Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 266 (2019) 2294–2326
Journal of Differential Equations
www.elsevier.com/locate/jde

Boundedness of semilinear Duffing equations at resonance in a critical situation

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Received 18 October 2017; revised 6 July 2018
Available online 22 August 2018

Abstract

In this paper, we propose a sufficient and necessary condition for the boundedness of all the solutions for the equation $\ddot{x} + n^2 x + g(x) = p(t)$ with the critical situation that $\left|\int_0^{2\pi} p(t)e^{-int}dt\right| = 2\left|g(+\infty) - g(-\infty)\right|$ on g and p, where $n \in \mathbb{N}^+$, $p(t)$ is periodic and $g(x)$ is bounded.

MSC: 34C15; 70H08

Keywords: Semilinear Duffing equations; Boundedness; Canonical transformation; At resonance; Critical situation; Moser's theorem

1. Introduction and the main results

The boundedness for semilinear equations at resonance is very complicate. It is well known that the linear equation

$$\ddot{x} + n^2 x = \sin nt, \qquad n \in \mathbb{N}^+$$

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https://doi.org/10.1016/j.jde.2018.08.031

has no bounded solutions, where $\ddot{x} = d^2x/dt^2$. Ding [4] constructed another example as follows:

$$\ddot{x} + n^2 x + \arctan x = 4\cos nt, \qquad n \in \mathbb{N}^+$$

for which each solution is unbounded. Due to these resonance phenomenons, the existence of bounded solutions and the boundedness of all the solutions for semilinear equation at resonance are very delicate.

In 1969, Lazer and Leach [7] studied the following semilinear equations:

$$\ddot{x} + n^2 x + g(x) = p(t), \qquad n \in \mathbb{N}^+, \tag{1.1}$$

where $p(t + 2\pi) = p(t)$ and g is continuous and bounded. They obtained the existence of a periodic solution of (1.1) if

$$\left| \int_0^{2\pi} p(t) e^{-int} dt \right| < 2(\liminf_{x \to +\infty} g - \limsup_{x \to -\infty} g). \tag{1.2}$$

In addition, they obtained that each solution of (1.1) is unbounded if

$$\left| \int_0^{2\pi} p(t) e^{-int} dt \right| \geq 2(\sup g - \inf g). \tag{1.3}$$

It implies that the Lazer-Leach condition (1.2) is also necessary for the existence of bounded solutions if

$$\lim_{x \to -\infty} g(x) = g(-\infty) \leq g(x) \leq g(+\infty) = \lim_{x \to +\infty} g(x), \quad \forall x \in R. \tag{1.4}$$

Later, Alonso and Ortega [1] studied the following equation:

$$\ddot{x} + n^2 x + g(x) + \psi'(x) = p(t), \quad n \in \mathbb{N}^+, \tag{1.5}$$

where g and p are the same as above and the perturbation $\psi'(x)$ is small at infinity in the following sense:

$$\lim_{|x| \to \infty} \frac{\psi(x)}{x} = 0.$$

They showed that unboundedness for each solution with a large initial condition if

$$\left| \int_0^{2\pi} p(t) e^{-int} dt \right| > 2(H - K),$$

where

$$H = \max\{\limsup_{x\to-\infty} g, \quad \limsup_{x\to+\infty} g\}, \qquad K = \min\{\liminf_{x\to-\infty} g, \quad \liminf_{x\to+\infty} g\}.$$

Ortega [16] obtained the first positive result on the boundedness of (1.1). He established a variant of Moser's small twist theorem, with which he proved the boundedness for the equation

$$\ddot{x} + n^2 x + h_L(x) = p(t), \quad p(t) \in \mathcal{C}^5(\mathbb{R}/2\pi\mathbb{Z}),$$

where $L > 0$ and $h_L(x)$ is a piecewise linear function of the form

$$h_L(x) = \begin{cases} L, & \text{if} \quad x \geq 1, \\ Lx, & \text{if} \quad -1 \leq x \leq 1, \\ -L, & \text{if} \quad x \leq -1, \end{cases}$$

and $p(t)$ satisfies

$$\left| \int_0^{2\pi} p(t) e^{-int} dt \right| < 4L.$$

Then Liu [9] studied the equation (1.1) with $p(t) \in \mathcal{C}^7(\mathbb{R}/2\pi\mathbb{Z})$, $g(x) \in \mathcal{C}^6(\mathbb{R})$ satisfying

$$g(\pm\infty) = \lim_{x\to\pm\infty} g(x) \ \text{ exist and are finite,} \tag{1.6}$$

and

$$\lim_{|x|\to+\infty} x^k g^{(k)}(x) = 0, \ 0 \leq k \leq 6. \tag{1.7}$$

He showed that the Lazer-Leach condition (1.2) implies the twist condition in Ortega's small twist theorem, by which the boundedness of (1.1) is obtained.

One can refer to [9–12,16,17] for the applications of Ortega's small twist theorem.

Recently Wang, Wang and Piao [18] studied the equation

$$\ddot{x} + n^2 x + g(x) + \psi'(x) = p(t), \quad n \in \mathbb{N}, \tag{1.8}$$

where $g(x)$ and $p(t)$ are similar to those in [9] and $\psi(x+T) = \psi(x)$.

They proved that the Lazer-Leach condition (1.2) on g and p is sufficient for the boundedness of (1.8) with the existence of an oscillating term ψ, in other words, the oscillating term does not play any role in the boundedness.

If

$$\left| \int_0^{2\pi} p(t) e^{-int} dt \right| > 2\big|g(+\infty) - g(-\infty)\big|, \tag{1.9}$$

then Alonso–Ortega's result [1] implies the existence of unbounded solutions for (1.8), one can also see [9]. Thus Wang, Wang and Piao [18] obtained that if

$$\left|\int_0^{2\pi} p(t)e^{-int}dt\right| \neq 2(g(+\infty)-g(-\infty)), \tag{1.10}$$

then (1.2) is sufficient and necessary for the boundedness of (1.8).

Other conditions for the existence of bounded and unbounded solutions are described in [1,2, 5,6,13,14] and their references.

We say (1.1) is in the critical situation if

$$\left|\int_0^{2\pi} p(t)e^{-int}dt\right| = 2\big|g(+\infty)-g(-\infty)\big|. \tag{1.11}$$

In this case, due to the absence of the Laser-Leach condition (1.2), the study on the validity of the twist condition becomes more subtle and the corresponding results on the boundedness solutions of (1.1) are very few. In fact, the only known work for equation (1.1) is about the unboundedness of each solution in [7] (see (1.3)) for the case

$$\min\{g(-\infty), g(+\infty)\} \le g(x) \le \max\{g(-\infty), g(+\infty)\}. \tag{1.12}$$

In this paper, we will study the boundedness of (1.1) in the critical situation if the condition (1.12) is violated.

Let $0<d\neq 1$. Suppose that $g(x)\in C^{\upsilon_1}(\mathbb{R}),\ p(t)\in C^{\upsilon_2}(\mathbb{R}/2\pi\mathbb{Z})$ and there exist two constants $c_\pm$ satisfying $c_\pm>0$ if $0<d<1$ and $c_\pm<0$ if $d>1$ such that

$$\lim_{|x|\to\pm\infty} x^{k-1+d}G_\pm^{(k)}(x)=0,\ \ 0<k\le \upsilon_1+1, \tag{1.13}$$

where

$$G_\pm(x)=\int_0^x (g(x)-g(\pm\infty))dx - c_\pm\cdot(1+x^2)^{\frac{1-d}{2}}. \tag{1.14}$$

Remark 1. From the assumptions (1.13) and (1.14), one can easily check that g does not satisfy the condition (1.12).

Let $\nu=5$. The main result of the paper is as follows.

Theorem 1.1. *Let $0<d\neq 1$ and $\upsilon_1=5+\nu$, $\upsilon_2=1+\nu$. Assume (1.6), (1.11), (1.13) and (1.14) hold true. Then the sufficient and necessary condition for the boundedness of (1.1) is $0<d<1$.*

Example 1. Let $g(x)=\arctan x+2x(1+x^2)^{-2/3}$ and $p(t)=2\cos(nt)$. Then the sufficient condition in Theorem 1.1 for boundedness are met with $g(\pm\infty)=\pm\pi/2,\ d=1/3<1,\ c_\pm=3$. On the other hand, let $p(t)$ be the same as above and $g(x)=\arctan x+x(1/3+x^2)^{-1}+x(1+x^2)^{-3/2}$ and $d=2>1,\ c_\pm=-1$. Then Theorem 1.1 implies that (1.1) has unbounded solutions.

This paper is organized as follows. We prove the boundedness result for $0 < d < 1$ in sections 2, 3 and 4. In Section 2, we define action-angle coordinates. In Section 3, we introduce a rotation transformation such that a nearly integrable sublinear system is obtained under a series of canonical transformations. In Section 4 we show that a twist condition in some weak way holds true which implies the boundedness of solutions in Theorem 1.1 by Moser's twist theorem. The last two sections are devoted to the unboundedness result for $d > 1$. In section 5, we first make a series of canonical transformations to obtain a normal form (for which the twist condition is violated). Then in last section we show the existence of an invariant set such that each solution starting from it is unbounded.

2. Action-angle coordinates

Let $y = \dot{x}/n$, equation (1.8) is equivalent to a Hamiltonian system with the Hamiltonian

$$H(x, y, t) = \frac{1}{2}n(x^2 + y^2) + \frac{1}{n}G(x) - \frac{1}{n}xp(t), \tag{2.15}$$

where $G(x) = \int_0^x g(s)ds$.

Under the action-angle coordinates transformation ($dx \wedge dy = dI \wedge d\theta$)

$$\begin{cases} x = x(I, \theta) = \sqrt{\frac{2}{n}}I^{\frac{1}{2}}\cos n\theta \\ y = y(I, \theta) = \sqrt{\frac{2}{n}}I^{\frac{1}{2}}\sin n\theta \end{cases}, \ (I, \theta) \in \mathbb{R}^+ \times \mathbb{R}/(\frac{2\pi}{n}\mathbb{Z}),$$

(2.15) is transformed into

$$H(I, \theta, t) = I + \frac{1}{n}G(\sqrt{\frac{2}{n}}I^{\frac{1}{2}}\cos n\theta) - \frac{1}{n}\sqrt{\frac{2}{n}}I^{\frac{1}{2}}\cos n\theta\, p(t). \tag{2.16}$$

Denote $f_1(I, \theta) = \frac{1}{n}G(\sqrt{\frac{2}{n}}I^{\frac{1}{2}}\cos n\theta)$, $f_2(I, \theta, t) = -\frac{1}{n}\sqrt{\frac{2}{n}}I^{\frac{1}{2}}\cos n\theta\, p(t)$, then (2.16) is rewritten by

$$H(I, \theta, t) = I + f_1(I, \theta) + f_2(I, \theta, t),\ (I, \theta, t) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1 \tag{2.17}$$

with $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$.

Remark 2. From the action-angle coordinates, the Hamiltonian (2.16) is $2\pi/n$ periodic in θ, then it is also 2π periodic in θ for $n \in \mathbb{N}_+$. Thus, for convenience, we assume H be defined in $\mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1$.

In this context, we denote $[f](\cdot) = \frac{1}{2\pi}\int_0^{2\pi} f(\cdot, \theta)d\theta$ be the average function of $f(\cdot, \theta)$ with respect to θ. Without loss of generality, $C > 1$, $c < 1$ are two universal positive constants not concerning their quantities, and j, k, l, ν, κ, etc., are non-negative integers.

Next, we give several lemmas about the estimates on $f_1(I, \theta)$, $f_2(I, \theta, t)$.

Lemma 2.1. *For I large enough, $\theta \in \mathbb{S}^1$, $k + j \le \upsilon_1 + 1$, we have the estimates on $f_1(I, \theta)$ as follows:*

$$\left|f_1(I,\theta)\right| \le CI^{\frac{1}{2}}, \quad \left|\partial_I^k \partial_\theta^j f_1(I,\theta)\right| \le CI^{\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)};$$

The estimates about Lemma 2.1 are classic and can be obtained by direct calculations. Thus we omit it. Readers can refer to [9].

Direct calculations can lead to the following conclusions:

Lemma 2.2. *For I large enough, $\theta,\ t \in \mathbb{S}^1$, $k + j \le \upsilon_1 + 1$ and $l \le \upsilon_2$, we have the estimates on $f_2(I, \theta, t)$ as follows:*

$$\left|\partial_I^k \partial_\theta^j \partial_t^l f_2(I,\theta,t)\right| \le CI^{\frac{1}{2}-k}.$$

3. A sublinear system and its normal form for $0 < d < 1$

In this section, we first transform the semilinear system into a sublinear one by a rotation transformation. Then by a series of canonical transformation, we obtain the normal form of the sublinear system, for which a weak twist condition holds true.

3.1. A rotation transformation

Since $\partial_I H > 1/2$ when I is sufficiently large, we can solve (2.17) for I as follows:

$$I = I(h,t,\theta) = h - R(h,t,\theta), \tag{3.18}$$

where $R(h, t, \theta)$ is determined implicitly by the equation

$$R = f_1(h - R, \theta) + f_2(h - R, \theta, t). \tag{3.19}$$

It is obvious that $h \to +\infty$ if and only if $I \to +\infty$. Since $Id\theta - hdt = -(hdt - Id\theta)$, we know that the new Hamiltonian system

$$\begin{cases} \dfrac{dt}{d\theta} = -\partial_h I(h,t,\theta), \\ \dfrac{dh}{d\theta} = \partial_t I(h,t,\theta) \end{cases}$$

is equivalent to the original one, see [3,8,9,19]. Note that in the new system, θ is the new time variable, thus it can be eliminated from f_1 by a canonical transformation, which is helpful to obtain a normal form later.

