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Local existence and uniqueness of strong solutions to the Navier–Stokes equations with nonnegative density

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Abstract

In this paper, we consider the initial-boundary value problem to the nonhomogeneous incompressible Navier–Stokes equations. Local strong solutions are established, for any initial data $(\rho_0, u_0) \in (W^{1,\gamma} \cap L^\infty) \times H^1_{0,\sigma}$, with $\gamma > 1$, and if $\gamma \geq 2$, then the strong solution is unique. The initial density is allowed to be nonnegative, and in particular, the initial vacuum is allowed. The assumption on the initial data is weaker than the previous widely used one that $(\rho_0, u_0) \in (H^1 \cap L^\infty) \times (H^1_{0,\sigma} \cap H^2)$, and no compatibility condition is required.

MSC: 76D03; 76D05

Keywords: Local existence and uniqueness; Density-dependent incompressible Navier–Stokes equations; Compatibility condition; Gronwall type inequality

1. Introduction

The motion of the incompressible fluid in a domain Ω is governed by the following nonhomogeneous incompressible Navier–Stokes equations

$$\partial_t \rho + u \cdot \nabla \rho = 0, \tag{1.1}$$

$$\rho(\partial_t u + (u \cdot \nabla)u) - \Delta u + \nabla p = 0, \tag{1.2}$$

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$$\operatorname{div} u = 0, \tag{1.3}$$

in $\Omega \times (0, \infty)$, where the nonnegative function ρ is the density of the fluid, the vector field u denotes the velocity of the flow, and the scaler function p presents the pressure.

Since Leray's pioneer work [20] in 1934, in which he established the global existence of weak solutions to the homogeneous incompressible Navier–Stokes equations, i.e. system (1.1)–(1.3) with positive constant density, there has been a considerable number of papers devoted to the mathematical analysis on the incompressible Navier–Stokes equations. A generalization of Leray's result to the corresponding nonhomogeneous system, i.e. system (1.1)–(1.3) with variable density, was first made by Antontsev–Kazhikov in [3], for the case that the initial density is away from vacuum, see also the book Antontsev–Kazhikov–Monakhov [4]. For the case that the initial density is allowed to have vacuum, the global existence of weak solutions to system (1.1)–(1.3) was proved by Simon [29,30] and Lions [24]. However, the uniqueness and smoothness of weak solutions to the nonhomogeneous Navier–Stokes equation, even for the two dimensional case, is still an open problem; note that it is well known that weak solutions to the two dimensional homogeneous incompressible Navier–Stokes equations are unique, and are smooth immediately after the initial time, see, e.g., Ladyzhenskaya [18] and Temam [31].

Local existence (but without uniqueness) of strong solutions to the nonhomogeneous incompressible Navier–Stokes equations was first established by Antontsev–Kazhikov [3], under the assumption that the initial density is bounded and away from zero and the initial velocity has H^1 regularity. Local in time strong solutions, which enjoy the uniqueness, were later obtained by Ladyzhenskaya–Solonnikov [19], Padula [25,26] and Itoh–Tani [17]. Some more advances concerning the existence and uniqueness of strong solutions, in the framework of the so-called critical spaces, to the nonhomogeneous incompressible Navier–Stokes equations have been made recently, see, e.g., [1,2,9–12,15,27,28]. It should be mentioned that in all the works [1–3,9–12, 15,17,19,25–28], the initial density is assumed to have positive lower bound, and thus no vacuum is allowed.

For the general case that the initial density is allowed to have vacuum, Choe–Kim [7] first proved the local existence and uniqueness of strong solutions to the initial-boundary value problem of system (1.1)–(1.3), with initial data (ρ_0, u_0) satisfying

$$0 \le \rho_0 \in H^1 \cap L^\infty, \quad u_0 \in H^2 \cap H^1_{0,\sigma}, \tag{1.4}$$

and the compatibility condition

$$\Delta u_0 - \nabla p_0 = \sqrt{\rho_0} g, \tag{1.5}$$

for some $(p_0, g) \in H^1 \times L^2$. Since the work [7], conditions (1.4)–(1.5) and their necessary modifications are widely used, as the standard assumptions, in many papers concerning the studies of the existence and uniqueness of strong solutions, with initial vacuum allowed, to the nonhomogeneous Navier–Stokes equations and some related systems, such as the magnetohydrodynamics (MHD) and liquid crystals, see, e.g., [6,8,13,14,16,21,32,33].

Noticing that, when the initial vacuum is taken into consideration, conditions (1.4)–(1.5) are so widely used in the literatures to study the existence and uniqueness of strong solutions to the nonhomogeneous Navier–Stokes equations and some related models, we may ask if one can reduce the regularities on the initial data stated in (1.4) and drop the compatibility condition (1.5), so that the result of existence and uniqueness of strong solutions to the corresponding systems still holds. As will be indicated in this paper, we can indeed reduce the regularities of the initial velocity and drop the compatibility condition, without destroying the existence and uniqueness, but the prices that we need to pay are the following: (i) the corresponding strong solutions do not have as high regularities as those in [7]; (ii) one can only ask for the continuity, at the initial time, of the momentum ρu, instead of the velocity u itself.

In this paper, we consider the initial-boundary value problem to system (1.1)–(1.3), defined on a smooth bounded domain Ω of $\mathbb{R}^3$, and the initial and boundary conditions are as follows

$$u(x,t)=0, \quad x\in\partial\Omega, \tag{1.6}$$

$$(\rho,\rho u)|_{t=0}=(\rho_0,\rho_0u_0). \tag{1.7}$$

Note that, instead of imposing the initial condition on the velocity u, we impose the initial condition on the momentum ρu. As will be explained in (iii) of Remark 1.2, below, generally one can not expect the continuity of the velocity u, up to the initial time, when the vacuum appears and the initial data is not sufficiently smooth.

Throughout this paper, for positive integer k and positive number $q\in[1,\infty]$, we use L^q and $W^{k,q}$ to denote the standard Lebesgue and Sobolev spaces, respectively, on the domain Ω. When $q=2$, we use H^k, instead of $W^{k,2}$. Spaces L^2_σ and $H^1_{0,\sigma}$ are the closures in L^2 and H^1, respectively, of the space $C^\infty_{0,\sigma}:=\{\varphi\in C_0^\infty(\Omega)\,|\,\operatorname{div}\varphi=0\}$. For simplicity, we usually use $\|f\|_q$ to denote $\|f\|_{L^q}$.

Strong solutions to system (1.1)–(1.3), subject to (1.6)–(1.7), are defined as follows.

Definition 1.1. Given a positive time $T\in(0,\infty)$, and the initial data (ρ_0,u_0), with $\rho_0\in W^{1,\gamma}\cap L^\infty$, $\gamma\in(1,\infty)$, and $u_0\in H^1_{0,\sigma}$. A pair (ρ,u) is called a strong solution to system (1.1)–(1.3), subject to (1.6)–(1.7), on $\Omega\times(0,T)$, if it has the regularities

$$\begin{aligned}
&\rho\in L^\infty(0,T;W^{1,\gamma}\cap L^\infty)\cap C([0,T];L^\gamma),\\
&u\in L^\infty(0,T;H^1_{0,\sigma})\cap L^2(0,T;H^2), \quad \rho u\in C([0,T];L^2),\\
&\sqrt{t}u\in L^\infty(0,T;H^2)\cap L^2(0,T;W^{2,6}), \quad \sqrt{t}\partial_t u\in L^2(0,T;H^1),
\end{aligned}$$

satisfies system (1.1)–(1.3) pointwisely, a.e. in $\Omega\times(0,T)$, for some associated pressure function $p\in L^2(0,T;H^1)$, and fulfills the initial condition (1.7).

Remark 1.1. Thanks to the regularities of the strong solutions stated in Definition 1.1, by equations (1.1) and (1.2), one can show that the strong solutions have the following additional regularities

$$\partial_t\rho\in L^4(0,T;L^\gamma), \quad \sqrt{\rho}\partial_t u\in L^2(0,T;L^2), \quad \sqrt{t}\sqrt{\rho}\partial_t u\in L^\infty(0,T;L^2).$$

The main result of this paper is the following theorem on the local existence and uniqueness of strong solutions to system (1.1)–(1.3), subject to (1.6)–(1.7).

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary. Suppose that the initial data (ρ_0,u_0) satisfies*

$$0 \le \rho_0 \le \bar{\rho}, \quad \rho_0 \in W^{1,\gamma}, \quad u_0 \in H^1_{0,\sigma},$$

for some $\gamma \in (1, \infty)$ *and* $\bar{\rho} \in (0, \infty)$.

Then, there is a positive time T_0*, depending only on* $\bar{\rho}, \Omega$ *and* $\|\nabla u_0\|_2$*, such that system (1.1)–(1.3), subject to (1.6)–(1.7), admits a strong solution* (ρ, u)*, on* $\Omega \times (0, T_0)$*. Moreover, if* $\gamma \in [2, \infty)$*, then the strong solution just established is unique.*

Remark 1.2. (i) Through we ask for the $W^{1,\gamma}$ regularity on the initial density, the only factor of the initial density that influences the existence time T_0 in Theorem 1.1 is the upper bound. As will be seen in the proof of Theorem 1.1, such higher regularity assumption on the initial density, i.e. $\rho_0 \in W^{1,\gamma}$, is used only to guarantee the continuity of the momentum at the initial time and the uniqueness of the solution.

(ii) The regularity assumptions on the initial data in Theorem 1.1 are weaker than those in [7], where the initial data was assumed to have the regularities stated in (1.4). Note that, the compatibility condition (1.5) plays an essential role in [7], while in Theorem 1.1, no compatibility condition on the initial data is required, for the local existence and unique of strong solutions.

