



Spatial dynamics for lattice differential equations with a shifting habitat

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Received 24 September 2014; revised 19 March 2015

Abstract

We study a lattice differential equation model that describes the growth and spread of a species in a shifting habitat. We show that the long term behavior of solutions depends on the speed of the shifting habitat edge c and a number $c^*(\infty)$ that is determined by the maximum linearized growth rate and the diffusion coefficient. We demonstrate that if $c > c^*(\infty)$ then the species will become extinct in the habitat, and that if $c < c^*(\infty)$ then the species will persist and spread along the shifting habitat at the asymptotic spreading speed $c^*(\infty)$. For our purpose the solutions to the model are formulated in the form of integral equations involving modified Bessel functions, for which new asymptotic estimates are provided. To the best of our knowledge, this is the first time that the classical Bessel functions are used to describe the solutions of lattice differential equations, and this approach possesses its own interest in further studying lattice differential equations.

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MSC: 34K31; 35B40; 92D40; 92D25

Keywords: Lattice differential equation; Modified Bessel function; Persistence; Spreading speed

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¹ This research was partially supported by the National Science Foundation under Grant DMS-1225693.

1. Introduction

Lattice differential equations are ordinary differential equations indexed by points on a spatial lattice. They are found in models arising from physics [30], chemistry [18], biology [4], and material science [6]. In the context of biology, lattice differential equations can be used to study the spatial spread of a species in a habitat consisting of infinitely many patches. The simplest nonlinear lattice differential equation model describing population growth and spread may take the form

$$u_t(x, t) = D[u(x + 1, t) - 2u(x, t) + u(x - 1, t)] + ru(x, t) - u^2(x, t), \quad x \in \mathbb{H}, \quad t > 0, \quad (1.1)$$

where $u(x, t)$ represents the density of the population at point x and time t , D is a positive number, r is the population growth rate, $-u^2$ accounts for density-dependent death (e.g. due to resource limitation), and \mathbb{H} is the habitat which can be the discrete space \mathbb{Z} consisting of all integers or the continuous space \mathbb{R} . $ru(x, t) - u^2(x, t)$ is the growth function. It is known that for the model (1.1), the number $c^* = \inf_{\mu > 0} \frac{4D \sinh^2(\mu/2) + r}{\mu}$ is the spreading speed and the slowest speed of a class of traveling wave solutions connecting 0 and r (see e.g., [13,20,34]). Traveling wave solutions have been extensively studied for lattice differential equations with various growth functions and functionals [5,7,8,10,11,13,15,16,21,22,35–37,39]. The lattice differential equation (1.1) is regarded as a discrete version of the reaction diffusion equation in a continuous habitat, i.e., the KPP–Fisher equation

$$u_t(x, t) = Du_{xx} + ru(x, t) - u^2(x, t), \quad x \in \mathbb{R}, \quad t > 0. \quad (1.2)$$

The dynamics of this equation and its extensions have been well studied (see e.g., [1,2,12,17]). It has been observed that in general a lattice differential equation could behave differently from the corresponding reaction–diffusion equation (see e.g., [16,35]).

Recently there has been renewed interest in spatial ecology, largely driven by the threats associated with global climate change [9,14,19,26–29,31,38]. An important effect of global climate change is shifts or translations in habitat ranges. This can be modeled by assuming that the population growth rate is described as $r(x - ct)$, where $r(x)$ is the spatially varying baseline, or historic rate of population growth. To explore the species spread in the context of climate change we consider the following lattice differential equation

$$u_t(x, t) = D[u(x + 1, t) - 2u(x, t) + u(x - 1, t)] + r(x - ct)u - u^2. \quad (1.3)$$

Here $c > 0$ and $r(\xi)$ is continuous, nondecreasing, and bounded with $r(-\infty) < 0$ and $r(\infty) > 0$. $r(x - ct)$ divides the spatial domain into two parts: the region with good quality habitat suitable for growth (i.e., $r(x - ct) > 0$), and the region with poor quality habitat unsuitable for growth (i.e., $r(x - ct) < 0$). The edge of the habitat suitable for species growth is shifting at a speed c . Model (1.3) is heterogeneous in space and time. The heterogeneity described by $r(x - ct)$ makes the existing theory on spreading speeds not applicable to (1.3).

The purpose of the present paper is to study the spatial dynamics of (1.3). We will determine the persistence and spreading speed for the model (1.3) by developing new techniques. We demonstrate that the long term behavior of solutions of (1.3) depends upon c and $c^*(\infty) =$

$\inf_{\mu>0} \frac{4D \sinh^2(\mu/2) + r(\infty)}{\mu}$. We show that if $c > c^*(\infty)$ then each solution will uniformly approach zero in space as $t \rightarrow \infty$, and that if $c < c^*(\infty)$ then a solution with initial compact data will spread rightward at an asymptotic spreading speed $c^*(\infty)$. The reaction–diffusion equation (1.2) with r replaced by $r(x - ct)$ was recently studied in [19] where the issues about persistence and spreading speeds are addressed. The methods given there largely depend on the formulation of heat kernel, i.e., $\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$, corresponding to the heat equation $u_t = Du_{xx}$. Extending the results in [19] to the model (1.3) is not trivial due to the lack of fundamental solutions to the linear lattice differential equations, like the heat kernel to the linear heat equation. In this paper we systematically develop the fundamental solution theory to the linear lattice differential equations by using the classical modified Bessel functions. To the best of our knowledge, this is the first time that the modified Bessel functions are used to express the solutions of lattice differential equations. This provides a new tool in the study of lattice differential equations.

This paper is organized as follows. In the next section, we introduce the modified Bessel functions and derive some useful asymptotic estimates for them. This section plays a fundamental role for analyzing (1.3). Section 3 discusses the well-posedness of the model (1.3) and establishes the comparison principle. Section 4 studies the non-persistence property of (1.3) in case of $c > c^*(\infty)$. Section 5 provides the spreading speed result in case of $c < c^*(\infty)$. The main results in this paper are contained in Theorem 4.1 and Theorem 5.1.

2. Linear lattice differential equations and modified Bessel functions

In this section we are concerned about the initial value problem to the linear lattice differential equations:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = D[u(x+1, t) - 2u(x, t) + u(x-1, t)] & t > 0, x \in \mathbb{H}, \\ u(0, x) = u_0(x), \end{cases} \quad (2.1)$$

where the initial data u_0 is assumed to be in $L^\infty(\mathbb{H})$. Define Δ_1 and Δ_{-1} to be the shifting operators as $(\Delta_1 u)(x) = u(x+1)$ and $(\Delta_{-1} u)(x) = u(x-1)$ for any $u \in L^\infty(\mathbb{H})$. Let \mathbb{I} be the identity operator in $L^\infty(\mathbb{H})$. Then obviously $\Delta_1 \cdot \Delta_{-1} = \Delta_{-1} \cdot \Delta_1 = \mathbb{I}$. The operator $\Delta_1 - 2\mathbb{I} + \Delta_{-1}$ is called the discretized Laplace operators. We perform a formal computation on the solutions $u(t, x)$ of (2.1) according to the above shifting operators,

$$\begin{aligned} u(t, x) &= [e^{D(\Delta_1 - 2\mathbb{I} + \Delta_{-1})t} u_0](x) \\ &= e^{-2Dt} \sum_{k, j \geq 0} \frac{(Dt)^{k+j}}{k!j!} [\Delta_1^k \cdot \Delta_{-1}^j u_0](x) \\ &= e^{-2Dt} \left[\sum_{m=k-j \geq 0} \sum_{j \geq 0} \frac{(Dt)^{m+2j}}{(m+j)!j!} [\Delta_1^m u_0](x) \right. \\ &\quad \left. + \sum_{m=j-k > 0} \sum_{k \geq 0} \frac{(Dt)^{m+2k}}{(m+k)!k!} [\Delta_{-1}^m u_0](x) \right] \\ &= e^{-2Dt} \sum_{m \in \mathbb{Z}} \mathbf{I}_m(2Dt) u_0(x-m), \end{aligned} \quad (2.2)$$

where \mathbf{I}_m , $m \geq 0$ are defined as

$$\mathbf{I}_m(t) = \sum_{k=0}^{\infty} \frac{(t/2)^{m+2k}}{k!(m+k)!},$$

and $\mathbf{I}_m(t) = \mathbf{I}_{-m}(t)$ for $m < 0$. The sequence $\{\mathbf{I}_m(t)\}_{m \geq 0}$ turns out to be the classical first kind modified Bessel functions of integer order m ; $\mathbf{I}_m(t)$ solves

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} - (t^2 + m^2)y = 0, \quad (2.3)$$

for $t > 0$ (see e.g. [24,32]) and satisfies the recurrence relation,

$$\mathbf{I}'_m(t) = \frac{1}{2}[\mathbf{I}_{m+1}(t) + \mathbf{I}_{m-1}(t)], \quad \forall t \geq 0, m \in \mathbf{Z}. \quad (2.4)$$

It is known that $\mathbf{I}_m(0) = 0$ for $m \neq 0$ while $\mathbf{I}_0(0) = 1$, and $\mathbf{I}_m(t) \geq 0$ for any integer m and $t \geq 0$. $\mathbf{I}_m(t)$ are increasing in x variable for any fixed integer m , and decreasing in $m \geq 0$ (see e.g. [24]). The modified Bessel functions, $\mathbf{I}_m(t)$, $m \geq 0$, $t \geq 0$ can be given in the integral form [24,32]:

$$\mathbf{I}_m(t) = \frac{1}{\pi} \int_0^{\pi} e^{t \cos \theta} \cos(m\theta) d\theta.$$

