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Spreading in a cone for the Fisher-KPP equation ☆

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Abstract

In this paper we consider the spreading phenomena in the Fisher-KPP equation in a high dimensional cone with Dirichlet boundary condition. We show that any solution starting from a nonnegative and compact supported initial data spreads and converges to the unique positive steady state. Moreover, the asymptotic spreading speeds of the front in all directions pointing to the opening are c_0 (which is the minimal speed of the traveling wave solutions of the 1-dimensional Fisher-KPP equation). Surprisingly, they do not depend on the shape of the cone, the propagating directions and the boundary condition.

MSC: 35K55; 35K57; 35B40

Keywords: Fisher-KPP equation; Cone; Spreading phenomena; Steady state; Traveling wave solution

1. Introduction

We consider the Fisher-KPP equation in a cone in $\mathbb{R}^N$:

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega,\ t > 0, \\ u(t,x) = 0, & x \in \partial\Omega,\ t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{P}$$

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where $f(u) \in C^1([0,\infty))$ is a Fisher-KPP type of nonlinearity:

$$f(0) = f(1) = 0, \quad f'(0) > 0 > f'(1), \quad f(u)/u \text{ is decreasing in } u > 0, \tag{F}$$

and the domain Ω is a cone in $\mathbb{R}^N$ which is constructed as follows: let E be a bounded and convex domain in the hyperplane $\{(1, x') \mid x' = (x_2, x_3, \cdots, x_N) \in \mathbb{R}^{N-1}\}$ with C^2 boundary ∂E, and $(1, 0, \cdots, 0) \in E$; let Ω_* be the cone with vertex 0 and directrix ∂E; $\Omega \subset \Omega_*$ is a convex cone-shaped domain obtained by smoothening Ω_* near the vertex 0.

We are interested in the spreading phenomena for the solutions of (P). In some applied fields like chemistry, ecology, etc., the spreading of a new or invasive species is an important topic. It is known that such phenomena can be described by the spreading for solutions of certain reaction diffusion equations. For example, in 1937, Fisher [9] used the equation $u_t = u_{xx} + u(1-u)$ to model the spreading of advantageous genetic trait in a population, and found that there are traveling wave solutions $u = \phi(x - ct)$ (see also Kolmogorov, et al. [11]). In 1970s, Aronson and Weinberger [1,2] studied systematically the spreading phenomena in the Cauchy problems of $u_t = \Delta u + f(u)$, where $f(u)$ can be a monostable (including the Fisher-KPP case) or bistable type of nonlinearity. In the monostable case, they found the so-called *hair-trigger effect*, that is, any solution starting from a nonnegative and compactly supported initial data, will converge to a positive steady state (to say, $u \equiv 1$). Moreover, they showed that the approximate spreading speeds of the level set $\{x \mid u(t,x) = \frac{1}{2}\}$ in any directions are c_0, which is the minimal speed of the traveling wave solutions of $u_t = u_{xx} + f(u)$. In the last decades, many authors also studied the spreading phenomena in the Cauchy problems of $u_t = \Delta u + f(u)$ with bistable or combustion type of nonlinearity (cf. [7,8,10,13–15], etc.). Among others, they gave sufficient conditions for spreading and used the traveling wave solutions to characterize the spreading solutions.

On the other hand, it is also interesting to study the spreading phenomena in unbounded domains with boundaries, like the half space, cones, cylinders with straight or undulating boundaries, etc. For example, Berestycki et al. [3–5] considered the following problem on the half plane:

$$\begin{cases} u_t - Du_{xx} = \nu v(t,x,0) - \mu u, & x \in \mathbb{R},\ t > 0, \\ v_t - d\Delta v = f(v), & x > 0,\ y \in \mathbb{R},\ t > 0, \\ -dv_y(t,x,0) = \mu u(t,x) - \nu v(t,x,0), & x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

where f is also a Fisher-KPP nonlinearity. This model is used to describe the spreading of a species in the field with a fast diffusion on its boundary $y = 0$ (like a road), and exchanges of populations taking place between the road and the field. They gave a function $c(\theta) \in C([-\pi/2, \pi/2])$ to characterize the approximate spreading speed in each direction $(\sin\theta, \cos\theta)$: $c(\theta) \geq c_0 := 2\sqrt{f'(0)}$ and, when $D > 2d$, there is $\theta_0 \in (0, \pi/2)$ such that $c(\theta) > c_0$ if $|\theta| > \theta_0$, $c(\theta) = c_0$ if $|\theta| \leq \theta_0$.

In this paper we consider the spreading phenomena for the solutions of (P). First we have the following result on the existence and uniqueness of the positive steady state of (P).

Theorem 1.1 *(Steady state). Assume* (F)*. Then the problem* (P) *has a unique steady state $V(x)$ which is positive in Ω. Moreover,*

$$V(x) - V^*(d(x)) \to 0, \quad \text{as } x_1 \to \infty, \tag{1.2}$$

where $d(x) := d(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$, and V^* is the unique solution of the one dimensional problem

$$v'' + f(v) = 0 \ \ (x > 0), \quad v(0) = 0, \quad v(\infty) = 1. \tag{1.3}$$

Denote by $c_0 := 2\sqrt{f'(0)}$ the minimal speed of traveling wave solutions of $u_t = u_{xx} + f(u)$. Our next result implies that any solution of (P), starting from a compactly supported nonnegative initial data, spreads and approaches the steady state with speed c_0.

Theorem 1.2. *Let u be the solution of* (P) *with compactly supported initial data $u_0 \geq 0$. Then*

$$u(\cdot, t) \to V(\cdot) \ \text{ as } \ t \to \infty, \quad \text{locally uniformly in } x \in \Omega. \tag{1.4}$$

Moreover, for any $c > c_0$ we have

$$\lim_{t\to\infty} \sup_{|x|>ct,\ x\in\Omega} u(t, x) = 0; \tag{1.5}$$

and for any $c \in (0, c_0)$ we have

$$\lim_{t\to\infty} \inf_{|x|<ct,\ x\in\Omega} u(t, x) = V(x). \tag{1.6}$$

The first half of this theorem shows that, like the Cauchy problem, our problem also has the *hair-trigger effect*: any nonnegative solution of (P) converges as $t \to \infty$ to the positive steady state. Since the domain Ω is a cone-shaped one, it is natural to ask: how do the shape of Ω (especially the vertex angle), the propagating directions $\overrightarrow{Oy}$ (for $y \in E$) and the Dirichlet boundary condition influence the spreading speeds? Our results show that, surprisingly, the asymptotic spreading speeds in all directions pointing to the opening are the same. They do *not* depend on any of the above mentioned factors. This is different from the conclusions for (1.1). The reason seems that, in our model (P), there is no exchange of populations between the road and the field.

The rest of the paper is arranged as follows. In Section 2 we present some preliminaries, including some positive steady states of $u_t = \Delta u + f(u)$ in bounded domains, in the half space $\mathbb{R}^N_+$, and in Ω, as well as traveling wave solutions with compact supports. We also prove Theorem 1.1 by using the properties of these solutions. In Section 3 we study the general convergence result and the asymptotic spreading speeds of the solution $u(t, x)$.

2. Steady states and traveling wave solutions

Let D be a connected domain in $\mathbb{R}^N$ with a smooth boundary. We call a function $v \in C^2(D) \cap C(\overline{D})$ as a *positive steady state* of $u_t = \Delta u + f(u)$ in D if v solves the following problem

$$\begin{cases} \Delta v + f(v) = 0, & x \in D, \\ v(x) > 0, & x \in D, \\ v(x) = 0, & x \in \partial D. \end{cases} \tag{2.1}$$

As it was seen in the previous section, we are interested in the positive steady state $V(x)$ in Ω (that is, the solution of (2.1) with $D = \Omega$). However, in our approach, we also need the solutions

$v(x; D)$ of (2.1) in bounded domains D, the solution $v = V^*(x)$ in the half space $D = \mathbb{R}^N_+$, and the traveling wave solutions $w^c(x - c\mathbf{e}t; D)$ in bounded domains (to be specified below). In this section we study these solutions.

2.1. Steady states in bounded domains

For any bounded domain $D \subset \mathbb{R}^N$ with a smooth boundary, consider the following eigenvalue problem:

$$\begin{cases} -\Delta\phi = \lambda\phi, & x \in D, \\ \phi(x) = 0, & x \in \partial D. \end{cases} \tag{2.2}$$

It is known that this problem has a principal eigenvalue (denoted by $\lambda_1(D)$), and the corresponding eigenfunction (denoted by $\phi_1(x; D)$) can be chosen positive in D and be normalized by $\|\phi_1(x; D)\|_{L^\infty(D)} = 1$. Moreover, $\lambda_1(D)$ is strictly decreasing in D in the sense that $\lambda_1(D_1) > \lambda_1(D_2)$ if $D_1 \subsetneqq D_2$. For any $X \in \mathbb{R}^N$ and $R > 0$, denote

$$B_R(X) := \{x \in \mathbb{R}^N \mid |x - X| < R\}.$$

When $D = B_R(X)$, it is easy to verify that $\lambda_1(B_R(X)) = \lambda_1^0/R^2$, where $\lambda_1^0 := \lambda_1(B_1(X)) = \lambda_1(B_1(0))$. Set

$$R^* := \left[\frac{\lambda_1^0}{f'(0)}\right]^{1/2}, \tag{2.3}$$

then $\lambda_1(B_R(X)) < f'(0)$ if and only if $R > R^*$.