(iii) Due to the insufficient smoothness and the absence of the compatibility conditions on the initial data, and the presence of vacuum, for the strong solutions (ρ, u) established in Theorem 1.1, the quantity $\partial_t u$, viewed as a vector valued function on the time interval $(0, T_0)$, is not generally integrable on $(0, T_0)$. As a result, one can not expect the continuity of u, up to the initial time. It is because of this that we impose the initial condition on ρu, in stead of u, in (1.7), and correspondingly ask for the continuity in time of ρu in Definition 1.1.

The key observation leading us to reduce the assumptions on the initial data, from those imposed in [7] and widely used in many other papers to the current version, stated in Theorem 1.1, is that the boundedness of the initial density and the H^1 regularity of the initial velocity is sufficient to guarantee the $L^1(0, T_0; W^{1,\infty})$ estimate on the velocity of the solutions to system (1.1)–(1.3). In order to achieve the $L^1(0, T; W^{1,\infty})$ estimate of the velocity, the main tool is to perform the t-weighted H^2 estimate to system (1.1)–(1.2) or its approximated system, see Proposition 3.3, below, obtaining

$$\sup_{0 \le t \le T_0} t(\|\nabla^2 u\|_2^2 + \|\sqrt{\rho}\partial_t u\|_2^2) + \int_0^{T_0} t\|\nabla \partial_t u\|_2^2 dt \le C.$$

Note that, thanks to the weighted factor t, the constant C in the above estimate is independent of the H^2 norm of the initial velocity. With the above estimate in hand, one can then successfully obtain the desired $L^1(0, T_0; W^{1,\infty})$ estimate on the velocity, and further the regularity estimates on the density, see Proposition 4.1, below, for the details. In proving the uniqueness of strong solutions, the idea of the t-weighted estimates is also used, but in a different manner from above, see the Gronwall type inequality in Lemma 2.5, below.

Remark 1.3. (i) The same argument can be adopted to other similar systems, including the nonhomogeneous incompressible magnetohydrodynamics (MHD) and the liquid crystals, in the presence of initial vacuum. Specifically, one can weaken the regularity assumptions and drop the compatibility conditions on the initial data stated in [6,8,13,14,16,21,32,33], without destroying the existence and uniqueness of strong solutions; however, the definitions of the strong solutions in those papers should be modified accordingly.

(ii) The idea of making use of the t-weighted estimate has been used in the study of several incompressible models, to weaken the regularity assumptions on the initial data, see, e.g., Paicu–Zhang–Zhang [28] for the inhomogeneous incompressible Navier–Stokes equations, i.e. system (1.1)–(1.3), in the absence of vacuum, Li–Titi [22] for the Boussinesq equations, Li–Titi [23] for a tropical atmosphere model, and Cao–Li–Titi [5] for the primitive equations. This idea can be also adopted to the compressible Navier–Stokes equations, but the argument will be different from and more complicated than the incompressible case. We will present the details of such kind result for the compressible Navier–Stokes equations in another paper.

The rest of this paper is arranged as follows: in Section 2, we collect some preliminary lemmas; in Section 3, we carry out the Galerkin approximation to system (1.1)–(1.3), and perform some uniform a priori estimates on the solutions to the approximated system; the proof of Theorem 1.1 is given in the last section.

2. Preliminaries

In this section, we state several preliminary lemmas which will be used in the rest of this paper. We start with the following compactness lemma due to DiPerna–Lions.

Lemma 2.1 *(cf. [24]). Let T be a positive time, and assume that $\{(\rho_N, u_N)\}_{N=1}^\infty$ satisfies*

$$\begin{gathered}\rho_N \in C([0,T]; L^1), \quad 0 \leq \rho_N \leq C, \text{ a.e. on } \Omega \times (0,T),\\ div\, u_N = 0, \text{ a.e. on } \Omega \times (0,T), \quad \|u_N\|_{L^2(0,T;H^1_{0,\sigma})} \leq C,\\ \partial_t \rho_N + div\,(\rho_N u_N) = 0, \text{ in } \mathcal{D}'(\Omega \times (0,T)),\\ \rho_N(0) \to \rho_0, \text{ in } L^1, \quad u_N \rightharpoonup u, \text{ in } L^2(0,T;H^1),\end{gathered}$$

where C is a positive constant independent of N.

Then, ρ_N converges in $C([0,T]; L^p)$, for $1 \leq p < \infty$, to the unique solution ρ, bounded on $\Omega \times (0,T)$, of

$$\begin{gathered}\partial_t \rho + div\,(\rho u) = 0, \quad \text{in } \mathcal{D}'(\Omega \times (0,T)),\\ \rho \in C([0,T]; L^1), \quad \rho(0) = \rho_0, \text{ a.e. in } \Omega.\end{gathered}$$

The next lemma about the existence, uniqueness and a priori estimates to the transport equations is standard, see, e.g., [19].

Lemma 2.2. *Let $v \in L^1(0,T; Lip)$ a vector field, such that $div\, v = 0$, and $v \cdot n = 0$ on $\partial\Omega$, where n denotes the outward normal vector on $\partial\Omega$. Let $\rho_0 \in W^{1,q}$, with $q \in [1,\infty]$.*

Then, the following system

$$\begin{cases}\rho_t + \mathrm{div}(\rho v) = 0, & \text{in } \Omega \times (0,T),\\ \rho|_{t=0} = \rho_0, & \text{in } \Omega,\end{cases}$$

has a unique solution in $L^\infty(0,T; W^{1,\infty}) \cap C([0,T]; \cap_{1\leq r<\infty} W^{1,r})$, if $q = \infty$, and in $C([0,T]; W^{1,q})$, if $1 \leq q < \infty$.

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Besides, the following estimate holds

$$\|\rho(t)\|_{W^{1,q}} \le e^{\int_0^t \|\nabla v(\tau)\|_\infty d\tau}\|\rho_0\|_{W^{1,q}},$$

for any $t \in [0, T]$.

To determine the pressure associated with the strong solutions, we will use the following two lemmas.

Lemma 2.3 *(cf. [31]). Let* Ω *be an open set in* $\mathbb{R}^d$, $d \ge 2$, *and* $f = \{f_1, \cdots, f_d\}$, *with* f_i *being distribution,* $i = 1, 2, \cdots, d$. *A necessary and sufficient condition for* $f = \nabla p$, *for some distribution* p, *is that* $\langle f, \phi\rangle = 0$, *for any* $\phi \in C^\infty_{0,\sigma}(\Omega)$.

Lemma 2.4 *(cf. [31]). Let* Ω *be a bounded Lipschitz open set in* $\mathbb{R}^d$, $d \ge 2$.

(i) If a distribution p *has all its first-order derivatives* $\partial_i p$, $1 \le i \le d$, *in* $L^2(\Omega)$, *then* $p \in L^2(\Omega)$ *and*

$$\|p - p_\Omega\|_{L^2(\Omega)} \le c(\Omega)\|\nabla p\|_{L^2(\Omega)},$$

where $p_\Omega = \frac{1}{|\Omega|}\int_\Omega p dx$.

(ii) If a distribution p *has all its derivatives* $\partial_i p$, $1 \le i \le d$, *in* $H^{-1}(\Omega)$, *then* $p \in L^2(\Omega)$ *and*

$$\|p - p_\Omega\|_{L^2(\Omega)} \le c(\Omega)\|\nabla p\|_{H^{-1}(\Omega)}.$$

In both cases, if Ω *is any open set in* $\mathbb{R}^d$, *then* $p \in L^2_{loc}(\Omega)$.

Finally, we state and prove a Gronwall type inequality which will be used to guarantee the uniqueness of strong solutions.

Lemma 2.5. *Given a positive time* T *and nonnegative functions* f, g, G *on* $[0, T]$, *with* f *and* g *being absolutely continuous on* $[0, T]$. *Suppose that*

$$\begin{cases} \frac{d}{dt}f(t) \le A\sqrt{G(t)}, \\ \frac{d}{dt}g(t) + G(t) \le \alpha(t)g(t) + \beta(t)f^2(t), \\ f(0) = 0, \end{cases}$$

a.e. on $(0, T)$, *where* A *is a positive constant,* α *and* β *are two nonnegative functions, satisfying*

$$\alpha \in L^1((0, T)) \quad \text{and} \quad t\beta(t) \in L^1((0, T)).$$

Then, the following estimates hold

$$f(t) \le A\sqrt{g(0)}\sqrt{t}e^{\frac{1}{2}\int_0^t(\alpha(s)+A^2 s\beta(s))ds},$$

and

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$$g(t)+\int_0^t G(s)ds \le g(0)e^{\int_0^t(\alpha(s)+A^2 s\beta(s))ds},$$

for $t\in[0,T]$, *which, in particular, imply* $f\equiv 0$, $g\equiv 0$ *and* $G\equiv 0$, *provided* $g(0)=0$.

Proof. It follows from the assumption and the Hölder inequality that

$$f(t)\le A\int_0^t \sqrt{G(s)}ds \le A\sqrt{t}\left(\int_0^t G(s)ds\right)^{\frac{1}{2}}, \tag{2.1}$$

which, along with the assumption, gives

$$\frac{dt}{dt}g(t)+G(t)\le \alpha(t)g(t)+A^2 t\beta(t)\int_0^t G(s)ds.$$

Setting $\eta(t)=g(t)+\int_0^t G(s)ds$, then it follows from the above inequality that

$$\eta'(t)\le(\alpha(t)+A^2 t\beta(t))\eta(t),$$

which, by the Gronwall inequality, implies

$$\eta(t)=g(t)+\int_0^t G(s)ds\le g(0)e^{\int_0^t(\alpha(s)+A^2 s\beta(s))ds}.$$

Thanks to the above estimate, and recalling (2.1), we have

$$f(t)\le A\sqrt{g(0)}\sqrt{t}e^{\frac{1}{2}\int_0^t(\alpha(s)+A^2 s\beta(s))ds}.$$

This completes the proof. □

3. Galerkin approximation

In this section, we preform the Galerkin approximation to system (1.1)–(1.3). We first present the approximation scheme, then prove the solvability of the approximated system, and finally carry out the uniform estimates to the approximated solutions.

