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Group-invariant solutions for the Ricci curvature equation and the Einstein equation

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Abstract

We consider the pseudo-Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, with coordinates $x = (x_1, \ldots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$, where at least one ϵ_i is positive, and also tensors of the form $A = \sum_{i,j} A_{ij} dx_i dx_j$, such that A_{ij} are differentiable functions of x. For such tensors, we use Lie point symmetries to find metrics $\overline{g} = \frac{1}{u^2} g$ that solve the Ricci curvature and the Einstein equations. We provide a large class of group-invariant solutions and examples of complete metrics $\overline{g}$ defined globally in $\mathbb{R}^n$. As consequences, for certain functions $\overline{K}$, we show complete metrics $\overline{g}$, conformal to the pseudo-Euclidean metric g, whose scalar curvature is $\overline{K}$.

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1. Introduction

In [13] Milnor considered the problem of understanding Ricci curvature as a fundamental problem in mathematics. One basic question is to determine which symmetric covariant tensors

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of rank two can be Ricci tensors of Riemannian metrics. Therefore we formulate the following problem:

(P1) Given a symmetric $(0,2)$-tensor R, defined on a manifold M^n, $n \geq 3$, does there exist a Riemannian metric g, such that $\operatorname{Ric} g = R$?

Studying problem (P1) corresponds to solving a system of nonlinear second-order differential equations. There are the same number of equations as unknowns in the system because g and R are both symmetric $n \times n$ matrices. However there is a complicating factor since, according to [8], any solution of $\operatorname{Ric} g = R$ must also satisfy the Bianchi identity given by

$$\operatorname{Bian}(g, R) = g^{ab}(R_{am;b} - \frac{1}{2}R_{ab;m}) = 0.$$

DeTurck showed in [5] that, whenever $n \geq 3$, problem (P1) admits a local solution when the given tensor R is nonsingular. Moreover, he presented examples of singular tensors R for which there is not a metric g satisfying $\operatorname{Ric} g = R$ even locally. When R is singular, but still has constant rank and satisfies certain appropriate conditions, the Ricci curvature equation also admits local solutions (see [7]). Once the existence of local solutions for the problem (P1) when the given tensor R is non-singular has been established, other aspects of the problem (P1) can be considered such as the existence and uniqueness of global solutions and completeness.

Results on the existence and uniqueness of solutions to problem (P1) when M^n is a two-dimensional manifold can be found in [2] and [8]. For compact manifolds some nonexistence results can be found in [9] and [10]. Cao and DeTurck in [3] studied the existence of global solutions on balls (including all of $\mathbb{R}^n$) and spheres for rotationally symmetric nonsingular tensors. For such tensors on balls, they showed that problem (P2) has a unique (up to homothety) local rotationally symmetric solution, and they studied the question of existence of complete metrics of this type. On the sphere S^n, they proved some nonexistence results, and they obtained necessary conditions on a rotationally symmetric tensor R for the existence of a metric g satisfying $\operatorname{Ric} g = R$ on S^n. Cao and DeTurck, in [3] also studied the system $\operatorname{Ric} g = R$ in $\mathbb{R}^n$ for rotationally symmetric tensors and proved that a metric g satisfying $\operatorname{Ric} g = R$ must be conformally flat.

On the other hand, in the problem section of the Seminar on Differential Geometry edited by Yau [24], the first problem on Ricci curvature is the following:

(P2) Find necessary and sufficient conditions on a symmetric tensor T on a manifold so that one can find a metric g to satisfy $\operatorname{Ric} g - \frac{1}{2}Kg = T$ where $\operatorname{Ric} g$ is the Ricci tensor and K is the scalar curvature of g.

If g is the Lorentz metric on a four-dimensional manifold and T represents a physical field such as electromagnetic field perfect fluid type, pure radiation field and vacuum ($T = 0$), the equation above equation is the Einstein field equation and has been studied in several papers, most of them dealing with solutions which are invariant under some symmetry group of the equation (see [6,11] for details).

Through the last years, Pina and Tenenblat have obtained results for problems (P1) and (P2) for special classes of tensors and conformal metrics. In [15,16], the authors considered metrics conformal to a pseudo-Euclidean standard metric that solve (P1) and (P2) for constant tensors. By considering metrics conformal to standard metrics g of the sphere $\mathbb{S}^n$, the hyperbolic space

$\mathbb{H}^n$ and a pseudo-Euclidean space, the authors have solved the problems when the corresponding tensor is a multiple of the metric [17,18]. In [19], Pina and Tenenblat considered the pseudo-Euclidean space $(\mathbb{R}^n, g)$, with $n \geq 3$, coordinates $x = (x_1, \ldots, x_n)$ and $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon = \pm 1$, where at least one ϵ_i is positive and solutions of (P1) and (P2) when the tensor A is given by $A = \sum_{i=1}^{n} A_i(x_k)\epsilon_i dx_i$, for a fixed k, while in [20], the tensor considered is non-diagonal, given by $A = \sum_{i,j} A_{ij}(x_i, x_j)dx_i dx_j$. In both cases, the solutions are metrics conformal to the metric g. Recently, Pulemotov [21,22] also has made progress on the problem of prescribed Ricci tensors. In [21], the problem is considered on homogeneous Riemannian manifolds $M = G/H$, where G is a compact Lie group and H is a closed subgroup of G, while in [22], the author considered cohomogeneity-one manifolds.

In this paper, we extend the results obtained by Pina and Tenenblat, by considering group invariant solutions for the problems (P1) and (P2). We apply the theory of Lie point symmetries to the system of partial differential equations that arises from (P1) and (P2) when the off-diagonal components are considered. Once the off-diagonal components of the metrics are obtained from group-invariant solutions of such a system, the diagonal components of the tensor will be determined. Besides a large class of solutions for more general tensors, we also consider particular cases that remained unsolved in [20]. Among these solutions, we present complete metrics in the Riemannian case, even when A is a singular tensor. The problem of prescribed scalar curvature is also considered, i.e., we show that for certain functions $\overline{K}$, there exist metrics $\overline{g}$, conformal to the pseudo-Euclidean metric g, whose scalar curvature is $\overline{K}$. Equivalently, we find C^∞ solutions, globally defined in $\mathbb{R}^n$, for the equation

$$\frac{4(n-1)}{n-2}\Delta_g w + \overline{K} w^{\frac{n+2}{n-2}} = 0, \tag{1.1}$$

where Δ_g denotes the Laplacian in the pseudo-Euclidean metric g.

