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Global existence of weak solutions to the three-dimensional Euler equations with helical symmetry ☆

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Abstract

In this paper, we mainly investigate the weak solutions of the three-dimensional incompressible Euler equations with helical symmetry in the whole space when the helical swirl vanishes. Specifically, we establish the global existence of weak solutions when the initial vorticity lies in $L^1 \cap L^p$ with $p > 1$. Our result extends the previous work [2], where the initial vorticity is compactly supported and belongs to L^p with $p > 4/3$. The key ingredient in this paper involves the explicit analysis of Biot–Savart law with helical symmetry in domain $\mathbb{R}^2 \times [-\pi, \pi]$ via the theories of singular integral operators and second order elliptic equations.

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1. Introduction

The three-dimensional unsteady incompressible Euler equations read as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \\ \mathrm{div} \mathbf{u} = 0, \end{cases} \tag{1.1}$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, where the unknowns $\mathbf{u} = (u_1, u_2, u_3)$ and $p = p(t, x)$ represent the velocity fields and the pressure of the fluid, respectively. The corresponding vorticity, $\boldsymbol{\omega} = \mathrm{curl} \mathbf{u}$ satisfies the equation

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u}. \tag{1.2}$$

As is well known, the global existence of weak solutions to the three-dimensional Euler equations remains an open problem due to the strong nonlinearity of the vortex stretching term $(\boldsymbol{\omega} \cdot \nabla) \mathbf{u}$ in (1.2) in contrast to the global well-posedness of the two-dimensional equations when the vortex stretching term vanishes. In fact, there are a number of literatures about global well-posedness results of the two-dimensional incompressible Euler equations when the initial vorticity $\boldsymbol{\omega}_0$ lies in various function spaces (see [22] and references therein). Specifically, when $\boldsymbol{\omega}_0 \in L^1 \cap L^\infty$, Yudovich [26] proved the global existence and uniqueness of weak solutions. Later, the result was improved by Vishik [24] and Yudovich [27] with $\boldsymbol{\omega}_0$ in a class slightly larger than L^∞. If $\boldsymbol{\omega}_0$ lies in $L^1 \cap L^p$ with $p > 1$, the global existence result was proved by Diperna–Majda [6]. When $\boldsymbol{\omega}_0$ is a finite Radon measure with one sign, the global existence result was studied by Delort [5], Majda [21], Evans–Müller [9] and Liu–Xin [18] via different approaches.

However, when it comes to the three-dimensional case, local well-posedness of classical solutions as studied in [22]. Some global well-posedness results were established when considering periodic domains or the solution with some symmetric property. When the domains are periodic, recently, based on the method introduced in [16,17], Szekelyhidi [23] and Wiedemann [25] constructed the global admissible and L^2 weak solutions respectively. If the flow is axisymmetric and the swirl component of velocity fields vanishes, the global well-posedness results similar to two-dimensional case were extensively studied (see [3,4,13–15,22] and references therein). If the swirl component appears, the global well-posedness issue is still open except [11] for Navier–Stokes equations with a class of large data.

Another attractive issue for global well-posedness issue in three-dimensional flow is helically symmetric case, which means that the flow is invariant under a superposition of a rotation around a fixed axis and a simultaneous translation along the rotation axis directions respectively (see Section 2 for more details). Under this situation, when the helical swirl vanishes (see (2.12) below) and x' lies in bounded domains, the global well-posedness results for smooth/strong solutions corresponding to the initial regular velocity filed/bounded vorticity had been established respectively by Dutrifoy [7] and Ettinger–Titi [8]. For the whole space case, when the third component ω_0 of initial vorticity is compactly supported and belongs to L^p with $p > \frac{4}{3}$, the global weak solutions were obtained by Bronzi–Lopes–Lopes in [2]. In addition, to our best knowledge, there is no any global well-posedness result when the helical swirl exists in three-dimensional case. Along this line, the vanishing viscosity limit for the three-dimensional helically symmetric

Navier–Stokes equations has been studied in [12] and the global well-posedness results of the Navier–Stokes equations with helical symmetry can be referred to [20].

In this paper we will focus on the global existence of weak solutions to the three-dimensional Euler equations with helical symmetry in the whole space when the helical swirl vanishes and the initial vorticity belongs to $L^1 \cap L^p$, $p > 1$. The motivation comes from the following facts. On the one hand, we recall that the helical flow is more close to two-dimensional flow instead of three-dimensional flow in some sense (see Lemma 2.1, [19]). On the other hand, as mentioned in [22], the global existence of classical weak solutions to Euler equations heavily relies on the uniform L^p-estimate of the vorticity, which inherits from the regularity of the initial vorticity. By virtue of the compactness method, it is possible to prove the desired result. It is well-known that the crucial value of index p is 1 when $W^{1,p} \hookrightarrow L^2$ compactly in two-dimensional case. Therefore, we naturally expect to prove the existence of global weak solutions with initial vorticity in $L^1 \cap L^p,\ p > 1$, which extends the result of [2].

However, the new challenge arises due to the gap between the *a priori* $W^{1,p}$-estimate of the velocity fields and the L^p-estimate of its vorticity. One of the useful tools to overcome the gap is the Biot–Savart law, which indicates the relation between the velocity fields and its corresponding vorticity, as stated in [22]. But the story becomes more subtle in helically symmetric case in comparison with the classical one in $\mathbb{R}^n$, where $n = 2, 3$ (see [22] for example). Roughly speaking, the Biot–Savart law with helical symmetry is related to a Green function split into two parts: the first part is same to the classical two-dimensional kernel and the second one is a Fourier series along the periodic direction, which delivers the difficulties to occupy the estimates of the velocity fields and especially its gradient. One of the new ingredients in the paper is the $W^{2,p}$-theory of second order elliptic equations as well as the techniques of the singular integral operator in [2], which give more information about *a priori* estimates of the gradient of the velocity fields.

Meanwhile, the other difficulty of this paper lies in the L^2 estimate of the velocity fields itself. It should be noted that in two-dimensional flow, to guarantee that the energy of the velocity field is finite ($\int_{\mathbb{R}^2} |\mathbf{u}|^2 dx < \infty$), the velocity must satisfy the strong restriction that $\int_{\mathbb{R}^2} \omega dx = 0$ because the decay behavior of the two-dimensional kernel at infinity is like $\frac{1}{r}$, which is not square integrable (see Proposition 2.3 of [22]). The standard strategy to explore the difficulty is to decompose the velocity fields into an explicit and steady solution, which is constructed by the average of vorticity [6,22] and the other part, which is recovered from the vorticity with the zero average by classical Biot–Savart law. In this paper, we obtain a L^2_{loc} estimate instead of L^2 one of the velocity fields, which is enough to prove the global existence of weak solutions. To do that, we verify rigorously the behaviors of the velocity field and its gradient at infinity when the corresponding initial vorticity belongs to $L^1 \cap L^p$ $(p > 1)$. In other words, we establish the global existence of weak solutions $\mathbf{u} \in L^2_{loc}$ to the three-dimensional Euler equations with helical symmetry directly, avoiding constructing steady helical solutions in comparison to [2] and [22], which is another new ingredient in this paper.

The paper is organized as follows. In Section 2, we introduce the mathematical preliminaries of helical flow and our main theorem. In Section 3, we make a rigorous analysis of the Biot–Savart law and obtain the required L^p $(p > 1)$ estimates of the velocity fields and its gradient. Furthermore, the construction of the approximate solutions and the proof of the main theorem will be stated in Section 4. In Appendix A, two elementary lemmas related to the Biot–Savart law are presented. In addition, we establish a key estimate associated with two-dimensional singular integral operator in Appendix B.

2. Mathematical preliminaries and main result

Helical flow means that the flow keeps invariant under a superposition of a rotation around the x_3-axis and a translation along the rotation axis at the same time. More precisely, let G^k be a one-parameter group of isometrics of $\mathbb{R}^3$, defined by

$$G^\kappa = \{S_\theta^\kappa : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 | \theta \in \mathbb{R}\}, \tag{2.1}$$

where

$$S_\theta^\kappa(x) = \begin{pmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \\ x_3 + \kappa\theta \end{pmatrix}, \tag{2.2}$$

and κ is a fixed nonzero constant. Then a helical function and a helical vector field are defined respectively as follows [8].

Definition 2.1 *(Helical function).* A scalar function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is helical if

$$f(S_\theta^\kappa(x)) = f(x), \ \forall\, \theta \in \mathbb{R}. \tag{2.3}$$

Namely, it is invariant under the action of G^κ.

Definition 2.2 *(Helical vector field).* The vector field $\boldsymbol{v} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is helical, if

$$\boldsymbol{v}(S_\theta^\kappa(x)) = R_\theta \boldsymbol{v}(x), \ \forall\, \theta \in \mathbb{R}, \tag{2.4}$$

where

$$R_\theta(x) = \begin{pmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \\ x_3 \end{pmatrix}. \tag{2.5}$$

It should be noted that $S_{2\pi}^\kappa$ has a translation by $2\pi\kappa$ along x_3-direction from (2.3) and (2.4), which implies that helical symmetry naturally inherits a periodic boundary condition in the x_3-direction with period $2\pi\kappa$. Consequently, to deal with the helical flow in $\mathbb{R}^3$, we essentially discuss with the flow defined on domain $\mathbb{R}^2 \times [-\kappa\pi, \kappa\pi]$ with a periodic boundary condition in x_3-direction. Without loss of generality, we take $\kappa \equiv 1$ and $S_\theta^1 = S_\theta$ throughout the rest paper. Therefore, the effective domain presented in the paper is $\mathbb{R}^2 \times [-\pi, \pi]$ instead of $\mathbb{R}^3$. In addition, we denote by $L^p_{per}(\mathbb{R}^2 \times [-\pi, \pi])$ or $L^p_{per}([-\pi, \pi]; L^p(\mathbb{R}^2))$ the L^p norm of function f in $\mathbb{R}^2 \times [-\pi, \pi]$ with periodicity in x_3 direction. In particular, we denote by $L^p_{c,per}(\mathbb{R}^2 \times [-\pi, \pi])$ or $L^p_{per}([-\pi, \pi]; L^p_c(\mathbb{R}^2))$ the L^p norm of function f with the compact support in $\mathbb{R}^2$ and periodicity in x_3-direction.

