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Prodi–Serrin condition for 3D Navier–Stokes equations via one directional derivative of velocity

Hui Chen [a], Wenjun Le [a], Chenyin Qian [b,*]

[a] School of Science, Zhejiang University of Science and Technology, Hangzhou, 310023, China
[b] Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, China

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Abstract

In this paper, we consider the conditional regularity of weak solution to the 3D Navier–Stokes equations. More precisely, we prove that if one directional derivative of velocity, say $\partial_3 \boldsymbol{u}$, satisfies $\partial_3 \boldsymbol{u} \in L^{p_0,1}(0,T;L^{q_0}(\mathbb{R}^3))$ with $\frac{2}{p_0}+\frac{3}{q_0}=2$ and $\frac{3}{2}<q_0<+\infty$, then the weak solution is regular on $(0,T]$. The proof is based on the new local energy estimates introduced by Chae-Wolf (2019) [4] and Wang-Wu-Zhang (2020) [21].

Keywords: Navier–Stokes equations; Regularity of weak solutions; Serrin-Prodi condition

1. Introduction

We consider the Cauchy problem for incompressible Navier–Stokes equations in $\mathbb{R}^3 \times (0,\infty)$.

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\pi = 0, \\ \nabla\cdot\boldsymbol{u} = 0\,, \\ u|_{t=0} = \boldsymbol{u}_0\,, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: chenhui@zust.edu.cn (H. Chen), le_wenjun@163.com (W. Le), qcyjcsx@163.com (C. Qian).

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where $\boldsymbol{u} = (u_1, u_2, u_3)$ and π stand for the velocity field and a scalar pressure of the viscous incompressible fluid, respectively.

For every $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ with div $\boldsymbol{u}_0 = 0$ in the sense of distribution, a global weak solution $\boldsymbol{u}$ to the Navier–Stokes equations (1.1), which satisfies the energy inequality

$$\|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \|\nabla\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}s \le \|\boldsymbol{u}_0\|^2_{L^2(\mathbb{R}^3)}, \qquad \text{for all } t>0, \tag{1.2}$$

was constructed by Leray [14] and Hopf [11]. However, the uniqueness and regularity of such weak solution is still one of the most challenging open problems in the field of mathematical fluid mechanics.

One essential work is usually referred as Prodi–Serrin (P–S) conditions (see [8,18–20] and the references therein.), i.e. if the weak solution $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1,\ 3\le q\le\infty, \tag{1.3}$$

then the weak solution is regular in $(0,T]$. Along with the pioneering works of Prodi and Serrin, Beirão da Veiga [1] established regularity criteria on the gradient of the velocity field, i.e.

$$\nabla\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2,\ \frac{3}{2}\le q\le\infty.$$

Later on, many efforts have been made to weakening the above criteria by imposing constraints only on partial components or directional derivatives of velocity field.

There are several notable results [5,6,10] based on one component of the velocity. For instance, B. Han etc. [10] proved that if $u_3 \in L^p(0,T;\dot{H}^{1/2+2/p}(\mathbb{R}^3))$ with $2\le p<+\infty$, the solution $\boldsymbol{u}$ is regular in $(0,T]$. Very recently, D. Chae and J. Wolf [4] made an important progress and obtained the regularity of solution to (1.1) under the condition

$$u_3 \in L^p\left(0,T;L^q\left(\mathbb{R}^3\right)\right), \quad \frac{2}{p}+\frac{3}{q}<1, \quad 3<q\le\infty. \tag{1.4}$$

W. Wang, D. Wu and Z. Zhang [21] improved to

$$u_3 \in L^{p,1}\left(0,T;L^q\left(\mathbb{R}^3\right)\right), \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3<q<\infty. \tag{1.5}$$

Throughout this paper, $L^{p,1}$ denotes the Lorentz space with respect to the time variable.

For the regularity criteria only involving one directional derivative of velocity, I. Kukavica and M. Zaine [12] get the result

$$\partial_3\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \frac{2}{p}+\frac{3}{q}=2, \quad \frac{9}{4}\le q\le 3.$$

There are also many efforts to extend the range of q for $\partial_3\boldsymbol{u}$, such as [3,17,12,22]. In particular, the first author of this paper and D. Fang and T. Zhang [7] proved that $\boldsymbol{u}$ is regular in $(0,T]$, if

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)),\ \frac{2}{p}+\frac{3}{q}=2,\ \frac{3}{2}<q\le 6.$$

Along this line of research, we obtain scaling invariant Prodi-Serrin criteria for $\partial_3 u$ with optimal range $\frac{3}{2}<q<\infty$. More precisely, we prove the following theorem:

Theorem 1.1. *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)\cap L^3(\mathbb{R}^3)$ and $(\boldsymbol{u},\pi)$ be a Leray-Hopf weak solution to the 3D Navier–Stokes equations (1.1). If $\boldsymbol{u}$ satisfies*

$$\partial_3 \boldsymbol{u} \in L^{p_0,1}(0,T;L^{q_0}(\mathbb{R}^3)),\ \frac{2}{p_0}+\frac{3}{q_0}=2,\ \frac{3}{2}<q_0<+\infty, \tag{1.6}$$

or

$$\partial_3 \boldsymbol{u} \in L^{p_0,1}(0,T;L^{\infty}(\mathbb{R}^3)),\ p_0>1, \tag{1.7}$$

then $\boldsymbol{u}$ is regular in $\mathbb{R}^3\times(0,T]$.

Remark 1.2. Theorem 1.1 is a direct consequence of Theorem 1.3 and Remark 1.4. Moreover, the initial data $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)\cap L^3(\mathbb{R}^3)$ implies the local-in-time regularity of weak solution, thus the weak solution is actually suitable weak solution.

A particular class of weak solution to (1.1) called *suitable weak solution* is introduced by Caffarelli, Kohn and Nirenberg [2]. We say that $(\boldsymbol{u},\pi)$ is a suitable weak solution of (1.1) in open domain $\Omega_T=\Omega\times(-T,0)$, if

(1) $\boldsymbol{u}\in L^\infty\left(-T,0;L^2(\Omega)\right)\cap L^2\left(-T,0;H^1(\Omega)\right)$ and $\pi\in L^{\frac{3}{2}}(\Omega_T)$;
(2) (1.1) is satisfied in the sense of distribution;
(3) the local energy inequality holds: for any nonnegative test function $\varphi\in C_c^\infty(\Omega_T)$ and $t\in(-T,0)$,

$$\begin{aligned}&\int_\Omega |\boldsymbol{u}(x,t)|^2\varphi\,\mathrm{d}x+2\int_{-T}^{t}\int_\Omega |\nabla\boldsymbol{u}|^2\varphi\,\mathrm{d}x\mathrm{d}s\\ &\le\int_{-T}^{t}\int_\Omega |\boldsymbol{u}|^2(\partial_s\varphi+\Delta\varphi)+\boldsymbol{u}\cdot\nabla\varphi\left(|\boldsymbol{u}|^2+2\pi\right)\,\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{1.8}$$

The important progress in [2] is that one-dimensional Hausdorff measure of the possible space-time singular points set for the suitable weak solution $(\boldsymbol{u},\pi)$ is zero. A simple proof is also given by F. Lin in [15].

Theorem 1.3. *Let $(\boldsymbol{u},\pi)$ be a suitable weak solution of (1.1) in $\mathbb{R}^3\times(-1,0)$. If $\boldsymbol{u}$ satisfies*

$$\partial_3 \boldsymbol{u} \in L^{p_0,1}\left(-1,0;L^{q_0}(B(2))\right),\ \frac{2}{p_0}+\frac{3}{q_0}=2,\ \frac{3}{2}<q_0<+\infty, \tag{1.9}$$

then for $0 < r \leq 1$,

$$r^{-2}\|\boldsymbol{u}\|^3_{L^3\left(B(r)\times(-r^2,0)\right)} \leq C. \tag{1.10}$$

Furthermore $\boldsymbol{u}$ *is regular at* $(0,0)$. *Here* $B(R)$ *is the ball in* $\mathbb{R}^3$ *with center at the origin and radius* R.

Remark 1.4. Our method fails in the case $q_0 = +\infty$. However, replacing (1.9) with the subcritical regularity criteria

$$\partial_3 \boldsymbol{u} \in L^{p_0}\left(-1,0; L^\infty(B(2))\right),\ p_0 > 1, \tag{1.11}$$

Theorem 1.3 still holds true. Actually, we can pick $1 < p_1 < p_0$ and $\frac{3}{2} < q_1 < +\infty$ such that $\frac{2}{p_1} + \frac{3}{q_1} = 2$. Therefore, we can prove it directly by the embedding inequality

$$\|\partial_3 \boldsymbol{u}\|_{L^{p_1,1}(-1,0;L^{q_1}(B(2)))} \leq C\, \|\partial_3 \boldsymbol{u}\|_{L^{p_0}((-1,0;L^\infty(B(2)))}\,.$$

The proof of Theorem 1.3 follows from the ideal of local energy estimates introduced in [2,4,21]. The non-trivial part is to establish the necessary *a priori* estimates of the quantity $J = \int_{-1}^{t}\int_{U_0(R)} \pi_0 u_3 \cdot \partial_3\left(\Phi_n \eta \psi\right) \mathrm{d}x\mathrm{d}s$ involving the "non-harmonic" part of the pressure π_0. The difficulty will be overcome by introducing the mean value function $\overline{(u_3)}_k$ defined in (3.6). More precisely, we adopt a new decomposition of the quantity J in (3.17) and (3.19). We should point out that the energy inequality in Lemma 2.1 and Poincaré's inequality in Lemma A.3 are crucial to deduce the estimates.

Our paper is organized as follows: in Section 2, we recall some notations and preliminary results; we establish the *a priori* estimates involving the convection term in Section 3.1 and the pressure term in Section 3.2; finally, we will complete the proof in Section 3.3.

2. Notations and preliminary

In this preparation section, we recall some usual notations and preliminary results.

For two comparable quantities, the inequality $X \lesssim Y$ stands for $X \leq CY$ for some positive constant C. The dependence of the constant C on other parameters or constants are usually clear from the context, and we will often suppress this dependence.

We also shall use the same notation as that in Chae-Wolf [4]. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we denote $x' = (x_1, x_2)$ the horizontal variable.

For $0 < R < +\infty$, we denote $B(R)$ the ball in $\mathbb{R}^3$ with center at origin and radius R, and $B'(R) \subset \mathbb{R}^2$ the ball in the horizontal plane with center at origin and radius R.

We set the spatial cylinder

$$U_n(R) \overset{\text{def}}{=} B'(R) \times (-r_n, r_n),$$

the parabolic cylinder

$$Q_n(R) \overset{\text{def}}{=} U_n(R) \times (-r_n^2, 0),$$

and

$$A_n(R) \overset{\text{def}}{=} Q_n(R) \backslash Q_{n+1}(R),$$

where $r_n = 2^{-n},\ n \in \mathbb{N}$.