This paper is organized as follows. In Section 2, we give a brief introduction of main concepts and techniques of Lie point symmetries. In Section 3, we present the main results and their consequences. A theorem regarding the infinitesimal generators of the group action will be given, as well as the invariant solutions under symmetry groups of the Ricci curvature and the Einstein equations. As consequences we provide complete metrics that solve such equations and solutions for two-dimensional hyperbolic equations that complete solutions given in [20]. Finally, the proofs of the main results is given in Section 4.

2. Lie point symmetries

The theory of Lie point symmetries is an important technique for the study of differential equations, developed by Lie at the end of the nineteen century [12]. In a few words, Lie point symmetries of a system of differential equations consist of a Lie group of transformations acting on the dependent and independent variables, that transform solutions of the system into solutions of the system.

Standard references for the theory of symmetry groups of differential equations are the books [14] and [23], where good approaches to the subject are given, with theoretical foundations and a large number of examples and techniques.

A system S of n-th order differential equations in p independent and q dependent variables is given as a system of equations

$$S_r(x, u, u_{,\alpha}) = 0, \ r = 1, \ldots, l, \tag{2.1}$$

involving $x = (x_1, \ldots, x_p)$, $u = (u_1, \ldots, u_q)$ and the derivatives

$$u_{,\alpha} := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_p^{\alpha_p}} \tag{2.2}$$

where α is a multi-index given by a p-tuple of non-negative integers $(\alpha_1, \ldots, \alpha_p)$ and $|\alpha| = \alpha_1 + \ldots + \alpha_p$, with $1 \leq |\alpha| \leq n$.

A symmetry group of the system S is a local Lie group of transformations G acting on an open subset $M \subset X \times U$ of the space of independent and dependent variables for the system, with the property that if $u = u(x)$ is a solution of S and $g \cdot u$ is defined for $g \in G$, then $\tilde{u} = g \cdot u(x)$ is also a solution of the system. A vector field V in the Lie algebra $\mathfrak{g}$ of the group G is called an infinitesimal generator.

The symmetry group of a system of differential equation is obtained by using the *prolongation* of a vector field and the *infinitesimal criterion* that are described as follows. Given a vector field V on $M \subset X \times U$,

$$V = \sum_{i=1}^{p} \xi^i(x, u)\partial_{x_i} + \sum_{j=1}^{q} \phi^j(x, u)\partial_{u_j},$$

the *n-th prolongation* of V is the vector field

$$\mathrm{pr}^{(n)} V = V + \sum_{j=1}^{q} \sum_{\beta} \phi^j_\beta(x, u, u_{,\alpha})\partial_{u_{j,\beta}},$$

defined on the corresponding jet space $M^{(n)} \subset X \times U \times U^{(1)} \times \ldots \times U^{(n)}$, whose coordinates represent the independent variables x, the dependent variables u and the derivatives $u_{,\alpha}$ up to order n. The second summation is taken over all multi-indices $\beta = (\beta_1, \ldots, \beta_p)$, with $1 \leq |\beta| \leq n$. The coefficient functions ϕ^j_β of $\mathrm{pr}^{(n)} V$ are given by the following formula:

$$\phi^j_\beta(x, u, u_{,\alpha}) = D_\beta \left(\phi^j - \sum_{i=1}^{p} \xi^i u_{j,x_i} \right) + \sum_{i=1}^{p} \xi^i (u_{j,\beta})_{,x_i},$$

where $u_{j,x_i} = \dfrac{\partial u_j}{\partial x_i}$, $(u_{j,\beta})_{,x_i} = \dfrac{\partial u_{j,\beta}}{\partial x_i}$ and D_β is given by the total derivatives

$$D_\beta = D_1^{\beta_1} D_2^{\beta_2} \ldots D_p^{\beta_p},$$

with $D_i \psi(x, u, u_{,\alpha}) = \dfrac{\partial \psi}{\partial x_i} + \sum_{j=1}^{q} \sum_{\beta} (u_{j,\beta})_{,x_i} \dfrac{\partial \psi}{\partial u_{j,\beta}}$. Here the expression "total derivatives" is used to distinguish D_i from the partial derivatives $\dfrac{\partial}{\partial x_i}$ of a function ψ that depends on x, u and the partial derivatives $u_{,\alpha}$ up to order n (see [14, Chapter 2]).

The set of all vectors fields V on M such that

$$\mathrm{pr}^{(n)} V[S_r(x,u,u_{,\alpha})] = 0,\ r = 1,\dots,l, \quad \text{whenever } S_r(x,u,u_{,\alpha}) = 0, \tag{2.3}$$

is a Lie algebra of infinitesimal generators of a symmetry group for the system. Equation (2.3) is known as the infinitesimal criterion of invariance. It is shown in [14] that the infinitesimal criterion is a necessary and sufficient condition for a group G to be a symmetry group of the system.

Since the prolongation formula is given in terms of ξ^i and ϕ^j and their partial derivatives with respect to x and u, the infinitesimal criterion provides a system of partial differential equations for the functions ξ^i and ϕ^j. This system is called the *determining equations*. By solving these equations, we obtain the vector field V that determines a Lie algebra $\mathfrak{g}$. The symmetry group G is then obtained by exponentiating this Lie algebra.

