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Bifurcation analysis of the Degond–Lucquin-Desreux–Morrow model for gas discharge

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Abstract

The main purpose of this paper is to investigate mathematically gas discharge. Townsend discovered α- and γ-mechanisms which are essential for ionization of gas, and then derived a threshold of voltage at which gas discharge can happen. In this derivation, he used some simplification such as discretization of time. Therefore, it is an interesting problem to analyze the threshold by using the Degond–Lucquin-Desreux–Morrow model and also to compare the results of analysis with Townsend's theory. Note that gas discharge never happens in Townsend's theory if γ-mechanism is not taken into account. In this paper, we study an initial–boundary value problem to the model with α-mechanism but no γ-mechanism. This problem has a trivial stationary solution of which the electron and ion densities are zero. It is shown that there exists a threshold of voltage at which the trivial solution becomes unstable from stable. Then we conclude that gas discharge can happen for a voltage greater than this threshold even if γ-mechanism is not taken into account. It is also of interest to know the asymptotic behavior of solutions to this initial–boundary value problem for the case that the trivial solution is unstable. To this end, we establish bifurcation of non-trivial stationary solutions by applying Crandall and Rabinowitz's Theorem, and show the linear stability and instability of those non-trivial solutions.

MSC: 35M33; 47A10; 70K50; 76X05; 82D10; 93D20

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Keywords: Hyperbolic-parabolic-elliptic coupled system; Stationary solutions; Nonlinear stability; Nonlinear instability; Stable manifold; Spectrum analysis

1. Introduction

We are interested in the mathematical analysis of gas ionization processes. At the beginning of the 1900s, Townsend discovered two essential mechanisms for ionization of gas. He experimented what happens in a chamber consisting of two planar parallel plates and filled with a gas, which is an insulator, when he apply a high-voltage to these two plates. Here the higher voltage plate is the anode, and another one is the cathode. If electrons are emitted in the tube, these initial electrons are accelerated from the cathode to the anode by high-voltage and simultaneously make ions and additional electrons by the collision with gas particles. This mechanism is called as α-mechanism. Another one is γ-mechanism which is the secondary emission of electrons caused by impact of positive ions with the cathode. These two mechanisms yield the electric multiplication which permit large current flow throughout the gas. This phenomenon is called as gas discharge or avalanche breakdown. From an observation taking α- and γ-mechanisms into account, Townsend also derived a threshold of voltage at which gas discharge can happen and continue. This threshold is called as sparking voltage. However, he used several simplification such as discretization of time and ignorance of advection in the derivation of sparking voltage (for more details, see [20]). Therefore, it is an interesting problem to analyze the sparking voltage by using a partial differential equation with no simplification and then compare the results of analysis with Townsend's theory. It should be noted here that gas discharge never happens in Townsend's theory if γ-mechanism is not taken into account. In this paper, we study an initial–boundary value problem of a partial differential equation with α-mechanism but no γ-mechanism, and also make clear whether we can have a sparking voltage or not.

Several mathematical models for gas discharge were proposed in [1,5,6,9–11,15]. These models vary with the constitutive equations of velocities. In this paper, we adopt the model derived by Morrow in [15]. This model has been widely used in a lot of numerical researches (for example, see [12,16,17]). Moreover, Degond and Lucquin-Desreux [4] formally derived it from the Euler–Maxwell system. Hence, it is reasonable to adopt this model from both physical and mathematical points of view. Throughout this paper, we call this model as the Degond–Lucquin-Desreux–Morrow model. It consists of two continuity equations for the densities of positive ions and of electrons, adopting constitutive velocity relations, coupled with the Poisson equation for the electrostatic potential:

$$\partial_t \rho_i + \partial_x(\rho_i u_i) = a \exp\left(-b|\partial_x \Phi|^{-1}\right) \rho_e |v_e|, \tag{1.1a}$$

$$\partial_t \rho_e + \partial_x(\rho_e u_e) = a \exp\left(-b|\partial_x \Phi|^{-1}\right) \rho_e |v_e|, \tag{1.1b}$$

$$\lambda \partial_{xx} \Phi = \rho_i - \rho_e, \tag{1.1c}$$

$$u_i := k_i \partial_x \Phi, \quad u_e := v_e - k_e \partial_x \rho_e / \rho_e, \quad v_e := -k_e \partial_x \Phi, \quad x \in I := (0, L), \ t > 0, \tag{1.1d}$$

where L is a width of the planar parallel plates. The unknown functions ρ_i, ρ_e, and $-\Phi$ denote the positive ion density, the electron density, and the electrostatic potential, respectively. The ion

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and electron velocities u_i and u_e are assumed to obey (1.1d). Moreover, k_i, k_e, a, b, and λ are positive constants. The right hand sides of (1.1a) and (1.1b) come from α-mechanism. In particular, $\alpha = a \exp\left(-b|\partial_x \Phi|^{-1}\right)$ is the first Townsend ionization coefficient expressing the number of ion–electron pairs generated per unit volume by the electron impact ionization. We notice that this model is a hyperbolic-parabolic-elliptic coupled system by substituting constitutive velocity relations (1.1d) into continuity equations (1.1a) and (1.1b).

We consider the initial–boundary value problem of (1.1) by prescribing the initial and boundary data

$$(\rho_i, \rho_e)(0, x) = (\rho_{i0}, \rho_{e0})(x), \quad \rho_{i0}(x) \geq 0, \quad \rho_{e0}(x) \geq 0, \quad x \in I = (0, L), \tag{1.1e}$$

$$\rho_i(t, 0) = \rho_e(t, 0) = \Phi(t, 0) = 0, \tag{1.1f}$$

$$\rho_e(t, L) = 0, \ \Phi(t, L) = V_c > 0. \tag{1.1g}$$

The boundaries $x = 0$ and $x = L$ correspond to the anode and cathode, respectively, since $-\Phi$ is the electrostatic potential. Boundary condition (1.1f) means that, in an instant, electrons are absorbed to the anode and ions are excluded near the anode. We emphasize that γ-mechanism is not taken into account on the cathode $x = L$, and thus the zero Dirichlet boundary condition is adopted. From physical point of view, it is reasonable to assume the non-negativity of initial densities ρ_{i0} and ρ_{e0}. For the compatibility, they are also assumed to satisfy

$$\rho_{i0}(0) = \rho_{e0}(0) = \rho_{e0}(L) = 0.$$

The first mathematical work for the Degond–Lucquin-Desreux–Morrow model was announced by the present authors [21]. They showed the time-local solvability of an initial boundary value problem over a domain $\Omega := \mathbb{R}^3_+ \setminus K$, where $\mathbb{R}^3_+$ is a half space, K is a simply connected open set, and the intersection of $\partial\mathbb{R}^3_+$ and K is the empty set. They also mentioned several remarks on the time-local solvability over other domains of which boundaries are two of plates and spheres, because the typical shapes of the cathode and anode are either a sphere or plate for the physical and numerical experiments.

In this paper, we study the Degond–Lucquin-Desreux–Morrow model only over a bounded interval to derive the sparking voltage in the same situation as in Townsend's theory. Initial–boundary value problem (1.1) has a trivial stationary solution of which densities ρ_i and ρ_e are zero. We will show that there exists a threshold of voltage at which the trivial solution becomes unstable from stable. This fact means that gas discharge can happen and continue for a voltage greater than the threshold. Therefore, we conclude that gas discharge can happen even if γ-mechanism is not taken into account, whereas it cannot happen without γ-mechanism in Townsend's theory. It is also of interest to know the asymptotic behavior of solutions to problem (1.1) for the case that the trivial solution is unstable. To do so, we establish bifurcation of non-trivial stationary solutions from the trivial stationary solution by applying Crandall and Rabinowitz's Theorem, and show the linear stability and instability of those non-trivial solutions.

Notation. For $1 \leq p \leq \infty$, $L^p(\Omega)$ is the Lebesgue space equipped with the norm $|\cdot|_p$. For a non-negative integer k, $H^k(I)$ is the k-th order Sobolev space in L^2 sense, equipped with the

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norm $\|\cdot\|_k$. Note that $H^0(I)=L^2(I)$ and $\|\cdot\|:=\|\cdot\|_0$. The inner product of $L^2(I)$ is denoted by $\langle f,g\rangle$ for $f,g\in L^2(I)$. Moreover, $H_0^1(I)$ and $H_{0l}^1(I)$ are closures of $C_0^\infty(I)$ and $C_0^\infty((0,L])$ with respect to H^1-norm, respectively. We denote by $C^m([0,T];X)$ the space of the m-times continuously differentiable functions on the interval $[0,T]$ with values in a Banach space X, and by $H^m(0,T;X)$ the space of H^m–functions on $(0,T)$ with values in a Banach space X. Furthermore, we denote by c and C generic positive constants and by $C[\alpha,\beta,\cdots]$ a generic positive constant depending on special parameters $\alpha,\beta,\ldots$.

