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Global Attractors and Steady State Solutions for a Class of Reaction–Diffusion Systems

Le Dung

*Department of Mathematics, Arizona State University, Tempe Arizona 85287**

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We show that weak L^p dissipativity implies strong L^∞ dissipativity and therefore implies the existence of global attractors for a general class of reaction–diffusion systems. This generalizes the results of Alikakos and Rothe. The results on positive steady states (especially for systems of three equations) in our earlier work (*J. Differential Equations* **130** (1996), 59–91) are improved.

Key Words: Sobolev inequalities; a priori estimates; reaction–diffusion systems; evolution operators; index theory.

1. INTRODUCTION

Reaction–diffusion systems have been studied extensively in different context and by various methods. A large part of literature devotes to the study the asymptotic behavior of the dynamics generated by the systems (see [21]). Many important and interesting information on the dynamics of solutions can be obtained if the systems generate dissipative semiflows on appropriate Banach spaces which are usually the spaces (or products) of non-negative continuous functions with supremum norms. To establish the dissipativeness we need a priori estimates on various norms of the solutions. In general, this problem is by no means trivial. Appropriate a priori estimates guarantee in turn the global existence of solutions and sometimes even the existence of a compact set that attracts all solutions eventually (see, for instance, [20, 32]). Such a set is called the global attractor and carries information on the asymptotic behavior of the solutions.

The problem to be considered in this paper is the system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \mathscr{A}_i(t, x, D) u_i + f_i(t, x, u) & t > 0, \quad x \in 0, \quad i = 1, ..., m, \\ \mathscr{B}_i(x, D) u_i = v_i^0 & \text{on } \partial\Omega, \quad t > 0, \\ u_1(0, x) = u_i^0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

* Current address: Georgia Institute of Technology, Center for Dynamical Systems and Nonlinear Studies, Atlanta, Georgia, 30332-0190.

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where $u=(u_1, ..., u_m)$, Ω is a bounded open set in R^N with smooth boundary $\partial\Omega$, and $\mathscr{A}_i(t, x, D)$'s are linear elliptic operators, and $\mathscr{B}_i$'s are regular elliptic boundary operators. In this form, (1.1) represents many reaction diffusion models in ecology, biology, chemistry, etc. Nonlinear diffusion systems (nondegenerate or degenerate) were studied in [8–10, 16].

Our first results in Sections 2 and 3 concern the strong L^∞-estimates and dissipativity of the solutions. We will show that such estimates (or dissipativity results) can be obtained if L^p estimates, with p sufficiently large, are known. This type of result is quite suitable for reaction diffusion systems encountered in applications. Because, in many cases, the components u_i of the solutions are usually nonnegative functions, and therefore by a simple integration over the domain we can obtain an ordinary differential equation (or differential inequality) for the spatial averages of u_i and derive estimates for their L^p norms from this simpler system. In [8–10], we deal with nonlinear diffusion systems, a different technique has been used to obtain results which are similar to those of this work.

The L^∞ estimates which imply only global existence results had been derived by using a Moser-type iterative method in the works of Alikakos (see [1, 2]) for *scalar equations* with homogeneous Neumann boundary condition and restricted structure (specifically, he consider equations whose diffusion terms are Laplacian and reaction terms are linear). In [30], F. Rothe devised an alternative technique using a “feedback” argument to obtain similar results. However, their estimates generally depend on the norms of the initial data and therefore are not sharp enough to give the dissipativeness and the compactness of the trajectories.

Alikakos' technique was refined and combined with an induction argument by Cantrell, Cosner, Hutson, and Schmitt in [6, 23] to establish the dissipativity of the semiflows generated by some ecological models. These authors then applied this estimate to systems of Lodka–Volterra type whose reaction terms satisfy the so-called food pyramid condition or its related versions so that they can reduce the problem to one equation.

Meanwhile, we should mention here the duality technique which was originally developed by Hollis, Martin, and Pierre [22] and then generalized by Morgan [27]. This entirely different approach has been quite sucessful in proving the global existence of the solutions. Roughly speaking, the method works well with systems satisfying some sort of generalized Lyapunov structure from which one can obtain the L^p estimates for certain Lyapunov functional of the components of the solution. The key idea is then to show that if the solution does not exist globally then the L^p norms of its components must blow up together and therefore is a contradiction. This method did not give explicit estimates for the L^∞ norms nor those for stronger norms of the solutions to obtain the dissipativity and compactness we need here.

Here we consider semilinear parabolic systems satisfying a general structure and boundary conditions (see (i)–(iii) of Section 3). The technique in this paper works directly with the whole system, and therefore allows us to drop this food pyramid condition on the nonlinearities. Actually, the use of the theory of evolution operators makes the proof simpler than those of the techniques mentioned above. In addition, this also gives us the estimates for the Hölder norms of solutions and thus the compactness of the trajectories, a crucial factor in the proof of the existence of the global attractor.

In many cases, the dissipativity of the system could not be seen explicitly from the reaction terms and so the food pyramid condition was unlikely. More often, there will be some sort of interaction among the reaction terms of the equations (and even between these terms and the diffusions) that still gives the dissipativeness. We call this *cancellation and growth* and formulate it in condition (F) (see also (Cp) and (Ap) at the end of Section 3).

Estimates which are uniform with respect to the initial data also play an important role in the study of steady-state solutions (especially when one uses the technique of index theory, see Theorem 4.3). We address another issue on the existence of steady-state solutions of (1.1) in the remainder of our study when the system is autonomous. Although the existence of the global attractor may guarantee that there is such a solution in that set, the solution can be the trivial one as it frequently occurs in applications. Therefore it is more interesting (and more difficult) to find conditions which ensure the existence of another nontrivial solution for the elliptic system associated to (1.1) (autonomous case)

$$\begin{cases} 0=\mathscr{A}_i(x,D)u_i+f_i(x,u) & x\in\Omega,\quad i=1,\ldots,m, \\ \mathscr{B}_i(x,D)u_i=v_i^0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This problem has been studied extensively in different contexts such as ecology, biology, etc., because of the interest in finding conditions for co-existence states of competing species in the models.

Here we use the index theory as in [13] to establish conditions for the existence of nontrivial solutions for (1.2) under very general structure conditions. For $m\geqslant 3$, this solution may be a semitrivial one (only two components are nonzero). Sufficient conditions for coexistence when $m=3$ will be derived without the uniqueness assumptions on semitrivial solutions as in [13].

We remark here that index theory was also used by Hadeler *et al.* and Rothe in [19, 29] to obtain existence results of at least one steady state. They assumed that there exists a bounded invariant region for the systems under consideration so that uniform estimates are thus easily obtained. As we mentioned above, such steady states could be the trivial one as in the models considered in this paper. The results of these authors should be compared with that of Corollary 3.6 of this work, where the existence of at least one steady state is a by-product of the existence of the global attractor. Our homotopy arguments to obtain the existence results of nontrivial steady states are, of course, completely different.

Besides the standard Schauder's and asymptotic fixed point theories, there is an interesting theory of permanence of dynamical systems which can be used to show the existence of nontrivial steady states for some models (for instance, see [6, 23] for models of Kolmogorov type). Having its own importance in understanding the dynamics, this theory is quite difficult to be applied in practice. One needs to either understand fairly well the dynamics of the boundary semiflows to establish their *acyclicity* or construct the so-called *averaged Lyapunov functions*. As far as we know this method has been used only for systems of two equations ($m=2$) for which, in some special cases, the boundary dynamics can be analyzed by studying those of scalar equations.

The sub- or super-solution technique as in [26, 28] require monotone structure on the system and therefore is more restricted. However, in some cases, this method can give valuable information on the stability of solutions.

We should mention that similar results for a 3-species competition with a diffusion of Lodka–Volterra type has been obtained in [7]. Our homotopy techniques are different and work for (1.2), which satisfies more general structures than those considered in [7] and the references therein.

2. L^p ESTIMATES

In this section, general conditions are described which ensure that one can obtain L^p-estimates for p arbitrarily large if a priori estimates of certain L^q norm are assumed. We consider the following general (nonlinear) reaction–diffusion system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \mathscr{A}_i u_i + f_i(t, x, u, Du_i) & t>0, \quad x\in\Omega, \quad i=1, ..., m, \\ \mathscr{B}_i u_i = v_i^0 & \text{on } \partial\Omega, \quad t>0, \\ u_i(0, x) = u_i^0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $u=(u_1, ..., u_m)$, Ω is a bounded open set in R^N with smooth boundary $\partial\Omega$. The differential operators are given by

$$\mathscr{A}_i v := D_k(a_k^i(t, x, v, Dv)), \qquad t>0, \quad x\in\Omega, \quad i=1, ..., m.$$

We can consider the mixed boundary conditions in (2.1). That is, $\partial\Omega$ may consist of two parts, $\partial\Omega_1$ and $\partial\Omega_2$, where our boundary conditions are either the Dirichlet condition

$$\mathscr{B}_i v := v = v_i^0, \qquad \text{on } \partial\Omega_1, \quad t > 0,$$

or the nonlinear Robin condition

$$\mathscr{B}_i v := \frac{\partial v}{\partial \mathscr{N}_i} + b_i(t, x, v) = v_i^0,$$

$$\text{with} \quad \frac{\partial v}{\partial \mathscr{N}_i} = a_k^i(t, x, v, Dv) \circ n_k(x) \qquad \text{on } \partial\Omega_2, \quad t > 0.$$

Here $n_k(x)$ denotes the cosine of the angle formed by the outward normal vector $n(x)$ and the x_k-axis. We follow the convention that repeated indices will be summed from 1 to N.

