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Renormalization group method applied to the primitive equations

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Dedicated to George Sell on the occasion of his 65th birthday

Abstract

In this article we study the limit, as the Rossby number ε goes to zero, of the primitive equations of the atmosphere and the ocean. From the mathematical viewpoint we study the averaging of a penalization problem displaying oscillations generated by an antisymmetric operator and by the presence of two time scales.

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1. Introduction

The study of the limit, as the Rossby number ε goes to zero, of the equations of the atmosphere and the ocean is a major physical and computational problem to which much effort has been devoted. In a more mathematical context, this problem is related to the averaging of oscillations using renormalization and other averaging procedures.

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In the mathematical literature, an important contribution is due to Schochet [18] who tackled similar problems by studying an asymptotics in the fast time variable; such problems have also been studied in the general framework of wave equations by Joly et al. [9], Grenier [8], and Gallagher [7]. For the equations of the atmosphere and the ocean, mathematical work includes the following: Embid and Majda [6], Babin et al. [1–3] or Warn et al. [23]. Many more articles on the subject are available in the physics and mathematical literature.

In the mathematical physics literature, a number of averaging problems and procedures have been studied or proposed; see e.g., the article [10] by van Kampen on the elimination of fast variables, or the averaging procedure by Bogolyubov and Mitropolsky [4]. Our work follows more closely the approach, based on renormalization theory, of Chen et al. [5] revisited by Ziane [25]. Here we also extend to infinite dimension part of the work by Temam and Wirosoetisno [22] valid in finite dimension.

As we said, the renormalization method that we use here was introduced in [5,25]. It was then applied to different types of partial differential equations by Moise, Temam, and Ziane (see [14,15]); the method was also applied to ordinary differential equations (see e.g., [13,21,25]).

This article is organized as follows: In the first part of Section 2 (Section 2.1), we present the PEs and recall a few facts on their mathematical setting, some well-known, and some borrowed from a companion paper [17]. See [16,24] for physics details regarding the primitive equations; those considered here are the PEs of the ocean; some slight changes are necessary for the atmosphere. In the second part of Section 2 (Section 2.2), we recall a few facts about renormalization following [5,15,21,25]. In Section 3 we study the properties of the renormalized system, starting with the existence of weak solutions and ending the section with the existence of very regular solutions. In Section 4 we show that we can approximate the exact solution of the primitive equations by an asymptotic solution which exists for all times and we estimate the difference between the exact and asymptotic solutions. We end the paper with three appendices: in Appendix A we give the details of the derivation of the renormalized system, in Appendix B.1 we give a result of number theory needed in Section 4 to bound some small denominators necessary for the error estimates, and in Appendix B.2 we present an alternate method for bounding the small denominators.

2. The initial and renormalized problems

In Section 2.1 we recall the primitive equations in a form suitable for our study. In Section 2.2 we recall a few facts about renormalization.

2.1. The PEs in space dimension two

We work in the two-dimensional space and consider the domain

$$\mathscr{M} = (0, L_1) \times (-L_3/2, L_3/2),$$

$0x$ being the west–east direction, and $0z$ being the vertical direction. All the quantities depend only on x, z and t. We consider the PEs written in the non-dimensional form (2.1) below; a description of the derivation of these equations and a study concerning the existence and regularity of their solutions is given in [17]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\frac{\partial p}{\partial x} = v_{\boldsymbol{v}}\Delta u + S_u, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + \frac{1}{\varepsilon}u = v_{\boldsymbol{v}}\,\Delta v + S_v, \tag{2.1b}$$

$$\frac{\partial p}{\partial z} = -N\rho, \tag{2.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2.1d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} - \frac{N}{\varepsilon}w = v_\rho\,\Delta\rho + S_\rho. \tag{2.1e}$$

Here u, v, w are the non-dimensional components of the three-dimensional velocity vector, p is the pressure, ρ is the density and ε is the Rossby number. In the more physical situation, the source terms S_u, S_v, and S_ρ usually vanish; they are introduced here for mathematical generality. Here $v_{\boldsymbol{v}}$ and v_ρ are the non-dimensional eddy viscosity coefficients, N is the Burgers number, and we set $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$. In the physical problem, the total pressure is

$$p_{\text{full}} = p_{\text{ref}} + \bar{p} + p',$$

and the total density is

$$\rho_{\text{full}} = \rho_{\text{ref}} + \bar{\rho} + \rho'.$$

Here p_{ref} is a hydrostatic pressure corresponding to the reference value of the density ρ_{ref}, $\bar{\rho}$ is the density stratification profile which is linear in z and $\bar{p}$ is the pressure in hydrostatic equilibrium with it; p' and ρ' are perturbations from these states. In (2.1) we do not work with the total pressure and the total density but with the perturbations p' and ρ' where the primes were dropped and ρ' has been replaced by ρ'/N. See [17] for more details regarding the derivation of this system.

We also assume that all the unknown functions are $\mathscr{M}$-periodic. The prognostic variables of this system are u, v, ρ and the diagnostic variables are p, w; as we will see below, p and w can, at each instant of time, be (essentially) determined in terms of the prognostic variables.

We recall that an $\mathscr{M}$-periodic function

$$u = \sum_{(k_1,k_3)\in\mathbb{Z}^2} u_{(k_1,\,k_3)} e^{2\pi i(k_1 x/L_1 + k_3 z/L_3)}$$

is in $H^m_{\text{per}}(\mathscr{M})$, $m>0$, if and only if

$$\sum_{k\in\mathbb{Z}^2} (1+|k|^2)^m |u_k|^2 < \infty,$$

where we denoted by k the pair (k_1, k_3). We denote by $\dot{H}^m_{\text{per}}(\mathscr{M})$ the functions from $H^m_{\text{per}}(\mathscr{M})$ with average zero on $\mathscr{M}$. In order to simplify the writing we will also set $k_i' = 2\pi k_i/L_i$. We easily notice that if (u, v, ρ, w, p) is a solution of (2.1) for $S = (S_u, S_v, S_\rho)$, then $(\tilde{u}, \tilde{v}, \tilde{\rho}, \tilde{w}, \tilde{p})$ is also a solution of (2.1) for $\tilde{S}_u$, $\tilde{S}_v$, $\tilde{S}_\rho$ where:

$$\tilde{u}(x,z,t) = u(x,-z,t), \quad \tilde{p}(x,z,t) = p(x,-z,t),$$

$$\tilde{v}(x,z,t) = v(x,-z,t), \quad \tilde{S}_u(x,z,t) = S_u(x,-z,t),$$

$$\tilde{w}(x,z,t) = -w(x,-z,t), \quad \tilde{S}_v(x,z,t) = S_v(x,-z,t),$$

$$\tilde{\rho}(x,z,t) = -\rho(x,-z,t), \quad \tilde{S}_\rho(x,z,t) = -S_\rho(x,-z,t).$$

Hence, assuming that S_u and S_v are even in z, and that S_ρ is odd in z,

$$S_u(x,z,t) = S_u(x,-z,t),$$

$$S_v(x,z,t) = S_v(x,-z,t),$$

$$S_\rho(x,z,t) = -S_\rho(x,-z,t),$$

it is natural to look for a solution where u, v and p are even in z and ρ, w odd in z,

$$u(x,z,t) = u(x,-z,t), \quad w(x,z,t) = -w(x,-z,t),$$

$$v(x,z,t) = v(x,-z,t), \quad p(x,z,t) = p(x,-z,t),$$

$$\rho(x,z,t) = -\rho(x,-z,t).$$

For more details regarding the motivation of this choice (symmetry and periodicity) we refer the reader to [17].