2. Main results

For mathematical convenience, let us rewrite initial–boundary value problem (1.1) by using the new unknown functions

$$R_i:=\rho_i e^{-\frac{L}{V_c}x},\quad R_e:=\rho_e e^{\frac{V_c}{2L}x}$$

and the new given functions

$$h(x):=a\exp\left(\frac{-b}{|x|}\right)|x|,\quad g(V_c):=h\left(\frac{V_c}{L}\right)-\frac{V_c^2}{4L^2}.$$

Note that the function g plays essential roles in our analysis. We also decompose the electrostatic potential as

$$\Phi=V+\frac{V_c}{L}x,$$

where V_cx/L is a solution to the equation $\partial_{xx}u=0$ with the boundary conditions $u(0)=0$ and $u(L)=V_c$. As a result, we have the following problem

$$\partial_t R_i+k_i\partial_x\left\{\left(\partial_x(V[R_i,R_e,V_c])+\frac{V_c}{L}\right)R_i\right\}+k_iR_i$$
$$=k_eh\left(\frac{V_c}{L}\right)e^{-\frac{L}{V_c}x-\frac{V_c}{2L}x}R_e+k_if_i[R_i,R_e,V_c], \tag{2.1a}$$
$$\partial_tR_e-k_e\partial_{xx}R_e-k_eg(V_c)R_e=k_ef_e[R_i,R_e,V_c], \tag{2.1b}$$
$$V[R_i,R_e,V_c]:=\frac{1}{\lambda}\int_0^L G(x,y)\left(e^{\frac{L}{V_c}y}R_i(t,y)-e^{-\frac{V_c}{2L}y}R_e(t,y)\right)\mathrm{d}y, \tag{2.1c}$$
$$(R_i,R_e)(0,x)=(R_{i0},R_{e0})(x),\quad R_{i0}(x)\geq 0,\ R_{e0}(x)\geq 0, \tag{2.1d}$$
$$R_i(t,0)=R_e(t,0)=R_e(t,L)=0, \tag{2.1e}$$

where $G(x,y)$ is the Green function of the Laplace operator ∂_{xx} with the Dirichlet zero condition, and the nonlinear terms f_i and f_e are defined as

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$$
\begin{aligned}
f_i[R_i, R_e, V_c] :=& -\frac{L}{V_c} R_i \partial_x (V[R_i, R_e, V_c]) \\
& - \frac{k_e}{k_i} \left\{ h\left(\frac{V_c}{L}\right) - h\left(\partial_x (V[R_i, R_e, V_c]) + \frac{V_c}{L}\right) \right\} e^{-\frac{L}{V_c}x - \frac{V_c}{2L}x} R_e, \qquad (2.2a)
\end{aligned}
$$

$$
\begin{aligned}
f_e[R_i, R_e, V_c] :=& \partial_x (V[R_i, R_e, V_c]) \partial_x R_e - \frac{V_c}{2L} R_e \partial_x (V[R_i, R_e, V_c]) + R_e \partial_{xx} (V[R_i, R_e, V_c]) \\
& - \left\{ h\left(\frac{V_c}{L}\right) - h\left(\partial_x (V[R_i, R_e, V_c]) + \frac{V_c}{L}\right) \right\} R_e. \qquad (2.2b)
\end{aligned}
$$

It is easy to check that the corresponding stationary problem has a trivial stationary solution

$$
(R_i, R_e) = (0, 0).
$$

The advantage of using the new unknown functions R_i and R_e lies in the following two facts. The first one is that the rewritten hyperbolic equation has the dissipative term $k_i R_i$, although the original hyperbolic equation does not have any dissipative structure. Secondly, the linear part of the rewritten parabolic equation is self-adjoint. These two facts play important roles in the proofs of both the nonlinear stability and instability of the trivial stationary solution.

We state the nonlinear stability and instability theorems for the trivial stationary solution.

Theorem 2.1. *Let $g(V_c) < \pi^2/L^2$. There exists $\varepsilon > 0$ such that if the initial data $(R_{i0}, R_{e0}) \in H^1_{0l}(I) \times H^1_0(I)$ satisfy $\|R_{i0}\|_1 + \|R_{e0}\|_1 < \varepsilon$, then problem (2.1) has a unique time-global solution (R_i, R_e) as*

$$
R_i \geq 0, \quad R_i \in C([0, \infty); H^1_{0l}(I)) \cap C^1([0, \infty); L^2(I)), \qquad (2.3a)
$$

$$
R_e \geq 0, \quad R_e \in C([0, \infty); H^1_0(I)) \cap L^2(0, \infty; H^2(I)) \cap H^1(0, \infty; L^2(I)). \qquad (2.3b)
$$

Moreover, it converges to zero exponentially fast in $H^1(I) \times H^1(I)$ as t goes to infinity.

Theorem 2.2. *Let $g(V_c) > \pi^2/L^2$ and $(\psi_i, \psi_e) \in H^1_{0l}(I) \times H^1_0(I)$ satisfy*

$$
\psi_i, \psi_e \geq 0, \quad \|\psi_i\|_1^2 + \|\psi_e\|_1^2 = 1, \quad \int_0^L \psi_e \sin \frac{\pi}{L} x \, dx > 0. \qquad (2.4)
$$

There exists $\varepsilon > 0$ such that for any sufficiently small $\delta > 0$, problem (2.1) with the initial data $(R_{i0}, R_{e0}) = (\delta\psi_i, \delta\psi_e)$ has a unique solution (R_i, R_e) satisfying $\|R_i(T)\|_1 + \|R_e(T)\|_1 \geq \varepsilon$ for some $T > 0$.

In this instability theorem, the last inequality in (2.4) is equivalent to that the initial datum R_{e0} is a non-zero function. It is of interest to know what happens for the case that R_{e0} is the zero function. For this question, Proposition 2.3 gives an answer that there exists a unique time-global solution, and it attains the trivial stationary solution at finite time.

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Proposition 2.3. *Let $V_c > 0$. There exists $\varepsilon > 0$ such that if the initial data $(R_{i0}, R_{e0}) \in H^1_{0l}(I) \times H^1_0(I)$ satisfy $R_{e0} = 0$ and $\|R_{i0}\|_{H^1} < \varepsilon$, then problem (2.1) has a unique time-global solution (R_i, R_e) as (2.3). Furthermore, there exists $T_0 > 0$ such that*

$$(R_i, R_e)(t, x) = (0, 0) \quad \textit{for} \ \ (t, x) \in [T_0, \infty) \times I. \tag{2.5}$$

The principal significance of this proposition is that a set $\{(R_{i0}, R_{e0}) \in H^1_{0l}(I) \times H^1_0(I); R_{e0} = 0\}$ is a local stable manifold of system (2.1a)–(2.1c) for any $V_c > 0$.

Now we mention physical observation from Theorems 2.1 and 2.2 and Proposition 2.3 in the next remark.

Remark 2.4. Townsend defined the sparking voltage as a threshold of voltage at which gas discharge happens and continues. In following his manner, it is reasonable from Theorems 2.1 and 2.2 to define the sparking voltage for the Degond–Lucquin-Desreux–Morrow model by $V_c^* > 0$ with

$$g(V_c^*) = \frac{\pi^2}{L^2}, \quad g'(V_c^*) > 0. \tag{2.6}$$

In fact, the solution never goes to the trivial stationary solution $(R_i, R_e) = (0, 0)$ in the case $V_c < V_c^*$; the solution converges to the trivial solution as t tends to infinity in the case $V_c > V_c^*$. These facts mean that V_c^* is a threshold of voltage at which gas discharge happens and continues. Therefore we conclude that gas discharge can happen even if γ-mechanism is not taken into account, whereas it cannot happen without γ-mechanism in Townsend's theory. On the other hand, in common with Townsend's theory, we cannot have gas discharge for the Degond–Lucquin-Desreux–Morrow model without α-mechanism, since there does not exist V_c^* with (2.6) in the case $a = 0$. We also remark that there exists the sparking voltage in (2.6) for some physical parameters, and moreover it is unique if it exists. For more details, see Appendix A.

Physically speaking, in the case that the electron density is zero, α-mechanism never happens and neither does gas discharge. Proposition 2.3 really coincides with this observation.

Next we are interested in finding the asymptotic behavior of solutions to problem (2.1) for the case $g(V_c) > \pi^2/L^2$. Then it is expected from Theorems 2.1 and 2.2 by regarding the voltage V_c as the bifurcation parameter that there is a non-trivial stationary solution curve near the point $(R_i, R_e, V_c) = (0, 0, V_c^*)$. The results on bifurcation are summarized in Theorem 2.5 and Corollary 2.6.

Theorem 2.5. *Let a positive number V_c^* satisfy (2.6). There exist $\eta > 0$, $V_c \in C^2([-\eta, \eta]; \mathbb{R})$, and $z \in C^2([-\eta, \eta]; H^1_{0l}(I) \times (H^1_0(I) \cap H^2(I)))$ such that $V_c(0) = V_c^*$, $z(0) = 0$, and stationary problem to (2.1) with $V_c = V_c(s)$ has a non-trivial solution $(R_i, R_e)(s) = s(\varphi_i, \varphi_e) + sz(s)$ for $s \in [-\eta, \eta]$, where*

$$\varphi_i(x) := a\frac{k_e}{k_i} e^{-\frac{bL}{V_c^*} - \frac{L}{V_c^*}x} \int_0^x e^{-\frac{V_c^*}{2L}y} \varphi_e(y)\,\mathrm{d}y, \quad \varphi_e(x) := \sin\frac{\pi}{L}x.$$

Moreover, $\dot{V}_c(0) \lesseqgtr 0$ holds, where "$\dot{}$" denotes the derivative with respect to s, if and only if

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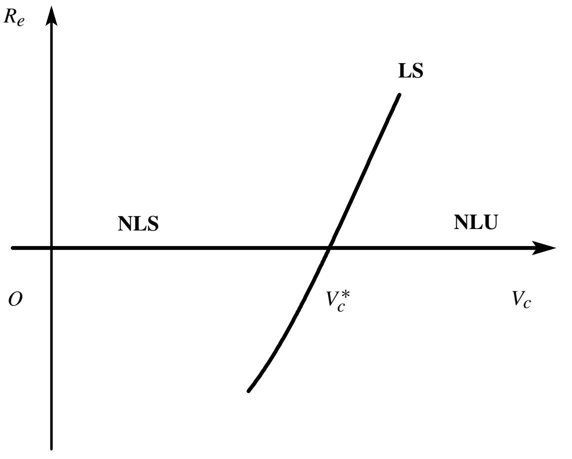

Fig. 1. Case $\dot{V}_c(0) > 0$.