Now we show that, when D is a large bounded domain, the problem (2.1) has a unique solution, which can be obtained by taking limit in the solution of the initial-boundary value problem

$$\begin{cases} \tilde{u}_t = \Delta\tilde{u} + f(\tilde{u}), & x \in D,\ t > 0, \\ \tilde{u}(t, x) = 0, & x \in \partial D,\ t > 0, \\ \tilde{u}(0, x) = \psi(x) \geq, \not\equiv 0, & x \in \overline{D}. \end{cases} \tag{2.4}$$

Lemma 2.1. *Let D be a connected bounded domain in $\mathbb{R}^N$ with a smooth boundary. Assume $B_{R^*}(X) \subsetneqq D$ for some $X \in \mathbb{R}^N$. Then*

(i) *the problem (2.1) has a unique positive solution $v(x; D) \leq 1$;*
(ii) *for any $\psi \in L^\infty(D)$ with $\psi(x) \geq, \not\equiv 0$, the solution $\tilde{u}(t, x)$ of (2.4) converges as $t \to \infty$ to $v(x; D)$, in $C^2(\overline{D})$ topology.*

Proof. (i). Since $B_{R^*}(X) \subsetneqq D$ we have $\lambda_1(D) < \lambda_1(B_{R^*}(X)) = f'(0)$. So, for small $\delta > 0$ we have

$$\Delta(\delta\phi_1(x; D)) + f(\delta\phi_1(x; D)) > \delta[\Delta\phi_1(x; D) + \lambda_1(D)\phi_1(x; D)] = 0, \quad x \in D.$$

This means that $\delta\phi_1(x; D)$ is a lower solution of (2.1). Clearly, $v \equiv 1$ is an upper solution of (2.1). Hence, the existence of a positive solution $v(x; D)$ of (2.1) can be obtained by the standard method of lower and upper solutions.

Now we prove the uniqueness of $v(x; D)$. Suppose by contradiction that (2.1) has two different positive solutions $v_1(x)$ and $v_2(x)$ (assume, without loss of generality, $v_1(\bar{x}) < v_2(\bar{x})$ for some $\bar{x} \in D$). Then, for any sufficiently small $\rho \in (0, 1)$ we have $v_1(x) \geq \rho v_2(x)$ in D. Taking ρ^* the maximum of such ρ, then $0 < \rho^* \leq \frac{v_1(\bar{x})}{v_2(\bar{x})} < 1$, and either

$$v_1(x) \geq \rho^* v_2(x) \text{ in } D \text{ and } v_1(y) = \rho^* v_2(y) \text{ for some } y \in D, \tag{2.5}$$

or,

$$v_1(x) > \rho^* v_2(x) \text{ in } D \text{ and } \frac{\partial}{\partial \nu} v_1(z) = \rho^* \cdot \frac{\partial}{\partial \nu} v_2(z) \text{ for some } z \in \partial D \tag{2.6}$$

holds, where ν denotes the outward unit normal vector on ∂D.

We claim that $f(\rho^* v_2(x)) \geq, \not\equiv \rho^* f(v_2(x))$. By the monotonicity of $f(s)/s$ in (F) we have $f(\rho^* v_2(x)) \geq \rho^* f(v_2(x))$. If $f(\rho^* v_2(x)) \equiv \rho^* f(v_2(x))$, then $f(\rho^* s) = \rho^* f(s)$ for all $s \in J := [0, \sup_D v_2(x)]$. This implies that $f'(s) = f'(0)$ or $f(s) = f'(0)s$ for all $s \in J$. Hence, the solution v_2 of (2.1) is actually an eigenfunction of $-\Delta$ corresponding to the eigenvalue $f'(0)$. Since v_2 is positive in D, $f'(0)$ should be the principal eigenvalue $\lambda_1(D)$, a contradiction. This proves the claim. Therefore,

$$\Delta(\rho^* v_2) + f(\rho^* v_2) = \rho^*[\Delta v_2 + f(v_2)] + [f(\rho^* v_2) - \rho^* f(v_2)] \geq, \not\equiv 0.$$

This implies, by the maximum principle, that the solution $\tilde{u}(x, t; \rho^* v_2)$ of the parabolic problem (2.4) with $\psi(x) = \rho^* v_2(x)$ satisfies

$$\tilde{u}(t, x) > \rho^* v_2(x) \text{ for } x \in D,\ t > 0; \qquad \frac{\partial \tilde{u}(t, x)}{\partial \nu} < \rho^* \frac{\partial v_2(x)}{\partial \nu} \text{ for } x \in \partial D,\ t > 0. \tag{2.7}$$

On the other hand, using the maximum principle to $v_1(x) - \tilde{u}(t, x)$ we have

$$v_1(x) \geq \tilde{u}(t, x) \text{ for } x \in D,\ t > 0; \qquad \frac{\partial v_1(x)}{\partial \nu} \leq \frac{\partial \tilde{u}(t, x)}{\partial \nu} \text{ for } x \in \partial D,\ t > 0. \tag{2.8}$$

Therefore, if (2.5) (resp. (2.6)) holds, the first (resp. the second) inequality in (2.7) contradicts that in (2.8). This proves the uniqueness of $v(x; D)$.

(ii). By the parabolic theory, the solution $\tilde{u}(x, t; \psi)$ of (2.4) exists globally in time and it is positive in D. Using Lyapunov functional in a standard way, one can show the convergence of $\tilde{u}(x, t; \psi)$ to $v(x; D)$. In particular, if we take $\psi(x) = \delta\phi_1(x; D)$ for small δ, then $\tilde{u}(x, t; \delta\phi_1)$ is monotonically increasing in $t > 0$ since $\delta\phi_1$ is a lower solution. Hence $\tilde{u}(x, t; \delta\phi_1)$ increases and converges as $t \to \infty$ to $v(x; D)$ from below. □

We now prove the monotonicity of $v(x; D)$ in D.

Lemma 2.2. *Assume that D_1 and D_2 are two connected domains in $\mathbb{R}^N$ with smooth boundaries, D_1 is a bounded one and $B_{R^*}(X) \subsetneq D_1$ for some $X \in D_1$. If $D_1 \subset D_2$ and if the problem (2.1) with $D = D_2$ has a solution $v(x; D_2)$ (no matter D_2 is bounded or not), then $v(x; D_1) < v(x; D_2)$ in D_1.*

Proof. Since $v(x; D_2) > 0$ in D_2, there exists a sufficiently small $\delta > 0$ such that $\delta\phi_1(x; D_1) < v(x; D_2)$ in D_1. As in the proof of the previous lemma, $\delta\phi_1(x; D_1)$ is a strict lower solution. Consider the problem (2.4) in D_1, with initial data $\psi(x) = \delta\phi_1(x; D_1)$, we have

$$\tilde{u}(x, t; \delta\phi_1) \le v(x; D_2), \quad x \in D_1,\ t > 0$$

by the comparison principle. Taking limit as $t \to \infty$ in this inequality and using Lemma 2.1 (ii) we conclude $v(x; D_1) \le v(x; D_2)$ in D_1. Moreover, the inequality is strict by the strong maximum principle. □

Remark 2.3. Since f is a Fisher-KPP type of nonlinearity, it is known that $v \equiv 1$ is the only positive solution of (2.1) in the entire space $\mathbb{R}^N$. As a consequence of the previous lemma, for any $X \in \mathbb{R}^N$, $v(x; B_R(X))$ is strictly increasing in R, and $v(x; B_R(X)) \to 1$ as $R \to \infty$, in $C^2_{loc}(\mathbb{R}^N)$ topology.