3.1. The scheme

Let $\{w_i\}_{i=1}^\infty$ be a sequence of eigenfunctions to the following eigenvalue problem of the Dirichlet problem to the Stokes equations in Ω:

$$\begin{cases} -\Delta w_i + \nabla p_i = \lambda_i w_i, \\ \operatorname{div} w_i = 0, \\ w_i|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, with $\lambda_i \to \infty$, as $i \to \infty$, are the eigenvalues. The sequence $\{w_i\}_{i=1}^\infty$ can be renormalized in such a way that it is an orthonormal basis in $L^2_\sigma(\Omega)$. One can further show that it is an orthogonal basis in $H^1_{0,\sigma}(\Omega)$, and a basis (but not necessary orthogonal) in $H^1_{0,\sigma}(\Omega) \cap H^2(\Omega)$, see, e.g., Ladyzhenskaya [18]. By the regularity theory of the Stokes equations, w_i is smooth on $\bar{\Omega}$.

For any positive integer N, we set $X_N = span\{w_1, w_2, \cdots, w_N\}$ as the linear space expanded by w_i, $i = 1, \cdots, N$. Denote by $\|\cdot\|_{X_N}$ the norm on X_N, given by

$$\|f\|_{X_N} = \left(\sum_{i=1}^N a_i^2\right)^{\frac{1}{2}}, \quad \text{for } f = \sum_{i=1}^N a_i w_i.$$

Recalling that $\{w_i\}_{i=1}^\infty$ is an orthonormal basis in $L^2_\sigma(\Omega)$, one can verify that $\|f\|_{X_N}$ is exactly the $L^2(\Omega)$ norm of f, for any $f \in X_N$. Note that X_N is a finite dimension space, all other norms on X_N are equivalent to the norm $\|\cdot\|_{X_N}$ defined above.

We are going to solve the following system:

$$\begin{cases} \partial_t \rho_N + u_N \cdot \nabla \rho_N = 0, \\ (\rho_N(\partial_t u_N + (u_N \cdot \nabla) u_N), w) + (\nabla u_N, \nabla w) = 0, \quad \forall w \in X_N, \\ \rho_N|_{t=0} = \rho_{0N}, \quad u_N|_{t=0} = u_{0N}, \end{cases} \tag{3.2}$$

where $\{\rho_{0N}\}_{N=1}^\infty$ is a sequence of functions from $C^2(\bar{\Omega})$, satisfying

$$0 < \underline{\rho} \leq \rho_{0N} \leq \bar{\rho}, \quad \rho_{0N} \in C^2(\bar{\Omega}), \quad \rho_{0N} \to \rho_0, \text{ in } W^{1,r}(\Omega), \tag{3.3}$$

for some $r \in (3, \infty)$, and u_{0N} is given as

$$u_{0N} = \sum_{i=1}^N (u_0, w_i) w_i. \tag{3.4}$$

For any $v_N \in C([0, T]; X_N)$, define $\Phi(\tau; x, t)$ as the particle path which goes along with the velocity field v_N and passes through point x at time t:

$$\frac{d}{d\tau}\Phi(\tau; x, t) = v_N(\Phi(\tau; x, t), \tau), \quad \Phi(t; x, t) = x.$$

Note that $v_N \in C([0, T]; C^3(\bar{\Omega}))$, by the standard theory of the ordinary differential equations, the particle pass $\Phi(\tau; x, t)$ is at least C^3 continuous in (x, t) and C^1 continuous in τ, on the domain $\{(\tau, x, t) | \tau, t \in [0, T], x \in \bar{\Omega}\}$, in other words we have $\partial^3_{x,t}\Phi, \partial_\tau \Phi \in C([0, T] \times \bar{\Omega} \times [0, T])$. Moreover, Φ depends continuously on the velocity v_N, and actually by considering the difference system of two particle paths $\bar{\Phi}$ and $\hat{\Phi}$, which pass through the same point x at the

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same time t, but go along two different velocity fields $\bar{v}_N$ and $\hat{v}_N$, respectively, by using the mean value theorem of differentials and the Gronwall inequality, one can explicitly deduce that

$$\|\bar{\Phi}-\hat{\Phi}\|_{C([0,T]\times\bar{\Omega}\times[0,T])}\le T\|\bar{v}_N-\hat{v}_N\|_{C(\bar{\Omega}\times[0,T])}e^{C\|\bar{v}\|_{L^1(0,T;Lip(\Omega))}},$$

for a constant depending only on the domain Ω.

Denote $\rho_N=\rho_{0N}(\Phi(0;x,t))$. Using the fact that $\Phi(0;x,t)=\Phi^{-1}(t;x,0)$, where the inverse is with respect to the spatial variable x, one can easily check that ρ_N is the unique solution to

$$\begin{cases}\partial_t\rho_N+v_N\cdot\nabla\rho_N=0,\\ \rho|_{t=0}=\rho_{0N}.\end{cases}\tag{3.5}$$

Recalling the regularities of Φ, it is straightforward that $\rho_N\in C^2(\bar{\Omega}\times[0,T])$. We define the map $S_N:C([0,T];X_N)\to C^2(\bar{\Omega}\times[0,T])$ as

$$v_N\mapsto\rho_N=S_N[v_N],\qquad \rho_N \text{ is the unique solution to (3.5).}$$

Recalling the continuous dependence on v_N of the particle pass Φ, the above solution mapping $\rho_N=S_N[v_N]$ is continuous, with respect to $v_N\in C([0,T];X_N)$.

In order to prove the solvability of system (3.2), it suffices to find a solution $u_N\in C([0,T];X_N)$ to the following system

$$\begin{cases}(S_N[u_N](\partial_t u_N+(u_N\cdot\nabla)u_N),w)+(\nabla u_N,\nabla w)=0,\quad \forall w\in X_N,\\ u_N|_{t=0}=u_{0N},\end{cases}\tag{3.6}$$

where $S_N[u_N]$, as defined before, is the unique solution to system (3.5), with v_N replaced by u_N. To this end, we consider the following linearized system

$$\begin{cases}(S_N[v_N](\partial_t u_N+(v_N\cdot\nabla)u_N),w)+(\nabla u_N,\nabla w)=0,\quad \forall w\in X_N,\\ u_N|_{t=0}=u_{0N},\end{cases}\tag{3.7}$$

where $v_N\in C([0,T];X_N)$ is given. We define a solution mapping $Q_N:C([0,T];X_N)\to C([0,T];X_N)$ as

$$v_N\mapsto u_N=Q_N[v_N],\qquad u_N \text{ is the unique solution to (3.7).}$$

As it will be shown later, the mapping Q_N is well-defined. Therefore, to prove the solvability of system (3.7), and consequently system (3.2), it suffices to look for a fixed point of the mapping Q_N in $C([0,T];X_N)$.

Given $v_N\in C([0,T];X_N)$, and denote by $\rho_N=S_N[v_N]$, as before, the unique solution to system (3.5). Then $\rho_N\in C^2(\bar{\Omega}\times[0,T])$ and $\underline{\rho}\le\rho\le\bar{\rho}$. Suppose that u_N has the form

$$u_N(x,t)=\sum_{i=1}^N f_{Ni}(t)w_i,$$

for some unknowns $f_{Ni} \in C([0,T])$, $i = 1, 2, \cdots, N$. Then (3.7) is equivalent to

$$\begin{cases} \sum_{j=1}^N a_{ij}^N(t) f'_{Nj}(t) + \sum_{j=1}^N b_{ij}^N(t) f_{Nj}(t) + \lambda_i f_{Ni} = 0, \\ f_{Ni}(0) = (u_0, w_i), \quad i = 1, 2, \cdots, N, \end{cases} \tag{3.8}$$

where the coefficients a_{ij}^N and b_{ij}^N are given by

$$a_{ij}^N(t) = (\rho_N w_j, w_i), \quad b_{ij}^N(t) = (\rho_N (v_N \cdot \nabla) w_j, w_i).$$

Rewrite the above system of ordinary differential equations in matrix form as

$$A_N(t) f'_N(t) + (B_N(t) + \Lambda_N) f_N(t) = 0, \quad f_N(t) = (f_{N1}(t), \cdots, f_{NN}(t))^T, \tag{3.9}$$

where $A_N(t) = (a_{ij}^N(t))_{N\times N}$, $B_N(t) = (b_{ij}^N(t))_{N\times N}$ and $\Lambda_N = diag\,(\lambda_1, \cdots, \lambda_N)$.