As an immediate consequence, $u(x, t)$ can be formally written as

$$u(x, t) = e^{-2Dt} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2Dt) u_0(x - m). \quad (2.5)$$

As $u(x, t) \equiv \text{constant}$ is a solution of (2.1), it is easy to see that

$$1 = e^{-2Dt} [\mathbf{I}_0(2Dt) + 2\mathbf{I}_1(2Dt) + 2\mathbf{I}_2(2Dt) + 2\mathbf{I}_3(2Dt) + \cdots]. \quad (2.6)$$

There are uniform asymptotic expansions in [23,24] concerning $\mathbf{I}_m(t)$ and $\mathbf{I}'_m(t)$ with respect to the t variable as the order parameter $m \rightarrow \infty$:

$$\mathbf{I}_m(t) \sim \frac{1}{\sqrt{2\pi m}} \frac{e^{m\eta}}{(1+z^2)^{1/4}} \sum_{k=0}^{\infty} \frac{u_k(s)}{m^k}, \quad (2.7)$$

and

$$\mathbf{I}'_m(t) \sim \frac{1}{\sqrt{2\pi m}} \frac{(1+z^2)^{1/4}}{z} e^{m\eta} \left[1 + \sum_{k=1}^{\infty} \frac{v_k(s)}{m^k} \right], \quad (2.8)$$

where

and
$$z = \frac{t}{m}, \quad \eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}, \quad s = \frac{1}{\sqrt{1+z^2}},$$

$$u_0(s) = 1, u_{k+1}(s) = \frac{1}{2}s^2(1-s^2)u'_k(s) + \frac{1}{8} \int_0^s (1-5\tau^2)u_k(\tau) d\tau, \quad k \geq 0.$$

$$v_k(s) = u_k(s) + s(s^2 - 1) \left[\frac{1}{2}u_{k-1}(s) + su'_{k-1}(s) \right], \quad k \geq 1.$$

By using (2.7) and (2.8), we have the following lemma.

Lemma 2.1. *We have the following uniform estimates. For $m \leq 2Dt$,*

$$e^{-2Dt} \mathbf{I}_m(2Dt) \leq \frac{C}{\sqrt{2\pi}} \frac{e^{-\alpha_1 \frac{m^2}{2Dt}}}{((2Dt)^2 + m^2)^{1/4}}, \quad (2.9)$$

$$e^{-2Dt} \mathbf{I}'_m(2Dt) \leq \frac{C}{\sqrt{2\pi}} \frac{((2Dt)^2 + m^2)^{1/4}}{2Dt} e^{-\alpha_1 \frac{m^2}{2Dt}}. \quad (2.10)$$

For $m \geq 2Dt$,

$$e^{-2Dt} \mathbf{I}_m(2Dt) \leq \frac{C}{\sqrt{2\pi}} \frac{e^{-\alpha_2 m}}{((2Dt)^2 + m^2)^{1/4}}, \quad (2.11)$$

$$e^{-2Dt} \mathbf{I}'_m(2Dt) \leq \frac{C}{\sqrt{2\pi}} \frac{((2Dt)^2 + m^2)^{1/4}}{2Dt} e^{-\alpha_2 m}, \quad (2.12)$$

where $\alpha_1 = \ln(\sqrt{2} + 1) - \frac{1}{2}$ and $\alpha_2 = \ln(\sqrt{2} + 1) + 1 - \sqrt{2}$, C is an absolute constant which may vary at each occurrence.

Proof. We first present an elementary exponential type inequality. For any $x \in [0, 1]$, $\exp(-x) \leq \sqrt{1+x^2} - x \leq \exp(-\alpha_0 x)$, where $\alpha_0 = \ln(\sqrt{2} + 1) \approx 0.8814$; see [3] for the proof. In the case of $m \leq 2Dt$,

$$\begin{aligned} \left(\frac{z}{1+\sqrt{1+z^2}} \right)^m &= \left(\frac{2Dt/m}{1+\sqrt{1+(2Dt/m)^2}} \right)^m \\ &= \left(\frac{1}{m/(2Dt) + \sqrt{1+(m/(2Dt))^2}} \right)^m \\ &= (\sqrt{1+(m/(2Dt))^2} - 1)^m \leq e^{-\alpha_0 \frac{m^2}{2Dt}}. \end{aligned}$$

We estimate the leading term of (2.7) and have

$$e^{-2Dt} \mathbf{I}_m(2Dt) \leq \frac{C}{\sqrt{2\pi}} \frac{e^{\frac{m^2}{\sqrt{m^2+x^2+x}} - \alpha_0 \frac{m^2}{2Dt}}}{(x^2 + m^2)^{1/4}} \leq \frac{C}{\sqrt{2\pi}} \frac{e^{-\alpha_1 \frac{m^2}{2Dt}}}{(x^2 + m^2)^{1/4}},$$

where $\alpha_1 = \alpha_0 - \frac{1}{2} \approx 0.3814$. So the inequality (2.9) is proved. In the case of $m > 2Dt$, the exponent of $e^{-2Dt} \mathbf{I}_m(2Dt)$ in (2.7) can be bounded as below,

$$\eta - \frac{2Dt}{m} = \sqrt{1 + \left(\frac{2Dt}{m}\right)^2} - \frac{2Dt}{m} + \ln \frac{2Dt}{m} - \ln \left(\sqrt{1 + \left(\frac{2Dt}{m}\right)^2} + 1 \right) \leq -\alpha_2,$$

where $\alpha_2 = \alpha_0 + 1 - \sqrt{2} \approx 0.4672$. Therefore (2.11) is confirmed. The inequalities (2.12) and (2.10) can be proved similarly, so the details are skipped. The proof is complete. \square

Remark 2.1. The estimate (2.9) implies that for small m , the kernel $e^{-2Dt} \mathbf{I}_m(2Dt)$ behaves like the Gaussian kernel, while (2.11) tells that $e^{-2Dt} \mathbf{I}_m(2Dt)$ is exponentially decaying for large m as the time t is fixed.

As an immediate consequence of Lemma 2.1, we have the following corollary.

Corollary 2.1. For any $u_0 \in L^\infty(\mathbb{H})$, $u(t, x)$ defined in (2.2) solves the lattice differential equation (2.1).

Proof. From Lemma 2.1, we are allowed to differentiate the series on t variable in (2.5). We have

$$\begin{aligned} \frac{\partial u}{\partial t}(x, t) &= -2De^{-2Dt} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2Dt) u_0(x-m) \\ &\quad + 2De^{-2Dt} \sum_{m=-\infty}^{\infty} \mathbf{I}'_m(2Dt) u_0(x-m) \\ &= -2Du(t, x) + De^{-2Dt} \sum_{m=-\infty}^{\infty} (\mathbf{I}_{m+1} + \mathbf{I}_{m-1})(2Dt) u_0(x-m) \\ &= D[u(x+1, t) - 2u(x, t) + u(x-1, t)], \end{aligned}$$

where we have used the recurrence relation (2.4). The proof is complete. \square

Continuing from Lemma 2.1, we have the useful tail estimates on $S(t)$.

Corollary 2.2. For any positive $\varepsilon \in (0, 1)$, there exists an $M > 1$ depending only on ε , such that,

$$\sum_{|m| \geq \max\{M, 2Dt\}} e^{-2Dt} \mathbf{I}_m(2Dt) \leq \varepsilon/2, \quad (2.13)$$

and

$$\sum_{|m| \geq \max\{M, \sqrt{2DMt}\}} e^{-2Dt} \mathbf{I}_m(2Dt) \leq \varepsilon. \quad (2.14)$$

Proof. We choose M satisfying

$$\frac{C}{\sqrt{2\pi M}} \int_M^\infty e^{-\alpha_1 m} dm \leq \varepsilon/2.$$

In case of $M \geq 2Dt$, by [Lemma 2.1](#), we have

$$\begin{aligned} \sum_{|m| \geq M} e^{-2Dt} \mathbf{I}_m(2Dt) &\leq \sum_{|m| \geq M} \frac{C}{\sqrt{2\pi}} \frac{e^{-\alpha_2 m}}{((2Dt)^2 + m^2)^{1/4}} \\ &\leq \sum_{|m| \geq M} \frac{C}{\sqrt{2\pi M}} e^{-\alpha_2 m} \\ &\leq \frac{C}{\sqrt{2\pi M}} \int_M^\infty e^{-\alpha_1 m} dm \leq \varepsilon/2. \end{aligned}$$

In case of $M \leq 2Dt$, similarly we have

$$\begin{aligned} \sum_{|m| \geq 2Dt} e^{-2Dt} \mathbf{I}_m(2Dt) &\leq \sum_{|m| \geq 2Dt} \frac{C}{\sqrt{2\pi M}} e^{-\alpha_2 m} \\ &\leq \frac{C}{\sqrt{2\pi M}} \int_M^\infty e^{-\alpha_1 m} dm \leq \varepsilon/2. \end{aligned}$$