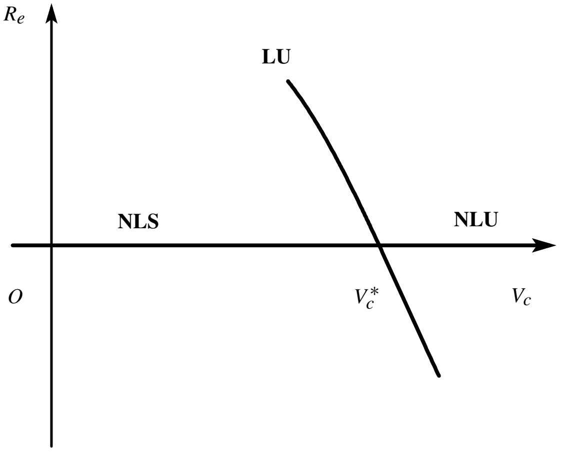

Fig. 2. Case $\dot{V}_c(0) < 0$.

$$-2Lg'(V_c^*)\int_0^L \varphi_e^2 \partial_x (V[\varphi_i, \varphi_e, V_c^*])\,\mathrm{d}x - \int_0^L \varphi_e^2 \partial_{xx}(V[\varphi_i, \varphi_e, V_c^*])\,\mathrm{d}x \lesseqgtr 0. \tag{2.7}$$

Corollary 2.6. *Let $\dot{V}_c(0)$ in Theorem 2.5 be nonzero. Then there exists a positive constant θ such that $(R_i(s), R_e(s))$ satisfy*

$$s\dot{V}_c(0)R_i(s,x) > 0, \quad s\dot{V}_c(0)R_e(s,x) > 0 \quad \textit{for} \ \ s \in [-\theta, \theta]\backslash\{0\}, \ \ x \in I. \tag{2.8}$$

Furthermore, the positive non-trivial solution is linearly stable if $\dot{V}_c(0) > 0$, and linearly unstable if $\dot{V}_c(0) < 0$.

From Theorem 2.5 and Corollary 2.6, we can draw the bifurcation diagram of stationary solutions as Figs. 1 and 2. Both diagrams are truly possible for some physical parameters k_i, k_e,

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a, b, and L (for details, see Appendix B). For the case $V_c > V_c^*$, the solution to problem (2.1) may approach to the positive non-trivial stationary solution as t tends to infinity if $\dot{V}_c(0) > 0$; the solution may either blow up or grow up as time goes by if $\dot{V}_c(0) < 0$.

Outline of paper. This paper is organized as follows. We show the nonlinear stability of the trivial solution in Subsection 3.1. The proof is based on the energy method with using the best constant in Poincaré inequality. Subsection 3.2 deals with the nonlinear instability of the trivial solution. Here we construct the Green function of the linearized equation of (2.1b) by using the eigenvalues and eigenfunctions, and then represent the solutions R_e to (2.1b) by applying Duhamel's principle. This formula enables us to find a growth mode of solutions. In Subsection 3.3, we prove Proposition 2.3 asserting that system (2.1a)–(2.1c) has a local stable manifold for any voltage $V_c > 0$. The main idea of the proof is to combine the energy method and Green's Theorem. Section 4 establishes the bifurcation of a non-trivial solution from the trivial solution by the application of Crandall and Rabinowitz's Theorem, and also provides some properties of the non-trivial solution such as the positivity, linear stability, and linear instability.

3. Trivial stationary solution

Section 3 deals with the stability and instability of the trivial stationary solution $(R_i, R_e) = (0, 0)$. Throughout this section, we use a new notation

$$N(T) := \sup_{0 \le t \le T} (\|R_i(t)\|_1 + \|R_e(t)\|_1).$$

Let us observe several fundamental properties of solutions (R_i, R_e) to problem (2.1). From formula (2.1c), one can obtain the elliptic estimate

$$\|V[R_i, R_e, V_c](t)\|_{2+k} \le C\|R_i(t)\|_k + C\|R_e(t)\|_k \quad \text{for } t \ge 0, \quad k = 0, 1. \tag{3.1}$$

This gives

$$\partial_x V[R_i, R_e, V_c](t, x) + \frac{V_c}{L} > \frac{V_c}{2L} > 0 \tag{3.2}$$

if $N(T)$ is sufficiently small. Then it follows from equation (2.1a) and boundary condition (2.1e) that

$$\partial_x R_i(t, 0) = 0 \quad \text{for } t \ge 0, \tag{3.3}$$

provided that $\partial_t R_i(t), \partial_x R_i(t) \in H^1(I)$. Furthermore, if $N(T) < 1$, the nonlinear terms f_i and f_e in (2.1) are estimated as

$$|f_e[R_i, R_e, V_c]| \le C(\|R_i\|_1 + \|R_e\|_1)(|R_e| + |\partial_x R_e|) \le CN(T)(|R_e| + |\partial_x R_e|), \tag{3.4}$$

$$\sum_{l=0}^{k} |\partial_x^l (f_i[R_i, R_e, V_c])| \le C[V_c^{-1}]N(T) \sum_{l=0}^{k} (|\partial_x^l R_i| + |\partial_x^l R_e|) \quad \text{for } k = 0, 1, \tag{3.5}$$

by (3.1) and Sobolev's inequality. Note that the above properties hold for any $V_c > 0$. The constant $C[V_c^{-1}]$ on the right hand side of (3.5) diverges if V_c tends to zero. This fact does not cause any issues in our proofs. Indeed we first fix $V_c > 0$ in all Theorems 2.1 and 2.2 and Proposition 2.3. Furthermore, the proofs in Subsections 3.1–3.3 work even if all constants C depend on fixed $V_c > 0$. Hereafter we do not write the dependence of C for V_c^{-1}.

We also use Poincaré's inequality

$$\|f\| \leq \frac{L}{\pi}\|\partial_x f\| \quad \text{for} \quad f \in H_0^1(I), \tag{3.6}$$

where L/π is the best constant in Poincaré's inequality.

3.1. Nonlinear stability

In this subsection we discuss the nonlinear stability of the trivial stationary solution $(R_i, R_e) = (0, 0)$. The time local solvability of problem (2.1) is summarized as follows. Here the smallness assumption for the initial data is required to determine the sign of characteristic of hyperbolic equation (2.1a) as (3.2).

Lemma 3.1. *For any $V_c > 0$, there exists $\varepsilon > 0$ such that if the initial data $(R_{i0}, R_{e0}) \in H_{0l}^1(I) \times H_0^1(I)$ satisfy $\|R_{i0}\|_1 + \|R_{e0}\|_1 < \varepsilon$, then problem (2.1) has a unique solution (R_i, R_e) as*

$$R_i \geq 0, \quad R_i \in C([0,T]; H_{0l}^1(I)) \cap C^1([0,T]; L^2(I)), \tag{3.7a}$$

$$R_e \geq 0, \quad R_e \in C([0,T]; H_0^1(I)) \cap L^2(0,T; H^2(I)) \cap H^1(0,T; L^2(I)), \tag{3.7b}$$

and $N(T) \leq 2(\|R_{i0}\|_1 + \|R_{e0}\|_1)$ for some $T > 0$ depending only on ε, L, a, b, k_i, k_e, and λ.

Proof. The proof is similar in spirit to those of Lemma 3.1 in [18] and Lemma 6.3 in [19]. □

The stability analysis of the trivial solution is completed by deriving the a priori estimate below.

Lemma 3.2. *Let $g(V_c) < \pi^2/L^2$. Suppose that (R_i, R_e) satisfying (3.7) is a solution to problem (2.1). There exists $\delta > 0$ such that if $N(T) \leq \delta$, then it holds for $t \in [0,T]$ that*

$$e^{\gamma t}(\|R_i(t)\|_1^2 + \|R_e(t)\|_1^2) + \int_0^t e^{\gamma\tau}(\|R_i(\tau)\|_1^2 + \|R_e(\tau)\|_2^2)\,d\tau \leq C(\|R_{i0}\|_1^2 + \|R_{e0}\|_1^2), \tag{3.8}$$

where γ and C are positive constants independent of t.

Notice that the time-global solution (R_i, R_e) with the properties in (2.3) can be constructed by the standard continuation argument using the time local solvability established in Lemma 3.1 and the a priori estimate in Lemma 3.2. Because this argument is well-known (for example, see [13,14]), we give briefly the proof. Let us take $\|R_{i0}\|_1 + \|R_{e0}\|_1$ so small that it is less than $\min\{\varepsilon/2, \delta/2\}$ and that the right hand side of (3.8) is less than $\min\{\varepsilon^2/4, \delta^2/4\}$ for ε and δ being in Lemmas 3.1 and 3.2. First we see from Lemma 3.1 that a time-local solution exists at $t = T$ and satisfies $N(T) \le \delta$. Then applying Lemma 3.2 to the time-local solution leads to the fact $N(T) \le \min\{\varepsilon/2, \delta/2\}$. Regarding T and $(R_i, R_e)(T)$ as an initial time and datum and then using Lemma 3.1, one can have a time-local solution with $N(2T) \le \delta$ until $t = 2T$. Again Lemma 3.2 ensures $N(2T) \le \min\{\varepsilon/2, \delta/2\}$. Repeating this argument, we conclude that a time-global solution exists and satisfies $N(S) \le \delta/2$ for any $S > 0$.

Once the global solution is constructed, it is obvious that the resulting global solution satisfies the estimate (3.8) for $t \in [0, \infty)$. This means that the global solution decays exponentially fast in $H^1 \times H^1$ as t goes to infinity. Thanks to this standard machinery for the time-asymptotic stability, it suffices to show Lemma 3.2 in order to prove Theorem 2.1. The rest of this subsection is devoted to proving Lemma 3.2.