We present some estimates on $R(h, t, \theta)$ in (3.18).

Lemma 3.1. *For h large enough, $\theta,\ t \in \mathbb{S}^1$, $k + j \le \upsilon_1 + 1$ and $l \le \upsilon_2$, we have estimates on $R(h, t, \theta)$ as follows:*

$$\left|\partial_h^k \partial_t^l \partial_\theta^j R\right| \le Ch^{\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)}.$$

The proof is given in the Appendix.

From the identity (3.19), R has the following form by Taylor's formula:

$$R = f_1(h,\theta) + f_2(h,t,\theta) - \int_0^1 \partial_I f_1(h-\mu R,\theta)Rd\mu - \int_0^1 \partial_I f_2(h-\mu R,\theta,t)Rd\mu. \quad (3.20)$$

(3.18) and (3.20) yield

$$I = h - f_1(h,\theta) - f_2(h,\theta,t) + R_0(h,t,\theta), \quad (3.21)$$

where $R_0(h,t,\theta) = \int_0^1 \partial_I f_1(h-\mu R,\theta)Rd\mu + \int_0^1 \partial_I f_2(h-\mu R,\theta,t)Rd\mu$. Moreover, for h large enough, $\theta,\ t \in \mathbb{S}^1$, $k+j \le \upsilon_1+1$ and $l \le \upsilon_2$, $R_0(h,t,\theta)$ satisfies

$$\left|\partial_h^k \partial_t^l \partial_\theta^j R_0\right| \le Ch^{-k+\frac{j}{2}} \quad (3.22)$$

by direct calculations.

Next we introduce a rotation transformation to eliminate the linear part of the Hamiltonian which helps us to obtain a sublinear function.

Define a rotation transformation $\Phi_1 : (h_1,t_1,\theta) \to (h,t,\theta)$ by

$$\begin{cases} h = h_1 \\ t = t_1 + \theta. \end{cases} \quad (3.23)$$

Under Φ_1, the original semilinear system determined by the Hamiltonian I is transformed into a sublinear system given by the following Hamiltonian:

$$I_1(h_1,t_1,\theta) = -f_1(h_1,\theta) - f_2(h_1,\theta,t_1+\theta) + R_1(h_1,t_1,\theta)$$

with $R_1(h_1,t_1,\theta) = R_0(h_1,t_1+\theta,\theta)$.

Lemma 3.2. *For h_1 large enough, $\theta,\ t_1 \in \mathbb{S}^1$, and $k+j \le \upsilon_1+1,\ l \le \upsilon_2$, it holds that:*

$$|\partial_{h_1}^k \partial_{t_1}^l \partial_\theta^j R_1| \le C{h_1}^{-k+\frac{j}{2}}.$$

Proof. It is obtained from formula (3.22). □

3.2. Normal form with a weak twist condition

Since θ is the new time variable, we can easily eliminate it from $f_1(h,\theta)$ by a canonical transformation $\Phi_2 : (h_2,t_2,\theta) \to (h_1,t_1,\theta)$ given by

$$\begin{cases} h_1 = h_2, \\ t_1 = t_2 - \partial_{h_2} S_2(h_2,\theta) \end{cases} \quad (3.24)$$

with the generating function $S_2(h_2,\theta)$ determined by

$$S_2(h_2,\theta)=\int_0^\theta \big(f_1(h_2,\theta)-[f_1](h_2)\big)d\theta. \tag{3.25}$$

Under Φ_2, the Hamiltonian I_1 is transformed into I_2 as follows

$$\begin{aligned} I_2(h_2,t_2,\theta) &= -f_1(h_2,\theta)-f_2(h_2,\theta,t_2+\theta-\partial_{h_2}S_2(h_2,\theta))+R_1(h_2,t_2-\partial_{h_2}S_2(h_2,\theta),\theta) \\ &= -[f_1](h_2)-f_2(h_2,\theta,t_2+\theta)+[f_1](h_2)-f_1(h_2,\theta)+\partial_\theta S_2(h_2,\theta) \\ &\quad +\int_0^1 \partial_{t_1} f_2(h_2,\theta,t_2+\theta-\mu\partial_{h_2}S_2(h_2,\theta))\partial_{h_2}S_2(h_2,\theta)d\mu \\ &\quad +R_1(h_2,t_2-\partial_{h_2}S_2(h_2,\theta),\theta). \end{aligned}$$

It is obvious that (3.25) implies

$$[f_1](h_2)-f_1(h_2,\theta)+\frac{\partial}{\partial\theta}S_2(h_2,\theta)=0.$$

Let

$$\begin{aligned} R_2(h_2,t_2,\theta) &= R_1(h_2,t_2-\partial_{h_2}S_2(h_2,\theta),\theta) \\ &\quad +\int_0^1 \partial_{t_1} f_2(h_2,\theta,t_2-\mu\partial_{h_2}S_2(h_2,\theta))\partial_{h_2}S_2(h_2,\theta)d\mu. \end{aligned} \tag{3.26}$$

Thus, I_2 is rewritten as

$$I_2(h_2,t_2,\theta)=-[f_1](h_2)-f_2(h_2,\theta,t_2+\theta)+R_2(h_2,t_2,\theta).$$

We have the following estimates:

Lemma 3.3. *For h_2 large enough, $\theta,\ t_2\in\mathbb{S}^1$, it holds that*

$$|\partial_{h_2}^k\partial_\theta^j S_2(h_2,\theta)|\le Ch_2^{\frac12-k+\frac j2},\ k+j\le \upsilon_1+1, \tag{3.27}$$

and

$$\begin{aligned} &|\partial_{h_2}t_1|\le Ch_2^{-\frac32},\quad \partial_{t_2}t_1=1,\quad |\partial_\theta t_1|\le Ch_2^{-\frac12}, \\ &|\partial_{h_2}^k\partial_{t_2}^l\partial_\theta^j t_1|\le Ch_2^{-\frac12-k+\frac j2},\ k+l+j\ge 2,\ k+j\le\upsilon_1, l\le \nu_2. \end{aligned}$$

Moreover, for $k+j\le\upsilon_1$ and $l\le\upsilon_2-1$, it holds that:

$$|\partial_{h_2}^k\partial_{t_2}^l\partial_\theta^j R_2|\le Ch_2^{-k+\frac j2}.$$

The proof is given in the Appendix.

Now we are in a position to eliminate θ from $f_2(h_2, \theta, t_2 + \theta)$. Without causing confusion, for convenience we still denote

$$[f_2](h,t) = \frac{1}{2\pi}\int_0^{2\pi} f_2(h,\theta,t+\theta)d\theta.$$

Then we have

Lemma 3.4. *For any $h \in \mathbb{R}^+$, $t \in \mathbb{S}^1$, it holds that*

$$[f_2](h,t) = -\frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} h^{\frac{1}{2}} \left\{ \cos(nt)\int_0^{2\pi} p(\tau)\cos(n\tau)d\tau + \sin(nt)\int_0^{2\pi} p(\tau)\sin(n\tau)d\tau \right\}. \quad (3.28)$$

Moreover,

$$\left|[f_2](h,t)\right| \le \frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} h^{\frac{1}{2}} \left| \int_0^{2\pi} p(\tau) e^{in\tau} d\tau \right|. \quad (3.29)$$

Proof.

$$\begin{aligned}[f_2](h,t) &= \frac{1}{2\pi}\int_0^{2\pi} f_2(h,\theta,t+\theta)d\theta = -\frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} h^{\frac{1}{2}} \int_0^{2\pi} \cos(n\theta) p(t+\theta)d\theta \\ &= -\frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} h^{\frac{1}{2}} \{\cos(nt)\int_0^{2\pi} p(\tau)\cos(n\tau)d\tau + \sin(nt)\int_0^{2\pi} p(\tau)\sin(n\tau)d\tau\}.\end{aligned}$$

Thus, (3.29) is obtained immediately. □

We make a transformation $\Phi_3 : (h_3, t_3, \theta) \to (h_2, t_2, \theta)$ implicitly given by

$$\begin{cases} h_2 = h_3 + \partial_{t_2} S_3(h_3, t_2, \theta) \\ t_3 \ = t_2 + \partial_{h_3} S_3(h_3, t_2, \theta) \end{cases} \quad (3.30)$$

with

$$S_3(h_3, t_2, \theta) = \int_0^{\theta} (f_2(h_3, \theta, t_2 + \theta) - [f_2](h_3, t_2))d\theta. \quad (3.31)$$

Under Φ_3, the Hamiltonian I_2 is transformed into I_3 as follows

$$
\begin{aligned}
I_3(h_3,t_3,\theta) &= -[f_1](h_3+\partial_{t_2}S_3) - f_2(h_3+\partial_{t_2}S_3,\theta,t_2+\theta) + R_2(h_3+\partial_{t_2}S_3, t_3-\partial_{h_3}S_3,\theta) \\
&\quad +\partial_\theta S_3 \\
&= -[f_1](h_3) - [f_2](h_3,t_3) + [f_2](h_3,t_2) - f_2(h_3,\theta,t_2+\theta) + \partial_\theta S_3 \\
&\quad - \int_0^1 [f_1]'(h_3+\mu\partial_{t_2}S_3)\partial_{t_2}S_3(h_3,t_2,\theta)d\mu \\
&\quad - \int_0^1 \partial_I f_2(h_3+\mu\partial_{t_2}S_3,\theta,t_2+\theta)\partial_{t_2}S_3(h_3,t_2,\theta)d\mu \\
&\quad + \int_0^1 \partial_t[f_2](h_3,t_3-\mu\partial_{h_3}S_3)\partial_{h_3}S_3 d\mu + R_2(h_3+\partial_{t_2}S_3,t_3-\partial_{h_3}S_3,\theta).
\end{aligned}
$$

(3.31) implies

$$
[f_2](h_3,t_2) - f_2(h_3,\theta,t_2+\theta) + \partial_\theta S_3 = 0.
$$

Let

$$
\begin{aligned}
\alpha(h_3,t_3) &= -[f_1](h_3) - [f_2](h_3,t_3); \\
R_3(h_3,t_3,\theta) &= R_2(h_3+\partial_{t_2}S_3,t_3-\partial_{h_3}S_3,\theta) - \int_0^1 [f_1]'(h_3+\mu\partial_{t_2}S_3)\partial_{t_2}S_3(h_3,t_2,\theta)d\mu \\
&\quad - \int_0^1 \partial_I f_2(h_3+\mu\partial_{t_2}S_3,\theta,t_2+\theta)\partial_{t_2}S_3(h_3,t_2,\theta)d\mu \\
&\quad + \int_0^1 \partial_t[f_2](h_3,t_3-\mu\partial_{h_3}S_3)\partial_{h_3}S_3 d\mu;
\end{aligned}
$$

Thus we have

$$
I_3(h_3,t_3,\theta) = \alpha(h_3,t_3) + R_3(h_3,t_3,\theta). \tag{3.32}
$$

Lemma 3.5. *For h_3 large enough, $\theta,\ t_3 \in \mathbb{S}^1$ and $k+j \le \upsilon_1,\ l \le \upsilon_2 - 1$, it holds that:*

$$
\left|\partial_{h_3}^k \partial_{t_3}^l \partial_\theta^j R_3\right| \le C h_3^{-k+\frac{j}{2}}. \tag{3.33}
$$

Proof. The proof is similar to the one of Lemma 3.3. □

Next we estimate the lower and upper bounds for derivatives of $\alpha(h_3, t_3)$ in (3.32), which shows that $\alpha(h_3, t_3)$ satisfies some weak twist condition. Denote

$$\beta(h_3) = [f_1](h_3) - \frac{\sqrt{2}}{\pi} n^{-\frac{3}{2}} h_3^{\frac{1}{2}} \cdot \left| g(+\infty) - g(-\infty) \right|$$

and

$$a(t_3) = \frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} \Big(2\left| g(+\infty) - g(-\infty) \right| - \big(\cos nt_3 \int_0^{2\pi} p(s) \cos nsds + \sin nt_3 \int_0^{2\pi} p(s) \sin nsds \big) \Big).$$

Then it holds that

$$-\alpha(h_3, t_3) = a(t_3) \cdot h_3^{\frac{1}{2}} + \beta(h_3). \tag{3.34}$$

Denote $A := \left| \int_0^{2\pi} p(t) e^{-int} dt \right| = 2\left| g(+\infty) - g(-\infty) \right|$, then we have that

$$a(t_3) = \frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} A(1 - \cos(nt_3 - \xi)) \tag{3.35}$$

with $\tan\xi = \frac{\int_0^{2\pi} p(s)\sin(ns)ds}{\int_0^{2\pi} p(s)\cos(ns)ds}$.