Once we have the infinitesimal generators, we will find the invariant solutions by solving a reduced system with fewer variables than the original system. In this paper, the original system will be reduced to another one of ordinary differential equations. More precisely, we will consider symmetry groups G that act projectably on $X \times U$, i.e., the transformations on the independent variables x do not depend on the variables u, such that their orbits have all the same dimension $n-1$. Under these assumptions, the reduced system of ordinary differential equations will depend on the variable ξ (called a basic invariant), where ξ is a solution of $V(\xi) = 0$ and V is an infinitesimal generator of G (see Olver [14, Chapter 3] for details regarding construction of group-invariant solutions).

3. Main results and their consequences

In this section will be presented the main results of the paper. The proofs are given in the next section.

We consider the pseudo-Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, with coordinates $x = (x_1, \dots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon = \pm 1$, where at least one ϵ_i is positive. We consider tensors of the form $A = \sum_{i,j} A_{ij} dx_i dx_j$, such that A_{ij} are differentiable functions of x. For such a tensor, we want to find metrics $\overline{g} = \frac{1}{u^2} g$, that solve the Ricci curvature equation or the Einstein equation. More precisely, we want to solve the following problems:

$$\begin{cases} \overline{g} = \frac{1}{u^2} g, \\ \mathrm{Ric}\,\overline{g} = R \end{cases} \tag{3.1}$$

for a given tensor R and

$$\begin{cases} \overline{g} = \frac{1}{u^2} g, \\ \mathrm{Ric}\,\overline{g} - \frac{\overline{K}}{2}\overline{g} = T \end{cases} \tag{3.2}$$

for a given tensor T, where $\overline{K}$ stands for the scalar curvature of $\overline{g}$.

Since g is the standard metric on $\mathbb{R}^n$, the second of equations (3.1) can be written as

$$R = \mathrm{Ric}\,\overline{g} = \frac{1}{u^2}\left\{(n-2)u\mathrm{Hess}_g(u) + \left[u\Delta_g u - (n-1)|\nabla_g u|^2\right] g\right\}.$$

This is equivalent to saying that u is a nonvanishing solution of the following system of equations:

$$u_{,x_i x_j} = \frac{R_{ij}}{n-2} u, \quad \text{for all } i \neq j, \tag{3.3}$$

$$R_{ii} = (n-2)\frac{u_{,x_i x_i}}{u} + \epsilon_i \frac{\Delta_g u}{u} - \epsilon_i (n-1)\frac{|\nabla_g u|^2}{u^2}, \quad \text{for all } i. \tag{3.4}$$

Similarly, when we consider the Einstein equation (3.2), we have that $\overline{g} = \frac{1}{u^2} g$ is such that $\mathrm{Ric}\,\overline{g} - \frac{\overline{K}}{2}\overline{g} = T$ if, and only if, the following equations hold

$$u_{,x_i x_j} = \frac{T_{ij}}{n-2} u, \quad \text{for all } i \neq j, \tag{3.5}$$

$$T_{ii} = (n-2)\left(\frac{u_{,x_i x_i}}{u} - \epsilon_i \frac{\Delta_g u}{u} + \epsilon_i (n-1)\frac{|\nabla_g u|^2}{u^2}\right), \quad \text{for all } i. \tag{3.6}$$

In this context, the strategy used for obtaining group-invariant solutions for the Ricci curvature and Einstein equations is the following. Given the off-diagonal elements of the tensor R (resp. T) in equation (3.1) (resp. (3.2)) that satisfy the invariance criterion for symmetry under the action of the group G, we first solve equations (3.3) (resp. (3.5)) to determine the conformal factor u. Once this solution is obtained, equation (3.4) (resp. (3.6)) will be used to determine the diagonal elements R_{ii} and T_{ii} and consequently, we will have a set of necessary and sufficient conditions to solve the Ricci curvature and Einstein equations by means of such invariant solutions.

3.1. Infinitesimal generators

In light of the previous discussion, the first theorem presents the determining equations to obtain a vector field in the Lie algebra of the symmetry group for a system of the type

$$u_{,x_i x_j} = \frac{f_{ij}}{n-2} u, \quad \text{for all } i \neq j, \tag{3.7}$$

for given functions f_{ij}, i.e., system of partial differential equations as given in (3.3) and (3.5).

Theorem 3.1. *Let V be an infinitesimal generator of the symmetry group of the system (3.7) given by*

$$V = \sum_{i=1}^{n} \xi^i \partial_{x_i} + \phi \partial_u \tag{3.8}$$

Then

a) the function ϕ is given by

$$\phi(x,u) = au + B(x), \tag{3.9}$$

where $B(x)$ is a solution of (3.7) and $a \in \mathbb{R}$.

b) the functions ξ^i depend only on x_i and they are solutions of the PDE system

$$\left(\xi^i_{,x_i} + \xi^j_{,x_j}\right) f_{ij} + \sum_{k=1}^{n} \xi^k f_{ij,x_k} = 0 \tag{3.10}$$

As was to be expected, the infinitesimal generators and consequently the symmetry subgroups of (3.7) will depend on the functions f_{ij}. Examples of this situation were considered by Bozhkov and Freire [1] for a classification of Lie point symmetries of the nonlinear Poisson equation, whereas Cicogna [4] considers symmetries of quasilinear equations containing arbitrary functions, and also Lahno and Zhdanov [25] present a classification of symmetries for a general evolution equation. Also, the result (3.9) about the function ϕ could have been anticipated, since the system (3.7) is linear.

In what follows, we consider only projectable actions, i.e., actions that do not depend on u, and consider just vector fields given as $V = \sum_{i=1}^{n} \xi^i \partial_{x_i}$. Theorem 3.1 gives us a way to find symmetry groups and invariant solutions for the system (3.7) by giving conditions on the functions f_{ij} that enable us to find the coefficients ξ^i of the vector field V. The next result present a first relation between the functions f_{ij} and basic invariants of (3.7).

Recall that a function h is invariant under the infinitesimal action generated by a vector field V if $V(h) = 0$. The following theorem states conditions on the invariance of the functions f_{ij}.