2.2. Steady states in the half space

Denote by $\mathbb{R}^N_+$ the half space:

$$\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \mid x_1 > 0\}.$$

We consider the problem (2.1) in this domain:

$$\begin{cases} \Delta v + f(v) = 0, & x \in \mathbb{R}^N_+, \\ v(x) > 0, & x_1 > 0, \\ v(x) = 0, & x_1 = 0. \end{cases} \tag{2.9}$$

Lemma 2.4. *The problem (2.9) admits a unique solution $v(x) = V^*(x_1)$, where $V^*(s)$ is the solution of the following one-dimensional problem*

$$v''(s) + f(v(s)) = 0 \ \ (s > 0), \quad v(0) = 0, \quad v(\infty) = 1, \quad v'(s) > 0 \ (s \ge 0). \tag{2.10}$$

Proof. Multiplying the equation in (2.10) by $2v'$ and integrating it over (s, ∞), one has

$$v'(s) = \sqrt{2\int_v^1 f(r)dr}.$$

Its solution $V^*(s)$ solves the problem (2.10). Clearly, $v(x) = V^*(x_1)$ is a solution of (2.9).

In what follows we prove the uniqueness of the solution of (2.9). First, we present some a priori estimates for any given solution V of (2.9).

Estimate 1. There exist $M > m > 0$ (depending on V) such that

$$m \leq \left.\frac{\partial V}{\partial x_1}\right|_{\partial \mathbb{R}^N_+} \leq M. \tag{2.11}$$

The second inequality follows from the boundary estimate for elliptic equations. On the other hand, for any $z \in \partial \mathbb{R}^N_+$, denote $X = X(z) := (2R^*, z_2, \cdots, z_N)$, then the ball $B_{2R^*}(X)$ lies in $\mathbb{R}^N_+$ and touches $\partial \mathbb{R}^N_+$ at exactly one point z. By Lemma 2.2 we have $V(x) > v(x; B_{2R^*}(X))$ in $B_{2R^*}(X)$ and so

$$\left.\frac{\partial V(x)}{\partial x_1}\right|_{x=z} > m := \left.\frac{\partial v(x; B_{2R^*}(X))}{\partial x_1}\right|_{x=z}.$$

Estimate 2. For any $l > 0$, set $D_l := \{x \in \mathbb{R}^N \mid 0 < x_1 < l\}$. For any $\alpha \in (0,1)$ and some positive constant C depending on V, α and l, by the boundary estimate we have

$$\|V\|_{C^{2+\alpha}(\overline{D_l})} \leq C. \tag{2.12}$$

Estimate 3. $V(x) \to 1$ as $x_1 \to \infty$, uniformly in $(0, x_2, \cdots, x_N) \in \mathbb{R}^{N-1}$. For any given small $\varepsilon > 0$, by Remark 2.3, there exists $R_\varepsilon > 0$ such that $v(0; B_{R_\varepsilon}(0)) > 1 - \varepsilon$. Hence, for any $x \in \mathbb{R}^N_+$ with $x_1 > R_\varepsilon$, we have $B_{R_\varepsilon}(x) \subset \mathbb{R}^N_+$ and so by Lemma 2.2

$$V(x) > v(x; B_{R_\varepsilon}(x)) = v(0; B_{R_\varepsilon}(0)) > 1 - \varepsilon.$$

The opposite estimate $V(x) < 1$ is clear since 1 in an upper solution.

We now prove the uniqueness based on the above estimates. Assume by contradiction that (2.9) has two different solutions V_1 and V_2. Without loss of generality, we assume $V_1(\bar{x}) < V_2(\bar{x})$ for some $\bar{x} \in \mathbb{R}^N_+$. By the above estimates we see that $\rho V_2(x) \leq V_1(x)$ in $\mathbb{R}^N_+$ provided $\rho > 0$ is sufficiently small (the first two estimates give the comparison near the boundary, and third estimate gives the comparison near $x_1 = \infty$). Taking ρ^* the supremum of such ρ, then $0 < \rho^* \leq \frac{V_1(\bar{x})}{V_2(\bar{x})} < 1$, and one of the following holds:

(a) $\rho^* V_2(x) \leq V_1(x)$ for $x \in \mathbb{R}^N_+$, and $\rho^* V_2(y) = V_1(y)$ for some $y \in \mathbb{R}^N_+$;

(b) $\rho^* V_2(x) < V_1(x)$ for $x \in R^N_+$, and $\rho^* \frac{\partial V_2(z)}{\partial x_1} = \frac{\partial V_1(z)}{\partial x_1}$ for some $z \in \partial \mathbb{R}^N_+$;

(a)′ $\rho^* V_2(x) < V_1(x)$ for $x \in \mathbb{R}^N_+$, and $\rho^* V_2(y^{(k)}) - V_1(y^{(k)}) \to 0$ as $k \to \infty$ for a sequence $\{y^{(k)}\} \subset \mathbb{R}^N_+$ with $|y^{(k)}| \to \infty$ $(k \to \infty)$;

(b)′ $\rho^* V_2(x) < V_1(x)$ for $x \in \mathbb{R}^N_+$, $\rho^* \frac{\partial V_2(z)}{\partial x_1} < \frac{\partial V_1(z)}{\partial x_1}$ for all $z \in \partial \mathbb{R}^N_+$, and $\rho^* \frac{\partial V_2(z^{(k)})}{\partial x_1} - \frac{\partial V_1(z^{(k)})}{\partial x_1} \to 0$ as $k \to \infty$ for a sequence $\{z^{(k)}\} \subset \partial \mathbb{R}^N_+$ with $|z^{(k)}| \to \infty$ $(k \to \infty)$.

In case (a) or (b) holds, we can derive a contradiction as in the proof of Lemma 2.1. Now we derive contradictions in case (a)′ or (b)′ holds.

In case (b)′ holds, we move the origin of the coordinate system to the point $z^{(k)}$ and define

$$\tilde{v}_k(x) := \rho^* V_2(x + z^{(k)}), \quad \hat{v}_k(x) := V_1(x + z^{(k)}).$$

Then, for each k, $\tilde{v}_k$ is a lower solution and $\hat{v}_k$ is a solution of the problem (2.9),

$$\tilde{v}_k(x) - \hat{v}_k(x) \to \rho^* - 1 \ \text{ as } x_1 \to \infty, \quad \text{uniformly in } k,$$

and

$$\frac{\partial \tilde{v}_k(0)}{\partial x_1} - \frac{\hat{v}_k(0)}{\partial x_1} \to 0 \ \text{ as } k \to \infty.$$

Using the elliptic estimate (like that in Estimate 2), we conclude that, there exist two C^2 functions $\widetilde{V}$ and $\widehat{V}$ and a subsequence of $\{k\}$ (denote it again by $\{k\}$) such that

$$\tilde{v}_k(x) \to \widetilde{V}(x), \quad \hat{v}_k(x) \to \widehat{V}(x) \quad \text{as } k \to \infty,$$

in $C^2_{loc}(\overline{\mathbb{R}^N_+})$ topology, and

$$\widetilde{V}(x) - \widehat{V}(x) \to \rho^* - 1 \ \text{ as } x_1 \to \infty, \qquad \frac{\partial \widetilde{V}(0)}{\partial x_1} = \frac{\partial \widehat{V}(0)}{\partial x_1}.$$

The first limit implies that $\widetilde{V}(x) \not\equiv \widehat{V}(x)$. Hence they satisfy the conditions in case (b) and so lead to a contradiction.

In case (a)$'$ holds, we first see by Estimate 3 that the sequence $\{y^{(k)}\}$ satisfies $0 < y_1^{(k)} \le M$ for some $M > 0$. Hence, a subsequence of $\{y^{(k)}\}$ (denoted it again by $\{y^{(k)}\}$) satisfies $y_1^{(k)} \to \bar{y}_1$ $(k \to \infty)$ for some $\bar{y}_1 \in [0, M]$. When $\bar{y}_1 = 0$, a contradiction can be derived as in case (b)$'$. When $\bar{y}_1 \in (0, M]$, as in case (b)$'$ we move the origin of the coordinate system to the point $Y^{(k)} := (0, y_2^{(k)}, \cdots, y_N^{(k)})$ and define

$$\tilde{w}_k(x) := \rho^* V_2(x + Y^{(k)}), \quad \hat{w}_k(x) := V_1(x + Y^{(k)}).$$

For each k, $\tilde{w}_k$ is a lower solution and $\hat{w}_k$ is a solution of the problem (2.9), and

$$\tilde{w}_k((y_1^{(k)}, 0, \cdots, 0)) - \hat{w}_k((y_1^{(k)}, 0, \cdots, 0)) \to 0 \quad \text{as } k \to \infty.$$

Using the elliptic estimate (like that in Estimate 2), we see that, there exist two C^2 functions $\widetilde{W}$ and $\widehat{W}$ and a subsequence of $\{k\}$ (denote it again by $\{k\}$) such that

$$\tilde{w}_k(x) \to \widetilde{W}(x), \quad \hat{w}_k(x) \to \widehat{W}(x) \quad \text{as } k \to \infty,$$

in $C^2_{loc}(\overline{\mathbb{R}^N_+})$ topology, and $\widetilde{W}((\bar{y}_1, 0, \cdots, 0)) = \widehat{W}((\bar{y}_1, 0, \cdots, 0))$. Since

$$\widetilde{W}(x) \to \rho^*, \ \ \widehat{W}(x) \to 1 \ \text{ as } x_1 \to \infty,$$

we have $\widetilde{W}(x) \le, \not\equiv \widehat{W}(x)$. Hence they satisfy the conditions in case (a) and so lead to a contradiction. $\square$

Remark 2.5. Note that the uniqueness of the solution of (2.9) has been proved in [6, Proposition 4.1] for one dimension problem, and in [12, Proposition 6.2] for two dimension problem. Here we use a different method to verify the uniqueness for the N-dimension problem. One will see that our method remains valid for the problems in general cones (see Theorem 2.8 below).