Obviously, $a(t_3) \geq 0$ and $a(t_3) = 0$ if and only if the following holds true:

$$(\cos(nt_3),\ \sin(nt_3)) = \frac{(\int_0^{2\pi} p(s)\cos(ns)ds,\ \int_0^{2\pi} p(s)\sin(ns)ds)}{\left| \int_0^{2\pi} p(t) e^{-int} dt \right|}.$$

We have the following estimate on β:

Lemma 3.6. *For h_3 large enough, $\beta(h_3)$ satisfies*

$$C \cdot h_3^{\frac{1-d}{2} - k} \geq \beta^{(k)}(h_3) \geq c \cdot h_3^{\frac{1-d}{2} - k} \text{ for } k = 0,\ 1, \qquad |\beta^{(2)}(h_3)| \geq c \cdot h_3^{\frac{1-d}{2} - 2},$$

and

$$|\beta^{(k)}(h_3)| \leq C \cdot h_3^{\frac{1-d}{2} - k}, \qquad k \leq \upsilon_1 + 1. \tag{3.36}$$

Proof. Without loss of generality, assume $g(+\infty) \geq g(-\infty)$. Note that

$$\beta(h_3) = \frac{1}{2n\pi} \int_0^{2\pi} G(\sqrt{\frac{2}{n}} h_3^{\frac{1}{2}} \cos n\theta) d\theta - \frac{\sqrt{2}}{\pi} n^{-\frac{3}{2}} \big(g(+\infty) - g(-\infty) \big) \cdot h_3^{\frac{1}{2}}.$$

Then we have

$$\beta'(h_3)=\frac{\sqrt{2}}{4\pi}n^{-\frac{3}{2}}h_3^{-\frac{1}{2}}\Big(\int_0^{2\pi}g(\sqrt{\frac{2}{n}}h_3^{\frac{1}{2}}\cos n\theta)\cos n\theta d\theta-2\big(g(+\infty)-g(-\infty)\big)\Big)$$
$$=\frac{\sqrt{2}}{4\pi}n^{-\frac{3}{2}}h_3^{-\frac{1}{2}}\sum_{k=1}^{n}\Big(J_{k+}(h_3)-J_{k-}(h_3)\Big) \tag{3.37}$$

where

$$J_{k+}(h_3)=\int_{\frac{2k\pi}{n}-\frac{\pi}{2n}}^{\frac{2k\pi}{n}+\frac{\pi}{2n}}\Big(g(\sqrt{\frac{2}{n}}h_3^{\frac{1}{2}}\cos n\theta)-g(+\infty)\Big)\cos n\theta d\theta, \tag{3.38}$$
$$J_{k-}(h_3)=\int_{\frac{2k\pi}{n}+\frac{\pi}{2n}}^{\frac{2k\pi}{n}+\frac{3\pi}{2n}}\Big(g(\sqrt{\frac{2}{n}}h_3^{\frac{1}{2}}\cos n\theta)-g(-\infty)\Big)\cos n\theta d\theta.$$

From (1.13),

$$(\frac{2}{n}h_3)^{\frac{d}{2}}J_{k+}(h_3)=\int_{\frac{2k\pi}{n}-\frac{\pi}{2n}}^{\frac{2k\pi}{n}+\frac{\pi}{2n}}\Big(g(\sqrt{\frac{2}{n}}h_3^{\frac{1}{2}}\cos n\theta)-g(+\infty)\Big)(\sqrt{\frac{2}{n}}h_3^{\frac{1}{2}}\cos n\theta)^d\cos^{1-d}n\theta d\theta$$
$$\to s(d)c_+,\ \text{as } h_3\to\infty, \tag{3.39}$$

with $s(d)$ some positive constant. Similarly, we have

$$(\frac{2}{n}h_3)^{\frac{d}{2}}J_{k-}(h_3)=\int_{\frac{2k\pi}{n}+\frac{\pi}{2n}}^{\frac{2k\pi}{n}+\frac{3\pi}{2n}}\Big(g(\sqrt{\frac{2}{n}}h_3^{\frac{1}{2}}\cos n\theta)-g(-\infty)\Big)(\sqrt{\frac{2}{n}}h_3^{\frac{1}{2}}\cos n\theta)^d\cos^{1-d}n\theta d\theta$$
$$\to -s(d)c_-,\ \text{as } h_3\to\infty. \tag{3.40}$$

Thus

$$(\frac{2}{n}h_3)^{\frac{1+d}{2}}\beta'(h_3)\to\frac{1}{2n\pi}s(d)(c_++c_-),\ \text{as } h_3\to\infty, \tag{3.41}$$

which means that

$$c\cdot h_3^{\frac{1-d}{2}-1}\le\beta'(h_3)\le C\cdot h_3^{\frac{1-d}{2}-1}.$$

With L'Hospital's rule, (3.41) implies

$$c \cdot h_3^{\frac{1-d}{2}} \le \beta(h_3) \le C \cdot h_3^{\frac{1-d}{2}}.$$

From (3.38) and (1.13), we have

$$\begin{aligned} 2\Big(\sqrt{\frac{2}{n}}\Big)^d h_3^{1+\frac{d}{2}} J'_{k+}(h_3) &= \int\limits_{\frac{2k\pi}{n}-\frac{\pi}{2n}}^{\frac{2k\pi}{n}+\frac{\pi}{2n}} \Big(g'(\sqrt{\frac{2}{n}}h_3^{\frac12}\cos n\theta)(\sqrt{\frac{2}{n}}h_3^{\frac12}\cos n\theta)^{1+d}\cos^{1-d} n\theta d\theta, \\ &\to -\tilde{s}(d)c_+ d(1-d), \text{ as } h_3 \to \infty, \end{aligned}$$

with $\tilde{s}(d)$ some positive constant. Similarly, we have

$$\begin{aligned} 2\Big(\sqrt{\frac{2}{n}}\Big)^d h_3^{1+\frac{d}{2}} J'_{k-}(h_3) &= \int\limits_{\frac{2k\pi}{n}+\frac{\pi}{2n}}^{\frac{2k\pi}{n}+\frac{3\pi}{2n}} \Big(g'(\sqrt{\frac{2}{n}}h_3^{\frac12}\cos n\theta)(\sqrt{\frac{2}{n}}h_3^{\frac12}\cos n\theta)^{1+d}\cos^{1-d} n\theta d\theta, \\ &\to \tilde{s}(d)c_- d(1-d), \text{ as } h_3 \to \infty. \end{aligned}$$

Thus

$$\Big(J'_{k+}(h_3) - J'_{k-}(h_3)\Big) \to -d(1-d)\tilde{s}(d)(c_+ + c_-)\frac{1}{2}\Big(\sqrt{\frac{2}{n}}\Big)^{-d} h_3^{-1-\frac{d}{2}}, \text{ as } h_3 \to \infty. \tag{3.42}$$

Note that

$$\beta''(h_3) = -\frac{1}{2}h_3^{-1}\beta'(h_3) + \frac{\sqrt{2}}{4\pi}n^{-\frac32}h_3^{-\frac12}\sum_{k=1}^{n}\Big(J'_{k+}(h_3) - J'_{k-}(h_3)\Big) \tag{3.43}$$

By this together with (3.41) and (3.42), we have $|\beta''(h_3)| \ge c \cdot h_3^{\frac{1-d}{2}-2}$.

For the remaining upper bound estimates, by direct calculations, we have for $m \le \upsilon_1 + 1$ that

$$h_3^{m+\frac{d}{2}} J_{k+}^{(m)}(h_3) = \int\limits_{\frac{2k\pi}{n}-\frac{\pi}{2n}}^{\frac{2k\pi}{n}+\frac{\pi}{2n}} \sum_{i=1}^{m} cg^{(i)}\Big(\sqrt{\frac{2}{n}}h_3^{\frac12}\cos n\theta\Big)\Big(\sqrt{\frac{2}{n}}h_3^{\frac12}\cos n\theta\Big)^{i+d}\cos^{1-d} n\theta d\theta,$$

which together with (1.13) implies

$$|J_{k+}^{(m)}(h_3)| < C \cdot h_3^{-\frac{d}{2}-m}. \tag{3.44}$$

Similarly, we have

$$|J_{k-}^{(m)}(h_3)| \le C \cdot h_3^{-m-\frac{d}{2}}, \text{ as } h_3 \to \infty. \tag{3.45}$$

Thus (3.37), (3.44) and (3.45) gives (3.36). □

Consequently, $\alpha(h_3, t_3)$ in (3.32) satisfies a weak twist condition:

$$|\partial_{h_3}^k \alpha(h_3, t_3)| \geq c \cdot h_3^{\frac{1-d}{2}-k}, \qquad k = 0,\ 1,\ 2, \tag{3.46}$$

and for $k \leq \upsilon_1 + 1,\ l \leq \upsilon_2$,

$$|\partial_{h_3}^k \partial_{t_3}^l \alpha(h_3, t_3)| \leq C \cdot h_3^{\frac{1}{2}-k}. \tag{3.47}$$

3.3. A nearly integrable system

Lemma 3.7. *Assume Hamiltonian*

$$I = \alpha(h, t) + R(h, t, \theta)$$

with α satisfying (3.46), (3.47) for $k \leq m,\ l \leq n$, and $R(h, t, \theta)$ satisfying

$$\left|\partial_h^k \partial_t^l \partial_\theta^j R\right| \leq C h^{-\frac{i}{2}-k+\max\{0, \frac{j-i}{2}\}},$$

for h large enough, $k + j \leq m_1$, $l \leq n_1 (m_1 \leq m,\ n_1 \leq n)$.

Then there exists a transformation $\Phi_+ : (h_+,\ t_+,\ \theta) \to (h,\ t,\ \theta)$, such that

$$I_+(h_+, t_+, \theta) = I \circ \Phi_+(h,\ t,\ \theta) = \alpha_+(h_+,\ t_+) + R_+(h_+,\ t_+,\ \theta),$$

with $\alpha_+(h_+,\ t_+) = \alpha(h_+,\ t_+) + [R](h_+,\ t_+)$ satisfying (3.46) and (3.47) for $k \leq m_1,\ l \leq n_1$. Moreover for $h_+ \gg 1$, $l \leq n_1 - 1$, $k + j \leq m_1 - 1$, it holds that

$$\left|\partial_{h_+}^k \partial_{t_+}^l \partial_\theta^j R_+\right| \leq C h_+^{-\frac{i+1}{2}-k+\max\{0, \frac{j-i-1}{2}\}}$$

Proof. Set $\Phi_+ : (h_+,\ t_+,\ \theta) \to (h,\ t,\ \theta)$ implicitly given by

$$\begin{cases} h = h_+ + \partial_t S_+(h_+, t, \theta) \\ t_+ \ = t + \partial_{h_+} S_+(h_+, t, \theta) \end{cases}$$

with the generating function $S_+(h_+, t, \theta)$ determined by

$$S_+(h_+, t, \theta) = -\int_0^\theta (R(h_+, t, \theta) - [R](h_+, t))d\theta.$$

It is easy to show that, for $k \leq m_1,\ l \leq n_1, j \leq i$,

$$\left|\partial_{h_+}^k \partial_t^l \partial_\theta^j S_+(h_+, t, \theta)\right| \leq C h_+^{-\frac{i}{2}-k},\ \ j = 0,\ 1,\ \cdots,\ i+1.$$

Under Φ_+, the Hamiltonian I is transformed into I_+ as follows

$$\begin{aligned} I_+(h_+,t_+,\theta) &= \alpha(h_+ + \partial_t S_+, t_+ - \partial_{h_+} S_+) + R(h_+ + \partial_t S_+, t, \theta) + \partial_\theta S_+ \\ &= \alpha(h_+,t_+) + R_+(h_+,t_+,\theta), \end{aligned}$$

where

$$\begin{aligned} R_+(h_+,t_+,\theta) =& \int_0^1 \partial_I \alpha(h_+ + \mu\partial_t S_+, t)\partial_t S_+ d\mu - \int_0^1 \partial_t \alpha(h_+, t_+ - \mu\partial_{h_+} S_+)\partial_{h_+} S_+ d\mu \\ & - \int_0^1 \partial_t [R](h_+, t_+ - \mu\partial_{h_+} S_+)\partial_{h_+} S_+ d\mu + \int_0^1 \partial_h R(h_+ + \mu\partial_t S_+, t, \theta)\partial_t S_+ d\mu. \end{aligned}$$

Then the estimates on R_+ can be obtained by direct computations. $\square$

Remark 3. Without loss of generality, α_+ is still denoted by α.