Now we give the equivalent definitions of a helical function and a helical vector field respectively.

Lemma 2.1. *[Claim 2.3 of [8]] A continuously differentiable function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is helical if and only if*

$$x_2 \partial_{x_1} f - x_1 \partial_{x_2} f + \partial_{x_3} f = \boldsymbol{\xi} \cdot \nabla f = \partial_{\boldsymbol{\xi}} f = 0 \tag{2.6}$$

with $\boldsymbol{\xi} = (x_2, -x_1, 1)^T$, the tangential direction of the flow along the helices (2.2) at $x = (x_1, x_2, x_3)$.

Lemma 2.2. *[Claim 2.5 of [8]] A continuously differentiable vector field $\boldsymbol{v} = (v_1, v_2, v_3)^T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is helical if and only if it obeys the following relations*

$$\partial_{\boldsymbol{\xi}} v_1 = v_2, \tag{2.7}$$

$$\partial_{\boldsymbol{\xi}} v_2 = -v_1, \tag{2.8}$$

$$\partial_{\boldsymbol{\xi}} v_3 = 0. \tag{2.9}$$

The following lemma tells that the helical flow can be essentially viewed as an extension of the two-dimensional one in some sense (see Lemma 2.1 of [19]), which makes it possible to improve the result of [2] to the two-dimensional case [6].

Lemma 2.3. *Let $\boldsymbol{v} = \boldsymbol{v}(x)$ ($p = p(x)$) be a smooth helical vector field (scalar function), then there exists a* unique $\boldsymbol{w} = (w_1, w_2, w_3)(y_1, y_2)$ ($q = q(y_1, y_2)$) *such that*

$$\boldsymbol{v}(x) = \left(R_{x_3}(\boldsymbol{w})\right)(\boldsymbol{y}(\boldsymbol{x})), \quad p = p(x) = q(\boldsymbol{y}(\boldsymbol{x})), \tag{2.10}$$

with

$$\boldsymbol{y}(\boldsymbol{x}) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} \cos x_3 & -\sin x_3 \\ \sin x_3 & \cos x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.11}$$

Conversely, if $\boldsymbol{v}$ and p are defined through (2.10) for some $\boldsymbol{w} = \boldsymbol{w}(y_1, y_2)$ and $q = q(y_1, y_2)$, then $\boldsymbol{v}$ is a helical vector field and p is a helical scalar function.

We define a function

$$\eta =: \mathbf{u} \cdot \boldsymbol{\xi} = x_2 u_1 - x_1 u_2 + u_3, \tag{2.12}$$

called the *helical swirl* of the velocity field $\mathbf{u}$, similar to the azimuthal component *swirl* of three-dimensional axisymmetric flow ([3,4,13] and references therein). By recalling that the assumption of zero azimuthal component for the axisymmetric flows shows the vorticity stretching term vanishing in (1.2), we similarly introduce a geometric requirement that $\eta \equiv 0$, which means an orthogonality of the velocity fields to the symmetry lines of the group G^1. It is remarkable that this constraint of zero helical swirl has also been adaptable in [2,7] and [8].

For a helical vector $\boldsymbol{u}$, which is the solution of (1.1), the corresponding helical swirl η solves

$$\partial_t \eta + \mathbf{u} \cdot \nabla \eta = 0. \tag{2.13}$$

Evidently, quantity η is conserved from (2.13). Furthermore, under the constraint that $\eta = 0$, vorticity $\boldsymbol{\omega}$ of helical flow has some special form, which will be stated in the following.

Lemma 2.4. *(see [8]) Let* $\mathbf{u}$ *be a* C^2 *helical vector field with zero helical swirl, i.e.,* $\eta = 0$. *Denote* $\boldsymbol{\omega} = \nabla \times \mathbf{u} = (\omega_1, \omega_2, \omega_3)$, *the vorticity of* $\boldsymbol{u}$, *then*

$$\boldsymbol{\omega} = \omega_3 \boldsymbol{\xi}. \tag{2.14}$$

Remark 2.1. Under the helically symmetric assumption, the vorticity equations (1.2) become

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - \omega_3 \mathcal{R}\mathbf{u} = 0, \tag{2.15}$$

where

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.16}$$

In particular, the third component ω_3 of the vorticity satisfies the scalar transport equation

$$\partial_t \omega_3 + \mathbf{u} \cdot \nabla \omega_3 = 0. \tag{2.17}$$

Obviously, the vorticity stretching term in (2.17) vanishes.

Now we are ready to state the definition of weak solutions to the Euler equation, which will be helpful to introduce the main theorem.

Definition 2.3 *(Weak solutions to the Euler equations).* Given $T > 0$, a vector field $\mathbf{u}(t,x) \in L^\infty(0,T; L^2_{loc,per}(\mathbb{R}^2 \times [-\pi,\pi]))$ with initial data $\mathbf{u}_0 \in L^2_{loc,per}(\mathbb{R}^2 \times [-\pi,\pi])$ is a weak solution of Euler equations (1.1) provided that

(i) For any vector field $\mathbf{\Phi} = \mathbf{\Phi}(t,x) \in C_0^\infty([0,T); C^\infty_{0,per}(\mathbb{R}^2 \times [-\pi,\pi]))$ with $\nabla \cdot \mathbf{\Phi} = 0$,

$$\int_0^T \int_\Omega (\mathbf{u} \cdot \mathbf{\Phi}_t + \mathbf{u} \otimes \boldsymbol{u} : \nabla \mathbf{\Phi}) dx dt = \int_\Omega \mathbf{u}_0 \cdot \mathbf{\Phi}(0,x) dx. \tag{2.18}$$

(ii) For any $\phi \in C_0^\infty((0,T); C^\infty_{0,per}(\mathbb{R}^2 \times [-\pi,\pi]))$,

$$\int_0^T \int_\Omega \mathbf{u} \cdot \nabla \phi \, dx = 0. \tag{2.19}$$

Eventually, we are in the position to present the main result as follows.

Theorem 2.5. *Given a scalar helical function $\omega_0 \in L^1_{per}(\mathbb{R}^2 \times [-\pi,\pi]) \cap L^p_{per}(\mathbb{R}^2 \times [-\pi,\pi])$ with some $p > 1$, for any $T > 0$, there exist weak solutions $\mathbf{u} = \mathbf{u}(t,x) \in L^\infty([0,T]; W^{1,p}_{loc}(\mathbb{R}^2 \times [-\pi,\pi]))$ to the three-dimensional Euler equations (1.1) with helical symmetry with the initial vorticity $\boldsymbol{\omega_0} = \omega_0 \boldsymbol{\xi}$ in the sense of Definition 2.3. Moreover, $\mathbf{u} \cdot \boldsymbol{\xi} = 0$.*

3. Biot–Savart law

In this section, we intend to establish the explicit vorticity–velocity formula for the helical flow in $\mathbb{R}^2 \times [-\pi,\pi]$, which is usually called the Biot–Savart law. It is well known that in $\mathbb{R}^n$ $(n = 2,3)$, the velocity field $\mathbf{u}$ is represented by

$$\mathbf{u}(\boldsymbol{x}) = \int_{\mathbb{R}^n} K_n(\boldsymbol{x} - \boldsymbol{y})\boldsymbol{\omega}(\boldsymbol{y})dy, \quad \boldsymbol{x} \in \mathbb{R}^n (n = 2,3), \tag{3.1}$$

where the kernel $K_2(\boldsymbol{x}) = \frac{1}{2\pi}(-\frac{x_2}{|\boldsymbol{x}|^2}, \frac{x_1}{|\boldsymbol{x}|^2})$ and $K_3(\boldsymbol{x})\boldsymbol{h} = \frac{1}{4\pi}\frac{\boldsymbol{x} \times \boldsymbol{h}}{|\boldsymbol{x}|^3}$, where $\boldsymbol{h}$ is a three-dimensional vector. Furthermore, it holds that for $\boldsymbol{\omega} \in L^p(\mathbb{R}^n)$

$$\|\nabla \mathbf{u}\|_{L^p(\mathbb{R}^n)} \le C\|\boldsymbol{\omega}\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{3.2}$$

(3.1) and (3.2) are derived from the fact that the related div-curl problem can be converted into a second order elliptic equations via determined potential function with appropriate behavior at infinity [22].