We assume that

(A) *The differential operators $\mathscr{A}_i$ are uniformly elliptic. That is, there exist positive constant ν_0, δ and non-negative measurable function μ_1, μ_2 such that for any $(t, x, u, p) \in R^+ \times \Omega \times R^m \times R^N$ and $i = 1, ..., m$,*

$$a_k^i(t, x, u, p)\, p_k \geqslant \nu_0 \, \|p\|^2 - \mu_1(t, x)\, |u|^\delta - \mu_2(t, x). \tag{2.2}$$

(B) *For the Robin boundary conditions, b_i's are continuous functions in their variables. In addition, there exist positive constants ν_1, ν_2 and $\beta \geqslant 1$ such that*

$$b_i(t, x, u)\, u \geqslant -\nu_1 \, |u|^{\beta+1} - \nu_2,$$

for all $(t, x) \in R^+ \times \partial\Omega_2$ and $u \in R$. Note that (A) *above implies that $\partial/\partial\mathscr{N}_i$ are regular oblique derivative boundary operators.*

Remark 2.1. The boundary operators $\partial/\partial\mathscr{N}_i$'s are not necessarily related to the operartors $\mathscr{A}_i$'s in the way described above. Other form of $\partial/\partial\mathscr{N}_i$ could be considered, provided that we still have similar estimates for the boundary integrals occuring from the use of integration by parts in the proof of Theorem 2.6.

Concerning the boundary and initial conditions, we assume that v_i^0, u_i^0 are bounded continuous functions on $R_+ \times \partial\Omega$ and Ω, respectively. We also denote $u_0 = (u_1^0, ..., u_m^0)$.

To obtain the L^p-estimates we need to impose the following *cancellation and growth conditions* on the nonlinearities f_i of (2.1).

(F) *There exists positive constants* α, σ *and non-negative measurable functions* k_1, k_2, k_3 *such that* $0 \leqslant \alpha < 2$ *and*

$$\sum_{i=1}^{m} f_i(t, x, u, \zeta_i)\, |u_i|^{p-1} u_i$$
$$\leqslant \qquad k_1(t, x) \sum_{i=1}^{m} |u_i|^{\sigma+p} + k_2(t, x) \sum_{i=1}^{m} |\zeta_i|^{\alpha}\, |u_i|^p + k_3(t, x), \tag{2.3}$$

for all $p \geqslant 1$ *and* $(t, x, u, \zeta) \in R_+ \times \Omega \times R^m \times R^N$. *Remember that* $u = (u_1, ..., u_m)$.

An easy consequence of the Young inequality implies that (2.3) holds if for $k = 1, ..., m$

$$|f_k(t, x, u, \zeta)| \leqslant c_1(t, x) \sum_{i=1}^{m} |u_i|^{\sigma} + c_2(t, x)\, |\zeta|^{\alpha} + c_3(t, x) \tag{2.4}$$

for some non-negative functions c_1, c_2, c_3. The functions k_i's are then just some linear combinations of the c_i's.

Concerning the functional spaces of the parameters in (A) and (F) we assume that

(P) *There exist real numbers* q, r *such that* $q > N/2$ *and* $r > N/(2-\alpha)$ *such that for each* $t \geqslant 0$ *the functions*

$$\mu_1, \mu_2, k_1, k_3 \in L^q(\Omega); \qquad k_2 \in L^r(\Omega),$$

with respect to the spatial variable $x \in \Omega$. *Furthermore, we assume that their corresponding* L^q, L^r *norms are uniformly bounded for all* $t \geqslant 0$. *That is, for some finite constant* M,

$$\|\mu_1, \mu_2, k_1, k_3(t, \bullet)\|_q, \qquad \|k_2(t, \bullet)\|_r \leqslant M, \qquad \textit{for all} \quad t \geqslant 0,$$

where $\|\bullet\|_p$ *denotes the* L^p *norm in* $L^p(\Omega)$.

Remark 2.2. We could allow all the constants in the hypotheses (A), (B), and (F) to belong to some weighted Lebesgue spaces. Our proof still works in this case by using the weighted Sobolev space inequalities developed in [11]. Fewer smoothness assumptions on a_k^i and f_i could be considered. Moreover, in many applications, special forms of some f_i's may directly give L^∞ bounds for the corresponding components of the solutions via comparison principles. This would relax the restrictions on the growth rates of these components in (2.3).

Our structure assumptions above allow us to apply the standard theory of quasilinear parabolic systems in divergence form (e.g. see [17, 25]) to assert the following local existence of solutions.

PROPOSITION 2.3. *Assume* (A), (B), *and* (F). *There is a positive number* $\tau(u_0)$ *such that there exists a unique solution for* (2.1) *on the maximal interval of existence* $(0, \tau(u^0))$.

In the proof, we will need the following consequence of the Nirenberg–Gagliardo inequality.

LEMMA 2.4. *Let* $p>0$, $\theta\geqslant 0$, $q>N/2$, *and* ϕ *is a non-negative measurable function in* $L^q(\Omega)$. *Suppose that*

$$\theta<p\left(\frac{2}{N}-\frac{1}{q}\right). \tag{2.5}$$

Let u *be a measurable function given on* Ω. *If* $w:=|u|^p\in W^{1,2}(\Omega)$, *then, for any given* $\varepsilon>0$, *there exists positive constant* C *depending only on* p, q, θ, $\|\phi\|_q$ *such that*

$$\int_\Omega \phi\,|u|^{2p+\theta}\,dx\leqslant\varepsilon\left(\int_\Omega |Dw|^2\,dx+\|w\|_1^2\right)+C(\varepsilon,p,q,\theta,\|\phi\|_q)\,\|w\|_1^l, \tag{2.6}$$

where $l=2+(\theta(2/N+1))/(p(2/N-1/q)-\theta)$.

Proof. Using the Holder inequality we have

$$\int_\Omega \phi\,|u|^{2p+\theta}\,dx\leqslant\|\phi\|_q\left(\int_\Omega |u|^{(2p+\theta)q'}\,dx\right)^{1/q'}=\|\phi\|_q\,\|w\|_{sq'}^s \tag{2.7}$$

where $1/q+1/q'=1$, $s=(2p+\theta)/p$. Apply the Nirenberg–Gagliardo inequality to the function w to get

$$\|\phi\|_q\,\|w\|_{sq'}^s\leqslant C\,\|\phi\|_q\,\|w\|_1^{s\beta}\,\|w\|_{W^{1,2}(\Omega)}^{s(1-\beta)} \tag{2.8}$$

where

$$\frac{1}{sq'}=\beta+\left(\frac{1}{2}-\frac{1}{N}\right)(1-\beta).$$

By simple calculations one can see that (2.5) is equivalent to the fact that $s(1-\beta)<2$. Therefore, we can apply the Young inequality to (2.8) and get

$$\|\phi\|_q\,\|w\|_{sq'}^s\leqslant\varepsilon\,\|u\|_{W^{1,2}(\Omega)}^2+C(\varepsilon,p,q,\theta,\|\phi\|_q)\,\|w\|_1^l \tag{2.9}$$

where $l = 2s\beta/(2 - s(1-\beta))$. Easy calculations show that l is given by the formula in the lemma. We use the following equivalent norm of $W^{1,2}(0)$ (see [33])

$$\|u\|_{W^{1,2}(\Omega)} = \left(\int_\Omega |Du|^2\, dx + \left(\int_\Omega |u|\, dx \right)^2 \right)^{1/2} \tag{2.10}$$

Then (2.6) follows from (2.7) and (2.9). ∎

The next lemma will be used to handle the boundary integrals.

LEMMA 2.5. *Let $\beta, \varepsilon > 0$ $\lambda \geqslant 1$ and $u \in W^{1,2}(\Omega)$. There exist positive constants ε, $C(\varepsilon)$ independent of u such that*

$$\int_{\partial\Omega} |u|^{\beta+\lambda}\, d\sigma \leqslant \varepsilon \int_\Omega |Du|^2\, |u|^{\lambda-1}\, dx + C(\beta, \varepsilon)\, \lambda^2 \int_\Omega (|u|^{\gamma+\lambda} + 1)\, dx, \tag{2.11}$$

where $\gamma = \max\{\beta, 2\beta - 1\}$.