We take Φ_n the fundamental solution of the backward heat equation

$$\partial_t \Phi_n + \partial_3^2 \Phi_n = 0,$$

with singularity at $(0, r_n^2)$. More explicitly, we consider Φ_n given by

$$\Phi_n(x,t) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi(-t+r_n^2)}} e^{-\frac{x_3^2}{4(-t+r_n^2)}}, \quad (x,t) \in \mathbb{R}^3 \times (-\infty, 0).$$

There exist absolute constants $c_1, c_2 > 0$ such that for $j = 1, \ldots, n-1$, it holds

$$\begin{aligned} \Phi_n &\le c_2 r_j^{-1}, & |\partial_3 \Phi_n| &\le c_2 r_j^{-2}, & \text{in} \quad & A_j(R), \\ c_1 r_n^{-1} \le \Phi_n &\le c_2 r_n^{-1}, & c_1 r_n^{-2} \le |\partial_3 \Phi_n| &\le c_2 r_n^{-2}, & \text{in} \quad & Q_n(R). \end{aligned} \tag{2.1}$$

We denote the energy

$$E_n(R) \overset{\text{def}}{=} \sup_{t \in (-r_n^2, 0)} \int_{U_n(R)} |\boldsymbol{u}(t)|^2 \mathrm{d}x + \int_{-r_n^2}^{0} \int_{U_n(R)} |\nabla \boldsymbol{u}|^2 \, \mathrm{d}x \mathrm{d}s,$$

$$\mathcal{E} \overset{\text{def}}{=} \sup_{t \in (-1,0)} \int_{\mathbb{R}^3} |\boldsymbol{u}(t)|^2 \, \mathrm{d}x + \int_{-1}^{0} \int_{\mathbb{R}^3} |\nabla \boldsymbol{u}|^2 \, \mathrm{d}x \mathrm{d}s.$$

Next, we denote $L_v^{q_v} L_h^{q_h} \big(B'(R) \times (-r, r)\big)$ the anisotropic Lebesgue space equipped with the norm

$$\|h(x)\|_{L_v^{q_v} L_h^{q_h}(B'(R) \times (-r,r))} = \left(\int_{-r}^{r} \|h(\cdot, x_3)\|_{L^{q_h}(B'(R))}^{q_v} \, \mathrm{d}x_3 \right)^{\frac{1}{q_v}}. \tag{2.2}$$

The following lemma ensures the anisotropic Lebesgue space obeys the energy estimates in [4, Lemma 3.1].

Lemma 2.1. *Let $R \ge \frac{1}{2}$. For $\forall\ 2 \le p \le \infty$, $2 \le q_h, q_v < +\infty$, $\frac{2}{p} + \frac{2}{q_h} + \frac{1}{q_v} = \frac{3}{2}$, we have*

$$\|\boldsymbol{u}\|_{L^p\left(-r_n^2, 0; L_v^{q_v} L_h^{q_h}(U_n(R))\right)}^2 \le C E_n(R). \tag{2.3}$$

Proof. For $0 < r \le 1$, set $U = B'(R) \times (-r, r)$, $U_1 = B'(R) \times (-2r, 2r)$, $U_2 = B'(R) \times (-4R, 4R)$ and $U_3 = B'(1) \times (-4, 4)$.

We claim that

$$\|\boldsymbol{u}\|_{L_v^{q_v} L_h^{q_h}(U)} \le C r^{-\frac{2}{p}} \|\boldsymbol{u}\|_{L^2(U)} + C \|\boldsymbol{u}\|_{L^2(U)}^{1-\frac{2}{p}} \|\nabla \boldsymbol{u}\|_{L^2(U)}^{\frac{2}{p}}. \tag{2.4}$$

In fact, given $\boldsymbol{u} \in W^{1,2}(U)$, we define the extension

$$\boldsymbol{v}_1(x) := \begin{cases} \boldsymbol{u}(x', x_3) & \text{if} \quad x_3 \in (-r, r), \\ \boldsymbol{u}\left(x', 2r - x_3\right) & \text{if} \quad x_3 \in [r, 2r), \\ \boldsymbol{u}\left(x', -2r - x_3\right) & \text{if} \quad x_3 \in (-2r, -r]. \end{cases}$$

Then $\boldsymbol{v}_1 \in W^{1,2}(U_1)$ and it holds

$$\|\nabla \boldsymbol{v}_1\|_{L^2(U_1)} \le 3 \|\nabla \boldsymbol{u}\|_{L^2(U)}, \quad \|\boldsymbol{v}_1\|_{L^2(U_1)} \le 3 \|\boldsymbol{u}\|_{L^2(U)}. \tag{2.5}$$

Let $\zeta \in C_c^\infty(-2r, 2r)$ denote a cut off function such that $\zeta = 1$ on $(-r, r)$ and $\left|\zeta'\right| \le 2r^{-1}$. Noting that by $r \le 2R$, it holds $U_1 \subset U_2$ and $\boldsymbol{v}_2 = \boldsymbol{v}_1 \zeta \in W^{1,2}(U_2)$.

Set $\boldsymbol{v}_3(x) = \boldsymbol{v}_2(Rx),\ x \in U_3$. There is an Sobolev extension $\tilde{\boldsymbol{v}}_3(x) \in W^{1,2}(\mathbb{R}^3)$ with $\tilde{\boldsymbol{v}}_3(x) = \boldsymbol{v}_3(x),\ x \in U_3$ and

$$\|\tilde{\boldsymbol{v}}_3\|_{L^2(\mathbb{R}^3)} \le C \|\boldsymbol{v}_3\|_{L^2(U_3)}, \quad \|\nabla \tilde{\boldsymbol{v}}_3\|_{L^2(\mathbb{R}^3)} \le C \|\boldsymbol{v}_3\|_{W^{1,2}(U_3)}. \tag{2.6}$$

By means of Sobolev's inequality and a simple scaling argument along with (2.5) and (2.6), we get

$$\begin{aligned} \|\boldsymbol{u}\|_{L_v^{q_v} L_h^{q_h}(U)} &\le \|\boldsymbol{v}_2\|_{L_v^{q_v} L_h^{q_h}(U_2)} \\ &\le R^{\frac{2}{q_h}+\frac{1}{q_v}} \|\boldsymbol{v}_3\|_{L_v^{q_v} L_h^{q_h}(U_3)} \\ &\le C R^{\frac{2}{q_h}+\frac{1}{q_v}} \|\tilde{\boldsymbol{v}}_3\|_{L_v^{q_v} L_h^{q_h}(\mathbb{R}^3)} \\ &\le C R^{\frac{2}{q_h}+\frac{1}{q_v}} \|\tilde{\boldsymbol{v}}_3\|_{L^2(\mathbb{R}^3)}^{\frac{2}{q_h}+\frac{1}{q_v}-\frac{1}{2}} \|\nabla \tilde{\boldsymbol{v}}_3\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}-\frac{2}{q_h}-\frac{1}{q_v}} \\ &\le C R^{\frac{2}{q_h}+\frac{1}{q_v}} \left(\|\boldsymbol{v}_3\|_{L^2(U_3)} + \|\boldsymbol{v}_3\|_{L^2(U_3)}^{\frac{2}{q_h}+\frac{1}{q_v}-\frac{1}{2}} \|\nabla \boldsymbol{v}_3\|_{L^2(U_3)}^{\frac{3}{2}-\frac{2}{q_h}-\frac{1}{q_v}} \right) \\ &\le C \|\boldsymbol{v}_2\|_{L^2(U_2)} + C \|\boldsymbol{v}_2\|_{L^2(U_2)}^{\frac{2}{q_h}+\frac{1}{q_v}-\frac{1}{2}} \|\nabla \boldsymbol{v}_2\|_{L^2(U_2)}^{\frac{3}{2}-\frac{2}{q_h}-\frac{1}{q_v}} \\ &\le C r^{-\frac{3}{2}+\frac{2}{q_h}+\frac{1}{q_v}} \|\boldsymbol{u}\|_{L^2(U)} + C \|\boldsymbol{u}\|_{L^2(U)}^{\frac{2}{q_h}+\frac{1}{q_v}-\frac{1}{2}} \|\nabla \boldsymbol{u}\|_{L^2(U)}^{\frac{3}{2}-\frac{2}{q_h}-\frac{1}{q_v}}, \end{aligned}$$

which gives rise to (2.4). By using of Hölder's inequality and (2.4) with $r = r_n$, we achieve

$$\|\boldsymbol{u}\|^2_{L^p\left(-r_n^2, 0; L_v^{q_v} L_h^{q_h}(U_n(R))\right)} \le C \|\boldsymbol{u}\|^2_{L^\infty(-r_n^2, 0; L^2(U_n))} + C \|\nabla \boldsymbol{u}\|^2_{L^2\left(-r_n^2, 0; L^2(U_n)\right)}.$$

This completes the proof of this lemma. $\square$

3. Proof of the main results

In this section, we apply a similar argument in Cafferalli-Kohn-Nirenberg [2], Chae-Wolf [4], or Wang-Wu-Zhang [21] to prove Theorem 1.3.

We assume that solution $\boldsymbol{u}$ satisfies

$$\partial_3 \boldsymbol{u} \in L^{p_0,1}\left(-1,0; L^{q_0}(B(2))\right), \ \frac{2}{p_0}+\frac{3}{q_0}=2, \ \frac{3}{2}<q_0<+\infty. \tag{3.1}$$

For fixed p_0, q_0, we can pick $\frac{4q_0}{4q_0-5} < p_0^* < p_0$. Set

$$\mathscr{B}_i = r_i^{2-\frac{2}{p_0^*}-\frac{3}{q_0}} \|\partial_3 \boldsymbol{u}\|_{L^{p_0^*}\left(-r_i^2,0;L^{q_0}(B(2))\right)}. \tag{3.2}$$

By Lemma A.5, we have

$$\sum_{i=0}^{+\infty} \mathscr{B}_i \le C \|\partial_3 \boldsymbol{u}\|_{L^{p_0,1}(-1,0;L^{q_0}(B(2)))}. \tag{3.3}$$

Let $\eta(x_3,t) \in C_c^\infty((-1,1)\times(-1,0])$ denote a cut-off function, $0\le\eta\le 1$, and $\eta=1$ on $\left(-\frac{1}{2},\frac{1}{2}\right)\times\left(-\frac{1}{4},0\right)$.

In addition, let $\frac{1}{2}\le\rho<R\le 1$ be arbitrarily chosen, but $|R-\rho|\le\frac{1}{2}$. Let $\psi=\psi\left(x'\right)\in C^\infty\left(\mathbb{R}^2\right)$ with $0\le\psi\le 1$ in $B'(R)$ satisfying

$$\psi(x')=\psi(|x'|)=\begin{cases} 1 \text{ in } B'(\rho) \\ 0 \text{ in } \mathbb{R}^2\backslash B'\left(\frac{R+\rho}{2}\right) \end{cases} \tag{3.4}$$

and

$$|D\psi|\le\frac{C}{R-\rho}, \quad \left|D^2\psi\right|\le\frac{C}{(R-\rho)^2}.$$

For $j=0,1,\cdots,n$, denote $\chi_j=\chi_{B'(R)}(x')\cdot\eta(2^j\cdot x_3, 2^{2j}\cdot t)$, where $\chi_{B'(R)}$ is the indicator function of the set $B'(R)$. Let

$$\phi_j=\begin{cases} \chi_j-\chi_{j+1}, & \text{if} \quad j=0,\dots,n-1, \\ \chi_n, & \text{if} \quad j=n, \end{cases} \tag{3.5}$$

and the mean value function

$$\overline{(u_3)}_k(x')=\frac{1}{2r_k}\int_{-r_k}^{r_k} u_3(x',\omega)\,\mathrm{d}\omega. \tag{3.6}$$

Taking the test function $\varphi = \Phi_n \eta \psi$ in (1.8), it yields that

$$\begin{aligned}&\int_{U_0(R)} |\boldsymbol{u}(x,t)|^2 \Phi_n \eta \psi \, \mathrm{d}x + 2 \int_{-1}^{t} \int_{U_0(R)} |\nabla \boldsymbol{u}|^2 \Phi_n \eta \psi \, \mathrm{d}x \mathrm{d}s \\ \leq & \int_{-1}^{t} \int_{U_0(R)} |\boldsymbol{u}|^2 (\partial_s + \Delta)(\Phi_n \eta \psi) \, \mathrm{d}x \mathrm{d}s + \int_{-1}^{t} \int_{U_0(R)} |\boldsymbol{u}|^2 \boldsymbol{u} \cdot \nabla (\Phi_n \eta \psi) \, \mathrm{d}x \mathrm{d}s \\ & + 2 \int_{-1}^{t} \int_{U_0(R)} \pi \boldsymbol{u} \cdot \nabla (\Phi_n \eta \psi) \, \mathrm{d}x \mathrm{d}s. \end{aligned} \tag{3.7}$$

Next, we shall handle the right side of (3.7) term by term.

