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Well-posedness and long-time behavior of a non-autonomous Cahn–Hilliard–Darcy system with mass source modeling tumor growth

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Abstract

In this paper, we study an initial boundary value problem of the Cahn–Hilliard–Darcy system with a non-autonomous mass source term S that models tumor growth. We first prove the existence of global weak solutions as well as the existence of unique local strong solutions in both 2D and 3D. Then we investigate the qualitative behavior of solutions in details when the spatial dimension is two. More precisely, we prove that the strong solution exists globally and it defines a closed dynamical process. Then we establish the existence of a minimal pullback attractor for translated bounded mass source S. Finally, when S is assumed to be asymptotically autonomous, we demonstrate that any global weak/strong solution converges to a single steady state as $t \to +\infty$. An estimate on the convergence rate is also given.
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1. Introduction

In this paper, we consider the following Cahn–Hilliard–Darcy (CHD in short) system that arises in the study of morphological evolution in solid tumor growth (see, e.g., [15,45]):

$$\phi_t + \operatorname{div}(\mathbf{u}\phi) = \Delta\mu + S, \quad \text{in } (\tau, T) \times \Omega, \tag{1.1}$$

$$\mu = -\epsilon^2 \Delta\phi + f'(\phi) \quad \text{with} \quad f(\phi) = \frac{1}{4}\phi^4 - \frac{1}{2}\phi^2, \tag{1.2}$$

$$\mathbf{u} = -\nabla p + \frac{\gamma}{\epsilon}\mu\nabla\phi, \quad \text{in } (\tau, T) \times \Omega, \tag{1.3}$$

$$\operatorname{div}\mathbf{u} = S, \quad \text{in } (\tau, T) \times \Omega. \tag{1.4}$$

Here, Ω is assumed to be a bounded domain in $\mathbb{R}^d$ ($d \in \{2, 3\}$). $\tau \in \mathbb{R}$ denotes the initial time and $T > \tau$ is any given number. The CHD system (1.1)–(1.4) is subject to the following boundary and initial conditions:

$$\partial_\nu \phi = \partial_\nu \mu = 0, \quad \text{on } \partial\Omega, \tag{1.5}$$

$$\mathbf{u} \cdot \nu = 0, \quad \text{on } \partial\Omega, \tag{1.6}$$

$$\phi(t, x)|_{t=\tau} = \phi_\tau(x), \tag{1.7}$$

where ν is the unit outward normal vector to the boundary $\partial\Omega$.

The CHD system (1.1)–(1.4) can be viewed as the simplest version of those general diffuse interface models for tumor growth, which were derived based on the principle of mass conservation together with the second law of thermodynamics [15,45]. In the diffuse-interface (or phase-field) framework, the tumor volume fraction is denoted by a scalar order parameter ϕ and the sharp tumor/host interfaces are replaced by narrow transition layers, whose thickness is approximately characterized by a small parameter $\epsilon > 0$. Instead of tracking the interfaces explicitly, the dynamics of interfaces (now recognized as zero level sets of the order parameter) can be simulated on a fixed grid. Therefore, the diffuse-interface model has the advantage that it can easily describe topological transitions of interfaces (e.g., pinch-off and reconnection for two-phase immiscible flow) in a natural way (see [2,21,22,24,25]).

Eq. (1.1) is a convective Cahn–Hilliard type equation, which is derived from the mass conservation. The vector $\mathbf{u}$ stands for the advective velocity field, while the scalar functions μ, S stand for the chemical potential and the mass source term accounting for cell proliferation (or the rate of change in tumor volume, see [15,45]), respectively. The chemical potential μ is the variational derivative of the free energy functional:

$$E(\phi) := \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla\phi|^2 + f(\phi) \right) dx,$$

in which the function f (see (1.2)) can be viewed as a smooth double-well polynomial approximation of the physically relevant logarithmic potential (see [6]). Eq. (1.3) for the advective velocity $\mathbf{u}$ follows from a generalized Darcy's law, in which γ is a positive constant measuring the excess adhesion force at the diffusive tumor/host tissue interfaces and p is the pressure that consists of a combination of certain generalized Gibbs free energy and the gravitational potential. Eq. (1.4) serves as a constraint for the velocity due to the possible mass exchange.

We recall some previous works in the literature that are related to our problem. In biological applications, e.g., the phase-field models for tumor growth and wound healing [15,28], the mass source term S may depend on the order parameter ϕ in a quadratic way such that $S = \alpha\phi(1-\phi)$ ($\alpha > 0$). When S has a linear dependence on ϕ, Eq. (1.1) (neglecting the velocity $\mathbf{u}$) is also known as the Cahn–Hilliard–Oono equation that accounts for long-range (nonlocal) interactions in the phase separation process [33]. Concerning the mathematical analysis for these generalized Cahn–Hilliard equations with mass source (with the convection under velocity $\mathbf{u}$ being neglected), we refer to the recent work [9,32,34], in which well-posedness and asymptotic behavior of the associated dynamical system have been investigated. When $S = 0$, the CHD system (1.1)–(1.4) is referred to as the Cahn–Hilliard–Hele-Shaw (CHHS) system that has been used to describe two-phase flows in the Hele-Shaw geometry [21,22] (see also [37] for a similar model for spinodal decomposition of a binary fluid in a Hele-Shaw cell). The CHHS system with zero mass source term has been studied by many authors in the literature, both numerically and mathematically. For instance, an unconditionally energy stable and solvable finite difference scheme based on convex-splitting was proposed in [46], see also [14] for an implicit Euler temporal scheme combined with a mixed finite element discretization in space. Concerning the analysis results, existence and uniqueness of global classical solutions in 2D torus and local classical solution in 3D torus were first established in [44]. Besides, some blow-up criteria were also obtained in the three-dimensional case. In [43], long-time behavior of global solutions and stability of local minimizers in both 2D and 3D periodic setting were proved based on the Łojasiewicz–Simon approach [39]. For the CHHS system in a 2D rectangle or in a 3D box under homogeneous Neumann boundary conditions, qualitative behaviors of strong solutions such as existence, uniqueness, regularity and asymptotic stability of the constant state $\frac{1}{|\Omega|}\int_\Omega \phi_\tau dx$ are studied in [29]. Quite recently, the connection between the Cahn–Hilliard–Brinkman (CHB) system and the CHHS system has been investigated in [4] such that a suitable weak solution to the CHHS system can be shown to be a limit of solutions to the CHB system as the fluid viscosity goes to zero. Moreover, we would like to remark that the CHHS system can be viewed as a simplification of the full Cahn–Hilliard–Navier–Stokes (CHNS) system (see e.g., [2,24,25]) in the Hele-Shaw geometry. We refer to [1,5,12,17,18,41,47] and the references therein for analytical results of the CHNS system on well-posedness as well as long-time behavior under various situations.

However, to the best of our knowledge, there seem no analytical results in the literature concerning the CHD system (1.1)–(1.4) with a non-zero mass source term S. This is the main goal of the present paper. In this paper, we shall confine ourselves to the situation that S is assumed to be a given source of mass, possibly depending on time t and position x, but not on the parameter ϕ. The case with more general mass source term will be treated in the future work.

We summarize the main results of this paper as follows. First, under suitable integrability conditions on the mass source term S, we apply the Galerkin method to prove the existence of global weak solutions as well as the existence and uniqueness of local strong solutions to the CHD system (1.1)–(1.7) in both 2D and 3D cases (see Theorem 2.1). Then we focus on the studies of qualitative behavior for solutions in the 2D case. It is shown that in 2D, problem (1.1)–(1.7) actually admits a unique global strong solution ϕ in $H^2_N(\Omega)$ which defines a family of *closed processes* $\{U(t,\tau)\}_{t\geq\tau}$ on $H^2_N(\Omega)$ (see Theorem 2.2). If the mass source S is further assumed to be a *translated bounded* function in $L^2_t L^2_x$ (see (2.4)), the family of processes $\{U(t,\tau)\}_{t\geq\tau}$

that are confined on the phase space $\mathcal{H}_M$ (see (2.3)) turns out to admit a minimal *pullback attractor* $\mathcal{A}$ (see Definition 5.3 and Theorem 2.3). In addition, we prove that under suitable decay assumption on S (see (2.5)), the dynamical process becomes *asymptotically autonomous*. In this specific case, the ω-limit set of each trajectory is actually a singleton. Namely, for arbitrary large initial datum, the global bounded solution will converge to a single steady state as $t \to +\infty$ and an estimate on the convergence rate is also given (see Theorem 2.4).

Before concluding the introduction part, we would like to stress some new features of the present paper. The presence of the mass source term S brings us several difficulties in the mathematical analysis. First, unlike in [14,29,43,44], the velocity field $\mathbf{u}$ is no longer divergence free. As a consequence, in order to prove the existence of weak/strong solutions, we use a modified Galerkin approximation different from that in [29]. Instead of solving the approximate velocity directly (by taking the Helmholtz–Leray orthogonal projection to eliminate the pressure term), we solve the pressure function that satisfies a Poisson type equation subject to homogeneous Neumann boundary condition (see (3.1)) and then obtain the velocity via the Darcy equation (1.3). Besides, some new estimates for the pressure p and its derivative (cf. [44]) are derived, which play an important role in the subsequent proofs for existence of global solutions (see Lemma 3.1).

Second, we study the long-time dynamics of problem (1.1)–(1.7) from the infinite dimensional dynamical system point of view [42]. The theory of global attractors has been generalized to the case of non-autonomous dynamical systems, for instance, the uniform attractors (see [8]) and pullback attractors (see [11,27] and the references therein). In this paper, we prove the existence of a pullback attractor for the CHD system (1.1)–(1.7) under rather general assumptions on the time dependent mass source term S in 2D. Due to the mass conservation property (2.2), we cannot expect an absorbing set for initial data varying in the whole space. Instead, we first confine the associated dynamical process $\{U(t,\tau)\}_{t\geq\tau}$ on a suitable phase space $\mathcal{H}_M$ (see (2.3)), which is a subset of $H^2_N(\Omega)$. Next, due to the highly nonlinear coupling of the CHD system, it seems difficult to obtain (strong) continuity of the process $\{U(t,\tau)\}_{t\geq\tau}$ in $\mathcal{H}_M$ but only a continuous dependence result in the lower-order space H^1 (see Lemma 4.2). This indicates that the process $\{U(t,\tau)\}_{t\geq\tau}$ is only *closed* (see Definition 4.1, cf. also [35] for the notion of closed semigroups). We then perform a nonstandard argument devised in [19] for closed processes to conclude our result (cf. [40] for the case with closed cocycles). For this purpose, we deduce a generalized Gronwall type inequality (see Lemma A.2) to obtain some uniform estimates that lead to the existence of a pullback absorbing set (see Proposition 5.1). We believe that Lemma A.2 may have its own interests and can be applied to other problems with highly nonlinear structure. Besides, since the mass source term S is only assumed to be translated bounded in $L^2_tL^2_x$, we are not able to obtain higher-order estimates of the solutions (and thus compactness) by taking derivatives of the PDEs. Instead, we use a continuity method for energy functions (see e.g., [19,30]) to obtain the pullback asymptotic compactness (see Proposition 5.2).

At last, we study the long-time behavior for any bounded global weak/strong solution of the CHD system (1.1)–(1.7) when the mass source S becomes asymptotically autonomous. This is nontrivial, since the topology of the set of steady states (see (6.7)) can be rather complicated in high dimensional case and it may form a continuum (see e.g., [36]). Moreover, since our problem (1.1)–(1.7) is now non-autonomous due to the presence of S, it no longer has a Lyapunov functional. Nevertheless, for global bounded solutions in H^2, it is possible to derive an energy inequality (see (6.8)), which enables us to characterize the corresponding ω-limit sets. Based on that energy inequality, we are able to apply the Łojasiewicz–Simon approach (cf. [10,13,26,39]) to obtain the convergence of $\phi(t)$ as time goes to infinity as well as an estimate on convergence rate. Our convergence result generalizes the previous one in [43] for the homogeneous CHHS system in periodic setting. Moreover, we do not need to impose any additional assumption either on the initial datum for ϕ (e.g., the average of initial datum $\frac{1}{|\Omega|}\int_\Omega \phi_\tau dx$ being outside the spinodal region) or on the size of domain (being 'small') like in [29] in order to obtain certain asymptotical stability.

The rest of this paper is organized as follows. In Section 2, we introduce the functional settings and state the main results of this paper. Section 3 is devoted to the proof of the existence of global weak solutions as well as the existence and uniqueness of local strong solutions to problem (1.1)–(1.7) in both 2D and 3D. In Section 4, we prove the existence of a unique global strong solution as well as the regularity of weak solutions in 2D. Then we show in Section 5 that the associated closed processes $\{U(t,\tau)\}_{t\geq\tau}$ on the phase space $\mathcal{H}_M$ admit a minimal pullback attractor $\mathcal{A}$, provided that the mass source S is translated bounded in $L^2_t L^2_x$. Finally, in Section 6, we prove the convergence of global weak/strong solutions to a single steady state as $t \to +\infty$ and obtain an estimate on the convergence rate.

2. Preliminaries and main results

We first introduce some notations on the functional spaces. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be either a smooth bounded domain or a convex polygonal or polyhedral domain. $L^q(\Omega)$, $1 \leq q \leq \infty$ denotes the usual Lebesgue space and $\|\cdot\|_{L^q(\Omega)}$ denotes its norm. Similarly, $W^{m,q}(\Omega)$, $m \in \mathbb{N}$, $1 \leq q \leq \infty$, denotes the usual Sobolev space with norm $\|\cdot\|_{W^{m,p}(\Omega)}$. When $q = 2$, we simply denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and denote the norms $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^m(\Omega)}$ by $\|\cdot\|$ and $\|\cdot\|_{H^m}$, respectively. The L^2-Bessel potential spaces are denoted by $H^s(\Omega)$, $s \in \mathbb{R}$, which are defined by restriction of distributions in $H^s(\mathbb{R}^d)$ to Ω. If X is a Banach space, we denote by X' its dual and by $\langle\cdot,\cdot\rangle$ the associated duality product. The inner product in L^2 will be denoted by $(\cdot,\cdot)$. If I is an interval of $\mathbb{R}^+$ and X a Banach space, we use the function space $L^p(I; X)$, $1 \leq p \leq +\infty$, which consists of p-integrable functions with values in X. Moreover, $C_w(I; X)$ denotes the topological vector space of all bounded and weakly continuous functions from I to X, while $W^{1,p}(I, X)$ $(1 \leq p < +\infty)$ stands for the space of all functions u such that $u, \frac{du}{dt} \in L^p(I; X)$, where $\frac{du}{dt}$ denotes the vector-valued distributional derivative of u. Bold characters will be used to denote vector spaces.

Given any function $v \in L^1(\Omega)$, we denote by $\bar{v} = |\Omega|^{-1}\int_\Omega v(x)dx$ its mean value. Then we define the space $\dot{L}^2(\Omega) := \{v \in L^2(\Omega) : \bar{v} = 0\}$ and $\dot{v} = \mathrm{P}_0 v := v - \bar{v}$ the orthogonal projection onto $\dot{L}^2(\Omega)$. Furthermore, we denote $\dot{H}^1(\Omega) = H^1(\Omega) \cap \dot{L}^2(\Omega)$, which is a Hilbert space with inner product $(u, v)_{\dot{H}^1} = \int_\Omega \nabla u \cdot \nabla v dx$ due to the classical Poincaré inequality for functions with zero mean. Its dual space is simply denoted by $\dot{H}^{-1}(\Omega)$. Denote the spaces $H^2_N = \{\varphi \in H^2(\Omega) \mid \partial_\nu \varphi = 0 \text{ on } \partial\Omega\}$ and $H^4_N = \{\varphi \in H^4(\Omega) \mid \partial_\nu \varphi = \partial_\nu \Delta\varphi = 0 \text{ on } \partial\Omega\}$. We can see that the operator $A = -\Delta$ with its domain $D(A) = H^2_N \cap \dot{L}^2(\Omega)$ is a positively defined, self-adjoint operator on $D(A)$ and the spectral theorem enables us to define powers A^s of A for $s \in \mathbb{R}$. Then space $(H^1(\Omega))'$ is endowed with the equivalent norm $\|v\|^2_{H^1(\Omega)'} = \|A^{-\frac{1}{2}}(v - \bar{v})\|^2 + |\bar{v}|^2$ and the norm on $\dot{H}^{-1}(\Omega)$ is given by $\|v\|^2_{\dot{H}^{-1}} = \|A^{-\frac{1}{2}}(v - \bar{v})\|^2$.

Throughout the paper, without loss of generality, we assume that $\gamma = \epsilon = 1$. $C \geq 0$ will stand for a generic constant and $\mathcal{Q}(\cdot)$ for a generic positive monotone increasing function. Special dependence will be pointed out in the text if necessary.

Following the constraint (1.4) and the boundary condition (1.6), we can easily see that a necessary condition for the external force S is that

$$\int_\Omega S(t,x)dx \equiv 0. \tag{2.1}$$

Below we introduce the definitions of weak solution as well as strong solution to the CHD system (1.1)–(1.4).

Definition 2.1. Assume $d = 2, 3$.