Since $\rho_N \in C^2(\bar{\Omega} \times [0,T])$ and $v_N \in C([0,T]; X_N) \subseteq C(\bar{\Omega} \times [0,T])$, it is clear that $A_N, B_N \in C([0,T])$. Besides, A_N is nonsingular. Otherwise, there are constants $\alpha_1, \cdots, \alpha_N$, not all zero, such that

$$A_N(t)\alpha = 0, \quad \alpha = (\alpha_1, \cdots, \alpha_N)^T,$$

that is

$$\sum_{j=1}^N a_{ij}^N(t)\alpha_j = \sum_{j=1}^N (\rho_N w_j, w_i)\alpha_j = \left(\rho_N \sum_{j=1}^N \alpha_j w_j, w_i\right) = 0, \quad i = 1, \cdots, N.$$

Multiplying by α_i the i-th equality of the above system, and summing up the resultants with respect to i yield

$$\left(\rho_N \sum_{j=1}^N \alpha_j w_j, \sum_{i=1}^N \alpha_i w_i\right) = 0,$$

wherefrom, recalling that $\rho_N \ge \underline{\rho} > 0$, we get $\sum_{i=1}^N \alpha_i w_i = 0$, which contradicts to the linearly independency of the basis $\{w_i\}_{i=1}^\infty$.

Thanks to the nondegeneracy of A_N, (3.9) can be reformed as

$$f'_N(t) + A_N^{-1}(t)(B_N(t) + \Lambda_N) f_N(t) = 0. \tag{3.10}$$

Since A_N and B_N are continuous on $[0,T]$, so is A_N^{-1}, the solvability of the initial value problem to the above system follows from the classical theory of the ordinary differential equations. Therefore, for any given $v_N \in C([0,T]; X_N)$, there is a unique solution $u_N \in C([0,T]; X_N)$ to (3.7), in other words, the solution mapping Q_N is well-defined. Moreover, noticing that the solution mapping $\rho_N = S_N[v_N]$ to system (3.5) is continuous in $v_N \in C([0,T]; X_N)$, it is straightforward that the matrices A_N and B_N, viewed as the functionals of v_N, are both continuous in v_N, so is A_N^{-1}. Therefore, in view of (3.10), f_N is continuous in v_N, and as a result the mapping $u_N = Q_N[v_N]$ is continuous with respect to $v_N \in C([0,T]; X_N)$.

3.2. *Solvability of (3.2)*

As mentioned before, in order to prove the solvability of (3.2), it suffices to find a fixed point to the solution mapping Q_N, with $u_N = Q_N[v_N]$ being the unique solution to the linearized system (3.7). Recall that in the previous subsection, we have shown that Q_N is a continuous mapping from $C([0,T]; X_N)$ to itself. We will apply the Brower fixed point theorem for compact continuous mappings to prove the existence of a fixed point to the mapping Q_N. To this end, recalling that the continuity of Q_N has been proven in the previous subsection, one still need to verify compactness of Q_N, which, noticing that X_N is a finite dimensional space, is guaranteed by the following proposition:

Proposition 3.1. *Let S_N and Q_N be the mappings defined as before. Then, for any $v_N \in C([0,T]; X_N)$, the following hold*

$$\sup_{0\le t\le T} \|\sqrt{S_N[v_N]}Q_N[v_N]\|_2^2 + 2\int_0^T \|\nabla Q_N[v_N]\|_2^2 dt \le \|\sqrt{\rho_{0N}}u_{0N}\|_2^2,$$

$$\sup_{0\le t\le T} \|\nabla Q_N[v_N]\|_2^2 + \int_0^T \|\sqrt{S_N[v_N]}\partial_t Q_N[v_N]\|_2^2 dt \le \|\nabla u_{0N}\|_2^2 e^{C_N T\|v_N\|^2_{C([0,T];X_N)}},$$

where C_N is a positive constant depending only on N, $\bar\rho$ and Ω.

Proof. Denote $\rho_N = S_N[v_N]$ and $u_N = Q_N[v_N]$. Taking $w = u_N$ in (3.7), then it follows from integration by parts and using equation (3.5) that

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho_N}u_N\|_2^2 + \|\nabla u_N\|_2^2 = 0,$$

from which, integrating in t yields the first conclusion.

Next, we prove the second conclusion. Choosing $w = \partial_t u_N$ in (3.7), and integration by parts, one obtains

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|_2^2 + \|\sqrt{\rho_N}\partial_t u_N\|_2^2 &= -\int_\Omega \rho_N(v_N\cdot\nabla)u_N\cdot\partial_t u_N dx \\
&\le \sqrt{\bar\rho}\|\sqrt{\rho_N}\partial_t u_N\|_2\|v_N\|_\infty\|\nabla u_N\|_2 \le C_N\|v_N\|_{X_N}\|\sqrt{\rho_N}\partial_t u_N\|_2\|\nabla u_N\|_2 \\
&\le \frac{1}{2}\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + C_N\|v_N\|_{X_N}^2\|\nabla u_N\|_2^2,
\end{aligned}$$

wherefrom, by the Gronwall inequality, the second conclusion follows. In the above, we have used the fact that the L^∞ norm and the norm $\|\cdot\|_{X_N}$ are equivalent, as X_N is a finite dimensional Banach space. □

Thanks to the above proposition, we can prove the global solvability of system (3.2), and we have the following:

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Corollary 3.1 *(Solvability of (3.2)). For any positive time T, there is a unique solution (ρ_N, u_N) to system (3.2), satisfying*

$$\rho_N \in C^2(\bar{\Omega} \times [0, T]), \quad u_N \in C^2([0, T]; X_N), \quad \underline{\rho} \le \rho \le \bar{\rho}.$$

Proof. As mentioned before, it suffices to find a fixed point to the mapping Q_N in $C([0, T]; X_N)$. The regularity of ρ_N has been mentioned several times in last subsection, while the regularity of u_N can be easily seen from the ordinary differential equations (3.10), in view of the fact that $A_N^{-1}, B_N \in C^2([0, T])$, which are guaranteed by the regularity of ρ_N. Thanks to the first conclusion in Proposition 3.1, we have the estimate

$$\|Q_N[v_N]\|_{C([0,T];X_N)} \le K, \quad \forall v_N \in C([0, T]; X_N),$$

where $K = K(N, \bar{\rho}, \|u_0\|_2^2)$ is a positive constant. By the second conclusion of Proposition 3.1, the following estimate holds

$$\|\partial_t Q_N[v_N]\|_{L^2(0,T;X_N)} \le C(N, T, K, \|u_0\|_{H^1}^2),$$

for any $v_N \in C([0, T]; X_N)$ subject to $\|v_N\|_{C([0,T];X_N)} \le K$. Recalling that X_N is a finite dimensional Banach space, by the Arzelá–Ascoli theorem, the above two estimates imply that Q_N is a compact mapping from $\mathcal{B}_K$ to itself, where $\mathcal{B}_K$ is the closed ball in $C([0, T]; X_N)$. Thanks to this fact, and recalling that Q_N is a continuous mapping from $C([0, T]; X_N)$ to itself, by the Brower fixed point theorem, there is a fixed point in $\mathcal{B}_K$ to the mapping Q_N. This completes the proof. □

3.3. Uniform in N estimates

In this subsection, we will establish some a priori estimates, which are uniform in N, in a short time, to the solution (ρ_N, u_N) established in Corollary 3.1.

Recall the expression of $u_N = \sum_{j=1}^N f_{Nj}(t) w_j$. On the one hand, choosing $w = w_i$ in (3.2), one obtains by integration by parts that

$$(-\Delta u_N, w_i) = (\nabla u_N, \nabla w_i) = (-\rho_N(\partial_t u_N + (u_N \cdot \nabla) u_N, w_i),$$

for $i = 1, 2, \cdots, N$. On the other hand, recalling (3.1), it follows from integration by parts that

$$\begin{aligned}(-\Delta u_N, w_i) &= \sum_{j=1}^N f_{Nj}(t)(-\Delta w_j, w_i) = \sum_{j=1}^N f_{Nj}(t)(\lambda_j w_j - \nabla p_j, w_i) \\ &= \sum_{j=1}^N f_{Nj}(t)\lambda_j \delta_{ij} = \lambda_i f_{Ni}(t).\end{aligned}$$

Thus, we have

$$f_{Ni}(t) = -\frac{1}{\lambda_i}(\rho_N(\partial_t u_N + (u_N \cdot \nabla) u_N, w_i).$$

Thanks to this, and using (3.1) again, one deduces

$$
\begin{aligned}
-\Delta u_N &= -\sum_{j=1}^{N} f_{Nj}\Delta w_j = \sum_{j=1}^{N}(\lambda_j w_j - \nabla p_j)(-\frac{1}{\lambda_j})(\rho_N \dot{u}_N, w_j) \\
&= \nabla\left(\sum_{j=1}^{N}\frac{1}{\lambda_j}(\rho_N\dot{u}_N, w_j)p_j\right) - \sum_{j=1}^{N}(\rho_N\dot{u}_N, w_j)w_j,
\end{aligned}
$$

with $\dot{u}_N = \partial_t u_N + (u_N\cdot\nabla)u_N$, or equivalently

$$
\Delta u_N + \nabla P_N = \sum_{j=1}^{N}(\rho_N(\partial_t u_N + (u_N\cdot\nabla)u_N), w_j)w_j, \tag{3.11}
$$

where the pressure P_N is given as $P_N = \sum_{j=1}^{N}\frac{1}{\lambda_j}(\rho_N\dot{u}_N, w_j)p_j$.

We first consider the H^1 estimate, that is the following proposition:

Proposition 3.2. *Let (ρ_N, u_N) be the solution established in Corollary 3.1. Then, there is a positive time T_0 depending only on $\bar{\rho}$, Ω and $\|\nabla u_0\|_2$, such that*

$$
\sup_{0\le t\le T_0}\|\nabla u_N\|_2^2 + \int_0^{T_0}(\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \|\nabla^2 u_N\|_2^2)dt \le C\|\nabla u_0\|_2^2,
$$

for a positive constant C depending only on $\bar{\rho}$ and Ω.