In accordance with these requirements of symmetry and periodicity, we introduce the following function spaces:

$$\boldsymbol{V} = \{(u, v, \rho) \in (\dot{H}^1_{\mathrm{per}}(\mathscr{M}))^3; u, v \text{ even in } z, \rho \text{ odd in } z, u_{(k_1,0)} = 0,\ \forall k_1 \in \mathbb{Z}\},$$

$$\boldsymbol{H} = \text{the closure of } \boldsymbol{V} \text{ in } (L^2(\mathscr{M}))^3,$$

$$\boldsymbol{V}_2 = \text{the closure of } \boldsymbol{V} \cap (H^2_{\mathrm{per}}(\mathscr{M}))^3 \text{ in } (H^2_{\mathrm{per}}(\mathscr{M}))^3.$$

The condition $u_{(k_1,0)} = 0,\ \forall k_1$, expresses condition (2.3) appearing below.

We can express the diagnostic variables w and p in terms of the prognostic variables u, v, and ρ. For each $U = (u, v, \rho) \in \boldsymbol{V}$ we can determine uniquely

$$w = w(U) = -\int_0^z u_x(x, z', t)\, dz'. \tag{2.2}$$

Note that $w = 0$ at $z = 0$ and $L_3/2$ by the requirements of w (periodicity and anti-symmetry); see more details in [17]. By (2.2), the fact that $w = 0$ at $z = L_3/2$ gives the constraint on u

$$\int_{-L_3/2}^{L_3/2} u_x\, dz = 0. \tag{2.3}$$

As for the pressure, it can be determined uniquely in terms of ρ up to p_{s}, writing

$$p(x, z, t) = p_{\mathrm{s}}(x, t) - \int_0^z \rho(x, z', t)\, dz'.$$

For U, $\tilde{U} \in \boldsymbol{V}$, we set

$$((U, \tilde{U})) = ((u, \tilde{u})) + ((v, \tilde{v})) + ((\rho, \tilde{\rho})), \quad ||U|| = ((U, U))^{1/2}, \tag{2.4}$$

where we have written $d\mathscr{M}$ for $dx\, dz$, and

$$((\phi, \tilde{\phi})) = \int_{\mathscr{M}} \left(\frac{\partial \phi}{\partial x} \frac{\partial \tilde{\phi}}{\partial x} + \frac{\partial \phi}{\partial z} \frac{\partial \tilde{\phi}}{\partial z} \right) d\mathscr{M}. \tag{2.5}$$

By the Poincaré inequality,

$$|U|_{L^2} \leqslant c_0 ||U||, \quad \forall U \in \boldsymbol{V}, \tag{2.6}$$

so that $||\cdot||$ is a Hilbert norm on $\boldsymbol{V}$.

The space $\boldsymbol{H}$ is endowed with the usual scalar product of $(L^2(\mathscr{M}))^3$.

Variational formulation of the problem: We introduce the following forms:

$$a(U, \tilde{U}) = v_{\boldsymbol{v}}((u, \tilde{u})) + v_{\boldsymbol{v}}((v, \tilde{v})) + v_\rho((\rho, \tilde{\rho})),$$

$$e(U, \tilde{U}) = \int_{\mathscr{M}} (-v\tilde{u} + u\tilde{v})\, d\mathscr{M} + N \int_{\mathscr{M}} \rho w\, d\mathscr{M} - N \int_{\mathscr{M}} w\tilde{\rho}\, d\mathscr{M},$$

$$b(U, U^{\#}, \tilde{U}) = \int_{\mathscr{M}} \left(u \frac{\partial u^{\#}}{\partial x} + w(U) \frac{\partial u^{\#}}{\partial z} \right) \tilde{u} \, d\mathscr{M} + \int_{\mathscr{M}} \left(u \frac{\partial v^{\#}}{\partial x} + w(U) \frac{\partial v^{\#}}{\partial z} \right) \tilde{v} \, d\mathscr{M}$$
$$+ \int_{\mathscr{M}} \left(u \frac{\partial \rho^{\#}}{\partial x} + w(U) \frac{\partial \rho^{\#}}{\partial z} \right) \tilde{\rho} \, d\mathscr{M}.$$

The variational form of the problem is obtained by multiplying (2.1a), (2.1b), (2.1e), by u, v and ρ, respectively, integrating over $\mathscr{M}$ and adding the resulting equations. After some easy calculations we arrive at this problem:

Given $t_\star > 0$ arbitrary, $U_0 \in \boldsymbol{H}$ and $S = (S_u, S_v, S_\rho) \in L^2(0, t_\star; \boldsymbol{H})$, we look for a function U from $(0, t_\star)$ into $\boldsymbol{V}$ such that

$$\frac{d}{dt}(U, \tilde{U})_{\boldsymbol{H}} + a(U, \tilde{U}) + b(U, U, \tilde{U}) + \frac{1}{\varepsilon} e(U, \tilde{U}) = (S, \tilde{U})_{\boldsymbol{H}}, \quad \forall \tilde{U} \in \boldsymbol{V} \tag{2.7}$$

and

$$U(0) = U_0. \tag{2.8}$$

We also define the linear operators

$$A : \boldsymbol{V} \to \boldsymbol{V}', \quad \langle AU, \tilde{U} \rangle_{\boldsymbol{V}', \boldsymbol{V}} = a(U, \tilde{U}), \quad \forall U, \tilde{U} \in \boldsymbol{V}', \tag{2.9}$$

$$L : \boldsymbol{V} \to \boldsymbol{V}', \quad \langle LU, \tilde{U} \rangle_{\boldsymbol{V}', \boldsymbol{V}} = e(U, \tilde{U}), \quad \forall U, \tilde{U} \in \boldsymbol{V}, \tag{2.10}$$

and the bilinear form

$$B : \boldsymbol{V} \times \boldsymbol{V}_2 \to \boldsymbol{V}', \quad \langle B(U, \tilde{U}), U^{\#} \rangle_{\boldsymbol{V}', \boldsymbol{V}} = b(U, \tilde{U}, U^{\#}), \quad \forall U, U^{\#} \in \boldsymbol{V}, \ \tilde{U} \in \boldsymbol{V}_2, \tag{2.11}$$

where $\boldsymbol{V}'$ denotes the dual space of $\boldsymbol{V}$; it is shown in [17] that b is trilinear continuous on $\boldsymbol{V} \times \boldsymbol{V}_2 \times \boldsymbol{V}$ and $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}_2$ so that B is bilinear continuous from $\boldsymbol{V} \times \boldsymbol{V}_2$ into $\boldsymbol{V}'$ and from $\boldsymbol{V} \times \boldsymbol{V}$ into $\boldsymbol{V}_2'$.