So the estimate [\(2.13\)](#) is proved. For [\(2.14\)](#) it will suffice to show the case $2Dt \geq M$, which gives $M \leq \sqrt{2MDt} \leq 2Dt$. We compute the tail of the series to obtain

$$\sum_{|m| \geq \sqrt{2MDt}} e^{-2Dt} \mathbf{I}_m(2Dt) = \sum_{\sqrt{2MDt} \leq |m| \leq 2Dt} e^{-2Dt} \mathbf{I}_m(2Dt) + \sum_{|m| > 2Dt} e^{-2Dt} \mathbf{I}_m(2Dt).$$

By [\(2.13\)](#), the second term on the right hand side is less than or equal to $\varepsilon/2$. For the first term,

$$\begin{aligned} \sum_{\sqrt{2MDt} \leq |m| \leq 2Dt} e^{-2Dt} \mathbf{I}_m(2Dt) &\leq \sum_{|m| \geq \sqrt{2MDt}} \frac{C}{\sqrt{2\pi}} \frac{e^{-\alpha_1 \frac{m^2}{2Dt}}}{((2Dt)^2 + m^2)^{1/4}} \\ &\leq \frac{C}{\sqrt{2\pi}} \int_{\sqrt{2MDt}}^\infty \frac{e^{-\alpha_1 \frac{m^2}{2Dt}}}{((2Dt)^2 + m^2)^{1/4}} dm \\ &\leq \frac{C}{\sqrt{2\pi M}} \int_M^\infty e^{-\alpha_1 m} dm \leq \varepsilon/2. \end{aligned}$$

So we have obtained [\(2.14\)](#). The proof is complete. \square

Remark 2.2. The estimate (2.14) shows that, for large time t , only terms of order $O(\sqrt{t})$ are needed to arrive at the desired accuracy, while (2.13) presents an order of $O(t)$. The order $O(\sqrt{t})$ obtained in (2.14) is crucial in the study of spreading speed.

With aid of the modified Bessel functions, we can express the solution to the initial value problem

$$\frac{du}{dt}(x, t) = D[u(x+1, t) - 2u(x, t) + u(x-1, t)], \quad u(0, x) = u_0(x),$$

as

$$u(x, t) = (S(t)u_0)(x) = e^{-2Dt} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2Dt) u_0(x-m). \quad (2.15)$$

Remark 2.3. The estimate (2.13) also implies uniform continuity of $S(t)$ for t in a bounded interval in the compact open topology with respect to the initial data u_0 , i.e., for any $\varepsilon > 0$, $K > 0$, $M > 0$, there exist $\delta(\varepsilon, K) > 0$, $M(\varepsilon, K) > 0$ such that if, for u_0^1, u_0^2 satisfying $0 \leq u_0^1(x)$, $u_0^2(x) \leq r(\infty)$ and $|u_0^1(x) - u_0^2(x)| < \delta$ for $x \in [-K-M, K+M]$,

$$|(S(t)u_0^1)(x) - (S(t)u_0^2)(x)| < \varepsilon, \quad \forall t \in [0, T], x \in [-K, K].$$

The following proposition characterizes $S(t)$ as a linear operator in L^p space for $1 \leq p \leq \infty$.

Proposition 2.1. *The family of bounded operators $\{S(t)\}_{t \geq 0}$, defined above is a uniformly continuous semigroup in L^p space for $1 \leq p \leq \infty$, and satisfies the estimate*

$$\|S(t)u_0\|_{L^p} \leq \|u_0\|_{L^p}, \quad t \geq 0. \quad (2.16)$$

For $u_0 \in L^\infty(\mathbb{H})$, $S(t)$ and \limsup (\liminf) commute with each other, i.e.,

$$\limsup_{x \rightarrow \infty} (S(t)u_0)(x) = \limsup_{x \rightarrow \infty} u_0(x); \quad \limsup_{x \rightarrow \infty} (S(t)u_0)(x) = \liminf_{x \rightarrow \infty} u_0(x), \quad (2.17)$$

Proof. By (2.15) we have

$$\begin{aligned} \|S(t)\|_{L^p} &\leq e^{-2Dt} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2Dt) \|u_0(\cdot + m)\|_{L^p} \\ &= e^{-2Dt} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2Dt) \|u_0\|_{L^p} = \|u_0\|_{L^p}, \end{aligned}$$

here we used the identity (2.6). (2.17) is a trivial consequence of Corollary 2.2, we omit the details. The proof is complete. \square

Remark 2.4. From (2.17), we have, if $\lim_{x \rightarrow \infty} u_0(x)$ exists, then

$$\lim_{x \rightarrow \infty} (S(t)u_0)(x) = \lim_{x \rightarrow \infty} u_0(x).$$

The same results hold true as $x \rightarrow -\infty$. This fact will be used in the proof of Lemma 4.1.

Remark 2.5. It can be shown that the solution $u(x, t) = (S(t)u_0)(x)$ admits exactly the same smoothness as the initial data u_0 when $x \in \mathbb{R}$. The Laplace operator $\Delta_1 - 2\mathbb{I} + \Delta_{-1}$ does not have the regularizing effect to the system (2.1).

Remark 2.6. For the nonhomogeneous system

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = D[u(x+1, t) - 2u(x, t) + u(x-1, t)] + f(t, x) & t > 0, x \in \mathbb{H}, \\ u(0, x) = u_0(x), \end{cases} \quad (2.18)$$

where $f \in L^\infty(\mathbb{R}^+, \mathbb{H})$. The Duhammel principle holds true, i.e., $u(t, x)$ can be expressed as

$$u(t, x) = e^{-2Dt} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2Dt)u_0(x-m) + \sum_{m=-\infty}^{\infty} \int_0^t e^{-2D(t-\tau)} \mathbf{I}_m(2D(t-\tau))f(\tau, x-m)d\tau.$$

3. Well-posedness of nonlinear equations

In this section we establish the well-posedness of the nonlinear model (1.3), and the comparison principle, which is the main tool in the proof of our major Theorems 4.1, 5.1. We make the standing hypothesis on $r(x)$.

Hypothesis 3.1. $r(x)$ is continuous, non-decreasing and bounded, piecewise continuously differentiable in x for $-\infty < x < \infty$, $0 < r(\infty) < \infty$, and $-\infty < x < 0$.

For $0 \leq u_1, u_2 \leq r(\infty)$, $-\infty < x < \infty$ and $t \geq 0$,

$$|u_1(r(x-ct) - u_1) - u_2(r(x-ct) - u_2)| \leq 3r(\infty)|u_1 - u_2|,$$

so that $f(x-ct, u) = u(r(x-ct) - u)$ is Lipschitz continuous in u . We will study the following equation obtained by adding a dominant linear term $\rho u(x, t)$ to both sides of (1.3),

$$\begin{aligned} u_t(x, t) + \rho u(x, t) &= D[u(x+1, t) - 2u(x, t) + u(x-1, t)] \\ &\quad + u(x, t)(\rho + r(x-ct) - u(x, t)), \end{aligned} \quad x \in \mathbb{H}, t > 0. \quad (3.1)$$

Here we choose $\rho \geq 3r(\infty)$. Then $u(\rho + r(x-ct) - u)$ is non-decreasing in u for $0 \leq u \leq r(\infty)$, i.e., for $0 \leq u_1 \leq u_2 \leq r(\infty)$, $0 \leq u_1(\rho + r(x-ct) - u_1) \leq u_2(\rho + r(x-ct) - u_2) \leq \rho r(\infty)$. It is clear that $u \equiv 0$ is the trivial lower solution of (3.1) and $u \equiv r(\infty)$ is an upper solution of (3.1).

To proceed, we first introduce the subset of $L^\infty(\mathbb{H})$,

$$\mathcal{L}_\alpha := \{u : 0 \leq u(x) \leq \alpha \text{ for all } x\}.$$

In view of Remark 2.6, the solution of (3.1) with $u(0, x) = u_0(x) \in \mathcal{L}_{r(\infty)}$ can be expressed as the fixed point of the nonlinear integral equation in $C(\mathbb{R}^+, \mathcal{L}_{r(\infty)})$,

$$\begin{aligned} u(t, x) &= (T[u])(t, x) \\ &= \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)t} \mathbf{I}_m(2Dt) u_0(x+m) \\ &\quad + \int_0^t \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) \\ &\quad \cdot u(x+m, \tau) (\rho + r(x+m-c\tau) - u(x+m, \tau)) d\tau. \end{aligned} \quad (3.2)$$

Consider the sequence $u^{(n)}(t, x)$ generated by

$$u^{(n+1)}(t, x) = (T[u^n])(t, x), \quad (3.3)$$

where $u^{(0)}(t, x) = 0$ or $u^{(0)}(t, x) = r(\infty)$.

Theorem 3.1. Assume that Hypothesis 3.1 is satisfied, and $u_0 \in \mathcal{L}_{r(\infty)}$. Then there exists a unique solution $u \in C(\mathbb{R}^+, \mathcal{L}_{r(\infty)})$ to (3.2). Moreover the comparison principle holds for the system (3.2), i.e., if $u_1(t, x)$ and $u_2(t, x)$ are two solutions of (3.2) associated with initial data $u_1^0, u_2^0 \in \mathcal{L}_{r(\infty)}$ respectively, and if $u_1^0(x) \leq u_2^0(x)$ for any $x \in \mathbb{H}$, then $u_1(t, x) \leq u_2(t, x)$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{H}$.