Proof of Lemma 3.2. Multiply (2.1b) by $2e^{\gamma_1 t} R_e$, integrate it by parts over $[0, t] \times I$, and use boundary condition (2.1e). The result is

$$\begin{aligned}
& e^{\gamma_1 t} \int_0^L R_e^2 \,\mathrm{d}x + 2k_e \int_0^t \int_0^L e^{\gamma_1 \tau} \left\{ (\partial_x R_e)^2 - g(V_c) R_e^2 \right\} \mathrm{d}x \mathrm{d}\tau \\
&= \int_0^L R_{e0}^2 \,\mathrm{d}x + \gamma_1 \int_0^t \int_0^L e^{\gamma_1 \tau} R_e^2 \,\mathrm{d}x \mathrm{d}\tau + 2k_e \int_0^t \int_0^L e^{\gamma_1 \tau} f_e R_e \,\mathrm{d}x \mathrm{d}\tau \\
&\le \|R_{e0}\|^2 + C(\gamma_1 + N(T)) \int_0^t e^{\gamma_1 \tau} \|R_e(\tau)\|_1^2 \,\mathrm{d}\tau,
\end{aligned} \tag{3.9}$$

where we have used (3.4). Applying Poincaré's inequality (3.6) to the second term of the left hand side of (3.9), using the assumption $g(V_c) < \pi^2/L^2$, and taking $\gamma_1 > 0$ and $N(T)$ sufficiently small, we have

$$e^{\gamma_1 t} \|R_e(t)\|^2 + c \int_0^t e^{\gamma_1 \tau} \|R_e(\tau)\|_1^2 \,\mathrm{d}\tau \le C \|R_{e0}\|^2, \tag{3.10}$$

where c and C are positive constants independent of t.

Multiply (2.1b) by $2e^{\gamma_1 t} \partial_{xx} R_e$, integrate the result by parts over $[0, t] \times I$, and use boundary condition (2.1e) to obtain

$$\begin{aligned}
& e^{\gamma_1 t} \int_0^L (\partial_x R_e)^2 \,\mathrm{d}x + 2k_e \int_0^t \int_0^L e^{\gamma_1 \tau} (\partial_{xx} R_e)^2 \,\mathrm{d}x \mathrm{d}\tau \\
&= \int_0^L (\partial_x R_{e0})^2 \,\mathrm{d}x + \gamma_1 \int_0^t \int_0^L e^{\gamma_1 \tau} (\partial_x R_e)^2 \,\mathrm{d}x \mathrm{d}\tau - 2k_e \int_0^t \int_0^L e^{\gamma_1 \tau} (g(V_c) R_e + f_e) \partial_{xx} R_e \,\mathrm{d}x \mathrm{d}\tau
\end{aligned}$$

$$\leq \|\partial_x R_{e0}\|^2 + \mu \int_0^t e^{\gamma_1 \tau} \|\partial_{xx} R_e(\tau)\|^2 \mathrm{d}\tau + C[\mu] \int_0^t e^{\gamma_1 \tau} \|R_e(\tau)\|^2 \,\mathrm{d}\tau. \tag{3.11}$$

Here μ is a positive constant to be determined later and we have used (3.4) with Hölder's and Gagliardo–Nirenberg–Sobolev's inequalities in deriving the above inequality. Then taking μ small enough and using (3.10) leads to

$$e^{\gamma_1 t} \|\partial_x R_e(t)\|^2 + c \int_0^t e^{\gamma_1 \tau} \|\partial_{xx} R_e(\tau)\|^2 \,\mathrm{d}\tau \leq C \|R_{e0}\|_1^2. \tag{3.12}$$

Multiply (2.1a) by $2e^{\gamma_2 t} R_i$, integrate it by parts over $[0, t] \times I$, and use boundary condition (2.1e). Moreover, differentiate (2.1a) with respect to x, multiply the result by $2e^{\gamma_2 t} \partial_x R_i$, and use boundary condition (3.3). Then summing up these two equalities gives

$$\begin{aligned}
&e^{\gamma_2 t} \int_0^L \{R_i^2 + (\partial_x R_i)^2\} \,\mathrm{d}x + 2k_i \int_0^t \int_0^L e^{\gamma_2 \tau} \{R_i^2 + (\partial_x R_i)^2\} \,\mathrm{d}x \mathrm{d}\tau \\
&\quad + k_i \int_0^t e^{\gamma_2 \tau} \left(\partial_x V + \frac{V_c}{L} \right) \{R_i^2 + (\partial_x R_i)^2\}(\tau, L) \,\mathrm{d}\tau \\
&= \int_0^L \{R_{i0}^2 + (\partial_x R_{i0})^2\} \,\mathrm{d}x + \gamma_2 \int_0^t \int_0^L e^{\gamma_2 \tau} \{R_i^2 + (\partial_x R_i)^2\} \,\mathrm{d}x \mathrm{d}\tau \\
&\quad - \int_0^t \int_0^L e^{\gamma_2 \tau} \left\{ k_i (\partial_{xx} V) R_i - 2k_e h \left(\frac{V_c}{L} \right) e^{-\frac{L}{V_c} x - \frac{V_c}{2L} x} R_e - 2k_i f_i \right\} R_i \,\mathrm{d}x \mathrm{d}\tau \\
&\quad - \int_0^t \int_0^L e^{\gamma_2 \tau} \Big\{ k_i (\partial_{xx} V)(\partial_x R_i) + 2k_i \partial_x ((\partial_{xx} V) R_i) \\
&\qquad\qquad - \partial_x \left(2k_e h \left(\frac{V_c}{L} \right) e^{-\frac{L}{V_c} x - \frac{V_c}{2L} x} R_e + 2k_i f_i \right) \Big\} \partial_x R_i \,\mathrm{d}x \mathrm{d}\tau \\
&\leq \|R_{i0}\|_1^2 + C(\gamma_2 + \mu + N(T)) \int_0^t e^{\gamma_2 \tau} \|R_i(\tau)\|_1^2 \,\mathrm{d}\tau + C[\mu] \int_0^t e^{\gamma_2 \tau} \|R_e(\tau)\|_1^2 \,\mathrm{d}\tau,
\end{aligned} \tag{3.13}$$

where we have used (3.1), (3.5), and Schwarz's and Sobolev's inequalities in deriving the above inequality. Owing to (3.2), the third term of the left hand side of (3.13) is non-negative and thus negligible. Then letting $\gamma_2 > 0$, μ, and $N(T)$ be sufficiently small, setting $\gamma := \min\{\gamma_1, \gamma_2\}$, and using (3.10), we conclude

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$$e^{\gamma t}\|R_i(t)\|_1^2 + c\int_0^t e^{\gamma\tau}\|R_i(\tau)\|_1^2\,\mathrm{d}\tau \le C(\|R_{i0}\|_1^2 + \|R_{e0}\|_1^2), \tag{3.14}$$

where c and C are positive constants independent of t. Then summing up (3.10), (3.12), and (3.14) completes the proof. □

In the proof of Lemma 3.2, we need to justify the formal computations in the derivation of (3.14). Because a standard mollifier technique is not applicable, we need to make an approximate sequence $\{R_i^j\}_{j\in\mathbb{N}}$ for R_i by solving an initial–boundary value problem, whose initial datum R_{i0} has higher regularity, and then prove R_i^j satisfies (3.14) for any $j\in\mathbb{N}$. We omit the detailed argument, since it is straightforward.

3.2. Nonlinear instability

This subsection provides the proof of Theorem 2.2 which ensures the instability of the trivial stationary solution $(R_i, R_e) = (0, 0)$. To this end, we use Duhamel's principle for R_e as

$$R_e(t,x) = \sum_{n=1}^{\infty} e^{k_e\mu_n t}\langle R_{e0}, \varphi_n\rangle\varphi_n(x) + k_e\sum_{n=1}^{\infty}\int_0^t e^{k_e\mu_n(t-\tau)}\langle f_e[R_i, R_e, V_c](\tau,\cdot), \varphi_n\rangle\varphi_n(x)\,\mathrm{d}\tau, \tag{3.15}$$

$$\varphi_n(x) := \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x, \quad \mu_n := g(V_c) - \frac{n^2\pi^2}{L^2},$$

where φ_n and μ_n are the eigenfunctions and eigenvalues of the operator $\partial_{xx} + g(V_c)$ with the zero Dirichlet boundary condition.

We begin by showing the following lemma for ψ_i and ψ_e defined in Theorem 2.2. Note that $\mu_1 > 0$ is equivalent to the condition $g(V_c) > \pi^2/L^2$.

Lemma 3.3. *Let $g(V_c) > \pi^2/L^2$. There exists $\varepsilon_0 > 0$ such that if the solution (R_i, R_e) to problem (2.1) with $(R_{i0}, R_{e0}) = (\delta\psi_i, \delta\psi_e)$ for any $\delta \in (0, \varepsilon_0)$ satisfies $N(T) \le \varepsilon_0$ and*

$$\left\| R_e(t) - \delta\sum_{n=1}^{\infty} e^{k_e\mu_n t}\langle\psi_e, \varphi_n\rangle\varphi_n \right\| \le \delta e^{k_e\mu_1 t} \quad \textit{for } t\in[0,T], \tag{3.16}$$

then it holds that

$$\|R_i(t)\|_1 + \|R_e(t)\|_1 \le C\delta e^{k_e\mu_1 t}, \tag{3.17}$$

$$\left\| R_e(t) - \delta\sum_{n=1}^{\infty} e^{k_e\mu_n t}\langle\psi_e, \varphi_n\rangle\varphi_n \right\| < m\delta^2 e^{2k_e\mu_1 t} \quad \textit{for } t\in[0,T], \tag{3.18}$$

where C and m are positive constants independent of t.

Proof. We first show (3.17). The triangle inequality together with (3.16) gives

$$\|R_e(t)\| \le C\delta e^{k_e\mu_1 t}. \tag{3.19}$$

Even for the case $g(V_c) > \pi^2/L^2$, the inequalities (3.11) and (3.13) hold. Set $\gamma_1 = 0$ in (3.11), take μ small enough, and use (3.19), $R_{e0} = \delta\psi_e$, and assumption (2.4) to obtain

$$\|\partial_x R_e(t)\| \le C\delta e^{k_e\mu_1 t}. \tag{3.20}$$

Furthermore, set $\gamma_2 = 0$ in (3.13), take $\mu + N(T)$ small enough, and use (3.2), (3.19), (3.20), $R_{i0} = \delta\psi_i$, and assumption (2.4). Then we have

$$\|R_i(t)\|_1 \le C\delta e^{k_e\mu_1 t}. \tag{3.21}$$

Consequently, inequalities (3.19)–(3.21) give (3.17).