Then problem (2.7) with initial condition (2.8) is equivalent to the abstract evolution equation:

$$\frac{dU}{dt} + AU + B(U, U) + \frac{1}{\varepsilon} LU = S, \quad \text{in } \boldsymbol{V}_2',$$

$$U(0) = U_0. \tag{2.12}$$

Regarding the existence and uniqueness of solutions of (2.7) we recall from [17] the following result:

Theorem 2.1. *Given $U_0 \in \boldsymbol{H}$ and $S \in L^\infty(\mathbb{R}_+; \boldsymbol{H})$, there exists at least one solution U of Eq.* (2.7) *with initial condition* (2.8) *such that*

$$U \in L^\infty(\mathbb{R}_+; \boldsymbol{H}) \cap L^2(0, t_\star; \boldsymbol{V}), \quad \textit{for all } t_\star > 0. \tag{2.13}$$

If $U_0 \in \boldsymbol{V}$ *and* $S \in L^\infty(\mathbb{R}_+; \boldsymbol{H})$, *there exists a unique solution* U *of* (2.7)–(2.8) *such that*

$$U \in L^\infty(\mathbb{R}_+; \boldsymbol{V}) \cap L^2(0, t_\star; (\dot{H}^2_{\text{per}}(\mathscr{M}))^3), \quad \forall t_\star > 0.$$

Moreover, for all $m \in \mathbb{N}$, $m \geqslant 2$, *if* $U_0 \in (\dot{H}^m_{\text{per}}(\mathscr{M}))^3 \cap \boldsymbol{V}$ *and* $S \in L^\infty(\mathbb{R}_+; (\dot{H}^{m-1}_{\text{per}}(\mathscr{M}))^3 \cap \boldsymbol{V})$, *then* $U \in L^\infty(\mathbb{R}_+; (\dot{H}^m_{\text{per}}(\mathscr{M}))^3) \cap L^2(0, t_\star; (\dot{H}^{m+1}_{\text{per}}(\mathscr{M}))^3)$, $\forall t_\star > 0$.

2.2. Asymptotics and renormalization group method

The aim of this article is to present an application of the renormalization group method (RG) to the 2D primitive equations described above. The RG method gives us an algorithm for finding approximate (averaged) solutions for a general equation of the form:

$$\frac{dU}{dt} + \frac{1}{\varepsilon} LU = \mathscr{F}(U),$$

$$U(0) = U_0, \tag{2.14}$$

where $\varepsilon > 0$ is a small parameter and L is an antisymmetric operator, so that the solutions of (2.14) display large oscillations for ε small. We assume that L is a diagonalizable, antisymmetric linear operator (not necessarily bounded) and $\mathscr{F}$ is a non-linear operator. Two natural time scales (at least) are present in (2.14), the slow time t, and the fast time $s = t/\varepsilon$. To implement the RG method, we imagine a formal asymptotic expansion for Eq. (2.14) written in the fast time variable:

$$\frac{d\check{U}}{ds} + L\check{U} = \varepsilon \mathscr{F}(\check{U}),$$

$$\check{U}(0) = U_0, \tag{2.15}$$

where we have set $\check{U}(s) = U(\varepsilon s)$. In what follows, we drop the checks and the formal expansion is written as

$$U = U^0 + \varepsilon U^1 + \varepsilon^2 U^2 + \cdots \ . \tag{2.16}$$

We formally substitute (2.16) into (2.15) and we find

$$\frac{dU^0}{ds} + LU^0 = 0, \tag{2.17}$$

$$\frac{dU^1}{ds} + LU^1 = \mathscr{F}(U^0), \tag{2.18}$$

$$\frac{dU^2}{ds} + LU^2 = \nabla_U \mathscr{F}(U^0) \cdot U^1, \tag{2.19}$$

and so on.

The solution of (2.17) can be written as

$$U^0(s) = e^{-Ls}U(0).$$

For Eq. (2.18) we apply the variation of constants formula and we obtain

$$U^1(s) = e^{-Ls}\int_0^s e^{Ls'}\mathscr{F}(e^{-Ls'}U_0)\,ds'. \tag{2.20}$$

For U^1 we choose the initial data to be zero, but other choices may be appropriate (see [21]).

We set $F(s,\cdot) = e^{Ls}\mathscr{F}(e^{-Ls}\cdot)$ and we split F into two parts: the resonant part $F_{\mathrm{r}}(\cdot)$ corresponding to the time-independent part of $F(s,\cdot)$ and the remaining non-resonant part $F_{\mathrm{n}}(s,\cdot)$. In our applications, $\mathscr{F}$ will be polynomial[1] in U and the definition of the time-independent part of F is not problematic. We thus have

$$F(s,U) = F_{\mathrm{r}}(U) + F_{\mathrm{n}}(s,U), \tag{2.21}$$

and we define the primitive of the non-resonant part by

$$F_{\mathrm{np}}(s,U) = \int_0^s F_{\mathrm{n}}(s',U)\,ds'. \tag{2.22}$$

Substituting these relations in (2.20) we find:

$$U^1(s) = e^{-Ls}\{sF_{\mathrm{r}}(U_0) + F_{\mathrm{np}}(s,U_0)\}. \tag{2.23}$$

The first-order RG equation, as discussed in [21], is of the form

$$\begin{aligned}\frac{d\bar{U}}{ds} &= \varepsilon F_{\mathrm{r}}(\bar{U}),\\ \bar{U}(0) &= U_0.\end{aligned} \tag{2.24}$$

For the details, see e.g. [15,18,21]. The first-order approximate solution is defined by

$$\tilde{U}^1(s) = e^{-Ls}\big\{\bar{U}(s) + \varepsilon F_{\mathrm{np}}(s,\bar{U}(s))\big\}, \tag{2.25}$$

and it is shown, e.g., in [18], that $\tilde{U}^1 - U$ is of order ε in an interval of time s of order $\mathcal{O}(1/\varepsilon)$ and in an interval of time t of order $\mathcal{O}(1)$.

The renormalized system (2.24)–(2.25) gives us an $\mathcal{O}(\varepsilon)$ approximation to the exact solution over a timescale $t \sim \mathcal{O}(1)$ without having to solve an oscillatory differential equation. Because of the computational difficulties, in this article we only derive the first-order approximate solution but we can apply the method to higher-order approximate solutions as described in [21] in the context of ordinary differential equations.

[1] Here we call polynomial function a function of the form $\mathscr{F}(U) = \sum_{j=0}^n \mathscr{F}_j(U,\dots,U)$, where n is finite arbitrary, and $\mathscr{F}_j$ is j-linear continuous on a suitable function space.

In this article, the polynomial $\mathscr{F}$ is taken to be of the form

$$\mathscr{F}(U) = S - A(U) - B(U, U),$$

where S is an external force, A is a linear coercive operator and B is a bilinear operator. In Appendix A, we explicitly construct the resonant parts of A and B. We will see that the resonant parts of A and B have the same properties as the original operators; this does not seem to happen at higher orders. In Appendices B.1 and B.2 we give two different methods to handle the small denominators, one result being a typical number theory result and the other is a more particular result, the method following [3].

3. Description of the renormalized system

We start this section by writing the initial system (2.1) in Fourier modes and by introducing a change of variables to facilitate the computation of the renormalized equation (Section 3.1). In the subsequent subsections we prove the existence of weak solutions (Section 3.2), of strong solutions (Section 3.3) and of even more regular solutions for the renormalized system (Section 3.4).