In the above lemma, the existence of the solution of (2.9) is verified directly. In fact, we can also construct the unique solution $V^*(x_1)$ of (2.9) in the following way. For any given $z = (0, z_2, \cdots, z_N) \in \partial\mathbb{R}^N_+$ and any $k > 0$, denoting

$$\mathbf{e}_1 := (1, 0, \cdots, 0), \quad X_z(k) := z + k\mathbf{e}_1 = (k, z_2, \cdots, z_N)$$

and considering the problem (2.1) in the ball $B_k(X_z(k))$. By Lemma 2.1, when k is large, the unique solution $v(x; B_k(X_z(k)))$ of this problem is positive in $B_k(X_z(k))$. Moreover, by Lemma 2.2, this solution is increasing in k, and so it converges as $k \to \infty$ to the unique solution $V^*(x_1)$, in $C^2_{loc}\big(\mathbb{R}^N_+\big)$ topology. Therefore, for any given $k_0 > 0$ and any small $\varepsilon > 0$, there exists $K = K(\varepsilon, k_0) > 0$ such that, when $k \geq K$,

$$v(x; B_k(X_z(k))) \geq V^*(x_1) - \varepsilon \quad \text{for } x \in B_{k_0}(X_z(k_0)).$$

On the other hand, we have $V^*(x_1) > v(x; B_k(X_z(k)))$ by Lemma 2.2. Thus we have the following lemma.

Lemma 2.6. *For any $k_0 > 0$ and any small $\varepsilon > 0$, there exists $K = K(\varepsilon, k_0) > 0$ such that*

$$V^*(x_1) \geq v(x; B_K(X_z(K))) \geq V^*(x_1) - \varepsilon \quad \textit{for } x \in B_{k_0}(X_z(k_0)). \tag{2.13}$$

2.3. Positive steady state in Ω

Now we study the solution of (2.1) with $D = \Omega$.

Theorem 2.7. *The problem (2.1) with $D = \Omega$ admits at least one solution V, which satisfies*

$$V(x) \to 1 \textit{ as } \operatorname{dist}(x, \partial\Omega) \to \infty, \quad \textit{and} \quad V(x) \leq V^*(d(x)) \textit{ for } x \in \Omega. \tag{2.14}$$

Proof. Let $\{D_k\}$ be a sequence of bounded subsets of Ω such that, they have smooth boundaries, $B_k(X_k) \subset D_k \subset D_{k+1}$ for some $X_k \in \Omega$, and $\bigcup_{k=1}^{\infty} D_k = \Omega$. For any large k, we see by Lemma 2.1 that the problem (2.1) with $D = D_k$ has a unique positive solution $v(x; D_k) \leq 1$. By Lemma 2.2 we have

$$v(x; D_k) \leq v(x; D_{k+1}) \leq 1, \quad x \in D_k.$$

Therefore, there exists a function $V(x)$ such that

$$v(x; D_k) \nearrow V(x) \text{ as } k \to \infty, \quad \text{in } C^2_{loc}(\Omega) \text{ topology}.$$

So, $V(x)$ is a solution of (2.1) with $0 < V(x) \leq 1$.

Moreover, for any small $\varepsilon > 0$, by Remark 2.3 there exists $K(\varepsilon) > 0$ large such that $v(0; B_{K(\varepsilon)}(0)) > 1 - \varepsilon$. Hence for any $x \in \Omega$ with $\text{dist}(x, \partial\Omega) > K(\varepsilon)$ and $B_{K(\varepsilon)}(x) \subset D_k$ we have

$$1 - \varepsilon < v(0; B_{K(\varepsilon)}(0)) = v(x; B_{K(\varepsilon)}(x)) < v(x; D_k) < V(x).$$

Combining with $V(x) \le 1$ we obtain the first limit in (2.14).

For any given $z \in \partial\Omega$, we can take a tangent plane $T(z)$ of $\partial\Omega$ at z such that the whole domain Ω lies on one side of this plane (this is possible since E is convex, so is Ω). Denote by $\mathbf{n}(z)$ the unit normal vector of $T(z)$ (or $\partial\Omega$) pointing into Ω, and denote

$$\mathbb{R}^N_+(\mathbf{n}(z)) := \{x = y + s\mathbf{n}(z) \mid y \in T(z),\ s > 0\}$$

the half space of $\mathbb{R}^N$ separated by $T(z)$, where Ω lies. Consider the problem

$$\begin{cases} \Delta v + f(v) = 0, & x \in \mathbb{R}^N_+(\mathbf{n}(z)), \\ v(x) > 0, & x \in \mathbb{R}^N_+(\mathbf{n}(z)), \\ v(x) = 0, & x \in T(z). \end{cases} \tag{2.15}$$

By Lemma 2.4, this problem has a unique positive solution

$$v = V^*\big((x - z) \cdot \mathbf{n}(z)\big).$$

For any bounded domain $D_k \subset \Omega \subset \mathbb{R}^N_+(\mathbf{n}(z))$, $v(x; D_k) \le V^*\big((x - z) \cdot \mathbf{n}(z)\big)$ by Lemma 2.2. We conclude that

$$V(x) \le V^*\big((x - z) \cdot \mathbf{n}(z)\big) \quad \text{for } x \in \Omega.$$

Note that this inequality holds for any given $z \in \partial\Omega$. In particular, fix an $x \in \Omega$, the inequality holds for $z = Z_x$, where Z_x is a point on $\partial\Omega$ such that $d(x) := \text{dist}(x, \partial\Omega) = |x - Z_x|$. Thus, $(x - Z_x) \cdot \mathbf{n}(Z_x) = d(x)$, and so (2.14) holds at this given x. Since $x \in \Omega$ can be chosen arbitrarily, we indeed obtain (2.14) for all $x \in \Omega$. □

We now prove the uniqueness of V.

Theorem 2.8. *V in the previous lemma is the unique positive solution of (2.1) with $D = \Omega$.*

Proof. The proof is similar to that in Lemma 2.4. Assume by contradiction that V_1 and V_2 are two different solutions of (2.1) with $D = \Omega$, and that $V_1(\bar{x}) < V_2(\bar{x})$ for some $\bar{x} \in \Omega$. By the standard theory of elliptic equations, similar estimates as in Lemma 2.4 hold for both V_1 and V_2. Hence, there exists $0 < \rho^* \le \frac{V_1(\bar{x})}{V_2(\bar{x})} < 1$ such that one of the following holds:

(a) $\rho^* V_2(x) \le V_1(x)$ for $x \in \Omega$, and $\rho^* V_2(y) = V_1(y)$ for some $y \in \Omega$;
(b) $\rho^* V_2(x) < V_1(x)$ for $x \in \Omega$, and $\rho^* \frac{\partial V_2(z)}{\partial \mathbf{n}(z)} = \frac{\partial V_1(z)}{\partial \mathbf{n}(z)}$ for some $z \in \partial\Omega$;
(a)′ $\rho^* V_2(x) < V_1(x)$ for $x \in \Omega$, and $\rho^* V_2(y^{(k)}) - V_1(y^{(k)}) \to 0$ as $k \to \infty$ for a sequence $\{y^{(k)}\} \subset \Omega$ with $|y^{(k)}| \to \infty$ $(k \to \infty)$;

(b)′ $\rho^* V_2(x) < V_1(x)$ for $x \in \Omega$, $\rho^* \frac{\partial V_2(z)}{\partial \mathbf{n}(z)} < \frac{\partial V_1(z)}{\partial \mathbf{n}(z)}$ for all $z \in \partial\Omega$, and $\rho^* \frac{\partial V_2(z^{(k)})}{\partial \mathbf{n}(z^{(k)})} - \frac{\partial V_1(z^{(k)})}{\partial \mathbf{n}(z^{(k)})} \to 0$ as $k \to \infty$ for a sequence $\{z^{(k)}\} \subset \partial\Omega$ with $|z^{(k)}| \to \infty$ $(k \to \infty)$.