Proof. Here we generalize the classical upper–lower solution method for reaction–diffusion equations in [25] to the lattice differential equations (3.1). Define $\underline{u}^{n+1} = (T[\underline{u}^n])(t, x)$ with $\underline{u}^0(t, x) = 0$, and $\bar{u}^{n+1} = (T[\bar{u}^n])(t, x)$ with $\bar{u}^0(t, x) = r(\infty)$. It is easy to check the following monotonicity of the two sequences $\{\underline{u}^n\}_{n=0}^{\infty}$ and $\{\bar{u}^n\}_{n=0}^{\infty}$,

$$0 \leq \underline{u}^1(t, x) \leq \underline{u}^2(t, x) \leq \cdots \leq \bar{u}^2(t, x) \leq \bar{u}^1(t, x) \leq r(\infty).$$

Then it can be derived $\underline{u}^n(t, x) \rightarrow \underline{u}(t, x)$ and $\bar{u}^n(t, x) \rightarrow \bar{u}(t, x)$ as $n \rightarrow \infty$. Moreover both \underline{u} and \bar{u} are in $C(\mathbb{R}^+, \mathcal{L}_{r(\infty)})$, and solutions to (3.2). To finish the proof, we just need to show the uniqueness, i.e., $\underline{u}(t, x) = \bar{u}(t, x)$. To this end, we compute $\bar{u}(t, x) - \underline{u}(t, x)$ to obtain,

$$\begin{aligned} 0 &\leq \bar{u}(t, x) - \underline{u}(t, x) \\ &\leq (\rho + 3r(\infty)) \int_0^t \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) \sup_{x \in \mathbb{H}} [\bar{u}(\tau, x) - \underline{u}(\tau, x)] d\tau. \end{aligned}$$

Taking $\sup_{x \in \mathbb{H}}$, we obtain

$$0 \leq \sup_{x \in \mathbb{H}} [\bar{u}(t, x) - \underline{u}(t, x)] \leq (\rho + 3r(\infty)) \int_0^t e^{-\rho(t-\tau)} \sup_{x \in \mathbb{H}} [\bar{u}(\tau, x) - \underline{u}(\tau, x)] d\tau.$$

Therefore the classical Gronwall's inequality implies that $\bar{u}(t, x) = \underline{u}(t, x)$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{H}$. The comparison principle is straightforward from the construction of solutions. The proof is complete. \square

We conclude this section with a comparison result which will be the principal tool in subsequent discussions.

Corollary 3.1. *Let $u, v \in C(\mathbb{R}^+, \mathcal{L}_{r(\infty)})$, such that $u(t, x) \geq (T[u])(t, x)$ and $v(t, x) \leq (T[v])(t, x)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{H}$, and $u(0, x) \geq v(0, x)$ for all $x \in \mathbb{H}$. Then $u(t, x) \geq v(t, x)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{H}$.*

Proof. With the choice of ρ in (3.1), it can be derived that the operator T is order preserving in the following sense,

$$u(t, x) \geq (T[u])(t, x) \geq (T^2[u])(t, x) \geq \cdots \geq \lim_{n \rightarrow \infty} (T^n[u])(t, x),$$

and

$$v(t, x) \leq (T[v])(t, x) \leq (T^2[v])(t, x) \leq \cdots \leq \lim_{n \rightarrow \infty} (T^n[v])(t, x).$$

Let $\tilde{u}(t, x) = \lim_{n \rightarrow \infty} (T^n[u])(t, x)$ and $\tilde{v}(t, x) = \lim_{n \rightarrow \infty} (T^n[v])(t, x)$. Then both \tilde{u} and \tilde{v} solve the nonlinear equation (3.2) with initial data $u(0, x)$ and $v(0, x)$ respectively. By Theorem 3.1, $u(t, x) \geq \tilde{u}(t, x) \geq \tilde{v}(t, x) \geq v(t, x)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{H}$. The proof is complete. \square

4. Non-persistences

In this section we shall prove that, if the edge of the habitat shifts relatively fast, then the species will become extinct in the long run. We introduce some notations. For $r(x) > 0$, define

$$\phi(x; \mu) = \frac{4D \sinh^2(\mu/2) + r(x)}{\mu}.$$

It is elementary to check that $\phi(x; \mu)$ has only one minimum, assumed to be $c^*(x)$, i.e.,

$$c^*(x) = \inf_{\mu > 0} \phi(x; \mu).$$

Let $\mu = \mu^*(x)$ be the unique point where the minimum occurs, i.e., $c^*(x) = \phi(x; \mu^*(x))$.

We first provide a useful lemma for the equation

$$\frac{du}{dt}(x, t) = D[u(x+1, t) - 2u(x, t) + u(x-1, t)] + u(x, t)(r(x) - u(x, t)). \quad (4.1)$$

Lemma 4.1. *Let $\bar{u}(t, x)$ be the solution of (4.1) with $\bar{u}(0, x) = r(\infty)$. Then $\bar{u}(t, x)$ is non-increasing in t and non-decreasing in x , $\lim_{t \rightarrow \infty} \bar{u}(t, -\infty) = 0$, and $\bar{u}(t, \infty) = r(\infty)$ for all $t > 0$.*

Proof. Let $u^{(n)}(t, x)$ be the sequence generated by the iteration (3.3) with $c = 0$, $u^{(0)}(t, x) = r(\infty)$. Then from Theorem 3.1, $\bar{u}(t, x) = \lim_{n \rightarrow \infty} u^{(n)}(t, x)$, with $\bar{u}(0, x) = r(\infty)$ is a solution of (4.1). According to the choice of ρ , it is easily seen that each term in the series in (3.2) is monotone with respect to u . Through induction, we claim that $u^n(t, x)$ is increasing in the x direction for $n \geq 1$, so is $\bar{u}(t, x)$. To study its monotonicity in t , we begin with the $u^{(1)}(t, x)$ in (3.3). Direct computations show that

$$\begin{aligned} \frac{\partial u^{(1)}}{\partial t}(t, x) &= -\rho e^{-\rho t} r(\infty) \\ &\quad + r(\infty) \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)t} \mathbf{I}_m(2Dt) (\rho + r(x+m) - r(\infty)) \\ &= r(\infty) \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)t} \mathbf{I}_m(2Dt) (r(x+m) - r(\infty)) \leq 0. \end{aligned}$$

We proceed by mathematical induction. Assume that $\frac{\partial u^{(n)}}{\partial t}(t, x) \leq 0$ for $n > 0$. Then

$$\begin{aligned} \frac{\partial u^{(n+1)}}{\partial t}(t, x) &= -\rho e^{-\rho t} r(\infty) \\ &\quad + \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)t} \mathbf{I}_m(2Dt) u^{(n)}(0, x+m) (\rho + r(x+m) - u^{(n)}(0, x+m)) \\ &\quad + \int_0^t \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)\tau} \mathbf{I}_m(2D\tau) \frac{\partial u^{(n)}}{\partial t}(x+m, t-\tau) \\ &\quad \cdot (\rho + r(x+m) - 2u^{(n)}(x+m, t-\tau)) d\tau. \end{aligned}$$

Since $\rho > 3r(\infty)$ and $0 \leq u^{(n)}(t-\tau, x+m) \leq r(\infty)$, we obtain that $\rho + r(x+m) - 2u^{(n)}(t, x) \geq 0$. From Eq. (3.3), it can be seen easily that $u^{(n)}(0, x) = r(\infty)$ for all $n \geq 0$. Therefore we have that

$$\begin{aligned} \frac{\partial u^{(n+1)}}{\partial t}(t, x) &= \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)t} \mathbf{I}_m(2Dt) r(\infty) (r(x+m) - r(\infty)) \\ &\quad + \int_0^t \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)\tau} \mathbf{I}_m(2D\tau) \frac{\partial u^{(n)}}{\partial t}(x+m, t-\tau) \\ &\quad \cdot (\rho + r(x+m) - 2u^{(n)}(x+m, t-\tau)) d\tau \leq 0. \end{aligned}$$

By induction, $\frac{\partial u^{(n)}}{\partial t}(t, x) \leq 0$ for all $n > 0$. Since $\bar{u}(t, x) = \lim_{n \rightarrow \infty} u^{(n)}(t, x)$, we have $\frac{\partial \bar{u}}{\partial t}(t, x) \leq 0$. Moreover \bar{u} satisfies

$$\begin{aligned}
 u(t, x) = & r(\infty)e^{-\rho t} + \int_0^t \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)\tau} \mathbf{I}_m(2D\tau) \\
 & \cdot u(x+m, t-\tau)(\rho+r(x+m)-u(x+m, t-\tau)) d\tau.
 \end{aligned} \tag{4.2}$$

Taking the limit as $x \rightarrow -\infty$ in (4.2) and applying the dominant convergence theorem and Proposition 2.1, we obtain

$$\begin{aligned}
 u(t, -\infty) = & r(\infty)e^{-\rho t} + \int_0^t \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)\tau} \mathbf{I}_m(2D\tau) \\
 & \cdot u(-\infty, t-\tau)(\rho+r(-\infty)-u(-\infty, t-\tau)) d\tau \\
 = & r(\infty)e^{-\rho t} + \int_0^t e^{-\rho\tau} u(-\infty, t-\tau)(\rho+r(-\infty)-u(-\infty, t-\tau)) d\tau,
 \end{aligned}$$

which implies that $\bar{u}(-\infty, t)$ satisfies the initial value problem,

$$\frac{\partial \bar{u}}{\partial t}(t, -\infty) = \bar{u}(t, -\infty)(r(-\infty) - \bar{u}(t, -\infty)), \quad \bar{u}(-\infty, 0) = r(\infty).$$

This is the classical logistic equation with a negative growth rate. Immediately it can be inferred that $\lim_{t \rightarrow \infty} \bar{u}(t, -\infty) = 0$. Similarly for $\bar{u}(t, \infty)$, we have

$$\frac{\partial \bar{u}}{\partial t}(t, \infty) = \bar{u}(t, \infty)(r(-\infty) - \bar{u}(t, \infty)), \quad \bar{u}(-\infty, 0) = r(\infty).$$

Since $\bar{u}(0, \infty) = r(\infty)$, which happens to be the positive equilibrium of the classical logistic equation, we have that $\bar{u}(t, \infty) \equiv r(\infty)$ for all $t \geq 0$. The proof is complete. \square

Now we are ready to prove the following theorem, which essentially shows that if $c^*(\infty) < c$, then the species will eventually become extinct in space under certain conditions.