(i) Let $T > \tau$, $\phi_\tau \in H^1(\Omega)$ and $S \in L^2(\tau, T; \dot{L}^2(\Omega))$ be given. A triplet $(\phi, \mathbf{u}, p)$ is a weak solution to the system (1.1)–(1.4) endowed with boundary and initial conditions (1.5)–(1.7), if

$$\phi \in C_w([\tau, T]; H^1(\Omega)) \cap L^2(\tau, T; H^3(\Omega)), \quad \partial_t\phi \in L^{\frac{8}{5}}(\tau, T; (H^1(\Omega))'),$$
$$\mathbf{u} \in L^2(\tau, T; \mathbf{L}^2(\Omega)), \quad p \in L^{\frac{8}{5}}(\tau, T; H^1(\Omega))$$

such that

$$\langle\phi_t, \psi\rangle + \langle\mathrm{div}(\mathbf{u}\phi), \psi\rangle + (\nabla\mu, \nabla\psi) = (S, \psi), \quad \forall\psi \in H^1(\Omega), \quad \text{a.e. } t \in [\tau, T],$$
$$(\nabla p, \nabla\varphi) = (S, \varphi) + (\mu\nabla\phi, \nabla\varphi), \quad \forall\varphi \in H^1(\Omega), \quad \text{a.e. } t \in [\tau, T],$$
$$(\mathbf{u}, \mathbf{v}) = (-\nabla p + \mu\nabla\phi, \mathbf{v}), \quad \forall\mathbf{v} \in \mathbf{L}^2(\Omega), \quad \text{a.e. } t \in [\tau, T],$$

with $\mu \in L^2(\tau, T; H^1(\Omega))$ given by (1.2), and

$$\partial_\nu\phi = 0, \quad \text{a.e. on } \partial\Omega \times (\tau, T),$$
$$\phi|_{t=\tau} = \phi_\tau, \quad \text{a.e. in } \Omega.$$

(ii) Let $T > \tau$, $\phi_\tau \in H^2_N(\Omega)$ and $S \in L^2(\tau, T; \dot{L}^2(\Omega))$ be given. A triplet $(\phi, \mathbf{u}, p)$ is a strong solution to the system (1.1)–(1.4) endowed with boundary and initial conditions (1.5)–(1.7), if

$$\phi \in C([\tau, T]; H^2_N(\Omega)) \cap L^2(\tau, T; H^4_N(\Omega)), \quad \phi_t \in L^2(\tau, T; L^2(\Omega)),$$
$$\mathbf{u} \in L^2(\tau, T; \mathbf{H}^1(\Omega)), \quad p \in L^2(\tau, T; H^2(\Omega)),$$
$$\mu \in C([\tau, T]; L^2(\Omega)) \cap L^2(\tau, T; H^2(\Omega)),$$

such that

$$\phi_t + \mathrm{div}(\mathbf{u}\phi) = \Delta\mu + S, \quad \text{in } L^2(\Omega) \quad \text{a.e. } t \in [\tau, T]$$

with μ given by (1.2),

$$-\Delta p = S - \mathrm{div}\,(\mu\nabla\phi), \quad \text{in } L^2(\Omega) \quad \text{a.e. } t \in [\tau, T],$$

(1.3) holds in $\mathbf{H}^1(\Omega)$ for a.e. $t \in [\tau, T]$ and

$$\partial_\nu \phi = \partial_\nu \mu = \partial_\nu p = 0, \quad \text{a.e. on } \partial\Omega \times (\tau, T),$$
$$\phi|_{t=\tau} = \phi_\tau, \quad \text{a.e. in } \Omega.$$

Remark 2.1. It is easy to see that the mean of any weak/strong solution ϕ over Ω is conserved in time, i.e.,

$$\bar{\phi}(t) := \frac{1}{|\Omega|} \int_\Omega \phi(t,x)dx \equiv \frac{1}{|\Omega|} \int_\Omega \phi_\tau dx := M. \tag{2.2}$$

Now we are in a position to state our main results.

Theorem 2.1. *Suppose that $d = 2, 3$.*

(i) For any $\phi_\tau \in H^1(\Omega)$ and $S \in L^2(\tau, T; \dot{L}^2(\Omega))$ with arbitrary $T \in (\tau, +\infty)$, problem (1.1)–(1.7) admits at least one global weak solution $(\phi, \mathbf{u}, p)$ on $[\tau, T]$.

(ii) For any $\phi_\tau \in H^2_N(\Omega)$, $S \in L^2(\tau, T; \dot{L}^2(\Omega)) \cap L^\infty(\tau, T; \dot{H}^{-1}(\Omega))$ with arbitrary $T \in (\tau, +\infty)$, there exists a time $T^* \in (\tau, T)$ such that problem (1.1)–(1.7) admits a strong solution $(\phi, \mathbf{u}, p)$ on $[\tau, T^*]$ that is unique up to an additive function of t to p.

When the spatial dimension is two, more comprehensive information about problem (1.1)–(1.7) can be achieved. First, we can prove the existence of a unique global strong solution, i.e.,

Theorem 2.2. *Suppose that $d = 2$. For any $\phi_\tau \in H^2_N(\Omega)$, $S \in L^2_{loc}(\mathbb{R}; \dot{L}^2(\Omega))$ and arbitrary $T \in (\tau, +\infty)$, problem (1.1)–(1.7) admits a global strong solution $(\phi, \mathbf{u}, p)$ on $[\tau, T]$ that is unique up to an additive function of t to p. The global strong solution defines a family of closed processes $\{U(t,\tau)\}_{t\geq\tau}$ on $H^2_N(\Omega)$ such that*

$$U(t,\tau)\phi_\tau = \phi(t), \quad \forall t \in [\tau, T].$$

Consider the following phase space:

$$\mathcal{H}_M = \left\{\phi \in H^2_N(\Omega),\ |\bar{\phi}| \leq M\right\}, \quad M \geq 0. \tag{2.3}$$

For the external source term S, we consider the Banach space $L^2_b(\mathbb{R}; \dot{L}^2(\Omega))$ defined by

$$L^2_b(\mathbb{R}; \dot{L}^2(\Omega)) = \left\{ S \in L^2_{loc}(\mathbb{R}; \dot{L}^2(\Omega)) \,:\, \|S\|^2_{L^2_b(\mathbb{R};\dot{L}^2(\Omega))} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|S(s)\|^2 ds < \infty \right\}, \tag{2.4}$$

which is the subspace of $L^2_{loc}(\mathbb{R}; \dot{L}^2(\Omega))$ of translation bounded functions.

Then we can prove that:

Theorem 2.3. *Let $d = 2$. For any $S \in L^2_b(\mathbb{R}, \dot{L}^2(\Omega))$, the family of closed processes $\{U(t,\tau)\}_{t\geq\tau}$ associated with problem (1.1)–(1.7) defined on the phase space $\mathcal{H}_M$ admits a minimal pullback attractor $\mathcal{A}$ in the sense of Definition 5.3.*

Furthermore, if the dynamical process becomes *asymptotically autonomous* under suitable assumptions on the external source S, we can prove that the global weak (or strong) solution converges to a single steady state as $t \to +\infty$ and obtain an estimate on the convergence rate.

Theorem 2.4. *Let $d = 2$. Assume that $S \in L^2(\tau, +\infty; \dot{L}^2(\Omega))$ and satisfies the following condition*

$$\sup_{t \geq \tau}(1+t)^{1+\rho} \int_t^{+\infty} \|S\|^2 ds < +\infty, \quad \text{for some } \rho > 0. \tag{2.5}$$

Let $(\phi, \mathbf{u}, p)$ be a global weak (or strong) solution to problem (1.1)–(1.7). Then there exists a steady state $\phi_\infty \in H^2_N(\Omega)$, which is a solution to the stationary Cahn–Hilliard equation

$$\begin{cases} -\Delta\phi_\infty + f'(\phi_\infty) = \fint_\Omega f'(\phi_\infty)dx, & \text{in } \Omega, \\ \partial_\nu \phi_\infty = 0, \quad \text{on } \partial\Omega, \\ \fint_\Omega \phi_\infty dx = \fint_\Omega \phi_\tau dx \end{cases} \tag{2.6}$$

such that as $t \to +\infty$

$$\begin{cases} \phi(t) \to \phi_\infty & \text{strongly in } H^s(\Omega),\ s < 2, \\ \phi(t) \rightharpoonup \phi_\infty & \text{weakly in } H^2(\Omega). \end{cases}$$

Moreover, the following convergence rate holds

$$\|\phi(t) - \phi_\infty\|_{H^s} \leq C(1+t)^{-\frac{2-s}{3}\min\{\frac{\theta}{1-2\theta}, \frac{\rho}{2}\}}, \quad \forall t \geq \tau + 1, \quad s \in [-1, 2). \tag{2.7}$$

Here C is a constant depending on $\|\phi_\tau\|_{H^1}$, $\int_\tau^{+\infty} \|S\|^2 d\tau$ and Ω, $\theta \in (0, \frac{1}{2})$ is a constant depending on ϕ_∞.

3. Well-posedness

In this section, we prove Theorem 2.1, namely, the existence of global weak solutions and (unique) local strong solutions to the system (1.1)–(1.7) in both 2D and 3D. For the sake of simplicity, we shall present the proofs in the 3D case, which are still valid for the 2D case with minor modifications due to different Sobolev embedding theorems and interpolation inequalities.

3.1. Pressure estimate

The following lemma on the estimate for the pressure p will be useful in the subsequent analysis:

Lemma 3.1. *Suppose $d = 2, 3$. For any given function $\phi \in H^3(\Omega) \cap H^2_N(\Omega)$, the pressure function p satisfies the following Poisson equation subject to a homogeneous Neumann boundary condition:*

$$\begin{cases} -\Delta p = S - \operatorname{div}(\mu\nabla\phi), & in\ \Omega, \\ \partial_\nu p = 0, & on\ \partial\Omega, \\ \int_\Omega p\,dx = 0. \end{cases} \tag{3.1}$$

Moreover, the following estimates hold:

$$\|\nabla p\| \le C\|S\| + C\|\mu\|_{L^6}\|\nabla\phi\|_{L^3}, \tag{3.2}$$

$$\|p\| \le C\|S\| + C\|\nabla\mu\|\|\nabla\phi\|_{L^{\frac{3}{2}}} + \left|\overline{\mu(\phi)}\right|\|\phi - \bar{\phi}\|, \tag{3.3}$$

where μ is given by $\mu = -\Delta\phi + \phi^3 - \phi$.

Proof. It follows from the assumption on ϕ and the Sobolev embedding theorem ($d = 3$) that $\mu = -\Delta\phi + \phi^3 - \phi \in H^1(\Omega)$. Multiplying (3.1) by p and integrating by parts, we get

$$\|\nabla p\|^2 = \int_\Omega (Sp + (\mu\nabla\phi)\cdot\nabla p)\,dx.$$

The above formula together with the Poincaré inequality and the Hölder inequality easily yields (3.2).

Next, we deduce from (3.1) that

$$\begin{aligned} p &= A^{-1}S - A^{-1}\operatorname{div}(\mu(\phi)\nabla\phi) \\ &= A^{-1}S - A^{-1}\operatorname{div}\Big((\mu(\phi) - \overline{\mu(\phi)})\nabla\phi\Big) - A^{-1}\operatorname{div}\Big(\overline{\mu(\phi)}\nabla\phi\Big) \\ &= A^{-1}S - A^{-1}\operatorname{div}\Big((\mu(\phi) - \overline{\mu(\phi)})\nabla\phi\Big) - \overline{\mu(\phi)}A^{-1}\operatorname{div}\big(\nabla(\phi - \bar{\phi})\big) \\ &= A^{-1}S - A^{-1}\operatorname{div}\Big((\mu(\phi) - \overline{\mu(\phi)})\nabla\phi\Big) + \overline{\mu(\phi)}(\phi - \bar{\phi}). \end{aligned} \tag{3.4}$$

Applying the Sobolev embeddings $L^{\frac{6}{5}}(\Omega) \hookrightarrow (H^1(\Omega))'$, $H^1 \hookrightarrow L^6$ ($d = 3$) and Hölder's inequality, we obtain that

$$\begin{aligned} \|p\| &\le \|A^{-1}S\| + \|A^{-1}\operatorname{div}\Big((\mu(\phi) - \overline{\mu(\phi)})\nabla\phi\Big)\| + \left|\overline{\mu(\phi)}\right|\|\phi - \bar{\phi}\| \\ &\le C(\|S\| + \|(\mu - \bar{\mu})\nabla\phi\|_{(H^1)'}) + \left|\overline{\mu(\phi)}\right|\|\phi - \bar{\phi}\| \\ &\le C(\|S\| + \|(\mu - \bar{\mu})\nabla\phi\|_{L^{\frac{6}{5}}}) + \left|\overline{\mu(\phi)}\right|\|\phi - \bar{\phi}\| \\ &\le C\|S\| + C\|\mu - \bar{\mu}\|_{L^6}\|\nabla\phi\|_{L^{\frac{3}{2}}} + \left|\overline{\mu(\phi)}\right|\|\phi - \bar{\phi}\| \\ &\le C\|S\| + C\|\mu - \bar{\mu}\|_{H^1}\|\nabla\phi\|_{L^{\frac{3}{2}}} + \left|\overline{\mu(\phi)}\right|\|\phi - \bar{\phi}\|, \end{aligned}$$

which together with the Poincaré inequality yields our conclusion (3.3). □

3.2. Global weak solutions

The existence of global weak solutions can be obtained by a suitable Galerkin procedure. We consider the eigenvalue problem $-\Delta w = \lambda w$ subject to the homogeneous Neumann boundary condition $\partial_\nu w = 0$. It is well known that there exist two sequences $\{\lambda_n\}_{n=1,2,\ldots}$ and $\{w_n\}_{n=1,2,\ldots}$ such that, for every $n \geq 1$, $\lambda_n \geq 0$ is an eigenvalue and $w_n \neq 0$ is a corresponding eigenfunction, the sequence λ_n is nondecreasing, tending to infinity as $n \to +\infty$, and the sequence $\{w_n\}$ is orthonormal and complete in $L^2(\Omega)$. We notice that $\lambda = 0$ is an eigenvalue, whence $\lambda_1 = 0$, and that any non-zero constant is an eigenfunction (i.e., $w_1 = 1$). For every $i > 1$, w_i cannot be a constant and $\int_\Omega w_i dx = 0$, whence $\lambda_i = \int_\Omega |\nabla w_i|^2 dx > 0$. Moreover, as $w_1 = 1$ is a constant and $\{w_n\}$ is orthonormal in $L^2(\Omega)$, we easily deduce that $A^{-1}w_i = \lambda_i^{-1} w_i$ for every $i > 1$.

For any $n \geq 1$, we introduce the finite-dimensional space $W_n = \text{span}\{w_1, \ldots, w_n\}$ and Π_n the orthogonal projection on W_n. Then we consider the Galerkin approximate problem (P_n):

Set

$$\phi_n(t, x) = \sum_{i=1}^{n} g_{ni}(t) w_i(x)$$

which satisfies the following approximation equation:

$$\begin{cases} \partial_t \phi_n = \Delta \mu_n + \Pi_n(S - \text{div}(\mathbf{u}_n \phi_n)), \\ \mu_n = -\Delta \phi_n + \Pi_n f(\phi_n), \\ \phi_n(\tau) = \Pi_n \phi_\tau, \end{cases} \tag{3.5}$$

where $f(\phi_n) = \phi_n^3 - \phi_n$ and

$$\mathbf{u}_n = -\nabla p_n + \mu_n \nabla \phi_n. \tag{3.6}$$

Here, p_n satisfies a Poisson equation with homogeneous Neumann boundary condition:

$$\begin{cases} -\Delta p_n = S - \text{div}\,(\mu_n \nabla \phi_n), \quad \text{in } \Omega, \\ \partial_\nu p_n = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Then p_n is uniquely determinate up to an arbitrary additive function that may only depend on t. For the sake of simplicity and without affecting the mathematical analysis, we require that $\int_\Omega p_n dx = 0$ and thus

$$p_n = A^{-1} S - A^{-1}\,\text{div}(\mu_n \nabla \phi_n).$$

Taking the inner product of (3.5) in $L^2(\Omega)$ with w_j, we infer that $g_{nj}(t)$ satisfies the following ODE system

$$\begin{cases} g_{nj}' + (\lambda_j^2 - \lambda_j) g_{nj} + G_j(g) = S_j(t), \quad j = 1, \cdots, n, \\ g_{nj}(\tau) = \xi_j := (\phi_\tau, w_j) \end{cases} \tag{3.8}$$

where

$$G_j(g) = \lambda_j \left((\sum_{i=1}^{n} g_{ni} w_i)^3, w_j \right) + \left(\operatorname{div}(\mathbf{u}_n \sum_{i=1}^{n} g_{ni} w_i), w_j \right),$$

and

$$S_j(t) = (S, w_j) \in L^2(\tau, T).$$

It is easy to verify that the nonlinearity G_j is locally Lipschitz in $g = (g_{n1}, \cdots, g_{nn})$ and as a consequence there exists $T_n \in (\tau, T)$ depending on $|\xi_j|$ such that (3.8) has a unique local solution $g_{nj}(t) \in C[\tau, T_n]$.

In what follows, we derive some a priori estimates on the approximate solutions that are valid in both 2D and 3D.