Let us prove (3.18). It follows from $R_{e0} = \delta\psi_e$, representation (3.15), and Parseval's equality that

$$\left\| R_e(t) - \delta\sum_{n=1}^{\infty} e^{k_e\mu_n t}\langle\psi_e,\varphi_n\rangle\varphi_n \right\|^2 = k_e^2\sum_{n=1}^{\infty}\left|\langle\int_0^t e^{k_e\mu_n(t-\tau)} f_e[R_i,R_e,V_c](\tau,\cdot)\,\mathrm{d}\tau,\varphi_n\rangle\right|^2.$$

Using Schwarz's inequality, (3.4), and (3.17), one can estimate this right hand side from above as

$$\begin{aligned}
(\mathrm{RHS}) &\le k_e^2\sum_{n=1}^{\infty}\left|\int_0^t e^{k_e\mu_n(t-\tau)}\|f_e[R_i,R_e,V_c](\tau,\cdot)\|\,\mathrm{d}\tau\right|^2 \\
&\le C\sum_{n=1}^{\infty}\left|\int_0^t e^{k_e\mu_n(t-\tau)}(\|R_i(\tau)\|_1^2 + \|R_e(\tau)\|_1^2)\,\mathrm{d}\tau\right|^2 \\
&< m^2\delta^4 e^{4k_e\mu_1 t},
\end{aligned}$$

where m is a positive constant independent of t. Hence, this inequality concludes (3.18). □

From now on we prove Theorem 2.2 by using Lemma 3.3.

Proof of Theorem 2.2. For ε_0 and m being in Lemma 3.3, we take positive constants ν, ε, δ, T as

$$\nu = \langle\psi_e,\varphi_1\rangle, \quad \varepsilon < \min\left\{1, \varepsilon_0, \frac{\nu}{3m}, \frac{\nu^2}{6m}\right\}, \quad \delta < \min\left\{\frac{\varepsilon_0}{2}, \varepsilon\right\},$$

$$T = \sup\left\{t\,;\, \left\| R_e(\tau) - \delta\sum_{n=1}^{\infty} e^{k_e\mu_n\tau}\langle\psi_e,\varphi_n\rangle\varphi_n \right\| \le \delta e^{k_e\mu_1\tau},\right.$$

$$\left.\|R_i(\tau)\|_1 + \|R_e(\tau)\|_1 < \varepsilon \text{ for } \tau \in (0,t)\right\},$$

where the last inequality in (2.4) ensures $\nu > 0$. Then we also take T_* as

$$2\varepsilon < \delta\nu e^{k_e\mu_1 T_*} < 3\varepsilon.$$

If $T = \infty$, it is obvious that $T_* < T$. For the case $T < \infty$, let us show that either $T_* < T$ or $N(T) = \varepsilon$ holds. Suppose, contrary to our claim, that $T_* \geq T$ and $N(T) < \varepsilon$ hold. From the definition of T, the following equality holds:

$$\begin{aligned}\delta e^{k_e\mu_1 T} &= \left\| R_e(T) - \delta \sum_{n=1}^{\infty} e^{k_e\mu_n T}\langle\psi_e,\varphi_n\rangle\varphi_n \right\| \\ &< m\delta^2 e^{2k_e\mu_1 T},\end{aligned}$$

where the above inequality holds by the virtue of Lemma 3.3. This gives $1/m < \delta e^{k_e\mu_1 T}$. On the other hand, the definitions of T_* and ε yield $\delta e^{k_e\mu_1 T_*} < 3\varepsilon/\nu \leq 1/m$. These two inequalities lead to $T_* < T$ which contradicts our assumption $T_* \geq T$.

If $N(T) = \varepsilon$ holds, the proof is complete. Thus, it remains to deal with the case $T_* < T$ for which (3.18) holds from Lemma 3.3. By (3.18), the triangle inequality, and Parseval's equality, we obtain

$$\begin{aligned}\|R_e(T_*)\| &\geq \delta \left\| \sum_{n=1}^{\infty} e^{k_e\mu_n T_*}\langle\psi_e,\varphi_n\rangle\varphi_n \right\| - m\delta^2 e^{2k_e\mu_1 T_*} \\ &= \delta \left(\sum_{n=1}^{\infty} e^{2k_e\mu_n T_*} |\langle\psi_e,\varphi_n\rangle|^2 \right)^{1/2} - m\delta^2 e^{2k_e\mu_1 T_*}.\end{aligned}$$

Use the definitions of ν, T_*, and ε to estimate this rightmost from below by

$$\delta e^{k_e\mu_1 T_*}\langle\psi_e,\varphi_1\rangle - m\delta^2 e^{2k_e\mu_1 T_*} = \delta\nu e^{k_e\mu_1 T_*}\left(1 - \frac{m}{\nu^2}\delta\nu e^{k_e\mu_1 T_*}\right) \geq 2\varepsilon\left(1 - 3\frac{m}{\nu^2}\varepsilon\right) > \varepsilon.$$

Consequently, we can conclude $\|R_i(T_*)\|_1 + \|R_e(T_*)\|_1 \geq \|R_e(T_*)\| \geq \varepsilon$. □

3.3. Stable manifold

This subsection is devoted to showing Proposition 2.3 which asserts that a set $\{(R_{i0}, R_{e0}) \in H^1_{0l}(I) \times H^1_0(I);\ R_{e0} = 0\}$ is a local stable manifold of system (2.1a)–(2.1c) for any $V_c > 0$. Similarly to the argument in subsection 3.1, we can establish a unique existence of time-global solutions to problem (2.1) by combining the time-local solvability stated in Lemma 3.1 and the following a priori estimate.

Lemma 3.4. *Let $V_c > 0$. Suppose that (R_i, R_e) satisfying (3.7) is a solution to problem (2.1) with the initial data $R_{e0} = 0$. There exists $\delta > 0$ such that if $N(T) \leq \delta$, then it holds*

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$$R_e(t,x) = 0 \quad \textit{for} \ \ (t,x) \in [0,T] \times I, \tag{3.22}$$

$$\|R_i(t)\|_1^2 \le \|R_{i0}\|_1^2 \quad \textit{for} \ \ t \in [0,T]. \tag{3.23}$$

Proof. We begin by proving (3.22). Multiply (2.1b) by $2R_e$, integrate it by parts over $[0,t] \times I$, and use boundary condition (2.1e) and $R_{e0} = 0$. The result is

$$\begin{aligned} \int_0^L R_e^2 \,\mathrm{d}x + 2k_e \int_0^t \int_0^L (\partial_x R_e)^2 \,\mathrm{d}x\mathrm{d}\tau &= 2k_e \int_0^t \int_0^L g(V_c) R_e^2 \,\mathrm{d}x\mathrm{d}\tau + 2k_e \int_0^t \int_0^L f_e R_e \,\mathrm{d}x\mathrm{d}\tau \\ &\le \mu \int_0^t \|\partial_x R_e(\tau)\|^2 \,\mathrm{d}\tau + C[\mu, N(T)] \int_0^t \|R_e(\tau)\|^2 \,\mathrm{d}\tau, \end{aligned} \tag{3.24}$$

where μ is a positive constant to be determined later and we have used (3.4) and Schwarz's inequality in deriving the above inequality. Then letting μ be small enough and applying Gronwall's inequality to (3.24) yield $\|R_e(t)\|^2 = 0$ which means (3.22).

Let us show (3.23). Even for the case $V_c > 0$, the inequality (3.13) holds. Hence, setting $\gamma_2 = R_e = 0$ in (3.13), letting $\mu + N(T)$ be sufficiently small, and using (3.2), we conclude (3.23). □

The proof of Proposition 2.3 is completed by showing (2.5).

Proof of Proposition 2.3. We have shown $R_e = 0$ in Lemma 3.4. Hence, it suffices to prove that there exists $T_0 > 0$ such that $R_i(T_0, x) = 0$ for any $x \in I$, because problem (2.1) with the initial time $t = T_0$ and the initial data $R_i(T_0, x) = R_e(T_0, x) = 0$ has a unique solution $(R_i, R_e) = (0, 0)$.

Substitute $R_e = 0$ into (2.1a), multiply the result by $2R_i$, and integrate over the domain

$$\Omega_{T_0} := \left\{ (\tau, x) \,\middle|\, 0 \le \tau \le T_0, \ 0 \le x \le \frac{k_i V_c}{2L} \tau \right\}, \quad \text{where} \ \ T_0 := \frac{2L^2}{k_i V_c}.$$

Then we have

$$\begin{aligned} \iint_{\Omega_{T_0}} \partial_t (R_i^2) + \partial_x \left\{ k_i \left(\partial_x V + \frac{V_c}{L} \right) R_i^2 \right\} \mathrm{d}x\mathrm{d}\tau &= \iint_{\Omega_{T_0}} k_i (2f_i - (\partial_{xx} V) R_i - 2R_i) R_i \,\mathrm{d}x\mathrm{d}\tau \\ &\le C \iint_{\Omega_{T_0}} R_i^2 \,\mathrm{d}x\mathrm{d}\tau, \end{aligned} \tag{3.25}$$

where we have used (3.1), (3.5) with $R_e = 0$, (3.23) and Schwarz's inequality in deriving the above inequality. To apply Green's theorem to the left hand side of (3.25), let us set curves c_1, c_2, and c_3 as

$$c_1 : \tau = s,\ x = 0,\ s \in [0, T_0], \quad c_2 : \tau = T_0,\ x = s,\ s \in [0, L],$$
$$c_3 : \tau = \frac{2L}{k_i V_c}(L - s),\ x = L - s,\ s \in [0, L].$$