Theorem 4.1. Assume that Hypothesis 3.1 is satisfied. Let $c^*(\infty) < c$. Then for every $\varepsilon > 0$ and for $0 \leq u_0(x) \leq r(\infty)$ and $u_0(x) \equiv 0$ for sufficiently large x , there exists $T > 0$ such that for $t \geq T$, the solution $u(t, x)$ of (1.3) satisfies that $u(t, x) \leq \varepsilon$ for all x .

Proof. Let $\bar{u}(t, x)$ be the solution of (4.1) with $\bar{u}(0, x) = r(\infty)$. Since $\lim_{t \rightarrow \infty} \bar{u}(t, -\infty) = 0$ from Lemma 4.1, for any $\varepsilon > 0$, there exists $T_1 > 0$ and $M > 0$ such that for $x \leq -M$, $\bar{u}(T_1, x) \leq \varepsilon$. Since $\bar{u}(t, x)$ is non-increasing in t , we have

$$\bar{u}(t, x) \leq \varepsilon, \quad \text{for } t \geq T_1, \text{ and } x \leq -M. \tag{4.3}$$

Define $\tilde{u}(t, x) = \bar{u}(t, x - ct)$. Then \tilde{u} satisfies

$$\frac{\partial \tilde{u}}{\partial t} = D[\tilde{u}(x+1, t) - 2\tilde{u}(t, x) + \tilde{u}(t, x-1)] + c \frac{\partial \tilde{u}}{\partial x} + \tilde{u}(r(x-ct) - \tilde{u}). \tag{4.4}$$

Since $\tilde{u}(t, x)$ is non-decreasing in x , $\frac{\partial \tilde{u}}{\partial x} \geq 0$, Eq. (4.4) implies that

$$\frac{\partial \tilde{u}}{\partial t} \geq D[\tilde{u}(x+1, t) - 2\tilde{u}(t, x) + \tilde{u}(t, x-1)] + \tilde{u}(r(x-ct) - \tilde{u}),$$

so that $\tilde{u} = \bar{u}(t, x-ct)$ with $\tilde{u}(0, x) = r(\infty)$ is an upper solution of (1.3). Since $u(0, x) = u_0(x) \leq r(\infty)$, we have

$$u(t, x) \leq \bar{u}(t, x-ct).$$

By (4.3), we can claim that

$$u(t, x) \leq \varepsilon, \quad \text{for } t \geq T_1, \text{ and } x \leq -M + ct. \quad (4.5)$$

Choose $0 < \delta < c - c^*(\infty)$. Let $\mu_\delta > 0$ be the smaller solution of equation $\phi(\infty, \mu) = c^*(\infty) + \delta/2$. It is easily seen that $\hat{u}(t, x) = Ae^{-\mu_\delta(x-(c^*(\infty)+\delta/2)t)}$, where A is a positive constant, is a solution of the linear equation

$$\frac{\partial u}{\partial t} = D[u(x+1, t) - 2u(x, t) + u(x-1, t)] + r(\infty)u.$$

Since $r(\infty)u \geq u(r(x-ct) - u)$, $\hat{u}(t, x)$ is an upper solution of (1.3). Choose A sufficiently large such that $u_0(x) \leq \hat{u}(0, x) = Ae^{-\mu_\delta x}$. Therefore

$$u(t, x) \leq Ae^{-\mu_\delta(x-(c^*(\infty)+\delta/2)t)},$$

which implies that for $x \geq (c^*(\infty) + \delta)t$, $u(t, x) \leq Ae^{-(\mu_\delta\delta/2)t}$. It follows that for above given $\varepsilon > 0$, there exists $T_2 > 0$, such that

$$u(t, x) < \varepsilon, \quad \text{for } t \geq T_2 \text{ and } x \geq (c^*(\infty) + \delta)t. \quad (4.6)$$

On the other hand, since $c > c^*(\infty)$, there exists $T_3 > 0$, such that for $t \geq T_3$, $-M + ct \geq (c^*(\infty) + \delta)t$, which, together with (4.5) and (4.6), implies that for $t \geq T := \max\{T_1, T_2, T_3\}$ and for all x , $u(t, x) < \varepsilon$. This completes the proof. \square

5. Spreading speed

In this section we shall establish the major results in this paper, which demonstrate, if the edge of habitat is moving at a speed c less than $c^*(\infty)$, the species will persist in the habitat space and spread rightward at a speed $c^*(\infty)$. First we need an auxiliary function that can be found in the pioneering work by Weinberger [33]. For $\gamma > 0$ and $\mu > 0$, define

$$v(\mu; x) = \begin{cases} e^{-\mu x} \sin \gamma x & \text{if } 0 \leq x \leq \pi/\gamma, \\ 0 & \text{elsewhere.} \end{cases} \quad (5.1)$$

$v(\mu; x)$ is a continuous function in x and its second order derivative in x exists and is continuous when $x \neq 0, \gamma$. The maximum of $v(\mu; x)$ occurs at $\sigma(\mu) = (1/\gamma) \tan^{-1}(\gamma/\mu)$. $\sigma(\mu)$ is a strictly decreasing function of μ . We also define

$$\psi(\mu, \gamma) = \frac{D \sin \gamma}{\gamma} (e^{\mu} - e^{-\mu}).$$

We note that $\gamma > 0$ will be assumed to be sufficiently small so that $\psi(\mu, \gamma) \sim D(e^{\mu} - e^{-\mu})$. Let

$$\phi_{\gamma}(\mu, l) = \frac{D[(e^{\mu} + e^{-\mu}) \cos \gamma - 2] + r(l)}{\mu},$$

and

$$c_{\gamma}^{*}(l) = \inf_{\mu > 0} \phi_{\gamma}(\mu, l).$$

It is easy to see that $\phi_{\gamma}(\mu, l) < \phi(\mu, l)$ and $\phi_{\gamma}(\mu, l)$ converges to $\phi(\mu, l)$ uniformly for μ in any bounded interval as $\gamma \rightarrow 0$. Also we have $c_{\gamma}^{*}(l) < c^{*}(l)$ and the convergence $c_{\gamma}^{*}(l) \rightarrow c^{*}(l)$ as $\gamma \rightarrow 0$. We have the following useful lemma.

Lemma 5.1. Assume that $c^{*}(\infty) > c \geq 0$. For any δ satisfying $0 < \delta < \frac{c^{*}(\infty) - c}{5}$, let l be the point such that $c^{*}(l) = c^{*}(\infty) - \delta$, and $\gamma > 0$ such that $c^{*}(l) - c_{\gamma}^{*}(l) \leq \delta$. Let $0 < \mu_1 < \mu_2 < \mu^{*}(l)$ with $\psi(\mu_1, \gamma) = c + \delta$ and $\psi(\mu_2, \gamma) = c_{\gamma}^{*}(l) - \delta$. Then for $\mu \in [\mu_1, \mu_2]$, and $0 < a \leq \delta/\mu_2$, $w(t, x) = av(\mu; x - l - \psi(\mu, \gamma)t)$ is a continuous lower solution of (3.1). Furthermore if $u(0, x) \geq av(\mu; x - l)$, then $u(t, x) \geq av(\mu; x - l - \psi(\mu)t)$ for all $t > 0$.