The rest proof is also similar as that in Lemma 2.4. We consider only the case (b)′. Define

$$\tilde{v}_k(x) := \rho^* V_2(x + z^{(k)}), \quad \hat{v}_k(x) := V_1(x + z^{(k)}).$$

Then, for each k, $\tilde{v}_k$ is a lower solution and $\hat{v}_k$ is a solution of the problem (2.1) in $D = \Omega^{(k)} := \{x = y - z^{(k)} \mid y \in \Omega\}$,

$$\tilde{v}_k(x) - \hat{v}_k(x) \to \rho^* - 1 \ \text{ as } \operatorname{dist}(x, \partial\Omega^{(k)}) \to \infty$$

by (2.14), and

$$\frac{\partial \tilde{v}_k(0)}{\partial \mathbf{n}(z^{(k)})} - \frac{\partial \hat{v}_k(0)}{\partial \mathbf{n}(z^{(k)})} \to 0 \ \text{ as } k \to \infty.$$

Since $\{\mathbf{n}(z) \mid z \in \partial\Omega\} \subset \mathbb{S}^{N-1}$, we see that a subsequence of $\{\mathbf{n}(z^{(k)})\}$ (denote it again by $\{\mathbf{n}(z^{(k)})\}$) converges to $\mathbf{n}_*$, and so $\Omega^{(k)} \to \mathbb{R}^N_+(\mathbf{n}_*)$ as $k \to \infty$. Then, using the elliptic estimate as in Lemma 2.4, we conclude that, there exist two C^2 functions $\widetilde{V}$ and $\widehat{V}$ and a subsequence of $\{k\}$ (denote it again by $\{k\}$) such that

$$\tilde{v}_k(x) \to \widetilde{V}(x), \quad \hat{v}_k(x) \to \widehat{V}(x) \quad \text{as } k \to \infty,$$

in $C^2_{loc}(\mathbb{R}^N_+(\mathbf{n}_*))$ topology, and

$$\widetilde{V}(x) - \widehat{V}(x) \to \rho^* - 1 \ \text{ as } x \cdot \mathbf{n}_* \to \infty, \qquad \frac{\partial \widetilde{V}(0)}{\partial \mathbf{n}_*} = \frac{\partial \widehat{V}(0)}{\partial \mathbf{n}_*}.$$

The first limit implies that $\widetilde{V}(x) \not\equiv \widehat{V}(x)$. Hence we can derive a contradiction as in the proof of Lemma 2.4, since $\widetilde{V}(x)$ is a lower solution and $\widehat{V}(x)$ is a solution of (2.9) with $\mathbb{R}^N_+$ being replaced by $\mathbb{R}^N_+(\mathbf{n}_*)$. $\square$

To study further properties of V, we give some notation. Denote $X^* := (1, 0, \cdots, 0) \in E$. We can find two real numbers $\theta_1, \theta_2 \in (0, \pi)$ with $\theta_2 > \theta_1$ such that the cones

$$\mathcal{C}_i := \{x \in \mathbb{R}^N \mid x \cdot X^* = |x| \cdot |X^*| \cos\theta_i\} \ \ (i = 1, 2)$$

satisfy $\mathcal{C}_1 \subset \Omega_* \subset \mathcal{C}_2$, where Ω_* is the cone in constructing the domain Ω. Therefore, for any $z^* \in \partial E$, the angle θ between the ray $\overrightarrow{Oz^*}$ and $\overrightarrow{OX^*}$ satisfies $\theta_1 \le \theta \le \theta_2$. Any point z on the ray $\overrightarrow{Oz^*}$ is also on $\partial\Omega$ when $|z|$ is large. For any $m > 0$, denote by

$$L(z, m) := \{z + s\mathbf{n}(z) \mid 0 < s \le m\} \tag{2.16}$$

the line segment with length m on the normal line $\ell(z) := \{x \mid x = z + s\mathbf{n}(z),\ s \in \mathbb{R}\}$. For any $a > 0$, denote

$$\Omega_a := \{x + aX^* \mid x \in \Omega\}, \qquad \Omega_a^c(R) := \{x \in \Omega \backslash \Omega_a \mid |x| > R\}. \tag{2.17}$$

For any given $m > 0$, if $x \in \Omega_a^c(R)$ and if R is sufficiently large, there exists a unique $Z_x \in \partial\Omega$ such that $x \in L(Z_x, m) \subset \ell(Z_x)$. Hence, for such x,

$$d(x) := \mathrm{dist}(x, \partial\Omega) = (x - Z_x) \cdot \mathbf{n}(Z_x) = |x - Z_x|. \tag{2.18}$$

Lemma 2.9. *For any $a > 0$ and $R > a + 1$, there exist $\sigma_2 > \sigma_1 > 0$ (independent of a and R) such that the distance function $d(x) := \mathrm{dist}(x, \partial\Omega)$ satisfies*

$$d(x) > a\sigma_1 \text{ for } x \in \Omega_a, \qquad d(x) < a\sigma_2 \text{ for } x \in \Omega_a^c(R).$$

Proof. When $a = 1$, we see that the domain Ω_1 is separated from $\partial\Omega$ by a distance $\sigma_1 > 0$. Hence the first inequality holds by proportionality.

Now we prove the second inequality. For any $z_a \in \partial\Omega_a$ with $|z_a| > R > a + 1$, by the definition of Ω_a, we have $z := z_a - aX^* \in \partial\Omega$. Denote by O the original point and $O_a := aX^*$, then the ray $\overrightarrow{O_a z_a}$ is parallel to $\overrightarrow{Oz}$, which implies that

$$d(z_a) \le \mathrm{dist}(z_a, \overrightarrow{Oz}) = \mathrm{dist}(O_a, \overrightarrow{Oz}) = a\sin\theta \le a\sin\theta_2,$$

where θ denotes the angle between the rays $\overrightarrow{Oz}$ and $\overrightarrow{OO_a}$. $\square$

By the above lemma and the first limit in (2.14) we have the following result.

Lemma 2.10. *For any small $\varepsilon > 0$, there exists A_ε such that when $a \ge A_\varepsilon$, $V(x) \ge 1 - \varepsilon$ in Ω_a.*

Proof of Theorem 1.1. The existence and uniqueness of V have been proved in Theorems 2.7 and 2.8. We now prove (1.2). Using the result in Theorem 2.7 we only need to prove

$$\limsup_{x_1 \to \infty} [V^*(d(x)) - V(x)] = 0. \tag{2.19}$$

For any small $\varepsilon > 0$, by Lemma 2.10 we have

$$V^*(d(x)) - V(x) < 1 - V(x) \le \varepsilon \quad \text{ for } x \in \Omega_a, \tag{2.20}$$

provided $a \ge A_\varepsilon$. Fix such an a, and take a $k_0 \ge a\sigma_2$, then by Lemma 2.6, there exists $K = K(\varepsilon, k_0) > k_0$ such that

$$v(x; B_K(X_z(K))) \ge V^*(x_1) - \varepsilon, \quad x \in B_{k_0}(X_z(k_0)). \tag{2.21}$$

Choose $R > 0$ sufficiently large such that when $z \in \Gamma_1 := \{z \mid z \in \partial\Omega,\ |z| > R\}$, the ball $B_K(Y_z(K))$ (where $Y_z(K) := z + K\mathbf{n}(z)$) lies in Ω and its closure touches $\partial\Omega$ at exactly one point z. Moreover, we can take R so large that the line segments $L(z, 2K)$ do not meet each other.

This implies that, for any $x \in \Omega_a^c(R+K)$, there is a unique $Z_x \in \Gamma_1$ such that $x \in L(Z_x, K)$ (note that $K > k_0 \geq a\sigma_2 > d(x)$ for $x \in \Omega_a^c(R+K)$), and

$$d(x) = \text{dist}(x, \partial\Omega) = |x - Z_x|.$$

For each $z \in \Gamma_1$, we identify

$$\mathbb{R}_+^N, \quad B_{k_0}(X_z(k_0)), \quad B_K(X_z(K)), \quad x_1$$

in Lemma 2.6 and (2.21) with

$$\mathbb{R}_+^N(\mathbf{n}(z)), \quad \mathcal{B}_1(z) := B_{k_0}(Y_z(k_0)), \quad \mathcal{B}_2(z) := B_K(Y_z(K)), \quad d(x) = |x - Z_x|,$$

respectively. Then $v(x; B_K(X_z(K)))$ is converted into a function $\tilde{v}(x; \mathcal{B}_2(z))$ and (2.21) is converted into

$$\tilde{v}(x; \mathcal{B}_2(z)) \geq V^*(d(x)) - \varepsilon, \quad x \in \mathcal{B}_1(z). \tag{2.22}$$

Since this inequality holds for all $z \in \Gamma_1$ and since $\tilde{v}(x; \mathcal{B}_2(z)) \leq V(x)$ by Lemma 2.2, we conclude that

$$V(x) \geq V^*(d(x)) - \varepsilon, \quad x \in \Omega_a^c(R+K) \subset \bigcup_{z \in \Gamma_1} \mathcal{B}_1(z).$$