Proof. Let $\zeta \in C^2(\bar{\Omega}, R^n)$ be any vector field satisfying $\zeta \cdot n = 1$ on $\partial\Omega$. We have

$$\begin{aligned}\int_{\partial\Omega} |u|^{\beta+\lambda}\, d\sigma &= \int_\Omega \operatorname{div}(|u|^{\beta+\lambda}\, \zeta)\, dx \\ &\leqslant C \int_\Omega \{(\beta+\lambda)\, |Du|\, |u|^{\beta+\lambda-1} + |u|^{\beta+\lambda}\}\, dx,\end{aligned}$$

where C is some positive constant depending on $|\zeta|$, $|D\zeta|$ (and thus, on $\partial\Omega$). Using the Young inequality we can majorize the first integrand on the right as follows.

$$\begin{aligned}&C(\beta+\lambda) \int_\Omega |Du|\, |u|^{\beta+\lambda-1}\, dx \\ &\qquad \leqslant \varepsilon \int_\Omega |Du|^2\, |u|^{\lambda-1}\, dx + C(\varepsilon, \beta)\, \lambda^2 \int_\Omega |u|^{2\beta+\lambda-1}\, dx.\end{aligned}$$

From these estimates we get

$$\int_{\partial\Omega} |u|^{\beta+\lambda}\, d\sigma \leqslant \varepsilon \int_\Omega |Du|^2\, |u|^{\lambda-1}\, dx + C(\varepsilon, \beta) \int_\Omega (\lambda^2\, |u|^{2\beta+\lambda-1} + |u|^{\beta+\lambda})\, dx.$$

Finally, we can use the Young inequality to combine the powers of $|u|$ in the last integral into $|u|^{\gamma+\lambda}$, with $\gamma = \max\{\beta, 2\beta - 1\}$. The proof is complete. ∎

We now ready to prove

THEOREM 2.6. *Let p_0 be such that*

$$p_0 > \max\left\{\left(\frac{2}{N} - \frac{1}{q}\right)^{-1} \max\{2\beta - 2, \sigma - 1, \delta - 2\}, (\alpha - 1)\left(\frac{2-\alpha}{N} - \frac{1}{r}\right)^{-1}\right\}. \tag{2.12}$$

Suppose that there exists a positive function $C_{p_0}(v^0, u^0)$ such that

$$\|u_i(t, \bullet)\|_{p_0} \leqslant C_{p_0}(v^0, u^0), \qquad \text{for all} \quad t \in (0, \tau(u^0)), \tag{2.13}$$

then for any $p \geqslant p_0$ there exists a positive function $C_p(v^0, u^0)$ such that

$$\|u_i(t, \bullet)\|_p \leqslant C_p(v^0, u^0), \qquad \text{for all} \quad t \in (0, \tau(u^0)). \tag{2.14}$$

Alternatively, if there is a number K_{p_0} independent of initial data such that

$$\limsup_{t \to \tau(u^0)} \|u_i(t, \bullet)\|_{p_0} \leqslant K_{p_0}, \tag{2.15}$$

then there exists a number K_p independent of initial data such that

$$\limsup_{t \to \tau(u^0)} \|u_i(t, \bullet)\|_p \leqslant K_p. \tag{2.16}$$

Proof. We shall prove by induction. Let us assume that (2.14) holds for some $p \geqslant p_0$ (it holds for $p = p_0$). Consider the equation for u_i. Multiply the equation by $u_i\,|u_i|^{2p-2}$ and integrate to get

$$\int_\Omega u_i\,|u_i|^{2p-2} \frac{\partial u_i}{\partial t} = \int_\Omega \mathscr{A}_i(u_i)\, u_i\,|u_i|^{2p-2}\,dx + \int_\Omega f(t, x, u, Du_i)\, u_i\,|u_i|^{2p-2}\,dx. \tag{2.17}$$

Put $w_i = |u_i|^p$ and notice that

$$\int_\Omega u_i\,|u_i|^{2p-2} \frac{\partial u_i}{\partial t}\,dx = \frac{1}{2p}\frac{d}{dt}\int_\Omega w_i^2\,dx.$$

Integration by parts and the boundary conditions give

$$
\begin{aligned}
&\int_\Omega \mathscr{A}_i(u_i)\, u_i\, |u_i|^{2p-2}\, dx \\
&\quad = \int_{\partial\Omega} u_i\, |u_i|^{2p-2} \frac{\partial u_i\, d\sigma}{\partial \mathscr{N}_i} - (2p-1) \int_\Omega a_i(\cdots)\, D_i u_i\, |u_i|^{2p-2}\, dx \\
&\quad \leqslant -\frac{\nu_0(2p-1)}{p^2} \int_\Omega |Dw_i|^2 + C_1(p) \int_\Omega (\mu_1\, |u_i|^{2p-2+\delta} + \mu_2\, |u_i|^{2p-2})\, dx \\
&\qquad + C(\nu_1, \nu_2, \nu_0) \int_{\partial\Omega} (|u_i|^{\beta+2p-1} + |u_i|^{2p-2} + |u_i|^{2p-1})\, d\sigma.
\end{aligned}
$$

Using these estimates in (2.17), summing over i, and taking into account (2.3) of (F) we find

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega \sum_{i=1}^m w_i^2\, dx \leqslant -2\nu_0 \int_\Omega \sum_{i=1}^m |Dw_i|^2\, dx + \int_\Omega \bar{k}_1 \sum_{i=1}^m |u_i|^{2p+\theta_1}\, dx \\
+ \int_\Omega k_2 \sum_{i=1}^m |Du_i|^\alpha\, |u_i|^{2p-1}\, dx + \int_\Omega \bar{k}_3\, dx,
\end{aligned} \tag{2.18}
$$

where the functions $\bar{k}_1, \bar{k}_3$ are some linear combinations of μ_1, μ_2, k_1, k_3. Here we have used Lemma 2.5 to convert the boundary integrals into the volume ones, and then the Young inequality to combine the powers of u_i to $2p + \theta_1$ with $\theta_1 = \max\{2\beta - 2, \delta - 2, \sigma - 1\}$.

We are now going to estimate the terms on the R.H.S of (2.18). For the third term, we have

$$
\begin{aligned}
\int_\Omega k_2\, |Du_i|^\alpha\, |u_i|^{2p-1}\, dx &= p \int_\Omega k_2\, |Dw_i|^\alpha\, |u_i|^{p(2-\alpha)+\alpha-1}\, dx \\
&\leqslant \varepsilon \int_\Omega |Dw_i|^2\, dx + C(\varepsilon, p) \int_\Omega \bar{k}_2\, |u_i|^{2p+\theta_2}\, dx,
\end{aligned} \tag{2.19}
$$

where $\bar{k}_2 := k_2^{2/(2-\alpha)} \in L^{r(2-\alpha)/2}(\Omega)$ and $\theta_2 = 2(\alpha - 1)/(2 - \alpha)$.

Simple calculations show that the functions $\bar{k}_1, \bar{k}_2$, the exponents q, $r(2-\alpha)/2$, and θ_1, θ_2 satisfy the assumptions of Lemma 2.4 if our conditions (P) and (2.12) are given. Thus, we can apply (2.6) in that lemma to majorize the integrals of u_i's in (2.18) and (2.19) by

$$
\varepsilon \left(\int_\Omega |Dw_i|^2 + \left(\int_\Omega w_i\, dx \right)^2 \right) + K(\varepsilon) \left(\int_\Omega w_i\, dx \right)^l
$$

for some positive constants l, $K(\varepsilon)$.

Putting these estimates together and choosing ε small enough we obtain from (2.18) that

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^m w_i^2\,dx \leqslant -v_0\int_\Omega \sum_{i=1}^m |Dw_i|^2\,dx + l_1\sum_{i=1}^m\left(\int_\Omega w_i\,dx\right)^2 + l_2\sum_{i=1}^m\left(\int_\Omega w_i\,dx\right)^l + l_3$$

where l_i's are positive constants independent of u_i. Applying (2.6) again with $p=1$, $\theta=0$, and $q=\infty$, we get

$$\int_\Omega w_i^2\,dx \leqslant \left(\int_\Omega |Dw_i|^2\,dx + \left(\int_\Omega w_i\,dx\right)^2\right) + C\left(\int_\Omega w_i\,dx\right)^2$$

for some $C>0$. Finally, we have

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^m w_i^2\,dx \leqslant -v_0\int_\Omega \sum_{i=1}^m w_i^2\,dx + \sum_{i=1}^m\left\{l_1\left(\int_\Omega w_i\,dx\right)^2 + l_2\left(\int_\Omega w_i\,dx\right)^l\right\} + l_3.$$

The asserted estimates now follow by applying the induction hypotheses, noting that $\int_\Omega w_i = \|u_i(t,\bullet)\|_p^p$ and $\int_\Omega w_i^2 = \|u_i(t,\bullet)\|_{2p}^{2p}$, and integrating the last inequality. ∎

One may try to follow an iterative argument similar to those in [2, 30] to obtain the L^∞-estimates from the limit induction process above. However, a direct calculation of the exponents which involve in the recursive relations reveals that the exponents all diverge as p goes to infinity. More works need to be done to control these exponents and to derive a better inequality which is suitable for this process (see [8–10]).