Note that $\partial\Omega_{T_0} = c_1 + c_2 + c_3$. Then Green's theorem with boundary condition (2.1e) yields

$$\begin{aligned}(\text{LHS}) &= - \oint\limits_{c_1+c_2+c_3} \left\{ k_i \left(\partial_x V + \frac{V_c}{L} \right) R_i^2 \, \mathrm{d}\tau - R_i^2 \, \mathrm{d}x \right\} \\ &= \int_0^L R_i^2 (T_0, x) \, \mathrm{d}x + \int_0^L \left\{ \frac{2L}{V_c} \left(\partial_x V + \frac{V_c}{L} \right) R_i^2 - R_i^2 \right\} \left(\frac{2L}{k_i V_c}(L - s), L - s \right) \mathrm{d}s.\end{aligned} \tag{3.26}$$

We notice that the rightmost of (3.26) is non-negative thanks to (3.2). Hence, substituting (3.26) into (3.25) leads to

$$\| R_i(T_0) \|^2_{L^2(0,L)} \le C \iint\limits_{\Omega_{T_0}} R_i^2 \, \mathrm{d}x \mathrm{d}\tau = C \int_0^{T_0} \| R_i(\tau) \|^2_{L^2(0,(k_i V_c/2L)\tau)} \, \mathrm{d}\tau.$$

Note that the function $\| R_i(\tau) \|_{L^2(0,(k_i V_c/2L)\tau)}$ of τ is continuous on $[0, T_0]$, and $(k_i V_c/2L)T_0 = L$ holds. Then applying Gronwall's lemma to the above inequality, we conclude $\| R_i(T_0) \|_{L^2(0,L)} = 0$ which means that $R_i(T_0, x) = 0$ for any $x \in I$. □

4. Non-trivial stationary solutions

In this section, we investigate a non-trivial stationary solution bifurcating from the trivial stationary solution $(R_i, R_e) = (0, 0)$ at $V_c = V_c^*$. For this purpose, we set up notation

$$X := H^1_{0l}(I) \times (H^1_0(I) \cap H^2(I)), \quad Y := L^2(I) \times L^2(I),$$
$$\boldsymbol{\rho} := (R_i, R_e) \in X, \quad U := \{ \boldsymbol{\rho} \in X\, ;\ \| \boldsymbol{\rho} \|_X < 1 \}, \quad J := (V_c^*/2, 2V_c^*)$$

and define the mapping $\boldsymbol{F} := (F_i, F_e) : X \times J \to Y$ as

$$\begin{aligned} F_i(\boldsymbol{\rho}, V_c) &:= k_i \partial_x \left\{ \left(\partial_x (V[R_i, R_e, V_c]) + \frac{V_c}{L} \right) R_i \right\} \\ &\qquad + k_i R_i - k_e h \left(\frac{V_c}{L} \right) e^{-\frac{L}{V_c} x - \frac{V_c}{2L} x} R_e - k_i f_i[R_i, R_e, V_c], \\ F_e(\boldsymbol{\rho}, V_c) &:= -k_e \partial_{xx} R_e - k_e g(V_c) R_e - k_e f_e[R_i, R_e, V_c]. \end{aligned}$$

Here $V[R_i, R_e, V_c]$, $f_i[R_i, R_e, V_c]$, $f_e[R_i, R_e, V_c]$, and V_c^* are defined in (2.1c), (2.2a), (2.2b), and (2.6), respectively. Note that $\boldsymbol{\rho} = (R_i, R_e)$ is a stationary solution to problem (2.1) for $V_c > 0$ if and only if $F(\boldsymbol{\rho}, V_c) = 0$.

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4.1. Bifurcation of non-trivial solutions

This subsection is devoted to showing Theorem 2.5 which ensures the bifurcation of non-trivial solution. The proof is based on the application of Crandall and Rabinowitz's Theorem to the mapping $\boldsymbol{F}$. For this theorem, we refer the reader to [2, Theorem 1.7] and [8, Theorem I.5.1].

Proof of Theorem 2.5. Let us check that the mapping $\boldsymbol{F}$ satisfies the assumptions of Crandall and Rabinowitz's Theorem. It is evident that

$$\boldsymbol{F} \in C^3(U \times J), \quad \boldsymbol{F}(0, V_c) = 0 \quad \text{for } V_c \in J, \tag{4.1}$$

$$D_{\boldsymbol{\rho}} F_i(0, V_c)[\boldsymbol{\rho}] = \frac{k_i V_c}{L} \partial_x R_i + k_i R_i - k_e h \left(\frac{V_c}{L} \right) e^{-\frac{L}{V_c}x - \frac{V_c}{2L}x} R_e,$$

$$D_{\boldsymbol{\rho}} F_e(0, V_c)[\boldsymbol{\rho}] = -k_e \partial_{xx} R_e - k_e g(V_c) R_e,$$

where $D_{\boldsymbol{\rho}}$ means the Fréchet derivative with respect to $\boldsymbol{\rho}$.

We denote by $N(D_{\boldsymbol{\rho}} \boldsymbol{F}(0, V_c^*))$ and $R(D_{\boldsymbol{\rho}} \boldsymbol{F}(0, V_c^*))$ the kernel and range of the linear operator $D_{\boldsymbol{\rho}} \boldsymbol{F}(0, V_c^*)$, respectively, and show

$$\dim N(D_{\boldsymbol{\rho}} \boldsymbol{F}(0, V_c^*)) = \operatorname{codim} R(D_{\boldsymbol{\rho}} \boldsymbol{F}(0, V_c^*)) = 1. \tag{4.2}$$

From a standard theory of ordinary differential equations, we see that solutions to the scalar equation $D_{\boldsymbol{\rho}} F_e(0, V_c^*)[\boldsymbol{\rho}] = 0$ are only $R_e = c\varphi_e$ for any $c \in \mathbb{R}$. Then the scalar equation $D_{\boldsymbol{\rho}} F_i(0, V_c^*)[\boldsymbol{\rho}] = 0$ with $R_e = c\varphi_e$ has a unique solution $R_i = c\varphi_i$. Here φ_e and φ_i are defined in Theorem 2.5. Hence, it holds that

$$N(D_{\boldsymbol{\rho}} \boldsymbol{F}(0, V_c^*)) = \{(c\varphi_i, c\varphi_e) \in X \,;\, c \in \mathbb{R}\}. \tag{4.3}$$

On the other hand, Fredholm's alternative theorem immediately ensures $R(D_{\boldsymbol{\rho}} F_e(0, V_c^*)[\boldsymbol{\rho}]) = \{\psi_e \in L^2(I) \,;\, \langle \psi_e, \varphi_e \rangle = 0\}$. Letting R_e be a solution to the scalar equation $D_{\boldsymbol{\rho}} F_e(0, V_c^*)[\boldsymbol{\rho}] = \psi_e \in R(D_{\boldsymbol{\rho}} F_e(0, V_c^*)[\boldsymbol{\rho}])$ and

$$R_i := \frac{L}{k_i V_c^*} e^{-\frac{L}{V_c^*}x} \int_0^x e^{\frac{L}{V_c^*}y} \psi_i(y) + k_e h \left(\frac{V_c^*}{L} \right) e^{-\frac{V_c^*}{2L}y} R_e(y)\, \mathrm{d}y \quad \text{for any } \psi_i \in L^2(I),$$

we see $D_{\boldsymbol{\rho}} F_i(0, V_c^*)[(R_i, R_e)] = \psi_i$ and thus

$$R(D_{\boldsymbol{\rho}} \boldsymbol{F}(0, V_c^*)) = \{(\psi_i, \psi_e) \in Y \,;\, \langle \psi_e, \varphi_e \rangle = 0\}. \tag{4.4}$$

This together with (4.3) concludes (4.2).

One can also see

$$D_{\boldsymbol{\rho} V_c} \boldsymbol{F}(0, V_c^*)[\boldsymbol{\varphi}] \notin R(D_{\boldsymbol{\rho}} \boldsymbol{F}(0, V_c^*)), \quad \boldsymbol{\varphi} := (\varphi_i, \varphi_e) \tag{4.5}$$

since it holds that

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$$\langle D_{\boldsymbol{\rho} V_c} F_e(0, V_c^*)[\boldsymbol{\varphi}], \varphi_e\rangle = -k_e g'(V_c^*)\langle \varphi_e, \varphi_e\rangle < 0. \tag{4.6}$$

Now Crandall and Rabinowitz's Theorem is applicable to the mapping $\boldsymbol{F}$ with aid of (4.1), (4.2), and (4.5). Then we conclude that there exist $\eta > 0$, $V_c \in C^2([-\eta, \eta]; \mathbb{R})$, and $\boldsymbol{z} \in C^2([-\eta, \eta]; X)$ such that $V_c(0) = V_c^*$, $z(0) = 0$, and the equation $\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s)) = 0$ has a non-trivial solution $\boldsymbol{\rho}(s) = (R_i(s), R_e(s)) = s(\varphi_i, \varphi_e) + sz(s)$ for $s \in [-\eta, \eta]$.