Theorem 3.2. *Let G a symmetry group of the system (3.7) with non-null f_{ij}, for all $i \neq j$. Then the functions f_{ij} are invariant under G if and only if the symmetry group G is a translation group.*

Therefore, if we want to look for symmetries other than translations, the individual component functions f_{ij} cannot have the same invariance as the solutions of the system. In this context, we have the following theorem:

Theorem 3.3. *Let G a symmetry group of the system (3.7) with functions $f_{ij} = U_i'(x_i)U_j'(x_j) \times f(\zeta)$, where f and U_i, for all i, are arbitrary, non-null, differentiable functions, U_i depending only on x_i with $U_i'(x_i) = \frac{dU_i}{dx_i}$, and ζ is a function invariant under the action of the flow generated by V. Then V is the infinitesimal generator of G if and only if $V = \sum_{i=1}^{n} \frac{a_i}{U_i'(x_i)} \partial_{x_i}$ and $\zeta = \sum_{i=1}^{n} U_i(x_i)$, with $\sum_{i=1}^{n} a_i = 0$.*

Besides providing very general infinitesimal generators, the functions U_i will allow a generalization of the results of Pina and Tenenblat [19] as we will see in the next section.

3.2. Invariant solutions for the Ricci curvature equation and the Einstein equation

In this subsection we provide necessary and sufficient conditions for the existence of invariant solutions under symmetry groups of the Ricci curvature equation (3.1) and the Einstein equation (3.2) for tensors with components of the form considered in Theorem 3.3.

First, we consider translation-invariant solutions of the Ricci curvature equation.

Theorem 3.4. *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \ldots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Consider a nondiagonal symmetric tensor $R = \sum_{i,j=1}^{n} R_{ij} dx_i dx_j$ and suppose that for $i \neq j$ the functions R_{ij} are given by $R_{ij} = \alpha_i \alpha_j f(\xi)$, where f is an arbitrary non-null differentiable function, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ is vector with $\alpha_i \neq 0$ for all i and $\xi = \alpha \cdot x$ is a basic invariant under a group of translations G_T generated by vectors perpendicular to α. Then G_T is a symmetry group of the system (3.7). Consequently, there exists a metric $\overline{g} = \frac{1}{u^2} g$ such that $\mathrm{Ric}\,\overline{g} = R$ if and only if $u(x_1, \ldots, x_n) = u(\xi)$ is a solution of the ODE*

$$u_{,\xi\xi} - \frac{f(\xi)}{n-2} u = 0 \tag{3.11}$$

and

$$R_{ii} = (n-2)\frac{u_{,x_i x_i}}{u} + \epsilon_i \frac{\Delta_g u}{u} - \epsilon_i (n-1)\frac{|\nabla_g u|^2}{u^2}, \text{ for all } i.$$

When Theorem 3.3 is applied to the Ricci curvature equation 3.1, we obtain a general solution invariant under a symmetry group.

Theorem 3.5. *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \ldots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Consider a nondiagonal symmetric tensor $R = \sum_{i,j=1}^{n} R_{ij} dx_i dx_j$ and suppose that for $i \neq j$ the functions R_{ij} are given by $R_{ij} = U_i'(x_i) U_j'(x_j) f(\zeta)$, where f and U_i, for all i, are arbitrary, non-null, differentiable functions, U_i depending only on x_i with $U_i'(x_i) = \frac{dU_i}{dx_i}$ and $\zeta = \sum_{i=1}^{n} U_i(x_i)$ is a basic invariant under the group of transformations G generated by $V = \sum_{i=1}^{n} \frac{a_i}{U_i'(x_i)} \partial_{x_i}$, with $\sum_{i=1}^{n} a_i = 0$. Then G is a symmetry group of the system (3.7). Consequently, there exists a metric $\overline{g} = \frac{1}{u^2} g$ such that $\mathrm{Ric}\,\overline{g} = R$ if and only if $u(x_1, \ldots, x_n) = u(\zeta)$ is a solution of the ODE*

$$u_{,\zeta\zeta} - \frac{f(\zeta)}{n-2} u = 0 \tag{3.12}$$

and

$$R_{ii} = (n-2)\frac{u_{,x_i x_i}}{u} + \epsilon_i \frac{\Delta_g u}{u} - \epsilon_i (n-1)\frac{|\nabla_g u|^2}{u^2}, \text{ for all } i.$$

The function ζ has an important role in the geometric description of the underlying semi-Riemannian manifold given by the metric $\overline{g} = \frac{1}{u^2} g$. Let $\Omega \subset \mathbb{R}^n$ be the open such that $u(x) \neq 0$,

then $\zeta : \omega \to \mathbb{R}$ provides a foliation of Ω by level hypersurfaces $\zeta^{-1}(c)$, c a regular value for ζ, where the metric is invariant along each hypersurface.

We obtain similar results for the Einstein equation 3.2. We use the observation that if $(\mathbb{R}^n, g)$ is a pseudo-Euclidean space and $\overline{g} = \frac{1}{u^2} g$ is a conformal metric, then the scalar curvature of $\overline{g}$ is given by

$$\overline{K} = (n-1)(2u\Delta_g u - n|\nabla_g u|^2). \tag{3.13}$$

We first consider translation-invariant solutions of the Einstein equations:

Theorem 3.6. *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \ldots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Consider a nondiagonal symmetric tensor $T = \sum_{i,j=1}^{n} T_{ij} dx_i dx_j$ and suppose that for $i \neq j$ the functions T_{ij} are given by $T_{ij} = \alpha_i \alpha_j f(\xi)$, where f is an arbitrary non-null differentiable function, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ is a vector with $\alpha_i \neq 0$ for all i and $\xi = \alpha \cdot x$ is a basic invariant under a group of translations G_T generated by vectors perpendicular to α. Then G_T is a symmetry group of the system (3.7). Consequently, there exists a metric $\overline{g} = \frac{1}{u^2} g$ such that $\operatorname{Ric} \overline{g} - \frac{\overline{K}}{2} \overline{g} = T$ if and only if $u(x_1, \ldots, x_n) = u(\xi)$ is a solution of the ODE*

$$u_{,\xi\xi} - \frac{f(\xi)}{n-2} u = 0 \tag{3.14}$$

and

$$T_{ii} = (n-2)\left(\frac{u_{,x_i x_i}}{u} - \epsilon_i \frac{\Delta_g u}{u} + \epsilon_i (n-1) \frac{|\nabla_g u|^2}{2u^2} \right), \text{ for all } i$$