Proof. According to Corollary 3.1, we need to prove $w(t, x) \leq (T[w])(t, x)$ for all $(t, x) \in \mathbb{R}^{+} \times \mathbb{H}$. First, for $t > 0$ and $x \leq l + \psi(\mu, \gamma)t$ or $x \geq l + \pi/\gamma + \psi(\mu, \gamma)t$, $w(t, x) \equiv 0$, so it is trivial. We need to consider the case when $t > 0, l + \psi(\mu, \gamma)t \leq x \leq l + \pi/\gamma + \psi(\mu, \gamma)t$. Notice that, for such (t, x) , $w(t, x)$ is differentiable with respect to t variable. We estimate the nonlinear part of (3.2) to obtain,

$$\begin{aligned} & \int_0^t \sum_{m=-\infty}^{\infty} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) w(x+m, \tau) (\rho + r(x+m - c\tau) - w(x+m, \tau)) d\tau \\ &= \int_0^t \sum_{l+\psi(\mu, \gamma)\tau \leq x+m \leq l+\pi/\gamma+\psi(\mu, \gamma)\tau} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) \frac{\partial w}{\partial \tau}(\tau, x+m) d\tau \\ &+ \int_0^t \sum_{l+\psi(\mu, \gamma)\tau \leq x+m \leq l+\pi/\gamma+\psi(\mu, \gamma)\tau} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) \\ &\cdot \left[-\frac{\partial w}{\partial \tau}(\tau, x+m) + w(x+m, \tau) (\rho + r(x+m - c\tau) - w(x+m, \tau)) \right] d\tau = I + II. \end{aligned}$$

For the term I, exchanging the order of \int and the summation \sum , we obtain by integration by parts

$$\begin{aligned}
 I &= \sum_{l \leq x+m \leq l+\pi/\gamma+\psi(\mu,\gamma)t} \int_{0 \vee \frac{x+m-l-\pi/\gamma}{\psi(\mu,\gamma)}}^{t \wedge \frac{x+m-l}{\psi(\mu,\gamma)}} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) \frac{\partial w}{\partial \tau}(\tau, x+m) d\tau \\
 &= -(\rho+2D) \sum_{l \leq x+m \leq l+\pi/\gamma+\psi(\mu,\gamma)t} \int_{0 \vee \frac{x+m-l-\pi/\gamma}{\psi(\mu,\gamma)}}^{t \wedge \frac{x+m-l}{\psi(\mu,\gamma)}} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) w(\tau, x+m) d\tau \\
 &\quad + 2D \sum_{l \leq x+m \leq l+\pi/\gamma+\psi(\mu,\gamma)t} \int_{0 \vee \frac{x+m-l-\pi/\gamma}{\psi(\mu,\gamma)}}^{t \wedge \frac{x+m-l}{\psi(\mu,\gamma)}} e^{-(\rho+2D)(t-\tau)} \mathbf{I}'_m(2D(t-\tau)) w(\tau, x+m) d\tau \\
 &\quad + \sum_{l \leq x+m \leq l+\pi/\gamma+\psi(\mu,\gamma)t} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) w(\tau, x+m) \Big|_{\tau=0 \vee \frac{x+m-l-\pi/\gamma}{\psi(\mu,\gamma)}}^{\tau=t \wedge \frac{x+m-l}{\psi(\mu,\gamma)}} \\
 &= -(\rho+2D) \int_0^t \sum_{l+\psi(\mu,\gamma)\tau \leq x+m \leq l+\pi/\gamma+\psi(\mu,\gamma)\tau} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) w(\tau, x+m) d\tau \\
 &\quad + D \int_0^t \sum_{l+\psi(\mu,\gamma)\tau \leq x+m \leq l+\pi/\gamma+\psi(\mu,\gamma)\tau} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) \\
 &\quad \cdot [w(\tau, x+m+1) + w(\tau, x+m-1)] d\tau \\
 &\quad + w(t, x) - e^{-(\rho+2D)t} \sum_m \mathbf{I}_m(2Dt) w(0, x+m),
 \end{aligned}$$

where we used the identity (2.4). Here we also followed the conventional notations, $a \wedge b = \min(a, b)$, and $a \vee b = \max(a, b)$. To claim that $w(t, x) \leq (T[w])(t, x)$, we need to justify that

$$\begin{aligned}
 &\int_0^t \sum_{l+\psi(\mu,\gamma)\tau \leq x+m \leq l+\pi/\gamma+\psi(\mu,\gamma)\tau} e^{-(\rho+2D)(t-\tau)} \mathbf{I}_m(2D(t-\tau)) \\
 &\quad \cdot \left[-\frac{\partial w}{\partial \tau}(\tau, x+m) + D[w(\tau, x+m+1) - 2w(\tau, x+m) + w(\tau, x+m-1)] \right. \\
 &\quad \left. + w(x+m, \tau)(r(x+m-c\tau) - w(x+m, \tau)) \right] d\tau \geq 0.
 \end{aligned}$$

It will suffice to prove that for $t \geq 0$, $l + \psi(\mu, \gamma) \leq x \leq l + \pi/\gamma + \psi(\mu, \gamma)t$,

$$\frac{\partial w}{\partial t}(t, x) \leq D[w(t, x+1) - 2w(t, x) + w(t, x-1)] + w(x, t)(r(x-ct) - w(x, t)).$$

It is clear that $w(t, x)$ is differentiable on t variable in the case of our interest. We notice that

$$\begin{aligned} & D[w(t, x+1) - 2w(t, x) + w(t, x-1)] \\ & \geq Da[e^{-\mu(x+1-l-\psi(\mu, \gamma)t}) \sin \gamma(x+1-l-\psi(\mu, \gamma)t) \\ & \quad + e^{-\mu(x-1-l-\psi(\mu, \gamma)t}) \sin \gamma(x-1-l-\psi(\mu, \gamma)t) \\ & \quad - 2e^{-\mu(x-l-\psi(\mu, \gamma)t}) \sin \gamma(x-l-\psi(\mu, \gamma)t)] \\ & = Dae^{-\mu(x-l-\psi(\mu, \gamma)t)} \{[(e^{-\mu} + e^{\mu}) \cos \gamma - 2] \sin \gamma(x-l-\psi(\mu, \gamma)t) \\ & \quad - \sin \gamma[e^{\mu} - e^{-\mu}] \cos \gamma(x-l-\psi(\mu, \gamma)t)\}, \end{aligned}$$

where $>$ holds true as either $x \pm 1 - l - \psi(\mu, \gamma)t \notin [0, \pi/\gamma]$. We find

$$\begin{aligned} \frac{\partial w}{\partial t} &= aD\mu\psi(\mu, \gamma)e^{-\mu(x-l-\psi(\mu, \gamma)t)} \sin \gamma(x-l-\psi(\mu, \gamma)t) \\ &\quad - aD\gamma\psi(\mu, \gamma)e^{-\mu(x-l-\psi(\mu, \gamma)t)} \cos \gamma(x-l-\psi(\mu, \gamma)t). \end{aligned}$$

Notice that, by the definition of $\psi(\mu, \gamma)$, the terms involving $\cos \gamma(x-l-\psi(\mu, \gamma)t)$ are cancelled on both sides in the above. In order for $w(t, x)$ to be a lower solution of (1.3), the following inequality should hold true,

$$\mu\psi(\mu, \gamma) \leq D[(e^{\mu} + e^{-\mu}) \cos \gamma - 2] + r(x-ct) - av(\mu; x-l-\psi(\mu, \gamma)t). \quad (5.2)$$

μ is chosen to satisfy

$$\mu\psi(\mu, \gamma) \leq D[(e^{\mu} + e^{-\mu}) \cos \gamma - 2] + r(l) - a. \quad (5.3)$$

By elementary computation, we can find the desired μ_1, μ_2 such that

$$\psi(\mu, \gamma) > c + \delta, \quad \psi(\mu, \gamma) \leq \phi_{\gamma}(\mu, l) - \delta,$$

for $\forall \mu \in [\mu_1, \mu_2]$. Hence with the choice of $a \leq \delta/\mu_2$, (5.3) is satisfied. The proof is now complete. \square

We define

$$w(t, x) = \begin{cases} \frac{\alpha}{v(\mu_1; \sigma(\mu_1))} v(\mu_1; x-l-\psi(\mu_1)t), & \text{if } l + \psi(\mu_1)t \leq x \\ & \leq l + \sigma(\mu_1) + \psi(\mu_1)t, \\ \alpha & \text{if } l + \sigma(\mu_1) + \psi(\mu_1)t \leq x \\ & \leq l + 3\pi/\gamma + \sigma(\mu_2) + \psi(\mu_2)t, \\ \frac{\alpha}{v(\mu_2; \sigma(\mu_2))} v(\mu_2; x-l-3\pi/\gamma-\psi(\mu_2)t), & \text{if } l + 3\pi/\gamma + \sigma(\mu_2) + \psi(\mu_2)t \leq x \\ & \leq l + 4\pi/\gamma + \psi(\mu_2)t, \\ 0, & \text{elsewhere} \end{cases} \quad (5.4)$$

Lemma 5.2. $w(t, x)$ is a lower solution of (3.2) in the sense that, if $v(t, x)$ is a solution of (3.2) with $v(0, x) \geq w(0, x)$, then $v(t, x) \geq w(t, x)$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{H}$.

Proof. To this end, we construct a family of lower solutions of (1.3) which are initially above $w(0, x)$. Let

$$\mu(\theta) = \begin{cases} \mu_1 & l \leq \theta < l + \pi/\gamma, \\ \mu_1 + \frac{\gamma}{\pi}(\theta - l - \pi/\gamma)(\mu_2 - \mu_1) & l + \pi/\gamma \leq \theta < 2\pi/\gamma, \\ \mu_2 & \theta \geq l + 2\pi/\gamma. \end{cases}$$

It is straightforward to notice that $w(0, x) \geq \max_{\theta \in [l, l+3\pi/\gamma]} \frac{\alpha}{v(\mu(\theta); \sigma(\mu(\theta)))} v(\mu(\theta); x - \theta)$. Then from Lemma 5.1, we obtain

$$v(t, x) \geq \max_{\theta \in [l, l+3\pi/\gamma]} \frac{\alpha}{v(\mu(\theta); \sigma(\mu(\theta)))} v(\mu(\theta); x - \theta - \psi(\mu(\theta), \gamma)t).$$

Notice that the right hand side of this inequality is exactly the function $w(t, x)$. So the proof is complete. \square

Remark 5.1. Lemma 5.2 implies that the species will establish at a low level of density in a moving interval, whose size is increasing linearly with respect to time t . This is the effect of the diffusion represented by the discretized Laplace operator $\Delta_1 - 2\mathbb{I} + \Delta_{-1}$ and positive growth rate.