On the other hand, to show the existence of the global attractor, we also need estimates on certain stronger norms (such as the Hölder norms) to obtain the compactness of the trajectories of bounded sets of initial data. This does not come from the iterative argument mentioned above. In the next section, when the system is semilinear, we will make a simple use of the theory of evolution operator in L^p spaces to derive estimates for both L^∞- and Hölder-norms. For the estimate of Hölder norms of solutions of nonlinear diffusion systems we refer to the works [14–16] where a more sophisticated technique has been developed to achieve this.

3. L^∞-ESTIMATES

In this section, we shall show that L^p-estimates (resp. weak L^p-dissipativity) for large p can be translated into L^∞-estimates (resp. strong L^∞-dissipativity) Since our primary interest here is to apply our results to the systems frequently encountered in applications we will restricted ourself to the case of *semilinear systems* with nonlinearities not depending on gradients.

On the other hand, we prefer to base our consideration on the easily accessible and well established theory of nonautonomous evolution equations with operators having constant domains rather than using more recent results, e.g., by Amann [4], to allow the boundary conditions depend also on t (but see also [8–10]).

Let us consider the following semilinear parabolic system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \mathscr{A}_i(t, x, D) u_i + f_i(t, x, u) & t>0, \quad x \in \Omega, \quad i=1, ..., m \\ \mathscr{B}^i(x, D) u_i = 0 & \text{on } \partial\Omega, \quad t>0 \\ u_i(0, x) = u_i^0(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where

$$\mathscr{A}_i(t, x, D) u = D_k(a^i_{kl}(t, x)\, D_l u) + a^i_k(t, x)\, D_k u, \qquad t>0, \quad x\in\Omega, \quad i=1, ..., m,$$

and

$$\mathscr{B}_i v := v, \qquad \text{on } \partial\Omega_1; \qquad \mathscr{B}_i v := \frac{\partial u}{\partial \mathscr{N}_i} + r_i(x) u, \qquad \text{on } \partial\Omega_2.$$

We impose the following smoothness conditions on (3.1).

(i) $a^i_{kl} \in C^{1+\theta}(R_+ \times \Omega)$, $a^i_k \in C^\theta(R_+ \times \Omega)$ for some positive θ; and α^i_{kl}'s, r_i's are continuous functions on $\partial\Omega_2$.

(ii) (*Ellipticity*). There are positive constants λ, Λ, such that

$$\lambda\, |\zeta|^2 \leqslant a^i_{kl}(t, x)\, \zeta_k \zeta_l \leqslant \Lambda\, |\zeta|^2$$

for all $x \in \Omega$, $\zeta \in R^N$ and $t \in R_+$.

(iii) (*Growth Condition*). There exist positive constant σ and nonnegative measurable functions k_1, k_2 such that

$$\sum_{i=1}^m f_i(t, x, u)\, |u_i|^{p-1}\, u_i \leqslant k_1(t, x) \sum_{i=1}^m |u_i|^{\sigma+p} + k_2(t, x), \tag{3.2}$$

for all $p \geqslant 1$ and for all $(t, x, u) \in R_+ \times \Omega \times R^m$. Assume that k_1, k_2 belong to $L^q(\Omega)$ for some $q > N/2$ and for some finite constant M

$$\|k_1(t, \bullet)\|_q, \qquad \|k_2(t, \bullet)\|_q \leqslant M, \qquad \text{for all} \quad t \geqslant 0.$$

PROPOSITION 3.1. *Assuming* (i)–(iii), *and let* $p_0 > ((2/N) - (1/q))^{-1} (\sigma - 1)$, *then the result of Theorem* 2.6 *hold for the system* (3.1).

Proof. We simply set $\bar{f}_i(t, x, u, \zeta) = a^i_k(t, x)\, \zeta_k + f_i(t, x, u)$ and observe that the $\bar{f}_i$'s satisfy the conditions (F) and (P) with $r \equiv \infty$, $\alpha \equiv 1$ and $\beta \equiv 1$. ∎

Let us consider bounded continuous initial data u_i^0's. We can regard our problem in larger class of measurable functions L^p, $1 < p < \infty$. Let $\mathscr{X} = L^p(0)$ and $A_i(t)$ be the realization of $(\mathscr{A}_i, \mathscr{B}_i)$ in $\mathscr{X}$. That is,

$$\begin{cases} \operatorname{dom}(A_i(t)) = W^{2,p}_{\mathscr{B}_i}(\Omega) = \{v \in W^{2,p}(0) : \mathscr{B}_i(v) = 0\} \\ A_i(t)v = \mathscr{A}_i(t, x, D)v. \end{cases}$$

Let $u = (u_1, ..., u_m)$ and $u_0 = (u_1^0, ..., u_m^0)$. We can write abstractly our system as

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = A(t)u + F(t, u) & \text{in } X = \mathscr{X} \times \cdots \times \mathscr{X} \ (m \text{ times}) \\ u(0) = u_0 \end{cases} \tag{3.3}$$

where $A(t) = \operatorname{diag}\{A_i(t)\}$ and $F(t, u) = (f_1(t, x, u), ..., f_m(t, x, u))$. Under the smoothness assumptions (i)–(iii) we easily see that $A(t)$ is a family of closed linear operators on X and satisfies all the conditions in [17] to ensure the existence of the *evolution operators*

$$U(t, \tau) \in \mathscr{L}(X) \qquad 0 \leqslant \tau \leqslant t < \infty. \tag{3.4}$$

So, the solution of (3.3) can be represented in the form

$$u(t) = U(t, 0)(u_0) + \int_0^t U(t, s)\, F(s, u(s))\, ds. \tag{3.5}$$

We have the following estimate concerning the operator $U(t, s)$: There exist positive numbers ω, C_γ such that for any $0 \leqslant \gamma \leqslant 1$ and $0 \leqslant s < t < \infty$ (see (16.38) of [17])

$$\|A^\gamma(t)\, U(t, s)\|_{\mathscr{L}(X)} \leqslant \frac{C_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma}. \tag{3.6}$$

Remark 3.2. We notice that (3.6) comes from Theorem 14.1 and (13.18), (13.19) in [17] which still hold if the function $\eta(\mu)$ in Lemma 13.1 (see (13.9)) in the reference is bounded in μ. This is satisfied here because of the uniform ellipticity conditions in (ii).

We are now ready to show that

THEOREM 3.3. *Let $p_0 > (2/N - 1/q)^{-1} (\sigma - 1)$. Suppose that* (i)–(iii) *hold and there exists a positive function $C_{p_0}(v_0, u_0)$ such that*

$$\|u_i(t, \bullet)\|_{p_0} \leqslant C_{p_0}(v_0, u_0) \qquad 0 \leqslant t < \tau(u_0). \tag{3.7}$$

then the solution exists for all time $(\tau(u_0) = \infty)$ *and there is a positive continuous function C_∞ such that*

$$\|u_i(t, \bullet)\|_\infty \leqslant C_\infty(v_0, u_0) \qquad 0 \leqslant t < \infty. \tag{3.8}$$

Alternatively, if there is a finite number K_{p_0} independent of initial data such that

$$\limsup_{t \to \tau(u^0)} \|u_i(t, \bullet)\|_{p_0} \leqslant K_{p_0} \tag{3.9}$$

then there exists a finite number K_∞ independent of initial data such that

$$\limsup_{t \to \infty} u_i(t, \bullet)\|_\infty \leqslant K_\infty \tag{3.10}$$

Remark 3.4. The fact that (3.7) implies (3.8) has been proved by Alikakos and Rothe [2, 30]. We are interested here the implication of (3.10) from (3.9).