3.1. Estimates for nonlinear terms

Lemma 3.1. *Let $(\boldsymbol{u}, \pi)$ be a suitable weak solution of (1.1) in $\mathbb{R}^3 \times (-1, 0)$. If $(\boldsymbol{u}, \pi)$ satisfies the assumption of Theorem 1.3. Then we have*

$$\int_{-1}^{t} \int_{U_0(R)} |\boldsymbol{u}|^2 (\partial_s + \Delta)(\Phi_n \eta \psi) \, \mathrm{d}x \mathrm{d}s \leq C \frac{\mathcal{E}}{(R-\rho)^2}. \tag{3.8}$$

Proof. The proof of this lemma is similar to Lemma 3.1 in [21], referring to the properties of Φ_n, η and ψ. We omit it here. □

Lemma 3.2. *Under the assumption of Lemma 3.1, we have*

$$\begin{aligned}&\int_{-1}^{t} \int_{U_0(R)} |\boldsymbol{u}|^2 \boldsymbol{u} \cdot \nabla (\Phi_n \eta \psi) \, \mathrm{d}x \mathrm{d}s \\ \leq & C \sum_{i=0}^{n} \mathscr{B}_i \left(r_i^{-1} E_i(R) \right) + C (R-\rho)^{-1} \mathcal{E}^{\frac{1}{2}} \sum_{i=0}^{n} r_i^{\frac{1}{2}} \left(r_i^{-1} E_i(R) \right), \end{aligned} \tag{3.9}$$

where $\mathscr{B}_i$ is defined in (3.2).

Proof. We first note that

$$\begin{aligned}&\int_{-1}^{t} \int_{U_0(R)} |\boldsymbol{u}|^2 \boldsymbol{u} \cdot \nabla (\Phi_n \eta \psi) \, \mathrm{d}x \mathrm{d}s \\ = & \int_{-1}^{t} \int_{U_0(R)} |\boldsymbol{u}|^2 u_3 \cdot \partial_3 (\Phi_n \eta) \psi \, \mathrm{d}x \mathrm{d}s + \sum_{\mu=1,2} \int_{-1}^{t} \int_{U_0(R)} |\boldsymbol{u}|^2 u_\mu \cdot \Phi_n \eta \partial_\mu \psi \, \mathrm{d}x \mathrm{d}s \end{aligned}$$

$$\stackrel{\text{def}}{=} I_1 + I_2. \tag{3.10}$$

By integration by parts, the estimates (2.1) for Φ_n, Hölder inequality, Lemma 2.1 and Lemma A.3, we have

$$\begin{aligned}
I_1 =& \sum_{k=0}^{n} \int_{Q_0(R)} \boldsymbol{u}\cdot\boldsymbol{u}\cdot u_3 \cdot \partial_3\left(\Phi_n\phi_k\right)\psi \,\mathrm{d}x\mathrm{d}s \\
=& \sum_{k=0}^{n} \int_{Q_0(R)} \boldsymbol{u}\cdot\boldsymbol{u}\left(u_3-\overline{(u_3)}_k\right)\cdot\partial_3\left(\Phi_n\phi_k\right)\psi\,\mathrm{d}x\mathrm{d}s \\
& -2\sum_{k=0}^{n} \int_{Q_0(R)} \partial_3\boldsymbol{u}\cdot\boldsymbol{u}\cdot\overline{(u_3)}_k\cdot\left(\Phi_n\phi_k\right)\psi\,\mathrm{d}x\mathrm{d}s \\
\lesssim& \sum_{k=0}^{n} r_k^{-2}\int_{-r_k^2}^{0} \|\boldsymbol{u}\|_{L^{\frac{2q_0}{q_0-1}}(U_k(R))}\|\boldsymbol{u}\|_{L_v^2 L_h^{\frac{2q_0}{q_0-1}}(U_k(R))}\|u_3-\overline{(u_3)}_k\|_{L_v^{2q_0}L_h^{q_0}(U_k(R))}\mathrm{d}s \\
& +\sum_{k=0}^{n} r_k^{-1}\int_{-r_k^2}^{0}\|\partial_3\boldsymbol{u}\|_{L^{q_0}(U_k(R))}\|\boldsymbol{u}\|_{L^{\frac{2q_0}{q_0-1}}(U_k(R))}\|\overline{(u_3)}_k\|_{L^{\frac{2q_0}{q_0-1}}(U_k(R))}\mathrm{d}s \\
\lesssim& \sum_{k=0}^{n} r_k^{-1-\frac{1}{2q_0}}\int_{-r_k^2}^{0}\|\boldsymbol{u}\|_{L^{\frac{2q_0}{q_0-1}}(U_k(R))}\|\boldsymbol{u}\|_{L_v^2L_h^{\frac{2q_0}{q_0-1}}(U_k(R))}\|\partial_3\boldsymbol{u}\|_{L^{q_0}(U_k(R))}\mathrm{d}s \\
\lesssim& \sum_{k=0}^{n} r_k^{1-\frac{2}{p_0^*}-\frac{3}{q_0}}\|\boldsymbol{u}\|_{L^{\frac{4q_0}{3}}\left(-r_k^2,0;L^{\frac{2q_0}{q_0-1}}(U_k(R))\right)}\|\boldsymbol{u}\|_{L^{2q_0}\left(-r_k^2,0;L_v^2L_h^{\frac{2q_0}{q_0-1}}(U_k(R))\right)} \\
& \times\|\partial_3\boldsymbol{u}\|_{L^{p_0^*}\left(-r_k^2,0;L^{q_0}(U_k(R))\right)} \\
\lesssim& \sum_{i=0}^{n}\mathscr{B}_i\left(r_i^{-1}E_i(R)\right).
\end{aligned}\tag{3.11}$$

For the second term, by (2.1) and Lemma 2.1 again, we have

$$\begin{aligned}
I_2 \lesssim& \frac{1}{(R-\rho)}\sum_{i=0}^{n} r_i^{-1}\int_{Q_i(R)}|\boldsymbol{u}|^3\,\mathrm{d}x\mathrm{d}s \\
\lesssim& \frac{1}{(R-\rho)}\sum_{i=0}^{n} r_i^{-1}r_i^{1/2}\|\boldsymbol{u}\|^3_{L^4\left(-r_i^2,0;L^3(U_i(R))\right)}
\end{aligned}$$

$$\lesssim \frac{\mathcal{E}^{\frac{1}{2}}}{(R-\rho)} \sum_{i=0}^{n} r_i^{\frac{1}{2}} \left(r_i^{-1} E_i(R) \right). \tag{3.12}$$

Combining (3.11) with (3.12), we obtain (3.9). □

3.2. Estimates for the pressure

This part is devoted to the estimates regarding the third term on the right side of (3.7). Compared with [4,21], the decomposition of the pressure is in a slightly different way.

Given 3×3 matrix valued function $f = \left(f_{\mu\nu} \right)$, we set

$$\widehat{\mathscr{J}(f)}(\xi) = - \sum_{\mu,\nu=1,2,3} \frac{\xi_\mu \xi_\nu}{|\xi|^2} \mathcal{F}\left(f_{\mu\nu} \cdot \chi_{U_0(R)} \right),$$

where the Fourier transform is defined by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-i(x\cdot\xi)} f(x)\, \mathrm{d}x.$$

Therefore, $\mathscr{J} : L_v^{q_v} L_h^{q_h} (U_0(R)) \to L_v^{q_v} L_h^{q_h} \left(\mathbb{R}^3 \right)$, $1 < q_h, q_v < +\infty$ defines a bounded linear operator [16], with

$$\| \mathscr{J}(f) \|_{L_v^{q_v} L_h^{q_h}(\mathbb{R}^3)} \leq C \| f \|_{L_v^{q_v} L_h^{q_h}(U_0(R))}. \tag{3.13}$$

Denote

$$f = \boldsymbol{u} \otimes \boldsymbol{u} \cdot \chi_0, \tag{3.14}$$

and

$$\pi_0 = \mathscr{J}(f), \ \pi_h = \pi - \pi_0. \tag{3.15}$$

It follows that

$$-\Delta \pi_0 = \nabla \cdot \nabla \cdot f, \quad \text{in} \quad \mathbb{R}^3 \times (-1, 0)$$

in the sense of distributions and π_h is harmonic in $Q_1(R)$. Then we have

$$\begin{aligned} &\int_{-1}^{t} \int_{U_0(R)} \pi \boldsymbol{u} \cdot \nabla \left(\Phi_n \eta \psi \right) \mathrm{d}x \mathrm{d}s \\ &= \int_{-1}^{t} \int_{U_0(R)} \pi_0 \boldsymbol{u} \cdot \nabla \left(\Phi_n \eta \psi \right) \mathrm{d}x \mathrm{d}s + \int_{-1}^{t} \int_{U_0(R)} \pi_h \boldsymbol{u} \cdot \nabla \left(\Phi_n \eta \psi \right) \mathrm{d}x \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \int_{-1}^{t} \int_{U_0(R)} \pi_0 u_3 \cdot \partial_3 (\Phi_n \eta \psi) \, \mathrm{d}x \mathrm{d}s + \sum_{\mu=1,2} \int_{-1}^{t} \int_{U_0(R)} \pi_0 u_\mu \cdot \Phi_n \eta \partial_\mu \psi \, \mathrm{d}x \mathrm{d}s \\
&\quad - \int_{-1}^{t} \int_{U_0(R)} \nabla \pi_h \cdot \boldsymbol{u} \cdot (\Phi_n \eta \psi) \, \mathrm{d}x \mathrm{d}s \\
&\overset{\text{def}}{=} J + K + H. \qquad (3.16)
\end{aligned}
$$

As to J, setting $\tau_0 = \mathscr{J}\left(\partial_3 (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \chi_0\right)$ and $\tau_h = \mathscr{J}\left(\boldsymbol{u} \otimes \boldsymbol{u} \cdot \partial_3 \chi_0\right)$, we have $\partial_3 \pi_0 = \tau_0 + \tau_h$ and

$$
\begin{aligned}
J &= \sum_{k=0}^{n} \int_{-1}^{t} \int_{U_0(R)} \pi_0 \cdot u_3 \cdot \partial_3 (\Phi_n \phi_k \psi) \, \mathrm{d}x \mathrm{d}s \\
&= - \sum_{k=4}^{n} \int_{-1}^{t} \int_{U_0(R)} (\pi_0 \cdot \partial_3 u_3 + \tau_0 \cdot u_3) \cdot (\Phi_n \phi_k \psi) \, \mathrm{d}x \mathrm{d}s \\
&\quad - \sum_{k=4}^{n} \int_{-1}^{t} \int_{U_0(R)} \tau_h \cdot u_3 \cdot (\Phi_n \phi_k \psi) \, \mathrm{d}x \mathrm{d}s \\
&\quad + \sum_{k=0}^{3} \int_{-1}^{t} \int_{U_0(R)} \pi_0 \cdot u_3 \cdot \partial_3 (\Phi_n \phi_k \psi) \, \mathrm{d}x \mathrm{d}s \\
&\overset{\text{def}}{=} J_1 + J_2 + J_3. \qquad (3.17)
\end{aligned}
$$

We will present the estimates of $J_i, i = 1, 2, 3$, K and H in the following several lemmas.