First, integrating (3.5) over $\Omega \times [\tau, T]$, it is easy to find that

$$\int_\Omega \phi_n(t) dx = \int_\Omega \phi_n(\tau) dx = \int_\Omega \phi_\tau dx, \quad \forall t \in [\tau, T]. \tag{3.9}$$

Multiplying Eq. (3.5) by μ_n and integrating by parts, we get

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \left(\frac{1}{2} |\nabla \phi_n|^2 + f(\phi_n) \right) dx + \|\nabla \mu_n\|^2 \\ &\quad = \int_\Omega S\mu_n (1 - \phi_n) dx - \int_\Omega (\mathbf{u}_n \cdot \nabla \phi_n) \mu_n dx. \end{aligned} \tag{3.10}$$

Taking L^2-inner product of (3.6) with $\mathbf{u}_n$, using integration by parts, we obtain that

$$\|\mathbf{u}_n\|^2 = \int_\Omega (-\nabla p_n + \mu_n \nabla \phi_n) \cdot \mathbf{u}_n dx = \int_\Omega p_n S + (\mu_n \nabla \phi_n) \cdot \mathbf{u}_n dx.$$

Summing it with (3.10), using (3.4) for p_n, Hölder's inequality and Poincaré's inequality, we deduce that

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \left(\frac{1}{2} |\nabla \phi_n|^2 + f(\phi_n) \right) dx + \|\nabla \mu_n\|^2 + \|\mathbf{u}_n\|^2 \\ &= \int_\Omega S\mu_n (1 - \phi_n) dx + \int_\Omega p_n S dx \\ &= \int_\Omega S(\mu_n - \overline{\mu_n})(1 - \phi_n) dx - \overline{\mu_n} \int_\Omega S\phi_n dx \\ &\quad + \int_\Omega S \left(A^{-1} S - A^{-1} \operatorname{div}\left((\mu_n - \overline{\mu_n}) \nabla \phi_n\right) + \overline{\mu_n} (\phi_n - \overline{\phi_n}) \right) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega S(\mu_n - \overline{\mu_n})(1-\phi_n)dx + \int_\Omega S\left(A^{-1}S - A^{-1}\operatorname{div}((\mu_n - \overline{\mu_n})\nabla\phi_n)\right)dx \\
&\le \|S\|\|\mu_n - \overline{\mu_n}\| + \|S\|_{L^{\frac{3}{2}}}\|\mu_n - \overline{\mu_n}\|_{L^6}\|\phi_n\|_{L^6} \\
&\quad + \|S\|(\|A^{-1}S\| + \|A^{-1}\operatorname{div}((\mu_n - \overline{\mu_n})\nabla\phi_n)\|) \\
&\le C\|S\|\|\nabla\mu_n\|(1+\|\phi_n\|_{H^1}) + C\|S\|\left(\|S\| + \|\nabla\mu_n\|\|\nabla\phi_n\|_{L^{\frac{3}{2}}}\right).
\end{aligned}
\tag{3.11}
$$

Thanks to Young's inequality and Poincaré's inequality, it holds

$$
\|\phi_n\|_{H^1}^2 = \|\nabla\phi_n\|^2 + \|\phi_n\|^2 \le C\left(\frac{1}{2}\|\nabla\phi_n\|^2 + \int_\Omega f(\phi_n)dx + 1\right). \tag{3.12}
$$

Denoting

$$
E_0(\phi_n) = \frac{1}{2}\|\nabla\phi_n\|^2 + \int_\Omega f(\phi_n)dx + 1,
$$

we infer from (3.11), (3.12) and Young's inequality that

$$
\frac{d}{dt}E_0(\phi_n) + \|\nabla\mu_n\|^2 + \|\mathbf{u}_n\|^2 \le \frac{1}{2}\|\nabla\mu_n\|^2 + C\|S\|^2E_0(\phi_n). \tag{3.13}
$$

Applying the Gronwall inequality, we obtain that

$$
\int_\Omega \left(\frac{1}{2}|\nabla\phi_n|^2 + f(\phi_n)\right)(t)dx + \int_\tau^T \|\nabla\mu_n\|^2dt + \int_\tau^T \|\mathbf{u}_n\|^2dt \le C \tag{3.14}
$$

where C depends on $\|\phi_\tau\|_{H^1}$, Ω and $\|S\|_{L^2(\tau,T;L^2)}$ but not T_n and n. This entails that

$$
\|\phi_n(t)\|_{H^1}^2 = \|(-\Delta+I)^{\frac{1}{2}}\phi_n\|^2 = \sum_{i=1}^n (1+\lambda_i)g_{ni}^2(t) \le C \qquad \text{for } \tau \le t \le T. \tag{3.15}
$$

Hence the local solution ϕ_n can be extended to $[\tau, T]$ for any fixed $T > \tau$.

The estimate (3.14) indicates that $\mathbf{u}_n$ is uniformly bounded in $L^2(\tau, T; L^2(\Omega))$. Since

$$
\left|\int_\Omega \mu_n dx\right| = \left|\int_\Omega f(\phi_n)dx\right| \le C(\|\phi_n\|_{L^1} + \|\phi_n\|_{L^3}^3) \le C, \tag{3.16}
$$

it follows from (3.14) and the Poincaré inequality that μ_n is uniformly bounded in $L^2(\tau, T; H^1(\Omega))$. Furthermore, by the Gagliardo–Nirenberg inequality ($d = 3$), we have

$$
\begin{aligned}
\|\nabla\Delta\phi_n\|^2 &\le C\left(\|\nabla\mu_n\|^2 + \int_\Omega \phi_n^4|\nabla\phi_n|^2dx + \|\nabla\phi_n\|^2\right) \\
&\le C(1+\|\nabla\mu_n\|^2 + \|\phi_n\|_{L^\infty}^4) \\
&\le C(1+\|\nabla\mu_n\|^2 + \|\phi_n\|_{L^6}^3\|\nabla\Delta\phi_n\| + \|\phi_n\|_{L^6}^4) \\
&\le \frac{1}{2}\|\nabla\Delta\phi_n\|^2 + C(1+\|\nabla\mu_n\|^2),
\end{aligned}
$$

which yields that

$$
\int_\tau^T \|\nabla\Delta\phi_n\|^2dt \le C.
$$

As a consequence, we obtain that ϕ_n is uniformly bounded in $L^\infty(\tau,T;H^1(\Omega))$ and also in $L^2(\tau,T;H^3(\Omega))$. By the following interpolation inequality ($d=3$)

$$
\|\phi_n\|_{L^\infty} \le C\|\phi_n\|_{L^6}^{\frac{3}{4}}\|\nabla\Delta\phi_n\|^{\frac{1}{4}} + C\|\phi_n\|_{L^6},
$$

it holds that for any $\varphi\in L^{\frac{8}{3}}(\tau,T;H^1(\Omega))$,

$$
\begin{aligned}
\left|\int_\tau^T\int_\Omega \operatorname{div}(\mathbf{u}_n\phi_n)\varphi dxdt\right| &\le \int_\tau^T \|\mathbf{u}_n\|\|\phi_n\|_{L^\infty}\|\nabla\varphi\|dt \\
&\le \left(\int_\tau^T\|\mathbf{u}_n\|^2dt\right)^{\frac{1}{2}}\left(\int_\tau^T\|\phi_n\|_{L^\infty}^8dt\right)^{\frac{1}{8}}\left(\int_\tau^T\|\varphi\|_{H^1}^{\frac{8}{3}}dt\right)^{\frac{3}{8}} \\
&\le C.
\end{aligned}
$$

Therefore, we have

$$
\operatorname{div}(\mathbf{u}_n\phi_n) \in L^{\frac{8}{5}}(\tau,T;(H^1(\Omega))'),
$$

which further implies that

$$
\partial_t\phi_n \in L^{\frac{8}{5}}(\tau,T;(H^1(\Omega))')
$$

is uniformly bounded.

By the interpolation inequality ($d=3$)

$$
\|\nabla\phi_n\|_{L^3} \le C\|\nabla\phi_n\|^{\frac{3}{4}}\|\nabla\Delta\phi_n\|^{\frac{1}{4}} + C\|\nabla\phi_n\|, \tag{3.17}
$$

we have for any $\mathbf{v}\in L^{\frac{8}{3}}(\tau,T;\mathbf{L}^2(\Omega))$, it holds

$$
\begin{aligned}
\left|\int_\tau^T (\mu_n \nabla\phi_n)\cdot \mathbf{v}dt\right| &\le \int_\tau^T \|\mu_n\|_{L^6}\|\nabla\phi_n\|_{L^3}\|\mathbf{v}\|dt \\
&\le C\left(\int_\tau^T \|\mu_n\|_{H^1}^2 dt\right)^{\frac12}\left(\int_\tau^T \|\nabla\phi_n\|_{L^3}^8 dt\right)^{\frac18}\left(\int_\tau^T \|\mathbf{v}\|^{\frac83} dt\right)^{\frac38} \\
&\le C. \qquad (3.18)
\end{aligned}
$$

As a consequence, $\mu_n\nabla\phi_n \in L^{\frac85}(\tau,T;\mathbf{L}^2(\Omega))$ and hence we have $\nabla p_n \in L^{\frac85}(\tau,T;\mathbf{L}^2(\Omega))$.

The above uniform estimates are enough to pass to the limit $n\to+\infty$ in the Galerkin scheme by standard compactness theorems to obtain the existence of global weak solutions to the system (1.1)–(1.7). The details are omitted here. One may refer to [4,43] for detailed argument for the simpler case $S=0$.

3.3. Local strong solutions

Now we proceed to prove the existence of local strong solutions. For this propose, we derive some higher order a priori estimates for the approximation solutions.

Testing (3.5) by $\Delta^2\phi_n$ and using integration by parts, we obtain that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\Delta\phi_n\|^2 + \|\Delta^2\phi_n\|^2 \\
&\quad = \int_\Omega \Delta(\phi_n^3-\phi_n)\Delta^2\phi_n dx + \int_\Omega S(1-\phi_n)\Delta^2\phi_n dx - \int_\Omega \mathbf{u}_n\cdot\nabla\phi_n\Delta^2\phi_n dx \\
&\quad \le \frac14\|\Delta^2\phi_n\|^2 + 3\int_\Omega \left(|\Delta(\phi_n^3-\phi_n)|^2 + S^2(1-\phi_n)^2 + |\mathbf{u}_n|^2|\nabla\phi_n|^2\right)dx. \qquad (3.19)
\end{aligned}
$$

By the three-dimensional Agmon's inequality $\|\phi_n\|_{L^\infty}\le C\|\phi_n\|_{H^1}^{\frac12}\|\phi_n\|_{H^2}^{\frac12}$ and the estimate (3.15), we can deduce that

$$
\begin{aligned}
&\int_\Omega \left|\Delta(\phi_n^3-\phi_n)\right|^2 dx \\
&\quad \le C\int_\Omega \left(\phi_n^2|\nabla\phi_n|^4 + \phi_n^4|\Delta\phi_n|^2 + |\Delta\phi_n|^2\right)dx \\
&\quad \le C\left(\|\phi_n\|_{L^6}^2\|\nabla\phi_n\|_{L^6}^4 + \|\phi_n\|_{L^\infty}^4\|\Delta\phi_n\|^2 + \|\Delta\phi_n\|^2\right) \\
&\quad \le C(\|\Delta\phi_n\|^2 + \|\Delta\phi_n\|^4 + 1), \qquad (3.20)
\end{aligned}
$$

and

$$
\int_\Omega S^2(1-\phi_n)^2 dx \le (1+\|\phi_n\|_{L^\infty})^2\|S\|^2 \le C(1+\|\Delta\phi_n\|)\|S\|^2. \qquad (3.21)
$$

For the third term on the right-hand side of (3.19), we have

$$\int_\Omega |\mathbf{u}_n|^2|\nabla\phi_n|^2 dx \le C\int_\Omega \left(|\nabla p_n|^2|\nabla\phi_n|^2 + |\mu_n|^2|\nabla\phi_n|^4\right)dx$$
$$\le C\|\nabla\phi_n\|_{L^\infty}^2\|\nabla p_n\|^2 + C\|\nabla\phi_n\|_{L^\infty}^4\|\mu_n\|^2. \tag{3.22}$$

Using the estimates (3.15), (3.17) together with Agmon's inequality for $\nabla\phi_n$

$$\|\nabla\phi_n\|_{L^\infty} \le C\|\phi_n\|_{H^2}^{\frac{1}{2}}\|\phi_n\|_{H^3}^{\frac{1}{2}}$$

and the fact

$$\|\nabla p_n\|^2 = \int_\Omega (Sp_n + (\mu_n\nabla\phi_n)\cdot\nabla p_n)\,dx$$

we have

$$\|\nabla\phi_n\|_{L^\infty}^2\|\nabla p_n\|^2 \le C\|\nabla\phi_n\|_{L^\infty}^2(\|S\|_{\dot H^{-1}}^2 + \|\mu_n\nabla\phi_n\|^2)$$
$$\le C\|\nabla\phi_n\|_{L^\infty}^2\|S\|_{\dot H^{-1}}^2 + C\|\nabla\phi_n\|_{L^\infty}^4\|\mu_n\|^2, \tag{3.23}$$

where

$$\|\nabla\phi_n\|_{L^\infty}^4\|\mu_n\|^2 \le C(1+\|\nabla\phi_n\|_{H^1}^2\|\nabla\phi_n\|_{H^2}^2)(1+\|\Delta\phi_n\|^2)$$
$$\le C(1+\|\Delta\phi_n\|^2\|\nabla\Delta\phi_n\|^2 + \|\Delta\phi_n\|^2 + \|\nabla\Delta\phi_n\|^2)(1+\|\Delta\phi_n\|^2)$$
$$\le \frac{1}{8}\|\Delta^2\phi_n\|^2 + C(\|\Delta\phi_n\|^{10}+1), \tag{3.24}$$

and

$$\|\nabla\phi_n\|_{L^\infty}^2\|S\|_{\dot H^{-1}}^2 \le C(1+\|\Delta\phi_n\|\|\nabla\Delta\phi_n\| + \|\Delta\phi_n\| + \|\nabla\Delta\phi_n\|)\|S\|_{\dot H^{-1}}^2$$
$$\le \frac{1}{8}\|\Delta^2\phi_n\|^2 + C(\|\Delta\phi_n\|^2+1). \tag{3.25}$$

As a consequence, we obtain from (3.19)–(3.24) that

$$\frac{d}{dt}\|\Delta\phi_n\|^2 + \|\Delta^2\phi_n\|^2 \le C\left(\|\Delta\phi_n\|^{10}+1\right). \tag{3.26}$$

Letting $y_n(t) = \|\Delta\phi_n\|^2 + 1$, we have

$$y_n'(t) \le C_0 y_n^5(t) \tag{3.27}$$

where the constant C_0 is independent of t. Solving this inequality implies that

$$y_n(t) \le \frac{y_n(\tau)}{(1-4C_0 y_n^4(\tau)t)^{\frac{1}{4}}}, \qquad \forall\, \tau \le t \le \min\left\{\frac{1}{4C_0 y_n^4(\tau)}, T\right\} := T_n.$$

Noticing that

$$y_n(\tau) \le y(\tau) = \|\Delta\phi_\tau\|^2 + 1,$$

we get

$$y_n(t) \le 2^{-\frac{1}{4}}(\|\Delta\phi_\tau\|^2 + 1), \qquad \text{whenever } \tau \le t \le \min\left\{\frac{1}{8C_0(\|\Delta\phi_\tau\|^2+1)^4}, T\right\} := T^*.$$

As a result, for any $t \in [\tau, T^*]$, the following estimate holds

$$\|\phi_n(t)\|_{H^2}^2 + \int_\tau^{T^*} \|\phi_n(t)\|_{H^4}^2 dt \le C. \tag{3.28}$$

The above estimate together with (3.21)–(3.24) yields

$$\int_\tau^{T^*} \|\mathrm{div}(\mathbf{u}_n\phi_n)\|^2 dt \le C.$$

Besides,

$$\int_\tau^{T^*} \|\mu_n\|_{H^2}^2 dt \le C\int_\tau^{T^*} (\|\Delta^2\phi_n\|^2 + \|\phi_n\|_{H^2}^2 + \|\phi_n\|_{H^2}^6) dt \le C. \tag{3.29}$$

As a consequence, we also have

$$\int_\tau^{T^*} \|\partial_t\phi_n\|^2 dt \le C \tag{3.30}$$

and

$$\begin{aligned}\int_\tau^{T^*} \|p\|_{H^2}^2 dt &\le C\int_\tau^{T^*} \left(\|S\|^2 + \|\mathrm{div}(\mu_n\nabla\phi_n)\|^2\right) dt\\ &\le C + \int_\tau^{T^*} (\|\nabla\mu_n\|_{L^3}^2\|\nabla\phi_n\|_{L^6}^2 + \|\mu_n\|_{L^\infty}^2\|\phi_n\|_{H^2}^2) dt\\ &\le C. \end{aligned} \tag{3.31}$$

Finally, from (3.29) and (3.31) we can easily derive that

$$\int_{\tau}^{T^*} \|\mathbf{u}_n\|_{H^1}^2 dt \le C. \tag{3.32}$$

Combining the above estimates together, we are able to prove the existence of local strong solution to the system (1.1)–(1.7) by the same argument as in [29]. Moreover, arguing exactly as in [29, Section 6], we can obtain the uniqueness of strong solutions. This completes the proof of Theorem 2.1.

4. Global strong solution in 2D

In this section, we focus on the study of the CHD system (1.1)–(1.7) in the 2D case and prove Theorem 2.2. Differently from the 3D case, the strong solution exists globally under weak assumption on the external source term S. Moreover, it defines a family of closed processes $\{U(t,\tau)\}_{t\ge\tau}$ in the space $H_N^2(\Omega)$.