With repeated applications of canonical transformations given in Lemma 3.7 by ν times, the Hamiltonian system with the Hamiltonian I_3 can be changed into the one with the following Hamiltonian

$$I_4 = \alpha(h_4,t_4) + R_4(h_4,t_4,\theta), \tag{3.48}$$

where $\alpha(h_4,t_4)$ satisfies

$$|\partial_{h_4}^k \alpha(h_4,t_4)| \geq c \cdot h_4^{\frac{1-d}{2}-k}, \qquad \text{for } k = 0,\ 1,\ 2, \tag{3.49}$$

$$|\partial_{h_4}^k \partial_{t_4}^l \alpha(h_4,t_4)| \leq C \cdot h_4^{\frac{1}{2}-k},\ \text{for } k \leq \upsilon_1 + 1 - \nu,\ l \leq \upsilon_2 - \nu; \tag{3.50}$$

and R_4 satisfies

$$|\partial_{h_4}^k \partial_{t_4}^l \partial_\theta^j R_4| \leq C \cdot h_4^{-\frac{\nu}{2}-k},\ \text{for } j \leq \nu,\ k + j \leq \upsilon_1 - \nu,\ l \leq \upsilon_2 - 1 - \nu. \tag{3.51}$$

4. The boundedness result for $0 < d < 1$

In (3.48), the leading term α depends on the angle variable t_4. Hence we have to exchange the roles of angle and time variables again. From (3.49) and (3.51), we have $\partial_{h_4} I_4 > c h_4^{\frac{-1-d}{2}} > 0$ as $h_4 \to \infty$, for large h_4 we can solve (3.48) for it with the following form:

$$h_4(I_4,\theta,t_4) = \mathcal{N}(I_4,t_4) + \mathcal{P}(I_4,\theta,t_4), \tag{4.52}$$

where $h_4 = \mathcal{N}(I_4,t_4)$ is the inverse function of $I_4 = \alpha(h_4,t_4)$. With (3.48) and (4.52), we have

$$
\begin{aligned}
I_4 &= \alpha(\mathcal{N}+\mathcal{P}, t_4) + R_4(\mathcal{N}+\mathcal{P}, t_4, \theta) \\
&= \alpha(\mathcal{N}, t_4) + \left(\int_0^1 \partial_{h_4}\alpha(\mathcal{N}+\mu\mathcal{P}, t_4)d\mu \right)\mathcal{P} + R_4(\mathcal{N}+\mathcal{P}, t_4, \theta).
\end{aligned} \tag{4.53}
$$

Note that $I_4 = \alpha(\mathcal{N}, t_4)$, then

$$
0 = \left(\int_0^1 \partial_{h_4}\alpha(\mathcal{N}+\mu\mathcal{P}, t_4)d\mu \right)\mathcal{P} + R_4(\mathcal{N}+\mathcal{P}, t_4, \theta).
$$

Implicitly,

$$
\mathcal{P} = -\frac{R_4(\mathcal{N}+\mathcal{P}, t_4, \theta)}{\int_0^1 \partial_{h_4}\alpha(\mathcal{N}+\mu\mathcal{P}, t_4)d\mu}. \tag{4.54}
$$

Exchanging the roles of time and angle variables, we obtain the new system determined by the Hamiltonian (4.52). Without leading to confusion, we denote (I_4, h_4, t_4) by $(\mathcal{I}, h, \tau)$, α_4 by α and R_4 by R, then the new Hamiltonian is as follows:

$$
h(\mathcal{I}, \theta, \tau) = \mathcal{N}(\mathcal{I}, \tau) + \mathcal{P}(\mathcal{I}, \theta, \tau). \tag{4.55}
$$

It holds that

Lemma 4.1. *For $\mathcal{I}$ large enough, $\theta, t \in \mathbb{S}^1$, it holds that:*

$$
c \cdot \mathcal{I}^2 \le \left|\mathcal{N}\right| \le C \cdot \mathcal{I}^{\frac{2}{1-d}}, \quad c\mathcal{N} \cdot \mathcal{I}^{-k} \le \left|\partial_{\mathcal{I}}^k \mathcal{N}\right| \le C\mathcal{N} \cdot \mathcal{I}^{-k}, \quad k = 1, 2 \tag{4.56}
$$

$$
\left|\partial_{\mathcal{I}}^k \mathcal{N}\right| \le C\mathcal{N}\mathcal{I}^{-k}, \quad k \le \upsilon_1 + 1 - \nu. \tag{4.57}
$$

Moreover, $\mathcal{P}$ satisfies

$$
|\partial_{\mathcal{I}}^k \partial_\theta^j \mathcal{P}| \le C \cdot \mathcal{I}^{-k-1}\mathcal{N} \cdot |R|. \tag{4.58}
$$

for $j \le \nu,\ k + j \le \upsilon_1 - \nu$.

The proof is similar to the one of Lemma 3.1. For the sake of readers, we provide the details of the proof in the Appendix.

4.1. The Poincaré map

Consider the system with Hamiltonian (4.55), that is

$$
\begin{cases}
\dfrac{d\theta}{d\tau} = \partial_I N(I, \tau) + \partial_I P(I, \theta, \tau), \\
\dfrac{dI}{d\tau} = -\partial_\theta P(I, \theta, \tau).
\end{cases} \tag{4.59}
$$

The Poincaré map $\mathcal{P}$ of (4.59) is of the form

$$\begin{cases} \theta_1 = \theta + r(I) + F_1(I,\theta), \\ I_1 = I + F_2(I,\theta), \end{cases} \tag{4.60}$$

where $(I,\theta) = (I(0),\theta(0))$, and

$$r(I) = \int_0^{2\pi} \partial_I N(I,\tau)d\tau;$$

$$F_1(I,\theta) = \int_0^{2\pi} \partial_I P(I(\tau)),\theta(\tau),\tau)d\tau + \int_0^{2\pi} \partial_I N(I(\tau),\tau)d\tau - \int_0^{2\pi} \partial_I N(I,\tau)d\tau;$$

$$F_2(I,\theta) = -\int_0^{2\pi} \partial_\theta P(I(\tau),\theta(\tau),\tau)d\tau.$$

From Lemma 4.1, we have the following estimates on the map $\mathcal{P}$:

Lemma 4.2. *Given I large enough and $\theta \in \mathbb{S}^1$, for $j \le \nu - 1,\ k + j \le \upsilon_1 - \nu - 1$, it holds that:*

$$|\partial_I^k \partial_\theta^j F_1(I,\theta)| \le C(\mathcal{N} \cdot \mathcal{I}^{-2} + 1) \cdot \mathcal{I}^{-k-1}\mathcal{N} \cdot |R|. \tag{4.61}$$

$$|\partial_I^k \partial_\theta^j F_2(I,\theta)| \le C \cdot \mathcal{I}^{-k-1}\mathcal{N} \cdot |R|. \tag{4.62}$$

Moreover, the following estimates hold true for $r(\mathcal{I})$:

$$c\mathcal{I} \le \left|r(\mathcal{I})\right| \le C\mathcal{I}^{\frac{1+d}{1-d}}, \qquad c \le \left|r'(\mathcal{I})\right| \le C\mathcal{I}^{\frac{2d}{1-d}}; \tag{4.63}$$

and

$$|r^{(k)}(\mathcal{I})| \le C\mathcal{I}^{\frac{1+d}{1-d}-k},\ k \le \upsilon_1 - \nu. \tag{4.64}$$

Let $\mathcal{I}(r)$ be the inverse function of $r(\mathcal{I})$. From (4.63) and (4.64), we obtain the following estimates on $\mathcal{I}(r)$:

$$c \cdot r^{\frac{1-d}{1+d}} \le \mathcal{I} \le C \cdot r, \tag{4.65}$$

$$|\mathcal{I}^{(k)}(r)| \le C \cdot r^{-k}|\mathcal{I}|, \quad k \le \upsilon_1 - \nu. \tag{4.66}$$

With a transformation: $(\theta,\mathcal{I}) \to (\theta,r)$, we obtain the following map:

$$\begin{cases} \theta_1 = \theta + r + \tilde{F}_1(r,\theta) \\ r_1 = r + \tilde{F}_2(r,\theta), \end{cases} \tag{4.67}$$

where

$$\tilde{F}_1(r,\theta) = F_1(\mathcal{I}(r),\theta), \quad \tilde{F}_2(r,\theta) = \int_0^1 r'(\mathcal{I} + \lambda F_2(\mathcal{I},\theta))F_2(\mathcal{I},\theta)d\lambda.$$

From Lemma 4.2 and (4.65), (4.66), we have that for $j \le \nu - 1,\ k + j \le \upsilon_1 - \nu - 1$,

$$\begin{aligned} |\partial_r^k \partial_\theta^j \tilde{F}_1| &\le \textstyle\sum_{i=1}^k |\partial_{\mathcal{I}}^i \partial_\theta^j F_1| \cdot |\mathcal{I}^{(k_1)}(r) \cdots \mathcal{I}^{(k_i)}(r)| \\ &\le C\mathcal{N}^2 \cdot I^{-i-3}|R| \cdot \mathcal{I}^i \cdot r^{-k} \\ &\le C\mathcal{I}^{-3} \cdot r^{-k} \cdot |R| \cdot \mathcal{N}^2, \end{aligned} \tag{4.68}$$

where $\sum_{k_1+\cdots+k_i} = k$. Similarly, we have that for $j \le \nu - 1,\ k + j \le \upsilon_1 - \nu - 1$,

$$|\partial_r^k \partial_\theta^j \tilde{F}_2| \le C\mathcal{I}^{-3} \cdot r^{-k} \cdot |R| \cdot \mathcal{N}^2. \tag{4.69}$$

Recall in (3.51) we have the estimate

$$|\partial_{h_4}^k \partial_{t_4}^l \partial_\theta^j R| \le C \cdot h_4^{-\frac{\nu}{2}-k}, \text{ for } j \le \nu,\ k + j \le \upsilon_1 - \nu,\ l \le \upsilon_2 - 1 - \nu.$$

Since $\mathcal{N} = h_4$, thus for $j \le \nu - 1,\ k + j \le \upsilon_1 - \nu - 1$, (4.68) and (4.69) yield that

$$|\partial_r^k \partial_\theta^j \tilde{F}_i| \le C\mathcal{I}^{-3} \cdot r^{-k} \cdot h^{-\frac{\nu}{2}+2} \le C\mathcal{I}^{-3}, \quad i = 1, 2$$

if $\nu \ge 4$. Assume

$$\begin{gathered} \nu \ge 4, \\ \nu - 1 \ge 4, \\ \upsilon_1 - \nu - 1 \ge 4, \\ \upsilon_2 - (\nu + 1) \ge 0. \end{gathered}$$

Therefore, let $\nu = 5,\ \upsilon_1 = 5 + \nu,\ \upsilon_2 = 1 + \nu$, then the map (4.67) is a $\mathcal{C}^4$-perturbation of the twist map, hence meets all the conditions of Moser's small twist Theorem [15]. Thus we obtain the boundedness results of Theorem 1.1.

5. Unbounded results for $d > 1$

In this section, we make a series of canonical transformations to obtain a normal form for which the twist condition is violated.

5.1. Some lemmas

We say a function $f(x)$ is of $O_m(|x|^{c_0})$ for $c_0 \in \mathbb{R}$ if $|f^{(k)}(x)| \le C|x|^{c_0-k}$ for x satisfying $|x| \gg 1$ and $0 \le k \le m$. Similarly, for a function $f : \mathbb{R}^+ \times \mathbb{S}^2 \to \mathbb{R}$, we say $f(I,\theta,t)$ is of $O_m(I^{c_0})$ for $c_0 \in \mathbb{R}$ if $|\partial_I^k \partial_\theta^j \partial_t^l f| \le C I^{c_0-k}$ for $j \le 1,\ j + k + l \le m$ and $I \gg 1$.

Since $G(x) = \int_0^x g(x)dx = x \cdot g(x) - \int_0^x xg'(x)dx := xg(x) - f_3(x)$, we have

$$f_1 = \frac{1}{n} G(\sqrt{\frac{2}{n}} I^{\frac{1}{2}} \cos n\theta) = \widetilde{f_1}(I,\theta) + f_3(I,\theta)$$

with $\widetilde{f_1}(I,\theta) = \frac{1}{n}\sqrt{\frac{2}{n}} I^{\frac{1}{2}} \cos n\theta \cdot g(\sqrt{\frac{2}{n}} I^{\frac{1}{2}} \cos n\theta)$, $f_3(I,\theta) = -\frac{1}{n} f_3(\sqrt{\frac{2}{n}} I^{\frac{1}{2}} \cos n\theta)$.

Therefore, (2.16) can be rewritten as

$$h = I + \widetilde{f_1}(I,\theta) + f_2(I,\theta,t) + f_3(I,\theta), \tag{5.69}$$

where $f_2(I,\theta,t) = -\frac{1}{n}\sqrt{\frac{2}{n}} I^{\frac{1}{2}} cosn\theta p(t)$. From the condition (1.13) and (1.14) on g, we have

$$f_3(x) = O_4(|x|^{1-d}). \tag{5.70}$$

Similarly, for $k + j \le 4$ and $j \le 1$ we have

$$|\partial_I^k \partial_\theta^j \widetilde{f_1}(I,\theta)|, |\partial_I^k \partial_\theta^j f_2(I,\theta,t)| \le C I^{\frac{1}{2}-k}. \tag{5.71}$$

Since $\partial_I H > 1/2$ when I is sufficiently large, we can solve (5.69) for large I as follows:

$$I = I(h,t,\theta) = h - R(h,t,\theta),$$

where $R(h,t,\theta)$ is determined implicitly by the equation

$$R = \widetilde{f_1}(h-R,\theta) + f_2(h-R,\theta,t) + f_3(h-R,\theta). \tag{5.72}$$

We present some estimates on $R(h,t,\theta)$ in (5.72).

Lemma 5.1. *For h large enough, θ, $t \in \mathbb{S}^1$, $j \le 1, k + j + l \le 5$, we have the estimates on $R(h,t,\theta)$ as follows:*

$$\left|\partial_h^k \partial_t^l \partial_\theta^j R\right| \le C h^{\frac{1}{2}-k}. \tag{5.73}$$

The proof is similar to the proof of Lemma 3.1. So we omit it.