3.1. The original equations in Fourier modes

We introduce the fast time $s = t/\varepsilon$ in system (2.1). Abusing the notation, new functions depending on x, z and s are denoted in the same way as before. We obtain the following system:

$$\begin{gathered}
\frac{\partial u}{\partial s} + \varepsilon u \frac{\partial u}{\partial x} + \varepsilon w \frac{\partial u}{\partial z} - v + \frac{\partial p}{\partial x} = \varepsilon \nu_{\boldsymbol{v}} \Delta u + \varepsilon S_u, \\
\frac{\partial v}{\partial s} + \varepsilon u \frac{\partial v}{\partial x} + \varepsilon w \frac{\partial v}{\partial z} + u = \varepsilon \nu_{\boldsymbol{v}} \Delta v + \varepsilon S_v, \\
\frac{\partial p}{\partial z} = -N\rho, \\
u_x + w_z = 0, \\
\frac{\partial \rho}{\partial s} + \varepsilon u \frac{\partial \rho}{\partial x} + \varepsilon w \frac{\partial \rho}{\partial z} - Nw = \varepsilon \nu_\rho \Delta \rho + \varepsilon S_\rho.
\end{gathered} \tag{3.1}$$

All the functions being periodic, they admit Fourier series expansions. Hence, for instance, for u we write

$$u = \sum_{(k_1,k_3)\in\mathbb{Z}^2} u_{(k_1,k_3)} e^{i(k_1' x + k_3' z)},$$

where $k'_j = 2\pi k_j/L_j$. Note here that, by periodicity of w, integration of the fourth equation of (3.1) yields

$$\int_{-L_3/2}^{L_3/2} u_x \, dz = 0. \tag{3.2}$$

In Fourier series, this is equivalent to the condition $u_{(k_1,0)} = 0$ for all $k_1 \in \mathbb{Z}$, which appears in the definition of the space $\boldsymbol{V}$. The fact that w is odd in z implies that $w_{(k_1,0)} = 0$, for all k_1. We use these properties in what follows.

We hereby assume that S_u, S_v, S_ρ are functions independent of time.

With primes denoting $\partial/\partial s$, we can write system (3.1) in Fourier modes as follows:

$$\begin{aligned}
u'_k + \varepsilon \sum_{j+l=k} (il'_1 u_j u_l + il'_3 w_j u_l) - v_k + ik'_1 p_k &= -\varepsilon \nu_{\boldsymbol{v}} |k'|^2 u_k + \varepsilon S_{u,k}, \\
v'_k + \varepsilon \sum_{j+l=k} (il'_1 u_j v_l + il'_3 w_j v_l) + u_k &= -\varepsilon \nu_{\boldsymbol{v}} |k'|^2 v_k + \varepsilon S_{v,k}, \\
ik'_3 p_k &= -N\rho_k, \\
k'_1 u_k + k'_3 w_k &= 0, \\
\rho'_k + \varepsilon \sum_{j+l=k} (il'_1 u_j \rho_l + il'_3 w_j \rho_l) - N w_k &= -\varepsilon \nu_\rho |k'|^2 \rho_k + \varepsilon S_{\rho,k}.
\end{aligned} \tag{3.3}$$

The zeroth-order system: We now make explicit for our problem the solution of the linear zeroth-order equation (2.17), whose solution will be used later on in the variation of constants formulas and in particular in the analogue of (2.20). With the same notation as before and with $U = (u, v, \rho)$, we have

$$\begin{aligned}
u'_k - v_k + ik'_1 p_k &= 0, \\
v'_k + u_k &= 0, \\
ik'_3 p_k &= -N\rho_k, \\
k'_1 u_k + k'_3 w_k &= 0, \\
\rho'_k - N w_k &= 0.
\end{aligned} \tag{3.4}$$

For $k_3 = 0$, we have $u_{(k_1,0)} = 0$, $w_{(k_1,0)} = 0$ and $\rho_{(k_1,0)} = 0$ from the definition of the space $\boldsymbol{V}$, so only the first two lines of system (3.4) are non-trivial:

$$\begin{aligned}
-v_k + ik'_1 p_k &= 0, \\
v'_k &= 0.
\end{aligned} \tag{3.5}$$

This gives us $v_{(k_1,0)}(s) = v_{(k_1,0)}(0)$ and (3.5) allows us to express p_k in terms of v_k.

For $k_3 \neq 0$ we can express the k-component of the diagnostic variables in terms of the prognostic variables:

$$p_k = -\frac{N}{ik'_3}\rho_k, \tag{3.6}$$

$$w_k = -\delta_k u_k, \tag{3.7}$$

where for notational conciseness we have set

$$\delta_k = \frac{k'_1}{k'_3} \text{ if } k'_3 \neq 0, \quad \text{and} \quad \delta_k = 0 \text{ if } k'_3 = 0. \tag{3.8}$$

Substituting (3.6) and (3.7) in (3.4) we find:

$$u'_k - v_k - \delta_k N \rho_k = 0,$$

$$v'_k + u_k = 0,$$

$$\rho'_k + \delta_k N u_k = 0. \tag{3.9}$$

To solve this system we introduce the following change of unknowns suggested by the diagonalization of system (3.9). We set

$$n_k = \frac{1}{\beta_k} v_k + \frac{\delta_k N}{\beta_k}\rho_k = (v_k, \rho_k) \cdot \vec{\phi}_k, \tag{3.10}$$

where we denoted

$$\beta_k = (1 + \delta_k^2 N^2)^{1/2} \tag{3.11}$$

and

$$\vec{\phi}_k = \left(\frac{1}{\beta_k}, \frac{\delta_k N}{\beta_k}\right). \tag{3.12}$$

We also define the following vector:

$$\vec{\gamma}_k = \left(-\frac{\delta_k N}{\beta_k}, \frac{1}{\beta_k}\right) \tag{3.13}$$

and we set $m_k = (v_k, \rho_k) \cdot \vec{\gamma}_k$. For notational conciseness we also set

$$\vec{m}_k = m_k \vec{\gamma}_k, \quad \vec{n}_k = n_k \vec{\phi}_k. \tag{3.14}$$

Note that $\vec{\phi}_k = (1, 0)$ and $\vec{\gamma}_k = (0, 1)$ when $k_3 = 0$.

Conversely, given m_k and n_k, the initial unknowns can be recovered using $v_k = (m_k, n_k) \cdot \vec{\gamma}_k$ and $\rho_k = (m_k, n_k) \cdot \vec{\phi}_k$.

In the new variables u_k, n_k, m_k, system (3.9) for $k_3 \neq 0$ can now be written as

$$u_k' - \beta_k n_k = 0,$$

$$n_k' + \beta_k u_k = 0,$$

$$m_k' = 0, \tag{3.15}$$

and this system is easy to solve.

Weak formulation (*in the new variables*): We denote by n and m the functions

$$n(x,z,s) = \sum\nolimits_k n_k(s) e^{i(k_1' x + k_3' z)}, \quad m(x,z,s) = \sum\nolimits_k m_k(s) e^{i(k_1' x + k_3' z)},$$

where here and elsewhere $\sum_k$ means the summation over $k = (k_1, k_3) \in \mathbb{Z}^2 \backslash \{0\}$.

We also consider S_n and S_m similarly defined by their Fourier series. Here we have set $S_{m,k} = (S_{v,k}, S_{\rho,k}) \cdot \vec{\gamma}_k$ and $S_{n,k} = (S_{v,k}, S_{\rho,k}) \cdot \vec{\phi}_k$.

As we saw before, $m_{(k_1,0)} = 0$. This motivates us to introduce the following spaces:

$$\tilde{\boldsymbol{V}} = \{(u,n,m) \in (\dot{H}^1_{\mathrm{per}}(\mathcal{M}))^3\colon\ u_{(k_1,0)} = 0, u, n \text{ are even in } z, m \text{ is odd in } z\},$$

$$\tilde{\boldsymbol{H}} = \text{the closure of } \tilde{\boldsymbol{V}} \text{ in } (\dot{L}^2(\mathcal{M}))^3.$$

Notice that technically the space $\tilde{\boldsymbol{V}}$ is the same as $\boldsymbol{V}$ but the components play different roles.