Lemma 3.3. *Under the assumption of Lemma 3.1, we have*

$$
J_1 \le C \sum_{i=0}^{n} \mathscr{B}_i \left(r_i^{-1} E_i(R) \right). \qquad (3.18)
$$

Proof. Let $\pi_{0,j} = \mathscr{J}\left(\boldsymbol{u} \otimes \boldsymbol{u} \cdot \phi_j\right)$ and $\tau_{0,j} = \mathscr{J}\left(\partial_3 (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \phi_j\right)$. By integration by parts, we have

$$
\begin{aligned}
J_1 &= - \sum_{k=4}^{n} \sum_{j=0}^{n} \int_{-1}^{t} \int_{U_0(R)} \left(\pi_{0,j} \cdot \partial_3 u_3 + \tau_{0,j} \cdot u_3\right) \cdot (\Phi_n \phi_k \psi) \, \mathrm{d}x \mathrm{d}s \\
&= - \sum_{k=4}^{n} \sum_{j=k-3}^{n} \int_{-1}^{t} \int_{U_0(R)} \left(\pi_{0,j} \cdot \partial_3 u_3 + \tau_{0,j} \cdot u_3\right) \cdot (\Phi_n \phi_k \psi) \, \mathrm{d}x \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{k=4}^{n}\sum_{j=0}^{k-4}\int_{-1}^{t}\int_{U_0(R)}\left(\pi_{0,j}\cdot\partial_3 u_3+\tau_{0,j}\cdot u_3\right)\cdot(\Phi_n\phi_k\psi)\,\mathrm{d}x\mathrm{d}s\\
=&-\sum_{k=4}^{n}\int_{-1}^{t}\int_{U_0(R)}\mathscr{J}\left(\boldsymbol{u}\otimes\boldsymbol{u}\cdot\chi_{k-3}\right)\cdot\partial_3 u_3\cdot(\Phi_n\phi_k\psi)\,\mathrm{d}x\mathrm{d}s\\
&-\sum_{k=4}^{n}\int_{-1}^{t}\int_{U_0(R)}\mathscr{J}\left(\partial_3\left(\boldsymbol{u}\otimes\boldsymbol{u}\right)\cdot\chi_{k-3}\right)\cdot u_3\cdot(\Phi_n\phi_k\psi)\,\mathrm{d}x\mathrm{d}s\\
&-\sum_{j=0}^{n-4}\sum_{k=j+4}^{n}\int_{-1}^{t}\int_{U_0(R)}\left(\pi_{0,j}\cdot\partial_3 u_3+\tau_{0,j}\cdot u_3\right)\cdot(\Phi_n\phi_k\psi)\,\mathrm{d}x\mathrm{d}s\\
=&\sum_{k=4}^{n}\int_{-1}^{t}\int_{U_0(R)}\mathscr{J}\left(\boldsymbol{u}\otimes\boldsymbol{u}\cdot\chi_{k-3}\right)\cdot\left(u_3-\overline{(u_3)}_k\right)\cdot\partial_3\left(\Phi_n\phi_k\psi\right)\mathrm{d}x\mathrm{d}s\\
&+\sum_{k=4}^{n}\int_{-1}^{t}\int_{U_0(R)}\mathscr{J}\left(\boldsymbol{u}\otimes\boldsymbol{u}\cdot\partial_3\chi_{k-3}\right)\cdot\left(u_3-\overline{(u_3)}_k\right)\cdot(\Phi_n\phi_k\psi)\,\mathrm{d}x\mathrm{d}s\\
&-\sum_{k=4}^{n}\int_{-1}^{t}\int_{U_0(R)}\mathscr{J}\left(\partial_3\left(\boldsymbol{u}\otimes\boldsymbol{u}\right)\cdot\chi_{k-3}\right)\cdot\overline{(u_3)}_k\cdot(\Phi_n\phi_k\psi)\,\mathrm{d}x\mathrm{d}s\\
&-\sum_{j=0}^{n-4}\sum_{k=j+4}^{n}\int_{-1}^{t}\int_{U_0(R)}\pi_{0,j}\cdot\partial_3 u_3\cdot(\Phi_n\phi_k\psi)\,\mathrm{d}x\mathrm{d}s\\
&-\sum_{j=0}^{n-4}\sum_{k=j+4}^{n}\int_{-1}^{t}\int_{U_0(R)}\tau_{0,j}\cdot\left(u_3-\overline{(u_3)}_j\right)\cdot(\Phi_n\phi_k\psi)\,\mathrm{d}x\mathrm{d}s\\
&-\sum_{j=0}^{n-4}\sum_{k=j+4}^{n}\int_{-1}^{t}\int_{U_0(R)}\tau_{0,j}\cdot\overline{(u_3)}_j\cdot(\Phi_n\phi_k\psi)\,\mathrm{d}x\mathrm{d}s\\
\overset{\text{def}}{=}&J_{11}+J_{12}+J_{13}+J_{14}+J_{15}+J_{16}.
\end{aligned}
\tag{3.19}
$$

By (2.1), (3.13), Lemma 2.1 and Lemma A.3, we have

$$
J_{11}\lesssim\sum_{k=4}^{n}r_k^{-2}\int_{-r_k^2}^{0}\|\mathscr{J}\left(\boldsymbol{u}\otimes\boldsymbol{u}\cdot\chi_{k-3}\right)\|_{L_v^{\frac{2q_0}{2q_0-1}}L_h^{\frac{q_0}{q_0-1}}(U_k(R))}\|u_3-\overline{(u_3)}_k\|_{L_v^{2q_0}L_h^{q_0}(U_k(R))}\mathrm{d}s
$$

$$\lesssim \sum_{k=0}^{n} r_k^{-1-\frac{1}{2q_0}} \int_{-r_k^2}^{0} \|\boldsymbol{u}\|_{L^{\frac{2q_0}{q_0-1}}(U_k(R))} \|\boldsymbol{u}\|_{L_v^2 L_h^{\frac{2q_0}{q_0-1}}(U_k(R))} \|\partial_3 \boldsymbol{u}\|_{L^{q_0}(U_k(R))} \mathrm{d}s$$

$$\lesssim \sum_{i=0}^{n} \mathscr{B}_i\left(r_i^{-1} E_i(R)\right), \tag{3.20}$$

which is analogous to (3.11). Similarly, for J_{12}, J_{13}, we have

$$J_{12} \lesssim \sum_{k=4}^{n} r_k^{-1} \int_{-r_k^2}^{0} \|\mathscr{J}\left(\boldsymbol{u} \otimes \boldsymbol{u} \cdot \partial_3 \chi_{k-3}\right)\|_{L_v^{\frac{2q_0}{2q_0-1}} L_h^{\frac{q_0}{q_0-1}}(U_k(R))} \|u_3 - \overline{(u_3)}_k\|_{L_v^{2q_0} L_h^{q_0}(U_k(R))} \mathrm{d}s$$

$$\lesssim \sum_{k=0}^{n} r_k^{-1-\frac{1}{2q_0}} \int_{-r_k^2}^{0} \|\boldsymbol{u}\|_{L^{\frac{2q_0}{q_0-1}}(U_k(R))} \|\boldsymbol{u}\|_{L_v^2 L_h^{\frac{2q_0}{q_0-1}}(U_k(R))} \|\partial_3 \boldsymbol{u}\|_{L^{q_0}(U_k(R))} \mathrm{d}s$$

$$\lesssim \sum_{i=0}^{n} \mathscr{B}_i\left(r_i^{-1} E_i(R)\right), \tag{3.21}$$

and

$$J_{13} \lesssim \sum_{k=4}^{n} r_k^{-1} \int_{-r_k^2}^{0} \|\mathscr{J}\left(\partial_3\left(\boldsymbol{u} \otimes \boldsymbol{u}\right) \cdot \chi_{k-3}\right)\|_{L^{\frac{2q_0}{q_0+1}}(U_k(R))} \|\overline{(u_3)}_k\|_{L^{\frac{2q_0}{q_0-1}}(U_k(R))} \mathrm{d}s$$

$$\lesssim \sum_{k=0}^{n} r_k^{-1-\frac{1}{2q_0}} \int_{-r_k^2}^{0} \|\boldsymbol{u}\|_{L^{\frac{2q_0}{q_0-1}}(U_k(R))} \|\boldsymbol{u}\|_{L_v^2 L_h^{\frac{2q_0}{q_0-1}}(U_k(R))} \|\partial_3 \boldsymbol{u}\|_{L^{q_0}(U_k(R))} \mathrm{d}s$$

$$\lesssim \sum_{i=0}^{n} \mathscr{B}_i\left(r_i^{-1} E_i(R)\right). \tag{3.22}$$

By (2.1), (A.6) and Lemma 2.1, we have

$$J_{14} = \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_k^{-1} \int_{-r_k^2}^{0} \|\pi_{0,j}\|_{L^{\frac{q_0}{q_0-1}}(U_k(R))} \|\partial_3 u_3\|_{L^{q_0}(U_k(R))} \mathrm{d}s$$

$$\lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_j^{-1-\frac{1}{2q_0}} r_k^{-\frac{1}{q_0}} \int_{-r_k^2}^{0} \|\boldsymbol{u}\|^2_{L^{\frac{4q_0}{2q_0-1}}(U_j(R))} \|\partial_3 \boldsymbol{u}\|_{L^{q_0}(U_k(R))} \mathrm{d}s$$

$$\lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_j^{-1-\frac{1}{2q_0}} r_k^{2-\frac{5}{2q_0}-\frac{2}{p_0^*}} \|\boldsymbol{u}\|^2_{L^{\frac{8q_0}{3}}\left(-r_k^2,0;L^{\frac{4q_0}{2q_0-1}}(U_j(R))\right)}$$

$$\times \|\partial_3 \boldsymbol{u}\|_{L^{p_0^*}(-r_k^2,0;L^{q_0}(U_k(R)))}$$
$$\lesssim \sum_{i=0}^{n} \mathscr{B}_i \left(r_i^{-1} E_i(R)\right). \tag{3.23}$$