Then we have

$$\begin{aligned} I &= h - \widetilde{f_1}(h-R,\theta) - f_2(h-R,\theta,t) - f_3(h-R,\theta) \\ &= h - \widetilde{f_1}(h,\theta) - f_2(h,\theta,t) - f_3(h,\theta) \\ &\quad + (\partial_I \widetilde{f_1}(h,\theta) + \partial_I f_2(h,\theta,t) + \partial_I f_3(h,\theta)) \cdot R(h,\theta,t) + R_{00}(h,\theta,t), \end{aligned}$$

where

$$R_{00}(h,\theta,t) = \int_0^1 \int_0^1 \partial_I^2 (\widetilde{f_1}(h - s\mu R,\theta) + f_2(h - s\mu R,\theta,t) + f_3(h - s\mu R,\theta)) \cdot R^2(h,\theta,t) ds d\mu.$$

Moreover, from (5.70), (5.71) and Lemma 5.1, it holds that

$$R_{00}(h,\theta,t) \in O_4(h^{-\frac{1}{2}}).$$

From the definition of $\widetilde{f}_1,\ f_2,\ f_3$, we claim that

$$(\partial_I \widetilde{f}_1(h,\theta) + \partial_I f_2(h,\theta,t) + \partial_I f_3(h,\theta)) \cdot R(h,t,\theta) := R_1(h,t,\theta) + O_4(h^{-\frac{1}{2}}), \tag{5.74}$$

where

$$R_1(h,t,\theta) = \sum_i f_{4,i}(g(\sqrt{\frac{2}{n}}h^{\frac{1}{2}}\cos n\theta)) \cdot f_{5,i}(t,\theta),$$

with $f_{4,i} \in \mathcal{C}^4(\mathbb{R}^1)$ and $f_{5,i} \in \mathcal{C}^4(\mathbb{T}^2)$ are finitely many smooth functions. In fact,

$$\begin{aligned}
&\Big(\partial_I \widetilde{f}_1(h,\theta) + \partial_I f_2(h,\theta,t) + \partial_I f_3(\sqrt{\tfrac{2}{n}}h^{\frac{1}{2}}\cos n\theta)\Big) \cdot R(h,\theta,t) \\
= \ &\Big(\partial_I \widetilde{f}_1(h,\theta) + \partial_I f_2(h,\theta,t) + \partial_I f_3(\sqrt{\tfrac{2}{n}}h^{\frac{1}{2}}\cos n\theta)\Big) \\
&\cdot\Big(\widetilde{f}_1(h-R,\theta) + f_2(h-R,\theta,t) + f_3(h-R,\theta)\Big) \\
= \ &\partial_I \widetilde{f}_1 \cdot \widetilde{f}_1 + \partial_I f_2 \cdot \widetilde{f}_1 + \partial_I f_3 \cdot \widetilde{f}_1 + \partial_I \widetilde{f}_1 \cdot f_2 + \partial_I f_2 \cdot f_2 + \partial_I f_3 \cdot f_2 \\
&+\partial_I \widetilde{f}_1 \cdot f_3 + \partial_I f_2 \cdot f_3 + \partial_I f_3 \cdot f_3.
\end{aligned}$$

From (5.70), (5.71) and Lemma 5.1, we have

$$\begin{aligned}
&\partial_I \widetilde{f}_1(I,\theta) \cdot \widetilde{f}_1(h-R,\theta) = n^{-3}g^2(\sqrt{\tfrac{2}{n}}h^{\frac{1}{2}}\cos n\theta) \cdot \cos^2 n\theta + O_4(h^{-\frac{1}{2}}), \\
&\partial_I f_2(h,\theta,t) \cdot \widetilde{f}_1(h-R,\theta) = -n^{-3}g(\sqrt{\tfrac{2}{n}}h^{\frac{1}{2}}\cos n\theta) \cdot \cos^2 n\theta \cdot p(t) + O_4(h^{-\frac{1}{2}}), \\
&\partial_I \widetilde{f}_1(h,\theta) \cdot f_2(h-R,\theta,t) = -n^{-3}g(\sqrt{\tfrac{2}{n}}h^{\frac{1}{2}}\cos n\theta) \cdot \cos^2 n\theta \cdot p(t) + O_4(h^{-\frac{1}{2}}), \\
&\partial_I f_2(h,\theta) \cdot f_2(h-R,\theta,t) = n^{-3}\cos^2 n\theta \cdot p^2(t) + O_4(h^{-\frac{1}{2}}), \\
&\partial_I f_3(h,\theta) \cdot \widetilde{f}_1(h-R,\theta) + \partial_I f_3(h,\theta) \cdot f_2(h-R,\theta,t) = O_4(h^{-\frac{1}{2}}), \\
&\partial_I \widetilde{f}_1(I,\theta) \cdot f_3(h-R,\theta) + \partial_I f_2(h,\theta,t) \cdot f_3(h-R,\theta) + \partial_I f_3(I,\theta) \cdot f_3(h-R,\theta) \\
&\quad = O_4(h^{-\frac{1}{2}}).
\end{aligned}$$

Denote

$$R_1 = n^{-3}\cos^2 n\theta\Big\{g^2(\sqrt{\frac{2}{n}}h^{\frac{1}{2}}\cos n\theta) - 2g(\sqrt{\frac{2}{n}}h^{\frac{1}{2}}\cos n\theta) \cdot p(t) + p^2(t)\Big\}. \tag{5.75}$$

Therefore, we get the conclusion. Meanwhile, again from (1.13) and (1.14), we have

$$R_1 \in O_4(1).$$

Now I can be rewritten as

$$I = h - \widetilde{f}_1(h,\theta) - f_2(h,\theta,t) - f_3(h,\theta) + R_1(h,t,\theta) + R_{02}(h,t,\theta), \tag{5.76}$$

where

$$R_{02}(h,t,\theta) = R_{00}(h,\theta,t) + O_4(h^{-\frac{1}{2}}) \in O_4(h^{-\frac{1}{2}}).$$

5.2. A sublinear system and its normal form

Still using the rotation transformation Φ_1 defined in (3.23), the Hamiltonian (5.76) is transformed into a sublinear system with the following Hamiltonian

$$I_1 = -\widetilde{f}_1(h_1,\theta) - f_2(h_1,\theta,t_1+\theta) - f_3(h_1,\theta) + R_{11}(h_1,t_1,\theta) + R_{12}(h_1,t_1,\theta),$$

where

$$R_{11}(h_1,t_1,\theta) = R_1(h_1,\theta,t_1+\theta) \in O_4(1) \tag{5.77}$$

and

$$R_{12}(h_1,t_1,\theta) = R_{02}(h_1,\theta,t_1+\theta) \in O_4(h_1^{-\frac{1}{2}}). \tag{5.78}$$

To eliminate the new time variable in $\widetilde{f}_1$ and f_2, we make a transformation $\Psi : (h_2,t_2,\theta) \to (h_1,t_1,\theta)$ implicitly given by

$$\begin{cases} h_1 &= h_2 + \partial_{t_1} S_1, \\ t_2 &= t_1 + \partial_{h_2} S_1, \end{cases}$$

with

$$S_1(h_2,t_1,\theta) = \int_0^\theta \{-\widetilde{f}_1(h_2,\theta) - f_2(h_2,\theta,t_1+\theta) + [\widetilde{f}_1](h_2) + [f_2](h_2,t_1)\}d\theta,$$

where $[\widetilde{f}_1] = \int_0^{2\pi} \widetilde{f}_1(h_2,\theta)d\theta$ and $[f_2](h_2,t_1) = \int_0^{2\pi} f_2(h_2,\theta,t_1+\theta)d\theta$ satisfying similar estimates as in (5.71).

Similar to Lemma 3.3, we have

Lemma 5.2. *For h_2 large enough, $\theta,\ t_2 \in \mathbb{S}^1$, it holds that:*

$$\left|\partial_{h_2}^k \partial_{t_1}^l \partial_\theta^j S_1(h_2,t_1,\theta)\right| \le C h_2^{\frac{1}{2}-k},\ \ j \le 1, k+l+j \le 4;$$

Moreover, the map Ψ *satisfies*

$$|\partial_{h_2} t_1| \le C h_3^{-\frac{3}{2}}, \quad \frac{1}{2} \le |\partial_{t_2} t_1| \le 2, \quad |\partial_\theta t_1| \le C h_2^{-\frac{1}{2}},$$
$$|\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j t_1| \le C h_2^{-\frac{1}{2}-k}, \; j \le 1, 2 \le k+l+j \le 3;$$
$$\frac{1}{2} \le |\partial_{h_2} h_1| \le 2, \quad |\partial_{t_2} h_1| \le C h_2^{\frac{1}{2}}, \quad |\partial_\theta h_1| \le C h_2^{\frac{1}{2}},$$
$$|\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j h_1| \le C h_2^{\frac{1}{2}-k}, j \le 1, \; 2 \le k+l+j \le 3.$$

Under Ψ, the Hamiltonian I_1 is transformed into $I_2(h_2, t_2, \theta)$ as follows

$$I_2 = -[\widetilde{f}_1](h_2) - [f_2](h_2, t_2) - f_3(h_2, \theta) + R_{21}(h_2, t_2, \theta) + R_{22u}(h_2, t_2, \theta),$$

where $R_{21} = R_{11}(h_2, t_2, \theta)$ and

$$\begin{aligned} R_{22u} &= -\int_0^1 \partial_{h_1}(\widetilde{f}_1(h_2 + \mu\partial_{t_1} S_1, \theta) + f_2(h_2 + \mu\partial_{t_1} S_1, \theta, t_1 + \theta) + f_3(h_2 + \partial_{t_1} S_1, \theta))\partial_{t_1} S_1 d\mu \\ &\quad + \int_0^1 \partial_{h_1} R_{11}(h_2 + \mu\partial_{t_1} S_1, \theta, t_2 - \partial_{h_2} S_1)\partial_{t_1} S_1 d\mu - \int_0^1 \partial_{t_1} R_{11}(h_2, \theta, t_2 - \partial_{h_2} S_1)\partial_{h_2} S_1 d\mu \\ &\quad + \int_0^1 \partial_{t_1}[f_2](h_2, t_2 - \mu\partial_{h_2} S_1)\partial_{h_2} S_1 d\mu + R_{12}(h_2 + \partial_{t_1} S_1, \theta, t_2 - \partial_{h_2} S_1) \\ &= -\partial_{h_1}(\widetilde{f}_1(h_2, \theta) + f_2(h_2, \theta, t_1 + \theta))\partial_{t_1} S_1 + \partial_{t_1}[f_2](h_2, t_2) \cdot \partial_{h_2} S_1 + R_{22yu}, \end{aligned}$$

where

$$\begin{aligned} R_{22yu} = &-\int_0^1 \int_0^1 \partial_{h_1}^2(\widetilde{f}_1(h_2 + s\mu\partial_{t_1} S_1, \theta) + f_2(h_2 + s\mu\partial_{t_1} S_1, \theta, t_1 + \theta))(\partial_{t_1} S_1)^2 ds d\mu \\ &- \int_0^1 \partial_{h_1} f_3(h_2 + \partial_{t_1} S_1, \theta)\partial_{t_1} S_1 d\mu - \int_0^1 \int_0^1 \partial_{t_1}^2[f_2](h_2, t_2 - s\mu\partial_{h_2} S_1)(\partial_{h_2} S_1)^2 d\mu ds \\ &- \int_0^1 \partial_{t_1} R_{11}(h_2, \theta, t_2 - \partial_{h_2} S_1)\partial_{h_2} S_1 d\mu + \int_0^1 \partial_{h_1} R_{11}(h_2 + \mu\partial_{t_1} S_1, \theta, t_2)\partial_{h_1} S_1 d\mu \\ &+ R_{12}(h_2 + \partial_{t_1} S_1, \theta, t_2 - \partial_{h_2} S_1). \end{aligned}$$

Obviously, $R_{22yu} \in O_3(h_2^{-\frac{1}{2}})$ follows from (5.70), (5.71), (5.77), (5.78) and Lemma 5.2.

From the definition of $\tilde{f}_1$, f_2, f_3 and Lemma 5.2, the only term in $\partial_{h_1}(\widetilde{f_1}+f_2)\cdot\partial_{t_1}S_1$ depending on $g'(\sqrt{\frac{2}{n}}h_2^{\frac{1}{2}}\cos n\theta)$ is

$$\begin{aligned}\frac{1}{2n}\sqrt{\frac{2}{n}}\cos^2 n\theta\cdot g'(\sqrt{\frac{2}{n}}h_2^{\frac{1}{2}}\cos n\theta)&=\frac{1}{2n}\sqrt{\frac{2}{n}}(h_2^{\frac{1}{2}}\cos n\theta)^2\cdot h_2^{-1}\cdot g'(\sqrt{\frac{2}{n}}I^{\frac{1}{2}}\cos n\theta)\\&\triangleq(\partial_{h_1}(\widetilde{f_1}+f_2)\cdot\partial_{t_1}S_1)_I,\end{aligned}$$

which is $O_3(h_1^{-\frac{1}{2}})$ since $g'(x)=((1+x^2)^{-\frac{1+d}{2}})''\cdot(1+o(1))$ for $|x|\gg 1$.

Denote

$$(\partial_{h_1}(\widetilde{f_1}+f_2)\cdot\partial_{t_1}S_1)_{II}=\partial_{h_1}(\widetilde{f_1}+f_2)\cdot\partial_{t_1}S_1-(\partial_{h_1}(\widetilde{f_1}+f_2)\cdot\partial_{t_1}S_1)_I.$$

Then one can easily obtain from the definition of $\tilde{f}_1$, f_2, f_3 that

$$(\partial_{h_1}(\widetilde{f_1}+f_2)\cdot\partial_{t_1}S_1)_{II}=\sum_i f_{6,i}(g(\sqrt{\frac{2}{n}}h^{\frac{1}{2}}\cos n\theta)))\cdot f_{7,i}(\theta,t_1)\tag{5.79}$$

with $f_{6,i}\in\mathcal{C}^4(\mathbb{R}^1)$ and $f_{7,i}\in\mathcal{C}^4(\mathbb{T}^2)$ are finitely many smooth functions.

