in porous media if we view the nonlinear Darcy–Brinkman model as the "true" model.

The convergence results are derived under the zeroth order compatibility assumption $\mathbf{v}_0|_{z=0,1} = 0$. Additional singular structures will emerge without this compatibility condition. In [40,34], the authors used semiclassical techniques and layer potentials to study the boundary layer. This approach does not rely on the Prandtl theory and does not require any type of compatibility conditions between the initial and boundary data. However, it yields only convergence in $L^{\infty}(L^p)$ with $p \in [1, +\infty]$ and does not provide any estimate on normal gradients at the boundary.

A closely related model is the Bénard convection problem in a porous media region bounded by two parallel plates saturated with fluids. The bottom plate is kept at temperature T_2 and the top plate is kept at temperature T_1 with $T_2 > T_1$. Then the governing equations are the so-called *Darcy–Brinkman–Oberbeck–Boussinesq system* in the non-dimensional form [42], see also [27]:

$$\gamma_a\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) + \mathbf{v} - \tilde{Da}\Delta\mathbf{v} + \nabla p = Ra_D\mathbf{k}T,$$
$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \Delta T,$$
$$\operatorname{div}\mathbf{v} = 0,$$
$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad T|_{t=0} = T_1,$$
$$\mathbf{v}|_{z=0,1} = 0, \quad T|_{z=0} = T_1, \quad T|_{z=1} = T_2,$$

where $\mathbf{k}$ is the unit normal vector directed upward (the positive z direction), Ra_D is the *Rayleigh–Darcy number*. One can also consider the vanishing Darcy number limit of the above system by taking $\gamma_a = \tilde{Da} = \epsilon$. We anticipate a similar result as Theorem 1.2 holds for this system. But the analysis would be more involved, and we leave it to a future work.

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