4.1. Existence

We show that under a slightly weaker assumption on S than in Theorem 2.1(ii), one can actually prove the existence of global strong solution to the system (1.1)–(1.7). Based on the Galerkin scheme described before, we only need to obtain proper global-in-time *a priori* estimates. For the sake of simplicity, below we shall just perform formal estimates for smooth solutions (i.e., drop the subscript 'n'), which can be rigorously justified by the Galerkin approximation in previous section.

Lemma 4.1. *Suppose that $d = 2$ and $S \in L^2(\tau, T; \dot{L}^2(\Omega))$. Let $(\phi, \mathbf{u}, p)$ be a smooth solution to problem (1.1)–(1.7). Then the following estimates hold*

$$\|\Delta\phi(t)\|^2 \le C_1\left(1 + \frac{1}{t-\tau}\right), \quad \forall t \in (\tau, T], \tag{4.1}$$

and

$$\|\Delta\phi(t)\|^2 + \int_{\tau}^{T} \|\Delta^2\phi(t)\|^2 dt \le C_2, \quad \forall t \in [\tau, T] \tag{4.2}$$

where the constant C_1 depends on $\|\phi_\tau\|_{H^1}$, Ω and $\|S\|_{L^2(\tau,T;L^2)}$, while the constant C_2 depends on $\|\phi_\tau\|_{H^2}$, Ω and $\|S\|_{L^2(\tau,T;L^2)}$.

Proof. Similar to (3.14), we have the following estimate

$$\sup_{t\in[\tau,T]} \|\phi(t)\|_{H^1}^2 + \int_{\tau}^{T} \|\nabla\mu\|^2 dt + \int_{\tau}^{T} \|\mathbf{u}\|^2 dt \le C \tag{4.3}$$

where C depends on $\|\phi_\tau\|_{H^1}$, Ω and $\|S\|_{L^2(\tau,T;L^2)}$. Next, it is similar to (3.19) that by testing (1.1) by $\Delta^2\phi$ and using integration by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta\phi\|^2+\frac{3}{4}\|\Delta^2\phi\|^2\\&\quad\le 3\int_\Omega\left(|\Delta(\phi^3-\phi)|^2+S^2(1-\phi)^2+|\mathbf{u}|^2|\nabla\phi|^2\right)dx,\end{aligned}\tag{4.4}$$

Using the two-dimensional Agmon's inequality $\|\phi\|_{L^\infty}\le C\|\phi\|^{\frac{1}{2}}\|\phi\|_{H^2}^{\frac{1}{2}}$ and the Gagliardo–Nirenberg inequality $\|\nabla\phi\|_{L^4}\le C\|\nabla\Delta\phi\|^{\frac{1}{4}}\|\nabla\phi\|^{\frac{3}{4}}+C\|\nabla\phi\|$, we can estimate the first two terms on the right-hand side of (4.4) as follows:

$$\begin{aligned}&3\int_\Omega\left|\Delta(\phi^3-\phi)\right|^2dx\\&\quad\le C\int_\Omega\left(\phi^2|\nabla\phi|^4+\phi^4|\Delta\phi|^2+|\Delta\phi|^2\right)dx\\&\quad\le C\left(\|\phi\|_{L^\infty}^2\|\nabla\phi\|_{L^4}^4+\|\phi\|_{L^\infty}^4\|\Delta\phi\|^2+\|\Delta\phi\|^2\right)\\&\quad\le C(\|\phi\|^2+\|\Delta\phi\|\|\phi\|)(\|\nabla\Delta\phi\|\|\nabla\phi\|^3+\|\nabla\phi\|^4)\\&\qquad+C(\|\Delta\phi\|^2\|\phi\|^2+\|\phi\|^4)\|\Delta\phi\|^2+C\|\Delta\phi\|^2\\&\quad\le C\|\phi\|_{H^1}^3(\|\phi\|_{H^1}^2+\|\Delta\phi\|^2)(\|\nabla\Delta\phi\|+\|\phi\|_{H^1})+C\|\Delta\phi\|^2,\end{aligned}\tag{4.5}$$

where we have used the interpolation $\|\Delta\phi\|^2\le\|\nabla\phi\|\|\nabla\Delta\phi\|$, which is a consequence of the fact that ϕ fulfils $\partial_\nu\phi=0$ on the boundary. Besides, it is easy to see that

$$3\int_\Omega S^2(1-\phi)^2\le C\|S\|^2(1+\|\phi\|_{L^\infty})^2\le C\|S\|^2(\|\Delta\phi\|\|\phi\|+\|\phi\|^2).\tag{4.6}$$

For the third term on the right-hand side of (4.4), we deduce from (3.4) that

$$\begin{aligned}&3\int_\Omega|\mathbf{u}|^2|\nabla\phi|^2dx\\&\quad\le C\int_\Omega|\nabla p|^2|\nabla\phi|^2dx+C\|\nabla\phi\|_{L^\infty}^4\|\mu\|^2\\&\quad\le C\int_\Omega|\nabla A^{-1}S|^2|\nabla\phi|^2dx+\int_\Omega|\nabla A^{-1}\operatorname{div}(\mu\nabla\phi)|^2|\nabla\phi|^2dx+C\|\nabla\phi\|_{L^\infty}^4\|\mu\|^2\\&\quad\le C\|\nabla A^{-1}S\|_{L^4}^2\|\nabla\phi\|_{L^4}^2+C\|\nabla\phi\|_{L^\infty}^4\|\mu\|^2\\&\quad\le C\|S\|^2(\|\nabla\phi\|^2+\|\nabla\phi\|\|\Delta\phi\|)\end{aligned}$$

$$
\begin{aligned}
&+C(\|\nabla\phi\|^4+\|\nabla\phi\|^2\|\Delta\phi\|^2+\|\nabla\phi\|^2\|\nabla\Delta\phi\|^2)(\|f'(\phi)\|^2+\|\Delta\phi\|^2)\\
\le{}& C\|S\|^2(\|\nabla\phi\|^2+\|\Delta\phi\|^2)\\
&+C\|\nabla\phi\|^2(\|\nabla\phi\|^2+\|\nabla\Delta\phi\|^2)(\|\phi\|_{H^1}^6+\|\phi\|_{H^1}^2+\|\Delta\phi\|^2).
\end{aligned}
\tag{4.7}
$$

Here we note that the constants C in (4.5)–(4.7) depend only on Ω and coefficient of the system.

As a consequence, we deduce from (4.4)–(4.7) and the uniform estimate (4.3) that

$$
\frac{d}{dt}\|\Delta\phi\|^2+\|\Delta^2\phi\|^2\le Ch(t)\|\Delta\phi\|^2+Ch(t),
\tag{4.8}
$$

where

$$
h(t)=1+\|S\|^2+\|\nabla\Delta\phi\|^2
$$

and the constant C in (4.8) depends on $\|\phi_\tau\|_{H^1}$, Ω and $\|S\|_{L^2(\tau,T;L^2)}$.

Besides, it easily follows from (4.3) that

$$
\sup_{t\in[\tau,T)}\int_t^{t+r}h(s)ds\le r+C,\quad \forall r\in(0,\min\{1,T-t\}).
\tag{4.9}
$$

Then by the uniform Gronwall inequality [42, Lemma III.1.1], we infer that

$$
\|\Delta\phi(t+\delta)\|^2\le C(1+\delta^{-1}),\quad \forall t\in[\tau,T),\ \delta\in(0,\min\{1,T-t\}),
\tag{4.10}
$$

where the constant C depends on $\|\phi_\tau\|_{H^1}$, Ω and $\|S\|_{L^2(\tau,T;L^2)}$.

On the other hand, by the classical Gronwall inequality, we also infer that

$$
\|\Delta\phi(t)\|^2\le(\|\Delta\phi_\tau\|^2+1)e^{C\int_\tau^T h(s)ds},
\tag{4.11}
$$

and then

$$
\int_\tau^T\|\Delta^2\phi(t)\|^2dt\le C,
\tag{4.12}
$$

where the constant C depends on $\|\phi_\tau\|_{H^2}$, Ω and $\|S\|_{L^2(\tau,T;L^2)}$. $\square$

The existence of global strong solutions to problem (1.1)–(1.7) is a direct consequence of the uniform estimates (4.2) and (4.3) (see [29, Section 4] for detailed argument with $S=0$). Thus, the proof is omitted here.

4.2. Continuous dependence on initial data

The strong solution to problem (1.1)–(1.7) satisfies the following continuous dependence property, which also yields the uniqueness:

Lemma 4.2. *Suppose that $d = 2$. Let $(\phi_i, \mathbf{u}_i, p_i)$ $(i = 1, 2)$ be the two global strong solutions corresponding to the initial data $\phi_{\tau i} \in H^2_N(\Omega)$. Then for $t \in [\tau, T]$, the following estimate holds:*

$$\|\phi_1(t) - \phi_2(t)\|^2_{H^1} + \int_\tau^T (\|\nabla\mu(s)\|^2 + \|\mathbf{u}(s)\|^2)ds \le C_T\|\phi_{\tau 1} - \phi_{\tau 2}\|^2_{H^1}, \tag{4.13}$$

where the constant C_T may depend on $\|\phi_{\tau 1}\|_{H^2}$, $\|\phi_{\tau 2}\|_{H^2}$, $\int_\tau^T \|S\|^2 ds$, Ω, τ and T.

Proof. The argument is similar to [29, Section 6] with minor modifications due to the appearance of the source term S. For the convenience of the readers, we sketch the proof here. Let us set $\phi = \phi_1 - \phi_2$, $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $p = p_1 - p_2$. Also denote $\mu_i = -\Delta\phi_i + f(\phi_i)$, $i = 1, 2$ and $\mu := \mu_1 - \mu_2 = -\Delta\phi + f(\phi_1) - f(\phi_2)$. Then $(\phi, \mathbf{u}, p)$ solves the system

$$\begin{cases} \phi_t + \mathrm{div}(\mathbf{u}\phi_1 + \mathbf{u}_2\phi) = \Delta\mu, \\ \mathbf{u} = -\nabla p + (\mu\nabla\phi_1 + \mu_2\nabla\phi), \\ \mathrm{div}\,\mathbf{u} = 0, \end{cases} \tag{4.14}$$

subject to boundary and initial conditions

$$\begin{cases} \partial_\nu\phi = \partial_\nu\mu = \mathbf{u}\cdot\nu = 0 \quad \text{on } \partial\Omega, \\ \phi(t, x)|_{t=\tau} = \phi_{\tau 1} - \phi_{\tau 2}. \end{cases}$$

Testing the first equation of (4.14) by ϕ, after integration by parts we obtain that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \|\Delta\phi\|^2 \\ &= \int_\Omega (f'(\phi_1) - f'(\phi_2))\Delta\phi dx - \frac{1}{2}\int_\Omega S\phi^2 dx + \int_\Omega \phi_1\mathbf{u}\cdot\nabla\phi dx \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{4.15}$$

Using the uniform estimates (4.3) and Agmon's inequality, the terms I_1, I_3 can be estimated as in [29, (6.9)] such that

$$\begin{aligned} I_1 &\le (1 + \|\phi_1^2 + \phi_1\phi_2 + \phi_2^2\|_{L^\infty})\|\phi\|\|\Delta\phi\| \\ &\le \frac{1}{4}\|\Delta\phi\|^2 + C\|\phi\|^2, \end{aligned} \tag{4.16}$$

$$I_3 \le \|\mathbf{u}\|\|\nabla\phi\|\|\phi_1\|_{L^\infty} \le \frac{1}{8}\|\mathbf{u}\|^2 + C\|\nabla\phi\|^2. \tag{4.17}$$

Concerning I_2, we have

$$\begin{aligned} I_2 &\le \frac{1}{2}\|S\|\|\phi\|\|\phi\|_{L^\infty} \le C\|S\|\|\phi\|^{\frac{3}{2}}\|\phi\|_{H^2}^{\frac{1}{2}} \\ &\le C\|S\|\|\phi\|^2 + C\|S\|\|\phi\|^{\frac{3}{2}}\|\Delta\phi\|^{\frac{1}{2}} \\ &\le \frac{1}{4}\|\Delta\phi\|^2 + C(\|S\|^2+1)\|\phi\|^2. \end{aligned} \tag{4.18}$$

As a consequence, we have

$$\frac{d}{dt}\|\phi\|^2 + \|\Delta\phi\|^2 \le \frac{1}{4}\|\mathbf{u}\|^2 + C(\|S\|^2+1)(\|\nabla\phi\|^2 + \|\phi\|^2). \tag{4.19}$$

Next, testing the first and the second equation of (4.14) by μ and $\mathbf{u}$ respectively, adding the results together, we obtain that

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\nabla\phi\|^2 - \frac{1}{2}\|\phi\|^2 + \frac{1}{4}\int_\Omega \phi^4 dx\right) + \|\nabla\mu\|^2 + \|\mathbf{u}\|^2 \\ &= \int_\Omega \mu_2\nabla\phi\cdot\mathbf{u}dx + \int_\Omega \phi\mathbf{u}_2\cdot\nabla\mu dx + 3\int_\Omega \phi_1\phi_2\phi\phi_t dx. \end{aligned} \tag{4.20}$$

The first two terms on the right-hand side of (4.20) can be estimated exactly like [29, (6.6)–(6.7)] that

$$\begin{aligned} &\int_\Omega \mu_2\nabla\phi\cdot\mathbf{u}dx + \int_\Omega \phi\mathbf{u}_2\cdot\nabla\mu dx \\ &\le \frac{1}{8}\|\nabla\mu\|^2 + \frac{1}{8}\|\mathbf{u}\|^2 + C(\|\phi_2\|_{H^4}^2 + \|\mathbf{u}_2\|_{H^1}^2)(\|\nabla\phi\|^2 + \|\phi\|^2). \end{aligned} \tag{4.21}$$

For the third term, we have

$$\begin{aligned} &3\int_\Omega \phi_1\phi_2\phi\phi_t dx \\ &= -3\int_\Omega \operatorname{div}(\mathbf{u}\phi_1 + \mathbf{u}_2\phi)\phi_1\phi_2\phi dx + 3\int_\Omega \phi_1\phi_2\phi\Delta\mu dx \\ &= -3\int_\Omega (\mathbf{u}\cdot\nabla\phi_1)\phi_1\phi_2\phi dx - 3\int_\Omega S\phi_1\phi_2\phi^2 dx - 3\int_\Omega (\mathbf{u}_2\cdot\nabla\phi)\phi_1\phi_2\phi dx \\ &\quad - 3\int_\Omega \nabla(\phi_1\phi_2\phi)\cdot\nabla\mu dx \\ &\le \frac{1}{8}\|\mathbf{u}\|^2 + C\|\nabla\phi_1\|_{L^4}^2\|\phi_1\|_{L^\infty}^2\|\phi_2\|_{L^\infty}^2\|\phi\|_{L^4}^2 + C\|S\|\|\phi_1\|_{L^\infty}\|\phi_2\|_{L^\infty}\|\phi\|_{L^4}^2 \end{aligned}$$

$$
\begin{aligned}
&+C\|\mathbf{u}_2\|_{L^4}\|\phi_1\|_{L^\infty}\|\phi_2\|_{L^\infty}\|\nabla\phi\|\|\phi\|_{L^4}+\frac{1}{8}\|\nabla\mu\|^2+C\|\phi_1\|_{L^\infty}^2\|\phi_2\|_{L^\infty}^2\|\nabla\phi\|^2\\
&+C\|\nabla\phi_1\|_{L^\infty}^2\|\phi_2\|_{L^\infty}^2\|\phi\|^2+C\|\phi_1\|_{L^\infty}^2\|\nabla\phi_2\|_{L^\infty}^2\|\phi\|^2\\
\le\ &\frac{1}{8}\|\mathbf{u}\|^2+\frac{1}{8}\|\nabla\mu\|^2\\
&+C(\|\phi_2\|_{H^3}^2+\|\phi_1\|_{H^3}^2+\|\mathbf{u}_2\|_{H^1}^2+\|S\|^2+1)(\|\nabla\phi\|^2+\|\phi\|^2). \qquad (4.22)
\end{aligned}
$$

As a consequence, we infer from (4.20)–(4.22) that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\nabla\phi\|^2-\frac{1}{2}\|\phi\|^2+\frac{1}{4}\int_\Omega\phi^4dx\right)+\frac{3}{4}\|\nabla\mu\|^2+\frac{3}{4}\|\mathbf{u}\|^2\\
&\quad\le C(\|\phi_2\|_{H^4}^2+\|\phi_1\|_{H^3}^2+\|\mathbf{u}_2\|_{H^1}^2+\|S\|^2+1)(\|\nabla\phi\|^2+\|\phi\|^2). \qquad (4.23)
\end{aligned}
$$

Adding (4.19) with (4.23), we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\nabla\phi\|^2+\frac{1}{2}\|\phi\|^2+\frac{1}{4}\int_\Omega\phi^4dx\right)+\frac{3}{4}\|\nabla\mu\|^2+\frac{1}{2}\|\mathbf{u}\|^2\\
&\quad\le Ch(t)\left(\|\nabla\phi\|^2+\|\phi\|^2+\frac{1}{2}\int_\Omega\phi^4dx\right), \qquad (4.24)
\end{aligned}
$$

where

$$
h(t)=\|\phi_2\|_{H^4}^2+\|\phi_1\|_{H^3}^2+\|\mathbf{u}_2\|_{H^1}^2+\|S\|^2+1.
$$

Due to (4.2),

$$
\int_\tau^t h(s)ds\le C,\quad \forall t\in[\tau,T],
$$

where the constant C depends on $\|\phi_2(\tau)\|_{H^2}$, $\int_\tau^T\|S\|^2ds$, τ and T. Thus by the Gronwall inequality, we deduce that for all $t\in[\tau,T]$

$$
\begin{aligned}
&\|\nabla\phi(t)\|^2+\|\phi(t)\|^2+\frac{1}{2}\int_\Omega\phi^4dx\\
&\quad\le e^{C\int_\tau^T h(s)ds}\left(\|\nabla(\phi_{\tau1}-\phi_{\tau2})\|^2+\|\phi_{\tau1}-\phi_{\tau2}\|^2+\frac{1}{2}\|\phi_{\tau1}-\phi_{\tau2}\|_{L^4}^4\right).
\end{aligned}
$$

Our conclusion (4.13) easily follows from the above estimate. The proof is complete. □

4.3. Associated process

Recall the following definition (see [19], we also refer to [35] for the definition of closed semigroups):

Definition 4.1. Let X be a metric space. The set class $\{U(t,\tau)\}_{t\geq\tau}$ that $U(t,\tau): X \to X$ is called a *process* on X, if (i) $U(\tau,\tau)x = x$ for any $x \in X$; (ii) $U(t,\tau)x = U(t,s)U(s,\tau)x$ for any $\tau \leq s \leq t$ and any $x \in X$.