Combining with (2.20) we obtain (2.19). □

2.4. *Traveling wave solutions with compact supports*

A special solution of $u_t = \Delta u + f(u)$ with the form $u = w(x - c\mathbf{e}t)$ for some $c > 0$ and $\mathbf{e} \in \mathbb{S}^{N-1}$ is called a traveling wave solution (with speed c in the direction $\mathbf{e}$). Clearly, the function w should be a solution of the following elliptic problem:

$$\begin{cases} \Delta w + c\mathbf{e} \cdot \nabla w + f(w) = 0, & x \in D \subset \mathbb{R}^N, \\ w(x) = 0, & x \in \partial D. \end{cases} \tag{2.23}$$

To study (2.23), we first consider the eigenvalue problem for the operator $-\Delta - c\mathbf{e} \cdot \nabla$:

$$\begin{cases} -\Delta \phi - c\mathbf{e} \cdot \nabla \phi = \lambda \phi, & x \in D, \\ \phi(x) = 0, & x \in \partial D. \end{cases} \tag{2.24}$$

Since, with $\psi(x) := \phi(x) e^{c\mathbf{e} \cdot x/2}$, the equation is equivalent to $-\Delta \psi = (\lambda - \frac{c^2}{4})\psi$, we see that the principal eigenvalue of (2.24) $\lambda_1^c(D) = \lambda_1(D) + \frac{c^2}{4}$, where $\lambda_1(D)$ is the principal eigenvalue of (2.2) in D. Denote $c_0 := 2\sqrt{f'(0)}$ as before. For any $c \in [0, c_0)$, due to $\frac{c^2}{4} < f'(0)$ we have $\lambda_1^c(D) = \lambda_1(D) + \frac{c^2}{4} < f'(0)$ when $\lambda_1(D)$ is sufficiently small. In particular, if $D = B_R(X)$ for some $X \in \mathbb{R}^N$, then there exists a constant R_*^c such that the principal eigenvalue $\lambda_1^c(B_R(X))$ of (2.24) in $D = B_R(X)$ satisfies $\lambda_1^c(B_R(X)) < f'(0)$ if and only if $R > R_*^c$.

Lemma 2.11. *Assume $c \in [0, c_0)$ and R_*^c is the positive real number given above.*

(i) *If $B_{R_*^c}(X) \subsetneqq D$ for some $X \in \mathbb{R}^N$, then the problem (2.23) has a unique positive solution $w^c(x; D)$.*
(ii) *$w^c(x; D)$ is strictly increasing in D in the sense that $w^c(x; D_2) > w^c(x; D_1)$ in D_1 if $B_{R_*^c}(X) \subsetneqq D_1 \subsetneqq D_2$ for some $X \in D_1$. In particular, $w^c(x; B_R(0)) \to 1$ as $R \to \infty$, in $C^2_{loc}(\mathbb{R}^N)$ topology.*
(iii) *If $D = \mathbb{R}^N_+ := \{x \in \mathbb{R}^N \mid x_1 > 0\}$ and $\mathbf{e} \perp \mathbf{e}_1 = (1, 0, \cdots, 0)$, then the unique solution of (2.23) is $V^*(x_1)$. Moreover, for any $k_0 > 0$ and any small $\varepsilon > 0$, there exists $K = K(\varepsilon, k_0) > 0$ such that, for any $z = (0, z_2, \cdots, z_N)$ and any $k \geq K$,*

$$V^*(x_1) \geq w^c(x; B_k(X_z(k))) \geq V^*(x_1) - \varepsilon \quad \textit{for } x \in B_{k_0}(X_z(k_0)), \tag{2.25}$$

where $X_z(k) := z + k\mathbf{e}_1$.

Proof. This lemma can be proved in a similar way as Lemmas 2.1, 2.2, 2.4 and 2.6. □

To end this section we show $w^c(x; B_1) < v(x; B_2)$ when $B_1 \subset B_2$ and B_2 is sufficiently large.

Lemma 2.12. *Assume $\mathbf{e} \in \mathbb{S}^{N-1}$ and $c \in (0, c_0)$. Let $w^c(x; B_m(0))$ be a positive solution of (2.23) in $D = B_m(0)$ and $v(x; B_M(0))$ be a solution of (2.1) in $D = B_M(0)$. Assume m is fixed. If M is sufficiently large then*

$$w^c(x - 2M\mathbf{e} + m\mathbf{e}; B_m(0)) < v(x - M\mathbf{e}; B_M(0)), \quad x \in B_m(2M\mathbf{e} - m\mathbf{e}). \tag{2.26}$$

Proof. Since $\bar{w} := \sup_{x \in B_m(0)} w^c(x; B_m(0)) < 1$, by Remark 2.3, there exists $M > m$ sufficiently large such that $v(x; B_M(0)) \geq \bar{w}$ in $B_m(0)$. Hence,

$$v(x - M\mathbf{e}; B_M(0)) \geq w^c(x - M\mathbf{e}; B_m(0)) \quad \text{for } x \text{ satisfying } |x - M\mathbf{e}| \leq m.$$

Both $u_1 := v(x - M\mathbf{e}; B_M(0))$ and $u_2 := w^c(x - M\mathbf{e} - c\mathbf{e}t; B_m(0))$ are solutions of $u_t = \Delta u + f(u)$, and so the comparison principle is applied in the time interval $t \in [0, (M - m)/c]$ (since in this period the domain of u_2 lies in that u_1). In particular, at $t = (M - m)/c$, we have (2.26). □

3. Spreading for the solutions of (P)

3.1. Convergence in $L^\infty_{loc}(\Omega)$ topology

We show that any solution u of (P) converges, in $L^\infty_{loc}(\Omega)$ topology, to the positive steady state V.

Theorem 3.1. *Let u be the solution of (P) with nonnegative and compactly supported initial data u_0. Then for any given $R > 0$ we have*

$$\|u(t, \cdot) - V(\cdot)\|_{L^\infty(\Omega \cap B_R)} \to 0 \textit{ as } t \to \infty,$$

where V is the positive steady state constructed in Section 2.

Proof. For any given small $\varepsilon > 0$, we first prove

$$\liminf_{t\to\infty} u(t,x) \geq V(x) - \varepsilon, \quad \text{uniformly in } x \in \Omega \cap B_R. \tag{3.1}$$

Recall the construction of V in the proof of Theorem 2.7 we see that there exists a large domain D with $\Omega \cap B_R \subset D \subset \Omega$ such that

$$V(x) > v(x; D) > V(x) - \varepsilon, \quad x \in \Omega \cap B_R.$$

On the other hand, by the maximum principle we have $u(1,x) > 0$ in D. Consider the following auxiliary problem

$$\begin{cases} \tilde{u}_t = \Delta \tilde{u} + f(\tilde{u}), & x \in D,\ t > 0, \\ \tilde{u}(t,x) = 0, & x \in \partial D,\ t > 0, \\ \tilde{u}(0,x) = u(1,x), & x \in D. \end{cases}$$

By the comparison principle we have

$$u(t+1,x) \geq \tilde{u}(t,x), \quad x \in D, t > 0.$$

Taking limit as $t \to \infty$ and noticing $\tilde{u}(t,x) \to v(x; D)$ (by Lemma 2.1) we obtain (3.1).

Next we prove

$$\limsup_{t\to\infty} u(t,x) \leq V(x) + \varepsilon, \quad \text{uniformly in } x \in \Omega \cap B_R. \tag{3.2}$$

Choose $M > 1$ sufficiently large such that $MV(x) \geq u_0(x)$. Denote by $u(t,x; MV)$ the solution of (P) with initial condition MV. By comparison we have

$$u(t,x; MV) \geq u(t,x) \text{ for } t > 0,\ x \in \Omega; \qquad u(t,x; MV) \geq V(x) \text{ for } x \in \Omega. \tag{3.3}$$

Moreover, by (F) we have $f(MV) \leq Mf(V)$, and so $u_t(t,x; MV) \leq 0$ due to

$$\Delta(MV) + f(MV) \leq M[\Delta V + f(V)] = 0.$$

This implies that

$$u(t,x; MV) \searrow \tilde{v}(x) \text{ in } C^2_{loc}(\Omega), \tag{3.4}$$

for some $\tilde{v}(x) \geq V(x)$. By the standard parabolic theory, $\tilde{v}$ is a positive steady state of $u_t = \Delta u + f(u)$, and so $\tilde{v} \equiv V$. Thus, (3.2) follows from (3.4) and the first inequality of (3.3). $\square$

3.2. Spreading speed

The limits (1.5) and (1.6) in Theorem 1.2 are given by the following two theorems, respectively.