Proof. Apply $A^\gamma(t)$ to both sides of (3.5) to have

$$A^\gamma(t)\, u(t) = A^\gamma(t)\, U(t, 0)(u_0) + \int_0^t A^\gamma(t)\, U(t, s)\, F(s, u(s))\, ds.$$

From the result of the previous section and the polynomial growth condition on f_i's we can find a positive continuous function C_p such that

$$\|F(t, u(t))\|_p \leqslant C_p(v_0, u_0), \qquad \forall t \in (0, \tau(u_0)).$$

Then,

$$\begin{aligned}\|A^\gamma(t)\,u(t)\|_p &\leqslant \|A^\gamma(t)\,U(t,0)(u_0)\|_p + \int_0^t \|A^\gamma(t)\,U(t,s)\|_{\mathscr{L}(X)}\,\|F(s,u(s))\|_p\,ds\\ &\leqslant C_\gamma t^{-\gamma}e^{-\omega t}\,\|u_0\|_p + \int_0^t C_\gamma(t-s)^{-\gamma}\,e^{-\omega(t-s)}\,\|F(s,u(s))\|_p\,ds\\ &\leqslant C_\gamma t^{-\gamma}e^{-\omega t}\,\|u_0\|_p + C_p(v_0,u_0)\int_0^t C_\gamma(t-s)^{-\gamma}\,e^{-\omega(t-s)}\,ds\\ &\leqslant C_\gamma t^{-\gamma}e^{-\omega t}\,\|u_0\|_p + C_p(v_0,u_0)\int_0^\infty C_\gamma r^{-\gamma}e^{-\omega r}\,dr. \qquad (3.11)\end{aligned}$$

Because of the uniform ellipticity condition (ii) of the operator $A(t)$, we see that

$$\sup_{0<t,\,s<\infty} \|A(t)\,A^{-1}(s)\|_{\mathscr{L}(X)} < \infty.$$

So, we can obtain from the above estimates that

$$\|A^\gamma(t_0)\,u(t)\|_p \leqslant CC_\gamma t^{-\gamma}e^{-\omega t}\,\|u_0\|_p + CC_p(v_0,u_0)\int_0^\infty C_\gamma r^{-\gamma}e^{-\omega r}\,dr \qquad (3.12)$$

for some fixed positive constants t_0, C. Consider the space $Y^\gamma \equiv D(A^\gamma(t_0))$ with the graph norm $\|u\|_{Y^\gamma} = \|A^\gamma(t_0)u\|_p$. We choose p such that $N/2p<\gamma<1$ and note the imbedding

$$Y^\gamma \to C^\nu, \qquad 0 \leqslant \nu < 2\gamma - N/p.$$

This imbedding and (3.12) show that

$$\|u(t)\|_{C^\nu} \leqslant C_\infty(v_0, u_0)$$

for $t \geqslant 1$. To bound the uniform norm of $u(t)$ for $t\in[0,1]$ we note that $\|U(t,0)\|_{\mathscr{L}(\mathscr{C}_m)}$ is uniformly bounded on $[0,1]$ ($\mathscr{C}_m \equiv \prod_1^m C(\Omega)$). The integral term in (3.5) can be estimated in the space Y^γ exactly as before. It follows immediately from this and Theorem 3.3.4 of [21] that $u(t)$ is defined for all $t\geqslant 0$ and we get (3.8).

To obtain (3.10) assuming (3.9), we see that there is a $\eta=\eta(u_0)$ and a positive constant c, independent of u_0, such that

$$\|F(s,u(s))\|_p \leqslant \begin{cases} C_p(v_0,u_0) & \text{for } \; 0\leqslant s\leqslant \eta,\\ c & \text{for } \; \eta<s<\tau(u_0).\end{cases}$$

Then by splitting the integral term in (3.11) into integrals on $(0, \eta)$ and $(\eta, \tau(u_0))$, we obtain similarly the following

$$\|A^\gamma(t)\, u(t)\|_p \leqslant C_\gamma t^{-\gamma} e^{-\omega t}\, \|u_0\|_p + C_p(v_0, u_0)\, C_\gamma \eta t^{-\gamma} e^{-\omega t}$$
$$+ c \int_0^\infty C_\gamma r^{-\gamma} e^{-\omega r}\, dr$$

This obviously gives (3.10). Our proof is complete. ∎

Remark 3.5. It is also possible to consider nonhomogeneous boundary conditions, that is, $\mathscr{B}_i(x, D) u = v_i^0(x, t)$ with $v_i^0 \not\equiv 0$. In this case, we consider the temporal variable t as a parameter and let $\{u_{i*}^t\}_{t>0}$ be the family of unique solution of the BVP

$$\begin{cases} D_k(a_{kl}^i(t, x)\, D_l u) + a_k^i(t, x)\, D_k u = 0 & \text{in } \Omega \\ \mathscr{B}_i(x, D) u = v_i^0(x, t) & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Let $u = (u_1, ..., u_m)$ and $u_*^t = (u_{1*}^t, ..., u_{m*}^t)$. Set $w = u - u_*$. By replacing u by w and u_0 by $u_0 - u_*^t$ in the above argument we conclude that (3.10) (or (3.8)) holds for w. If we assume that the smoothness conditions (i) and (ii) are uniform with respect to the variable t then, from the Schauder estimates for elliptic equations (e.g., see [18]), we can see that $\|u_*^t\|_\infty$ and $\|u_*^t\|_{C^\nu}$ are bounded uniformly with respect to t. Therefore, the estimates on w imply similar estimates on u.

COROLLARY 3.6. *Suppose that the hypotheses of Theorem* 3.3 *hold and the system* (3.1) *is autonomous. Then system* (3.1) *generates a nonlinear semidynamical system*

$$T(t)\colon \mathscr{C}_m := \prod_1^m C(\Omega) \to \mathscr{C}_m$$
$$(u_1^0, ..., u_m^0) \mapsto (u_1(t), ..., u_m(t)),$$

where $\mathscr{C}_m$ is equiped with the supremum norm.

Moreover, if (3.9) *holds then there exists a compact, connected, invariant global attractor A which attracts every bounded set in $\mathscr{C}_m$. In addition, A has finite Hausdorff dimension and contains at least one steady state solution of* (3.1).

Proof. The estimates of Theorem 3.3 imply the global existence of the solutions of (3.1) and therefore $T(t)$ is well defined for all $t \geqslant 0$. From the proof of that theorem, one can see that the uniform norm can be replaced by the C^ν norm, for some $\nu > 0$, in the estimates. On the other hand, the imbedding $C^\nu(0) \to C(0)$ is compact so that $T(t)$ is compact for $t > 0$.

Therefore, if (3.9) holds then $T(t)$ satisfies the following:

(a) $T(t)$ is point dissipative;

(b) orbits of bounded sets are bounded, that is, $\{T(t)\,B : t > 0\}$ is bounded for any bounded subset $B \subset \mathscr{C}_m$.

These facts and the general theory for dissipative dynamical systems and asymptotic fixed point theorems in [20, Theorem 3.4.7] give the last assertion. ∎

We close this section by presenting a slight generalization of a technique that was used in [13] to obtain a priori L^p-estimates. We assume the following.

(Cp) *There exist positive constants* h_i *and real constants* k, c *such that for all* $u \in R^{m+1}$

$$\sum_{i=0}^{m} h_i u_i^{p-1} f_i(t, x, u) \leqslant k \sum_{i=0}^{m} h_i u_i^p + c \tag{3.14}$$

for some $p \geqslant 1$.

(Ap) *There exist a positive function* ϕ *in* $\bar{\Omega}$ *and positive constants* μ, C *such that for* $i = 0, ..., m$ *we have*

$$\int_\Omega \mathscr{A}_i(t, x, D)\, u_i^{p-1} \phi\, dx \leqslant -\mu \int_\Omega u_i^p \phi\, dx + C \tag{3.15}$$

and that $\alpha = \mu - k > 0$, *for* $p \geqslant 1$ *and* k *given in* (Cp).

In practice, the function ϕ in (Ap) can be chosen as the principal eigenfunction of some linear elliptic operator relative to the $\mathscr{A}_i$'s. We refer to [13, 16] for concrete examples.

We have the following L^p-estimates.

Proposition 3.7. *Suppose that the system* (3.1) *satisfies the condition* (Cp) *and* (Ap). *There exist positive constants* C_1, C_2 *independent of* u_{i0}*'s such that*

$$\sum_{i=0}^{m} \|u_i(t, \bullet)\|_p \leqslant C_1(1 - e^{-p(\mu - k)t}) + C_2 \sum_{i=0}^{m} \|u_{i0}\|_p\, e^{-p(\mu - k)t}. \tag{3.16}$$

So, (3.7) *and* (3.9) *are verified.*

Proof. Multiply the equation for u_i by $h_i u_i^{p-1}\phi$ and integrate over Ω to obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega h_i u_i^p \phi\, dx = h_i \int_\Omega \mathscr{A}_i(t, x, D)\, u_i(u_i^{p-1}\phi)\, dx + \int_\Omega h_i u_i^{p-1}\phi f_i(t, x, u)\, dx. \tag{3.17}$$

Now set $H(u)=\sum_{i=0}^m h_i u_i^p$, use (3.15) in (3.17), and add the resulting inequalities to get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega H(u)\,\phi\, dx &\leqslant -\mu \int_\Omega H(u)\,\phi\, dx + \int_\Omega \phi \sum_{i=0}^m h_i u_i^{p-1} f_i(t, x, u)\, dx + C\\ &\leqslant -\alpha \int_\Omega H(u)\,\phi\, dx + c + C\end{aligned}$$

where we have also used (3.14) and the fact that $\alpha = \mu - k > 0$. Integrating the inequality gives

$$\int_\Omega H(u)\,\phi\, dx \leqslant e^{-p\alpha t}\int_\Omega H(u)\,\phi\, dx + \frac{C+c}{\alpha}(1-e^{-p\alpha t}).$$

As $\phi(x)>0$ in $\bar{\Omega}$, (3.16) follows. ∎

4. APPLICATION TO THE EXISTENCE OF EQUILIBRIA

In this section we consider the steady-state problem of (3.1). The equations are

$$\begin{cases} -\mathscr{A}_i(x, D)\, u_i = f_i(x, u), & x \in \Omega, \quad i = 0, ..., m.\\ \mathscr{B}_i(x, D)\, u_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

We assume that the assumptions of Theorem 3.3 hold for (4.1) so that there exists at least one solution for (4.1) according to Corollary 3.6. However, in many applications, $u \equiv 0$ is usually a trivial solution of (4.1) and thus the above conclusion is not interesting enough. We want to study the existence of nontrivial solutions to (4.1) in these cases.