Moreover, a process $\{U(t,\tau)\}_{t\geq\tau}$ is said to be *closed* on X, if for any $\tau \leq t$, and any sequence $\{x_n\} \in X$ with $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.

Then we infer from Lemma 4.2 that:

Proposition 4.1. *For any $S \in L^2_{loc}(\mathbb{R}; \dot{L}^2(\Omega))$, we are able to define a family of closed processes $\{U(t,\tau)\}_{t\geq\tau}$ on $\mathcal{H} = H^2_N(\Omega)$ as follows:*

$$U(t,\tau)\phi_\tau = \phi(t;\tau,\phi_\tau), \quad \forall \phi_\tau \in H^2_N(\Omega), \quad \forall \tau \leq t,$$

where $\phi(t)$ is the unique global strong solution to problem (1.1)–(1.6).

5. Pullback attractor in 2D

In this section, we study the long-time dynamics of the family of processes $\{U(t,\tau)\}_{t\geq\tau}$ defined by the global strong solution to CHD problem (1.1)–(1.7) in terms of the *pullback attractor*. To this end, we first introduce some basic definitions and abstract results about pullback attractors for closed processes adopted from [19] (cf. [40] for the case of closed cocycles).

5.1. Preliminaries

Consider a metric space (X, d_X). We denote by $\mathrm{dist}_X(B_1, B_2)$ the Hausdorff semi-distance in X between two sets $B_1, B_2 \subset X$ defined as $\mathrm{dist}_X(B_1, B_2) = \sup_{x\in B_1} \inf_{y\in B_2} \mathrm{d}_X(x,y)$. $\mathcal{P}(X)$ stands for the family of all nonempty subsets of X. Let $\mathcal{D}$ be a nonempty class of families parameterized in time $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ is called a *universe* in $\mathcal{P}(X)$ (see [31]).

We recall now some definitions that will be useful in the subsequent analysis (see e.g., [7,19]):

Definition 5.1. A family of nonempty sets $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be *pullback $\mathcal{D}$-absorbing* for the process $\{U(t,\tau)\}_{t\geq\tau}$, if for any $\hat{D} \in \mathcal{D}$ and any $t \in \mathbb{R}$, there exists a $\tau_0(t,\hat{D}) \leq t$ such that $U(t,\tau)D(\tau) \subset D_0(t)$ for any $\tau \leq \tau_0(t,\hat{D})$.

Definition 5.2. The process $\{U(t,\tau)\}_{t\geq\tau}$ is said to be *pullback $\mathcal{D}$-asymptotically compact*, if for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$ and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}_{n=1}^\infty$ is relatively compact in X.

Definition 5.3. A family $\mathcal{A}_\mathcal{D} = \{A_\mathcal{D}(t) : t \in \mathbb{R}\}$ of nonempty subsets of X is said to be *a pullback $\mathcal{D}$-attractor* for the process $\{U(t,\tau)\}_{t\geq\tau}$ in X, if

(i) $A_{\mathcal{D}}(t)$ is compact in X for any $t \in \mathbb{R}$,
(ii) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e., $U(t,\tau)A_{\mathcal{D}}(\tau) = A_{\mathcal{D}}(t)$ for any $\tau \le t$,
(iii) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e., for any $t \in \mathbb{R}$ and any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, it holds

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), A_{\mathcal{D}}(t)) = 0.$$

The following abstract result on the existence of minimal pullback attractors for closed processes is proved in [19] (see also [40] for the case of closed cocycles):

Lemma 5.1. *Consider a closed process $\{U(t,\tau)\}_{t\ge\tau}$ in X. Let $\mathcal{D}$ be a universe in $\mathcal{P}(X)$. If the following conditions are satisfied:*

(1) there exists a family $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ such that $\hat{D}_0$ is pullback $\mathcal{D}$-absorbing for $\{U(t,\tau)\}_{t\ge\tau}$,
(2) $\{U(t,\tau)\}_{t\ge\tau}$ is pullback $\mathcal{D}$-asymptotically compact,

then there exists a minimal pullback $\mathcal{D}$-attractor $\mathcal{A}_{\mathcal{D}} = \{A_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ in X given by

$$A_{\mathcal{D}}(t) = \overline{\bigcup_{\hat{D}\in\mathcal{D}} \Lambda(\hat{D}, t)}^X,$$

where

$$\Lambda(\hat{D}, t) = \bigcap_{s\le t} \overline{\bigcup_{\tau\le s} U(t,\tau)D(\tau)}^X, \quad \hat{D} \in \mathcal{D}.$$

Remark 5.1. (i) Such a family $\mathcal{A}_{\mathcal{D}}$ is *minimal* in the sense that if $\hat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed subsets such that for any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), C(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

(ii) In the definition above, $\hat{D}_0$ does not necessarily belong to the class $\mathcal{D}$. Furthermore, if $\hat{D}_0 \in \mathcal{D}$, then we have $A_{\mathcal{D}}(t) = \Lambda(\hat{D}_0, t) \subset \overline{D_0(t)}^X$.

5.2. Existence of pullback $\mathcal{D}_F^{\mathcal{H}_M}$-absorbing sets

Since our system (1.1)–(1.4) preserves the spatial average of ϕ (see (2.2)), it seems impossible to construct a suitable absorbing set for the process $\{U(t,\tau)\}_{t\ge\tau}$ on the whole space $\mathcal{H} := H_N^2(\Omega)$. Instead, we shall study the dynamics of problem (1.1)–(1.7) confined on the phase space $\mathcal{H}_M$ (see (2.3) for its definition).

For the sake of simplicity, in the subsequent text, we denote by $\mathcal{D}_F^{\mathcal{H}_M}$ the class of families $\hat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D being a nonempty fixed bounded subset of $\mathcal{H}_M$ (i.e., $\hat{D} \subset \mathcal{P}(\mathcal{H}_M)$ and D is parameterized in time but constant for all $t \in \mathbb{R}$, see [11]). Then $\mathcal{D}_F^{\mathcal{H}_M}$ is the universe we shall work on.

First, we prove the existence of a pullback $\mathcal{D}_F^{\mathcal{H}_M}$-absorbing family of sets for the process $\{U(\tau,t)\}_{t\geq\tau}$:

Proposition 5.1. *Let $d=2$. Suppose that $S\in L_b^2(\mathbb{R};\dot{L}^2(\Omega))$. Then there is a family $\hat{D}_0\subset\mathcal{D}_F^{\mathcal{H}_M}$ that is pullback $\mathcal{D}_F^{\mathcal{H}_M}$-absorbing for the processes $\{U(t,\tau)\}_{t\geq\tau}$ associated with problem (1.1)–(1.7).*

Proof. In the subsequent proof, C, C_i denote constants that may depend on Ω, M, but are independent of the initial datum for ϕ. $\mathcal{Q}_i(\cdot)$ stand for certain monotone increasing functions.

Multiplying (1.1) by μ and (1.3) by $\mathbf{u}$, integrating over Ω then adding the resultants together (comparing with (3.11) for the approximate solutions), we deduce from the Hölder inequality and the Poincaré inequality that

$$\begin{aligned}
&\frac{d}{dt}E(\phi)+\|\nabla\mu\|^2+\|\mathbf{u}\|^2\\
&=\int_\Omega S(\mu-\overline{\mu})(1-\phi)dx+\int_\Omega S\left(A^{-1}S-A^{-1}\left(\operatorname{div}\left((\mu-\overline{\mu})\nabla\phi\right)\right)\right)dx\\
&\leq\|S\|\|\mu-\overline{\mu}\|+\|S\|\|\mu-\overline{\mu}\|_{L^4}\|\phi\|_{L^4}\\
&\quad+\|S\|\left(\|A^{-1}S\|+\|A^{-1}\left(\operatorname{div}\left((\mu-\overline{\mu})\nabla\phi\right)\right)\|\right)\\
&\leq C\|S\|\|\nabla\mu\|(1+\|\phi\|_{L^4})+C\|S\|\left(\|A^{-1}S\|+\|\nabla\mu\|\|\nabla\phi\|_{L^{\frac{3}{2}}}\right),
\end{aligned}\tag{5.1}$$

where

$$E(\phi)=\int_\Omega\left(\frac{1}{2}|\nabla\phi|^2+f(\phi)\right)dx.$$

By the two-dimensional Gagliardo–Nirenberg inequality and Young's inequality, we have

$$\begin{aligned}
&C\|S\|\|\nabla\mu\|(1+\|\phi\|_{L^4})+C\|S\|\left(\|A^{-1}S\|+\|\nabla\mu\|\|\nabla\phi\|_{L^{\frac{3}{2}}}\right)\\
&\leq\frac{1}{4}\|\nabla\mu\|^2+C\|S\|^2(1+\|\phi\|_{L^4}^2+\|\nabla\phi\|_{L^{\frac{3}{2}}}^2)\\
&\leq\frac{1}{4}\|\nabla\mu\|^2+C\|S\|^2\left(1+\|\nabla\phi\|^{\frac{2}{3}}\|\phi\|_{L^4}^{\frac{4}{3}}+\|\phi\|_{L^4}^2\right)\\
&\leq\frac{1}{4}\|\nabla\mu\|^2+C\|S\|^2\left(1+\|\phi\|_{L^4}^{\frac{8}{3}}+\|\nabla\phi\|^{\frac{4}{3}}\right).
\end{aligned}\tag{5.2}$$

From estimates (5.1)–(5.2) and Young's inequality we infer that

$$\frac{d}{dt}\int_\Omega\left(\frac{1}{2}|\nabla\phi|^2+f(\phi)\right)dx+\frac{1}{2}\|\nabla\mu\|^2+\|\mathbf{u}\|^2\leq C_1\|S\|^2\left(1+\|\phi\|_{L^4}^{\frac{8}{3}}+\|\nabla\phi\|^{\frac{4}{3}}\right).\tag{5.3}$$

Recalling the mass conservation property (2.2), we rewrite Eq. (1.1) in the following form

$$(\phi-\bar{\phi})_t+\Delta^2(\phi-\bar{\phi})-\Delta(f'(\phi)-\overline{f'(\phi)})=S-\mathrm{div}(\mathbf{u}\phi). \tag{5.4}$$

Multiplying the above equation by $A^{-1}(\phi-\bar{\phi})$, integrating by parts, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A^{-\frac{1}{2}}(\phi-\bar{\phi})\|^2+\|\nabla\phi\|^2+\int_\Omega(f'(\phi)-\overline{f'(\phi)})(\phi-\bar{\phi})dx\\&=\int_\Omega(S-\mathrm{div}(\mathbf{u}\phi))\,A^{-1}(\phi-\bar{\phi})dx.\end{aligned} \tag{5.5}$$

By Young's inequality, we have

$$\begin{aligned}\int_\Omega(f'(\phi)-\overline{f'(\phi)})(\phi-\bar{\phi})dx&=\int_\Omega f'(\phi)(\phi-\bar{\phi})dx\\&=\int_\Omega(\phi^3-\phi)(\phi-\bar{\phi})dx\\&=\int_\Omega(\phi^4-\phi^2)dx-|\bar{\phi}|\int_\Omega\phi^3dx+|\Omega||\bar{\phi}|^2\\&=\int_\omega(2f(\phi)+\frac{1}{2}\phi^4)dx-|\bar{\phi}|\int_\Omega\phi^3dx+|\Omega||\bar{\phi}|^2\\&\geq 2\int_\Omega f(\phi)dx-C_2.\end{aligned} \tag{5.6}$$

Moreover, by Young's inequality and Poincaré's inequality, the right-hand side of (5.5) can be estimated as follows

$$\begin{aligned}&\int_\Omega(S-\mathrm{div}(\mathbf{u}\phi))\,A^{-1}(\phi-\bar{\phi})dx\\&\leq\int_\Omega SA^{-1}(\phi-\bar{\phi})dx+\int_\Omega\phi\mathbf{u}\cdot\nabla A^{-1}(\phi-\bar{\phi})dx\\&\leq\|A^{-1}(\phi-\bar{\phi})\|\|S\|+\|\mathbf{u}\|\|\phi\|_{L^4}\|\nabla A^{-1}(\phi-\bar{\phi})\|_{L^4}\\&\leq\frac{1}{2}\|\nabla\phi\|^2+C\|S\|^2+\frac{1}{2\eta}\|\mathbf{u}\|^2+\frac{\eta}{2}\|\phi\|_{L^4}^2\|\nabla A^{-1}(\phi-\bar{\phi})\|_{L^4}^2\\&\leq\frac{1}{2}\|\nabla\phi\|^2+\frac{1}{2\eta}\|\mathbf{u}\|^2+C\eta\|\phi\|_{L^4}^2(\|\phi\|_{L^4}^2+|\bar{\phi}|^2)+C\|S\|^2\\&\leq\frac{1}{2}\|\nabla\phi\|^2+\frac{1}{2\eta}\|\mathbf{u}\|^2+(C_3\eta\|\phi\|_{L^4}^4+C_3M^2\eta\|\phi\|_{L^4}^2)+C_3\|S\|^2,\end{aligned}$$

where $\eta > 0$ is a constant to be specified later. Since

$$C_3\eta\|\phi\|_{L^4}^4 + C_3M^2\eta\|\phi\|_{L^4}^2 \le C_3\eta\left(1+\frac{M^2}{4}\right)\|\phi\|_{L^4}^4 + C_3M^2\eta$$
$$\le C_3\eta(8+2M^2)\int_\Omega f(\phi)dx + C_3(4+2M^2)\eta,$$

we take $\eta = \frac{1}{C_3(8+2M^2)}$ and deduce that

$$\frac{d}{dt}\|A^{-\frac{1}{2}}(\phi-\bar{\phi})\|^2 + \|\nabla\phi\|^2 + 2\int_\Omega f(\phi)dx \le C\|S\|^2 + C_3(8+2M^2)\|\mathbf{u}\|^2 + C_4. \quad (5.7)$$

Multiplying (5.7) by $C_5 = \frac{1}{C_3(16+4M^2)}$ and adding the resultant up with (5.3) gives

$$\frac{d}{dt}\left(E(\phi) + C_5\|A^{-\frac{1}{2}}(\phi-\bar{\phi})\|^2\right)$$
$$+\frac{1}{2}\|\nabla\mu\|^2 + \frac{1}{2}\|\mathbf{u}\|^2 + C_5\|\nabla\phi\|^2 + 2C_5\int_\Omega f(\phi)dx$$
$$\le C_6\|S\|^2\left(1+\|\phi\|_{L^4}^{\frac{8}{3}} + \|\nabla\phi\|^{\frac{4}{3}}\right) + C_7. \quad (5.8)$$

It is easy to see that there exist constants C_8, C_9 that are independent of ϕ such that

$$C_8(\|\nabla\phi\|^2 + \|\phi\|_{L^4}^4) - C_9 \le E(\phi) + C_5\|A^{-\frac{1}{2}}(\phi-\bar{\phi})\|^2 \le C_8(\|\nabla\phi\|^2 + \|\phi\|_{L^4}^4) + C_9.$$

Then we define $\Psi_1(t) := E(\phi) + C_5\|A^{-\frac{1}{2}}(\phi-\bar{\phi})\|^2 + C_9 + 1$, which satisfies

$$\Psi_1(t) \ge \max\left\{1, C_8(\|\nabla\phi\|^2 + \|\phi\|_{L^4}^4)\right\}. \quad (5.9)$$

Then it follows from (5.8) and Young's inequality that

$$\frac{d}{dt}\Psi_1(t) + C_{10}\Psi_1(t) + \frac{1}{2}\|\nabla\mu\|^2 + \frac{1}{2}\|\mathbf{u}\|^2 \le C_{11}\|S\|^2\Psi_1^{\frac{2}{3}}(t) + C_{11}(1+\|S\|^2). \quad (5.10)$$

Since $S \in L_b^2(\mathbb{R}; \dot{L}^2(\Omega))$, then applying Lemma A.2 in Appendix A with $n = 1$ and $\omega = a_1 = \frac{2}{3}$, we obtain the following dissipative estimates

$$\Psi_1(t) \le C_{13}\Psi_1(\tau)e^{-\frac{3}{4}C_{10}(t-\tau)} + \mathcal{Q}_1\left(\|S\|^2_{L_b^2(\mathbb{R};\dot{L}^2(\Omega))}\right), \quad \forall t \ge \tau. \quad (5.11)$$