By (2.1), (A.7), Lemma 2.1 and Lemma A.3, we have

$$\begin{aligned}
J_{15} &\lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_k^{-1} \int_{-r_k^2}^{0} \|\tau_{0,j}\|_{L^{\frac{q_0}{q_0-1}}(U_k(R))} \|u_3 - \overline{(u_3)}_j\|_{L^{q_0}(U_k(R))} ds \\
&\lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_j^{-1-\frac{3}{2q_0}} \int_{-r_k^2}^{0} \|\boldsymbol{u}\|^2_{L^{\frac{4q_0}{2q_0-1}}(U_j(R))} \|\partial_3 \boldsymbol{u}\|_{L^{q_0}(U_j(R))} ds \\
&\lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_j^{-1-\frac{3}{2q_0}} r_k^{2-\frac{3}{2q_0}-\frac{2}{p_0^*}} \|\boldsymbol{u}\|^2_{L^{\frac{8q_0}{3}}\left(-r_k^2,0;L^{\frac{4q_0}{2q_0-1}}(U_j(R))\right)} \\
&\qquad \times \|\partial_3 \boldsymbol{u}\|_{L^{p_0^*}(-r_k^2,0;L^{q_0}(U_j(R)))} \\
&\lesssim \sum_{i=0}^{n} \mathscr{B}_i \left(r_i^{-1} E_i(R)\right).
\end{aligned} \tag{3.24}$$

By (2.1), (A.8), Lemma 2.1 and Lemma A.3, we have

$$\begin{aligned}
J_{16} &\lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_k^{-1} \int_{-r_k^2}^{0} \|\tau_{0,j}\|_{L^2(U_k(R))} \|\overline{(u_3)}_j\|_{L^2(U_k(R))} ds \\
&\lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_j^{-1-\frac{3}{2q_0}} \int_{-r_k^2}^{0} \|\partial_3 \boldsymbol{u} \otimes \boldsymbol{u}\|_{L^{\frac{2q_0}{q_0+1}}(U_j(R))} \|\boldsymbol{u}\|_{L^2(U_j(R))} ds \\
&\lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_j^{-1-\frac{3}{2q_0}} \int_{-r_k^2}^{0} \|\boldsymbol{u}\|_{L^{\frac{2q_0}{q_0-1}}(U_j(R))} \|\partial_3 \boldsymbol{u}\|_{L^{q_0}(U_j(R))} \|\boldsymbol{u}\|_{L^2(U_j(R))} ds \\
&\lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_j^{-1-\frac{3}{2q_0}} r_k^{2-\frac{3}{2q_0}-\frac{2}{p_0^*}} \|\boldsymbol{u}\|_{L^{\frac{4q_0}{3}}\left(-r_k^2,0;L^{\frac{2q_0}{q_0-1}}(U_j(R))\right)} \|\boldsymbol{u}\|_{L^\infty(-r_k^2,0;L^2(U_j(R)))} \\
&\qquad \times \|\partial_3 \boldsymbol{u}\|_{L^{p_0^*}(-r_k^2,0;L^{q_0}(U_j(R)))} \\
&\lesssim \sum_{i=0}^{n} \mathscr{B}_i \left(r_i^{-1} E_i(R)\right).
\end{aligned} \tag{3.25}$$

Summing up all the estimates of (3.19), (3.20), (3.21), (3.22), (3.23), (3.24) and (3.25), we have (3.18). □

Lemma 3.4. *Under the assumption of Lemma 3.1, we have*

$$J_2 + J_3 + H \le \frac{C}{(R-\rho)^{\frac{5}{2}}}\mathcal{E}^{\frac{3}{2}}. \tag{3.26}$$

Proof. The proof of this lemma is similar with Lemma 3.3 in [21]. By using of integration by parts and $\nabla \cdot \boldsymbol{u} = 0$, we have

$$\begin{aligned}
H =& -\sum_{k=0}^{3} \int_{Q_0(R)} \nabla \pi_h \cdot \boldsymbol{u} \cdot (\Phi_n \phi_k \psi)\, \mathrm{d}x\mathrm{d}s - \sum_{k=4}^{n} \int_{Q_0(R)} \nabla \pi_h \cdot \boldsymbol{u} \cdot (\Phi_n \phi_k \psi)\, \mathrm{d}x\mathrm{d}s \\
=& \sum_{k=0}^{3} \int_{Q_0(R)} \pi_h \cdot \boldsymbol{u} \cdot \nabla (\Phi_n \phi_k \psi)\, \mathrm{d}x\mathrm{d}s - \sum_{k=4}^{n} \int_{Q_0(R)} \nabla \pi_h \cdot \boldsymbol{u} \cdot (\Phi_n \phi_k \psi)\, \mathrm{d}x\mathrm{d}s \\
\overset{\text{def}}{=}& H_1 + H_2.
\end{aligned} \tag{3.27}$$

Applying (2.1) and Hölder's inequality, we have

$$\begin{aligned}
J_3 + H_1 &\lesssim \int_{Q_0(R)} |\pi_0||\boldsymbol{u}|\, \mathrm{d}x\mathrm{d}s + \frac{1}{R-\rho} \int_{Q_0(R)} |\pi_h||\boldsymbol{u}|\, \mathrm{d}x\mathrm{d}s \\
&\lesssim \frac{1}{R-\rho} \|\boldsymbol{u}\|^3_{L^3(\mathbb{R}^3 \times (-1,0))} \\
&\lesssim \frac{1}{R-\rho} \mathcal{E}^{\frac{3}{2}}.
\end{aligned} \tag{3.28}$$

Moreover, we may choose a finite family of points $\{x'_\nu\}$ in $B'\left(\frac{R+\rho}{2}\right)$ such that $\left\{B'(x'_\nu; \frac{R-\rho}{4})\right\}$ cover the ball $B'\left(\frac{R+\rho}{2}\right)$ and

$$\sum_\nu \chi_{B'(x'_\nu; \frac{R-\rho}{2})} \le C. \tag{3.29}$$

$$\begin{aligned}
J_2 + H_2 &\lesssim \sum_{k=4}^{n} \sum_\nu r_k^{-1} \int_{Q_k(R) \cap \left\{|x'-x'_\nu| < \frac{R-\rho}{4}\right\}} (|\nabla \pi_h| + |\tau_h|) \cdot |\boldsymbol{u}|\, \mathrm{d}x\mathrm{d}s \\
&\lesssim \sum_{k=4}^{n} \sum_\nu r_k^{-1} \||\nabla \pi_h| + |\tau_h|\|_{L^{\frac{3}{2}}\left(Q_k(R) \cap \left\{|x'-x'_\nu| < \frac{R-\rho}{4}\right\}\right)} \|\boldsymbol{u}\|_{L^3\left(Q_k(R) \cap \left\{|x'-x'_\nu| < \frac{R-\rho}{4}\right\}\right)} \\
&\lesssim (R-\rho)^{\frac{4}{3}} \sum_{k=4}^{n} \sum_\nu r_k^{-\frac{1}{3}} \||\nabla \pi_h| + |\tau_h|\|_{L^{\frac{3}{2}}\left(-r_k^2, 0; L^\infty\left(U_k(R) \cap \left\{|x'-x'_\nu| < \frac{R-\rho}{4}\right\}\right)\right)}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \|u\|_{L^3\left(Q_k(R)\cap\left\{|x'-x'_\nu|<\frac{R-\rho}{4}\right\}\right)} \\
&\lesssim (R-\rho)^2 \sum_{k=4}^{n}\sum_{\nu} r_k^{-\frac{1}{3}} \||\nabla\pi_h|+|\tau_h|\|^{\frac{3}{2}}_{L^{\frac{3}{2}}\left(-r_k^2,0;L^\infty\left(U_k(R)\cap\left\{x'-x'_\nu|<\frac{R-\rho}{4}\right\}\right)\right)} \\
&\quad + \sum_{k=2}^{n} r_k^{-\frac{1}{3}} \|u\|^3_{L^3(Q_k(R))}.
\end{aligned}
\tag{3.30}
$$

For any

$$
x^* \in U_k(R) \cap \left\{x = \left(x', x_3\right); \left|x' - x'_\nu\right| < \frac{R-\rho}{4}\right\},
$$

we have

$$
x^* \in B\left(x^*; \frac{R-\rho}{4}\right) \subset U_1(R) \cap \left\{x = (x', x_3); \left|x' - x'_\nu\right| < \frac{R-\rho}{2}\right\},
$$

due to $k \geq 4$ and $|R-\rho| \leq \frac{1}{2}$. Since π_h, τ_h are harmonic in $U_1(R)$, using the mean value property, we have

$$
\begin{aligned}
|\nabla\pi_h|\left(x^*\right) + |\tau_h|\left(x^*\right) &\lesssim \frac{1}{|R-\rho|^4} \int_{B\left(x^*;\frac{R-\rho}{4}\right)} |\pi_h| + |\tau_h| \,\mathrm{d}x \\
&\lesssim \frac{1}{(R-\rho)^3} \||\pi_h| + |\tau_h|\|_{L^{\frac{3}{2}}\left(U_1(R)\cap\left\{|x'-x'_\nu|<\frac{R-\rho}{2}\right\}\right)},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\||\nabla\pi_h| + |\tau_h|\|^{\frac{3}{2}}_{L^{\frac{3}{2}}\left(-r_k^2,0;L^\infty\left(U_k(R)\cap\left\{|x'-x'_\nu|<\frac{R-\rho}{4}\right\}\right)\right)} \\
&= \int_{-r_k^2}^{0} \||\nabla\pi_h| + |\tau_h|\|^{\frac{3}{2}}_{L^\infty\left(U_k(R)\cap\left\{|x'-x'_\nu|<\frac{R-\rho}{4}\right\}\right)} \,\mathrm{d}s \\
&\lesssim \frac{1}{(R-\rho)^{\frac{9}{2}}} \int_{-r_k^2}^{0}\int_{U_1(R)} (|\pi_h| + |\tau_h|)^{\frac{3}{2}} \chi_{B'(x'_\nu;\frac{R-\rho}{2})} \,\mathrm{d}x\mathrm{d}s.
\end{aligned}
\tag{3.31}
$$

Hence, combining (3.30) and (3.31) and taking sum over ν, we have

$$
J_2 + H_2 \lesssim \frac{1}{(R-\rho)^{\frac{5}{2}}} \sum_{k=2}^{n} r_k^{-\frac{1}{3}} \left(\||\pi_h| + |\tau_h|\|^{\frac{3}{2}}_{L^{\frac{3}{2}}\left(-r_k^2,0;L^{\frac{3}{2}}(U_1(R))\right)} + \|u\|^3_{L^3(-r_k^2,0;L^3(U_k(R)))} \right)
$$

$$
\begin{aligned}
&\lesssim \frac{1}{(R-\rho)^{\frac{5}{2}}} \sum_{k=2}^{n} r_k^{\frac{1}{6}} \|u\|^3_{L^4(-r_k^2,0;L^3(\mathbb{R}^3))} \\
&\lesssim \frac{1}{(R-\rho)^{\frac{5}{2}}} \mathcal{E}^{\frac{3}{2}}.
\end{aligned} \tag{3.32}
$$

Summing up (3.27), (3.28) and (3.32), we have (3.26). □

Lemma 3.5. *Under the assumption of Lemma 3.1, we have*

$$
K \le C \frac{\mathcal{E}^{\frac{1}{2}}}{R-\rho} \sum_{i=0}^{n} r_i^{\frac{1}{2}} \left(r_i^{-1} E_i(R) \right). \tag{3.33}
$$

Proof. The proof is rather similar as Lemma 3.3 in [21]. We omit the detail here. □

Lemma 3.6. *Under the assumption of Lemma 3.1, we have*

$$
\begin{aligned}
&\int_{-1}^{t} \int_{U_0(R)} \pi \boldsymbol{u} \cdot \nabla \left(\Phi_n \eta \psi \right) \mathrm{d}x \mathrm{d}s \\
&\le C \sum_{i=0}^{n} \mathscr{B}_i \left(r_i^{-1} E_i(R) \right) + C \frac{\mathcal{E}^{\frac{1}{2}}}{R-\rho} \sum_{i=0}^{n} r_i^{\frac{1}{2}} \left(r_i^{-1} E_i(R) \right) + \frac{C}{(R-\rho)^{\frac{5}{2}}} \mathcal{E}^{\frac{3}{2}}.
\end{aligned} \tag{3.34}
$$

Proof. Combining the estimates (3.16), (3.17), (3.18), (3.26) and (3.33), we have (3.34). □

3.3. The proof of Theorem 1.1 and Theorem 1.3

On the basis of the estimates of the nonlinear term and the pressure in subsection 3.1–3.2, we are in position to give the detail proof of Theorem 1.3.