Theorem 3.2. *Let u be the solution of* (P) *with compactly supported initial data u_0. Then, for any $c_1 > c_0 := 2\sqrt{f'(0)}$,*

$$\lim_{t\to\infty} \sup_{|x|>c_1 t,\ x\in\Omega} u(t,x) = 0.$$

Proof. We choose a large constant $R > 0$ such that the support of initial data $\mathrm{spt}(u_0) \subset B_R$, and consider the Cauchy problem

$$\begin{cases} \tilde{u}_t = \Delta \tilde{u} + f(\tilde{u}), & x \in \mathbb{R}^N,\ t > 0, \\ \tilde{u}(0,x) = \|u_0\|_{L^\infty} \cdot \chi_{B_R}, & x \in \mathbb{R}^N, \end{cases}$$

where χ_D denotes the characteristic function over D. Then the solution $\tilde{u}(t,x)$ is a radially symmetric one and $\lim_{t\to\infty} \sup_{|x|>c_1 t} \tilde{u}(t,x) = 0$ (cf. [2]). Clearly, $\tilde{u}$ is an upper solution of (P), and $u(t,x) \le \tilde{u}(t,x)$ by the comparison principle. This reduces to the conclusion. $\square$

We now prove the lower estimate for the spreading speed.

Theorem 3.3. *Let u, c_0 be the same as in the previous theorem, $V(x)$ be the unique positive steady state of* (P)*. Then, for any $c_2 \in (0, c_0)$*

$$\lim_{t\to\infty} \inf_{|x|\le c_2 t,\ x\in\Omega} u(t,x) = V(x). \tag{3.5}$$

Remark 3.4. We first state the idea of the proof. The precise values of V depend on the shape of Ω and are not easy to be specified in detail, hence, to prove (3.5) it is convenient to substitute V by some simpler approximate functions. More precisely, we divide Ω into three parts Ω_a, $\Omega_a^c(R)$ and $\Omega \cap B_R$. The convergence in the last bounded domain follows from Theorem 3.1. In the first two domains, we use 1 to approximate V in Ω_a (for large a), and use $V^*(d(x))$ to approximate $V(x)$ in $\Omega_a^c(R)$ (for large R). Clearly, both 1 and V^* are simpler than V since they do not depend on the shape of Ω. Moreover, they can be estimated from below by the traveling wave solution $w^c(x - c\mathbf{e}t; B_{R_0})$ for some $R_0 > 0$. In fact, we will show that, for any small $\varepsilon > 0$,

$$u(t,x) \ge w^c(x - c\mathbf{e}t; B_{R_0}) \ge 1 - \varepsilon > V(x) - \varepsilon, \quad \text{in } \Omega_a, \tag{3.6}$$

and

$$u(t,x) \ge w^c(x - c\mathbf{e}t; B_{R_0}) \ge V^*(d(x)) - \varepsilon \approx V(x) - \varepsilon, \quad \text{in } \Omega_a^c(R), \tag{3.7}$$

provided R_0, R and t are sufficiently large. Therefore, besides u and V, the functions V^* and $w^c(x - c\mathbf{e}t; B_{R_0})$ are also involved in our proof. This point is more complicated than that in Theorem 3.2, where only one uniform upper solution $\tilde{u}$ is used.

Proof of Theorem 3.3. We use the notation given in the previous section. Let $\varepsilon > 0$ be any given small number, we only need to prove

$$\inf_{|x|\le c_2 t,\ x\in\Omega} \big[u(t,x) - V(x)\big] \ge -r\varepsilon, \qquad \text{when } t \text{ is sufficiently large}, \tag{3.8}$$

for some integer $r > 0$. We prove it in three domains Ω_a, $\Omega_a^c(R)$ and $\Omega \cap B_R$, respectively.

Step 1. To prove (3.8) in $\Omega_a := \{x + aX^* \mid x \in \Omega\}$.

(1). By Lemma 2.10, there exists $a_1 = a_1(\varepsilon)$ such that $V(x) > 1 - \varepsilon$ in Ω_{a_1}.

(2). For any given $c \in (c_2, c_0)$ and any $\mathbf{e} \in \mathbb{S}^{N-1}$, by Lemma 2.11, the problem

$$\begin{cases} \Delta w + c\mathbf{e}\cdot\nabla w + f(w) = 0, & x \in B_{R_0} := B_{R_0}(0), \\ w(x) = 0, & x \in \partial B_{R_0} \end{cases} \tag{3.9}$$

has a unique positive solution provided $R_0 > 0$ is large. We use $w^c(x; \mathbf{e}, B_{R_0})$ to denote this solution. Moreover, one can choose a suitable $R_0 = R_0(\varepsilon)$ such that

$$1 - 3\varepsilon < w^c(\bar{x}; \mathbf{e}, B_{R_0}) := \max_{x\in B_{R_0}} w^c(x; \mathbf{e}, B_{R_0}) < 1 - 2\varepsilon, \tag{3.10}$$

for some $\bar{x} = \bar{x}(\mathbf{e}) \in B_{R_0}$.

(3). Choose $\tilde{x} \in \Omega_{a_1}$ such that $B_{R_0}(\tilde{x}) \subset \Omega_{a_1}$. By Theorem 3.1, $u(t,x) \to V(x)$ $(t \to \infty)$ uniformly in $B_{R_0}(\tilde{x})$. So, there exists $T_1 > 0$ such that, when $t \ge T_1$,

$$u(t,x) > V(x) - \varepsilon > 1 - 2\varepsilon > w^c(x - \tilde{x}; \mathbf{e}, B_{R_0}), \quad x \in B_{R_0}(\tilde{x}). \tag{3.11}$$

(4). Choose $a > a_1$ sufficiently large such that $B_{R_0}(\tilde{x}) \cap \Omega_a = \emptyset$, $\mathrm{dist}(\Omega_a, \partial\Omega) > R_0$ and

$$x - y - \tilde{x} \in \Omega \quad \text{for all } x \in \Omega_a \text{ and } y \in B_{R_0}.$$

Then, for any $y^* \in B_{R_0}(\tilde{x})$ and $y^{**} \in \Omega_a$, when the ball $B_{R_0}(\tilde{x})$ moves along the line segment $\overline{y^*y^{**}}$, it remains in Ω.

(5). Set

$$T_2 := \frac{cT_1 + R_0 + |\tilde{x}|}{c - c_2}.$$

Now we prove a claim:

Claim 1. *For any $t > T_2$ and any $x \in \Omega_a$ with $|x| \le c_2 t$, we have $u(t,x) \ge 1 - 3\varepsilon$.*

Otherwise, there exists $\hat{t} > T_2$ and $\hat{x} \in \Omega_a$ with $|\hat{x}| < c_2\hat{t}$ such that $u(\hat{t}, \hat{x}) < 1 - 3\varepsilon$. Since the time moment

$$T(\mathbf{e}) := \hat{t} - \frac{|\hat{x} - \bar{x}(\mathbf{e}) - \tilde{x}|}{c} \ge \hat{t} - \frac{c_2\hat{t} + |\bar{x}(\mathbf{e})| + |\tilde{x}|}{c} > \frac{(c - c_2)T_2 - R_0 - |\tilde{x}|}{c} = T_1,$$

we have by (3.11),

$$u(T(\mathbf{e}), x) > 1 - 2\varepsilon \geq w^c(x - \tilde{x}; \mathbf{e}, B_{R_0}) \quad \text{for all } \mathbf{e} \in \mathbb{S}^{N-1},\ x \in B_{R_0}(\tilde{x}). \tag{3.12}$$

Recall that the set E in the construction of Ω is a convex domain. Define a map $P : \overline{E} \to \Gamma := \mathbb{S}^{N-1} \cap \overline{\Omega}$ as follows: for any $x \in \overline{E}$, let Px be the contact point between the line $\overline{Ox}$ and Γ. Then P is a homeomorphism from $\overline{E}$ to Γ. Define another map $A :\ \Gamma \to \Gamma$ as

$$A\mathbf{e} := \frac{\hat{x} - \bar{x}(\mathbf{e}) - \tilde{x}}{|\hat{x} - \bar{x}(\mathbf{e}) - \tilde{x}|}, \quad \mathbf{e} \in \Gamma.$$

Then $B := P^{-1} \circ A \circ P$ is a continuous map from the bounded, closed and convex domain $\overline{E}$ into itself. Using the Brouwer theorem we see that the map B has a fixed point $x^* \in \overline{E}$, and so $\mathbf{e}^* := Px^* \in \Gamma$ is a fixed point of A: $A\mathbf{e}^* = \mathbf{e}^*$. Using $\mathbf{e}^*$ to replace the direction $\mathbf{e}$ in the above argument, then $w^c(x - \tilde{x} - c\mathbf{e}^* t; \mathbf{e}^*, B_{R_0})$ is a traveling wave solution of (P) in direction $\mathbf{e}^*$ (with compact support). By (3.12) and the comparison principle we have

$$u(T(\mathbf{e}^*) + t, x) > w^c(x - \tilde{x} - c\mathbf{e}^* t; \mathbf{e}^*, B_{R_0}) \ \text{ for } t > 0 \text{ and } x \text{ with } |x - \tilde{x} - c\mathbf{e}^* t| \leq R_0.$$

In particular, at $t = \hat{t} - T(\mathbf{e}^*)$ and $x = \hat{x}$, we deduce a contradiction:

$$1 - 3\varepsilon > u(\hat{t}, \hat{x}) > w^c(\bar{x}(\mathbf{e}^*); \mathbf{e}^*, B_{R_0}) > 1 - 3\varepsilon.$$

(6). The estimate (3.8) with $r = 3$ follows from Claim 1 directly.