We also introduce the space

$$\tilde{\boldsymbol{V}}_2 = \text{the closure of } \tilde{\boldsymbol{V}} \cap (\dot{H}^2_{\mathrm{per}}(\mathcal{M}))^3 \text{ in } (\dot{H}^2_{\mathrm{per}}(\mathcal{M}))^3. \tag{3.16}$$

We now define the linear operators $\tilde{A}$, $\tilde{L}$ from $\tilde{\boldsymbol{V}}$ into the dual $\tilde{\boldsymbol{V}}'$ of $\tilde{\boldsymbol{V}}$, and the bilinear operator $\tilde{B}$ from $\tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}}$ into $\tilde{\boldsymbol{V}}_2'$. These operators are the expressions of A and B in the new variables. With $V = (u,n,m)$, they are defined by their Fourier series components $\tilde{A}_k$, $\tilde{B}_k$ as follows:

$$\tilde{A}V = \sum\nolimits_k \tilde{A}_k(V) e^{i(k_1' x + k_3' z)},$$

$$\tilde{B}(V, V^\flat) = \sum\nolimits_k \tilde{B}_k(V, V^\flat) e^{i(k_1' x + k_3' z)},$$

$$\tilde{L}V = \sum\nolimits_k \tilde{L}_k(V) e^{i(k_1' x + k_3' z)}.$$

More explicitly, for $\tilde{A}_k$ we have

$$\tilde{A}_k V_k = \begin{pmatrix} |k'|^2 v_{\boldsymbol{v}} u_k \\ |k'|^2 v_{\boldsymbol{v}} n_k + (v_\rho - v_{\boldsymbol{v}})|k'|^2 (N\delta_k/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k \\ |k'|^2 v_{\boldsymbol{v}} m_k + |k'|^2 (1/\beta_k)(v_\rho - v_{\boldsymbol{v}})(m_k, n_k) \cdot \vec{\phi}_k \end{pmatrix} \quad \text{for all } k,$$

while for $\tilde{L}_k$ we have

$$\tilde{L}_k = 0 \quad \text{for } k_3 = 0,$$

$$\tilde{L}_k = \begin{pmatrix} 0 & -\beta_k & 0 \\ \beta_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } k_3 \neq 0,$$

and similarly for $\tilde{B}$,

$$\tilde{B}_k(V, V^\flat) = \begin{pmatrix} 0 \\ i \sum^k k'_1 u_j (\vec{m}^\flat_l + \vec{n}^\flat_l) \cdot \vec{\phi}_k \\ 0 \end{pmatrix} \quad \text{for } k_3 = 0,$$

$$\tilde{B}_k(V, V^\flat) = \begin{pmatrix} i \sum^k (l'_1 - l'_3 \delta_j) u_j u^\flat_l \\ i \sum^k (l'_1 - l'_3 \delta_j) u_j (\vec{m}^\flat_l + \vec{n}^\flat_l) \cdot \vec{\phi}_k \\ i \sum^k (l'_1 - l'_3 \delta_j) u_j (\vec{m}^\flat_l + \vec{n}^\flat_l) \cdot \vec{\gamma}_k \end{pmatrix} \quad \text{for } k_3 \neq 0.$$

Here and elsewhere in this paper $\sum^k$ means that the sum is taken over j, l in $\mathbb{Z}^2 \backslash \{0\}$, for $j + l = k$.

The resulting system from this change of variables can be written in the form

$$V' + \tilde{L} V = \varepsilon \tilde{\mathscr{G}}(V), \tag{3.17}$$

where $\tilde{S} = (S_u, S_n, S_m)$ and

$$\tilde{\mathscr{G}}(V) = -\tilde{A} V - \tilde{B}(V, V) + \tilde{S}.$$

We also define the bilinear forms $\tilde{a}(V, V^\flat) = \langle \tilde{A} V, V^\flat \rangle_{\tilde{\boldsymbol{V}}', \tilde{\boldsymbol{V}}}$ and $\tilde{e}(V, V^\flat) = \langle \tilde{L} V, V^\flat \rangle_{\tilde{\boldsymbol{V}}', \tilde{\boldsymbol{V}}}$ where V and $V^\flat$ belong to $\tilde{\boldsymbol{V}}$. We also introduce $\tilde{b}(V, V^\flat, V^\#) = \langle \tilde{B}(V, V^\flat), V^\# \rangle_{\tilde{\boldsymbol{V}}', \tilde{\boldsymbol{V}}}$ where V, $V^\#$ belong to $\tilde{\boldsymbol{V}}$ and $V^\flat$ belongs to $\tilde{\boldsymbol{V}}_2$. Writing explicitly the trilinear form $\tilde{b}$ we find

$$\begin{aligned} \tilde{b}(V, V^\flat, V^\#) = {} & i \sum^{\mathrm{c}} (l'_1 - l'_3 \delta_j) u_j u^\flat_l u^\#_k \\ & + i \sum^{\mathrm{c}} (l'_1 - l'_3 \delta_j) u_j (\vec{m}^\flat_l + \vec{n}^\flat_l) \cdot (\vec{m}^\#_k + \vec{n}^\#_k). \end{aligned}$$

Here and elsewhere in this paper, $\sum^{c}$ means that the sum is taken over j, l, k, for $j+l+k=0$.

The variational formulation of the problem in the new variables now reads:

Given $t_\star > 0$ arbitrary, $V_0 \in \tilde{\boldsymbol{H}}$ and $\tilde{S} = (S_u, S_n, S_m) \in L^2(0, t_\star; \tilde{\boldsymbol{H}})$, we look for a function V from $(0, t_\star)$ into $\tilde{\boldsymbol{V}}$ such that

$$\frac{d}{dt}(V, V^\flat)_{\tilde{\boldsymbol{H}}} + \tilde{a}(V, V^\flat) + \tilde{b}(V, V, V^\flat) + \tilde{e}(V, V^\flat) = (\tilde{S}, V^\flat)_{\tilde{\boldsymbol{V}}}, \quad \forall V^\flat \in \tilde{\boldsymbol{V}} \tag{3.18}$$

and

$$V(0) = V_0. \tag{3.19}$$

The first-order system: We write the full non-linear system (3.3) in terms of the new variables.

For $k_3 \neq 0$, system (3.3) in the new variables reads

$$u_k' - \beta_k n_k = -\varepsilon \nu_{\boldsymbol{v}} |k'|^2 u_k - i\varepsilon \sum\nolimits^k (l_1' - l_3'\delta_j) u_j u_l + \varepsilon S_{u,k},$$

$$\begin{aligned} n_k' + \beta_k u_k = & -\varepsilon \nu_{\boldsymbol{v}} |k'|^2 n_k - \varepsilon |k'|^2 (\nu_\rho - \nu_{\boldsymbol{v}}) \frac{\delta_k N}{\beta_k} (m_k, n_k) \cdot \vec{\phi}_k \\ & - i\varepsilon \sum\nolimits^k (l_1' - l_3'\delta_j) u_j (\vec{m}_l + \vec{n}_l) \cdot \vec{\phi}_k + \varepsilon S_{n,k}, \end{aligned}$$

$$\begin{aligned} m_k' = & -\varepsilon \nu_{\boldsymbol{v}} |k'|^2 m_k - \varepsilon |k'|^2 (\nu_\rho - \nu_{\boldsymbol{v}}) \frac{1}{\beta_k} (m_k, n_k) \cdot \vec{\phi}_k \\ & - i\varepsilon \sum\nolimits^k (l_1' - l_3'\delta_j) u_j (\vec{m}_l + \vec{n}_l) \cdot \vec{\gamma}_k + \varepsilon S_{m,k}. \end{aligned} \tag{3.20}$$

For the case $k_3 = 0$ we note that $u_k = 0$ and $m_k = 0$ because of the definitions of the spaces.