It is left to prove that $\dot{V}_c(0) \lesseqgtr 0$ holds if and only if (2.7) holds. Differentiating $F_e(\boldsymbol{\rho}(s), V_c(s)) = 0$ twice with respect to s and evaluating the result at $s = 0$, we obtain

$$\begin{aligned} &D_{\boldsymbol{\rho}\boldsymbol{\rho}} F_e(0, V_c^*)[\dot{\boldsymbol{\rho}}(0), \dot{\boldsymbol{\rho}}(0)] + D_{\boldsymbol{\rho}} F_e(0, V_c^*)[\ddot{\boldsymbol{\rho}}(0)] + 2D_{\boldsymbol{\rho} V_c} F_e(0, V_c^*)[\dot{\boldsymbol{\rho}}(0)]\dot{V}_c(0) \\ &\quad + D_{V_c V_c} F_e(0, V_c^*)(\dot{V}_c(0))^2 + D_{V_c} F_e(0, V_c^*)\ddot{V}_c(0) = 0. \end{aligned} \tag{4.7}$$

Here $D_{\boldsymbol{\rho}\boldsymbol{\rho}}$, $D_{\boldsymbol{\rho} V_c}$ $D_{V_c V_c}$, and D_{V_c} mean the Fréchet derivatives with respect to $\boldsymbol{\rho}$ and V_c. Furthermore, "$\dot{}$" and "$\ddot{}$" denote the derivatives with respect to s. It is easy to check $\dot{\boldsymbol{\rho}}(0) = \boldsymbol{\varphi}$ and $D_{V_c} F_e(0, V_c^*) = D_{V_c V_c} F_e(0, V_c^*) = 0$. Substituting these two equalities into (4.7) leads to

$$D_{\boldsymbol{\rho}\boldsymbol{\rho}} F_e(0, V_c^*)[\boldsymbol{\varphi}, \boldsymbol{\varphi}] + D_{\boldsymbol{\rho}} F_e(0, V_c^*)[\ddot{\boldsymbol{\rho}}(0)] + 2D_{\boldsymbol{\rho} V_c} F_e(0, V_c^*)[\boldsymbol{\varphi}]\dot{V}_c(0) = 0.$$

Then taking the L^2-inner product of this equation with φ_e and using (4.4), we have

$$\dot{V}_c(0) = \frac{k_e^{-1}\langle D_{\boldsymbol{\rho}\boldsymbol{\rho}} F_e(0, V_c^*)[\boldsymbol{\varphi}, \boldsymbol{\varphi}], \varphi_e\rangle}{-2k_e^{-1}\langle D_{\boldsymbol{\rho} V_c} F_e(0, V_c^*)[\boldsymbol{\varphi}], \varphi_e\rangle}, \tag{4.8}$$

where the denominator of the right hand side is positive owing to (4.6). We also see from integration by parts and $(V_c^*/2L) - h'(V_c^*/L) = -Lg'(V_c^*)$ that the numerator of the right hand side equals to the left hand side of (2.7). Hence, $\dot{V}_c(0) \lesseqgtr 0$ holds if and only if (2.7) holds. □

4.2. *Linear stability and instability of positive non-trivial solutions*

This subsection is devoted to the proof of Corollary 2.6. Let us denote $\boldsymbol{\rho}(s) = (R_i(s), R_e(s)) = s\boldsymbol{\varphi} + s\boldsymbol{z}(s)$ by the non-trivial solution in Theorem 2.5.

Proof of Corollary 2.6. We discuss only in the case $\dot{V}_c(0) > 0$ and $s > 0$, since the other cases are shown similarly. We can assume by taking $\theta_1 > 0$ small enough that $\boldsymbol{z}(s) = (z_i(s), z_e(s))$, $(R_i(s), R_e(s))$, and $V_c(s)$ satisfy

$$\begin{aligned} &\sup_{s\in[-\theta_1,\theta_1]} (|z_e(s)|_\infty + |\partial_x z_e(s)|_\infty + |\partial_x(V[R_i(s), R_e(s), V_c(s)])|_\infty) \\ &\quad \le \min\left\{\frac{\sqrt{2}}{4}, \frac{\sqrt{2}\pi}{4L}, \frac{V_c^*}{4L}\right\}, \end{aligned} \tag{4.9}$$

$$\inf_{s\in[-\theta_1,\theta_1]} V_c(s) \ge \frac{V_c^*}{2}, \tag{4.10}$$

and the bounded linear operator $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s)) + \boldsymbol{I}$ has a bounded inverse for $s \in [-\theta_1, \theta_1]$, where $\boldsymbol{I}$ denotes the identity operator on X.

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The proof of (2.8) is completed by showing $R_i(s)$, $R_e(s) > 0$. Let us first prove $R_e(s) > 0$. It is straightforward to see from (4.9) that

$$\begin{aligned}
&R_e(s,x) = s\varphi_e(x) + s z_e(s,x) \geq s(\sqrt{2}/2 - \sqrt{2}/4) > 0 \\
&\quad \text{for } x \in [L/4, 3L/4], \\
&\partial_x R_e(s,x) = s\partial_x\varphi_e(x) + s\partial_x z_e(s,x) \geq s(\sqrt{2}\pi/2L - \sqrt{2}\pi/4L) > 0 \\
&\quad \text{for } x \in (0, L/4), \\
&\partial_x R_e(s,x) = s\partial_x\varphi_e(x) + s\partial_x z_e(s,x) \leq s(-\sqrt{2}\pi/2L + \sqrt{2}\pi/4L) < 0 \\
&\quad \text{for } x \in (3L/4, L).
\end{aligned}$$

These inequalities together with the boundary condition (2.1e) leads to $R_e(s,x) > 0$ for $x \in I$. We next show $R_i(s) > 0$. It is seen from $F_i(\boldsymbol{\rho}(s), V_c(s)) = 0$ that

$$\begin{aligned}
R_i(s,x) = &\frac{k_e}{k_i} e^{-\frac{L}{V_c(s)}x} \left(\partial_x(V[R_i(s), R_e(s), V_c(s)]) + \frac{V_c(s)}{L}\right)^{-1} \\
&\times \int_0^x h\left(\partial_x(V[R_i(s), R_e(s), V_c(s)]) + \frac{V_c(s)}{L}\right) e^{-\frac{V_c(s)}{2L}y} R_e(s,y)\,\mathrm{d}y.
\end{aligned}$$

Furthermore, (4.9) and (4.10) yields $\partial_x(V[R_i(s), R_e(s), V_c(s)]) + V_c(s)/L > 0$. Hence, $R_i(s,x) > 0$ holds for $x \in I$.

It remains to prove that this non-trivial solution is linearly stable. We begin by analyzing the perturbation of the critical zero eigenvalue of $D_{\boldsymbol{\rho}}\boldsymbol{F}(0, V_c^*)$ to an eigenvalue of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}, V_c(s))$. It is obvious that $(\boldsymbol{\rho}(0), V_c(0)) = (0, V_c^*)$ and $\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s)) = 0$. We also see from (4.3) and (4.4) that the zero eigenvalue of $D_{\boldsymbol{\rho}}\boldsymbol{F}(0, V_c^*)$ is simple. Then Corollary 1.13 in [3] (see also [8, Theorem I.7.2]) ensures that there exist $\theta_2 > 0$, $\mu \in C^1([-\theta_2, \theta_2]; \mathbb{R})$, and $\boldsymbol{w} = (w_i, w_e) \in C^1([-\theta_2, \theta_2]; X)$ such that $\mu(0) = 0$, $\boldsymbol{w}(0) = 0$, and $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))[\boldsymbol{\varphi} + \boldsymbol{w}(s)] = \mu(s)(\boldsymbol{\varphi} + \boldsymbol{w}(s))$. Differentiate this equation with respect to s, evaluate the result at $s = 0$, and use $\dot{\boldsymbol{\rho}}(0) = \boldsymbol{\varphi}$ to get

$$D_{\boldsymbol{\rho}\boldsymbol{\rho}}F_e(0, V_c^*)[\boldsymbol{\varphi}, \boldsymbol{\varphi}] + D_{\boldsymbol{\rho}V_c}F_e(0, V_c^*)[\boldsymbol{\varphi}]\dot{V}_c(0) + D_{\boldsymbol{\rho}}F_e(0, V_c^*)[\dot{\boldsymbol{w}}(0)] = \dot{\mu}(0)\varphi_e.$$

Taking the L^2-inner product of the result with φ_e, and use (4.4) and (4.8) to obtain

$$\dot{\mu}(0)\|\varphi_e\|^2 = -\dot{V}_c(0)\langle D_{\boldsymbol{\rho}V_c}F_e(0, V_c^*)[\dot{\boldsymbol{\rho}}(0)], \varphi_e\rangle.$$

This together with $\dot{V}_c(0) > 0$ and (4.6) leads to $\dot{\mu}(s) > 0$. Hence, there exists $\theta_3 > 0$ such that $\mu(s) > 0$ holds for $s \in (0, \theta_3]$ owing to $\mu(0) = 0$ (for the case $\dot{V}_c(0) < 0$ and $s < 0$, the proof above gives $\mu(s) < 0$ which means that the positive non-trivial solution is linearly unstable).

We next prove that the real parts of all other eigenvalues of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$ are also positive. For this purpose, we denote $\boldsymbol{G}(s)$ by the inverse operator of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s)) + \boldsymbol{I}$ and show that the eigenvalues of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$ is continuous on $s \in [-\theta_1, \theta_1]$. The eigenvalue problem of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$ can be reduced to that of $\boldsymbol{G}(s)$, since $\boldsymbol{G}(s)[\boldsymbol{\psi}] = (\mu + 1)^{-1}\boldsymbol{\psi}$

holds if and only if $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))[\boldsymbol{\psi}] = \mu\boldsymbol{\psi}$. Note here that $\mu = -1$ is not an eigenvalue of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$. Therefore, it suffices to investigate $\boldsymbol{G}(s)$. It is straightforward to check $\boldsymbol{G}(s) \in C([-\theta_1, \theta_1]; \mathcal{K}(Y))$, where $\mathcal{K}(Y)$ denotes the space of the compact operators on Y. Then Kato's perturbation theory in [7] ensures that the eigenvalues of $\boldsymbol{G}(s)$ are continuous on $[-\theta_1, \theta_1]$ and so are those of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$. On the other hand, the nonzero eigenvalues of $D_{\boldsymbol{\rho}}\boldsymbol{F}(0, V_c^*)$ are only $(n^2 - 1)(\pi/L)^2$ for $n = 2, 3, 4, \ldots$. Consequently, by taking $\theta(\leq \min\{\theta_1, \theta_3\})$ sufficiently small, we deduce that the real parts of all eigenvalues of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$ are positive.