When it comes to the helically symmetric case, the related kernel becomes more complicated to establish the vorticity–velocity estimate (3.2) in contrast to the whole space case. In addition, the loss of the L^p integrability of $\boldsymbol{\omega}$ makes it impossible to directly arrive at the estimate of (3.2) as a result of the special form that $\boldsymbol{\omega} = \omega_3 \boldsymbol{\xi}$ with $\omega_3 \in L^1 \cap L^p$. Other than the result of [2], we have the new observation about the properties of the G_1 (see (3.18) below) in this paper, which helps us to establish the L^p-estimate of the velocity and then obtain its related $W^{1,p}$-estimates (3.40) directly in virtue of the regularity theory of second order elliptic equations. Specifically, for the vorticity field $\boldsymbol{\omega} = \omega_3 \boldsymbol{\xi}$ with the helical function

$$\omega_3 \in L^p_{per}([-\pi,\pi]; L^p(\mathbb{R}^2)) \cap L^1_{per}([-\pi,\pi]; L^1(\mathbb{R}^2)),$$

we search for the corresponding helical velocity field $\mathbf{u}$ with $\mathbf{u} \cdot \boldsymbol{\xi} = 0$, satisfying

$$\begin{cases} \mathrm{curl}\mathbf{u} = \boldsymbol{\omega}, \ \mathrm{div}\mathbf{u} = 0, \\ |\mathbf{u}(x)| = o(|x'|), \ \text{as } |x'| \to \infty, \\ \mathbf{u}(x', x_3 + 2\pi) = \mathbf{u}(x', x_3), \end{cases} \tag{3.3}$$

where $x' = (x_1, x_2)$ and $o(\delta)$ means that $o(\delta)/\delta \to 0$ when $\delta \to \infty$. Here we remark that the behavior of $\mathbf{u}$ at infinity in (3.3) gets involved in the helical symmetry of $\boldsymbol{\omega}$, which is also a different point from the classical case in $\mathbb{R}^n$. For the classical case in $\mathbb{R}^n$, if $\boldsymbol{\omega} \in L^1 \cap L^p$ $(1 < p < \infty)$, $\mathbf{u}$ converges to zero at infinity, and then the div-curl equation similar to (3.3) has a unique solution. In the helically symmetric case the vorticity has the special form of $\boldsymbol{\omega} = \omega_3 \boldsymbol{\xi}$. Then when $\omega_3 \in L^p$ $(p > 1)$ with compact support, the velocity field $\mathbf{u}$ decays like $1/|x'|$ when

$|x'| \to \infty$ (see Remark 3.3). However, when $\omega_3 \in L^1 \cap L^p$ without compact support and due to the special expressions (3.41) of $\mathbf{u}$ and $\nabla\Psi$ (see (3.33) for example), the behavior of $\mathbf{u}$ in (3.3) should be $o(|x'|)$ as $|x'| \to \infty$ in our paper (see Lemma 3.3 for more details).

To solve (3.3), we first investigate an auxiliary problem for a given function f as

$$\begin{cases} -\Delta\psi = f, \\ |\nabla\psi| \to 0, \ \text{as } |x'| \to \infty, \\ \psi(x', x_3 + 2\pi) = \psi(x). \end{cases} \tag{3.4}$$

Based on the theories of Fourier series and elliptic equations, we have the following lemma.

Lemma 3.1. *Given $f \in L^1_{per}([-\pi,\pi]; L^1(\mathbb{R}^2)) \cap L^p_{per}([-\pi,\pi]; L^p(\mathbb{R}^2))$ for any $p > 1$, the system (3.4) has a unique solution ψ (up to a constant), with the form*

$$\begin{aligned} \psi &= \psi^0 + \psi^1 \\ &=: -\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \ln|x'-y'| f(y', y_3)dy'dy_3 \\ &\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} K_0(n|x'-y'|)\cos(n(x_3-y_3)) f(y', y_3)dy'dy_3, \end{aligned} \tag{3.5}$$

where $K_0(z) = \int_0^\infty \frac{\cos(zt)}{\sqrt{1+t^2}}dt\,(z>0)$, the modified Bessel function of the second kind. Moreover, for $1 < r \le p$

$$\begin{aligned} &\|\nabla^2\psi\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))} \lesssim \|f\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))}, \\ &\|\nabla\psi\|_{L^p_{per}([-\pi,\pi];L^q(\mathbb{R}^2))} \lesssim \|f\|_{L^1_{per}([-\pi,\pi];L^1(\mathbb{R}^2))} + \|f\|_{L^p_{per}([-\pi,\pi];L^p(\mathbb{R}^2))}, \end{aligned} \tag{3.6}$$

where $q = \begin{cases} \frac{2p}{2-p} & 1 < p < 2, \\ \frac{3p}{3-p} & 2 \le p < 3, \\ p+3, & p \ge 3. \end{cases}$ Furthermore, here and in the following, "$\lesssim$" means less and equal C (a constant may dependent on p or q) times.

Proof. The proof is divided into three steps. Without loss of generality, we assume $f \in C^\infty_{c,per}(\mathbb{R}^2 \times [-\pi,\pi])$. For the general case, one can treat by the standard density argument.

Step 1. Existence of ψ.

First, we employ the Fourier series extension on the function f with respect to x_3-variable

$$f(x) = f_0(x') + \sum_{n=1}^{\infty}\left(f_n^1(x')\cos(nx_3) + f_n^2(x')\sin(nx_3)\right), \tag{3.7}$$

where

$$\left(f_0, f_n^1, f_n^2\right)(x') = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x', x_3)(1/2, \cos(nx_3), \sin(nx_3))dx_3. \tag{3.8}$$

Similarly, ψ could be extended as

$$\psi = \psi^0 + \psi^1 =: \psi^0(x') + \sum_{n=1}^{\infty}\left(\psi_n^1(x')\cos(nx_3) + \psi_n^2(x')\sin(nx_3)\right), \tag{3.9}$$

where ψ^0 and ψ_n^i $(i = 1, 2)$ satisfy

$$\begin{cases} -\Delta_{x'}\psi^0 = f_0(x'), \\ \nabla_{x'}\psi^0 \to 0 \text{ as } |x'| \to \infty, \end{cases} \tag{3.10}$$

and

$$\begin{cases} -\Delta_{x'}\psi_n^i + n^2\psi_n^i = f_n^i(x'), \\ \nabla_{x'}\psi_n^i \to 0 \text{ as } |x'| \to \infty, \end{cases} \tag{3.11}$$

respectively.

Up to a constant, system (3.10) has a unique solution

$$\psi^0(x') = -\int_{\mathbb{R}^2} G_0(x' - y')f_0(y')dy', \tag{3.12}$$

with $G_0(x') = \dfrac{1}{2\pi}\ln|x'|$, where $G_0(x')$ is the fundamental solution of the two-dimensional Laplacian equation.

Applying the Fourier transformation to x'-variable in (3.11), we yield

$$\widehat{\psi_n^i}(\xi) = \frac{\widehat{f_n^i}(\xi)}{n^2 + |\xi|^2}. \tag{3.13}$$

Then the inverse transformation gives that

$$\psi_n^i(x') = K_0(n|x'|) * f_n^i(x') \ (n \geq 1,\ i = 1, 2) \tag{3.14}$$

with $K_0(z) = \displaystyle\int_0^{\infty} \frac{\cos(z\tau)}{\sqrt{1+\tau^2}}d\tau$ $(z \geq 0)$, the modified Bessel function of the second kind. Substituting the expression (3.8) into (3.14), one has

$$\psi_n^1(x') = \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} K_0(n|x'-y'|)\cos(ny_3) f(y',y_3)dy, \tag{3.15}$$

and

$$\psi_n^2(x') = \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} K_0(n|x'-y'|)\sin(ny_3) f(y',y_3)dy. \tag{3.16}$$

Then taking (3.15) and (3.16) back to (3.9), we arrive that

$$\psi^1(x) = \int_{-\pi}^{\pi}\int_{\mathbb{R}^2} G_1(x-y) f(y)dy, \tag{3.17}$$

with

$$G_1(x) = \frac{1}{\pi}\sum_{n=1}^{+\infty} K_0(n|x'|)\cos(nx_3). \tag{3.18}$$

Therefore, (3.5) is verified in view of (3.12) and (3.17).

Step 2. Estimates of ψ^0.

For ψ^0 defined in (3.12), utilizing Lemma B.1 in Appendix B, we have,

$$\|\nabla^2\psi^0\|_{L^r(\mathbb{R}^2)} \lesssim \|f_0\|_{L^r(\mathbb{R}^2)}, \quad \|\nabla\psi^0\|_{L^q(\mathbb{R}^2)} \lesssim \sum_{l=1,p} \|f_0\|_{L^l(\mathbb{R}^2)}. \tag{3.19}$$

Noting that $f_0(x') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x',y_3)dy_3$ and ψ^0 is independent of x_3, it holds from (3.19) that

$$\begin{aligned} &\|\nabla^2\psi^0\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))} \lesssim \|f\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))}, \\ &\|\nabla\psi^0\|_{L^p_{per}([-\pi,\pi];L^q(\mathbb{R}^2))} \lesssim \sum_{l=1,p} \|f\|_{L^l_{per}([-\pi,\pi];L^l(\mathbb{R}^2))}, \end{aligned} \tag{3.20}$$

where r and p are given as the assumption of the Lemma.

Step 3. Estimates of ψ^1.