Proof. Taking $w = \partial_t u_N$ in (3.2), then it follows from integration by parts and the Young inequality that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|_2^2 + \|\sqrt{\rho_N}\partial_t u_N\|_2^2 &= -\int_\Omega \rho_N(u_N\cdot\nabla)u_N\cdot\partial_t u_N dx \\
&\le \frac{1}{2}\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \frac{1}{2}\int_\Omega \rho_N|u_N|^2|\nabla u_N|^2 dx,
\end{aligned}
$$

and thus

$$
\frac{d}{dt}\|\nabla u_N\|_2^2 + \|\sqrt{\rho_N}\partial_t u_N\|_2^2 \le \int_\Omega \rho_N|u_N|^2|\nabla u_N|^2 dx. \tag{3.12}
$$

Applying the H^2 estimate to (3.11), and noticing that $\left\|\sum_{j=1}^{N}(g, w_i)w_i\right\|_2 \le \|g\|_2$, for any $g\in L^2(\Omega)$, we deduce

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$$\begin{aligned}
\|\nabla^2 u_N\|_2^2 &\le C \left\| \sum_{j=1}^{N} (\rho_N(\partial_t u_N + u_N\cdot\nabla u_N), w_j) w_j \right\|_2^2 \\
&\le C\|\rho_N(\partial_t u_N + u_N\cdot\nabla u_N)\|_2^2 \\
&\le C\bar\rho\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + C\bar\rho \int_\Omega \rho_N |u_N|^2 |\nabla u_N|^2 dx \\
&\le M_1\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + C \int_\Omega \rho_N |u_N|^2 |\nabla u_N|^2 dx,
\end{aligned} \tag{3.13}$$

where M_1 and C are positive constants depending only on $\bar\rho$ and Ω.

Multiplying (3.12) by $2M_1$, and summing the resultant with (3.13), we obtain

$$2M_1 \frac{d}{dt}\|\nabla u_N\|_2^2 + M_1\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \|\nabla^2 u_N\|_2^2 \le C\int_\Omega \rho_N|u_N|^2|\nabla u_N|^2 dx, \tag{3.14}$$

for a positive constant C depending only on $\bar\rho$ and Ω.

We have to estimate the term $\int_\Omega \rho_N|u_N|^2|\nabla u_N|^2dx$. By the Hölder, Sobolev and Poincaré inequalities, we deduce

$$\begin{aligned}
C\int_\Omega \rho_N|u_N|^2|\nabla u_N|^2 dx &\le C\|u_N\|_6^2\|\nabla u_N\|_2\|\nabla u_N\|_6 \\
&\le C\|\nabla u_N\|_2^3\|\nabla^2 u_N\|_2 \le \frac12\|\nabla^2 u_N\|_2^2 + C\|\nabla u_N\|_2^6,
\end{aligned} \tag{3.15}$$

which, substituted into (3.14), gives

$$2M_1\frac{d}{dt}\|\nabla u_N\|_2^2 + (M_1\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \frac12\|\nabla^2 u_N\|_2^2) \le C\|\nabla u_N\|_2^6, \tag{3.16}$$

for a positive constant C depending only on $\bar\rho$ and Ω.

Set

$$F_N(t) = 2M_1\|\nabla u_N\|_2^2(t) + \int_0^t (M_1\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \frac12\|\nabla^2 u_N\|_2^2)ds.$$

Then, it follows from (3.16) that

$$F_N'(t) \le C_1 F_N^3(t), \quad t\in[0,T],$$

where C_1 is a positive constant depending only on $\bar\rho$ and Ω. Simple calculations to the above ordinary differential inequality yields

$$F_N(t) \le \frac{F_N(0)}{\sqrt{1-2C_1F_N^2(0)t}} = \frac{2M_1\|\nabla u_{0N}\|_2^2}{\sqrt{1-8C_1M_1^2\|\nabla u_{0N}\|_2^4 t}},$$

for any $t \in [0, (16C_1M_1^2\|\nabla u_{0N}\|_2^4)^{-1}]$, from which, noticing that $\|\nabla u_{0N}\|_2 \le \|\nabla u_0\|_2$, one obtains

$$F_N(t) \le 2\sqrt{2}M_1\|\nabla u_{0N}\|_2^2 \le 2\sqrt{2}M_1\|\nabla u_0\|_2^2, \quad t \in [0, (16C_1M_1^2\|\nabla u_0\|_2^4)^{-1}].$$

This completes the proof of Proposition 3.2. □

Next, we study the t-weighted H^2 estimate, which is stated in the next proposition.

Proposition 3.3. *Let (ρ_N, u_N) be the solution established in Corollary 3.1 and T_0 the number in Proposition 3.2. Then, the following estimate holds*

$$\sup_{0\le t\le T_0} t(\|\nabla^2 u_N\|_2^2 + \|\sqrt{\rho_N}\partial_t u_N\|_2^2) + \int_0^{T_0} t\|\nabla\partial_t u_N\|_2^2 dt \le C,$$

for a positive constant C depending only on $\bar\rho$, T_0, Ω and $\|\nabla u_0\|_2$.

Proof. Differentiating $(3.2)_2$ with respect to t, and using $(3.2)_1$ yield

$$\begin{aligned}
&(\rho_N(\partial_t^2 u_N + (u_N\cdot\nabla)\partial_t u_N), w) + (\nabla\partial_t u_N, \nabla w)\\
=&(\mathrm{div}\,(\rho_N u_N)(\partial_t u_N + (u_N\cdot\nabla)u_N), w) - (\rho_N(\partial_t u_N\cdot\nabla)u_N, w),
\end{aligned}$$

for all $w \in X_N$. Taking $w = \partial_t u_N$ in the above equality, then it follows from integration by parts and using equation $(3.2)_1$ that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \|\nabla\partial_t u_N\|_2^2\\
&= (\mathrm{div}\,(\rho_N u_N)(\partial_t u_N + (u_N\cdot\nabla)u_N) - \rho_N\partial_t u_N\cdot\nabla u_N, \partial_t u_N)\\
&\le \int_\Omega [\rho_N|u_N|(2|\partial_t u_N||\nabla\partial_t u_N| + |u_N||\nabla u_N||\nabla\partial_t u_N|\\
&\quad + |u_N||\nabla^2 u_N||\partial_t u_N| + |\nabla u_N|^2|\partial_t u_N|) + \rho_N|\partial_t u_N|^2|\nabla u_N|]dx\\
&= 2\int_\Omega \rho_N|u_N||\partial_t u_N||\nabla\partial_t u_N|dx + \int_\Omega \rho_N|u_N|^2|\nabla u_N||\nabla\partial_t u_N|dx\\
&\quad + \int_\Omega \rho_N|u_N|^2|\nabla^2 u_N||\partial_t u_N|dx + \int_\Omega \rho_N|u_N||\nabla u_N|^2|\partial_t u_N|dx\\
&\quad + \int_\Omega \rho_N|\partial_t u_N|^2|\nabla u_N|dx =: \sum_{i=1}^5 I_i. \qquad (3.17)
\end{aligned}$$

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We estimate $I_i, i = 1, 2, \cdots, 5$, as follows. By the Gagliardo–Nirenberg inequality, $\|f\|_\infty \le C\|f\|_6^{\frac12}\|f\|_{H^2}^{\frac12}$, it follows from the Sobolev and Poincaré inequalities that

$$\|u_N\|_\infty \le C\|u_N\|_6^{\frac12}\|u_N\|_{H^2}^{\frac12} \le C\|\nabla u_N\|_2^{\frac12}\|\nabla^2 u_N\|_2^{\frac12}. \tag{3.18}$$

Thanks to this, by the Hölder inequality, we can estimate I_1 and I_2 as

$$\begin{aligned} I_1 &\le 2\|\sqrt{\rho_N}\|_\infty\|u_N\|_\infty\|\sqrt{\rho_N}\partial_t u_N\|_2\|\nabla\partial_t u_N\|_2 \\ &\le C\|\nabla u_N\|_2^{\frac12}\|\nabla^2 u_N\|_2^{\frac12}\|\sqrt{\rho_N}\partial_t u_N\|_2\|\nabla\partial_t u_N\|_2, \end{aligned}$$

and

$$I_2 \le \|\rho_N\|_\infty\|u_N\|_\infty^2\|\nabla u_N\|_2\|\nabla\partial_t u_N\|_2 \le C\|\nabla u_N\|_2^2\|\nabla^2 u_N\|_2\|\nabla\partial_t u_N\|_2,$$

respectively. By the Hölder, Sobolev and Poincaré inequalities, we deuce

$$\begin{aligned} I_3 &\le \|\rho_N\|_\infty\|u_N\|_6^2\|\nabla^2 u_N\|_2\|\partial_t u_N\|_6 \le C\|\nabla u_N\|_2^2\|\nabla^2 u_N\|_2\|\nabla\partial_t u_N\|_2, \\ I_4 &\le \|\rho_N\|_\infty\|u_N\|_6\|\nabla u_N\|_2\|\nabla u_N\|_6\|\partial_t u_N\|_6 \le C\|\nabla u_N\|_2^2\|\nabla^2 u_N\|_2\|\nabla\partial_t u_N\|_2, \end{aligned}$$

and

$$\begin{aligned} I_5 \le&\|\sqrt{\rho_N}\|_\infty\|\sqrt{\rho_N}\partial_t u_N\|_2\|\nabla u_N\|_3\|\partial_t u_N\|_6 \\ \le& C\|\sqrt{\rho_N}\partial_t u_N\|_2\|\nabla u_N\|_2^{\frac12}\|\nabla^2 u_N\|_2^{\frac12}\|\nabla\partial_t u_N\|_2. \end{aligned}$$