Study of the new variational problem: We can see, after some elementary computations, that a is a bilinear and coercive form on $\tilde{\boldsymbol{V}}$, so it remains to prove the properties of $\tilde{b}$.

Lemma 3.1. *The form $\tilde{b}$ is trilinear continuous from $\tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}}_2 \times \tilde{\boldsymbol{V}}$ to $\mathbb{R}$ and from $\tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}}_2$ to $\mathbb{R}$, and*

$$\tilde{b}(V, V^\flat, V^\flat) = 0, \quad \forall V \in \tilde{\boldsymbol{V}}, \ \forall V^\flat \in \tilde{\boldsymbol{V}}_2,$$

$$\tilde{b}(V, V^\flat, V^\#) = -\tilde{b}(V, V^\#, V^\flat) \quad \forall V, V^\flat, V^\# \in \tilde{\boldsymbol{V}}, \text{ with } V^\flat \text{ or } V^\# \in \tilde{\boldsymbol{V}}_2. \tag{3.21}$$

Furthermore,

$$|\tilde{b}(V, V^\flat, V^\#)| \leqslant c\,|V|_{H^1}|V^\flat|_{H^1}^{1/2}|V^\flat|_{H^2}^{1/2}|V^\#|_{L^2}^{1/2}|V^\#|_{H^1}^{1/2}, \tag{3.22}$$

for all V, $V^\#$ in $\tilde{\boldsymbol{V}}$ and $V^\flat$ in $\tilde{\boldsymbol{V}}_2$.

Proof. To prove the continuity of the bilinear form and (3.22), we estimate for example the second term of $\tilde{b}(V, V^\flat, V^\#)$, the estimates being similar for all the terms:

$$\begin{aligned}
&\left| i\sum\nolimits^{\mathrm{c}} (l_1' - l_3'\delta_j)u_j(\vec{m}_l^\flat + \vec{n}_l^\flat)\cdot(\vec{m}_k^\# + \vec{n}_k^\#)\right| \\
&\quad \leqslant \sum\nolimits^{\mathrm{c}} |l'|\,|j'|\,|u_j|(|m_l^\flat| + |n_l^\flat|)(|m_k^\#| + |n_k^\#|) \\
&\quad \leqslant \int_{\mathscr{M}} \eta_1\eta_2\eta_3\, d\mathscr{M} \leqslant |\eta_1|_{L^2}|\eta_2|_{L^4}|\eta_3|_{L^4} \\
&\quad \leqslant c|\eta_1|_{L^2}|\eta_2|_{L^2}^{1/2}|\eta_2|_{H^1}^{1/2}|\eta_3|_{L^2}^{1/2}|\eta_3|_{H^1}^{1/2} \\
&\quad \leqslant c|V|_{H^1}|V^\flat|_{H^1}^{1/2}|V^\flat|_{H^2}^{1/2}|V^\#|_{L^2}^{1/2}|V^\#|_{H^1}^{1/2},
\end{aligned}$$

here we wrote

$$\eta_1 = \sum_j |j'||u_j|e^{i(x\cdot j')}, \quad \eta_2 = \sum_j |j'|(|m_j^\flat| + |n_j^\flat|)e^{i(x\cdot j')},$$
$$\eta_3 = \sum_j (|m_j^\#| + |n_j^\#|)e^{i(x\cdot j')}.$$

It remains to prove the orthogonality property (3.21). For $V^\flat = V^\#$ we have

$$\begin{aligned}
\tilde{b}(V, V^\flat, V^\flat) = {}& i\sum\nolimits^{\mathrm{c}} (l_1' - l_3'\delta_j)u_j u_l^\flat u_k^\flat \\
&+ i\sum\nolimits^{\mathrm{c}} (l_1' - l_3'\delta_j)u_j(\vec{m}_l^\flat + \vec{n}_l^\flat)\cdot(\vec{m}_k^\flat + \vec{n}_k^\flat).
\end{aligned} \tag{3.23}$$

Interchanging k and l and adding the resulting equations to (3.23), we find

$$\begin{aligned}
\tilde{b}(V, V^\flat, V^\flat) = {}& \frac{i}{2}\sum\nolimits^{\mathrm{c}} [l_1' + k_1' - (l_3' + k_3')\delta_j]u_j u_l^\flat u_k^\flat \\
&+ \frac{i}{2}\sum\nolimits^{\mathrm{c}} [l_1' + k_1' - (l_3' + k_3')\delta_j]u_j(\vec{m}_l^\flat + \vec{n}_l^\flat)\cdot(\vec{m}_k^\flat + \vec{n}_k^\flat) \\
= {}& 0.
\end{aligned}$$

We have used here the fact that

$$l_1' + k_1' - (l_3' + k_3')\delta_j = -j_1' + j_3'\frac{j_1'}{j_3'} = 0. \qquad \square$$

Remark 3.2. Because of the algebraic way we changed the variables and the conservation of the properties for the linear and bilinear operators, we have exactly the same result as Theorem 2.1 for the new system.

3.2. The renormalized equation. Existence of weak solutions

We turn now to the renormalized system [the analogue of (2.24) for (3.17)],

$$\frac{d\bar{V}}{dt} + \tilde{A}_{\mathrm{r}}(\bar{V}) + \tilde{B}_{\mathrm{r}}(\bar{V}, \bar{V}) = \tilde{S}_{\mathrm{r}}. \tag{3.24}$$

The computation of $\tilde{A}_{\mathrm{r}}$, $\tilde{B}_{\mathrm{r}}$ and $\tilde{S}_{\mathrm{r}}$ is given in Appendix A. It is established there that $\tilde{a}_{\mathrm{r}}(V, V^{\#}) = \langle \tilde{A}_{\mathrm{r}} V, V^{\#} \rangle_{\tilde{\boldsymbol{V}}', \tilde{\boldsymbol{V}}}$ is a bilinear continuous form in $\tilde{\boldsymbol{V}}$ satisfying

$$\tilde{a}_{\mathrm{r}}(\bar{V}, \bar{V}) \geqslant c_1 ||\bar{V}||^2, \tag{3.25}$$

and that $\tilde{b}_{\mathrm{r}}(V, V^{\#}, V^{\flat}) = \langle \tilde{B}_{\mathrm{r}}(V, V^{\#}), V^{\flat} \rangle_{\tilde{\boldsymbol{V}}', \tilde{\boldsymbol{V}}}$ is trilinear continuous on $\tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}}_2 \times \tilde{\boldsymbol{V}}$ satisfying

$$\tilde{b}_{\mathrm{r}}(\bar{V}, \bar{V}, \bar{V}) = 0. \tag{3.26}$$

The variational formulation of the renormalized problem (3.24): *Given* $t_\star > 0$ *arbitrary and*

$$\bar{V}_0 \in \tilde{\boldsymbol{H}}, \quad \tilde{S}_{\mathrm{r}} \in \tilde{\boldsymbol{H}},$$