Similarly, the only terms in $\partial_{t_1}[f_2]\cdot\partial_{h_2}S_1$ depending on g' are

$$\partial_{t_1}[f_2]\cdot\int_0^\theta(-\frac{1}{2}\cos^2 n\theta\cdot g'(\cdot)+[\frac{1}{2}\cos^2 n\theta\cdot g'(\cdot)])d\theta\triangleq(\partial_{t_1}[f_2]\cdot\partial_{h_2}S_1)_I\in O_3(h_2^{-\frac{1}{2}}),$$

where $g'(\cdot)=g'(\sqrt{\frac{2}{n}}h_2^{\frac{1}{2}}\cos n\theta)$. Let $(\partial_{t_1}[f_2]\cdot\partial_{h_2}S_1)_{II}=(\partial_{t_1}[f_2]\cdot\partial_{h_2}S_1)-(\partial_{t_1}[f_2]\cdot\partial_{h_2}S_1)_I$.

Then a direct computation shows that

$$\begin{aligned}(\partial_{t_1}[f_2]\cdot\partial_{h_2}S_1)_{II}&=\int_0^{2\pi}f_2(h_2,\theta,t_1+\theta)d\theta\cdot\int_0^\theta(\tfrac{1}{2}h_2^{-\frac{1}{2}}\cos n\theta\cdot g(\cdot)-[\tfrac{1}{2}h_2^{-\frac{1}{2}}\cos n\theta\cdot g(\cdot)])d\theta\\&=-\int_0^{2\pi}\tfrac{1}{n}\sqrt{\tfrac{2}{n}}\cos\theta p(t_1+\theta)d\theta\cdot\int_0^\theta(\tfrac{1}{2}\cos n\theta\cdot g(\cdot)-[\tfrac{1}{2}\cos n\theta\cdot g(\cdot)])d\theta.\end{aligned}$$

Thus we obtain a Hamiltonian as follows:

$$I_2=-[\widetilde{f_1}](h_2)-[f_2](h_2,t_2)-f_3(h_2,\theta)+f_{6\cdot 7}(h_2,t_2,\theta)+R_2(h_2,t_2,\theta),$$

where

$$\begin{aligned}f_{6\cdot 7}=&\sum_i f_{6,i}(g(\sqrt{\frac{2}{n}}h^{\frac{1}{2}}\cos n\theta))\cdot f_{7,i}(\theta,t_2)\\&+\int_0^{2\pi}\frac{1}{2n}\sqrt{\frac{2}{n}}\cos\theta p(t_2+\theta)d\theta\cdot\int_0^\theta(\cos n\theta\cdot g(\cdot)-[\cos n\theta\cdot g(\cdot)])d\theta\end{aligned}$$

and

$$R_2(h_2, t_2, \theta) = R_{22yu} + O_3(h_2{}^{-\frac{1}{2}}) \in O_3(h_2^{-\frac{1}{2}}). \tag{5.80}$$

Next, to eliminate the term $f_{6\cdot 7}$, we make the following canonical transformation Ψ_1:

$$\begin{cases} h_2 & = h_3 + \partial_{t_2} S_2, \\ t_3 & = t_2 + \partial_{h_3} S_2 \end{cases}$$

with the generating function S_2 determined by

$$S_2(h_3, t_2, \theta) = \int_0^\theta \{-f_{6\cdot 7} + [f_{6\cdot 7}](h_3, t_2)\} d\theta$$

with

$$[f_{6\cdot 7}](h_3, t_2) = \frac{1}{2\pi} \int_0^{2\pi} f_{6\cdot 7} d\theta.$$

Under Ψ_1, Hamiltonian I_2 is transformed into I_3 as follows

$$I_3 = -[\widetilde{f_1}](h_3) - [f_2](h_3, t_3) - f_3(h_3 + \partial_{t_2} S_2, \theta) + [f_{6\cdot 7}] + R_3,$$

where

$$\begin{aligned} R_3 = & -\int_0^1 ([\widetilde{f_1}]'(h_3 + s\partial_{t_2} S_2) + \partial_{h_2}[f_2](h_3 + s\partial_{t_2} S_2, t_2))\partial_{t_2} S_2 ds \\ & + \int_0^1 \partial_{t_2}[f_2](h_3, t_3 - \partial_{h_3} S_2)\partial_{h_3} S_2 ds + R_2 \end{aligned}$$

We have the following estimates:

Lemma 5.3. *For h_3 large enough, $\theta,\ t_3 \in \mathbb{S}^1$, we have the estimates on $R_3(h_3, t_3, \theta)$ as follows: for $j \le 1, k + l + j \le 2$,*

$$\left| \partial_{h_3}^k \partial_{t_3}^l \partial_\theta^j R_3(h_3, t_3, \theta) \right| \le C h_3{}^{-\frac{1}{2} - k};$$

Proof. Similar to the proof of Lemma 3.3, the estimates are obtained by direct calculations from (5.80). □

Note that $\widetilde{f_1}(I, \theta) = f_1(I, \theta) - f_3(I, \theta)$. From (3.35) and (5.70), we have

$$[\widetilde{f_1}](h_3) + [f_2](h_3, t_3) = -\frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} A(1 - \cos(nt_3 + \xi))\sqrt{h_3} + O_2(h_3{}^{\frac{1-d}{2}}).$$

In conclusion, the new Hamiltonian I_3 can be rewritten as

$$I_3 = -\frac{\sqrt{2}}{2\pi}n^{-\frac{3}{2}}A(1-\cos(nt_3-\xi))\sqrt{h_3} - f_3(\sqrt{\frac{2}{n}}h_3{}^{\frac{1}{2}}\cos n\theta) + [f_{6\cdot 7}] + O_2(h_3^{\max\{-\frac{1}{2},\frac{1-d}{2}\}}).$$

Similarly, we can construct a canonical transformation to eliminate the term $f_3(\sqrt{\frac{2}{n}}h_3{}^{\frac{1}{2}}\cos n\theta)$ by the following canonical transformation Ψ_2:

$$\begin{cases} h_3 & = \rho, \\ \tau & = t_3 + \partial_\rho S_3 \end{cases}$$

with

$$S_3(\rho,\theta) = \int_0^\theta \{f_3(\sqrt{\frac{2}{n}}\rho^{\frac{1}{2}}\cos n\theta) - [f_3](\rho)\}d\theta,$$

where $[f_3](\rho) = \int_0^{2\pi} f_3(\sqrt{\frac{2}{n}}\rho^{\frac{1}{2}}\cos n\theta)d\theta$.

Under Ψ_2, the Hamiltonian I_3 is transformed into $I_4(\rho,\tau,\theta)$ as follows

$$I_4 = -\frac{\sqrt{2}}{2\pi}n^{-\frac{3}{2}}A(1-\cos(n\tau+\xi))\sqrt{\rho} - [f_3](\rho) + f_8(\tau) + O_2(h_3^{\max\{-\frac{1}{2},\frac{1-d}{2}\}}).$$

With the help of (5.70), we have

$$|[f_3]^{(k)}(\rho)| \le C\cdot\rho^{-k-\max\{-\frac{1}{2},\ \frac{1-d}{2}\}}, \quad k=0,1. \tag{5.81}$$

In fact, for ρ large enough,

$$\begin{aligned}
\Big|[f_3](\rho)\Big| &\le 4\int_0^{\rho^{-\frac{1}{2}}} \Big|f_3(\sqrt{\frac{2}{n}}\rho^{\frac{1}{2}}\sin n\theta)\Big|d\theta + 4\int_{\rho^{-\frac{1}{2}}}^{\frac{\pi}{2}} \Big|f_3(\sqrt{\frac{2}{n}}\rho^{\frac{1}{2}}\sin n\theta)\Big|d\theta \\
&\le C_1\cdot\rho^{-\frac{1}{2}} + 4\int_{\rho^{-\frac{1}{2}}}^{\frac{\pi}{2}} |x|^{d-1}\Big|f_3(x)\Big|\rho^{\frac{1-d}{2}}\sin^{1-d}\theta d\theta \\
&\le C_1\cdot\rho^{-\frac{1}{2}} + C_2\rho^{\frac{1-d}{2}}\int_{\rho^{-\frac{1}{2}}}^{\frac{\pi}{2}} \sin^{1-d}\theta d\theta \\
&\le C_1\cdot\rho^{-\frac{1}{2}} + \frac{2}{\pi}C_2\rho^{\frac{1-d}{2}}\int_{\rho^{-\frac{1}{2}}}^{\frac{\pi}{2}} \theta^{1-d}d\theta \le C\cdot\max\{\rho^{-\frac{1}{2}},\ \rho^{\frac{1-d}{2}}\},
\end{aligned}$$

where $x = \sqrt{\frac{2}{n}}\rho^{\frac{1}{2}}\sin n\theta$. Moreover, the estimate of $[f_3]'(\rho)$ is similar. Therefore, (5.81) holds.

Similarly, from (1.13) and (1.14) we have that

$$[f_{6\cdot 7}](h_3, t_3) = f_8(t_3) + O_2(h_3{}^{-\frac{d}{10}}), \tag{5.82}$$

where $f_8(t_3) = \sum_i (\int_{\cos n\theta \geq 0} f_{6,i}(g(+\infty)) \cdot f_{7,i}(\theta, t_3) d\theta - \int_{\cos n\theta \leq 0} f_{6,i}(g(-\infty)) \cdot f_{7,i}(\theta, t_3) d\theta) \in \mathcal{C}^2(\mathbb{S}^1)$.

From (5.81) and (5.82), we have that

$$h(\rho, \tau, \theta) = -\frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} A(1 - \cos(n\tau + \xi))\sqrt{\rho} + f_8(\tau) + O_1(\rho^{-\delta}), \tag{5.83}$$

where $-\delta = max\{-\frac{1}{2}, \frac{1-d}{2}, -\frac{d}{10}\} < 0$ for $d > 1$.

6. The existence of unbounded solutions for $d > 1$

Finally, we prove that

Lemma 6.1. *The Hamiltonian system with Hamiltonian (5.83) has unbounded solutions.*

Proof. The system with Hamiltonian (5.83) is given by

$$\begin{cases} \dfrac{d\tau}{d\theta} = \partial_\rho h(\rho, \tau, \theta), \\ \dfrac{d\rho}{d\theta} = -\partial_\tau h(\rho, \tau, \theta). \end{cases} \tag{6.84}$$

Let $\rho^* \gg 1$ determined later. Assume τ^* satisfy $1 - \cos(n\tau^* + \xi) = 0$, i.e., $n\tau^* + \xi = 0$. Denote $\zeta = \tau - \tau^*$, then we have

$$\begin{cases} \dfrac{d\zeta}{d\theta} = -\dfrac{\sqrt{2}}{4\pi} n^{-\frac{3}{2}} A(1 - \cos(n\zeta))\rho^{-\frac{1}{2}} + O_0(\rho^{-\delta-1}), \\ \dfrac{d\rho}{d\theta} = \dfrac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} A \sin(n\zeta)\rho^{\frac{1}{2}} - f_8'(\tau^* + \zeta) + O_0(\rho^{-\delta}). \end{cases} \tag{6.85}$$

Assume $\rho(0) \geq \rho^*$. The second equation in (6.85) implies that $\frac{d\rho}{d\theta} = O(\rho^{1/2} + 1)$, it holds that $\rho(\theta) \geq \frac{1}{2}\rho^*$ for any $\theta \in [0, 1]$ if $\rho^* \gg \max\{A, n, \|f_8'\|\}$, which is true if $\rho^* \gg \max\{n, \|g\|, \|p\|\}$.

Next we further assume that $(\zeta(0), \rho(0)) \in D = \{(\zeta, \rho) | \rho^{-\frac{1+\delta}{4}} \leq \zeta \leq \rho^{-\frac{1}{20}}\}$. We claim that $(\zeta(\theta), \rho(\theta)) \in D$ for any $\theta \in [0, 1]$.