It follows from the above estimate and (5.9) that

$$\|\phi(t)\|_{H^1}^2 \le \mathcal{Q}_2(\|\phi_\tau\|_{H^1}^2)e^{-C_{14}(t-\tau)} + \mathcal{Q}_3\left(\|S\|^2_{L_b^2(\mathbb{R};\dot{L}^2(\Omega))}\right). \quad (5.12)$$

As a consequence, we deduce from (5.12) that for any $t \in \mathbb{R}$, $\hat{D} \in \mathcal{D}_F^{\mathcal{H}_M}$, there exists a time $\tau_1(\hat{D}, t) < t - 3$ such that

$$\|\phi(r; \tau, \phi_\tau)\|_{H^1}^2 \le \rho_1, \quad \forall r \in [t-3, t],\ \tau \le \tau_1(\hat{D}, t),\ \phi_\tau \in D \in \hat{D}, \tag{5.13}$$

where

$$\rho_1 = 1 + \mathcal{Q}_3\left(\|S\|^2_{L^2_b(\mathbb{R}; \dot{L}^2(\Omega))}\right).$$

Besides, integrating (5.10), we infer that

$$\sup_{r \in [t-2, t]} \int_{r-1}^{r} \left(\|\nabla \mu(s)\|^2 + \|\mathbf{u}(s)\|^2\right) ds \le \mathcal{Q}_4\left(\rho_1, \|S\|^2_{L^2_b(\mathbb{R}; \dot{L}^2(\Omega))}\right). \tag{5.14}$$

for $\tau \le \tau_1(\hat{D}, t)$ and $\phi_\tau \in D \in \hat{D}$, which together with (5.12) and the Sobolev embedding theorem yields

$$\sup_{r \in [t-2, t]} \int_{r-1}^{r} \|\phi\|_{H^3}^2 ds \le \mathcal{Q}_5\left(\rho_1, \|S\|^2_{L^2_b(\mathbb{R}; \dot{L}^2(\Omega))}\right). \tag{5.15}$$

Next, testing (1.4) by $\Delta^2\phi$, using the estimate (5.12) and a similar argument in Lemma 4.1, we can still obtain the differential inequality (4.8) for $\|\Delta\phi\|^2$, namely,

$$\frac{d}{ds}\|\Delta\phi(s)\|^2 + \|\Delta^2\phi(s)\|^2 \le Ch(s)\|\Delta\phi\|^2 + Ch(s), \tag{5.16}$$

for a.e. $s \in [t-3, t]$, $\tau \le \tau_1(\hat{D}, t)$ and $\phi_\tau \in D \in \hat{D}$, here $h(s) = 1 + \|S\|^2 + \|\nabla\Delta\phi\|^2$, and the constant C now depends on ρ_1, Ω and $\|S\|_{L^2_b(\mathbb{R}; L^2)}$.

Using (5.15), (5.16) and the uniform Gronwall inequality [42, Lemma III.1.1], we can deduce that

$$\|\Delta\phi(r)\|^2 \le \mathcal{Q}_6\left(\rho_1, \|S\|^2_{L^2_b(\mathbb{R}; \dot{L}^2(\Omega))}\right), \quad \forall r \in [t-2, t]. \tag{5.17}$$

Thus, it follows from (5.12) and (5.17) that

$$\|\phi(r; \tau, \phi_\tau)\|_{H^2}^2 \le \rho_2, \quad \forall r \in [t-2, t],\ \tau \le \tau_1(\hat{D}, t),\ \phi_\tau \in D \in \hat{D} \tag{5.18}$$

where ρ_2 depends on ρ_1, $\|S\|^2_{L^2_b(\mathbb{R}; \dot{L}^2(\Omega))}$, M and Ω.

In summary, we can take the family

$$\hat{D}_0 = \left\{D_0(t) = \mathcal{B}_M(0, \rho_2^{\frac{1}{2}}),\ t \in \mathbb{R}\right\} \in \mathcal{D}_F^{\mathcal{H}_M},$$

where $\mathcal{B}_M(0, \rho_2^{\frac{1}{2}})$ is the closed ball in $\mathcal{H}_M$ of center zero and radius $\rho_2^{\frac{1}{2}}$. Then $\hat{D}_0$ satisfies that for any $t \in \mathbb{R}$ and any family $\hat{D} \in \mathcal{D}_F^{\mathcal{H}_M}$, there exists a time $\tau_0(\hat{D}, t) < t$ such that

$$U(t,\tau)D(\tau) \subset D_0(t), \quad \forall \tau \le \tau_0(\hat{D}, t),\ D(t) \in \hat{D}.$$

This completes the proof. □

Using the uniform estimates obtained in the above proposition and the Sobolev embedding theorem, indeed we can also prove the following

Corollary 5.1. *For any $t \in \mathbb{R}$ and any family $\hat{D} \in \mathcal{D}_F^{\mathcal{H}_M}$, there exists a time $\tau_0(\hat{D}, t) < t$ such that*

$$\sup_{r \in [t-1,t]} \int_{r-1}^{r} \left(\|\phi(s)\|_{H^4}^2 + \|\mathbf{u}(s)\|_{H^1}^2 + \|\phi_t(s)\|^2 \right) ds \le \rho_3, \quad \forall \tau \le \tau_0(\hat{D}, t),\ \phi_\tau \in D(\tau).$$

5.3. Pullback $\mathcal{D}_F^{\mathcal{H}_M}$-asymptotic compactness

Now we proceed to prove the pullback $\mathcal{D}_F^{\mathcal{H}_M}$-asymptotic compactness for the universe $\mathcal{D}_F^{\mathcal{H}_M}$ in $\mathcal{H}_M$.

Proposition 5.2. *Suppose that $S \in L_b^2(\mathbb{R}; \dot{L}^2(\Omega))$. Then the family of processes $\{U(t,\tau)\}_{t \ge \tau}$ is pullback $\mathcal{D}_F^{\mathcal{H}_M}$-asymptotically compact.*

Proof. Consider $t \in \mathbb{R}$, a family $\hat{D} \in \mathcal{D}_F^{\mathcal{H}_M}$, a sequence of time $\tau_n \to -\infty$ and a sequence of initial data $\phi_{\tau_n} \in D(\tau_n) \in \hat{D}$ (recall from the definition that here the set $D(t)$ is indeed time independent). For the sake of simplicity, below we just denote

$$\phi^n(s) = \phi(s; \tau_n, \phi_{\tau_n}) = U(s, \tau_n)\phi_{\tau_n}.$$

It follows from Proposition 5.1 and Corollary 5.1 that there exists a $\tau_0(\hat{D}, t) < t - 3$ such that the subsequence $\{\phi^n : \tau_n \le \tau_0(\hat{D}, t)\} \subset \{\phi^n\}$ is uniformly bounded in $L^\infty(t-2, t; H^2(\Omega) \cap L^2(t-2, t; H^4(\Omega))$ and correspondingly, $\{\phi_t^n\}$ is uniformly bounded in $L^2(t-2, t; L^2(\Omega))$.

Recall the following compactness lemma (see e.g., [38]):

Lemma 5.2. *Let $X \subset Y \subset Z$ be three Hilbert spaces, $T \in (0, +\infty)$. Suppose that the embedding $X \hookrightarrow Y$ is compact. Then:*

(1) For any $p, q \in (1, +\infty)$, the embedding $\{\phi \in L^p(0, T; X),\ \ \phi_t \in L^q(0, T; Z)\} \hookrightarrow L^p(0, T; Y)$ is compact.
(2) For any $q \in (1, +\infty)$, the embedding $\{\phi \in L^\infty(0, T; X),\ \phi_t \in L^q(0, T; Z)\} \hookrightarrow C([0, T]; Y)$ is compact.
(3) The embedding $\{\phi \in L^2(0, T; X),\ \phi_t \in L^2(0, T; Y)\} \hookrightarrow C([0, T]; [X, Y]_{\frac{1}{2}})$ is continuous.

We deduce that there exists a subsequence still denoted by $\{\phi^n\}$ and a function $\phi \in L^\infty([t-2,t]; H^2(\Omega) \cap L^2(t-2,t; H^4(\Omega))$ with $\phi_t \in L^2(t-2,t; L^2(\Omega))$ such that

$$
\begin{aligned}
&\phi^n \rightharpoonup \phi, \quad \text{weakly star in} \quad L^\infty(t-2,t; H^2(\Omega)),\\
&\phi^n \rightharpoonup \phi, \quad \text{weakly in} \quad L^2(t-2,t; H^4(\Omega)),\\
&\phi^n_t \rightharpoonup \phi_t, \quad \text{weakly in} \quad L^2(t-2,t; L^2(\Omega)),\\
&\phi^n \to \phi, \quad \text{strongly in} \quad L^2(t-2,t; H^2(\Omega)) \text{ and } C([t-2,t], H^1(\Omega)), \qquad (5.19)\\
&\phi^n(s) \to \phi(s), \quad \text{strongly in} \quad H^2(\Omega), \text{ for a.e. } s \in (t-2,t). \qquad (5.20)
\end{aligned}
$$

Moreover, we have $\phi \in C([t-2,t], H^2(\Omega))$ and it satisfies the system (1.1)–(1.4) a.e. on $(t-2,t)$.

From the fact that $\{\phi^n\}$ is uniformly bounded in $C([t-2,t], H^2(\Omega))$, we infer that for any sequence $\{s_n\} \subset [t-2,t]$ satisfying $s_n \to s_* \in [t-2,t]$, it holds (up to a subsequence)

$$
\phi^n(s_n) \rightharpoonup \phi(s_*) \quad \text{weakly in } H^2(\Omega). \tag{5.21}
$$

In what follows, we prove that the sequence $\{\phi^n(t)\}$ is relatively compact in $\mathcal{H}$ (see Definition 5.2), which is a direct consequence of the following result such that up to a subsequence, it holds

$$
\phi^n \to \phi \quad \text{strongly in } C([t-1,t]; H^2(\Omega)). \tag{5.22}
$$

To proceed, first we need to derive proper energy estimates. For every ϕ^n, recalling (4.4) and the computations in (4.5)–(4.7), using the interpolation inequality $\|\nabla\Delta\phi^n\|^2 \le \|\Delta\phi^n\|\|\Delta^2\phi^n\|$ and Young's inequality, after a straightforward but tedious calculation, we can re-estimate the three terms on the right-hand side of (4.4) (now in terms of ϕ^n, cf. (4.5)–(4.7)) and deduce that

$$
\frac{d}{dt}\|\Delta\phi^n\|^2 + \|\Delta^2\phi^n\|^2 \le C_\Omega(F_1(\phi^n) + F_2(\phi^n) + F_3(\phi^n)), \tag{5.23}
$$

where C_Ω is a constant that depends only on Ω. In particular, it is independent of ϕ^n. The functions F_i are given by

$$
\begin{aligned}
F_1(\phi^n) &= \|\phi^n\|^4_{H^1}\|\Delta\phi^n\|^6,\\
F_2(\phi^n) &= (\|\phi^n\|^{16}_{H^1} + \|S\|^2 + 1)\|\Delta\phi^n\|^2,\\
F_3(\phi^n) &= \|\phi^n\|^{10}_{H^1} + \|S\|^2\|\phi^n\|^2_{H^1} + 1.
\end{aligned}
$$

In a similar manner, we have for ϕ

$$
\frac{d}{dt}\|\Delta\phi\|^2 + \|\Delta^2\phi\|^2 \le C_\Omega(F_1(\phi) + F_2(\phi) + F_3(\phi)), \tag{5.24}
$$

where C_Ω is the same as in (5.23).

As a consequence, for ϕ^n and ϕ, $t-2 \le s_1 \le s_2 \le t$, we infer from the above inequalities that

$$
\begin{aligned}
&\|\Delta\phi^n(s_2)\|^2 + \int_{s_1}^{s_2} \|\Delta^2\phi^n(\xi)\|^2 d\xi \\
&\quad \le \|\Delta\phi^n(s_1)\|^2 + C_\Omega \int_{s_1}^{s_2} (F_1(\phi^n(\xi)) + F_2(\phi^n(\xi)) + F_3(\phi^n(\xi)))d\xi, \qquad (5.25)
\end{aligned}
$$

$$
\begin{aligned}
&\|\Delta\phi(s_2)\|^2 + \int_{s_1}^{s_2} \|\Delta^2\phi(\xi)\|^2 d\xi \\
&\quad \le \|\Delta\phi(s_1)\|^2 + C_\Omega \int_{s_1}^{s_2} (F_1(\phi(\xi)) + F_2(\phi(\xi)) + F_3(\phi(\xi)))d\xi. \qquad (5.26)
\end{aligned}
$$

Define

$$
J_n(s) = \|\Delta\phi^n(s)\|^2 - C_\Omega \int_{t-2}^{s} (F_1(\phi^n(\xi)) + F_2(\phi^n(\xi)) + F_3(\phi^n(\xi)))d\xi,
$$

$$
J(s) = \|\Delta\phi(s)\|^2 - C_\Omega \int_{t-2}^{s} (F_1(\phi(\xi)) + F_2(\phi(\xi)) + F_3(\phi(\xi)))d\xi.
$$

Since $\phi^n, \phi \in C([t-2,t]; H^2(\Omega))$, the functions $J_n(s)$ and $J(s)$ are continuous for $s \in [t-2,t]$. Moreover, they are non-increasing with respect to $s \in [t-2,t]$. To this end, we infer from (5.25) that

$$
\begin{aligned}
&J_n(s_2) - J_n(s_1) \\
&\quad = \|\Delta\phi^n(s_2)\|^2 - \|\Delta\phi^n(s_1)\|^2 - C_\Omega \int_{s_1}^{s_2} (F_1(\phi^n(\xi)) + F_2(\phi^n(\xi)) + F_3(\phi^n(\xi)))d\xi \\
&\quad \le -\int_{s_1}^{s_2} \|\Delta^2\phi^n(\xi)\|^2 d\xi \\
&\quad \le 0, \quad \text{for all } t-2 \le s_1 \le s_2 \le t.
\end{aligned}
$$

Similar result holds for $J(s)$. From the strong convergence results (5.19) and (5.20), we have for a.e. $s \in (t-2,t)$, $\|\Delta\phi^n(s)\| \to \|\Delta\phi(s)\|$ and $\|\phi^n(s)\|_{H^1} \to \|\phi(s)\|_{H^1}$. As a consequence,

$$
F_i(\phi^n(s)) \to F(\phi(s)), \quad \text{a.e. for } s \in (t-2,t), \quad i = 1,2,3. \qquad (5.27)
$$

Since ϕ^n is uniformly bounded in $L^\infty(t-2,t;H^2(\Omega))$, then $F_i(\phi^n)$ is also bounded $L^\infty(t-2,t)$. It follows from the Lebesgue dominated convergence theorem that

$$\int_{t-2}^{s} F_i(\phi^n(\xi))d\xi \to \int_{t-2}^{s} F_i(\phi(\xi))d\xi, \quad \forall s \in [t-2,t],\ i=1,2,3, \tag{5.28}$$

which implies

$$J_n(s) \to J(s), \quad \text{a.e. } s \in (t-2,t). \tag{5.29}$$

Now we proceed to prove the strong convergence property (5.22) by a contradiction argument introduced in [19,30]. Assume that (5.22) is not true, then there exists a constant $\kappa > 0$ and a sequence $\{t_n\}_{n=1}^\infty \subset [t-1,t]$ that without loss of generality, converges to a certain point $t^* \in [t-1,t]$ (otherwise, we can take a convergent subsequence) such that

$$\|\phi^n(t_n) - \phi(t^*)\|_{H^2} \geq 2\kappa.$$

From the elliptic estimate, here we can simply use the equivalent norm on $H^2(\Omega)$ given by $\|\cdot\|_{H^2} = \|\cdot\|_{H^1} + \|\Delta\cdot\|$. Then it follows from (5.19) that there exists $n_0 \in \mathbb{N}$ depending on κ such that

$$\|\Delta\phi^n(t_n) - \Delta\phi(t^*)\| \geq \kappa, \quad \forall n \geq n_0. \tag{5.30}$$

On the other hand, from (5.29), we can take a monotone increasing sequence $\{r_j\} \subset (t-2,t^*)$ that satisfies

$$\lim_{j\to+\infty} r_j = t^* \quad \text{and} \quad \lim_{n\to+\infty} J_n(r_j) = J(r_j), \quad \forall j \in \mathbb{N}. \tag{5.31}$$

For any $\delta > 0$, it follows from the continuity of $J(s)$ that there exists a constant $j_0 \in \mathbb{N}$ depending on δ such that

$$|J(r_j) - J(t^*)| < \frac{\delta}{2}, \quad \forall j \geq j_0(\delta). \tag{5.32}$$

Due to (5.31), for j_0, there exists an integer n_1 depending on j_0 and satisfying $n_1 \geq n_0$ such that

$$t_n \geq r_{j_0}, \quad \text{and} \quad |J_n(r_{j_0}) - J(r_{j_0})| < \frac{\delta}{2}, \quad \forall n \geq n_1. \tag{5.33}$$