we look for a function $\bar{V}$ *from* $(0, t_\star)$ *into* $\tilde{\boldsymbol{V}}$, *such that, for every test function* $V^{\#} \in \tilde{\boldsymbol{V}}$,

$$\left(\frac{d}{dt}\bar{V}, V^{\#}\right) + \tilde{a}_{\mathrm{r}}(\bar{V}, V^{\#}) + \tilde{b}_{\mathrm{r}}(\bar{V}, \bar{V}, V^{\#}) = (\tilde{S}_{\mathrm{r}}, V^{\#}), \tag{3.27}$$

with

$$\bar{V}(0) = \bar{V}_0. \tag{3.28}$$

As usual, in order to solve this problem we need to obtain some a priori estimates. For that purpose, for arbitrary fixed $t > 0$, we set $V^{\#} = \bar{V}(t)$ in Eq. (3.27). Taking into account coercivity (3.25) and orthogonality (3.26) properties, we obtain

$$\frac{1}{2}\frac{d}{dt}|\bar{V}|_{L^2}^2 + c_1||\bar{V}||^2 \leqslant (\tilde{S}_{\mathrm{r}}, \bar{V})_{L^2} \leqslant \frac{c_1}{2}||\bar{V}||^2 + c_1'|\tilde{S}_{\mathrm{r}}|_{L^2}^2.$$

This gives

$$\frac{d}{dt}|\bar{V}|_{L^2}^2 + c_1||\bar{V}||^2 \leqslant 2c_1'|\tilde{S}_{\mathrm{r}}|_{L^2}^2. \tag{3.29}$$

Applying Poincaré's inequality (2.6) we find,

$$\frac{d}{dt}|\bar{V}|_{L^2}^2 + c_1 c_0|\bar{V}|_{L^2}^2 \leqslant 2c_1'|\tilde{S}_{\mathrm{r}}|_{L^2}^2, \tag{3.30}$$

and, using the Gronwall lemma,

$$|\bar{V}(t)|_{L^2}^2 \leqslant e^{-c_1 c_0 t}|\bar{V}(0)|_{L^2}^2 + \frac{2c_1'}{c_0 c_1}|\tilde{S}_{\mathrm{r}}|_{L^2}^2(1 - e^{-c_0 c_1 t}). \tag{3.31}$$

This bounds $\bar{V}(t)$ for all t by its initial data,

$$|\bar{V}(t)|_{L^2}^2 \leqslant |\bar{V}(0)|_{L^2}^2 + \frac{2c_1'}{c_0 c_1}|\tilde{S}_{\mathrm{r}}|_{L^2}^2.$$

Eq. (3.31) also gives us a bound on $\bar{V}(t)$ independent of the initial data: Setting $r_0^2 \coloneqq (2c_1'/c_0 c_1)|\tilde{S}_{\mathrm{r}}|_{L^2}^2$, we obtain by classical computations (see e.g. [20]) that any ball $B(0, r_0')$ with $r_0' > r_0$ is an absorbing ball and that $|\bar{V}(t)|_{L^2}^2 \leqslant r_0^2$ for all $t \geqslant t_0(|\bar{V}_0|_{L^2})$.

Using the previous estimates and the Galerkin method we can establish the existence of weak solutions of (3.27) and (3.28) exactly as for the original problem (Theorem 2.1):

Theorem 3.3. *Given* $t_\star > 0$, $\tilde{S}_{\mathrm{r}} \in \tilde{\boldsymbol{H}}$ *and* $\bar{V}_0 \in \tilde{\boldsymbol{H}}$, *problem* (3.27)–(3.28) *has at least one solution*

$$\bar{V} \in L^\infty(\mathbb{R}_+; \tilde{\boldsymbol{H}}) \cap L^2(0, t_\star; \tilde{\boldsymbol{V}}).$$

3.3. *Strong solutions for the renormalized equation*

We derive the appropriate a priori estimates. Setting $V^\# = \Delta\bar{V}(t)$ in (3.27) with $t > 0$ arbitrary, we find

$$\frac{1}{2}\frac{d}{dt}||\bar{V}||^2 + c_1|\Delta\bar{V}|_{L^2}^2 \leqslant |\tilde{b}_{\mathrm{r}}(\bar{V}, \bar{V}, \Delta\bar{V})| + c_3'|\tilde{S}_{\mathrm{r}}|_{L^2}^2 + \frac{c_1}{4}|\Delta\bar{V}|_{L^2}^2. \tag{3.32}$$

Bounding the trilinear form on the r.h.s. using Lemma A.1,

$$|\tilde{b}_{\mathrm{r}}(\bar{V}, \bar{V}, \Delta\bar{V})| \leqslant 2c_2|\bar{V}|_{L^2}^{1/2}||\bar{V}|||\Delta\bar{V}|_{L^2}^{3/2},$$

we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\bar{V}||^2 + c_1|\Delta\bar{V}|_{L^2}^2 &\leqslant 2c_2|\bar{V}|_{L^2}^{1/2}||\bar{V}|||\Delta\bar{V}|_{L^2}^{3/2} + c_3'|\tilde{S}_{\mathrm{r}}|_{L^2}^2 + \frac{c_1}{4}|\Delta\bar{V}|_{L^2}^2 \\ &\leqslant c_4'|\bar{V}|_{L^2}^2||\bar{V}||^4 + c_3'|\tilde{S}_{\mathrm{r}}|_{L^2}^2 + \frac{c_1}{2}|\Delta\bar{V}|_{L^2}^2\end{aligned}$$

or

$$\frac{d}{dt}||\bar{V}||^2 + c_1|\Delta\bar{V}|_{L^2}^2 \leqslant 2c_4'|\bar{V}|_{L^2}^2||\bar{V}||^4 + 2c_3'|\tilde{S}_{\mathrm{r}}|_{L^2}^2. \tag{3.33}$$

Existence follows from applying the classical Gronwall lemma, giving us a bound on $\bar{V}$ in $L^\infty(0, t_\star; H^1)$.

A bound uniform in time is obtained in the following manner: We pick $r>0$ arbitrary and integrate (3.29) from t to $t+r$ with $t\geqslant 0$,

$$c_1\int_t^{t+r}||\bar{V}(t')||^2\,dt'\leqslant rc_2'|\tilde{S}_{\mathrm{r}}|_{L^2}^2+|\bar{V}(t)|_{L^2}^2. \tag{3.34}$$

This and the fact that $|\bar{V}|_{L^2}$ is bounded in $L^\infty(\mathbb{R}_+)$ allows us to apply the uniform Gronwall lemma to (3.33) (as in [20]). Computations similar to those in [17] give us estimates uniform in time and we have that $||\bar{V}||$ is bounded in $L^\infty(\mathbb{R}_+)$.

Integrating (3.33) from 0 to $t_\star$ we obtain a bound of $\bar{V}$ in $L^2(0,t_\star;\tilde{\boldsymbol{V}}\cap(H^2_{\mathrm{per}}(\mathscr{M}))^3)$. For later purposes, we note that integrating (3.33) from t to $t+r$ gives us

$$\int_t^{t+r}|\Delta\bar{V}(t')|_{L^2}^2\,dt'\leqslant k(r,\tilde{S}_{\mathrm{r}}),\quad \forall t\geqslant t_1(|\bar{V}_0|_{L^2},r). \tag{3.35}$$

These a priori estimates give the following:

Theorem 3.4. *Given $\tilde{S}_{\mathrm{r}}\in\tilde{\boldsymbol{H}}$ and $\bar{V}_0\in\tilde{\boldsymbol{V}}$, problem* (3.27) *has a unique solution*

$$\bar{V}\in L^\infty(\mathbb{R}_+;\tilde{\boldsymbol{V}})\cap L^2(0,t_\star;\tilde{\boldsymbol{V}}\cap(H^2_{\mathrm{per}}(\mathscr{M}))^3),\quad \forall t_\star>0. \tag{3.36}$$

Remark 3.5. (i) Uniqueness in Theorem 3.4 is proved in a classical way.