The following theorem shows that if $c^*(\infty) > c$, then the species persists in space and spreads to the right at the asymptotic spreading speed $c^*(\infty)$.

Theorem 5.1. Assume Hypothesis 3.1 is satisfied. Let $c^*(\infty) > c \geq 0$. Then the following statements are valid:

i. for any $\varepsilon > 0$ and for $0 \leq u(0, x) \leq r(\infty)$,

$$\lim_{t \rightarrow \infty} \left[\sup_{x \leq t(c-\varepsilon)} u(t, x) \right] = 0;$$

ii. for any $\varepsilon > 0$ and for $0 \leq u(0, x) \leq r(\infty)$, and $u(0, x) \equiv 0$ for all sufficiently large x ,

$$\lim_{t \rightarrow \infty} \left[\sup_{x \geq t(c^*(\infty)+\varepsilon)} u(t, x) \right] = 0;$$

iii. for any $\varepsilon > 0$ with $0 < \varepsilon < (c^*(\infty) - c)/2$ and for $0 \leq u(0, x) \leq r(\infty)$, and $u(0, x) > 0$ on a closed interval,

$$\lim_{t \rightarrow \infty} \left[\sup_{t(c+\varepsilon) \leq x \leq t(c^*(\infty)+\varepsilon)} |r(\infty) - u(t, x)| \right] = 0.$$

Proof. The proof of statements (i) and (ii) is identical to the proof of Theorem 4.1, so we skip them. We now prove the statement (iii). Choose δ sufficiently small with $0 < \delta < \min\{r(\infty), \frac{c^*(\infty)-c}{5}\}$ so that Lemma 5.1 applies. Since $u(0, x) \geq 0$ and $u(0, x) \not\equiv 0$, $u(t, x) > 0$

for all x and $t > 0$. Choose $t_0 > 0$, $\alpha > 0$ and $\gamma > 0$ sufficiently small such that $u(t_0, x) \geq w(t_0, x)$. From [Corollary 3.1](#), we obtain

$$u(t, x) \geq w(t - t_0, x),$$

for any $t \geq t_0$, $x \in \mathbb{H}$. By [Lemma 2.1](#), for fixed $\eta > 0$, there exists $M \geq 1$ such that

$$\sum_{|m| \leq \max\{M, \sqrt{2DMs}\}} e^{-2Ds} I_m(2Ds) \geq 1 - \eta.$$

Set sufficiently large $t_1 \geq t_0$ as the initial time, then $u(t, x)$ satisfies the integral equation

$$\begin{aligned} u(t, x) = & \sum_{-\infty}^{\infty} e^{-(\rho+2D)(t-t_1)} \mathbf{I}_m(2D(t-t_1)) u(t_1, x+m) \\ & + \int_{t_1}^t \sum_{-\infty}^{\infty} e^{-(\rho+2D)(t-s)} \mathbf{I}_m(2D(t-s)) u(s, x+m) \\ & \cdot [\rho + r(x+m-cs) - u(s, x+m)] ds, \end{aligned} \quad (5.5)$$

where $t > t_1$. From [Lemma 5.1](#), it can be inferred from above for $t > t_1$,

$$\begin{aligned} u(t, x) \geq & \sum_{-\infty}^{\infty} e^{-(\rho+2D)(t-t_1)} \mathbf{I}_m(2D(t-t_1)) w(t_1 - t_0, x+m) \\ & + \int_{t_1}^t \sum_{-\infty}^{\infty} e^{-(\rho+2D)(t-s)} \mathbf{I}_m(2D(t-s)) w(s - t_0, x+m) \\ & \cdot [\rho + r(x+m-cs) - w(s - t_0, x+m)] ds, \end{aligned} \quad (5.6)$$

For $t \geq t_1$ and x satisfying

$$\begin{aligned} l + \sigma(\mu_1) + \psi(\mu_1, \gamma)(t - t_0) + \max\{M, \sqrt{2DM(t-t_1)}\} \\ \leq x \leq l + 3\pi/\gamma + \sigma(\mu_2) + \psi(\mu_2, \gamma)(t - t_0) - \max\{M, \sqrt{2DM(t-t_1)}\}. \end{aligned} \quad (5.7)$$

and m satisfying

$$\max\{M, \sqrt{2DM(t-t_1)}\} \leq m \leq -\max\{M, \sqrt{2DM(t-t_1)}\}, \quad (5.8)$$

we have that

$$l + \sigma(\mu_1) + \psi(\mu_1, \gamma)(t - t_0) \leq x + m \leq l + 3\pi/\gamma + \sigma(\mu_2) + \psi(\mu_2, \gamma)(t - t_0) \quad (5.9)$$

and

$$x + m - ct \geq l + \sigma(\mu_1) + \psi(\mu_1, \gamma)(t - t_0) - ct \geq l + \sigma(\mu_1) + \delta t - \psi(\mu_1)t_0 \geq l.$$

For the linear part in (5.6), we have for x satisfying (5.7),

$$\begin{aligned} & \sum_{-\infty}^{\infty} e^{-(\rho+2D)(t-t_1)} \mathbf{I}_m(2D(t-t_1)) w(t_1-t_0, x+m) \\ & \geq \alpha e^{-\rho(t-t_1)} \sum_{|m| \leq \max\{M, \sqrt{2DM(t-t_1)}\}} e^{-2D(t-t_1)} \mathbf{I}_m(2D(t-t_1)) \geq (1-\eta)\alpha e^{-\rho(t-t_1)}. \end{aligned} \quad (5.10)$$

For the nonlinear part, we have

$$\begin{aligned} & \int_{t_1}^t \sum_{-\infty}^{\infty} e^{-(\rho+2D)(t-s)} \mathbf{I}_m(2D(t-s)) w(s-t_0, x+m) [\rho + r(x+m-cs) - w(s-t_0, x+m)] ds \\ & \geq (1-\eta)\alpha[\rho + r(\infty) - \delta - \alpha] e^{-\rho(t-s)}, \end{aligned} \quad (5.11)$$

where we have used the fact that for x satisfying (5.7), and m satisfying (5.8),

$$r(x+m-cs) \geq r(l) = r(\infty) - \delta.$$

It follows that from (5.6), (5.10), and (5.11), that for $t \geq t_1$ and x satisfying (5.7),

$$u(t, x) \geq \tilde{u}^1(t),$$

where

$$\tilde{u}^1(t) = (1-\eta)\alpha e^{-\rho(t-t_1)} + (1-\eta) \int_{t_1}^t e^{-\rho(t-s)} \alpha[\rho + r(\infty) - \delta - \alpha] ds.$$

From (5.6), by induction we derive that for $t \geq t_1$ and x described by a sub-interval of (5.7), i.e.,

$$\begin{aligned} & l + \sigma(\mu_1) + \psi(\mu_1, \gamma)(t-t_0) + n \max\{M, \sqrt{2DM(t-t_1)}\} \\ & \leq x \leq l + 3\pi/\gamma + \sigma(\mu_2) + \psi(\mu_2, \gamma)(t-t_0) - n \max\{M, \sqrt{2DM(t-t_1)}\} \end{aligned} \quad (5.12)$$

$$u(t, x) \geq \tilde{u}^n(t),$$

where $\tilde{u}^n(t)$ is defined recursively by

$$\begin{aligned} & \tilde{u}^n(t) = (1-\eta)\alpha e^{-\rho(t-t_1)} \\ & + (1-\eta) \int_{t_1}^t e^{-\rho(t-s)} \tilde{u}^{n-1}(s) [\rho + r(\infty) - \delta - \tilde{u}^{n-1}(s)] ds, \quad n \geq 1. \end{aligned} \quad (5.13)$$

It is clear that $0 < \tilde{u}^n(t) < r(\infty)$ for $n \geq 1$. We are concerned about the asymptotic behavior of $\tilde{u}^n(t)$ as $t \rightarrow \infty$. To this end, we rewrite (5.13) in its differential form,

$$\begin{cases} \frac{d}{dt}\tilde{u}^n(t) = -\rho\tilde{u}^n(t) + (1-\eta)\tilde{u}^{n-1}(t)[\rho + r(\infty) - \delta - \tilde{u}^{n-1}(t)], \\ \tilde{u}^n(0) = (1-\eta)\alpha, \quad n \geq 1. \end{cases} \quad (5.14)$$

Apparently, (5.14) can be regarded as a linear first order ODE on \tilde{u}^n with a non-homogeneous term involving \tilde{u}^{n-1} . We can derive from the classical theory of particular solutions to first order ODE, and induction that,

$$\tilde{u}^n(t) = \tilde{u}^n(\infty) + b_n(t)e^{-\rho(t-t_1)},$$

where $\tilde{u}^n(\infty) = \lim_{t \rightarrow \infty} \tilde{u}^n(t)$, and $b_n(t)$ is a sum of polynomials of t , and products of polynomials of t and exponential functions in the form of $e^{-j\rho(t-t_1)}$, where j is a positive integer. Then it can be seen that $\lim_{t \rightarrow \infty} \frac{d}{dt}\tilde{u}^n(t) = 0$, which, together with (5.14), implies the following recurrence relation,

$$\tilde{u}^n(\infty) = (1-\eta)\tilde{u}^{n-1}(\infty)[\rho + r(\infty) - \delta - \tilde{u}^{n-1}(\infty)]/\rho, \quad n \geq 1.$$