By (3.17), the generalized Young inequality for $1 < r \le p$ gives that

$$\|\psi^1\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))} \lesssim \|G_1\|_{L^1_{per}([-\pi,\pi];L^1(\mathbb{R}^2))} \|f\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))}. \tag{3.21}$$

It follows from (P1)–(P2) in Appendix A that

$$\|G_1\|_{L^1_{per}([-\pi,\pi];L^1(\mathbb{R}^2))} \lesssim 1. \tag{3.22}$$

Combining (3.21) with (3.22), we get

$$\|\psi^1\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))} \lesssim \|f\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))}. \tag{3.23}$$

Since ψ^1 satisfies the equation

$$-\Delta\psi^1 = f - f_0, \tag{3.24}$$

according to the interior estimate technique for second order elliptic equation (Theorem 9.11 of [10]), together with (3.23) and the technique of the regional coverage of domain (for the details, see Remark 3.2), we obtain for all $1 < r \leq p$

$$\begin{aligned} \|\psi^1\|_{W^{2,r}([-\pi,\pi]\times\mathbb{R}^2)} &\lesssim \|\psi^1\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))} + \|f - f_0\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))} \\ &\lesssim \|f\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))}. \end{aligned} \tag{3.25}$$

With the help of Sobolev inequality (see Proposition 10.7 in [22]), (3.25) implies that

$$\begin{aligned} &\|\nabla\psi^1\|_{L^p_{per}([-\pi,\pi];L^q(\mathbb{R}^2))} \\ &\quad \lesssim \|f\|_{L^1_{per}([-\pi,\pi];L^1(\mathbb{R}^2))} + \|f\|_{L^p_{per}([-\pi,\pi];L^p(\mathbb{R}^2))}. \end{aligned} \tag{3.26}$$

Then estimate (3.6) is available together with (3.5), (3.20), (3.25) and (3.26). □

Remark 3.1. For $f \in L^p_{c,per}(\mathbb{R}^2 \times [-\pi,\pi])$, ψ defined in (3.5) occupies the decay estimates

$$|\nabla_{x'}\psi| \leq \frac{C}{|x'|}, \quad |\nabla^2_{x'}\psi| \leq \frac{C}{|x'|^2} \quad \text{as} \quad |x'| \to \infty, \tag{3.27}$$

where the constant C depends on the compact support of f. In fact, (3.27) follows from that

$$\begin{aligned} &|\nabla_{x'}G_0(x')| \lesssim \frac{1}{|x'|}, \quad |\nabla_x G_1(x)| \lesssim \frac{1}{|x'|^2} \quad \text{as} \quad |x'| \to \infty, \\ &|\nabla^2_{x'}G_0(x')| \lesssim \frac{1}{|x'|^2}, \quad |\nabla^2_x G_1(x)| \lesssim \frac{1}{|x'|^3} \quad \text{as} \quad |x'| \to \infty, \end{aligned} \tag{3.28}$$

where the estimates of G_0 and G_1 come from (3.12) and the rigorous equation (A.4) and (A.7), respectively.

Remark 3.2. (The technique of the regional coverage of domain.) For integers i, j and a parameter $\lambda > 0$, defining

$$\Omega^\lambda_{ij} = [i - \lambda, i + \lambda] \times [j - \lambda, j + \lambda],$$

using Theorem 9.11 of [10] to the equation (3.24), it yields

$$\begin{aligned}\|\psi^1\|_{W^{2,r}([-\pi,\pi]\times\Omega_{ij}^{1/2})} &\lesssim \|\psi^1\|_{L^r([-2\pi,2\pi];L^r(\Omega_{ij}^1))} + \|f-f_0\|_{L^r([-2\pi,2\pi];L^r(\Omega_{ij}^1))} \\ &\lesssim \|\psi^1\|_{L^r_{per}([\pi,\pi];L^r(\Omega_{ij}^1))} + \|f-f_0\|_{L^r([-\pi,\pi];L^r(\Omega_{ij}^1))}.\end{aligned} \tag{3.29}$$

Summing up i, j from $-\infty$ to ∞ in (3.29) obtains (3.25).

Now, we turn back to consider the related system to (3.4) with $\boldsymbol{\omega} = \omega_3\boldsymbol{\xi}$

$$\begin{cases} -\Delta\Psi = \boldsymbol{\omega}, \\ |\nabla\Psi| = o(|x'|), \text{ as } |x'| \to \infty, \\ \Psi(x', x_3 + 2\pi) = \Psi(x). \end{cases} \tag{3.30}$$

From the analysis of Lemma 3.1, we deduce the following conclusions.

Lemma 3.2. *For $\boldsymbol{\omega} = \omega_3\boldsymbol{\xi}$ with $\omega_3 \in L^1_{per}([-\pi,\pi]; L^1(\mathbb{R}^2)) \cap L^p_{per}([-\pi,\pi]; L^p(\mathbb{R}^2))$ for any $p > 1$, system (3.30) has a solution $\Psi = (\psi_1, \psi_2, \psi_3)$ such that for any $M > 0$*

$$\begin{aligned}&\|\nabla^2\Psi\|_{L^r_{per}([-\pi,\pi];L^r(B_M(0))))} + \|\nabla\Psi\|_{L^p_{per}([-\pi,\pi];L^q(B_M(0))))} \\ &\quad\le C(M)\sum_{l=1,p}\|\omega_3\|_{L^l_{per}([-\pi,\pi];L^l(\mathbb{R}^2))},\end{aligned} \tag{3.31}$$

where r and q defined in Lemma 3.1.

Proof. Based on the idea of Lemma 3.1, we similarly define $\Psi = (\psi_1, \psi_2, \psi_3)$ as

$$\Psi =: \Psi^0 + \Psi^1 = \int_{-\pi}^{\pi} G_0(x') * (\boldsymbol{\xi}\omega_3)(x', y_3)dy_3 + G_1(x) * (\boldsymbol{\xi}\omega_3)(x). \tag{3.32}$$

Noticing that $\boldsymbol{\xi} = (x_2, -x_1, 1)^t$, ψ_3 satisfies the similar system to (3.4) and the corresponding estimate of (3.6), which therefore meets (3.31). Next, we focus on the desired estimates of ψ_1 and ψ_2. In fact, we will only discuss with ψ_1 for convenience since ψ_2 can be treated in the same way.

Similar to Lemma 3.1, we decompose ψ_1 into two parts ψ_1^0 and ψ_1^1, i.e., $\psi_1 = \psi_1^0 + \psi_1^1$. Then direct calculations to ψ_1^0 leads to

$$\begin{aligned}\nabla_{x'}\psi_1^0 &= \int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \nabla_{x'}G_0(x'-y')(y_2\omega_3)(y', y_3)dy'dy_3 \\ &= \int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \nabla_{x'}G_0(x'-y')(y_2 - x_2)\omega_3(y', y_3)dy'dy_3 \\ &\quad + x_2\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \nabla_{x'}G_0(x'-y')\omega_3(y', y_3)dy'dy_3 \\ &= J_1 + x_2J_2.\end{aligned} \tag{3.33}$$

It follows from the definition of J_1 in (3.33) that

$$|\nabla_{x'} J_1(x')| \lesssim \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \frac{1}{|x'-y'|} |\omega_3(y', y_3)| dy' dy_3.$$

Combing this with the proof process of (B.2) in Lemma B.1 that

$$\|\nabla_{x'} J_1(x')\|_{L^q(\mathbb{R}^2)} \lesssim \sum_{l=1,p} \|\omega_3\|_{L^l_{per}([-\pi,\pi]\times\mathbb{R}^2)}. \tag{3.34}$$

The estimate of J_2 is very similar to that of $\nabla_{x'}\psi^0$ in (3.12). Therefore,

$$\begin{aligned} \|\nabla J_2\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))} &\lesssim \|\omega_3\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))}, \\ \|J_2\|_{L^p_{per}([-\pi,\pi];L^q(\mathbb{R}^2))} &\lesssim \sum_{l=1,p} \|\omega_3\|_{L^l_{per}([-\pi,\pi];L^l(\mathbb{R}^2))}. \end{aligned} \tag{3.35}$$

Then it follows from (3.33)–(3.35) that[1]

$$\|\nabla^2\psi_1^0\|_{L^r_{per}([-\pi,\pi];L^r(B_M(0)))} \le C(M) \sum_{l=1,p} \|\omega_3\|_{L^l_{per}([-\pi,\pi];L^l(\mathbb{R}^2))}. \tag{3.36}$$

As for ψ_1^1, it can be rewritten as

$$\begin{aligned} \nabla_{x'}\psi_1^1 &= \int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \nabla_{x'} G_1(x-y)(y_2-x_2)\omega_3(y)dy \\ &\quad + x_2 \int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \nabla_{x'} G_1(x-y)\omega_3(y)dy \\ &= J_3 + x_2 J_4. \end{aligned} \tag{3.37}$$

Due to (P3) in Lemma A.2, we know that $\nabla_{x'} G_1(x)x_2 \in W^{1,1}([-\pi,\pi]\times\mathbb{R}^2)$ with

$$\|J_3\|_{W^{1,r}(\mathbb{R}^2\times[-\pi,\pi])} \le C \sum_{l=1,p} \|\omega_3\|_{L^l_{per}([-\pi,\pi];L^l(\mathbb{R}^2))}. \tag{3.38}$$

Similar to estimate (3.25), we obtain

---

[1] For the readers' interest, we point out that based on (3.30) and (3.33), one can also use L^p theory of second order elliptic equation to obtain this estimate (3.36) (see Theorem 14.1$'$ and the note below it in Page 700 of [1]).

$$\begin{aligned}\|J_4\|_{W^{1,r}(\mathbb{R}^2\times[-\pi,\pi])} &\leq C\|\omega_3\|_{L^r_{per}([-\pi,\pi];L^r(\mathbb{R}^2))} \\ &\leq C\sum_{l=1,p}\|\omega_3\|_{L^l_{per}([-\pi,\pi];L^l(\mathbb{R}^2))}. \end{aligned} \tag{3.39}$$

Taking (3.34), (3.35) into (3.33), and (3.38), (3.39) into (3.37), ψ_1 satisfies (3.31). Then we complete the proof of Lemma 3.2. □

Next we will state the Biot–Savart law with helical symmetry in $\mathbb{R}^2\times[-\pi,\pi]$. The main idea of the proof is also adaptable to the general case.

Lemma 3.3. (The Biot–Savart law) *Under the assumptions of Lemma 3.2 and $\boldsymbol{\omega}=\omega_3\boldsymbol{\xi}$ with ω_3 being helical and $\boldsymbol{\xi}(x)=(x_2,-x_1,1)$, the system (3.3) has a unique solution $\mathbf{u}$, which is a helical vector fields without helical swirl. Moreover, for any $M>0$*

$$\begin{aligned}&\|\nabla\mathbf{u}\|_{L^p_{per}([-\pi,\pi];L^p(B_M(0)))}+\|\mathbf{u}\|_{L^p_{per}([-\pi,\pi];L^p(B_M(0)))} \\ &\leq C(M)\sum_{l=1,p}\|\omega_3\|_{L^l_{per}([-\pi,\pi];L^l(\mathbb{R}^2))}. \end{aligned} \tag{3.40}$$

Proof. The proof can be divided into the following steps.