Assume that

(F1). *The functions* f_i: $\Omega \times R_+^{m+1} \to R$ *satisfy*:

(i) $f_i(x, u) = 0$ *whenever* $u_i = 0$;

(ii) *for* $0 \leqslant i \leqslant m$, *there exist constants* $k_i \geqslant 0$ *such that* $f_i(x, u) + k_i u_i \geqslant 0$ *for* $u \in R_+^{m+1}$.

Without lost of generality we may also suppose that $(\partial f_i/\partial u_i)(x, 0) + k_i > 0$ for $x \in \bar{\Omega}$, by choosing k_i larger if necessary. This condition also guarantees that the non-negative cone R_+^{m+1} remains positively invariant under the dynamics of (3.1) (see e.g. [31]).

Note that certain models in applications may seem not to satisfy the above condition (F1) at first glance. But we shall see below that they can be reduced to our case by using a simple change of variables. For example, there may be a special role played by the first component u_0 in the model. In biological applications, u_0 usually stands for nutrient or prey densities and u_i, $1 \leqslant i \leqslant m$, denote the concentrations of bacteria or predators consuming u_0. We may have the following natural assumptions on the model: (i) If no bacteria u_i is present then none can be produced and if no nutrient is present then no consumption of nutrient occurs; (ii) if all bacteria are absent from the bio-reactor then no consumption of nutrient takes place; (iii) if there is no nutrient, then there can be no growth of bacteria; (iv) the system is open in the sense that fresh nutrient is supplied from an external reservoir while growth medium, including unused nutrient and bacteria, is removed. This interaction with the external environment occurs at the boundary of the domain.

Mathematically, these facts impose the followings on the model

(F1′) *The functions* $f_i \colon \Omega \times R_+^{m+1} \to R$ *satisfy*:

(i) $f_i(x, u) = 0$ *whenever* $u_i = 0$;

(ii) $f_0(x, u) \leqslant 0$ *for all* $u \in R_+^{m+1}$ *and* $f_0(x, u_0, 0, ..., 0) = 0$ *for* $u_0 \geqslant 0$;

(iii) $f_i(x, 0, u_1, ..., u_m) \leqslant 0$ *and* $f_i(x, u) \geqslant 0$ *for* $1 \leqslant i \leqslant m$;

(iv) $\mathscr{B}_0(x, D) u_i = v_0^0$ *on* $\partial\Omega$ *for some nonnegative functions* v_0^0 *given on* $\partial\Omega$.

Now let u_* be the unique solution of

$$\begin{cases} \mathscr{A}_0(x, D)\, u = 0 & \text{in } \Omega \\ \mathscr{B}_0(x, D)\, u = v_0^0 & \text{on } \partial\Omega. \end{cases}$$

By the comparison theorem and using (ii) and (iii) of (F1′), it is easy to see that $u_0 \leqslant u_{0*}$ in Ω. We then define $w := u_* - u_0$ and $\bar{f}_0(x, w, u_1, ..., u_m) := -f_0(x, u_* - w, u_1, ..., u_m)$. The system satisfied by w, u_i is of the form (4.1) and has the property (i) of (F1).

Similarly, we could consider nonhomogenuous boundary conditions in (4.1) but by a simple change of variables as above we can reduce the problem to homogeneous one.

4.1. *A Fixed Point Problem*

We now go back to (4.1) and assume (F1). Observe that $u=(0, ..., 0)$ is an equilibrium solution of (4.1) by virtue of (F1). We seek solutions of (4.1) in the positive cone X_+ of the Banach space $\mathscr{C}_{m+1} := \prod_1^{m+1} C(\Omega)$. For $0 \leqslant i \leqslant m$, let k_i be as in (F1) and let K_i: $C(\bar{\Omega}) \to C(\bar{\Omega})$ be the bounded linear operator inverse to $-\mathscr{A}_i(x, D)+k_i I$, together with the corresponding boundary conditions as in (4.1), where I is the identity. That is, given $h \in C(\bar{\Omega})$, $v=K_i(h)$ is the unique solution of the boundary value problem

$$-\mathscr{A}_i(x, D)v+k_i v=h, \qquad \mathscr{B}_i(x, D)v=0.$$

It is well-known (see e.g. [3]) that K_i is a strongly positive, compact operator on $C(\bar{\Omega})$. System (4.1) is equivalent to the fixed point problem on X_+ given by

$$U=F(U) \equiv K \circ \mathscr{F}(U) \tag{4.2}$$

where $\mathscr{F}=(f_i+k_i u_i)_{i=0}^m$: $X_+ \to X_+$ (by (ii) of (F1)) and K: $X_+ \to X_+$ is given by $K=\operatorname{diag}\{K_0, ..., K_m\}$. Observe that K is a compact, positive linear operator on X_+ and $\mathscr{F}$ is continous on X_+ and satisfies $G(0)=0$. Therefore, F: $X_+ \to X_+$ is a completely continuous (nonlinear) map. Obviously, $U=0$ is a trivial fixed point of F. We are interested in finding nontrivial fixed points of F in X_+.

As the f_i's are continuously differentiable functions it follows that F has a derivative $F'_+(0)$ at $U=0$ in the direction of the cone X_+ (see [3]) and $F'_+(0)$ is a positive, compact linear operator. An easy calculation using (F1) shows that if $\lambda \neq 0$ is an eigenvalue of

$$F'_+(0)\Phi=\lambda\Phi$$

for $\Phi=(\phi_0, ..., \phi_m)$, then λ is an eigenvalue of

$$-\mathscr{A}_0(x, D)\phi_0=\lambda^{-1}\sum_{i=1}^m \phi_i \frac{\partial f_0}{\partial u_i}(x, 0, ..., 0),$$

and

$$-\mathscr{A}_i(x, D)\phi_i+k_i\phi_i=\lambda^{-1}\phi_i\left[\frac{\partial f_i}{\partial u_i}(x, 0, ..., 0)+k_i\right] \tag{4.3}$$

for $1 \leqslant i \leqslant m$, with the boundary conditions as in (4.1). Note that we have used (i) of (F1) to get $(\partial f_i/\partial u_j)(x, 0, ..., 0) \equiv 0$ if $i \neq j$.

4.2. *Existence of Semitrivial Steady States*

Our principal assumption concerns the eigenvalue problems (4.3). We assume that

(E_i) *The largest eigenvalue of* (4.3) *is greater than* 1. *We say that* (E) *holds if* (E_i) *holds for* $1 \leqslant i \leqslant m$.

(E_i) is equivalent to the assumption that the largest (principal) eigenvalue of

$$\begin{cases} \lambda\phi = \mathscr{A}_i(x, D)\phi + \dfrac{\partial f_i}{\partial u_i}(x, 0, ..., 0)\phi, \\ \mathscr{B}_i(x, D)\phi = 0 \end{cases} \tag{4.4}$$

is positive. Similarly, the largest eigenvalue of (4.3) is less than one if and only if the largest eigenvalue of (4.4) is negative. The proofs of these assertions follow from [3, Theorems 4.3–4.5; and 24, Theorem 2.5, p. 67] and are well-known.

The proof of the following is simple and therefore will be omitted (see [13, Lemma 3.2]).

LEMMA 4.1. *If* (E) *holds, then one is not an eigenvalue of* $F'_+(0)$ *corresponding to an eigenvector in* X_+ *and* $F'_+(0)$ *has an eigenvalue larger than one with a corresponding eigenvector in* X_+.

LEMMA 4.2. *There is an* $R > 0$ *such that*

$$F(U) = \lambda U, \qquad \lambda \geqslant 1$$

has no solution $U \in X_+$ *satisfying* $\|U\| = R$.