Substituting the estimates on $I_i, i = 1, 2, \cdots, 5$, into (3.17), and using the Young inequality, one obtains

$$\begin{aligned} &\frac12\frac{d}{dt}\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \|\nabla\partial_t u_N\|_2^2 \\ \le& C\|\nabla u_N\|_2^{\frac12}\|\nabla^2 u_N\|_2^{\frac12}\|\sqrt{\rho_N}\partial_t u_N\|_2\|\nabla\partial_t u_N\|_2 + C\|\nabla u_N\|_2^2\|\nabla^2 u_N\|_2\|\nabla\partial_t u_N\|_2 \\ \le& \frac12\|\nabla\partial_t u_N\|_2^2 + C\|\nabla u_N\|_2\|\nabla^2 u_N\|_2\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + C\|\nabla u_N\|_2^4\|\nabla^2 u_N\|_2^2, \end{aligned}$$

which implies

$$\begin{aligned} &\frac{d}{dt}\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \|\nabla\partial_t u_N\|_2^2 \\ \le& C(\|\nabla u_N\|_2\|\nabla^2 u_N\|_2\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \|\nabla u_N\|_2^4\|\nabla^2 u_N\|_2^2). \end{aligned} \tag{3.19}$$

Multiplying the above inequality by t yields

$$\frac{d}{dt}(t\|\sqrt{\rho_N}\partial_t u_N\|_2^2) + t\|\nabla\partial_t u_N\|_2^2 \le C(t\|\nabla u_N\|_2^4\|\nabla^2 u_N\|_2^2 + \|\sqrt{\rho_N}\partial_t u_N\|_2^2)$$
$$+C\|\nabla u_N\|_2\|\nabla^2 u_N\|_2 t\|\sqrt{\rho_N}\partial_t u_N\|_2^2,$$

from which, by the Gronwall inequality, and using Proposition 3.2, we obtain

$$\sup_{0\le t\le T_0}(t\|\sqrt{\rho_N}\partial_t u_N\|_2^2(t)) + \int_0^{T_0} t\|\nabla\partial_t u_N\|_2^2(t)dt \le C, \tag{3.20}$$

for a positive constant C depending only on $\bar\rho$, T_0, Ω and $\|\nabla u_0\|_2$.

Substituting (3.15) into (3.13), one obtains

$$\|\nabla^2 u_N\|_2^2 \le C(\|\sqrt{\rho_N}\partial_t u_N\|_2^2 + \|\nabla u_N\|_2^6), \tag{3.21}$$

for a positive constant C depending only on $\bar\rho$ and Ω, which along with (3.20) yields the conclusion. □

4. Local existence and uniqueness

This section is devoted to proving the local existence and uniqueness of strong solutions to system (1.1)–(1.3), subject to (1.6)–(1.7), in other words, we will give the proof of Theorem 1.1.

Let us first consider the case that the initial density has positive lower bound, and we have the following proposition:

Proposition 4.1. *Suppose that the initial data $(\rho_0, u_0) \in (W^{1,\gamma}\cap L^\infty)\times H^1_{0,\sigma}$, for some $\gamma\in[1,\infty)$, and that $\underline{\rho}\le\rho_0\le\bar\rho$, for two positive constants $\underline{\rho}$ and $\bar\rho$, and let T_0 be the positive time stated in Proposition 3.3.*

Then, there is a strong solution (ρ, u) to system (1.1)–(1.3), subject to (1.6)–(1.7), in $\Omega\times(0,T_0)$, such that $\underline{\rho}\le\rho\le\bar\rho$, and

$$\sup_{0\le t\le T_0}[\|\nabla u\|_2^2 + \|\nabla\rho\|_\gamma^\gamma + t(\|\nabla^2 u\|_2^2 + \|\sqrt{\rho}\partial_t u\|_2^2)] + \int_0^{T_0}[\|\nabla^2 u\|_2^2$$
$$+\|\sqrt{\rho}\partial_t u\|_2^2 + t(\|\nabla^2 u\|_6^2 + \|\nabla\partial_t u\|_2^2) + \|\nabla u\|_\infty + \|\partial_t\rho\|_\gamma^4]dt \le C,$$

for a positive constant C depending only on $\bar\rho$, Ω, $\|\nabla u_0\|_2$ and $\|\nabla\rho_0\|_\gamma$.

Proof. Choose a sequence of $\rho_{0N}\in C^2(\bar\Omega)$, such that

$$\underline{\rho}\le\rho_{0N}\le\bar\rho,\quad \rho_{0N}\to\rho_0,\ \text{in } W^{1,\gamma}(\Omega),\quad \|\nabla\rho_{0N}\|_\gamma\le\|\nabla\rho_0\|_\gamma.$$

Let $\{w_i\}_{i=1}^\infty$ be the sequence of eigenfunctions to (3.1), as stated in the previous section, Section 3. For any positive integer N, we set $u_{0N}=\sum_{i=1}^N(u_0,w_i)w_i$. Then, $u_{0N}\to u_0$ in $H^1(\Omega)$.

By Corollary 3.1 and Propositions 3.2–3.3, for any positive integer N, there is a solution (ρ_N, u_N) to system (3.2), such that $\underline{\rho}\le\rho_N\le\bar\rho$, and

$$\sup_{0\le t\le T_0}[\|\nabla u_N\|_2^2+t(\|\nabla^2u_N\|_2^2+\|\sqrt{\rho_N}\partial_tu_N\|_2^2)]$$
$$+\int_0^{T_0}(\|\sqrt{\rho_N}\partial_tu_N\|_2^2+\|\nabla^2u_N\|_2^2+t\|\nabla\partial_tu_N\|_2^2)dt\le C,$$

where T_0 is the positive time stated in Proposition 3.2, and C is a positive constant depending only on $\bar\rho$, T_0, Ω and $\|\nabla u_0\|_2$.

Thanks to the above estimate, using the Cantor diagonal argument, and applying Lemma 2.1, there is a subsequence of $\{(\rho_N,u_N)\}_{N=1}^\infty$, still denoted by $\{(\rho_N,u_N)\}_{N=1}^\infty$, and a pair (ρ,u), with $\underline{\rho}\le\rho_N\le\bar\rho$ and

$$\sup_{0\le t\le T_0}[\|\nabla u\|_2^2+t(\|\nabla^2u\|_2^2+\|\sqrt{\rho}\partial_tu\|_2^2)]$$
$$+\int_0^{T_0}(\|\sqrt{\rho}\partial_tu\|_2^2+\|\nabla^2u\|_2^2+t\|\nabla\partial_tu\|_2^2)dt\le C, \tag{4.1}$$

for a positive constant C depending only on $\bar\rho$, T_0, Ω and $\|\nabla u_0\|_2$, such that

$$\rho_N\to\rho,\ \text{in } C([0,T_0];L^q),\quad q\in[1,\infty),$$
$$u_N\overset{*}{\rightharpoonup}u,\ \text{in } L^\infty(0,T_0;H^1)\cap L^\infty(\tau,T_0;H^2),$$
$$u_N\rightharpoonup u,\ \text{in } L^2(0,T_0;H^2),\quad \partial_tu_N\overset{*}{\rightharpoonup}\partial_tu,\ \text{in } L^\infty(\tau,T_0;L^2),$$
$$\partial_tu_N\rightharpoonup\partial_tu,\ \text{in } L^2(0,T_0;L^2)\cap L^2(\tau,T_0;H^1),$$

for any $\tau=\frac{T_0}{k}$, $k=2,3,\cdots$, and thus for any $\tau\in(0,T_0)$, where $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ denote the weak and weak-* convergences, respectively. Noticing that $H^2\hookrightarrow\hookrightarrow H^1\hookrightarrow L^2$, by the Aubin–Lions compactness lemma, we have $u_N\to u$ in $C([0,T_0];L^2)\cap L^2(0,T_0;H^1)$. Therefore, we have $(\rho,u)|_{t=0}=(\rho_0,u_0)$.

Thanks to the previous convergences, it is clear that (ρ,u) satisfies (1.1), in the sense of distribution, and moreover, since ρ has the regularities $\rho\in L^\infty(0,T_0;W^{1,\gamma})$ and $\partial_t\rho\in L^4(0,T_0;L^\gamma)$, which will be proven in the below, (ρ,u) satisfies equation (1.1) pointwisely, a.e. in $\Omega\times(0,T_0)$. The previous convergences also imply

$$\rho_N\partial_tu_N\rightharpoonup\rho\partial_tu,\ \text{in } L^2(0,T_0;L^2),$$
$$\rho_N(u_N\cdot\nabla)u_N\rightharpoonup\rho(u\cdot\nabla)u,\ \text{in } L^2(0,T_0;L^2).$$

Consequently, one can take the limit $N\to\infty$ in the momentum equation in (3.2) to deduce that

$$(\rho(\partial_tu+(u\cdot\nabla)u),w_i)+(\nabla u,\nabla w_i)=0,\quad \text{for any positive integer } i,$$

or equivalently, by integration by parts, that

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$$(\rho(\partial_t u + (u\cdot\nabla)u) - \Delta u, w_i) = 0, \quad \text{for any positive integer } i.$$

Since $\{w_N\}_{N=1}^\infty$ is a basis in $L^2_\sigma(\Omega)$, and noticing that

$$\rho(\partial_t u + (u\cdot\nabla)u) - \Delta u \in L^2(0,T_0;L^2),$$

a density argument yields

$$(\rho(\partial_t u + (u\cdot\nabla)u) - \Delta u, \phi) = 0, \quad \forall w \in L^2_\sigma(\Omega).$$

Thanks to this, by Lemma 2.3 and Lemma 2.4, there is a function $\Phi \in L^2(0,T_0;H^1)$, such that

$$\rho(\partial_t u + (u\cdot\nabla)u) - \Delta u = \nabla\Phi,$$

which is exactly the momentum equation (1.2), by setting $p = -\Phi$.