Step 1. Existence of $\boldsymbol{u}$.

As shown in Lemma 3.2, for given $\boldsymbol{\omega}$, there exists the $\boldsymbol{\Psi}$ satisfying (3.30). Define

$$\mathbf{u}=:\nabla\times\boldsymbol{\Psi}+\left(0,0,-\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\omega_3(y)dy\right). \tag{3.41}$$

Then we easily verify that $\mathbf{u}$ is a solution of (3.3) and satisfies estimate (3.40). It should be mentioned that the second constant vector defined in (3.41) ensures the cancellation of the terms related to G_0 kernel in $\boldsymbol{u}\cdot\boldsymbol{\xi}$.

In particular, $\boldsymbol{u}=(u_1,u_2,u_3)$ in (3.41) has an explicit expression as

$$\begin{aligned}u_1=&\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\frac{x_2-y_2}{|x'-y'|^2}\omega_3(y)dy \\ &+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\frac{n(x_2-y_2)}{|x'-y'|}K_0'(n|x'-y'|)\cos(n(x_3-y_3))\omega_3(y)dy \\ &-\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}nK_0(n|x'-y'|)\sin(n(x_3-y_3))y_1\omega_3(y)dy,\end{aligned} \tag{3.42}$$

$$
\begin{aligned}
u_2 = &-\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\frac{x_1-y_1}{|x'-y'|^2}\omega_3(y)dy \\
&-\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\frac{n(x_1-y_1)}{|x'-y'|}K_0'(n|x'-y'|)\cos(n(x_3-y_3))\omega_3(y)dy \\
&-\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}nK_0(n|x'-y'|)\sin(n(x_3-y_3))y_2\omega_3(y)dy,
\end{aligned}
\tag{3.43}
$$

and

$$
\begin{aligned}
u_3 = &-\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\frac{(x_1-y_1)x_1+(x_2-y_2)x_2}{|x'-y'|^2}\omega_3(y)dy \\
&-\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\frac{n(x_1-y_1)}{|x'-y'|}K_0'(n|x'-y'|)\cos(n(x_3-y_3))y_1\omega_3(y)dy \\
&-\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\frac{n(x_2-y_2)}{|x'-y'|}K_0'(n|x'-y'|)\cos(n(x_3-y_3))y_2\omega_3(y)dy.
\end{aligned}
\tag{3.44}
$$

Step 2. Vanishing helical swirl of u.

In this step, we will focus on the property that $\mathbf{u}\cdot\boldsymbol{\xi}=0$. First, a direct verification tells that

$$
-\Delta(\boldsymbol{\xi}\cdot\mathbf{u}) = -\Delta\mathbf{u}\cdot\boldsymbol{\xi} - 2\nabla u_i\cdot\nabla\boldsymbol{\xi}_i = curl\boldsymbol{\omega}\cdot\boldsymbol{\xi} - 2\omega_3 \equiv 0. \tag{3.45}
$$

Furthermore, it comes from (3.42)–(3.44) that

$$
\begin{aligned}
&\pi(\boldsymbol{\xi}\cdot\mathbf{u}) \\
&= \int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\sum_{n=1}^{\infty}nK_0'(n|x'-y'|)|x'-y'|\cos(n(x_3-y_3))\omega_3(y)dy \\
&+ x_1\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\sum_{n=1}^{\infty}nK_0(n|x'-y'|)(y_2-x_2)\sin(n(x_3-y_3))\omega_3(y)dy \\
&- x_2\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\sum_{n=1}^{\infty}nK_0(n|x'-y'|)(y_1-x_1)\sin(n(x_3-y_3))\omega_3(y)dy \\
&=: J_5 + x_1J_6 + x_2J_7.
\end{aligned}
\tag{3.46}
$$

Due to (A.11) in Appendix A, we have

$$\sum_{i=5}^{7} \|J_i\|_{L^p_{per}([-\pi,\pi];L^p(\mathbb{R}^2))} \lesssim \|\omega_3\|_{L^p_{per}([-\pi,\pi];L^p(\mathbb{R}^2))}. \tag{3.47}$$

Then according to L^p interior estimate theory of elliptic equation (see Theorem 9.19 of [10]) and Sobolev inequality, it is shown from (3.45)–(3.47) that $\eta = \boldsymbol{\xi} \cdot \mathbf{u}$ satisfies

$$\begin{aligned}
\|\eta\|_{L^\infty(B_1(x')\times[-\pi,\pi])} &\lesssim \|\eta\|_{W^{[3/p]+1,p}(B_1(x')\times[-\pi,\pi])} \\
&\lesssim \|\eta\|_{L^p_{per}(B_2(x')\times[-\pi,\pi])} \\
&\lesssim (1+|x'|)\sum_{i=5}^{7} \|J_i\|_{L^p_{per}(B_2(x')\times[-\pi,\pi])}.
\end{aligned}$$

Combing this with (3.47) implies that $\eta = o(|x'|)$ as $|x'| \to \infty$ and then it follows from (3.45) and (3.47) that η satisfies

$$\begin{cases} -\Delta \eta = 0, \\ \eta = o(|x'|), \ \text{as } |x'| \to \infty, \\ \eta \in L^p_{per}([-\pi,\pi]; L^p_{loc}(\mathbb{R}^2)). \end{cases} \tag{3.48}$$

By the help of Lemma A.1, we have the conclusion that η is a constant in (3.48). Then integrating both sides of (3.46) on $Q = [-1,1]^2 \times [-\pi,\pi]$, we derive that

$$\int_Q \eta dx = 0, \tag{3.49}$$

which implies that $\eta = 0$, i.e., the velocity field $\boldsymbol{u}$ given in (3.41) has zero helical swirl.

Step 3. Helical property of u.

In this step, we say that $\mathbf{u}$ in (3.41) is helical. According to Definition 2.2, for any $\theta \in \mathbb{R}$, it is enough to verify that

$$\begin{cases} u_1(S_\theta(x)) = u_1(x)\cos\theta + u_2(x)\sin\theta, \\ u_2(S_\theta(x)) = -u_1(x)\sin\theta + u_2(x)\cos\theta, \\ u_3(S_\theta(x)) = u_3(x). \end{cases} \tag{3.50}$$

The verification of (3.50) is very standard, therefore we only state the proof that u_1 is helical for the convenience.

For any $\theta \in \mathbb{R}$, the following fact holds

$$\begin{cases} \partial_2 G_i(x-y)\cos\theta - \partial_1 G_i(x-y)\sin\theta \\ \qquad\qquad = \partial_2 G_i(S_\theta(x) - S_\theta(y))\ (i=0,1), \\ \partial_3 G_1(x-y) = \partial_3 G_1(S_\theta(x) - S_\theta(y)). \end{cases} \tag{3.51}$$

By mean of (3.51) and the property that ω_3 is a helical function, a direct computation shows

$$\begin{aligned} & u_1(x)\cos\theta + u_2(x)\sin\theta \\ &= \sum_{i=0}^{1} (\partial_2 G_i(x)\cos\theta - \partial_1 G_i(x)\sin\theta) * \omega_3 + \partial_3 \left(G_1(x) * ((x_1\cos\theta + x_2\sin\theta)\omega_3(x))\right) \\ &= \sum_{i=0}^{1} \int_{\mathbb{R}^2\times[-\pi,\pi]} \partial_2 G_i(S_\theta(x) - S_\theta(y))\omega_3(S_\theta(y))dy \\ &\quad + \int_{\mathbb{R}^2\times[-\pi,\pi]} \partial_3 G_1(S_\theta(x) - S_\theta(y))(y_1\cos\theta + y_2\sin\theta)\omega_3(S_\theta(y))dy \\ &= \sum_{i=0}^{1} \int_{\mathbb{R}^2\times[-\pi,\pi]} \partial_2 G_i(S_\theta(x) - y)\omega_3(y)dy + \int_{\mathbb{R}^2\times[-\pi,\pi]} \partial_3 G_1(S_\theta(x) - y)y_1\omega_3(y)dy \\ &= u_1(S_\theta(x)), \end{aligned} \tag{3.52}$$

which ensure that $\mathbf{u}$ given in (3.41) (or see (3.42)–(3.44) for its explicit form) is a helical vector field.

Step 4. Uniqueness of u.