We complete the proof by showing that the spectrum of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$ consists of only eigenvalues above. It suffices to show that if μ is not an eigenvalue of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$, then μ belongs to the resolvent set. We first see from a standard theory of elliptic equations that $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$ is a closed operator. Let us next find the inverse of $(D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s)) - \mu I)$. We know that $G(s)$ defined above is a compact operator on Y and thus the spectrum of $G(s)$ consists of only eigenvalues and zero. By this fact, the inverse operator $\{G(s) - (\mu + 1)^{-1}I\}^{-1}$ is well-defined as a bounded operator on Y since μ is not an eigenvalue of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))^{-1}$, that is, $(\mu + 1)^{-1}$ is not an eigenvalue of $G(s)$. Using this operator, we can write explicitly the inverse of $(D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s)) - \mu I)$ as

$$(D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s)) - \mu I)^{-1} = \frac{-1}{\mu + 1}\left\{G(s) - \frac{1}{\mu + 1}I\right\}^{-1} G(s).$$

It is straightforward to check that this inverse is a bounded operator on Y. Therefore, we conclude that μ belongs the resolvent set of $D_{\boldsymbol{\rho}}\boldsymbol{F}(\boldsymbol{\rho}(s), V_c(s))$. The proof is complete. □

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Appendix A. Graph of g and well-definedness of V_c^*

Appendix A is devoted to studying the properties of the function g subject to the physical parameters a, b, and L. There exist two cases such as Figs. 3 and 4. For the first case, g has only one local minimum and only one global maximum. For the second case, g is strictly decreasing. Furthermore, we show that the sparking voltage is well-defined for suitable choice of physical parameters.

Let us explain why we have the above graphs. It is straightforward to check that $g'(V_c) \lesseqgtr 0$ holds for $V_c > 0$ if and only if $q(V_c) := -(bL/V_c) - 2\log V_c + \log(bL + V_c) + \log(2aL) \lesseqgtr 0$. By the first derivative test, we also see that $q(V_c)$ is strictly increasing on the interval $(0, (-1 + \sqrt{5})bL/2]$ and strictly decreasing on the interval $[(-1 + \sqrt{5})bL/2, \infty)$. This fact together with $g(0) = 0$ gives the graph in Fig. 3 if the maximum of $q(V_c)$ is positive, and the graph in Fig. 4 if the maximum is negative. It is also obvious from these two graphs that the sparking voltage defined in (2.6) is unique if it exists.

We discuss the well-definedness of sparking voltage provided the condition $ae^{-1} - 4^{-1}b > 0$ holds. From the equality $L^2 g(bL) = L^2 b(ae^{-1} - 4^{-1}b)$, it follows that $g(bL) > \pi^2/L^2$ holds if either L or a is large enough. Then the graph of the function g must be drown as in Fig. 5. Consequently, the sparking voltage is well-defined for some physical parameters.

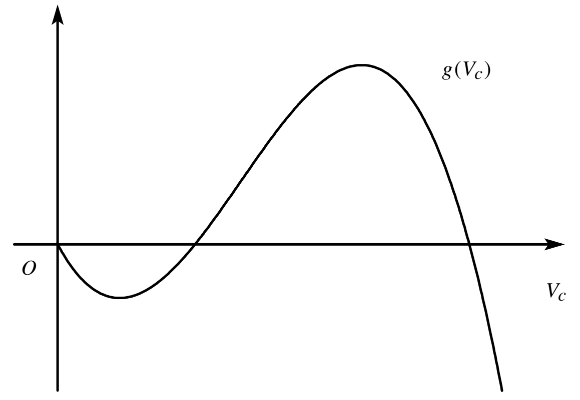

Fig. 3. Case 1.

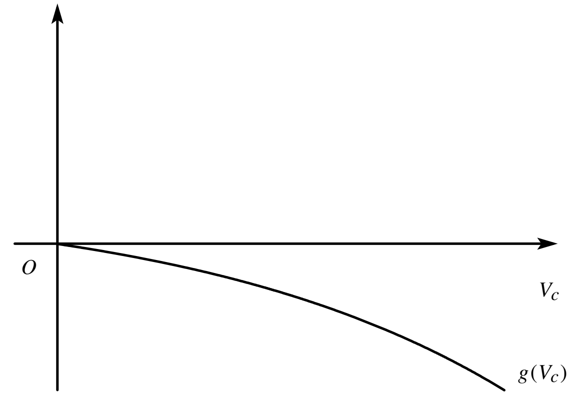

Fig. 4. Case 2.

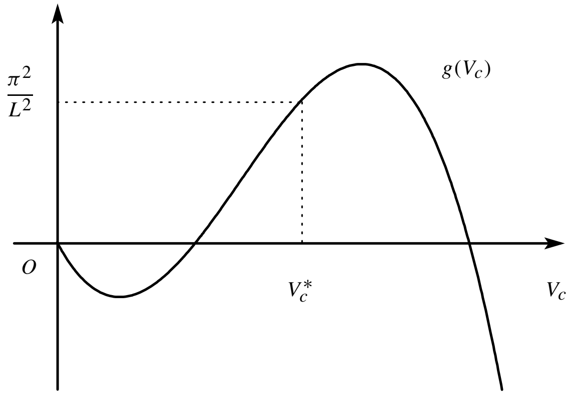

Fig. 5. Well-definedness of V_c^*.

Appendix B. Sign of $\dot{V}_c(0)$

In Appendix B we investigate the sign of $\dot{V}_c(0)$ being in Theorem 2.5. We emphasize once again that this sign is same as that of the quantity

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$$l := -2Lg'(V_c^*)\int_0^L \varphi_e^2 \partial_x (V[\varphi_i, \varphi_e, V_c^*])\,\mathrm{d}x - \int_0^L \varphi_e^2 \partial_{xx} (V[\varphi_i, \varphi_e, V_c^*])\,\mathrm{d}x.$$

First let us show $\dot{V}_c(0) > 0$ in the case that k_e/k_i is sufficiently small. The quantity l can be written as

$$\begin{aligned} l = &- 2Lg'(V_c^*)\int_0^L \varphi_e^2 \partial_x (V[\varphi_i, 0, V_c^*])\,\mathrm{d}x - \int_0^L \varphi_e^2 \partial_{xx} (V[\varphi_i, 0, V_c^*])\,\mathrm{d}x \\ &- 2Lg'(V_c^*)\int_0^L \varphi_e^2 \partial_x (V[0, \varphi_e, V_c^*])\,\mathrm{d}x - \int_0^L \varphi_e^2 \partial_{xx} (V[0, \varphi_e, V_c^*])\,\mathrm{d}x. \end{aligned}$$

The first and second terms of this right hand side can be taken arbitrarily small for suitably small k_e/k_i, since V_c^* defined in (2.6) is independent of k_i and k_e. On the other hand, the third and fourth terms are positive and independent of k_e and k_i. Hence, $\dot{V}_c(0) > 0$ holds provided that k_e/k_i is small enough.

Next, setting $L = 1$ and $a = 2\sqrt{e}b/3$, we prove $\dot{V}_c(0) < 0$ for suitable choice of b and k_e/k_i. It holds that

$$\begin{aligned} l\frac{k_i}{k_e} = &- 2g'(V_c^*)\int_0^1 \varphi_e^2 \partial_x (V[(k_i/k_e)\varphi_i, 0, V_c^*])\,\mathrm{d}x - \int_0^1 \varphi_e^2 \partial_{xx} (V[(k_i/k_e)\varphi_i, 0, V_c^*])\,\mathrm{d}x \\ &- 2\frac{k_i}{k_e}g'(V_c^*)\int_0^1 \varphi_e^2 \partial_x (V[0, \varphi_e, V_c^*])\,\mathrm{d}x - \frac{k_i}{k_e}\int_0^1 \varphi_e^2 \partial_{xx} (V[0, \varphi_e, V_c^*])\,\mathrm{d}x. \end{aligned}$$

If k_e/k_i is sufficiently large, one can make the third and forth terms of this right hand side arbitrarily small. We also notice that the first and second terms are independent of k_e and k_i. Therefore, the proof of $\dot{V}_c(0) < 0$ is completed by showing that the sum of the first and second terms is negative. For a moment, we assume that $|V_c^* - 2m|$ and $g'(V_c^*) > 0$ can be taken arbitrarily small, where

$$m := \sqrt{3}\pi.$$

Then the second term is uniformly negative with respect to V_c^* and the first term is arbitrarily small and thus negligible. Therefore, the sum of the first and second terms is negative.

What is left is to obtain the desired sparking voltage V_c^* satisfying the above assumption for some b. Note that V_c^* depends only on b since $L = 1$ and $a = 2\sqrt{e}b/3$. We define the functions $H_1(b, V_c) := g(V_c) - \pi^2$ and $H_2(b, V_c) := V_c - 2b$. Then it holds at the point $(b, V_c) = (m, 2m)$ that

$$H_1(m, 2m) = H_2(m, 2m) = 0,$$

$$H_{1b}(m, 2m) > 0, \quad H_{2b}(m, 2m) < 0, \quad H_{1V_c}(m, 2m) = 0, \quad H_{2V_c}(m, 2m) > 0,$$

since $H_2(b, V_c) = 0$ holds if and only if $g'(V_c) = 0$ and $g''(V_c) < 0$ hold. Therefore the equations $H_1(b, V_c) = 0$ and $H_2(b, V_c) = 0$ give two curves parameterized by b and crossing each other at the point $(m, 2m)$. The important point to note here is that $H_1 = 0$ and $H_2 = 0$ ensure the first condition in (2.6) and $g'(V_c) = 0$, respectively. Then it is possible to take a point (b, V_c^*) on the curve $H_1(b, V_c) = 0$ near the point $(m, 2m)$ so that $|V_c^* - 2m|$ and $g'(V_c^*) > 0$ are arbitrarily small. Consequently, we have the desired sparking voltage V_c^*.

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