In order to complete the proof of Proposition 4.1, one still need to verify

$$\sup_{0\le t\le T_0}\|\nabla\rho\|_\gamma^\gamma + \int_0^{T_0}(t\|\nabla^2 u\|_6^2 + \|\nabla u\|_\infty + \|\partial_t\rho\|_\gamma^4)dt \le C,$$

for a positive constant C depending only on $\bar\rho$, T_0, Ω, $\|\nabla u_0\|_2$ and $\|\nabla\rho_0\|_\gamma$.

Similar to (3.18), one has $\|u\|_\infty^2 \le C\|\nabla u\|_2\|\nabla^2 u\|_2$. Thanks to this, by the elliptic estimate for the Stokes equations, and using the Sobolev inequality, we deduce

$$\begin{aligned}\|\nabla^2 u\|_6 &\le C\|\rho(\partial_t u + u\cdot\nabla u)\|_6 \le C(\|\partial_t u\|_6 + \|u\|_\infty\|\nabla u\|_6)\\ &\le C(\|\nabla\partial_t u\|_2 + \|\nabla u\|_2^{\frac12}\|\nabla^2 u\|_2^{\frac32}),\end{aligned}$$

and thus, recalling (4.1), we obtain

$$\int_0^{T_0} t\|\nabla^2 u\|_6^2 dt \le C\int_0^{T_0} t(\|\nabla\partial_t u\|_2^2 + \|\nabla u\|_2\|\nabla^2 u\|_2^3)dt \le C,$$

for a positive constant C depending only on $\bar\rho$, T_0, Ω and $\|\nabla u_0\|_2$.

By the Gagliardo–Nirenberg inequality, $\|f\|_\infty \le C(\Omega)\|f\|_6^{\frac12}\|f\|_{W^{1,6}(\Omega)}^{\frac12}$, and using the Sobolev and Poincaré inequalities, we have

$$\|\nabla u\|_\infty \le C\|\nabla u\|_6^{\frac12}\|\nabla u\|_{W^{1,6}}^{\frac12} \le C\|\nabla^2 u\|_2^{\frac12}\|\nabla^2 u\|_6^{\frac12}.$$

Thus, it follows from the Hölder inequality and (4.1) that

$$\int_0^{T_0} \|\nabla u\|_\infty dt \le C \int_0^{T_0} \|\nabla^2 u\|_2^{\frac{1}{2}} \|\nabla^2 u\|_6^{\frac{1}{2}} dt = C \int_0^{T_0} \|\nabla^2 u\|_2^{\frac{1}{2}} (t\|\nabla^2 u\|_6^2)^{\frac{1}{4}} t^{-\frac{1}{4}} dt$$

$$\le C \left(\int_0^{T_0} \|\nabla^2 u\|_2^2 dt \right)^{\frac{1}{4}} \left(\int_0^{T_0} t\|\nabla^2 u\|_6^2 dt \right)^{\frac{1}{4}} \left(\int_0^{T_0} t^{-\frac{1}{2}} dt \right)^{\frac{1}{2}} \le C,$$

for a positive constant C depending only on $\bar\rho$, T_0, Ω and $\|\nabla u_0\|_2$. Thanks to this estimate, and applying Lemma 2.2, one obtains

$$\sup_{0\le t\le T_0} \|\nabla\rho\|_\gamma^\gamma \le C,$$

for a positive constant C depending only on $\bar\rho$, T_0, Ω, $\|\nabla u_0\|_2$ and $\|\nabla\rho_0\|_\gamma$.

Recall that $\|u\|_\infty \le C(\Omega)\|\nabla u\|_2^{\frac{1}{2}} \|\nabla^2 u\|_2^{\frac{1}{2}}$, it follows from the continuity equation (1.1) that

$$\int_0^{T_0} \|\partial_t\rho\|_\gamma^4 dt \le \int_0^{T_0} \|u\|_\infty^4 \|\nabla\rho\|_\gamma^4 dt \le C \int_0^{T_0} \|\nabla u\|_2^2 \|\nabla^2 u\|_2^2 \|\nabla\rho\|_\gamma^4 dt \le C,$$

for a positive constant C depending only on $\bar\rho$, T_0, Ω, $\|\nabla u_0\|_2$ and $\|\nabla\rho_0\|_\gamma$. This completes the proof of Proposition 4.1. $\square$

We are now ready to prove our main result, Theorem 1.1.

Proof of Theorem 1.1. Take a sequence $\{\rho_{0n}\}_{n=1}^\infty$, such that

$$\frac{1}{n} \le \rho_{0n} \le \bar\rho + 1, \quad \rho_{0n} \to \rho_0 \text{ in } W^{1,\gamma}, \quad \|\nabla\rho_{0n}\|_\gamma \le \|\nabla\rho_0\|_\gamma + 1.$$

By Proposition 4.1, there is a positive time T_0 depending only on $\bar\rho$, Ω, $\|\nabla u_0\|_2$, such that, for each n, there is a strong solution (ρ_n, u_n) to system (1.1)–(1.3), subject to (1.6)–(1.7), with initial data (ρ_{0n}, u_0), in $\Omega\times(0, T_0)$, satisfying $\frac{1}{n} \le \rho_{0n} \le \bar\rho + 1$ and

$$\sup_{0\le t\le T_0} [\|\nabla u_n\|_2^2 + \|\nabla\rho_n\|_\gamma^\gamma + t(\|\nabla^2 u_n\|_2^2 + \|\sqrt{\rho_n}\partial_t u_n\|_2^2)] + \int_0^{T_0} [\|\nabla^2 u_n\|_2^2$$
$$+ \|\sqrt{\rho_n}\partial_t u_n\|_2^2 + t(\|\nabla^2 u_n\|_6^2 + \|\nabla\partial_t u_n\|_2^2) + \|\nabla u_n\|_\infty + \|\partial_t\rho_n\|_\gamma^4] dt \le C, \tag{4.2}$$

for a positive constant C depending only on $\bar\rho$, T_0, Ω, $\|\nabla u_0\|_2$ and $\|\nabla\rho_0\|_\gamma$.

Thanks to the above estimates, by the Cantor diagonal argument, there is a subsequence of $\{(\rho_n, u_n)\}_{n=1}^\infty$, still denoted by $\{(\rho_n, u_n)\}_{n=1}^\infty$, and a pair (ρ, u), such that

$$
\begin{aligned}
u_n &\overset{*}{\rightharpoonup} u, \text{ in } L^\infty(0,T_0;H^1)\cap L^\infty(\tau,T_0;H^2),\\
u_n &\rightharpoonup u, \text{ in } L^2(0,T_0;H^2)\cap L^2(\tau,T_0;W^{2,6}),\\
\partial_t u_n &\rightharpoonup \partial_t u, \text{ in } L^2(\tau,T_0;H^1),\\
\rho_n &\overset{*}{\rightharpoonup} \rho, \text{ in } L^\infty(0,T_0;W^{1,\gamma}\cap L^\infty), \quad \partial_t\rho_n \rightharpoonup \partial_t\rho, \text{ in } L^4(0,T_0;L^\gamma),
\end{aligned}
$$

for any $\tau = \frac{T_0}{k}$, $k=1,2,\cdots$, and thus for any $\tau\in(0,T_0)$. Therefore, by the Aubin–Lions compactness lemma, the following two strong convergences hold

$$
\begin{aligned}
u_n &\to u, \text{ in } C([\tau,T_0];H^1\cap L^6)\cap L^2(\tau,T_0;C(\bar\Omega)),\\
\rho_n &\to \rho, \text{ in } C([0,T_0];L^q), \quad \text{for any } 1\le q<\infty.
\end{aligned}
$$

Claim 1: (ρ,u) has the regularities stated in Theorem 1.1, and satisfies system (1.1)–(1.3) pointwisely, a.e. in $\Omega\times(0,T_0)$.

The regularities of (ρ,u) stated in Theorem 1.1, except $\rho u\in C([0,T_0];L^2)$, which will be proven in Claim 2, below, follow from the previous weakly convergences. Besides, thanks to the previous convergences, one can check that

$$
\begin{aligned}
&u_n\cdot\nabla\rho_n \rightharpoonup u\cdot\nabla\rho, \text{ in } L^2(\tau,T_0;L^\gamma),\\
&\rho_n\partial_t u_n \rightharpoonup \rho\partial_t u, \text{ in } L^2(\tau,T_0;L^2),\\
&\rho_n(u_n\cdot\nabla)u_n \rightharpoonup \rho(u\cdot\nabla)u, \text{ in } L^2(\tau,T_0;L^2),
\end{aligned}
$$

for any $\tau\in(0,T_0)$. Therefore, one can take the limit $n\to\infty$ to see that (ρ,u) satisfies equations (1.1)–(1.3), in the sense of distribution, and further a.e. in $\Omega\times(0,T_0)$, by the regularities of (ρ,u). This proves claim 1.

Claim 2: $\rho u\in C([0,T_0];L^2)$ and (ρ,u) satisfies the initial condition (1.7).

Since $\rho_n\to\rho$ in $C([0,T_0];L^2)$, $\rho_n|_{t=0}=\rho_{0n}$ and $\rho_{0n}\to\rho_0$ in $W^{1,\gamma}$, one has $\rho|_{t=0}=\rho_0$. Recall that $u_n\to u$ in $C([\tau,T_0];H^1)$ and $\rho_n\to\rho$ in $C([0,T_0];L^q)$, for any $q\in[1,\infty)$, it is clear that $\rho_nu_n\to\rho u$, in $C([\tau,T_0];L^2)$, for any $\tau\in(0,T_0)$. It remains to verify the continuity of ρu at the original time and the initial data of ρu.