Based on Step 1–Step 3, the uniqueness of $\mathbf{u}$ is reduced to prove the following system

$$\begin{cases} -\Delta \mathbf{u} = 0,\ \boldsymbol{\xi}\cdot\boldsymbol{u} = 0, \\ |\mathbf{u}| = o(|x'|),\ \text{as } |x'| \to \infty, \\ \mathbf{u}(x', x_3 + 2\pi) = \mathbf{u}(x), \end{cases} \tag{3.53}$$

only has a trivial solution in $L^p_{per}([-\pi,\pi]; L^p_{loc}(\mathbb{R}^2))$. Due to the conclusion of Lemma A.1, we obtain that solution $\boldsymbol{u}$ of system (3.53) is a constant, i.e.,

$$\mathbf{u} = (c_1, c_2, c_3)$$

for some constant c_i $(i = 1, 2, 3)$. Moreover, the property of zero helical swirl of $\mathbf{u}$ tells us that

$$\boldsymbol{\xi}\cdot\mathbf{u} = c_1 x_2 - c_2 x_1 + c_3 = 0,\ x' \in \mathbb{R}^2,$$

which furthermore infers that $c_i = 0$ $(i = 1, 2, 3)$. Therefore, $\mathbf{u} \equiv 0$, which ends the proof of Lemma 3.3. □

Remark 3.3. Under the assumption that ω_3 is compactly supported, the velocity field $\mathbf{u}$ constructed in (3.41) has the decay property at infinity from Remark 3.1

$$|\mathbf{u}| \le \frac{C}{|x'|}, \quad |\nabla_{x'}\mathbf{u}| \le \frac{C}{|x'|^2} \text{ as } |x'| \to \infty.$$

4. Proof of main theorem

In this section, we will state the proof of Theorem 2.5, which is divided into Proposition 4.1 and Proposition 4.2. Our strategy begins with designing an approximate helical solutions of (1.1) without helical swirl for which we can easily prove a global existence of weak solutions and also an analogous energy estimate that is independent of the regularization parameter, which strongly depend on the two-dimensional property of helical flow and the Biot–Savart law with helical symmetry.

Before the construction of approximate solutions, we recall some notations about mollifier. Given any radial function $\rho_1(|x'|) \in C_c^\infty(\mathbb{R}^2)$ satisfying that $\rho_1 \ge 0$ and $\int_{\mathbb{R}^2} \rho_1(x')dx' = 1$, and a nonnegative periodic smooth function $\rho_2(x_3)$ in $[-\pi, \pi]$ with $\int_{-\pi}^{\pi} \rho_2(x_3)dx = 1$, we define $\omega_0^\epsilon = \rho^\epsilon * \omega_0$, where $\rho^\epsilon(x) = \epsilon^{-3}\rho(\frac{x}{\epsilon})$ and $\rho(x) = \rho_1(x')\rho_2(x_3)$. According to Definition 2.1, ω_0^ϵ is a helical function in terms of the fact that ω_0 is a helical function. Furthermore, $\mathbf{u}_\mathbf{0}^\epsilon$, recovered from its vorticity $\boldsymbol{\omega}_\mathbf{0}^\epsilon =: \omega_0^\epsilon \boldsymbol{\xi}$ by Biot–Savart law is a helical velocity fields without helical swirl. Then we correspondingly construct the approximate solutions of (1.1) with initial data $\mathbf{u}_\mathbf{0}^\epsilon$ as follows.

Proposition 4.1. *Let $\omega_0 \in L^1_{per}([-\pi, \pi] \times \mathbb{R}^2) \cap L^p_{per}([-\pi, \pi] \times \mathbb{R}^2)$ for some $p > 1$ and $\omega_0^\epsilon = \rho^\epsilon * \omega_0$. Then for any $T > 0$ there exists a smooth helical solution $\mathbf{u}^\epsilon \in L^\infty(0, T; W^{1,p}_{loc,per}(\mathbb{R}^2 \times [-\pi, \pi]))$ of (1.1) without helical swirl when initial data $\mathbf{u}_\mathbf{0}^\epsilon \in W^{1,p}_{loc,per}(\mathbb{R}^2 \times [-\pi, \pi])$, which is mentioned above.*

Proof of Proposition 4.1. For convenience, we always assume that ω_0^ϵ has the compact support about x' variable. Otherwise, we introduce a cut-off function χ defined on a ball in $\mathbb{R}^2$. Then we redefine $\omega_0^\epsilon = \rho^\epsilon * (\chi\omega_0)$ with compact support.

For each $\epsilon > 0$, we define $\mathbf{u}_\mathbf{0}^\epsilon = \nabla \times (G * (\boldsymbol{\xi}\omega_0^\epsilon))$. Then the helical vector $\mathbf{u}_0^\epsilon \in C_c^\infty(\mathbb{R}^2 \times [-\pi, \pi])$ and $\mathbf{u}_0^\epsilon \cdot \boldsymbol{\xi} = 0$ in terms of the Biot–Savart law. It is mentioned that the idea to construct the approximate solutions $\mathbf{u}^\epsilon \in C^\infty(\mathbb{R}^2 \times [-\pi, \pi])$ with initial data $\mathbf{u}_\mathbf{0}^\epsilon$ is very routine, which has been explicitly stated in [22] and Theorem 3.11 of [2]. We skip it here for simplicity. In addition, it holds that the initial data $\mathbf{u}_\mathbf{0}^\epsilon \in W^{1,p}_{loc,per}(\mathbb{R}^2 \times [-\pi, \pi])$ with the energy estimates for any $M > 0$

$$\|\mathbf{u}_\mathbf{0}^\epsilon\|_{W^{1,p}(B_M(0)\times[-\pi,\pi])} \le C(M)\|\omega_0\|_{L^p(\mathbb{R}^2\times[-\pi,\pi])},$$

where C is a constant independent of ϵ, due to the Biot–Savart law in Lemma 3.3. Correspondingly, we derive that $\mathbf{u}^\epsilon \in L^\infty(0, T; W^{1,p}_{loc,per}(\mathbb{R}^2 \times [-\pi, \pi])$ for any $T > 0$. Recalling that $\mathbf{u}_\mathbf{0}^\epsilon$ is a helical vector field, we obviously reduce that $\mathbf{u}^\epsilon$ is also a helical vector due to the invariance of Euler equations under the rotation and translation transformations and the uniqueness of solutions. In addition, we easily verify that $\mathbf{u}^\epsilon \cdot \boldsymbol{\xi} = 0$ according to the facts that $\mathbf{u}_0^\epsilon \cdot \boldsymbol{\xi} = 0$ and that η is conserved from (2.13). □

Now we are ready to prove Theorem 2.5. It remains to prove the L^2 strong convergence of the approximate sequence $\{\mathbf{u}^\epsilon\}$ constructed in Proposition 4.1, which is the key point to prove the existence of weak solutions in the sense of Definition 2.3. We invoke the two-dimensional property of $\mathbf{u}^\epsilon$, that solutions can be transformed to two-dimensional vector fields $\boldsymbol{w}^\epsilon$ with three components according to Lemma 2.3. That is, we have

Proposition 4.2. *Let $B_R \times [-\pi, \pi] = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \sqrt{x_1^2 + x_2^2} \le R, -\pi \le x_3 \le \pi\}$ for any $R > 0$. Given the helical approximate-solution sequence $\mathbf{u}^\epsilon$, stated in Proposition 4.1, there exists a limit $\mathbf{u}$ of a subsequence of $\mathbf{u}^\epsilon$ (still denote by $\mathbf{u}^\epsilon$ for simplicity), which is a weak solution for the Euler equations with helical symmetry in the sense of Definition 2.3, such that*

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^\infty(0,T;L^2(B_R\times[-\pi,\pi]))} \to 0 \text{ as } \epsilon \to 0, \tag{4.1}$$

with

$$\mathbf{u} \cdot \boldsymbol{\xi} = 0, \quad \operatorname{div}\mathbf{u} = 0. \tag{4.2}$$

Proof of Proposition 4.2. By the statement of Proposition 4.1, it holds that $\mathbf{u}^\epsilon \in L^\infty(0, T; W^{1,p}_{loc}(\mathbb{R}^2 \times [-\pi, \pi]))$. Utilizing the equation of $\mathbf{u}^\epsilon$, it holds that $\partial_t \mathbf{u}^\epsilon \in L^\infty(0, T; H^{-1}_{loc}(\mathbb{R}^2 \times [-\pi, \pi]))$. Based on the two-dimensional property of Lemma 2.3, there exists the corresponding vector fields $\boldsymbol{w}^\epsilon \in L^\infty(0, T; W^{1,p}_{loc}(\mathbb{R}^2))$ by virtue of (2.10). Then direct computations indicate that each component of $\nabla_y \boldsymbol{w}^\epsilon$ is a composition of the components of $\partial_{x_1}\mathbf{u}^\epsilon$, $\partial_{x_2}\mathbf{u}^\epsilon$ and the trigonometric functions about x_3 according to the formulae of (2.10) and (2.11). Without loss of generality, we take the expression of $\partial_{y_1} w_1^\epsilon$ for instance, i.e.,

$$\partial_{y_1} w_1^\epsilon = \partial_{x_1} u_1^\epsilon \cos^2 x_3 - \partial_{x_1} u_2^\epsilon \cos x_3 \sin x_3 - \partial_{x_2} u_1^\epsilon \cos x_3 \sin x_3 + \partial_{x_2} u_2^\epsilon \sin^2 x_3.$$

Therefore, it easily deduced that for $R > 0$,

$$\|\nabla_y \boldsymbol{w}^\epsilon\|_{L^\infty(0,T;L^p(B_R))} \le C\|\nabla_{x'}\mathbf{u}^\epsilon\|_{L^\infty(0,T;L^p(B_R\times[-\pi,\pi]))},$$

where C is a generic constant and $\nabla_{x'} = (\partial_{x_1}, \partial_{x_2})^t$. Moreover,

$$\|\boldsymbol{w}^\epsilon\|_{L^\infty(0,T;L^p(B_R))} \le C\|\mathbf{u}^\epsilon\|_{L^\infty(0,T;L^p(B_R\times[-\pi,\pi]))}.$$

That is, $\{\boldsymbol{w}^\epsilon\}$ is uniformly bounded in $L^\infty([0, T]; W^{1,p}(B_R))$. Similarly, $\partial_t \boldsymbol{w}^\epsilon$ is also uniformly bounded in $L^\infty(0, T; H^{-1}(B_R))$. By the Aubin–Lions compactness theorem and the two-dimensional compact embedding $W^{1,p}(B_R) \hookrightarrow\hookrightarrow L^2(B_R)$ for $p > 1$, there exists a limit $\boldsymbol{w} \in L^2(0, T; L^2(B_R))$ with $\|\boldsymbol{w}^\epsilon - \boldsymbol{w}\|_{L^2(0,T;L^2(B_R))} \to 0$ as $\epsilon \to 0$. Again, by Lemma 2.3, we recover a helical vector $\mathbf{u}$ from $\boldsymbol{w}$ by virtue of (2.10). Especially, direct calculations tell us that