The following theorem supplies a version of Theorem 3.5 for the Einstein equation:

Theorem 3.7. *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \ldots, x_n)$, and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Consider a nondiagonal symmetric tensor $T = \sum_{i,j=1}^{n} T_{ij} dx_i dx_j$ and suppose that for $i \neq j$ the functions T_{ij} are given by $T_{ij} = U_i'(x_i) U_j'(x_j) f(\zeta)$, where f and U_i, for all i, are arbitrary, non-null, differentiable functions, U_i depending only on x_i with $U_i'(x_i) = \frac{dU_i}{dx_i}$ and $\zeta = \sum_{i=1}^{n} U_i(x_i)$ is a basic invariant under the group of transformations G generated by $V = \sum_{i=1}^{n} \frac{a_i}{U_i'(x_i)} \partial_{x_i}$, with $\sum_{i=1}^{n} a_i = 0$. Then G is a symmetry group of the system (3.7). Consequently, there exists a metric $\overline{g} = \frac{1}{u^2} g$ such that $\operatorname{Ric} \overline{g} - \frac{\overline{K}}{2} \overline{g} = T$ if and only if $u(x_1, \ldots, x_n) = u(\zeta)$ is a solution of the ODE*

$$u_{,\zeta\zeta} - \frac{f(\zeta)}{n-2} u = 0 \tag{3.15}$$

and

$$T_{ii} = (n-2)\left(\frac{u_{,x_i x_i}}{u} - \epsilon_i \frac{\Delta_g u}{u} + \epsilon_i (n-1)\frac{|\nabla_g u|^2}{2u^2}\right), \text{ for all } i$$

Theorems 3.4 through 3.7 generalize corresponding results given by Pina and Tenenblat in ([20]), which concerns the Ricci curvature equation:

Theorem 3.8 *([20]). Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \ldots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Suppose the non-diagonal $(i \neq j)$ elements of the symmetric tensor $R = \sum_{i,j=1}^{n} R_{ij} dx_i dx_j$ have been prescribed and that (up to reordering the independent variables) they satisfy one of the following hypotheses:*

a) $f_{12}(x_1, x_2)$ is any nonzero differentiable function, $f_{ij} = 0$, for all $i \neq j$, such that $i \geq 3$ or $j \geq 3$ and $u = u(x_1, x_2)$ is a nonvanishing solution of the hyperbolic equation

$$(n-2)u_{,x_1 x_2} - f_{12}u = 0. \tag{3.16}$$

b) There exists an integer $p, 3 \leq p \leq n$ such that $f_{ij} = 0$, if $i \neq j$, $i \geq p+1$ or $j \geq p+1$. Moreover, there exist nonconstant differentiable functions, $U_j(x_j)$, for $1 \leq j \leq p$, such that for all i, j, $1 \leq i \neq j \leq p$,

$$f_{ij} = (n-2)U_i' U_j', \text{ and } u = a\exp\left(\sum_{j=1}^{p} U_j(x_j)\right) + b\exp\left(-\sum_{j=1}^{p} U_j(x_j)\right) \tag{3.17}$$

or

$$f_{ij} = -(n-2)U_i' U_j', \text{ and } u = a\cos\left(\sum_{j=1}^{p} U_j(x_j)\right) + b\sin\left(-\sum_{j=1}^{p} U_j(x_j)\right), \tag{3.18}$$

where a and b are real numbers such that $a^2 + b^2 \neq 0$. Moreover, in each case u is defined on an open connected subset of $\mathbb{R}^n$, where it does not vanish.

Then the metric $\overline{g} = \frac{1}{u^2} g$ satisfies $\operatorname{Ric} \overline{g} = R$ if and only if the diagonal elements of R are defined to be

$$f_{ii} = (n-2)\frac{u_{,x_i x_i}}{u} + \epsilon_i \frac{\Delta_g u}{u} - \epsilon_i (n-1)\frac{|\nabla_g u|^2}{u^2}, \text{ for all } i.$$

A similar result for the Einstein equation is also presented. The functions f_{ij} in our results generalizes those given in (3.17) and (3.18) and so we obtain results for a larger class of functions U_i and solutions invariant under a greater variety of symmetry groups.

By choosing different functions U_i, we can obtain symmetry groups with well known actions. For instance, one can consider actions given by rotation or by dilations:

(i) If p of the ϵ_i are positive and q of them are negative, then the symmetry group $SO(p,q)$ is obtained by choosing $f_{ij} = \epsilon_i \epsilon_j x_i x_j f(r)$, where f is an arbitrary, not-identically-zero, differentiable function and $r = \sum_{i=1}^{n} \epsilon_i x_i^2$.

(ii) To obtain dilations on the domain where $x_k > 0$ for all k, choose $f_{ij} = \frac{a_i a_j}{x_i x_j} f(\eta)$ where f is an arbitrary not-identically-zero differentiable function and $\eta = \sum_{i=1}^{n} a_i \ln(x_i)$.

Remark 3.9. Theorems 3.4–3.7 can be easily generalized to locally conformally flat manifolds in the same way that Pina and Tenenblat did in [20].

3.3. Particular cases and complete metrics

In this subsection, we consider particular choices of the function f in the ODE

$$u_{,\xi\xi} - \frac{f(\xi)}{(n-2)} u = 0 \tag{3.19}$$

that occurs in the solution of the Ricci curvature and Einstein equations in both the translation-invariant ((3.12) and (3.15) respectively) and more generally invariant ((3.11) and (3.14) respectively) cases.

Particular choices of f transform the ODE (3.19) into well-known equations for which there are extensive results concerning the qualitative properties of their solutions. The cases presented here are not exhaustive, but they illustrate how powerful the method of invariant solutions can be for the Ricci curvature and Einstein equations since it connects these equations, whose solutions are difficult to describe, with well known ODEs.