Similar to (3.18), one has $\|u_n\|_\infty^2\le C\|\nabla u_n\|_2\|\nabla^2u_n\|_2$. Thanks to this, recalling that $\rho_n\le\bar\rho+1$, and using (4.2), it follow from the Hölder that

$$
\|(\rho_nu_n)(t)-\rho_{0n}u_0\|_1=\left\|\int_0^t\partial_t(\rho_nu_n)d\tau\right\|_1
$$

$$
=\left\|\int_0^t(\rho_n\partial_tu_n+\partial_t\rho_nu_n)d\tau\right\|_1\le\int_0^t(\|\rho_n\partial_tu_n\|_1+\|\partial_t\rho_nu_n\|_1)d\tau
$$

$$
\le C\int_0^t(\|\sqrt{\rho_n}\|_\infty\|\sqrt{\rho_n}\partial_tu_n\|_2+\|\partial_t\rho_n\|_\gamma\|u_n\|_\infty)d\tau
$$

$$
\begin{aligned}
&\le C\int_0^t (\|\sqrt{\rho_n}\partial_t u_n\|_2 + \|\partial_t\rho_n\|_\gamma \|\nabla u_n\|_2^{\frac12}\|\nabla^2 u_n\|_2^{\frac12})d\tau \\
&\le C\sqrt{t}\left[\left(\int_0^t \|\sqrt{\rho_n}\partial_t u_n\|_2^2 d\tau\right)^{\frac12} + \left(\int_0^t \|\partial_t\rho_n\|_\gamma^4 d\tau\right)^{\frac14}\left(\int_0^t \|\nabla^2 u_n\|_2^2 d\tau\right)^{\frac14}\right] \\
&\le C(\gamma, \bar\rho, T_0, \Omega, \|\nabla u_0\|_2, \|\nabla\rho_0\|_\gamma)\sqrt{t},
\end{aligned}
$$

for any $t \in (0, T_0)$. With the aid of the above estimate, and using (4.2) again, it follows from the Hölder and Sobolev inequalities that

$$
\begin{aligned}
\|(\rho_n u_n)(t) - \rho_{0n}u_0\|_2 &\le \|(\rho_n u_n)(t) - \rho_{0n}u_n\|_1^{\frac25}\|(\rho_n u_n)(t) - \rho_{0n}u_n\|_6^{\frac35} \\
&\le Ct^{\frac15}(\|\nabla u_n\|_2(t) + \|\nabla u_0\|_2)^{\frac35} \le Ct^{\frac15},
\end{aligned}
$$

for any $t \in (0, T_0)$, where C is a positive constant independent of n.

Thanks to the above estimate, for any $t \in (0, T_0)$, we have

$$
\begin{aligned}
&\|(\rho u)(t) - \rho_0 u_0\|_2 \\
&\le \|(\rho u)(t) - (\rho_n u_n)(t)\|_2 + \|(\rho_n u_n)(t) - \rho_{0n}u_0\|_2 + \|\rho_{0n}u_0 - \rho_0 u_0\|_2 \\
&\le \|(\rho u)(t) - (\rho_n u_n)(t)\|_2 + \|\rho_{0n}u_0 - \rho_0 u_0\|_2 + Ct^{\frac15},
\end{aligned}
$$

for any positive integer n, where C is a positive constant independent of n. Recall that $\rho_n u_n \to \rho u$ in $C([\tau, T_0]; L^2)$, for any $\tau \in (0, T_0)$, and $\rho_n \to \rho$ in $C([0, T]; L^q)$, for any $q \in [1, \infty)$. Thus, we have

$$
\|(\rho u)(t) - \rho_0 u_0\|_2 \le \varliminf_{n\to\infty} (\|(\rho u)(t) - (\rho_n u_n)(t)\|_2 + \|\rho_{0n}u_0 - \rho_0 u_0\|_2) + Ct^{\frac15} = Ct^{\frac15},
$$

for any $t \in (0, T_0)$, which implies that ρu is continuous at the original time and satisfies the initial condition $\rho u|_{t=0} = \rho_0 u_0$. This proves Claim 2, and thus further the existence part of Theorem 1.1.

We now prove the uniqueness part of Theorem 1.1. Let $(\tilde\rho, \tilde u)$ and $(\hat\rho, \hat u)$ be two local strong solutions to system (1.1)–(1.3), subject to (1.6)–(1.7), on $\Omega \times (0, T)$, for a positive time T, with the same initial data (u_0, ρ_0). Then, we have following regularities

$$
\tilde\rho, \hat\rho \in L^\infty(0, T; H^1), \quad \tilde u, \hat u \in L^2(0, T; H^2), \tag{4.3}
$$

$$
\sqrt{t}\tilde u, \sqrt{t}\hat u \in L^\infty(0, T; H^2), \quad \sqrt{t}\partial_t\tilde u, \sqrt{t}\hat u \in L^2(0, T; H^1). \tag{4.4}
$$

Setting

$$
\rho = \tilde\rho - \hat\rho, \quad u = \tilde u - \hat u,
$$

then (ρ, u) satisfies

$$\partial_t \rho + \tilde{u} \cdot \nabla \rho + u \cdot \nabla \hat{\rho} = 0, \tag{4.5}$$

$$\operatorname{div} u = 0, \tag{4.6}$$

$$\tilde{\rho}(\partial_t u + \tilde{u} \cdot \nabla u) - \Delta u + \nabla p = -\tilde{\rho} u \cdot \nabla \hat{u} - \rho(\partial_t \hat{u} + \hat{u} \cdot \nabla \hat{u}), \tag{4.7}$$

a.e. in $\Omega \times (0, T)$.

Multiplying equation (4.5) by $|\rho|^{-\frac{1}{2}}\rho$, and integration by parts, then it follows from the Hölder, Sobolev and Poincaré inequalities that

$$\begin{aligned} \frac{2}{3}\frac{d}{dt}\|\rho\|_{3/2}^{3/2} &= -\int_\Omega u \cdot \nabla \hat{\rho} |\rho|^{-\frac{1}{2}} \rho dx \le \|u\|_6 \|\nabla \hat{\rho}\|_2 \|\rho\|_{3/2}^{1/2} \\ &\le C\|\nabla u\|_2 \|\nabla \hat{\rho}\|_2 \|\rho\|_{3/2}^{1/2}, \end{aligned}$$

from which, recalling (4.3), one obtains

$$\frac{d}{dt}\|\rho\|_{3/2} \le A\|\nabla u\|_2, \tag{4.8}$$

for a positive constant A.

Same to (3.18), we have $\|\hat{u}\|_\infty^2 \le C\|\nabla \hat{u}\|_2 \|\nabla^2 \hat{u}\|_2$. Multiplying equation (4.7) by u, and using equation (4.5), it follows from integration by parts, the Hölder, Sobolev, Poincaré and Young inequalities that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\sqrt{\tilde{\rho}}u\|_2^2 + \|\nabla u\|_2^2 = -\int_\Omega [\tilde{\rho} u \cdot \nabla \hat{u} - \rho(\partial_t \hat{u} + \hat{u} \cdot \nabla \hat{u})] \cdot u dx \\ &\le \|\sqrt{\tilde{\rho}}\|_\infty \|\sqrt{\tilde{\rho}}u\|_2 \|u\|_6 \|\nabla \hat{u}\|_3 + \|\rho\|_{3/2}(\|\partial_t \hat{u}\|_6 \|u\|_6 + \|\hat{u}\|_\infty \|\nabla \hat{u}\|_6 \|u\|_6) \\ &\le C\|\sqrt{\tilde{\rho}}u\|_2 \|\nabla u\|_2 \|\nabla \hat{u}\|_{H^1} + C\|\rho\|_{3/2}(\|\nabla \partial_t \hat{u}\|_2 \|\nabla u\|_2 + \|\nabla \hat{u}\|_2^{1/2} \|\nabla \hat{u}\|_{H^1}^{3/2} \|\nabla u\|_2) \\ &\le \frac{1}{2}\|\nabla u\|_2^2 + C\|\nabla \hat{u}\|_{H^1}^2 \|\sqrt{\tilde{\rho}}u\|_2^2 + C(\|\nabla \partial_t \hat{u}\|_2^2 + \|\nabla \hat{u}\|_2 \|\nabla \hat{u}\|_{H^1}^3)\|\rho\|_{3/2}^2, \end{aligned}$$

from which, one obtains

$$\frac{d}{dt}\|\sqrt{\tilde{\rho}}u\|_2^2 + \|\nabla u\|_2^2 \le \alpha(t)\|\sqrt{\tilde{\rho}}u\|_2^2 + \beta(t)\|\rho\|_{3/2}^2, \tag{4.9}$$

where

$$\alpha(t) = C\|\nabla \hat{u}\|_{H^1}^2(t), \quad \beta(t) = C(\|\nabla \partial_t \hat{u}\|_2^2 + \|\nabla \hat{u}\|_2 \|\nabla \hat{u}\|_{H^1}^3)(t).$$

Recalling (4.4), it is clear that $\alpha \in L^1((0, T))$ and $t\beta(t) \in L^1((0, T))$. As a result, combining (4.8) and (4.9), and applying Lemma 2.5, one obtains $\|\rho\|_{3/2} \equiv \|\sqrt{\tilde{\rho}}u\|_2 \equiv \|\nabla u\|_2 \equiv 0$. Thus $\rho \equiv u \equiv 0$, proving the uniqueness part of Theorem 1.1. □

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