Proof. The above equation is equivalent to

$$-\mathscr{A}_i(x, D)u_i = \lambda^{-1} f_i(x, u) + k_i(\lambda^{-1} - 1)u_i, \qquad 0 \leqslant i \leqslant m$$

together with the boundary conditions of (4.1). Define f_λ for $\lambda \geqslant 1$ by $f_\lambda = (\hat{f}_0, ..., \hat{f}_m)$ where $\hat{f}_i(x, u) = \lambda^{-1} f_i(x, u) + k_i(\lambda^{-1} - 1)u_i$, $0 \leqslant i \leqslant m$. Then it is easy to check that if f satisfies (iii) of Section 3 and (F1), which we are assuming, then f and f_λ also satisfy these assumptions with a common set of constants h_i and a common set of functions k, c in (Cp) and exponents

σ in (iii) of Section 3, which are independent of $\lambda \geqslant 1$. Consequently, we may take $R = K_\infty$ where K_∞ is defined by Theorem 3.3. ∎

These two lemmas allow us to apply Lemmas 3.2 and 3.3 and Theorem 13.2i and its proof in [3] to conclude that

THEOREM 4.3. *For* $r > 0$, *let* $P_r = \{u \in X_+ : \|u\| < r\}$. *If* (E) *holds, then there exists* r_0 *such that* $0 < r_0 < R$ *and for any* $r \in (0, r_0]$

$$\operatorname{ind}(F, P_R \backslash \bar{P}_r) = +1.$$

In particular, there is a fixed point of F *in* $P_R \backslash \bar{P}_r$.

COROLLARY 4.4. *If for some* i, $1 \leqslant i \leqslant m$, (E_i) *holds then there exists a semitrivial (single-population) equilibrium of* (4.1).

Proof. Assume without loss of generality that $i = 1$. We take $m = 1$ in Theorem 4.3 by dropping the equations for u_j for $j \neq 0, 1$ and setting $u_j = 0$ in the appropriate arguments in f_0 and f_1. Now note that (F1) continues to hold for this reduced system. Conditions (i) of (F1) assert that the solution given by Theorem 4.3 must have both components positive. ∎

4.3. *Existence of Positive Steady States*

We now turn attention to the case of three equations, that is $m = 2$. It is assumed that for $i = 1, 2$, the principal eigenvalue of the eigenvalue problem (4.4) is positive. Corollary 4.4 then implies the existence of at least one single-population equilibrium for each of the two populations. We then define Z_1 (resp. Z_2) to be the set of single-population equilibria for which $u_1 > 0$ (resp. $u_2 > 0$). The above results imply these sets are nonempty. Moreover,

LEMMA 4.5. *Assume* (E_i), $i = 1, 2$, *the sets* Z_i's *are compact bounded sets and bounded away from the origin.*

Proof. The first assertion comes from the compactness of the operators $(-\mathscr{A}_i(x, D))^{-1}$ and the boundedness result of Lemma 4.2. We need only to prove that Z_i is isolated from 0. Take $i = 1$ and suppose that there is a sequence $\{(u_0^n, u_1^n, 0)\}$ in Z_1 converging to $(0, 0, 0)$. Set $w_1^n := u_1^n / \|u_1^n\|$ and observe that w_1^n satisfies the following equation

$$-\mathscr{A}_1(x, D)\, w_1^n = w_1^n \int_0^1 \frac{\partial f_1}{\partial u_1}(x, u_0^n, s u_1^n, 0)\, ds.$$

Since u_i^n, w_1^n are bounded, so is the right-hand side of the above equation. Hence $\{w_1^n\}$ is compact. Passing to a subsequence we may assume that $w_1^n \to w$ for some positive function w which satisfies

$$-\mathscr{A}_1(x, D)w = w\frac{\partial f_1}{\partial u_1}(x, 0, 0, 0).$$

This contradicts to (E_1). ∎

Let us denote the elements of Z_i by $\hat{U}_i$, that is, $\hat{U}_1 = (\hat{u}_0, \hat{u}_1, 0)$ and $\hat{U}_2 = (\hat{u}_0, 0, \hat{u}_2)$. The following assumption roughly says whether Z_i's are repelling (unstable) or attracting (stable) in their complementary directions.

(E_+) *For $i \neq j$, $i, j = 1, 2$, and for any $\hat{U}_j \in Z_j$, the largest eigenvalue of*

$$\begin{cases} -\mathscr{A}_i(x, D)\phi + k_i\phi = \lambda^{-1}\phi\left\{\dfrac{\partial f_i}{\partial u_i}(\hat{U}_j) + k_i\right\}, \\ \mathscr{B}_i(x, D)\phi = 0 \end{cases} \tag{4.5}$$

is greater than 1.

(E_-) *Otherwise, that is, these eigenvalues are all less than* 1.

Remark 4.6. As before, (E_+) (resp. (E_-)) is equivalent to the fact that the principal eigenvalue of

$$\lambda\phi = \mathscr{A}_i(x, D)\,\phi + \phi\,\frac{\partial f_i}{\partial u_i}(\hat{U}_j),$$

is positive (resp. negative). Note also that (4.5) is not the full linearization of the system (4.1) at $\hat{U}_j$.

THEOREM 4.7. *Let $m = 2$ and assume (E_i), $i = 1, 2$ and either (E_+) or (E_-). The system* (4.1) *has at least one nontrivial positive solution.*

Proof. For the sake of brevity, we will drop the constants k_i from our equations (otherwise, one can simply replace $\mathscr{A}_i$, f_i in the argument by $\mathscr{A}_i + k_iI$, $f_i + k_iu_i$, respectively). Let us consider the following family of systems with parameter $t \in [0, 1]$.

$$\begin{cases} -\mathscr{A}_0(x, D)u_0 = f_0(x, u_0, u_1, tu_2) \\ -\mathscr{A}_1(x, D)u_1 = f_1(x, u_0, u_1, tu_2) \\ -\mathscr{A}_2(x, D)u_2 = u_2\displaystyle\int_0^1 \frac{\partial f_2}{\partial u_2}(x, u_0, u_1, tsu_2)\,ds. \end{cases} \tag{4.6}$$

The equivalent fixed point problem will be denoted by

$$U = H(t, U).$$

Because of (i) of (F1) and the fact that for any t,

$$f_2(x, u_0, u_1, tu_2) = tu_2 \int_0^1 \frac{\partial f_2}{\partial u_2}(x, u_0, u_1, tsu_2)\, ds, \tag{4.7}$$

we see that $H(1, U) \equiv F(U)$. By a positive solution of (4.1), or equivalently, of $F(U) = U$, we mean a solution for which $u_1 > 0$, $u_2 > 0$.

For $i, j = 1, 2$, choose a neighborhood $E_i = V_i \times W_i$ of Z_i in $P_R \backslash \bar{P}_r$ where V_i is a neighborhood in $C(\bar{\Omega}) \times C(\bar{\Omega})$ of the projection of Z_i onto this space, and W_i is a small neighborhood of 0 in $C(\bar{\Omega})$ such that E_i defined as above does not intersect Z_j, $j \neq i$ (see Lemma 4.5). Below, we will construct a chain of homotopic mappings and the reader should keep in mind that the domain of each is the neighborhood E_1.

We will show that either

(a) $F(U) = U$ has at least one positive solution in $P_R \backslash \bar{P}_r$, or

(b) the fixed point indices satisfy $\mathrm{ind}(F, E_1) = \mathrm{ind}(F, E_2) \in \{0, 1\}$.

As $\mathrm{ind}(F, P_R \backslash \bar{P}_r) = 1$ by Theorem 4.3, it follows from the additivity property of the fixed point index that (a) holds if (b) holds. Henceforth, we assume that (a) does not hold.

If there exists $t \in (0, 1]$ such that $H(t, U) = U$ has a solution $U = (u_0, u_1, u_2)$ on ∂E_1 (relative to X_+), then $u_2 \neq 0$ since otherwise $U \in Z_1$ and then U does not belong to the boundary of E_1. Therefore, $u_2 > 0$ and (u_0, u_1, tu_2) is a positive fixed point of F (note (4.7)), in contradiction to our assumption that (a) does not hold. If $H(0, U) = U$ has a solution $U = (u_0, u_1, u_2)$ on ∂E_1, then $(u_0, u_1, 0) \in Z_1$. If $u_2 = 0$, then $U \in Z_1$ but the latter does not belong to ∂E_1. Therefore, $u_2 > 0$ by the maximum principle and consequently we have a contradiction to our assumption that the principal eigenvalue of (4.5) is not 1 (when $t = 0$ and $\lambda = 1$, the third equation in (4.6) is exactly (4.5)). We conclude that $H(t, U) = U$ has no solutions (t, U) with $0 \leqslant t \leqslant 1$ and $U \in \partial E_1$. Consequently, by the homotopy invariance of the degree

$$\mathrm{ind}(F, E_1) = \mathrm{ind}(H(1, \bullet), E_1) = \mathrm{ind}(H(0, \bullet), E_1).$$

Consider now the system corresponding to $U = H(0, U)$.

$$\begin{cases} -\mathscr{A}_0(x, D)u_0 = f_0(x, u_0, u_1, 0), \\ -\mathscr{A}_1(x, D)u_1 = f_1(x, u_0, u_1, 0), \end{cases}$$

and

$$-\mathscr{A}_2(x, D)u_2 = u_2 \frac{\partial f_2}{\partial u_2}(x, u_0, u_1, 0).$$

Note that this system is already decoupled. We consider separately two cases.