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^2(0,T;L^2(B_R\times(-\pi,\pi)))} \le C\|\boldsymbol{w}^\epsilon - \boldsymbol{w}\|_{L^2(0,T;L^2(B_R))} \to 0 \tag{4.3}$$

as $\epsilon \to 0$, which implies that for $\mathbf{\Phi} \in C_c^\infty([0,T); C_{0,per}^\infty(\mathbb{R}^2 \times [-\pi,\pi]))$,

$$\int \int_{\mathbb{R}^2\times[-\pi,\pi]} \mathbf{\Phi}_t \cdot \mathbf{u}^\epsilon dxdt \to \int \int_{\mathbb{R}^2\times[-\pi,\pi]} \mathbf{\Phi}_t \cdot \mathbf{u} dxdt \quad \text{as } \epsilon \to 0. \tag{4.4}$$

Then the decomposition

$$\begin{aligned}&\int \int_{\mathbb{R}^2\times[-\pi,\pi]} (\nabla\mathbf{\Phi} : \mathbf{u}^\epsilon \otimes \mathbf{u}^\epsilon - \nabla\mathbf{\Phi} : \mathbf{u}\otimes\mathbf{u}) dxdt \\ &= \int \int_{\mathbb{R}^2\times[-\pi,\pi]} \nabla\mathbf{\Phi} : (\mathbf{u}^\epsilon - \mathbf{u}) \otimes \mathbf{u}^\epsilon dxdt \\ &\quad + \int \int_{\mathbb{R}^2\times[-\pi,\pi]} \nabla\mathbf{\Phi} : \mathbf{u} \otimes (\mathbf{u}^\epsilon - \mathbf{u}) dxdt \end{aligned} \tag{4.5}$$

together with (4.3) shows

$$\int \int_{\mathbb{R}^2\times[-\pi,\pi]} \nabla\mathbf{\Phi} : \mathbf{u}^\epsilon \otimes \mathbf{u}^\epsilon dxdt \to \int \int_{\mathbb{R}^2\times[-\pi,\pi]} \nabla\mathbf{\Phi} : \mathbf{u}\otimes\mathbf{u} dxdt \;\text{ as }\; \epsilon \to 0,$$

which together with (4.4) implies that $\mathbf{u}$ is a weak solution of the Euler equation in the sense of Definition 2.3. It is also holds that $\mathbf{u} \cdot \boldsymbol{\xi} = 0$ from the property of no helical swirl of $\mathbf{u}^\epsilon$. □

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Appendix A. Two elementary lemmas

In this section, we will present two elementary lemmas which are very helpful to investigate the Biot–Savart law in Section 3. We begin with the following system

$$\begin{cases} -\Delta\mathbf{u} = 0, \\ |\mathbf{u}| = o(|x'|) \text{ as } |x'| \to \infty, \\ \mathbf{u}(x', x_3 + 2\pi) = \mathbf{u}(x', x_3). \end{cases} \tag{A.1}$$

Lemma A.1. *If* $\mathbf{u} \in L^p_{per}([-\pi,\pi]; L^p_{loc}(\mathbb{R}^2))(p > 1)$ *is the solution to system* (A.1)*,* $\mathbf{u}$ *is a constant.*

Proof. The lemma can be viewed as a conclusion of the generalized Liouville Theorem. For the readers' convenience, we give a proof here.

Since $\mathbf{u} \in L^p_{per}([-\pi,\pi]; L^p_{loc}(\mathbb{R}^2))$ is the solution of (A.1), we improve the regularity of $\mathbf{u}$, i.e., $\mathbf{u} \in C^\infty_{per}([-\pi,\pi]; C^\infty(\mathbb{R}^2))$ via the elliptic theory (see Theorem 9.19 of [10]). Then by means of Theorem 2.10 of [10], it holds that for any $x_0 \in \mathbb{R}^3$ and $R > 0$,

$$|\nabla \mathbf{u}|(x_0) \leq \frac{3}{R} \sup_{|x-x_0|=R} |\mathbf{u}|.$$

Then according to the fact that $\mathbf{u} = o(|x|)$ as $|x| \to \infty$ ($u = o(|x'|)$ as $|x'| \to \infty$ and u is periodic in x_3 direction), we have $\nabla \mathbf{u} \equiv 0$ by taking limit $R \to \infty$ in the above inequality. This further indicates that $\mathbf{u}$ is a constant. The proof is complete. □

In the following part, we pay more attention to the properties of the function G_1 given in (3.18).

Lemma A.2. *For $G_1(x)$ given in (3.18), it holds that*

$(P1)$. $\|G_1(x',\cdot)\|_{L^1([-\pi,\pi])} \lesssim \frac{1}{|x'|}$;

$(P2)$. $|G_1(x)| \lesssim \frac{1}{|x'|^k}$, $k \geq 3$;

$(P3)$. $\|\nabla_{x'}(G_1(x))x\|_{W^{1,1}(\mathbb{R}^2\times[-\pi,\pi])} \lesssim 1$.

Proof. First, we prove $(P1)$. Using integration by parts, we easily have

$$K_0(n|x'|) = \frac{1}{n|x'|} \int_0^\infty \frac{t\sin(n|x'|t)}{(1+t^2)^{\frac{3}{2}}} dt. \tag{A.2}$$

Then we rewrite the form of $G_1(x)$ as

$$G_1(x) = \frac{1}{\pi|x'|} \sum_{n=1}^\infty \int_0^\infty \frac{t}{(1+t^2)^{\frac{3}{2}}} \frac{\sin(n|x'|t)}{n} \cos(nx_3) dt. \tag{A.3}$$

Utilizing the orthogonal property of the bases $\{\cos(nx_3)\}$ in $L^2[-\pi,\pi]$ and the Parseval equality, it follows from (A.3) that

$$\|G_1(x',\cdot)\|_{L^1[-\pi,\pi]} \lesssim \|G_1(x',\cdot)\|_{L^2[-\pi,\pi]} \lesssim \frac{1}{|x'|} \int_0^\infty \frac{t}{(1+t^2)^{\frac{3}{2}}} dt \lesssim \frac{1}{|x'|},$$

which guarantee inequality (P1).

For $(P2)$, we rewrite kernel $K_0(n|x'|)$ on account of integration by parts as

$$
\begin{aligned}
K_0(n|x'|) &= \frac{1}{n|x'|}\int_0^\infty \frac{t}{(1+t^2)^{\frac{3}{2}}}\sin(n|x'|t)dt \\
&= \frac{1}{n^2|x'|^2}\int_0^\infty \frac{1-2t^2}{(1+t^2)^{\frac{5}{2}}}\cos(n|x'|t)dt \\
&= \frac{1}{n^3|x'|^3}\int_0^\infty \frac{9t-6t^3}{(1+t^2)^{\frac{7}{2}}}\sin(n|x'|t)dt.
\end{aligned}
\tag{A.4}
$$

Direct calculations indicate that

$$
|K_0(n|x'|)| \lesssim \frac{1}{n^k|x'|^k}, \quad k \geq 3. \tag{A.5}
$$

Substituting (A.5) into (3.18), $(P2)$ holds true since

$$
|G_1(x)| \lesssim \sum_{n=1}^{+\infty} \frac{1}{n^k|x'|^k} \lesssim \frac{1}{|x'|^k}, \quad k \geq 3. \tag{A.6}
$$

For $(P3)$, we use $\partial_{x_1}G_1(x)x_2$ as an example since the other terms of $\nabla_{x'}G_1(x)x$ can be coped with in the same way. First,

$$
\begin{aligned}
&\partial_{x_1}G_1(x)x_2 = \frac{1}{\pi}\sum_{n=1}^{+\infty}\frac{nx_1x_2}{|x'|}K_0'(n|x'|)\cos(nx_3), \\
&\partial_{x_1}\left(\partial_{x_1}G_1(x)x_2\right) \\
&\quad = \frac{1}{\pi}\sum_{n=1}^{+\infty}\left(\frac{nx_2^3}{|x'|^3}K_0'(n|x'|)\cos(nx_3) + \frac{n^2x_1^2x_2}{|x'|^2}K_0''(n|x'|)\cos(nx_3)\right), \\
&\partial_{x_2}\left(\partial_{x_1}G_1(x)x_2\right) \\
&\quad = \frac{1}{\pi}\sum_{n=1}^{+\infty}\left(\frac{nx_1^3}{|x'|^3}K_0'(n|x'|)\cos(nx_3) + \frac{n^2x_1x_2^2}{|x'|^2}K_0''(n|x'|)\cos(nx_3)\right), \\
&\partial_{x_3}\left(\partial_{x_1}G_1(x)x_2\right) = -\frac{1}{\pi}\sum_{n=1}^{+\infty}\frac{n^2x_1x_2}{|x'|}K_0'(n|x'|)\sin(nx_3).
\end{aligned}
\tag{A.7}
$$

Since $K_0(z) = \frac{1}{z}\int_0^\infty \frac{t}{(1+t^2)^{\frac{3}{2}}}\sin(zt)dt$, we derive that

$$
\begin{aligned}
z^2 K_0'(z) &= -\int_0^\infty \frac{t}{(1+t^2)^{\frac{3}{2}}} \sin(zt)dt - \int_0^\infty \frac{t(2-t^2)}{(1+t^2)^{\frac{5}{2}}} \sin(zt)dt, \\
z^3 K_0''(z) &= -2z^2 K_0'(z) + \int_0^\infty \frac{t(2-t^2)}{(1+t^2)^{\frac{5}{2}}} \sin(zt)dt \\
&\quad + \int_0^\infty \frac{4t-10t^3+t^5}{(1+t^2)^{\frac{7}{2}}} \sin(zt)dt.
\end{aligned} \tag{A.8}
$$

Here, we only show the crucial estimate $\|\partial_{x_3}(\partial_{x_1} G_1(x)x_2)\|_{L^1(\mathbb{R}^2\times[-\pi,\pi])}$ since other terms can be treated in the same way.