Since $J_n(s)$ is non-increasing for $s \in [t-2,t]$, we infer from (5.32) and (5.33) that for all $n \geq n_1$, it holds

$$J_n(t_n) - J(t^*) \leq J_n(r_{j_0}) - J(t^*) \leq |J_n(r_{j_0}) - J(r_{j_0})| + |J(r_{j_0}) - J(t^*)| < \delta, \tag{5.34}$$

which implies

$$\limsup_{n\to+\infty} J_n(t_n) \leq J(t^*). \tag{5.35}$$

It follows from (5.28) and the boundedness of F_i that

$$\begin{aligned}
&\lim_{n\to+\infty}\left|\int_{t-2}^{t_n} F_i(\phi^n(\xi))d\xi-\int_{t-2}^{t^*} F_i(\phi(\xi))d\xi\right| \\
&\le \lim_{n\to+\infty}\left|\int_{t-2}^{t^*} F_i(\phi^n(\xi))d\xi-\int_{t-2}^{t^*} F_i(\phi(\xi))d\xi\right|+\lim_{n\to+\infty}\left|\int_{t^*}^{t_n} F_i(\phi^n(\xi))d\xi\right| \\
&=0, \quad i=1,2,3.
\end{aligned} \tag{5.36}$$

Then from the definition of J_n, J, and (5.35)–(5.36), we can see that

$$\limsup_{n\to+\infty}\|\Delta\phi^n(t_n)\|\le\|\Delta\phi(t^*)\|. \tag{5.37}$$

On the other hand, the weak convergence (5.21) implies that

$$\liminf_{n\to+\infty}\|\Delta\phi^n(t_n)\|\ge\|\Delta\phi(t^*)\|. \tag{5.38}$$

As a consequence, we have the norm convergence

$$\lim_{n\to+\infty}\|\Delta\phi^n(t_n)\|=\|\Delta\phi(t^*)\|, \tag{5.39}$$

which together with the weak convergence (5.21) yields the strong convergence such that

$$\lim_{n\to+\infty}\|\Delta\phi^n(t_n)-\Delta\phi(t^*)\|=0. \tag{5.40}$$

This leads to a contradiction with our assumption (5.30). Therefore, (5.22) holds and the sequence $\{\phi^n(t)\}$ is relatively compact in $\mathcal{H}$. The proof is complete. □

5.4. Proof of Theorem 2.3

For any $S\in L^2_b(\mathbb{R};\dot{L}^2(\Omega))$, we know from Proposition 4.1 that the global strong solution ϕ to problem (1.1)–(1.7) defines a closed process $\{U(t,\tau)\}_{t\ge\tau}$ in the phase space $\mathcal{H}_M$. Observing Propositions 5.1 and 5.2, also noticing that the pullback $\mathcal{D}_F^{\mathcal{H}_M}$-absorbing family $\hat{D}_0$ constructed in Proposition 5.1 indeed belongs to the universe $\mathcal{D}_F^{\mathcal{H}_M}$, then we are able to apply the abstract results in Lemma 5.1 and Remark 5.1 to conclude that the process $\{U(t,\tau)\}_{t\ge\tau}$ admits a minimal pullback $\mathcal{D}_F^{\mathcal{H}_M}$-attractor $\mathcal{A}_{\mathcal{D}_F^{\mathcal{H}_M}}=\{A_{\mathcal{D}_F^{\mathcal{H}_M}}(t):t\in\mathbb{R}\}$ in $\mathcal{H}_M$, which is given by

$$A_{\mathcal{D}_F^{\mathcal{H}_M}}(t)=\Lambda(\hat{D}_0,t)=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}U(t,\tau)D_0(\tau)}^{H^2(\Omega)}.$$

The proof of Theorem 2.3 is complete.

Remark 5.2. We remark that in the current particular case under consideration, i.e., $\hat{D}$ is parameterized in time but constant for all $t \in \mathbb{R}$, the corresponding minimal pullback $\mathcal{D}_F^{\mathcal{H}_M}$-attractor for the process $\{U(t,\tau)\}_{t\geq\tau}$ is just the pullback attractor defined in [11]. One can also apply the abstract results in [19] to treat more general case that the family $\hat{D}$ is time dependent, under suitable assumptions on its element D and the external source term S. We leave this to the interested reader.

6. Convergence to steady states in 2D

In this section, we investigate the long-time behavior of a single trajectory $\phi(t)$ when the associated dynamical process becomes *asymptotically autonomous* as time goes to infinity.

6.1. Uniform-in-time estimates

Hereafter, we assume that the external source term S satisfies

$$S \in L^2(\tau, +\infty; \dot{L}^2(\Omega)). \tag{6.1}$$

We recall the inequality (3.13) which implies that

$$\frac{d}{dt}E_0(\phi_n) + \frac{1}{2}\|\nabla\mu_n\|^2 + \|\mathbf{u}_n\|^2 \leq C\|S\|^2 E_0(\phi_n), \tag{6.2}$$
$$E_0(\phi_n(t)) \leq E_0(\phi_\tau)e^{\int_\tau^t \|S\|^2 ds}, \quad \forall t \geq \tau.$$

The above estimate easily yields the following uniform-in-time estimates for global weak (or strong) solutions to problem (1.1)–(1.7) such that

$$\sup_{t\in[\tau,+\infty)} \|\phi(t)\|_{H^1}^2 + \int_\tau^{+\infty} \|\nabla\mu\|^2 dt + \int_\tau^{+\infty} \|\mathbf{u}\|^2 dt \leq C, \tag{6.3}$$

and

$$\sup_{t\geq\tau} \int_t^{t+1} \|\phi\|_{H^3}^2 ds \leq C, \tag{6.4}$$

where the constant C depends only on $\|\phi_\tau\|_{H^1}$, $\int_\tau^{+\infty} \|S\|^2 ds$ and Ω.

Next, recalling the differential inequality (4.8), by the uniform Gronwall inequality [42, Lemma III.1.1], we can deduce that

$$\|\Delta\phi(t+1)\|^2 \leq C, \quad \forall t \geq \tau, \tag{6.5}$$

where the constant C depends on $\|\phi_\tau\|_{H^1}$, Ω and $\int_\tau^{+\infty} \|S\|^2 ds$. If in addition, $\phi_\tau \in H^2(\Omega)$, then by the classical Gronwall inequality, we have

$$\|\Delta\phi(t)\|^2 \le (\|\Delta\phi_\tau\|^2+1)e^{C\int_\tau^{\tau+1}h(s)ds} \le C, \quad \forall t\in[\tau,\tau+1]. \tag{6.6}$$

The above uniform-in-time estimates (6.5)–(6.6) imply that:

Proposition 6.1. *Assume that $S \in L^2(\tau,+\infty;\dot{L}^2(\Omega))$. Then the global strong solution to problem (1.1)–(1.7) is uniformly bounded in H^2 for all $t \ge \tau$. Moreover, the global weak solution to problem (1.1)–(1.7) will become a strong one after a positive time and it is also uniformly bounded in H^2.*

6.2. *The ω-limit set*

Since we are interested in the long-time behavior of ϕ as $t \to +\infty$, Proposition 6.1 enables us to focus on the study of uniformly bounded global strong solution of problem (1.1)–(1.7).

For any initial datum $\phi_\tau \in H^2_N(\Omega)$. We define the ω-limit set as follows

$$\omega(\phi_\tau) = \{\phi_\infty \in H^2_N(\Omega) \mid \exists\{t_n\} \nearrow +\infty \text{ s.t. } \phi(t_n)\to\phi_\infty \text{ in } H^1, \text{ as } t_n \to +\infty\}.$$

Besides, we introduce the set of steady states associated with the initial datum

$$\mathcal{S} = \left\{\psi \in H^2_N(\Omega) \mid -\Delta\psi + f'(\psi) = \frac{1}{|\Omega|}\int_\Omega f'(\psi)dx, \text{ a.e. in } \Omega, \int_\Omega \psi dx = \int_\Omega \phi_\tau dx\right\}. \tag{6.7}$$

Using the classical variational method and the elliptic regularity theorem, we can easily deduce that (see [43, Proposition 3.5] for the case with periodic boundary condition):

Proposition 6.2. *The set $\mathcal{S}$ is nonempty. Any element $\psi \in \mathcal{S}$ is a critical point of $E(\phi)$, which satisfies $\psi \in C^\infty$ and its H^m-norms ($m \ge 0$) are bounded by a constant depending on $|\overline{\phi_\tau}|$ and Ω.*

Using the fact that the strong solution ϕ is uniformly bounded in H^2 for $t \ge \tau$, similar to the calculations in (3.10)–(3.11) for the approximate solution, we can apply Young's inequality to obtain the following energy inequality for ϕ:

$$\frac{d}{dt}E(\phi(t)) + \frac{1}{2}\|\nabla\mu\|^2 + \|\mathbf{u}\|^2 \le K_1\|S\|^2, \quad \text{for a.e. } t \ge \tau, \tag{6.8}$$

where

$$E(\phi) = \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + f(\phi)\right)dx \tag{6.9}$$

and K_1 is a constant depending on $\|\phi_\tau\|_{H^2}$, $\int_\tau^{+\infty}\|S\|^2 ds$ and Ω.

The above type of energy inequality plays an important role in studying the long-time behavior of global solutions to non-autonomous system (cf. [10,26]). First, we can prove the following relationship between the ω-limit set and set $\mathcal{S}$.

Proposition 6.3. *For any $\phi_\tau \in H^2_N(\Omega)$, its corresponding ω-limit set is a nonempty bounded subset in $H^2(\Omega)$ such that $\omega(\phi_\tau) \subset \mathcal{S}$. Moreover, $E(\phi)$ is a constant on $\omega(\phi_\tau)$.*

Proof. Due to the uniform H^2-estimate for ϕ and the compact embedding $H^2 \hookrightarrow H^1$, there exists certain function $\phi_\infty \in H^2_N(\Omega)$ and an unbounded increasing sequence $t_n \to +\infty$ that $\|\phi(t_n) - \phi_\infty\|_{H^1} \to 0$ as $n \to +\infty$. Hence, $\omega(\phi_\tau)$ is a nonempty, bounded subset in $H^2(\Omega)$.

It follows from (6.8) that

$$E(\phi(t_1)) - E(\phi(t_2)) \le K_1 \int_{t_2}^{t_1} \|S\|^2 dt, \quad \forall \tau \le t_2 \le t_1 < +\infty. \tag{6.10}$$

Thus, $E(\phi(t))$ is continuous in time (and it is bounded from below from its definition (6.9)).

Denote $\tilde{E}(t) = E(\phi(t)) + K_1 \int_t^\infty \|S\|^2 ds$. Then it follows from (6.8) that

$$\frac{d}{dt}\tilde{E}(t) + \frac{1}{2}\|\nabla\mu\|^2 + \|\mathbf{u}\|^2 \le 0, \quad \text{for } t \ge \tau.$$

Hence, $\tilde{E}(t)$ is non-increasing in t. Since $\tilde{E}$ is also bounded from below, we may infer that as $t \to +\infty$, $\tilde{E}(t) \to E_\infty$ for some constant E_∞. Recalling the fact $\lim_{t\to+\infty} \int_t^{+\infty} \|S\|^2 ds = 0$, we get

$$\lim_{t\to+\infty} E(\phi(t)) = E_\infty. \tag{6.11}$$

By the definition of $\omega(\phi_\tau)$, it is easy to see that $E(t)$ equals E_∞ on $\omega(\phi_\tau)$.

Next, for any cluster point $\phi_\infty \in \omega(\phi_\tau)$, it easily follows that $\phi_\infty \in H^2_N(\Omega)$ and $\overline{\phi_\infty} = \overline{\phi_\tau}$. In order to show that $\phi_\infty \in \mathcal{S}$, we apply the argument introduced in [26]. Consider the unbounded increasing sequence $t_n \to +\infty$ such that $\|\phi(t_n) - \phi_\infty\|_{H^1} \to 0$ as $n \to +\infty$. Without loss of generality, we assume $t_{n+1} \ge t_n + 1$, $n \in \mathbb{N}$. Integrating (6.8) on the time interval $[t_n, t_{n+1}]$, we obtain that

$$E(\phi(t_{n+1})) - E(\phi(t_n)) + \int_{t_n}^{t_{n+1}} \left(\frac{1}{2}\|\nabla\mu(s)\|^2 + \|\mathbf{u}(s)\|^2\right) ds \le K_1 \int_{t_n}^{t_{n+1}} \|S\|^2 ds. \tag{6.12}$$

It follows from (6.11) and (6.12) that as $n \to +\infty$, it holds

$$\int_0^1 \left(\frac{1}{2}\|\nabla\mu(t_n+s)\|^2 + \|\mathbf{u}(t_n+s)\|^2\right) ds \le \int_{t_n}^{t_{n+1}} \left(\frac{1}{2}\|\nabla\mu(s)\|^2 + \|\mathbf{u}(s)\|^2\right) ds \to 0. \tag{6.13}$$

Besides, by Eq. (1.1), the uniform H^2-estimate for ϕ and Agmon's inequality, we have (cf. [1])

$$\begin{aligned} \|\phi_t\|_{(H^1(\Omega))'} &\le C(\|\mathbf{u}\phi\| + \|\nabla\mu\| + \|S\|) \le C(\|\mathbf{u}\|\|\phi\|_{L^\infty} + \|\nabla\mu\| + \|S\|) \\ &\le K_2\left(\|\mathbf{u}\| + \|\nabla\mu\| + \|S\|\right), \end{aligned} \tag{6.14}$$

where K_2 is a constant depending on $\|\phi_\tau\|_{H^2}$, $\int_\tau^{+\infty}\|S\|^2 ds$ and Ω. By (6.14) and (6.13), we have

$$\lim_{n\to+\infty}\int_0^1 \|\phi_t(t_n+s)\|^2_{(H^1(\Omega))'}ds = 0. \tag{6.15}$$

As a consequence,

$$\|\phi(t_n+s_1)-\phi(t_n+s_2)\|_{(H^1(\Omega))'} \to 0, \quad \text{uniformly for all } s_1, s_2 \in [0,1].$$

From the precompactness of $\phi(t)$ in $H^1(\Omega)$ and the sequential convergence of $\phi(t_n)$ in H^1, we infer that

$$\lim_{n\to\infty}\|\phi(t_n+s)-\phi_\infty\|_{H^1} = 0, \quad \forall s\in[0,1]. \tag{6.16}$$

For any $\xi \in H^1(\Omega)$, using Lebesgue's dominated convergence theorem, the Poincaré inequality, (6.13) and (6.16), we deduce that

$$\begin{aligned}
&\left|\int_\Omega (\nabla\phi_\infty\cdot\nabla\xi + f'(\phi_\infty)\xi - \overline{f'(\phi_\infty)}\xi)dx\right| \\
&= \lim_{n\to+\infty}\left|\int_0^1\int_\Omega \left(\nabla\phi(t_n+s)\cdot\nabla\xi + f'(\phi(t_n+s))\xi - \overline{f'(\phi(t_n+s))}\xi\right)dxds\right| \\
&= \lim_{n\to+\infty}\left|\int_0^1\int_\Omega (\mu(t_n+s)-\overline{\mu}(t_n+s))\xi dxds\right| \\
&\le \lim_{n\to+\infty}\int_0^1 \|\mu(t_n+s)-\overline{\mu}(t_n+s)\|\|\xi\|ds \\
&\le \lim_{n\to+\infty}\left(\int_0^1 \|\mu(t_n+s)-\overline{\mu}(t_n+s)\|^2 ds\right)^{\frac{1}{2}}\|\xi\| \\
&\le \lim_{n\to+\infty} C\left(\int_0^1 \|\nabla\mu(t_n+s)\|^2 ds\right)^{\frac{1}{2}}\|\xi\| \\
&= 0
\end{aligned}$$

which enables us to conclude that $\phi_\infty \in \mathcal{S}$. The proof is complete. $\square$

Remark 6.1. Indeed, from (6.12), we can also obtain the decay of velocity $\mathbf{u}$ in the following weak sense

$$\lim_{t\to+\infty}\int_0^1 \|\mathbf{u}(t+s)\|^2 ds = 0.$$

6.3. Convergence of trajectory $\phi(t)$

The precompactness of the trajectory $\phi(t)$ in $H^1(\Omega)$ only yields a sequential convergence result for $\phi(t)$. Next, we demonstrate that the ω-limit set $\omega(\phi_\tau)$ consists of a single point, namely, we show that each bounded global strong solution converges to a single steady state as time goes to infinity. For this purpose, we assume in addition that

$$\sup_{t\geq\tau}(1+t)^{1+\rho}\int_t^{+\infty} \|S\|^2 ds < +\infty, \quad \text{for some } \rho > 0. \tag{6.17}$$

First, we introduce the following Łojasiewicz–Simon type inequality, which easily follows from the abstract result in [16]:

Lemma 6.1. *Let $\psi \in H^2_N(\Omega)$ be a critical point of $E(\phi)$. Then there exist constants $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ depending on ψ such that for any $\phi \in H^2_N(\Omega)$ satisfying $\int_\Omega \phi dx = \int_\Omega \psi dx$ and $\|\phi-\psi\|_{H^1} \leq \beta$, it holds that*

$$\|\mathrm{P}_0(-\Delta\phi + f'(\phi))\| \geq |E(\phi) - E(\psi)|^{1-\theta}. \tag{6.18}$$

The proof for convergence of the whole trajectory $\phi(t)$ follows from the so-called Łojasiewicz–Simon approach (see e.g., [10,13,17,26,47]). By Lemma 6.1, for each element $\phi_\infty \in \omega(\phi_\tau)$, there exist $\beta_{\phi_\infty} > 0$ and $\theta_{\phi_\infty} \in (0, \frac{1}{2})$ such that the inequality (6.18) holds for

$$\phi \in \mathbf{B}_{\beta_{\phi_\infty}}(\phi_\infty) := \left\{\phi \in H^2_N(\Omega) : \int_\Omega \phi dx = \int_\Omega \phi_\tau dx, \ \ \|\phi - \phi_\infty\|_{H^1} < \beta_{\phi_\infty}\right\}.$$