We consider the following cases:

(1) $f = (n-2)(c\xi + d)$: Using the substitution $\overline{\xi} = c^{-\frac{2}{3}}(c\xi + d)$ one obtains the Airy equation

$$u_{,\overline{\xi}\overline{\xi}} - \overline{\xi} u = 0.$$

(2) $f = (n-2)(c\xi^2 + d)$: Using the substitution $z = \xi^2 \sqrt{a}$ and $v = e^{z/2} u$ one obtains the Weber equation

$$z v_{,zz} + \left(\frac{1}{2} - z\right) v_{,z} - \frac{1}{4}\left(\frac{d}{\sqrt{c}} + 1\right) v = 0$$

(3) $f = (n-2)(c - 2d\cos(2\xi))$, one has the Mathieu equation.
(4) $f = (n-2)(c - 2d\cosh(2\xi))$, one has the modified Mathieu equation.
(5) $f = (n-2)(c\,\mathrm{sn}^2(\xi) + d)$, with $\mathrm{sn}(\xi)$ is the Jacobi elliptic function, one has the Lamé Equation in the form of Jacobi.
(6) $f = (n-2)(c\wp(\xi) + d)$, with $\wp(\xi)$ is the Weierstrass function, one has the Lamé Equation in the form of Weierstrass.

In what follows, we will consider the Riemannian case and functions that provide complete metrics globally defined in $\mathbb{R}^n$ and also examples where the corresponding tensor can become non-invertible on a subset of $\mathbb{R}^n$.

In a situation more general than the previously presented for the function f, if $f(\xi) = (n-2)(h^2(\xi) + h'(\xi))$, the ODE (3.19) has a particular solution $u_0 = \exp(\int h d\xi)$ and a general solution given by

$$u = C_1 u_0 + C_2 u_0 \int \frac{d\xi}{u_0^2}.$$

Let us consider the particular solution $u = \exp(\int h d\xi)$ for the Riemannian case and for a unit vector $\alpha = (\alpha_1, \dots \alpha_n)$, normal to the planes where ξ is constant. In this case, one has

$$\begin{aligned} f_{ij} &= (n-2)\alpha_i\alpha_j(h^2 + h'),\ i \neq j \\ f_{ii} &= (n-2)\alpha_i^2(h^2 + h') + h' - (n-2)h^2, \end{aligned}$$

which can be rewritten as

$$f_{ij} = (n-2)\alpha_i\alpha_j(h^2 + h') + \delta_{ij}(h' - (n-2)h^2). \tag{3.20}$$

By taking functions h such that $u = \exp(\int h d\xi)$ is globally defined in $\mathbb{R}^n$ and bounded, one constructs complete metrics $\overline{g}$ defined globally in $\mathbb{R}^n$.

A large class of solutions that provide complete metrics on $\mathbb{R}^n$ that also have a non-empty locus where the tensor fails to be invertible is given by functions $h(\xi) = -\xi^{2k-1}$ for $k \in \mathbb{N}$, with $k \geq 2$. In this case $u = \exp(-\xi^{2k}/2k)$ is a bounded function, which implies that the metric $\overline{g} = \dfrac{1}{u^2} g$ is complete, and all the components of the tensor will vanish along $\xi = 0$.

3.4. The 2-dimensional case and hyperbolic equations

A particular case remained unsolved in [20] and we use the results of symmetries groups to solve it. In Theorem 3.8, the authors have not studied solutions for the hyperbolic equation (3.16). This equation is of interest in its own right, since for example it is the Klein–Gordon equation when f_{12} is constant. In this case, one can also consider the following basic invariants:

(1) *invariance under the scaling group*

$$(x_1,\ x_2) \mapsto (\lambda x_1, \lambda^{-1} x_2),\ \lambda > 0.$$

Let η be a basic invariant given by $\eta = x_1 x_2$:

a) If $f_{12}(\eta) = -\dfrac{b}{\eta}$, equation (3.16) becomes

$$u_{,\eta\eta}\eta + u_{,\eta} + \frac{b}{\eta} u = 0$$

which is equivalent to the Euler equation $u_{,\eta\eta}\eta^2 + u_{,\eta}\eta + bu = 0$.

b) If $f_{12}(\eta) = -\dfrac{\eta^2 - a^2}{\eta}$ equation (3.16) becomes

$$u_{,\eta\eta}\eta + u_{,\eta} + \frac{\eta^2 - a^2}{\eta}u = 0$$

which is equivalent to the Bessel equation $u_{,\eta\eta}\eta^2 + u_{,\eta}\eta + (\eta^2 - a^2)u = 0$.

c) If $f_{12}(\eta) = \dfrac{\eta^2 + a^2}{\eta}$ equation (3.16) becomes

$$u_{\eta\eta}\eta + u_{\eta} - \frac{\eta^2 + a^2}{\eta}u = 0$$

which is equivalent to the modified Bessel equation $u_{,\eta\eta}\eta^2 + u_{,\eta}\eta - (\eta^2 + a^2)u = 0$.