Immediately we notice that

$$\lim_{n \rightarrow \infty} \tilde{u}^n(\infty) = r(\infty) - \delta - \frac{\eta\rho}{1-\eta}.$$

For an arbitrarily small $\kappa > 0$, we fix N as the integer such that $\tilde{u}^N(\infty) \geq r(\infty) - \delta - \frac{\eta\rho}{1-\eta} - \kappa$. We now choose t_1 sufficiently large, such that N as the largest integer satisfying

$$\begin{aligned} l + \sigma(\mu_1) + \psi(\mu_1, \gamma)(t - t_0) + N \max\{M, \sqrt{2DM(t - t_1)}\} \\ \leq x \leq l + \sigma(\mu_2) + \psi(\mu_2, \gamma)(t - t_0) - N \max\{M, \sqrt{2DM(t - t_1)}\} \end{aligned} \quad (5.15)$$

From the construction process for \tilde{u}^n we derive

$$\lim_{t \rightarrow \infty} \inf_{(t,x) \text{ satisfying (5.15)}} u(x, t) \geq \tilde{u}^N(\infty).$$

For any given ε with $0 < \varepsilon < (c^*(\infty) - c)/2$, since $\psi(\mu_1, \gamma) = c + \delta$ and $c^*(\infty) \leq \psi(\mu_2, \gamma) + 2\delta$ by Lemma 5.1, for sufficiently large $t > t_1$, $(c + \varepsilon)t \leq x \leq (c^*(\infty) - \varepsilon)t$ is a subset of the interval defined by (5.15), so we claim that

$$\lim_{t \rightarrow \infty} \inf_{(c+\varepsilon)t \leq x \leq (c^*(\infty)-\varepsilon)t} u(t, x) \geq \tilde{u}^N(\infty) \geq r(\infty) - \delta - \frac{\eta\rho}{1-\eta} - \kappa. \quad (5.16)$$

Since δ, η and κ can be chosen arbitrarily small, are independent of the left hand side of (5.16), we obtain

$$\lim_{t \rightarrow \infty} \inf_{(c+\varepsilon)t \leq x \leq (c^*(\infty)-\varepsilon)t} u(t, x) \geq r(\infty).$$

Obviously from Theorem 3.1, $u(t, x) \leq r(\infty)$ for all x and t . So the statement (iii) holds true. The proof is complete. \square

References

- [1] D.G. Aronson, H. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve impulse propagation, in: *Partial Differential Equations and Related Topics*, in: *Lecture Notes in Math.*, vol. 446, Springer Verlag, 1975, pp. 5–49.
- [2] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math.* 30 (1978) 33–76.
- [3] P. Balachandran, W. Vilesa, E. Kolaczyk, Exponential-type inequalities involving ratios of the modified Bessel function of the first kind and their applications, arXiv:1311.1450, 2013.
- [4] J. Bell, Some threshold results for models of myelinated nerves, *Math. Biosci.* 54 (1981) 181–190.
- [5] X. Chen, J.S. Guo, Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations, *J. Differential Equations* 184 (2002) 549–569.
- [6] J.W. Cahn, Theory of crystal growth and interface motion in crystal line materials, *Acta Metall.* 8 (1960) 554–562.
- [7] J.W. Cahn, J. Mallet-Paret, E.S. van Vleck, Traveling wave solutions for systems of ODEs on a two-dimensional spatial lattice, *SIAM J. Appl. Math.* 59 (1998) 455–493.
- [8] J. Carr, A. Chmaj, Uniqueness of traveling waves for nonlocal monostable equations, *Proc. Amer. Math. Soc.* 132 (2004) 2433–2439.
- [9] H. Berestycki, O. Diekmann, C.J. Nagelkerke, P.A. Zegeling, Can a species keep pace with a shifting climate?, *Bull. Math. Biol.* 71 (2009) 399–429.
- [10] H. Chi, J. Bell, B. Hassard, Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction theory, *J. Math. Biol.* 24 (1986) 583–601.
- [11] S.N. Chow, J. Mallet-Paret, W. Shen, Traveling waves in lattice dynamical systems, *J. Differential Equations* 149 (1998) 248–291.
- [12] R.A. Fisher, The wave of advance of Advantageous genes, *Annu. Eugen.* 7 (1937) 355–369.
- [13] J. Fang, J. Wei, X.-Q. Zhao, Spreading speeds and travelling waves for non-monotone time-delayed lattice equations, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 466 (2010) 1919–1934.
- [14] P. Gonzalez, R.P. Neilson, J.M. Lenihan, R.J. Drapek, Global patterns in the vulnerability of ecosystems to vegetation shifts due to climate change, *Glob. Ecol. Biogeogr.* 19 (2010) 755–768.
- [15] G. Iooss, Travelling waves in the Fermi–Pasta–Ulam lattice, *Nonlinearity* 13 (2000) 849–866.
- [16] J.P. Keener, Propagation and its failure in coupled systems of discrete excitable cells, *SIAM J. Appl. Math.* 47 (1987) 556–572.
- [17] A. Kolmogorov, I. Petrovskii, N. Piscounov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Moscow Univ. Math. Bull.* 1 (1937) 126.
- [18] J.P. Laplante, T. Erneux, Propagation failure in arrays of coupled bistable chemical reactors, *J. Phys. Chem.* 96 (1992) 4931–4934.
- [19] B. Li, S. Bewick, J. Shang, W.F. Fagan, Persistence and spread of s species with a shifting habitat edge, *SIAM J. Appl. Math.* 5 (2014) 1397–1417.
- [20] X. Liang, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.* 60 (2007) 1–40;
X. Liang, X.-Q. Zhao, *Comm. Pure Appl. Math.* 61 (2008) 137–138 (Erratum).
- [21] S. Ma, X. Zou, Propagation and its failure in a lattice delayed differential equation with global interaction, *J. Differential Equations* 212 (2005) 129–190.
- [22] J. Mallet-Paret, The global structure of traveling waves in spatially discrete dynamical systems, *J. Dynam. Differential Equations* 11 (1999) 49–128.
- [23] F. Olver, The asymptotic expansion of Bessel functions of larger order, *Philos. Trans. R. Soc. Lond. Ser. A* 247 (1954) 328–368.
- [24] F. Olver, Bessel functions of integer order, in: *Handbook of Mathematical Functions*, in: *Natl. Bur. Stand. Appl. Math. Ser.*, vol. 55, U.S. Government Printing Office, Washington, DC, 1964, pp. 355–433.
- [25] C.V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.
- [26] C.L. Parr, E.F. Gray, W.J. Bond, Cascading biodiversity and functional consequences of a global change-induced biome switch, *Divers. Distrib.* 18 (2012) 493–503.
- [27] J.J. Polovina, J.P. Dunne, P.A. Woodworth, E.A. Howell, Projected expansion of the subtropical biome and contraction of the temperate and equatorial upwelling biomes in the North Pacific under global warming, *ICES J. Mar. Sci.* 68 (2011) 986–995.
- [28] M. Scheffer, M. Hirota, M. Holmgren, E.H. Van Nes, F.S. Chapin, Thresholds for boreal biome transitions, *Proc. Natl. Acad. Sci. USA* 109 (2012) 21384–21389.

- [29] S. Scheiter, S.I. Higgins, Impacts of climate change on the vegetation of Africa: an adaptive dynamic vegetation modelling approach, *Glob. Change Biol.* 15 (2009) 2224–2246.
- [30] A. Scott, *Nonlinear Science: Emergence and Dynamics of Coherent Structures*, Oxf. Texts Appl. Eng. Math., vol. 8, Oxford University Press, 2003.
- [31] D.Y.P. Tng, B.P. Murphy, E. Weber, G. Sanders, G.J. Williamson, J. Kemp, D.M.J.S. Bowman, Humid tropical rain forest has expanded into eucalypt forest and savanna over the last 50 years, *Ecol. Evol.* 2 (2012) 34–45.
- [32] G.N. Watson, *Theory of Bessel Functions*, Cambridge at university Press, 1966.
- [33] H.F. Weinberger, Long-time behavior of a class of biological models, *SIAM J. Math. Anal.* 13 (1982) 353–396.
- [34] P.X. Weng, H.X. Huang, J.H. Wu, Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction, *IMA J. Appl. Math.* 68 (2003) 409–439.
- [35] J. Wu, X. Zou, Asymptotic and periodic boundary value problems of mixed FDEs and wave solutions of lattice differential equations, *J. Differential Equations* 135 (1997) 315–357.
- [36] L. Zhang, S. Guo, Existence and multiplicity of wave trains in 2D lattices, *J. Differential Equations* 257 (2014) 759–783.
- [37] B. Zinner, Existence of traveling wave front solutions for the discrete Nagumo equation, *J. Differential Equations* 96 (1992) 1–27.
- [38] Y. Zhou, M. Kot, Discrete-time growth-dispersal models with shifting species ranges, *Theor. Ecol.* 4 (2011) 13–25.
- [39] B. Zinner, G. Harris, W. Hudson, Traveling wave fronts for the discrete Fisher’s equation, *J. Differential Equations* 105 (1992) 46–62.