Proof. Gathering (3.7) and the estimates in Lemma 3.1, 3.2 and 3.6, we have

$$
\begin{aligned}
r_n^{-1} E_n(\rho) &\le C \sum_{i=0}^{n} \mathscr{B}_i \left(r_i^{-1} E_i(R) \right) + C \frac{\mathcal{E}^{\frac{1}{2}}}{R-\rho} \sum_{i=0}^{n} r_i^{\frac{1}{2}} \left(r_i^{-1} E_i(R) \right) + C \frac{1+\mathcal{E}^{\frac{3}{2}}}{(R-\rho)^{\frac{5}{2}}} \\
&\le C \sum_{i=0}^{n} \mathscr{B}_i \left(r_i^{-1} E_i(R) \right) + C \frac{\mathcal{E}^{\frac{3}{4}}}{R-\rho} \sum_{i=0}^{n} r_i^{-\frac{5}{8}} E_i(R)^{\frac{3}{4}} r_i^{\frac{1}{8}} + C \frac{1+\mathcal{E}^{\frac{3}{2}}}{(R-\rho)^{\frac{5}{2}}} \\
&\le C \sum_{i=0}^{n} \mathscr{B}_i \left(r_i^{-1} E_i(R) \right) + C \frac{\mathcal{E}^{\frac{3}{4}}}{R-\rho} \left(\sum_{i=0}^{n} r_i^{-\frac{5}{6}} E_i(R) \right)^{\frac{3}{4}} \left(\sum_{i=0}^{n} r_i^{\frac{1}{2}} \right)^{\frac{1}{4}} \\
&\quad + C \frac{1+\mathcal{E}^{\frac{3}{2}}}{(R-\rho)^{\frac{5}{2}}}
\end{aligned}
$$

$$\leq C_0 \sum_{i=0}^{n} \left(\mathscr{B}_i + r_i^{\frac{1}{6}}\right)\left(r_i^{-1} E_i(R)\right) + C_0 \frac{1+\mathcal{E}^3}{(R-\rho)^4}.$$

In view of (3.3), there exists a sufficient large number $n_0 \geq 1$ such that

$$C_0 \sum_{i=n_0}^{\infty} \left(\mathscr{B}_i + r_i^{\frac{1}{6}}\right) \leq \frac{1}{2}. \tag{3.35}$$

Then for $n \geq n_0$ we have

$$\begin{aligned} r_n^{-1} E_n(\rho) \leq & C_0 \sum_{i=n_0}^{n} \left(\mathscr{B}_i + r_i^{\frac{1}{6}}\right)\left(r_i^{-1} E_i(R)\right) \\ & + C_0 \sum_{i=0}^{n_0-1} \left(\mathscr{B}_i + r_i^{\frac{1}{6}}\right)\left(r_i^{-1} E_i(R)\right) + C_0 \frac{1+\mathcal{E}^3}{(R-\rho)^4} \\ \leq & C_0 \sum_{i=n_0}^{n} \left(\mathscr{B}_i + r_i^{\frac{1}{6}}\right)\left(r_i^{-1} E_i(R)\right) + \frac{A_0}{(R-\rho)^4}, \end{aligned} \tag{3.36}$$

where the constant $A_0 = C_1 \cdot 2^{n_0}\left(1 + \mathcal{E}^2 \cdot \|\partial_3 \boldsymbol{u}\|_{L^{p_0,1}(-1,0;L^{q_0}(B(2)))} + \mathcal{E}^3\right)$.

As the iteration argument in [9, V. Lemma 3.1], we introduce the sequence $\{\rho_k\}_{k=0}^{+\infty}$ which satisfies

$$\rho_0 = \frac{1}{2}, \quad \rho_{k+1} - \rho_k = \frac{1-\theta}{2}\theta^k,$$

with $\frac{1}{2} < \theta^4 < 1$ and $\lim_{k\to\infty} \rho_k = 1$. For $n_0 \leq j \leq n$ and $k \geq 1$, we have

$$\begin{aligned} r_j^{-1} E_j(\rho_k) \leq & C_0 \sum_{i=n_0}^{j} \left(\mathscr{B}_i + r_i^{\frac{1}{6}}\right)\left(r_i^{-1} E_i(\rho_{k+1})\right) + A_0 \cdot \frac{16}{(1-\theta)^4}\theta^{-4k} \\ \leq & C_0 \sum_{i=n_0}^{n} \left(\mathscr{B}_i + r_i^{\frac{1}{6}}\right)\left(r_i^{-1} E_i(\rho_{k+1})\right) + A_0 \cdot \frac{16}{(1-\theta)^4}\theta^{-4k}. \end{aligned} \tag{3.37}$$

By using iteration argument from (3.36), and then applying (3.35) and (3.37), we obtain that for $n \geq n_0$,

$$\begin{aligned} r_n^{-1} E_n(\rho_0) \leq & C_0 \sum_{j=n_0}^{n} \left(\mathscr{B}_j + r_j^{\frac{1}{6}}\right)\left(r_j^{-1} E_j(\rho_1)\right) + A_0 \cdot \frac{16}{(1-\theta)^4} \\ \leq & \frac{1}{2} C_0 \sum_{j=n_0}^{n} \left(\mathscr{B}_j + r_j^{\frac{1}{6}}\right)\left(r_j^{-1} E_j(\rho_2)\right) + A_0 \cdot \frac{16}{(1-\theta)^4}\left(1 + \frac{1}{2\theta^4}\right) \end{aligned}$$

$$\le \frac{1}{2^{k-1}} C_0 \sum_{j=n_0}^{n} \left(\mathscr{B}_j + r_j^{\frac{1}{6}}\right)\left(r_j^{-1} E_j(\rho_k)\right) + A_0 \cdot \frac{16}{(1-\theta)^4} \sum_{j=0}^{k-1} \left(\frac{1}{2\theta^4}\right)^j$$

$$\le \frac{1}{2^{k-1}} C_0 \sum_{j=n_0}^{n} \left(\mathscr{B}_j + r_j^{\frac{1}{6}}\right)\left(r_j^{-1} E_j(1)\right) + A_0 \cdot \frac{16}{(1-\theta)^4} \cdot \frac{2\theta^4}{2\theta^4 - 1}.$$

Let $k \to \infty$, we obtain that for $n \ge n_0$,

$$r_n^{-1} E_n(\rho_0) \le A_0 \cdot \frac{16}{(1-\theta)^4} \cdot \frac{2\theta^4}{2\theta^4 - 1}. \tag{3.38}$$

For $0 < r \le r_{n_0}$, there exists $n_1 \ge n_0$, such that

$$r_{n_1+1} < r \le r_{n_1},$$

which together with (3.38) ensures that

$$r^{-2} \|\boldsymbol{u}\|^3_{L^3(B(r)\times(-r^2,0))} \le C r^{-\frac{3}{2}} \|\boldsymbol{u}\|^3_{L^4(-r^2,0;L^3(B(r)))} \le C \left(r_{n_1}^{-1} E_{n_1}(\frac{1}{2})\right)^{\frac{3}{2}} \le C.$$

For $r_{n_0} < r \le 1$,

$$r^{-2} \|\boldsymbol{u}\|^3_{L^3(B(r)\times(-r^2,0))} \le C 2^{2n_0} \|\boldsymbol{u}\|^3_{L^3(-1,0;L^3(B(1)))} \le C.$$

Thus, (1.10) is proved. Similarly, by (3.38), we have for $0 < r \le 1$,

$$\sup_{t\in(-r^2,0)} r^{-1} \|\boldsymbol{u}(\cdot,t)\|^2_{L^2(B(r))} + r^{-1} \|\nabla \boldsymbol{u}\|^2_{L^2(B(r)\times(-r^2,0))} \le C. \tag{3.39}$$

Next, we recall the Theorem 1.1 in [23].

Lemma 3.7. *Let $(\boldsymbol{u}, \pi)$ be a suitable weak solution of (1.1) in $Q(1)$. Given $M > 0$, there exist positive constants $\varepsilon_0 = \varepsilon_0(M)$ such that if the two conditions*

$$R^{-2} \int_{Q(R)} |\boldsymbol{u}|^3 \,\mathrm{d}x\,\mathrm{d}t < \varepsilon_0 \tag{3.40}$$

and

$$R^{-2} \int_{Q(R)} |\pi - (\pi)_{B(R)}|^{\frac{3}{2}} \,\mathrm{d}x\,\mathrm{d}t < M, \tag{3.41}$$

hold for some $0 < R \le 1$, then $\boldsymbol{u}$ is Hölder continuous in $Q(\frac{R}{2})$. Here $Q(r) = B(r) \times (-r^2, 0)$ and $(\pi)_{B(r)} = \fint_{B(r)} \pi(y,t)\,\mathrm{d}y$.