Otherwise, let $\theta_1 \triangleq \sup\{\theta \mid (\zeta(s), \rho(s)) \in D, \ 0 \leq s \leq \theta\} < 1$. Obviously, it holds that $(\zeta(\theta_1), \rho(\theta_1)) \in \partial D$. Thus $\rho(\theta_1)^{-\iota} = \zeta(\theta_1)$ with $\iota = \frac{1}{20}$ or $\frac{1+\delta}{4}$, that is, $\rho(\theta_1)^\iota \cdot \zeta(\theta_1) = 1$ (note that $\rho(s) \geq \frac{1}{2}\rho^* \gg 1$ for any $0 \leq s \leq 1$). By a direct computation, we have

$$\begin{aligned} &(\rho(\theta)^\iota \cdot \zeta(\theta))' \mid_{\theta=\theta_1} \\ &= \iota\rho(\theta)^{\iota-1} \cdot \zeta(\theta) \cdot (\tfrac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} A \sin(n\zeta(\theta))\rho(\theta)^{\frac{1}{2}} - f_8'(\tau^* + \zeta(\theta)) + O_0(\rho(\theta)^{-\delta})) \mid_{\theta=\theta_1} \\ &\quad + \rho(\theta)^\iota \cdot (-\tfrac{\sqrt{2}}{4\pi} n^{-\frac{3}{2}} A(1 - \cos(n\zeta(\theta))\rho(\theta)^{-\frac{1}{2}} + O_0(\rho(\theta)^{-\delta-1})) \mid_{\theta=\theta_1} \\ &\triangleq J_1 + J_2. \end{aligned}$$

From $\rho(\theta_1)^\iota \cdot \zeta(\theta_1) = 1$, $\rho(\theta_1) \geq \frac{1}{2}\rho^* \gg \max\{n, \|g\|, \|p\|\}$ and the facts that $\sin x = x + O_0(|x|^3)$ for small x, we have that

$$\sin(n\zeta(\theta_1)) = n \cdot \rho(\theta_1)^{-\iota} + O_0(\rho(\theta_1)^{-3\iota}), \quad 1 - \cos(n\zeta(\theta_1)) = \frac{n^2}{2} \cdot \rho(\theta_1)^{-2\iota} + O_0(\rho(\theta_1)^{-4\iota}).$$

Thus we obtain

$$\begin{aligned} J_1 &= \iota\rho(\theta_1)^{-1} \cdot (\tfrac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} A \sin(n\zeta(\theta_1))\rho(\theta_1)^{\frac{1}{2}} - f_8'(\tau^* + \zeta(\theta_1)) + O_0(\rho(\theta_1)^{-\delta})) \\ &= \tfrac{\sqrt{2}A}{2\pi} n^{\frac{1}{2}} \cdot \iota \cdot \rho(\theta_1)^{-\frac{1}{2}-\iota} + O_0(\rho(\theta_1)^{-1}) \end{aligned}$$

and

$$J_2 = -\frac{\sqrt{2}A}{8\pi} n^{\frac{1}{2}} \cdot \rho(\theta_1)^{-\frac{1}{2}-\iota} + O_0(\rho(\theta_1)^{\max\{-\frac{1}{2}-3\iota,\ \iota-1-\delta\}}).$$

Hence for $\iota = \frac{1}{20}$, we have

$$(\rho(\theta)^\iota \cdot \zeta(\theta))' \,|_{\theta=\theta_1} = \frac{\sqrt{2}A}{2\pi} n^{\frac{1}{2}} \cdot (\iota - \frac{1}{4}) \cdot \rho(\theta_1)^{-\frac{1}{2}-\iota} \cdot (1 + o(1)) < 0,$$

which implies that there is a $\theta_2 > \theta_1$ such that $\rho(\theta)^{\frac{1}{20}} \cdot \zeta(\theta) \leq 1$ for $\theta \in [\theta_1, \theta_2]$. It contradicts the definition of θ_1. Thus for all $\theta \in [0, 1]$, it holds that $\zeta(\theta) \leq (\rho(\theta))^{-\frac{1}{20}}$.

Similarly for $\iota = \frac{1+\delta}{4} > \frac{1}{4}$, we have $-\frac{1}{2} - \iota > \max\{-\frac{1}{2} - 3\iota,\ \iota - 1 - \delta\}$. Then we have

$$(\rho(\theta)^\iota \cdot \zeta(\theta))' \,|_{\theta=\theta_1} = \frac{\sqrt{2}A}{2\pi} n^{\frac{1}{2}} \cdot (\iota - \frac{1}{4}) \cdot \rho(\theta_1)^{-\frac{1}{2}-\iota} \cdot (1 + o(1)) > 0,$$

which implies that there is a $\theta_2 > \theta_1$ such that $\rho(\theta)^{\frac{1+\delta}{4}} \cdot \zeta(\theta) \geq 1$ for $\theta \in [\theta_1, \theta_2]$. It contradicts the definition of θ_1. Thus $\zeta(\theta) \geq (\rho(\theta))^{-\frac{1+\delta}{4}}$ for all $\theta \in [0, 1]$. The proof of the claim is thus completed.

From the claim and $\delta \leq \frac{1}{2}$, we obtain $\zeta(\theta) \geq \rho(\theta)^{-\frac{1+\delta}{4}} \geq \rho(\theta)^{-\frac{3}{8}}$ for all $\theta \in [0, 1]$. Thus again from the second equation in (6.85), we have that

$$\frac{d\rho}{d\theta} = \frac{\sqrt{2A}}{2\pi} n^{\frac{1}{2}} (\zeta(\theta) + O_0(|\zeta(\theta)|^3))\rho(\theta)^{\frac{1}{2}} + O_0(1) \geq \frac{\sqrt{A}}{2\pi} n^{\frac{1}{2}} \cdot \zeta(\theta) \cdot \rho(\theta)^{\frac{1}{2}} \geq \frac{\sqrt{A}}{2\pi} n^{\frac{1}{2}} \cdot \rho(\theta)^{\frac{1}{8}}$$

for $0 \leq \theta \leq 1$, which implies that $\rho(1) > \rho(0) + 1 > \rho^*$. In a word, if $\rho(0) \geq \rho^*$ and $(\zeta(0), \rho(0)) \in D$, then $\rho(1) \geq \rho(0) + 1 \geq \rho^*$ and $(\zeta(1), \rho(1)) \in D$.

Since the system (6.85) is periodic in θ, using the above argument repeatedly, we obtain that if $(\zeta(0), \rho(0))$ satisfies the initial conditions stated above, then $\rho(i+1) \geq \rho(i) + 1$ for any i. It leads that the solution $(\zeta(\theta), \rho(\theta))$ is unbounded. □

Let the map $t = t(\tau, \rho, \theta), \quad h = h(\tau, \rho, \theta)$ is determined by $\Psi_2 \circ \Psi_1 \circ \Psi \circ \Phi$. For a solution $(\tau, \rho) = (\tau(\theta), \rho(\theta))$ of (6.84), let $(t, h) = (t(\tau(\theta), \rho(\theta), \theta), h(\tau(\theta), \rho(\theta), \theta)) \triangleq (t(\theta), h(\theta))$ be the corresponding solution for the system determined by the Hamiltonian (5.76).

Let $\theta(t)$ be the inverse function of $t(\theta)$, then $(\theta, I) = (\theta(t), I(\theta(t), h(t), t)) \triangleq (\theta(t), I(t))$ is a solution of (1.1).

From (5.69), one can easily see that $\theta \to +\infty$ is equivalent to $t \to +\infty$. Moreover, from the constructed canonical transformations, we know that $I = I(\rho, \theta, \tau) = \rho + o(\rho)$. Thus for $d > 1$, if a solution $(\tau, \rho) = (\tau(\theta), \rho(\theta))$ of (6.84) is unbounded, i.e., $\limsup_\theta \rho(\theta) = \infty$, then $(\theta, I) = (\theta(t), I(t))$ is an unbounded solution of (1.1), that is, $\limsup_t I(t) = \infty$. This completes the proof for the instability of (1.1) for $d > 1$.

Acknowledgments

The first author was supported by The Open Foundation of the '13th Five-Year' Discipline (Mathematics) of Xinjiang Uygur Autonomous Region (No. XJZDXK-M2017010). The third author was supported by the National Natural Science Foundation of China grant no. 11771205.

Appendix A. Proof of Lemma 3.1

Proof. Suppose $k + j \le \upsilon_1 + 1$ and $l \le \upsilon_2$.

i) When $k + j + l = 0$, the conclusion follows from Lemmas 2.1 and 2.2.

ii) When $k + j + l = 1$, define

$$\begin{aligned} g_1(h,t,\theta) &= \partial_I f_1(h-R,\theta) + \partial_I f_2(h-R,t,\theta); \\ g_2(h,t,\theta) &= \partial_t f_2(h-R,t,\theta); \\ g_3(h,t,\theta) &= \partial_\theta f_1(h-R,\theta) + \partial_\theta f_2(h-R,t,\theta); \\ \Delta(h,t,\theta) &= 1 + \partial_I f_1(h-R,\theta) + \partial_I f_2(h-R,t,\theta). \end{aligned}$$

Obviously, $\Delta(h,t,\theta) \ge 1/2$ for $h \gg 1$ and

$$\Delta \cdot \partial_h R(h,t,\theta) = g_1(h,t,\theta), \quad \Delta \cdot \partial_t R(h,t,\theta) = g_2(h,t,\theta), \quad \Delta \cdot \partial_\theta R(h,t,\theta) = g_3(h,t,\theta). \tag{A.1}$$

From Lemmas 2.1–2.2, we obtain

$$\begin{aligned} \frac{1}{2}\left|\partial_h R(h,t,\theta)\right| &\le \left|\Delta \cdot \partial_h R(h,t,\theta)\right| \\ &= \left|\partial_I f_1(h-R,\theta) + \partial_I f_2(h-R,t,\theta)\right| \\ &\le C(h-R)^{-\frac{1}{2}} \le Ch^{-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\left|\partial_t R(h,t,\theta)\right| &\le \left|\Delta \cdot \partial_t R(h,t,\theta)\right| = \left|\partial_t f_2(h-R,t,\theta)\right| \\ &\le C(h-R)^{\frac{1}{2}} \le Ch^{\frac{1}{2}}, \end{aligned}$$

and

$$
\begin{aligned}
\frac{1}{2}\Big|\partial_\theta R(h,t,\theta)\Big| \le \Big|\Delta\cdot\partial_\theta R(h,t,\theta)\Big| \\
&= \Big|\partial_\theta f_1(h-R,\theta)+\partial_\theta f_2(h-R,t,\theta)\Big| \\
&\le C(h-R)^{\frac{1}{2}} \le Ch^{\frac{1}{2}}.
\end{aligned}
$$

iii) When $k+j+l=2$, from i) and ii), we have

$$
\begin{aligned}
&\Big|\partial_h g_1(h,t,\theta)\Big| \le Ch^{-\frac{3}{2}}, \quad \Big|\partial_t g_1(h,t,\theta)\Big| \le Ch^{-\frac{1}{2}}, \quad \Big|\partial_\theta g_1(h,t,\theta)\Big| \le Ch^{-\frac{1}{2}};\\
&\Big|\partial_h g_2(h,t,\theta)\Big| \le Ch^{-\frac{1}{2}}, \quad \Big|\partial_t g_2(h,t,\theta)\Big| \le Ch^{\frac{1}{2}}, \quad \Big|\partial_\theta g_2(h,t,\theta)\Big| \le Ch^{\frac{1}{2}};\\
&\Big|\partial_h g_3(h,t,\theta)\Big| \le Ch^{0}, \quad \Big|\partial_t g_3(h,t,\theta)\Big| \le Ch^{\frac{1}{2}}, \quad \Big|\partial_\theta g_3(h,t,\theta)\Big| \le Ch^{\frac{1}{2}};
\end{aligned}
$$

and

$$
\Big|\partial_h \Delta(h,t,\theta)\Big| \le Ch^{-\frac{3}{2}}, \quad \Big|\partial_t \Delta(h,t,\theta)\Big| \le Ch^{-\frac{1}{2}}, \quad \Big|\partial_\theta \Delta(h,t,\theta)\Big| \le Ch^{0}.
$$

From (A.1), differentiating on both sides of the equations, we obtain:

$$
\begin{aligned}
\Delta\cdot\partial_h^2 R(h,t,\theta) &= \partial_h g_1(h,t,\theta) - \partial_h\Delta\cdot\partial_h R(h,t,\theta),\\
\Delta\cdot\partial_t^2 R(h,t,\theta) &= \partial_t g_2(h,t,\theta) - \partial_t\Delta\cdot\partial_t R(h,t,\theta),\\
\Delta\cdot\partial_\theta^2 R(h,t,\theta) &= \partial_\theta g_3(h,t,\theta) - \partial_\theta\Delta\cdot\partial_\theta R(h,t,\theta),\\
\Delta\cdot\partial_h\partial_t R(h,t,\theta) &= \partial_t g_1(h,t,\theta) - \partial_t\Delta\cdot\partial_h R(h,t,\theta),\\
\Delta\cdot\partial_h\partial_\theta R(h,t,\theta) &= \partial_\theta g_1(h,t,\theta) - \partial_\theta\Delta\cdot\partial_h R(h,t,\theta),\\
\Delta\cdot\partial_t\partial_\theta R(h,t,\theta) &= \partial_\theta g_2(h,t,\theta) - \partial_\theta\Delta\cdot\partial_t R(h,t,\theta).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{1}{2}\Big|\partial_h^2 R(h,t,\theta)\Big| &\le \Big|\partial_h g_1(h,t,\theta)\Big| + \Big|\partial_h\Delta\cdot\partial_h R(h,t,\theta)\Big| \le Ch^{-\frac{3}{2}},\\
\frac{1}{2}\Big|\partial_t^2 R(h,t,\theta)\Big| &\le \Big|\partial_t g_2(h,t,\theta)\Big| + \Big|\partial_t\Delta\cdot\partial_t R(h,t,\theta)\Big| \le Ch^{\frac{1}{2}},\\
\frac{1}{2}\Big|\partial_\theta^2 R(h,t,\theta)\Big| &\le \Big|\partial_\theta g_3(h,t,\theta)\Big| + \Big|\partial_\theta\Delta\cdot\partial_\theta R(h,t,\theta)\Big| \le Ch^{1},\\
\frac{1}{2}\Big|\partial_h\partial_t R(h,t,\theta)\Big| &\le \Big|\partial_t g_1(h,t,\theta)\Big| + \Big|\partial_t\Delta\cdot\partial_h R(h,t,\theta)\Big| \le Ch^{-\frac{1}{2}},\\
\frac{1}{2}\Big|\partial_h\partial_\theta R(h,t,\theta)\Big| &\le \Big|\partial_\theta g_1(h,t,\theta)\Big| + \Big|\partial_\theta\Delta\cdot\partial_h R(h,t,\theta)\Big| \le Ch^{-\frac{1}{2}},\\
\frac{1}{2}\Big|\partial_t\partial_\theta R(h,t,\theta)\Big| &\le \Big|\partial_\theta g_2(h,t,\theta)\Big| + \Big|\partial_\theta\Delta\cdot\partial_t R(h,t,\theta)\Big| \le Ch^{\frac{1}{2}}.
\end{aligned}
$$

Generally, if

$$\left|\partial_h^k \partial_t^l \partial_\theta^j R(h,t,\theta)\right| \le Ch^{\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)}, \text{ for } 1 \le j+k+l \le m,$$

then

$$\left|\partial_h^k \partial_t^l \partial_\theta^j g_1(h,t,\theta)\right| \le Ch^{-\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)}, \qquad \left|\partial_h^k \partial_t^l \partial_\theta^j g_2(h,t,\theta)\right| \le Ch^{\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)},$$
$$\left|\partial_h^k \partial_t^l \partial_\theta^j g_3(h,t,\theta)\right| \le Ch^{\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)}, \qquad \left|\partial_h^k \partial_t^l \partial_\theta^j \Delta(h,t,\theta)\right| \le Ch^{-\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)}.$$

The proof of these estimates is based on Leibniz's rule and direct calculations. Consequently, by induction and Leibniz's rule again to (A.1), we obtain

$$\left|\partial_h^k \partial_t^l \partial_\theta^j R(h,t,\theta)\right| \le Ch^{\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)}, \text{ for } j+k+l \le m+1. \qquad \square$$

Appendix B. Proof of Lemma 3.3

Proof. (3.27) follows from (3.25) and Lemma 2.1.