(2) *Invariance under the dilation group*

a) $\zeta = \dfrac{x_1}{x_2}$. If f_{12} is given by $\dfrac{1}{x_2^2}g(\zeta)$, we have the equation

$$u_{,\zeta\zeta} - u_{,\zeta} - g(\zeta)u = 0$$

If $g(\zeta) = h(\zeta) + h^2(\zeta) - h'(\zeta)$, then u is given by

$$u = u_0\left(C_1 + C_2\int \frac{e^{x+C_3}}{u_0^2}dx\right)$$

where $u_0 = \exp\left(-\int h\right)$

b) $\zeta = \dfrac{x_1x_2}{x_1^2 + x_2^2}$. If f_{12} is given by $\dfrac{f(\zeta)}{x_1^2 + x_2^2}g(\zeta)$, we have the equation

$$u_{,\zeta\zeta}\left(4\zeta^2 - 1\right)\zeta + u_{,\zeta}\left(8\zeta^2 - 1\right) - f(\zeta)u = 0.$$

By choosing different types of f, one has explicit solutions for u. By performing the change of variables $z = 2\zeta$, the equation above becomes

$$u_{,zz} + \left(\frac{1}{z} + \frac{1}{2(z-1)} + \frac{1}{2(z+1)}\right)u_{,z} - \frac{f(z/2)}{2z(z^2-1)}u = 0.$$

If $f(z/2) = 2\alpha\beta z + q$, with α, β, q real constants and $\alpha + \beta = 1$, one obtains Heun's Equation

$$u_{,zz} + \left(\frac{1}{z} + \frac{1}{2(z-1)} + \frac{1}{2(z+1)}\right)u_{,z} - \frac{\alpha\beta z + q}{z(z^2-1)}u = 0. \tag{3.21}$$

On the other hand, the substitution $z = 4\xi^2$ leads to the hypergeometric equation

$$z(1-z)u_{,zz} + \frac{1}{2}(2-3z)u_{,z} - \frac{1}{8}u = 0. \tag{3.22}$$

(3) *Invariance under dilations and translations*
Suppose $\eta = \dfrac{n_1x_1 + n_2x_2}{d_1x_1 + d_2x_2}$. If $f(x_1, x_2) = \dfrac{1}{(x_1^2 + x^2)^2 p(\eta)}$, where $p(\eta) = (d_1\eta - n_1)(d_2\eta - n_2)$, then u satisfies the following equation

$$u_{,\eta\eta}p + u_{,\eta}p_\eta - \frac{1}{p}u = 0,$$

which has a solution given by

$$u(\eta) = C_1 \exp\left(\int \frac{1}{p(\eta)}d\eta\right) + C_2 \exp\left(-\int \frac{1}{p(\eta)}d\eta\right)$$

3.5. Elliptic problems and prescribed scalar curvature

By considering $w = u^{-(n-2)/2}$ and the expression of the scalar curvature obtained from the Ricci tensor T, one gets the following corollary from Theorem 3.4, where the function f is the one presented at the end of subsection 3.3, that provides complete metrics as solutions of Ricci curvature equation.

Corollary 3.10. *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \ldots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Let $\overline{K}(x) : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$\overline{K}(x) = (n-1)e^{\int h(\xi)d\xi}\left((2-n)h^2(\xi) + 2h'(\xi)\right) \tag{3.23}$$

where h is an arbitrary differentiable function of $\xi = \sum \alpha_i x_i$, with $\sum \alpha_i^2 = 1$. Then the differential equation

$$\frac{4(n-1)}{n-2}\Delta_g w + \overline{K}(x)w^{\frac{n+2}{n-2}} = 0 \tag{3.24}$$

where Δ_g denotes the Laplacian in the metric g, has a solution globally defined in $\mathbb{R}^n$ given by

$$w = \exp\left(\frac{(2-n)\int h(\xi)d\xi}{2}\right) \tag{3.25}$$

The geometric interpretation of the above result is a solution of the following problem of prescribed scalar curvature:

Corollary 3.11. *Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, $n \geq 3$ with coordinates $x = (x_1, \ldots, x_n)$ and metric $g_{ij} = \delta_{ij}\varepsilon_i$, $\varepsilon_i = \pm 1$. Let $\overline{K}(x) : \mathbb{R}^n \to \mathbb{R}$ be given by (3.23). Then there exists a metric $\overline{g} = w^{\frac{4}{n-2}}g$, where w is given by (3.25) whose scalar curvature is $\overline{K}$. In particular, if $(\mathbb{R}^n, g)$ is Euclidean space and w is a bounded function then $\overline{g}$ is a complete metric.*

4. Proofs of the main results

Proof of Theorem 3.1. Let $V(x,u)$ the infinitesimal generator given by $V=\sum_{i=1}^{n}\xi^i\partial_{x_i}+\phi\partial_u$ and the prolongation of V given by $\mathrm{pr}^{(2)}V=V+\sum_{\beta}\phi_\beta\partial_{u_{,\beta}}$, where $\beta=(\beta_1,\ldots,\beta_n)$ is a multi-index with $1\leq|\beta|\leq 2$. Applying prolongation to the system (3.7) one obtains

$$(n-2)\phi_{\beta_{ij}}-(\sum_{k=1}^{n}\xi^k f_{ij,x_k})u-f_{ij}\phi=0, \tag{4.1}$$

where β_{ij} is a multi-index such that $\beta_i=\beta_j=1$ and the remaining indices are zero. Using the prolongation formulas, with the substitution $u_{,x_ix_j}=\dfrac{f_{ij}u}{n-2}$, $i\neq j$, one obtains the following equation

$$\begin{aligned}
&\phi_{,x_ix_j}+u_{,x_i}\left(\phi_{,x_ju}-\xi^i_{,x_ix_j}-\xi^i_{,u}\frac{f_{ij}u}{n-2}-\sum_{k\neq j}\xi^k_{,u}\frac{f_{kj}u}{n-2}\right)+\\
&+u_{,x_j}\left(\phi_{,x_iu}-\xi^j_{,x_ix_j}-\xi^j_{,u}\frac{f_{ij}u}{n-2}-\sum_{k\neq i}\xi^k_{,u}\frac{f_{ki}u}{n-2}\right)+\\
&+u_{,x_i}u_{,x_j}\left(\phi_{,uu}-\xi^j_{,x_ju}-\xi^i_{,x_iu}\right)-\sum_{k\neq i,k\neq j}\left(\xi^k_{,u}\frac{f_{ij}u}{n-2}\right)u_{,x_k}-\\
&-\sum_{k\neq j}\left(\xi^k_{,x_ju}\right)u_{,x_i}u_{,x_k}-\sum_{k\neq i}\left(\xi^k_{,x_iu}\right)u_{,x_i}u_{,x_k}+\\
&+\frac{u}{n-2}\left(\phi_{,u}f_{ij}-\sum_{k\neq j}\xi^k_{,x_i}f_{kj}-\sum_{k\neq i}\xi^k_{,x_j}f_{ki}-\sum_{k}\xi^k f_{ij,x_k}\right)-\\
&-\sum_{k}\xi^k_{,uu}u_{,x_i}u_{,x_j}u_{,x_k}-\xi^j_{,u}u_{,x_i}u_{,x_jx_j}-\xi^j_{,x_i}u_{,x_jx_j}-\xi^i_{,x_j}u_{,x_ix_i}-\xi^i_{,u}u_{,x_j}u_{,x_ix_i}-\\
&-\frac{1}{n-2}f_{ij}\phi=0.
\end{aligned}$$