Step 2. To prove (3.8) in $\Omega_a^c(R)$ for some large R to be determined.
(1). Once a is fixed, there exists $k_0 > 0$ such that $\text{dist}(x, \partial\Omega) < k_0$ for all $x \in \Omega \backslash \Omega_a$.
(2). For any given $c \in (c_2, c_0)$ and any $z \in \partial\mathbb{R}^N_+$, by (2.25) we have

$$w^c(x; B_K(X_z(K))) \geq V^*(x_1) - \varepsilon, \quad x \in B_{k_0}(X_z(k_0)). \tag{3.13}$$

(3). By taking $\mathbf{e} = -\mathbf{e}_1$ and $m = K$ in (2.26), we see that for sufficiently large $M > K$,

$$v(x; B_M(X_z(M))) > w^c(x; B_K(X_z(K))), \quad x \in B_K(X_z(K)). \tag{3.14}$$

(4). Take R_1 sufficiently large such that, the line segments $L(z, M)$ for $z \in \Gamma_1 := \{z \in \partial\Omega \mid |z| \geq R_1\}$ do not meet each other, and $B_M(Y_z(M)) \subset \Omega$ for all $z \in \Gamma_1$, where $Y_z(k) := z + k\mathbf{n}(z)$ for $k > 0$. Thus, for any $x \in \Omega$ satisfying $|x| \geq R_1 + M$ and $\text{dist}(x, \partial\Omega) \leq M$, there is a unique $Z_x \in \Gamma_1$ such that $x \in L(Z_x, M)$ and

$$d(x) = \text{dist}(x, \partial\Omega) = |x - Z_x|.$$

For any given $z \in \Gamma_1^0 := \{z \in \partial\Omega \mid |z| = R_1\}$, denote by $\mathbb{R}^N_+(\mathbf{n}(z))$ the half space separated by the tangent plane $T(z)$ with $\Omega \subset \mathbb{R}^N_+(\mathbf{n}(z))$. We identify

$$\mathbb{R}^N_+, \ \ \mathbf{e}_1, \ \ \mathbf{e}, \ \ B_{k_0}(X_z(k_0)), \ \ B_K(X_z(K)), \ \ B_M(X_z(M)), \ \ x_1$$

in Lemmas 2.11 and 2.12 with

$$\mathbb{R}^N_+(\mathbf{n}(z)),\quad \mathbf{n}(z),\quad \mathbf{r}(z)=\tfrac{z}{|z|},\quad \mathcal{B}_1(z):=B_{k_0}(Y_z(k_0)),$$
$$\mathcal{B}_2(z):=B_K(Y_z(K)),\quad \mathcal{B}_3(z):=B_M(Y_z(M)),\quad d(x)=|x-Z_x|,$$

respectively. Then $w^c(x;B_K(X_z(K)))$ is also converted into a new function $\tilde{w}(x;\mathcal{B}_2(z))$ and the inequality (3.13) becomes

$$\tilde{w}(x;\mathcal{B}_2(z))\geq V^*(d(x))-\varepsilon,\qquad x\in\mathcal{B}_1(z). \tag{3.15}$$

$v(x;B_M(X_z(M)))$ is converted into a new function $\tilde{v}(x;\mathcal{B}_3(z))$ and (3.14) reduces to

$$\tilde{v}(x;\mathcal{B}_3(z))>\tilde{w}(x;\mathcal{B}_2(z)),\qquad x\in\mathcal{B}_2(z). \tag{3.16}$$

Since $\mathcal{B}_3(z)\subset\Omega$ and $\tilde{v}(x;\mathcal{B}_3(z))$ is a solution of (2.1) in $\mathcal{B}_3(z)$, by Lemma 2.2 we have

$$V(x)>\tilde{v}(x;\mathcal{B}_3(z)),\qquad x\in\mathcal{B}_3(z).$$

By Theorem 3.1, $u(t,x)\to V(x)$ $(t\to\infty)$ uniformly in $\mathcal{B}_3(z)$. Hence, there exists $T_3(z)>0$ such that, when $t\geq T_3(z)$,

$$u(t,x)\geq\tilde{v}(x;\mathcal{B}_3(z)),\qquad x\in\mathcal{B}_3(z).$$

Combining with (3.16) we have

$$u(t,x)>\tilde{w}(x;\mathcal{B}_2(z))\qquad\text{for } x\in\mathcal{B}_2(z),\ t\geq T_3(z). \tag{3.17}$$

Taking $T_3:=\max\{T_3(z)\mid z\in\Gamma_1^0\}$, then (3.17) holds for any $z\in\Gamma_1^0$, $x\in\mathcal{B}_2(z)$ and $t\geq T_3$.

(5). Take a $R>R_1$ large such that $B_M(Y_z(M))\cap\Omega_a^c(R)=\emptyset$ for all $z\in\Gamma_1^0$. Set

$$T_4:=\frac{cT_3+2k_0+R_1}{c-c_2}.$$

Then we can prove

Claim 2. *For any $t>T_4$ and any $x\in\Omega_a^c(R)$ with $|x|\leq c_2t$, we have $u(t,x)-V(x)\geq-\varepsilon$.*

In fact, for any given $\tilde{t}>T_4$ and $\tilde{x}\in\Omega_a^c(R)$ with $|\tilde{x}|<c_2\tilde{t}$, denote

$$\tilde{z}:=Z_{\tilde{x}},\qquad \mathbf{r}:=\mathbf{r}(\tilde{z})=\frac{\tilde{z}}{|\tilde{z}|},\qquad \check{z}:=R_1\mathbf{r}\in\Gamma_1^0,\qquad \check{x}:=\tilde{x}-\tilde{z}+\check{z}\in\mathcal{B}_1(\check{z}).$$

Then $\check{z}=Z_{\check{x}}$, $\mathbf{r}=\frac{\tilde{x}-\check{x}}{|\tilde{x}-\check{x}|}$ and $d(\tilde{x})=|\tilde{x}-\tilde{z}|=|\check{x}-\check{z}|=d(\check{x})$. Since the time moment

$$\tau:=\tilde{t}-\frac{|\tilde{x}-\check{x}|}{c}\geq\tilde{t}-\frac{|\tilde{x}|+2k_0+R_1}{c}\geq\tilde{t}-\frac{c_2\tilde{t}+2k_0+R_1}{c}>\frac{(c-c_2)T_4-2k_0-R_1}{c}=T_3,$$

by (3.17) we have

$$u(\tau, x) \geq \tilde{w}(x; \mathcal{B}_2(\check{z})), \quad x \in \mathcal{B}_2(\check{z}).$$

Since $\tilde{w}(x - c\mathbf{r}t; \mathcal{B}_2(\check{z}))$ is a traveling wave solution of (P) (with compact support), by comparison principle we have

$$u(\tau + t, x) \geq \tilde{w}(x - c\mathbf{r}t; \mathcal{B}_2(\check{z})) \text{ for } t > 0 \text{ and } x \text{ with } x - c\mathbf{r}t \in \mathcal{B}_2(\check{z}).$$

In particular, at $t = \tilde{t} - \tau$ and $x = \tilde{x}$, we have

$$u(\tilde{t}, \tilde{x}) \geq \tilde{w}(\check{x}; \mathcal{B}_2(\check{z})).$$

Since $\check{x} \in \mathcal{B}_1(\check{z})$, by (3.15) we have

$$\tilde{w}(\check{x}; \mathcal{B}_2(\check{z})) \geq V^*(d(\check{x})) - \varepsilon = V^*(d(\tilde{x})) - \varepsilon \geq V(\tilde{x}) - \varepsilon.$$

The last inequality follows from (2.14). This proves Claim 2, and then the estimate (3.8) holds in $\Omega_a^c(R)$.

Step 3. The convergence of $u(t, x) \to V(x)$ in bounded domain $\Omega \cap B_R$ follows from Theorem 3.1.

Combining the results in these three steps we obtain (3.8). This completes the proof for Theorem 3.3. □

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