From (3.24), it is easy to see

$$|\partial_{h_2} t_1| \le C{h_2}^{-\frac{3}{2}}, \quad \partial_{t_2} t_1 = 1, \quad |\partial_\theta t_1| \le C{h_2}^{-\frac{1}{2}}.$$

By direct calculations, for $k+l+j \ge 2$ and $k+j \le \upsilon_1$,

$$\begin{aligned}|\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j t_1| &= |\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j (t_2 - \partial_{h_2} S_2(h_2,\theta))| = |\partial_{h_2}^{k+1} \partial_{t_2}^l \partial_\theta^j S_2(h_2,\theta)| \\ &\le C{h_2}^{-\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)}.\end{aligned}$$

Next, we consider the estimates on R_{21}. Firstly, it holds that $|R_{21}| \le C$.

Suppose $k+j \le \upsilon_1 - 1$.

i) Consider $\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j R_1(h_2, t_2 - \partial_{h_2} S_2(h_2,\theta), \theta)$. From Lemma 3.2 and by Leibniz's rule, it is the sum of terms

$$(\partial_{h_1}^p \partial_{t_1}^q \partial_\theta^r R_1)(\Pi_{i=1}^q \partial_{h_2}^{k_i} \partial_{t_2}^{l_i} \partial_\theta^{j_i} t_1)$$

with $1 \le p+q+r \le j+k+l$, $p + \sum_{i=1}^q k_i = k$, $\sum_{i=1}^q l_i = l$, $r + \sum_{i=1}^q j_i = j$, and $k_i + j_i + l_i \ge 1$, $i = 1, \ldots, q$, which implies that

$$|\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j R_1(h_2, t_2 - \partial_{h_2} S_2(h_2,\theta), \theta)| \le C{h_2}^{-k+\frac{j}{2}}, \text{ for } l \le \upsilon_2.$$

ii) Similar to part i), with Lemma 2.2 we have

$$|\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j (\partial_{t_1} f_2(h_2, \theta, t_2 + \theta - \mu \partial_{h_2} S_2))| \le C{h_2}^{-k+\frac{1}{2}(max\{1,j\}-1)+\frac{1}{2}}, \text{ for } l \le \upsilon_2 - 1.$$

By Leibniz's rule and the estimates on $\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j S_2(h_2,\theta)$, it holds that

$$|\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j (\partial_{t_1} f_2(h_2,\theta,t_2-\mu\partial_{h_2}S_2)\partial_{h_2}S_2)| \le C{h_2}^{-\frac{1}{2}-k+\frac{1}{2}(max\{1,j\}-1)}, \text{ for } l \le \upsilon_2 - 1,$$

which, together with part i) and part ii), implies that

$$|\partial_{h_2}^k \partial_{t_2}^l \partial_\theta^j R_2| \le C{h_2}^{-k+\frac{j}{2}}, \text{ for } l \le \upsilon_2 - 1. \qquad \square$$

Appendix C. Proof of Lemma 4.1

(i) Firstly, we estimate $N(I,\tau)$. Note that $\alpha(N(I,\tau),\tau) \equiv I$, then

$$cI^2 \le \Big|N\Big| \le CI^{\frac{2}{1-d}}, \tag{C.1}$$

and

$$\partial_{h_4}\alpha \cdot \partial_{I_4} N = 1, \quad \partial_{h_4}\alpha \cdot \partial_{t_4} N + \partial_{t_4}\alpha = 0.$$

Thus from (3.34) and (3.35) and Lemma 3.6, it follows that

$$c\cdot\alpha \le h\cdot\partial_h\alpha = \frac{a(t)}{2}h^{\frac{1}{2}} + \frac{1-d}{2}h^{\frac{1-d}{2}} \le C\cdot\alpha$$

Since $\alpha(h,t) = I$ and $N(I,\tau) = h$,

$$\partial_I h = \partial_I N = (\partial_{h_4}\alpha)^{-1} \in [chI^{-1},\ ChI^{-1}] \sim [cI^{-1}N,\ CI^{-1}N]. \tag{C.2}$$

It together with (C.1) implies

$$cI \le \partial_I N \le CI^{\frac{1+d}{1-d}}. \tag{C.3}$$

A direct computation shows that

$$\partial_I^2 N = \partial_I(\frac{1}{\partial_h\alpha}) = -\frac{\partial_I h\cdot\partial_h^2\alpha}{(\partial_h\alpha)^2} = -\frac{\partial_I N\cdot\partial_h^2\alpha}{(\partial_h\alpha)^2}.$$

Since $\partial_h^2\alpha = -\frac{a(t)}{4}h^{-\frac{1}{2}} - \frac{1-d^2}{4}h^{-\frac{3+d}{2}}$, we have $ch^{-1}|\partial_h\alpha| \le |\partial_h^2\alpha| \le Ch^{-1}|\partial_h\alpha|$. Together with (C.2) and (C.3), we have

$$|\partial_I^2 N| \le |\frac{\partial_I N\cdot h^{-1}}{(\partial_h\alpha)}| \le |\partial_I h\cdot h^{-1}\cdot\partial_I N| \le CI^{-1}\cdot|\partial_I N| \le CI^{-2}\cdot|N|. \tag{C.4}$$

Similarly, we have

$$|\partial_I^2 N| \ge cI^{-1}\cdot|\partial_I N| \ge cI^{-2}\cdot|N|. \tag{C.5}$$

Generally, for $2 \leq k \leq \upsilon_1 + 1 - \nu$, using Leibniz's rule, $\partial_I^k N$ is the sum of terms

$$\partial_{h_4}^u \left(\frac{\partial_h^2 \alpha}{(\partial_h \alpha)^2} \right) \Pi_{i=1}^u \partial_I^{k_i} N$$

with $0 \leq u \leq k$, $\sum_{i=1}^u k_i = k - 1$, and $k_i \geq 1$, $i = 1, \dots, u$. Since $|h^k \partial_h^k \alpha| \leq C|\alpha|$ for any k, the following are obtained inductively with the help of (C.2):

$$|\partial_I^k N| \leq C \cdot I^{-k} |N|, \ k \geq 1. \tag{C.6}$$

(ii) Secondly, from (4.54), we obtain

$$\big|P\big| \leq \frac{1}{2}\big|N\big|. \tag{C.7}$$

Furthermore, it holds that

$$|P| \leq |R(N + P, t, \theta)| \cdot \left| \int_0^1 \partial_{h_4} \alpha(\mathcal{N} + \mu \mathcal{P}, t_4) d\mu \right|^{-1} \leq C I^{-1} N |R| \leq C \partial_I N \cdot R. \tag{C.8}$$

For the estimate of $\partial_I P$, we have

$$\partial_I P = \frac{-(\partial_h R + P \int_0^1 \partial_{h_4}^2 \alpha(\mathcal{N} + \mu \mathcal{P}, t_4)) \cdot \partial_I N}{\int_0^1 \partial_{h_4} \alpha(\mathcal{N} + \mu \mathcal{P}, t_4) d\mu + P \int_0^1 \partial_{h_4}^2 \alpha(\mathcal{N} + \mu \mathcal{P}, t_4) \mu d\mu + \partial_h R}.$$

From (C.8) and $|h_4 \partial_{h_4}^2 \alpha| \leq C |\partial_{h_4} \alpha|$, $|P| \ll N = h_4$ and $|\partial_h R| \ll |\partial_h \alpha|$, it holds for $I \gg 1$ that

$$\begin{aligned}
&| \textstyle\int_0^1 \partial_{h_4} \alpha(\mathcal{N} + \mu \mathcal{P}, t_4) d\mu + \lambda P \int_0^1 \partial_{h_4}^2 \alpha(\mathcal{N} + \mu \mathcal{P}, t_4) d\mu + \partial_h R| \\
&\geq \tfrac{1}{2} | \textstyle\int_0^1 \partial_{h_4} \alpha(\mathcal{N} + \mu \mathcal{P}, t_4) d\mu| \\
&\geq c h^{-1} I.
\end{aligned}$$

From $|\partial_h R| \leq C h^{-1} |R|$ and (C.8) which implies $|P \int_0^1 \partial_{h_4}^2 \alpha(\mathcal{N} + \mu \mathcal{P}, t_4) \leq C \cdot \partial_I N \cdot h^{-2} I |R| \leq C h^{-1} |R|$, it follows that

$$|\partial_I P| \leq C h^{-1} |R| \cdot |\partial_I N| \cdot h I^{-1} \leq C I^{-1} |R| \cdot |\partial_I N|.$$

Similar to the proof of (C.6), inductively for any k we can obtain

$$|\partial_I^k P| \leq C I^{-k} |R| \cdot |\partial_I N|.$$

A direct computation shows that

$$\partial_\theta P = \frac{-\partial_\theta R}{\int_0^1 \partial_{h_4} \alpha(\mathcal{N} + \mu \mathcal{P}, t_4) d\mu + P \int_0^1 \partial_{h_4}^2 \alpha(\mathcal{N} + \mu \mathcal{P}, t_4) \mu d\mu + \partial_h R},$$

from (3.51) we have

$$|\partial_\theta P| \le C \cdot I^{-1} h \cdot |\partial_\theta R| \le C \cdot \partial_I N \cdot |R|.$$

Similar to the above argument, we have

$$|\partial_I^k \partial_\theta^j P| \le C \cdot I^{-k} |R| \cdot |\partial_I N|. \qquad \square$$

References

[1] J.M. Alonso, R. Ortega, Unbounded solutions of semilinear equations at resonance, Nonlinearity 9 (1996) 1099–1111.
[2] J.M. Alonso, R. Ortega, Roots of unity and unbounded motions of an asymmetric oscillator, J. Differential Equations 143 (1998) 201–220.
[3] V.I. Arnold, On the behavior of an adiabatic invariant under slow periodic variation of the Hamiltonian, Sov. Math., Dokl. 3 (1962) 136–139.
[4] T. Ding, Nonlinear oscillations at a point of resonance, Sci. Sin. 25 (1982) 918–931.
[5] R.E. Gaines, J. Mawhin, Coincidence Degree, and Nonlinear Differential Equations, Lecture Notes in Math, vol. 568, Springer-Verlag, Berlin, 1977.
[6] A.M. Krssnosel'skii, J. Mawhin, Periodic solutions of equations with oscillating nonlinearities, Math. Comput. Modelling 32 (2000) 1445–1455.
[7] A.C. Lazer, D.E. Leach, Bounded perturbations of forced harmonic oscillators at resonance, Ann. Mat. Pura Appl. 82 (1969) 49–68.
[8] M. Levi, Quasiperiodic motions in superquadratic time-periodic potentials, Comm. Math. Phys. 143 (1991) 43–83.
[9] B. Liu, Boundedness in nonlinear oscillations at resonance, J. Differential Equations 153 (1999) 142–174.
[10] B. Liu, Boundedness in asymmetric oscillations, J. Math. Anal. Appl. 231 (2) (1999) 355–373.
[11] B. Liu, Quasi-periodic solutions of a semilinear Liénard equation at resonance, Sci. China Ser. A 48 (9) (2005) 1234–1244.
[12] B. Liu, Quasi-periodic solutions of forced isochronous oscillators at resonance, J. Differential Equations 246 (2009) 3471–3495.
[13] J. Mawhin, Resonance and nonlinearity: a survey, Ukrainian Math. J. 59 (2) (2007) 197–214.
[14] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences, vol. 74, Springer-Verlag, New York, 1989.
[15] J.K. Moser, On invariant curves of area preserving mappings of an annulus, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1962) 1–20, 325–342.
[16] R. Ortega, Boundedness in a piecewise linear oscillator and a variant of the small twist theorem, Proc. Lond. Math. Soc. 79 (1999) 381–413.
[17] X. Wang, Invariant tori and boundedness in asymmetric oscillations, Acta Math. Sin. (Engl. Ser.) 19 (4) (2003) 765–782.
[18] Z. Wang, Y. Wang, D. Piao, A new method for the boundedness of semilinear Duffing equations at resonance, Discrete Contin. Dyn. Syst. Ser. A 36 (7) (2016) 3961–3991.
[19] X. Xing, Y. Wang, Boundedness for semilinear Duffing equations at resonance, Taiwanese J. Math. 16 (5) (2012) 1923–1949.