Proceeding with analysis of the coefficients will provide the determining equations for ξ^i and ϕ. The coefficients of $u_{,x_ix_i}$, $u_{,x_jx_j}$, $u_{,x_i}u_{,x_jx_j}$ and $u_{,x_j}u_{,x_ix_i}$ indicate that the function ξ^i depends only on x_i, for each i. Together with the determining equations given by the coefficients of $u_{,x_i}$ and $u_{,x_j}$ this implies that $\phi(x,u)=A(u)+B(x)$. Additionally, the coefficient of $u_{,x_i}u_{,x_j}$ shows that $\phi_{,uu}=0$. Therefore,

$$\phi(x,u)=au+B(x), \tag{4.2}$$

where $a\in\mathbb{R}$.

After all these considerations, the Equation (4.1) is reduced to

$$B_{,x_i x_j} + \frac{u}{n-2}\left(af_{ij} - (\xi^i f_{ij})_{,x_i} - (\xi^j f_{ij})_{,x_j} - \sum_{k\neq i, k\neq j} \xi^k f_{ij,x_k}\right) - \frac{f_{ij}\phi}{n-2} = 0.$$

By taking derivatives relative to u one obtains the following system,

$$\begin{aligned}(\xi^i f_{ij})_{,x_i} + (\xi^j f_{ij})_{,x_j} + \sum_{k\neq i, k\neq j} \xi^k f_{ij,x_k} &= 0,\\ B_{,x_i x_j} - \frac{B f_{ij}}{n-2} &= 0.\end{aligned}$$

The system above and (4.2) imply Equations (3.9) and (3.10) and the Theorem is proved. □

Proof of Theorem 3.2. The functions f_{ij} are invariant under a symmetry group of (3.7) if and only if $V(f_{ij}) = 0$, where V an infinitesimal generator given by $V(x) = \sum_{i=1}^{n} \xi^i \partial_{x_i}$, i.e., f_{ij} satisfy

$$\sum_{i=1}^{n} \xi^i f_{ij,x_k} = 0. \tag{4.3}$$

Therefore, by Equation (3.10) f_{ij} are invariant under a symmetry group of (3.7) if and only if

$$\left(\xi^i_{,x_i} + \xi^j_{,x_j}\right) f_{ij} = 0. \tag{4.4}$$

Since Theorem 3.1 states that each function ξ^i depends only on x_i and we are assuming that f_{ij} are not identically zero, the system (4.4) implies that $\xi^i = a_i x_i + b_i$, where a_i, b_i are real constants and $a_i + a_j = 0$. Since $n \geq 3$, the system $a_i + a_j = 0$ has only the zero solution, i.e., $\xi^i(x_i) = b_i$, which is equivalent to the symmetry group being a translation group. □

Proof of Theorem 3.3. The proof is similar to the proof of Theorem 3.2. Let $V = \xi^i \partial_{x_i}$ be the infinitesimal generator for a symmetry group of the system (3.7). If $f_{ij} = U_i'(x_i)U_j'(x_j) f(\zeta)$, where ζ a basic invariant under the action generated by V, then $V(f) = 0$ and the Equation (3.10) becomes:

$$\xi^i_{,x_i} + \xi^j_{,x_j} + \xi^i \frac{U_i''}{U_i'} + \xi^j_{,x_j} \frac{U_j''}{U_j'} = 0. \tag{4.5}$$

Since the functions ξ_i and U_i' depend only on x_i, equation (4.5) is equivalent to a linear system $c_i + c_j = 0$, where $\xi^i_{,x_i} U_i' + \xi^i U_i'' = c_i U_i'$. As $n \geq 3$, the system has only the zero solution. Therefore, $\xi^i_{,x_i} U_i' + \xi^i U_i'' = 0$, which implies in $\xi^i = \dfrac{a_i}{U_i}$ and consequently $V = \sum_{i=1}^{n} \dfrac{a_i}{U_i'(x_i)} \partial_{x_i}$

and $\zeta = \sum_{i=1}^{n} U_i(x_i)$ is a basic invariant for the action, with $\sum_{i=1}^{n} a_i = 0$. The converse is a straightforward verification that the hypothesis satisfy equation (3.10). □

Proofs of Theorems 3.4 to 3.7. The proofs of Theorems 3.4 to 3.7 are essentially the same, therefore we will give them with the same set of arguments. As we explained before, the Ricci curvature equation 3.1 is equivalent to equations (3.7) and (3.4), while the Einstein equation is given by (3.7) and (3.6). Consequently, the proofs of these theorems consist in applying Theorems 3.2 and 3.3 in order to find solutions invariant under the symmetry groups of (3.7). With such invariant solutions, equations (3.4) and (3.6) provide necessary and sufficient conditions on the diagonal elements f_{ii} for the existence of solutions.

In all four theorems, the invariant solutions of (3.7) are given by the ODE

$$u_{,tt} - \frac{f(t)}{n-2}u = 0, \tag{4.6}$$

where t is a basic invariant for the considered action. For Theorems 3.4 and 3.5, there exist solutions for the Ricci curvature equation if and only if u is a solution of (4.6) and the diagonal elements are given by (3.4). Similarly for the Einstein equation, considered in Theorems 3.6 and 3.7, there exist solutions if and only if u is a solution of (4.6) and the diagonal elements are determined by (3.6). □

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