It follows from the first equality in (A.8) by integration by parts that

$$
\begin{aligned}
z^2 K_0'(z) &= -\int_0^\infty \frac{t}{(1+t^2)^{\frac{3}{2}}} \sin(zt)dt - \int_0^\infty \frac{t(2-t^2)}{(1+t^2)^{\frac{5}{2}}} \sin(zt)dt \\
&= \frac{1}{z}\int_0^\infty \frac{3-12t^2}{(1+t^2)^{\frac{7}{2}}} \cos(zt)dt.
\end{aligned}
$$

This implies that there exists a generic constant C, such that

$$
|z^2 K_0'(z)| \le C, \quad |z^3 K_0'(z)| \le C. \tag{A.9}
$$

Similarly, one also has

$$
|z^4 K_0'(z)| \le C. \tag{A.10}
$$

Then it follows from the fourth equality in (A.7) that

$$
\begin{aligned}
&\|\partial_{x_3}(\partial_{x_1} G_1(x)x_2)\|_{L^1(\mathbb{R}^2\times[-\pi,\pi])} \\
&\lesssim \int_{\mathbb{R}^2} \|\partial_{x_3}(\partial_{x_1} G_1(x)x_2)\|_{L^1([-\pi,\pi])} \\
&\lesssim \int_{\mathbb{R}^2} \|\partial_{x_3}(\partial_{x_1} G_1(x)x_2)\|_{L^2([-\pi,\pi])} \\
&\lesssim \int_{\mathbb{R}^2} \frac{1}{|x'|}\left(\sum_{n=1}^\infty (n^2|x'|^2 K_0'(n|x'|))^2\right)^{1/2} dx'.
\end{aligned}
$$

Here we use the fact that $\{\sin(nx_3)\}$ are orthogonal in the sense of $L^2([-\pi,\pi])$. Thus, with (A.9)–(A.10), one has

$$
\begin{aligned}
&\|\partial_{x_3}(\partial_{x_1}G_1(x)x_2)\|_{L^1(\mathbb{R}^2\times[-\pi,\pi])}\\
&\lesssim \int_{B_1(0)} \frac{1}{|x'|}\left(\sum_{n=1}^{\infty}(n^2|x'|^2K_0'(n|x'|))^{\frac{3}{2}}\right)^{1/2} dx' + \int_{\mathbb{R}^2\setminus B_1(0)} \frac{1}{|x'|^3}dx'\\
&\lesssim \int_{B_1(0)} \frac{1}{|x'|^{\frac{7}{4}}}dx' + \int_{\mathbb{R}^2\setminus B_1(0)} \frac{1}{|x'|^3}dx'\\
&\lesssim \int_0^1 \frac{1}{r^{3/4}}dr + \int_1^{\infty}\frac{1}{r^2}dr\\
&\lesssim C.
\end{aligned}
$$

Thus we complete the proof of (P3). □

Remark A.1. According to (P1)–(P2) and (A.8), we can also obtain that

$$
\begin{aligned}
&\|\sum_{n=1}^{\infty} nK_0'(x')|x'|\cos(nx_3)\|_{L^1(\mathbb{R}^2\times[-\pi,\pi])} \le C,\\
&\|\sum_{n=1}^{\infty} nK_0(n|x'|)x_i\|_{L^1(\mathbb{R}^2\times[-\pi,\pi])} \le C,\ i=1,2.
\end{aligned}
\tag{A.11}
$$

Appendix B. Sobolev type estimate

In this section, we establish a kind of Sobolev type estimate for Newton potential, which is stated in the following lemma:

Lemma B.1. *If $f \in \bigcap_{l=1,p} L^l(\mathbb{R}^2)$ for $p > 1$ and $u(x) = \int_{\mathbb{R}^2} \ln|x-y|f(y)dy$, then one has*

$$
\|\nabla^2 u\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}. \tag{B.1}
$$

Moreover,

$$
\|\nabla u\|_{L^q(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^1(\mathbb{R}^2)}, \tag{B.2}
$$

with $q = \frac{2p}{2-p}$ when $1 < p < 2$ and $2 < q < +\infty$ when $p \ge 2$.

Proof. We only prove (B.2) since (B.1) is the standard result of the singular integral operator theory (see Theorem 9.9 in [10]). In addition, we only need to prove (B.2) for the case $1 < p < 2$ since the case of $p \ge 2$ can be reduced to the case $1 < p < 2$ in the following way: If (B.2) stands for $1 < p < 2$, then for any $2 < q < +\infty$, there exists $1 < q_0 < 2$ with $q = \frac{2q_0}{2-q_0}$, such that

$$\begin{aligned}\|\nabla u\|_{L^q(\mathbb{R}^2)} &\lesssim \|f\|_{L^{q_0}(\mathbb{R}^2)} + \|f\|_{L^1(\mathbb{R}^2)} \\ &\lesssim \|f\|^{\lambda}_{L^1(\mathbb{R}^2)}\|f\|^{1-\lambda}_{L^p(\mathbb{R}^2)} + \|f\|_{L^1(\mathbb{R}^2)} \\ &\lesssim \|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

Here the last two inequalities come from the interpolation inequality (see Lemma 10.5 in [22]) with $1/q_0 = \lambda + (1-\lambda)/p$ and the Young inequality.

To derive the estimate (B.2) for the case $1 < p < 2$, we first prove that

$$\|\nabla u\|_{L^{q_1}(\mathbb{R}^2)} \le C(q_1)\left(\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^p(\mathbb{R}^2)}\right), \tag{B.3}$$

for $q_1 = \frac{2p}{2-p+\delta} > 2$ with a fixed constant δ satisfying $0 < \delta < \min\{p, 2(p-1)\}$. Second, we set $\varrho(x)$ be a smooth radial symmetric function, such that $0 \le \varrho \le 1$ with $\varrho = 1, |x| \le 1$ and $\varrho = 0, |x| \ge 2$. Then

$$\begin{aligned}|\nabla u|(x) &\le \int_{\mathbb{R}^2} \frac{\varrho(x-y)}{|x-y|}|f(y)|dy + \int_{\mathbb{R}^2} \frac{1-\varrho(x-y)}{|x-y|}|f(y)|dy \\ &:= I_1(x) + I_2(x).\end{aligned} \tag{B.4}$$

Since

$$\frac{\varrho(x)}{|x|} \in L^{l_1}(\mathbb{R}^2), \quad \frac{1-\varrho(x)}{|x|} \in L^{l_2}(\mathbb{R}^2),$$

with $l_1 = \frac{2p}{p+\delta} \in (1,2)$ and $l_2 = \frac{2p}{2-p+\delta} \in (2,+\infty)$, then it follows from (B.4) and the Young inequality that

$$\|\nabla u\|_{L^{q_1}(\mathbb{R}^2)} \le \|I_1\|_{L^{q_1}(\mathbb{R}^2)} + \|I_2\|_{L^{q_1}(\mathbb{R}^2)} \le C(q_1)\left(\|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^1(\mathbb{R}^2)}\right),$$

which completes the estimate (B.3).

Next, we prove the estimate (B.2). Since $1 < p < 2 < q_1$, then for each $m \in \mathbb{N}$, it follows from (B.1) and (B.3) that

$$\varrho(x/m)\nabla u(x) \in W_0^{1,p}(\mathbb{R}^2). \tag{B.5}$$

Then it is derived from the Sobolev embedding theorem (see Proposition 10.7 in [21]) that

$$\begin{aligned}\|\varrho(x/m)\nabla u\|_{L^q(\mathbb{R}^2)} &\lesssim \|\varrho(x/m)\nabla^2 u\|_{L^p(\mathbb{R}^2)} + \frac{1}{m}\|\varrho'(x/m)\nabla u\|_{L^p(\mathbb{R}^2)} \\ &\lesssim \|\nabla^2 u\|_{L^p(\mathbb{R}^2)} + \frac{1}{m}\|\nabla u\|_{L^p(m\le|x|\le 2m)}.\end{aligned} \tag{B.6}$$

Here, the last inequality comes from the support of $\varrho'(x/m)$.

In addition, (B.6) with (B.1) and (B.3) yields

$$\begin{aligned}\|\varrho(x/m)\nabla u\|_{L^q(\mathbb{R}^2)} &\lesssim \|f\|_{L^p(\mathbb{R}^2)} + \frac{1}{m}\|\nabla u\|_{L^{q_1}(\mathbb{R}^2)} m^{2/l_3} \\ &\lesssim (1+m^{2/l_3-1})\left(\|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^1(\mathbb{R}^2)}\right),\end{aligned} \tag{B.7}$$

with $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{l_3}$. Thus, $l_3 = \frac{2p}{p-\delta} > 2$ and it follows from (B.7) that for each $m \in \mathbb{N}$,

$$\|\varrho(x/m)\nabla u\|_{L^q(\mathbb{R}^2)} \leq C\left(\|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^1(\mathbb{R}^2)}\right), \tag{B.8}$$

where C is an universal positive constant, independent of m and ϱ. Thus the estimate (B.2) is proved by taking limit in the left side of (B.8) as $m \to +\infty$. □

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