The union of balls $\{\mathbf{B}_{\beta_{\phi_\infty}}(\phi_\infty) : \phi_\infty \in \omega(\phi_\tau)\}$ forms an open cover of $\omega(\phi_\tau)$ and because of the compactness of $\omega(\phi_\tau)$ in H^1, we can find a finite sub-cover $\{\mathbf{B}_{\beta_i}(\phi^i_\infty) : i = 1, 2, \dots, m\}$ of $\omega(\phi_\tau)$ in H^1, where the constants β_i, θ_i corresponding to ϕ^i_∞ in Lemma 6.1 are indexed by i. From the definition of $\omega(\phi_\tau)$, there exists a sufficient large $t_0 > \max\{\tau, 0\}$ such that

$$\phi(t) \in \mathcal{U} := \bigcup_{i=1}^m \mathbf{B}_{\beta_i}(\psi_i), \quad \text{for } t \geq t_0.$$

Taking $\theta = \min_{i=1}^m\{\theta_i\} \in (0, \frac{1}{2})$, using Lemma 6.1 and the convergence of energy (6.11), we deduce that for all $t \geq t_0$,

$$\|\mathrm{P}_0(-\Delta\phi + f'(\phi))\| \geq |E(\phi(t)) - E_\infty|^{1-\theta}. \tag{6.19}$$

It follows from (6.8) and (6.14) that

$$\begin{aligned}&\frac{d}{dt}E(\phi(t))+\frac{1}{4K_2}\|\phi_t\|^2_{(H^1(\Omega))'}+\frac{1}{4}\|\nabla\mu\|^2+\frac{3}{4}\|\mathbf{u}\|^2\\&\le\left(K_1+\frac{1}{4}\right)\|S\|^2,\quad\text{for a.e. } t\ge\tau.\end{aligned}\tag{6.20}$$

Introduce the auxiliary functions

$$\mathcal{Y}(t)^2=\frac{1}{4K_2}\|\phi_t\|^2_{(H^1(\Omega))'}+\frac{1}{4}\|\nabla\mu\|^2+\frac{3}{4}\|\mathbf{u}\|^2,\quad z(t)=\left(K_1+\frac{1}{4}\right)\int_t^\infty\|S\|^2ds.$$

The assumption (6.17) implies that

$$z(t)\le C(1+t)^{-(1+\rho)},\quad\forall t\ge t_0.$$

Then the energy inequality (6.20) yields that for $t\ge t_0$,

$$\begin{aligned}E(\phi(t))-E_\infty&\ge\int_t^\infty\mathcal{Y}(s)^2ds-z(t)\\&\ge\int_t^\infty\mathcal{Y}(s)^2ds-C(1+t)^{-(1+\rho)}.\end{aligned}\tag{6.21}$$

Set the exponent

$$\zeta=\min\left\{\theta,\frac{\rho}{2(1+\rho)}\right\}\in(0,\frac{1}{2}).$$

We infer from (6.19) and the uniform H^2-bound for ϕ that

$$\begin{aligned}|E(\phi(t))-E_\infty|&\le\|\mathrm{P}_0(-\Delta\phi+f'(\phi))\|^{\frac{1}{1-\theta}}\\&\le C\|\mathrm{P}_0(-\Delta\phi+f'(\phi))\|^{\frac{1}{1-\zeta}}\\&\le C\|\nabla\mu\|^{\frac{1}{1-\zeta}}\le C\mathcal{Y}(t)^{\frac{1}{1-\zeta}},\quad\forall t\ge t_0.\end{aligned}\tag{6.22}$$

On the other hand, it is easy to verify that

$$\int_t^\infty(1+s)^{-2(1+\rho)(1-\zeta)}ds\le\int_t^\infty(1+s)^{-(2+\rho)}ds\le(1+t)^{-(1+\rho)},\quad\forall t\ge t_0.\tag{6.23}$$

Now we denote

$$Z(t) = \mathcal{Y}(t) + (1+t)^{-(1+\rho)(1-\zeta)}.$$

It follows from (6.21)–(6.23) that

$$\int_t^\infty Z(s)^2 ds \le C\mathcal{Y}(t)^{\frac{1}{1-\zeta}} + C(1+t)^{-(1+\rho)} \\ \le CZ(t)^{\frac{1}{1-\zeta}}, \quad \forall t \ge t_0. \tag{6.24}$$

Thanks to the technical Lemma A.3, we conclude from (6.24) that

$$\int_{t_0}^{+\infty} Z(t)dt < +\infty. \tag{6.25}$$

Since $\rho > 0$, we also have

$$\int_{t_0}^{+\infty} (1+t)^{-(1+\rho)(1-\zeta)} dt \le \int_{t_0}^{\infty} (1+t)^{-\frac{2+\rho}{2}} dt = \frac{2}{\rho}(1+t_0)^{-\frac{\rho}{2}} < +\infty, \quad \text{for } t_0 > 0,$$

which together with (6.25) yields

$$\int_{t_0}^{+\infty} \|\phi_t\|_{(H^1(\Omega))'} dt < +\infty.$$

As a consequence, $\phi(t)$ converges strongly in $(H^1(\Omega))'$ as $t \to +\infty$. Together with the compactness of the trajectory in $H^s(\Omega)$, $s \in (0, 2)$, we finally obtain that there exists $\phi_\infty \in \mathcal{S}$ such that

$$\lim_{t\to+\infty} \|\phi(t) - \phi_\infty\|_{H^s} = 0 \quad \text{and} \quad \phi(t) \rightharpoonup \phi_\infty \text{ weakly in } H^2(\Omega).$$

Next, we proceed to prove the estimate on convergence rate. Let

$$\mathcal{K}(t) = E(t) - E_\infty + z(t).$$

It follows from (6.20) that

$$\frac{d}{dt}\mathcal{K}(t) + \mathcal{Y}(t)^2 \le 0, \quad \text{for } t \ge t_0. \tag{6.26}$$

Thus, $\mathcal{K}(t)$ is decreasing on $[t_0, +\infty)$ and due to (6.11) and (6.17), $\mathcal{K}(t) \to 0$ as $t \to +\infty$. Besides, we deduce from (6.17), (6.22) that

$$\mathcal{K}(t)^{2(1-\theta)} \le C\mathcal{Y}(t)^2 + C(1+t)^{-2(1-\theta)(1+\rho)}$$
$$\le -C\frac{d}{dt}\mathcal{K}(t) + C(1+t)^{-2(1-\theta)(1+\rho)}.$$

Then by [3, Lemma 2.6], we obtain that

$$\mathcal{K}(t) \le C(1+t)^{-\kappa}, \quad \forall t \ge t_0,$$

with the exponent given by

$$\kappa = \min\left\{\frac{1}{1-2\theta}, 1+\rho\right\}.$$

We infer from (6.26) that for any $t \ge t_0$,

$$\int_t^{2t} \mathcal{Y}(s)ds \le t^{\frac{1}{2}}\left(\int_t^{2t} \mathcal{Y}^2(s)ds\right)^{\frac{1}{2}} \le Ct^{\frac{1}{2}}\mathcal{K}^{\frac{1}{2}}(t) \le C(1+t)^{\frac{1-\kappa}{2}}.$$

Thus, we have

$$\int_t^{+\infty} \mathcal{Y}(s)ds \le \sum_{j=0}^{+\infty} \int_{2^j t}^{2^{j+1}t} \mathcal{Y}(s)ds \le C\sum_{j=0}^{+\infty}(2^j t)^{-\lambda} \le C(1+t)^{-\lambda}, \quad \forall t \ge t_0,$$

where

$$\lambda = \frac{\kappa-1}{2} = \min\left\{\frac{\theta}{1-2\theta}, \frac{\rho}{2}\right\} > 0. \tag{6.27}$$

Therefore,

$$\int_t^{+\infty} \|\phi_t\|_{(H^1(\Omega))'}ds \le C\int_t^{+\infty} \mathcal{Y}(s)ds \le C(1+t)^{-\lambda}, \quad \forall t \ge t_0,$$

which yields the convergence rate of ϕ in $(H^1(\Omega))'$:

$$\|\phi(t) - \phi_\infty\|_{(H^1(\Omega))'} \le C(1+t)^{-\lambda}, \quad \forall t \ge t_0.$$

Using the interpolation inequality and the uniform H^2-estimates for ϕ, we have for any $s \in [-1, 2]$,

$$\|\phi(t) - \phi_\infty\|_{H^s} \le C\|\phi(t) - \phi_\infty\|_{(H^1(\Omega))'}^{\frac{2-s}{3}}\|\phi(t) - \phi_\infty\|_{H^2}^{\frac{s+1}{3}}$$
$$\le C(1+t)^{-\frac{2-s}{3}\lambda}, \quad \forall t \ge t_0. \tag{6.28}$$

The proof of Theorem 2.4 is complete.

Remark 6.2. If the external source term S is more regular, further decay property can be obtained. For instance, if in addition $S \in L^2(\tau, +\infty; \dot{H}^1(\Omega)) \cap H^1(\tau, +\infty; \dot{H}^{-2}(\Omega))$, then using the energy method (see e.g., [23,43,47]), we can prove

$$\lim_{t\to+\infty} (\|\phi(t) - \phi_\infty\|_{H^3} + \|\mathbf{u}(t)\| + \|p(t)\|_{H^1}) = 0.$$

Moreover, the convergence rate (6.28) can be improved such that

$$\|\phi(t) - \phi_\infty\|_{H^2} \le C(1+t)^{-\lambda}, \quad \forall t \ge t_0,$$

where the exponent λ is given in (6.27).

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Appendix A

We first recall the following Gronwall-type inequality (see [20, Lemma 2.5]):

Lemma A.1. *Let $y(t)$, $f(t)$ and $g(t)$ be nonnegative locally integrable functions on $[\tau, +\infty)$ which satisfy, for some $\gamma > 0$*

$$\frac{d}{dt}y(t) + \gamma y(t) \le f(t)y^{\frac{1}{2}}(t) + g(t) \qquad \text{for a.e. } t \in [\tau, +\infty). \tag{A.1}$$

Then

$$y(t) \le 2y(\tau)e^{-\gamma(t-\tau)} + \left(\int_\tau^t f(s)e^{-\frac{\gamma}{2}(t-s)}ds\right)^2 + 2\int_\tau^t g(s)e^{-\gamma(t-s)}ds \tag{A.2}$$

for any $t \in [\tau, +\infty)$. Moreover, the inequality

$$\int_\tau^t m(s)e^{-\gamma(t-s)}ds \le \frac{e^\gamma}{1-e^{-\gamma}} \sup_{r\ge\tau} \int_r^{r+1} m(s)ds \tag{A.3}$$

holds for any nonnegative locally integrable function m on $[\tau, +\infty)$ and any $\gamma > 0$.

The above lemma easily yields the following result:

Corollary A.1. *Let $y(t)$, $f(t)$ and $g(t)$ be the nonnegative locally integrable functions on $[\tau, +\infty)$ that satisfy the assumptions in Lemma A.1. Assume, in addition that*

$$\sup_{t \geq \tau} \int_t^{t+1} f(s)ds \leq A_1 \quad \text{and} \quad \sup_{t \geq \tau} \int_t^{t+1} g(s)ds \leq A_2 \tag{A.4}$$

for some positive constants A_1, A_2. Then

$$y(t) \leq 2y(\tau)e^{-\gamma(t-\tau)} + Q(\gamma, A_1, A_2) \tag{A.5}$$

where

$$Q(\gamma, A_1, A_2) = \left(\frac{e^{\frac{\gamma}{2}}}{1 - e^{-\frac{\gamma}{2}}} A_1\right)^2 + \frac{2e^{\gamma}}{1 - e^{-\gamma}} A_2. \tag{A.6}$$

The result in Corollary A.1 can be generalized. Namely, we have

Lemma A.2. *Let $y(t)$, $f(t)$ and $g(t)$ be nonnegative locally integrable functions on $[\tau, +\infty)$ which satisfy, for some $\gamma > 0$ and some $\omega \in \{a_n\}_{n=0}^{\infty}$ with $a_n := \frac{n+1}{n+2}$ $(n = 0, 1, 2, \dots)$*

$$\frac{d}{dt} y(t) + \gamma y(t) \leq f(t) y^{\omega}(t) + g(t) \qquad \text{for a.e. } t \in [\tau, +\infty) \tag{A.7}$$

and such that

$$\sup_{t \geq \tau} \int_t^{t+1} f(s)ds \leq A_1 \quad \text{and} \quad \sup_{t \geq \tau} \int_t^{t+1} g(s)ds \leq A_2$$

for some positive constants A_1, A_2. Then

$$y(t) \leq 4\left(4^{\alpha_n} 2^{\beta_n} y(\tau) e^{-\theta_n \gamma (t-\tau)} + Q^{\beta_n}(\frac{\gamma}{2}, A_1, A_2)\right) \tag{A.8}$$

for any $t \in [\tau, +\infty)$, where

$$\alpha_n = \begin{cases} 0, & \text{if } n = 0, \\ (n+2)\displaystyle\sum_{j=2}^{n+1} \frac{1}{j}, & \text{if } n \geq 1, \end{cases} \qquad \beta_n = \frac{n+2}{2}, \qquad \theta_n = \frac{n+2}{2^{n+1}},$$

and Q is the same as in Lemma A.1.

Proof. Without loss of generality, we suppose that $y(t) \geq 1$. Otherwise, we can simply set $\tilde{y}(t) = y(t) + 1$. Using the fact $y^{\omega} < \tilde{y}^{\omega}$, we obtain a differential inequality for $\tilde{y}$ that has the same form as for y.

Then we prove the result by induction. The case $\omega = a_0 = \frac{1}{2}$ corresponds to (A.5) in Corollary A.1, with $\alpha_0 = 0$, $\beta_0 = 1$ and $\theta_0 = \frac{1}{2}$. Supposing that (A.8) holds for $\omega = a_n$ $(n \geq 0)$, we consider the case $\omega = a_{n+1}$. Denote $\varphi(t) = y^{\omega}(t)$. Then $y(t) = \varphi^{\frac{1}{\omega}}(t)$ and it holds that

$$\frac{d}{dt}\varphi(t) + \omega\gamma\varphi(t) \leq \omega f(t)\varphi^{2-\frac{1}{\omega}}(t) + \omega h(t),$$

where

$$h(t) = \varphi^{1-\frac{1}{\omega}}(t)g(t).$$

Noticing that $\omega \in [\frac{1}{2}, 1)$, $\varphi(t) \geq 1$ and $2 - \frac{1}{a_{n+1}} = a_n$, we have

$$h(t) \leq g(t)$$

and

$$\frac{d}{dt}\varphi(t) + \frac{\gamma}{2}\varphi(t) \leq f(t)\varphi^{a_n}(t) + \omega g(t).$$

Then it follows from the case $\omega = a_n$ that

$$\varphi(t) \leq 4\left(4^{\alpha_n}2^{\beta_n}\varphi(\tau)e^{-\frac{\theta_n\gamma(t-\tau)}{2}} + Q^{\beta_n}(\frac{\gamma}{2}, A_1, A_2)\right)$$

i.e.,

$$y^{\omega}(t) \leq 4\left(4^{\alpha_n}2^{\beta_n}y^{\omega}(\tau)e^{-\frac{\theta_n\gamma(t-\tau)}{2}} + Q^{\beta_n}(\frac{\gamma}{2}, A_1, A_2)\right).$$

Applying the elementary inequality

$$(x+y)^{\theta} \leq 4(x^{\theta} + y^{\theta}), \qquad \text{for } x, y > 0,\ 1 \leq \theta \leq 2$$

and noticing that $\frac{1}{\omega} \in (1, 2]$, we get

$$y(t) \leq 4\left(4^{\frac{(1+\alpha_n)}{a_{n+1}}}2^{\frac{\beta_n}{a_{n+1}}}y(\tau)e^{-\frac{(t-\tau)\gamma\theta_n}{2a_{n+1}}} + Q^{\frac{\beta_n}{a_{n+1}}}(\frac{\gamma}{2}, A_1, A_2)\right),$$

with

$$\alpha_{n+1} = \frac{1+\alpha_n}{a_{n+1}}, \quad \beta_{n+1} = \frac{\beta_n}{a_{n+1}}, \quad \theta_{n+1} = \frac{\theta_n}{2a_{n+1}},$$

such that (A.8) holds for $\omega = a_{n+1}$. This completes the proof. □

Remark A.1. Since $a_n \nearrow 1$ as $n \to +\infty$, the above lemma enables us to deal with the general case $\omega \in (\frac{1}{2}, 1)$ in (A.7). On the other hand, when $\omega \in (0, \frac{1}{2})$, we can also employ Lemma A.1, thanks to Young's inequality such that $y^{\omega} \leq 2\omega y^{\frac{1}{2}} + (1 - 2\omega)$.

The following lemma (cf. [13,26]) will be used to study the long-time behavior of global solutions to problem (1.1)–(1.7):

Lemma A.3. *Let $\zeta \in (0, \frac{1}{2})$. Assume that $Z \geq 0$ is a measurable function on $(\tau, +\infty)$, $Z \in L^2(\tau, +\infty)$ and there exist $C > 0$ and $t_0 \geq \tau$ such that*

$$\int_t^\infty Z^2(s)ds \leq C Z(t)^{\frac{1}{1-\zeta}}, \quad \text{for a.e. } t \geq t_0.$$

Then $Z \in L^1(t_0, +\infty)$.

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