We will finish the proof of Theorem 1.3 by contradiction argument as done in [4]. For the constant M given in Lemma A.6, we claim that there exists some $0 < R_1 \le 1$,

$$R_1^{-2} \int_{Q(R_1)} |\boldsymbol{u}|^3 \,\mathrm{d}x\,\mathrm{d}t < \varepsilon_0. \tag{3.42}$$

Otherwise, we may have that for all $0 < r \le 1$,

$$r^{-2} \int_{Q(r)} |\boldsymbol{u}|^3 \,\mathrm{d}x\,\mathrm{d}t \ge \varepsilon_0. \tag{3.43}$$

For $k \ge 1$, denote

$$\boldsymbol{u}_k(x,t) = r_k \boldsymbol{u}\left(r_k x, r_k^2 t\right), \; \pi_k(x,t) = r_k^2 \left(\pi\left(r_k x, r_k^2 t\right) - \left(\pi(r_k^2 t)\right)_{B(r_k)}\right).$$

Thus, by (3.43), we have that for $k \ge 1$ and $0 < r \le 1$,

$$r^{-2} \int_{Q(r)} |\boldsymbol{u}_k|^3 \,\mathrm{d}x\,\mathrm{d}t \ge \varepsilon_0. \tag{3.44}$$

Moreover, (3.39) and Lemma A.6 imply that

$$\sup_{t\in(-1,0)} \|\boldsymbol{u}_k(\cdot,t)\|^2_{L^2(B(1))} + \|\nabla \boldsymbol{u}_k\|^2_{L^2(Q(1))} + \|\pi_k\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(1))} \le C, \tag{3.45}$$

and $(\boldsymbol{u}_k, \pi_k)$ is a suitable weak solution to the Navier-Stokes equations (1.1) in $Q(1)$. By (3.45), $(\boldsymbol{u}_k, \pi_k)$ converges weakly (by taking subsequences if needed) to some $(\boldsymbol{v}, \Pi)$ with

$$\sup_{t\in(-1,0)} \|\boldsymbol{v}(\cdot,t)\|^2_{L^2(B(1))} + \|\nabla \boldsymbol{v}\|^2_{L^2(Q(1))} + \|\Pi\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(1))} \le C. \tag{3.46}$$

By a similar argument as [15, Theorem 2.2], we can prove that $(\boldsymbol{v}, \Pi)$ is a suitable weak solution to the Navier-Stokes equations in $Q(1)$ and $\boldsymbol{u}_k \underset{k\to\infty}{\longrightarrow} \boldsymbol{v}$ strongly in $L^3(Q(1))$. Therefore, by (3.44), we have that for $0 < r \le 1$,

$$r^{-2} \int_{Q(r)} |\boldsymbol{v}|^3 \,\mathrm{d}x\,\mathrm{d}t \ge \varepsilon_0. \tag{3.47}$$

On the other hand, by (1.9), we have $\partial_3 \boldsymbol{v} = 0$. Thus, we can apply the localized regularity condition on $\partial_3 \boldsymbol{v}$ (see [13, Theorem 2.1]) to deduce that $\boldsymbol{v}$ is bounded function in $Q(1/4)$. Therefore,

$$\epsilon_0 \le r^{-2} \int_{Q(r)} |\boldsymbol{v}|^3 \,\mathrm{d}x\,\mathrm{d}t \le C r^3.$$

The above inequality leads to a contradiction when we send $r \to 0$. The claim (3.42) is true. By Lemma 3.7, we get that $(0, 0)$ is a regular point. □

At last, it remains for us to prove Theorem 1.1.

Proof. By Theorem 1.3, we obtain that the suitable weak solution $(\boldsymbol{u}, \pi)$ is regular at all the points $(x_0, t_0) \in \mathbb{R}^3 \times (0, T)$. We will prove the following Lemma 3.8. Thus, Theorem 1.1 can be proved directly.

Lemma 3.8. *Let $(\boldsymbol{u}, \pi)$ be a suitable weak solution of (1.1) in $\mathbb{R}^3 \times (-1, 0)$. The solution is regular at all the points $z_0 = (x_0, 0),\ x_0 \in \mathbb{R}^3$. In particular, there exist two positive constants ρ and C such that*

$$\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3\times(-\rho^2,0))} \le C. \tag{3.48}$$

We recall the result in [15, Theorem 3.1].

Lemma 3.9. *Let $(\boldsymbol{u}, \pi)$ be a suitable weak solution of (1.1) in $Q(1)$. There are two positive constants ε_2 and C_2 such that*

$$\int_{Q(1)} \left(|\boldsymbol{u}|^3 + |\pi|^{\frac{3}{2}}\right) \mathrm{d}x\, \mathrm{d}t < \varepsilon_2, \tag{3.49}$$

implies for some $\alpha > 0$,

$$\|\boldsymbol{u}\|_{C^\alpha(Q(\frac{1}{2}))} \le C_2. \tag{3.50}$$

We can choose a $\tilde{R}$ large enough such that

$$\int_{Q(z_0,1)} \left(|\boldsymbol{u}|^3 + |\pi|^{\frac{3}{2}}\right) \mathrm{d}x\, \mathrm{d}t < \varepsilon_2 \quad \text{for any } z_0 = (x_0, 0) \text{ with } |x_0| > \tilde{R},$$

where $Q(z_0, r) = \{(x, t), |x - x_0| < r, -r^2 < t < 0\}$. By Lemma 3.9, one has

$$\|\boldsymbol{u}\|_{L^\infty(Q(z_0,\frac{1}{2}))} \le C_2,$$

which implies that

$$|\boldsymbol{u}(x, t)| \le C_2 \quad \text{for } |x| \ge \tilde{R},\ -\frac{1}{4} < t < 0. \tag{3.51}$$

On the other hand, for every $|x_0| \le \tilde{R}$, the solution is regular at $(x_0, 0)$. Thus, there exist two positive constants ρ_{x_0} and C_{x_0} depending on x_0 such that

$$\|\boldsymbol{u}\|_{L^\infty(Q(x_0,\rho_{x_0}))} \le C_{x_0}. \tag{3.52}$$

We can choose finite open sets, say $B(x_0^k, \rho_{x_0^k}), k = 1, \cdots, N$, to cover $\overline{B(\tilde{R})}$ such that

$$\|\boldsymbol{u}\|_{L^\infty(Q(x_0^k, \rho_{x_0^k}))} \leq C_{x_0^k}.$$

Accordingly,

$$|\boldsymbol{u}(x,t)| \leq \tilde{C}_2, \quad \text{for } |x| \leq \tilde{R},\ -\tilde{\rho}_2^2 < t < 0, \tag{3.53}$$

where $\tilde{C}_2 = \max\{C_{x_0^1}, \cdots, C_{x_0^N}\}$ and $\tilde{\rho}_2 = \min\{\rho_{x_0^1}, \cdots, \rho_{x_0^N}\}$. Combining (3.51) with (3.53), we prove the Lemma 3.8. $\square$

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Appendix A

Lemma A.1. *Let $0 < r \leq R < +\infty$ and $h : B'(2R) \times (-r, r) \to \mathbb{R}$ be harmonic. Then for all $0 < \rho \leq \frac{r}{4}$ and $1 \leq \ell \leq q < +\infty$, we get*

$$\|\nabla^m h\|^q_{L^q(B'(R)\times(-\rho,\rho))} \leq C\rho r^{2-mq-\frac{3q}{\ell}} \|h\|^q_{L^\ell(B'(2R)\times(-r,r))}, \ m \in \mathbb{N}, \tag{A.1}$$

where C stands for a positive constant depending only on q, m and ℓ.

Proof. The proof is similar as Lemma A.2 in [4]. We choose a finite family of points $\{x'_\nu\}$ in $B'(R)$ such that $\{B'(x'_\nu, r/4)\}$ is a covering of $\overline{B'(R)}$, and it holds

$$\sum_\nu \chi_{B'(x'_\nu, r)} \leq C. \tag{A.2}$$

Setting $x_\nu = (x'_\nu, 0)$, we see that

$$B'(x'_\nu, r/4) \times (-r/4, r/4) \subset B(x_\nu, r/2).$$

With this notation, we have

$$\begin{aligned}
\|\nabla^m h\|^q_{L^q(B'(R)\times(-\rho,\rho))} &\leq \sum_\nu \|\nabla^m h\|^q_{L^q(B'(x'_\nu, r/4)\times(-\rho,\rho))} \\
&\leq Cr^2\rho \sum_\nu \|\nabla^m h\|^q_{L^\infty(B'(x'_\nu, r/4)\times(-r/4,r/4))} \\
&\leq Cr^2\rho \sum_\nu \|\nabla^m h\|^q_{L^\infty(B(x_\nu, r/2))}.
\end{aligned} \tag{A.3}$$

Since h is harmonic, using the mean value property and taking the sum over ν, we obtain

$$\sum_{\nu} \|\nabla^m h\|^q_{L^\infty(B(x_\nu, r/2))} \leq C r^{-mq-\frac{3q}{\ell}} \|h\|^q_{L^\ell(B'(2R)\times(-r,r))}. \tag{A.4}$$

Combining (A.3) and (A.4), we get

$$\|\nabla^m h\|^q_{L^q(B'(R)\times(-\rho,\rho))} \leq C\rho r^{2-mq-\frac{3q}{\ell}} \|h\|^q_{L^\ell(B'(2R)\times(-r,r))}. \quad \square \tag{A.5}$$

Corollary A.2. *For $1 < \ell \leq q < +\infty$ and $k \geq j+4$, we have*

$$\|\pi_{0,j}\|_{L^q(U_k(R))} \leq C r_k^{\frac{1}{q}} r_j^{\frac{2}{q}-\frac{3}{\ell}} \|\boldsymbol{u}\|^2_{L^{2\ell}(U_j(R))}, \tag{A.6}$$

$$\|\tau_{0,j}\|_{L^q(U_k(R))} \leq C r_k^{\frac{1}{q}} r_j^{\frac{2}{q}-\frac{3}{\ell}-1} \|\boldsymbol{u}\|^2_{L^{2\ell}(U_j(R))}, \tag{A.7}$$

and

$$\|\tau_{0,j}\|_{L^q(U_k(R))} \leq C r_k^{\frac{1}{q}} r_j^{\frac{2}{q}-\frac{3}{\ell}} \|\partial_3 \boldsymbol{u} \otimes \boldsymbol{u}\|_{L^\ell(U_j(R))}. \tag{A.8}$$

Proof. We recall that $\pi_{0,j} = \mathscr{J}\left(\boldsymbol{u}\otimes\boldsymbol{u}\cdot\phi_j\right)$ and $\tau_{0,j} = \mathscr{J}\left(\partial_3\left(\boldsymbol{u}\otimes\boldsymbol{u}\right)\cdot\phi_j\right)$. Hence

$$\tau_{0,j} = \partial_3\pi_{0,j} - \tau_{h,j} \tag{A.9}$$

with $\tau_{h,j} = \mathscr{J}\left(\boldsymbol{u}\otimes\boldsymbol{u}\cdot\partial_3\phi_j\right)$. From the definition of ϕ_j in (3.5), it follows that the functions $\pi_{0,j}, \tau_{0,j}, \tau_{h,j}$ are harmonic in $\mathbb{R}^2 \times (-r_{j+2}, r_{j+2}) \times (-r^2_{j+2}, 0)$. Applying (3.13) and Lemma A.1 with $r = r_{j+2}$, $\rho = r_k$, we have (A.6), (A.7) and (A.8) directly. $\square$

Lemma A.3 *(Poincaré's inequality). Set $\overline{(h)}_j(x') = \frac{1}{2r_j}\int_{-r_j}^{r_j} h(x',\omega)\, d\omega$. For $k \geq j$, it holds*

$$\|h - \overline{(h)}_j\|_{L^{q_v}_v L^{q_h}_h(B'(R)\times(-r_k,r_k))} \leq C r_k^{\frac{1}{q_v}} r_j^{1-\frac{1}{\ell}} \|\partial_3 h\|_{L^\ell_v L^{q_h}_h(B'(R)\times(-r_j,r_j))}, \tag{A.10}$$

and

$$\|\overline{(h)}_j\|_{L^{q_v}_v L^{q_h}_h(B'(R)\times(-r_k,r_k))} \leq C r_k^{\frac{1}{q_v}} r_j^{-\frac{1}{\ell}} \|h\|_{L^\ell_v L^{q_h}_h(B'(R)\times(-r_j,r_j))}. \tag{A.11}$$