Assume (E_-). Consider the following homotopy

$$\begin{cases} -\mathscr{A}_0(x, D)u_0 = f_0(x, u_0, u_1, 0) \\ -\mathscr{A}_1(x, D)u_1 = f_1(x, u_0, u_1, 0) \\ -\mathscr{A}_2(x, D)u_2 = tu_2 \dfrac{\partial f_2}{\partial u_2}(x, u_0, u_1, 0). \end{cases}$$

In fixed point form, this becomes $G(t, U) = U$. If $G(t, U) = U$ for some $t \in [0, 1]$ and $U = (\hat{u}_0, \hat{u}_1, \hat{u}_2) \in \partial E_1$ then obviously $(\hat{u}_0, \hat{u}_1, 0)$ belongs to Z_1 and $t > 0$ and $\hat{u}_2 > 0$. But this means that $\hat{u}_2$ is a positive eigenfunction to the eigenvalue $t^{-1} \geqslant 1$ of (4.5). By the uniqueness of eigenvalue having positive eigenfunction, $t^{-1} \geqslant 1$ is the largest eigenvalue and this contradicts to (E_-). Again, by the homotopy invariance of the degree,

$$\mathrm{ind}(F, E_1) = \mathrm{ind}(H(0, \bullet), E_1) = \mathrm{ind}(G(1, \bullet), E_1) = \mathrm{ind}(G(0, \bullet), E_1).$$

However, $G(0, \bullet)$ can be viewed as the product of two maps G_1 on V_1 and $G_2 \equiv 0$ on W_1. Now, $\mathrm{ind}(G_1, V_1) = +1$ (by applying Theorem 4.3 to the case $m = 1$ as in Corollary 4.4) and $\mathrm{ind}(G_2, W_1) = \mathrm{ind}(0, W_1) = +1$. So that by the product theorem of Leray (Theorem 13.F in [33]),

$$\mathrm{ind}(F, E_1) = \mathrm{ind}(G_1, V_1) \times \mathrm{ind}(G_2, W_1) = +1. \tag{4.8}$$

Similarly, we also have $\mathrm{ind}(F, E_2) = +1$.

Assume (E_+). Let $\Phi = (-\mathscr{A}_2(x, D))^{-1}(1) > 0$ be a fixed function in $\bar{\Omega}$, we consider the following homotopy $U = G(t, U)$ associated to the following family of systems

$$\begin{cases} -\mathscr{A}_0(x, D)u_0 = f_0(x, u_0, u_1, 0), \\ -\mathscr{A}_1(x, D)u_1 = f_1(x, u_0, u_1, 0), \end{cases} \tag{4.9}$$

and

$$u_2 = (-\mathscr{A}_2(x, D))^{-1}\left(u_2 \frac{\partial f_2}{\partial u_2}(x, u_0, u_1, 0)\right) + t\Phi, \tag{4.10}$$

with the parameter $t \geqslant 0$.

If $U=(u_0, u_1, u_2) \in \partial E_1$ is a solution of $U=G(t, U)$ then $(u_0, u_1, 0) \in Z_1$ and $u_2>0$ satisfies

$$u_2 = T(u_0, u_1, 0)\, u_2 + t\Phi \tag{4.11}$$

where

$$T(u_0, u_1, 0)\phi := (-\mathscr{A}_2(x, D))^{-1}\left(\phi \frac{\partial f_2}{\partial u_2}(x, u_0, u_1, 0)\right)$$

is a strongly positive compact operator. But (E_+) simply means that the spectral radius $r(T(u_0, u_1, 0))>1$. Therefore (4.11) contradicts to (ii) of [3, Theorem 3.2] if $t=0$ and (iv) of that theorem if $t>0$. Thus, the above homotopy is well-defined on E_1 for all $t \geqslant 0$.

However, for t very large, obviously (4.11) does not have any solution in the bounded set E_1 and therefore $\operatorname{ind}(G(t, \bullet), E_1)$ must be zero for t large. By the homotopy invariance of the degree $\operatorname{ind}(F, E_1)=0$. Similarly, we have $\operatorname{ind}(F, E_2)=0$.

We have shown (b). In either case, the fixed point index of F on $P_R \backslash \bar{P}_r$ is not the sum of the indices on the two sets E_1 and E_2 whose union contains all semitrivial steady states. By the additivity property of the fixed point index, there must be another fixed point of F which must be a positive fixed point of F in $P_R \backslash \bar{P}_r$, a contradiction. ∎

Remark 4.8. If Z_i consist of one element for $i=1, 2$ then a simpler homotopy can be devised similarly as in [13] to obtain (b). We will show that the uniqueness of single population for the case $N=1$ in the next section.

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We consider the following boundary value problem

$$\begin{aligned} S'' - 2\alpha(x)\, S' &= f(x, S, u), \\ u'' - 2\beta(x)\, u' &= -ug(x, S), \qquad 0<x<1, \end{aligned} \tag{5.1}$$

with boundary conditions

$$\begin{aligned} -S'(0) + 2\alpha(0)\, S(0) &= c_0, \qquad & -u'(0) + 2\beta(0)\, u(0) &= 0, \\ S'(1) + 2\alpha(1)\, S(1) &= c_1, \qquad & u'(1) + 2\beta(1)\, u(1) &= 0. \end{aligned}$$

Here, $f\colon (0,1)\times R_+\times R_+\to R_+$ and $g\colon (0,1)\times R_+\to R$ are increasing continuously differentiable functions in S and u; $\alpha(x), \beta(x)\geqslant 0$ for $x=0, 1$; $c_i\geqslant 0$.

We assume further that $\alpha^2(x)-\alpha'(x)\geqslant 0$ for $0<x<1$.

We will then show that

THEOREM 5.1. *There is at most one solution of* (5.1) *satisfying* $S(x)$, $u(x)>0$ *for* $0\leqslant x\leqslant 1$.

Put

$$\bar{S} := \exp\left(-\int_0^x \alpha(s)\,ds\right) S(x),$$

$$\bar{u} := \exp\left(-\int_0^x \beta(s)\,ds\right) u(x).$$

By calculation, we see that $\bar{S}$ and $\bar{u}$ satisfy the following

$$\begin{aligned}\bar{S} &= \exp\left(-\int_0^x \alpha(s)\,ds\right) f(x, S, u) + (\alpha^2(x)-\alpha'(x))\,\bar{S} =: \bar{f}(x, \bar{S}, \bar{u}),\\ \bar{u} &= -\exp\left(-\int_0^x \beta(s)\,ds\right) u(x)(g(x, S)-(\beta^2(x)-\beta'(x))) =:\ -\bar{u}\bar{g}(x, \bar{S}),\end{aligned} \tag{5.2}$$

with boundary conditions

$$-\bar{S}'(0)+\alpha(0)\,\bar{S}(0)=c_0, \qquad -\bar{u}'(0)+\beta(0)\,\bar{u}(0)=0,$$

$$\bar{S}'(1)+3\alpha(1)\,\bar{S}(1)=\exp\left(-\int_0^1 \alpha\right) c_1, \qquad \bar{u}'(1)+3\beta(1)\,\bar{u}(1)=0.$$

Observe that $\bar{f}(x, \bar{S}, \bar{u})$ is an increasing function in $\bar{S}$ and $\bar{u}$ (because $\alpha^2(x)-\alpha'(x)\geqslant 0$); $\bar{g}(x, \bar{S})$ is an increasing function in $\bar{S}$.

We need only to prove the uniqueness of positive solution to (5.2). This result has been obtained in [13] for the special case when $\bar{f}(x, S, u)=uF(S)$ and $\bar{g}(x, S)=G(S)$ for some increasing functions F, G. However, inspecting the proof of Proposition 4.1 in [13], we can see that it can carry over to our situation without any change. The essential ingredient in the proof is the monotonicity of the nonlinearities with respect to the unknown variables.

Essentially, we consider two positive distinct solutions (S_1, u_1) and (S_2, u_2) of (5.2) and show that the functions $S_1(x)$ and $S_2(x)$ can agree at at most finitely many points of $(0, 1)$. Therefore, there is an $n\geqslant 2$ and points x_i satisfying $1=x_0<x_1<\cdots<x_n=1$ such that $S_1(x)-S_2(x)$

changes sign at x_i for $1 \leqslant i \leqslant n-1$. We then show that in each interval the following pattern will be formed by the two solutions: if $S_i \geqslant S_j$ on $[x_m, x_{m+1}]$ and $u_i(x_m) \geqslant u_j(x_m)$ and if $m < n-1$, then $u_j(x_{m+1}) \geqslant u_i(x_{m+1})$ and, of course, $S_j \geqslant S_i$ on $[x_{m+1}, x_{m+2}]$. But then in the last interval $[x_{n-1}, x_n]$, this pattern and the boundary condition at 1 will give a contradiction and complete the proof. We refer to [13] for more details.

Finally, we remark that one can prove [12, 16] the following monotonicity result for solutions S, u of (5.1): $S(x)$ is decreasing and $u(x)$ is increasing on $(0, 1)$. More properties and numerical results on the systems with three equations can be found in [5].

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