Proof. For (A.10), we see that

$$\begin{aligned}|h(x',x_3) - \overline{(h)}_j| &\leq \frac{1}{2r_j}\int_{-r_j}^{r_j} \left|h(x',x_3) - h(x',\omega)\right| \,\mathrm{d}\omega \\ &\leq \frac{1}{2r_j}\int_{-r_j}^{r_j} \left|\int_{\omega}^{x_3} \partial_3 h(x',\xi)\,\mathrm{d}\xi\right| \mathrm{d}\omega\end{aligned}$$

$$\leq \int_{-r_j}^{r_j} \left|\partial_3 h(x', \xi)\right| \, \mathrm{d}\xi.$$

Applying Hölder's and Minkowski inequality, we have

$$\begin{aligned}
\|h - \overline{(h)}_j\|_{L_v^{q_v} L_h^{q_h}(B'(R)\times(-r_k,r_k))} &\lesssim r_k^{\frac{1}{q_v}} \cdot \int_{-r_j}^{r_j} \|\partial_3 h(\cdot, \xi)\|_{L_h^{q_h}(B'(R))} \, \mathrm{d}\xi \\
&\lesssim r_k^{\frac{1}{q_v}} r_j^{1-\frac{1}{\ell}} \|\partial_3 h\|_{L_v^{\ell} L_h^{q_h}(B'(R)\times(-r_j,r_j))}.
\end{aligned}$$

As to (A.11), we apply Hölder's and Minkowski inequalities again to get

$$\begin{aligned}
\|\overline{(h)}_j\|_{L_v^{q_v} L_h^{q_h}(B'(R)\times(-r_k,r_k))} &\lesssim r_k^{\frac{1}{q_v}} \cdot \frac{1}{2r_j} \int_{-r_j}^{r_j} \|h(\cdot, \omega)\|_{L^q(B'(R))} \, \mathrm{d}\omega \\
&\lesssim r_k^{\frac{1}{q_v}} r_j^{-\frac{1}{\ell}} \|h\|_{L_v^{\ell} L_h^{q_h}(B'(R)\times(-r_j,r_j))}.
\end{aligned}$$

The proof of this lemma is completed. $\square$

Lemma A.4 *(Lemma A.2 in [21]). Let $0 < p, \sigma < +\infty$. Then for any $h(s) \in L^{p,\sigma}(\mathbb{R})$, there exists a sequence $\{c_n\}_{n\in\mathbb{Z}} \in \ell^\sigma$ and sequence of functions $\{h_n\}_{n\in\mathbb{Z}}$ with each h_n bounded by $2^{\frac{n}{p}}$ and supported on a set of measure 2^{-n}. Moreover,*

$$h = \sum_{n\in\mathbb{Z}} c_n h_n$$

and

$$c(p,\sigma) \|\{c_n\}\|_{\ell^\sigma} \leq \|h\|_{L^{p,\sigma}} \leq C(p,\sigma) \|\{c_n\}\|_{\ell^\sigma} ,$$

where the constant $c(p,\sigma)$ and $C(p,\sigma)$ only depend on p, σ.

Lemma A.5. *For any*

$$1 \leq p^* < p = \frac{2q}{2q-3}, \quad \frac{3}{2} < q < +\infty,$$

we have

$$\sum_{k=0}^{+\infty} r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \left(\int_{-r_k^2}^{0} \|\partial_3 \boldsymbol{u}(\cdot, s)\|_{L^q(B(2))}^{p^*} \, ds \right)^{\frac{1}{p^*}} \leq C \, \|\partial_3 \boldsymbol{u}\|_{L^{p,1}(-1,0;L^q(B(2)))} .$$

Proof. Let $h(s) = \|\partial_3 \boldsymbol{u}(\cdot, s)\|_{L^q(B(2))}$, $-1 < s < 0$. By Lemma A.4, we know that

$$h = \sum_{j=0}^{+\infty} c_j h_j, \quad \|h\|_{L^{p,1}} \approx \sum_{j=0}^{+\infty} |c_j|,$$

where

$$|h_j| \le 2^{\frac{j}{p}}, \quad |D_j = \operatorname{supp} h_j| \le 2^{-j}.$$

We can restrict $j \ge 0$ here, since $s \in (-1, 0)$ and the construction of the function h_j in Lemma A.4 by standard atomic decomposition. Then we have

$$\begin{aligned}
&\sum_{k=0}^{+\infty} r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \left(\int_{-r_k^2}^{0} \|\partial_3 \boldsymbol{u}(\cdot, s)\|_{L^q(B(2))}^{p^*} \, ds \right)^{\frac{1}{p^*}} \\
&= \sum_{k=0}^{+\infty} r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \left(\int_{I_k} |h|^{p^*} \, ds \right)^{\frac{1}{p^*}} \\
&\le \sum_{k=0}^{+\infty} r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \sum_{j=0}^{+\infty} |c_j| \, 2^{\frac{j}{p}} \, |D_j \cap I_k|^{\frac{1}{p^*}} \\
&\le \sum_{j=0}^{+\infty} |c_j| \, 2^{\frac{j}{p}} \sum_{k=0}^{+\infty} r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \, |D_j \cap I_k|^{\frac{1}{p^*}} \\
&\le \sum_{j=0}^{+\infty} |c_j| \, 2^{\frac{j}{p}} \sum_{k \le \frac{j}{2}} r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \, |D_j \cap I_k|^{\frac{1}{p^*}} + \sum_{j=0}^{+\infty} |c_j| \, 2^{\frac{j}{p}} \sum_{k > \frac{j}{2}} r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \, |D_j \cap I_k|^{\frac{1}{p^*}},
\end{aligned}$$

where $I_k = \left(-r_k^2, 0\right)$. On the other hand, we notice that for any $k \le \frac{j}{2}$

$$r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \, |D_j \cap I_k|^{\frac{1}{p^*}} \le C 2^{-\frac{j}{p^*}} 2^{-k\left(2-\frac{2}{p^*}-\frac{3}{q}\right)}, \tag{A.12}$$

and for any $\frac{j}{2} < k < +\infty$

$$r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \, |D_j \cap I_k|^{\frac{1}{p^*}} \le C 2^{-\frac{2k}{p^*}} 2^{-k\left(2-\frac{2}{p^*}-\frac{3}{q}\right)} = C 2^{-k\left(2-\frac{3}{q}\right)}. \tag{A.13}$$

Accordingly,

$$
\begin{aligned}
&\sum_{k=0}^{+\infty} r_k^{2-\frac{2}{p^*}-\frac{3}{q}} \left(\int_{-r_k^2}^{0} \|\partial_3 \boldsymbol{u}(\cdot,s)\|_{L^q(B(2))}^{p^*} \, ds \right)^{\frac{1}{p^*}} \\
\lesssim &\sum_{j=0}^{+\infty} |c_j| 2^{\frac{j}{p}} \sum_{k\le \frac{j}{2}} 2^{-\frac{j}{p^*}} 2^{-k\left(2-\frac{2}{p^*}-\frac{3}{q}\right)} + \sum_{j=0}^{+\infty} |c_j| 2^{\frac{j}{p}} \sum_{k>\frac{j}{2}} 2^{-k\left(2-\frac{3}{q}\right)} \\
\lesssim &\sum_{j=0}^{+\infty} |c_j| \approx \|\partial_3 \boldsymbol{u}\|_{L^{p,1}(-1,0;L^q(B(2)))}.
\end{aligned}
$$

This completes the proof of this lemma. □

Lemma A.6 *(Lemma 3.2 in [4]). Let $f = \boldsymbol{u} \otimes \boldsymbol{u} \in L^{\frac{3}{2}}(Q(1))$ satisfy*

$$
r^{-2} \int_{Q(r)} |f|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}s \le K_0, \quad \forall\, 0 < r \le 1, \tag{A.14}
$$

and $\pi \in L^{\frac{3}{2}}(Q(1))$ solve $-\Delta \pi = \nabla \cdot \nabla \cdot f$ in the sense of distributions. Then there exists a positive constant M such that for all $0 < r \le 1$,

$$
r^{-2} \int_{Q(r)} \left|\pi - (\pi)_{B(r)}\right|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}s \le M, \tag{A.15}
$$

with $(\pi)_{B(r)} = \fint_{B(r)} \pi(y,t) \, \mathrm{d}y$.

Proof. We write

$$
\pi - (\pi)_{B(r)} = \pi_{0,r} + \pi_{h,r},
$$

where $\pi_{0,r} = \mathscr{J}\left(f \cdot \chi_{B(r)}\right)$ and $\chi_{B(r)}$ the indicator function on the set $B(r)$. Thus, $\Delta \pi_{h,r} = 0$ on $B(r)$ and

$$
\left\|\pi_{0,r}\right\|_{L^{\frac{3}{2}}(B(r))} \le C \|f\|_{L^{\frac{3}{2}}(B(r))}.
$$

Let $\theta \in (0, \frac{1}{2}]$. Using Poincaré's inequality and the mean value property of $\pi_{h,r}$, we obtain

$$
\begin{aligned}
&\left\|\pi_{h,r} - \left(\pi_{h,r}\right)_{B(\theta r)}\right\|_{L^{\frac{3}{2}}(B(\theta r))} \\
\lesssim &(\theta r)^3 \left\|\nabla \pi_{h,r}\right\|_{L^\infty(B(\theta r))} \\
\lesssim &\left(\frac{\theta}{1-\theta}\right)^3 \left\|\pi_{h,r}\right\|_{L^{\frac{3}{2}}(B(r))}
\end{aligned}
$$

$$\lesssim \theta^3 \left\| \pi - (\pi)_{B(r)} \right\|_{L^{\frac{3}{2}}(B(r))} + \|f\|_{L^{\frac{3}{2}}(B(r))}.$$

Accordingly, for $0 < r \leq 1$,

$$\begin{aligned}
&\left\| \pi - (\pi)_{B(\theta r)} \right\|_{L^{\frac{3}{2}}(B(\theta r))} \\
\lesssim &\left\| \pi_{h,r} - \left(\pi_{h,r}\right)_{B(\theta r)} \right\|_{L^{\frac{3}{2}}(B(\theta r))} + \left\| \pi_{0,r} - \left(\pi_{0,r}\right)_{B(\theta r)} \right\|_{L^{\frac{3}{2}}(B(\theta r))} \\
\lesssim &\theta^3 \left\| \pi - (\pi)_{B(r)} \right\|_{L^{\frac{3}{2}}(B(r))} + \|f\|_{L^{\frac{3}{2}}(B(r))},
\end{aligned}$$

and

$$\left\| \pi - (\pi)_{B(\theta r)} \right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(\theta r))} \leq C_3 \cdot \theta^{\frac{5}{2}} \cdot \theta^2 \left\| \pi - (\pi)_{B(r)} \right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(r))} + C_3 \|f\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(r))}. \quad \text{(A.16)}$$

We choose θ such that $C_3 \cdot \theta^{\frac{5}{2}} = \frac{1}{2}$. For $\theta < r \leq 1$,

$$r^{-2} \left\| \pi - (\pi)_{B(r)} \right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(r))} \leq C_4 \theta^{-2} \|\pi\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(1))},$$

and by (A.14) and (A.16),

$$(\theta r)^{-2} \left\| \pi - (\pi)_{B(\theta r)} \right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(\theta r))} \leq C_4 \theta^{-2} \left(\|\pi\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(1))} + K_0 \right),$$

with $C_4 = 2C_3 + 100$. This together with a standard iteration yields that for $0 < r \leq 1$,

$$r^{-2} \left\| \pi - (\pi)_{B(r)} \right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(r))} \leq C_4 \theta^{-2} \left(\|\pi\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q(1))} + K_0 \right).$$

The proof is completed. □

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