(ii) The proof Theorem 3.4 for the renormalized system (3.24)–(3.28) is simpler than for the original system (2.7) due to the fact that the analogue of (3.33) for the latter is of the form (see [17]):

$$\frac{d}{dt}||U||^2+c_1|\Delta U|_{L^2}^2\leqslant c_1'|\Delta U|_{L^2}^2||U||+c_2'||U||^2, \tag{3.37}$$

which does not lead immediately to the appropriate estimates in $L^\infty(0,t_1;\boldsymbol{V})$. The difference between the r.h.s. of (3.34) and (3.37) arises because the renormalized system does not contain problematic terms that are present in the original system.

3.4. More regular solutions for the renormalized system

It is desirable to establish the existence of more regular solutions for the renormalized equation. We do this by induction. For simplicity we take the forcing S independent of time and $S,\ \bar{V}_0\in\bigcap_m \dot{H}^m_{\mathrm{per}}$.

Suppose that for a fixed arbitrary $m\in\mathbb{N}$, $m\geqslant 2$, we have

$$\bar{V}\in L^\infty(\mathbb{R}_+;\tilde{\boldsymbol{V}}\cap(H^{m-1}_{\mathrm{per}}(\mathscr{M}))^3),$$

$$\int_t^{t+r}|\bar{V}(t')|_{H^m}^2\,dt'\leqslant K_m, \tag{3.38}$$

for all $t > t_{m-1}(\bar{V}_0)$, where by K_m we denote as before a constant independent of the initial condition.

We seek to prove that

$$\bar{V} \in L^{\infty}(\mathbb{R}_+; \tilde{\boldsymbol{V}} \cap (H^m_{\mathrm{per}}(\mathscr{M}))^3),$$

$$\int_t^{t+r} |\bar{V}(t')|^2_{H^{m+1}}\, dt' \leqslant K_{m+1}.$$

First we derive the a priori estimates: We set in (3.27)

$$\bar{V}_1 = (-1)^m \Delta^m \bar{V}(t) = \sum_{k \in \mathbb{Z}^2} |k'|^{2m} \bar{V}_k(t)\, e^{i(k' \cdot x)},$$

with $t > 0$ fixed, to get

$$\frac{1}{2}\frac{d}{dt}|\bar{V}|^2_{H^m} + c_1 |\bar{V}|^2_{H^{m+1}} \leqslant |\tilde{b_{\mathrm{r}}}(\bar{V}, \bar{V}, \Delta^m \bar{V})| + |(\tilde{S}_{\mathrm{r}}, \Delta^m \bar{V})_{L^2}|. \tag{3.39}$$

We estimate $|\tilde{b_{\mathrm{r}}}(\bar{V}, \bar{V}, \Delta^m \bar{V})|$ which, using (A.29), reads

$$\begin{aligned}
\tilde{b_{\mathrm{r}}}(\bar{V}, \bar{V}, \Delta^m \bar{V}) = &-\frac{i}{2} \sum^{\mathrm{c}}_{\substack{j_3 \neq 0,\ k_3 = 0 \\ \beta_j = \beta_l}} k'_1 |k'_1|^{2m} (\bar{n}_l \bar{u}_j - \bar{u}_l \bar{n}_j) \bar{n}_k \vec{\phi}_l \cdot \vec{\phi}_k \\
&- \frac{i}{2} \sum^{\mathrm{c}}_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_k}} |k'|^{2m} (l'_1 - l'_3 \delta_j) \bar{u}_k \bar{n}_j \bar{m}_l \vec{\phi}_k \cdot \vec{\gamma}_l \\
&- \frac{i}{2} \sum^{\mathrm{c}}_{\substack{j_3 \neq 0,\ l_3 = 0 \\ \beta_j = \beta_k}} l'_1 |k'_1|^{2m} \bar{n}_j \bar{n}_l \bar{u}_k \vec{\phi}_l \cdot \vec{\phi}_k \\
&+ \frac{i}{2} \sum^{\mathrm{c}}_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_k}} |k'|^{2m} (l'_1 - l'_3 \delta_j) \bar{u}_j \bar{m}_l \bar{n}_k \vec{\gamma}_l \cdot \vec{\phi}_k \\
&+ \frac{i}{2} \sum^{\mathrm{c}}_{\substack{j_3 \neq 0,\ l_3 = 0 \\ \beta_j = \beta_k}} l'_1 |k'_1|^{2m} \bar{u}_j \bar{n}_l \bar{n}_k \vec{\phi}_l \cdot \vec{\phi}_k \\
&+ \frac{i}{2} \sum^{\mathrm{c}}_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_l}} |k'|^{2m} (l'_1 - l'_3 \delta_j)(\bar{n}_l \bar{u}_j - \bar{n}_j \bar{u}_l) \bar{m}_k \vec{\phi}_l \cdot \vec{\gamma}_k.
\end{aligned} \tag{3.40}$$

The first term of (3.40) is bounded as

$$\left| \frac{i}{2} \sum^{\mathrm{c}}_{\substack{j_3 \neq 0,\ k_3 = 0 \\ \beta_j = \beta_l}} k'_1 |k'_1|^{2m} (\bar{n}_l \bar{u}_j - \bar{u}_l \bar{n}_j) \bar{n}_k \vec{\phi}_l \cdot \vec{\phi}_k \right|$$

$$
\begin{aligned}
&\leqslant c_1' \sum_{\substack{j_3 \neq 0,\ k_3=0 \\ \beta_j=\beta_l}}^{\mathrm{c}} |k'|^{2m+1}(|\bar{n}_l||\bar{u}_j| + |\bar{n}_j||\bar{u}_l|)|\bar{n}_k| \\
&\leqslant c_2' \sum_{\substack{j_3 \neq 0,\ k_3=0 \\ \beta_j=\beta_l}}^{\mathrm{c}} (|\bar{n}_l||\bar{u}_j| + |\bar{n}_j||\bar{u}_l|)|\bar{n}_k|(|j'|^m + |l'|^m)|k'|^{m+1} \\
&\leqslant c_3' \int_{\mathscr{M}} q_1 q_2 q_3 \, d\mathscr{M} + c_3' \int_{\mathscr{M}} q_3 q_4 q_5 \, d\mathscr{M} \\
&\leqslant c_3' |q_1|_{L^4} |q_2|_{L^4} |q_3|_{L^2} + c_3' |q_3|_{L^2} |q_4|_{L^4} |q_5|_{L^4} \\
&\leqslant c_4' |q_1|_{L^2}^{1/2} ||q_1||^{1/2} |q_2|_{L^2}^{1/2} ||q_2||^{1/2} |q_3|_{L^2} + c_4' |q_3|_{L^2} |q_4|_{L^2}^{1/2} ||q_4||^{1/2} |q_5|_{L^2}^{1/2} ||q_5||^{1/2} \\
&\leqslant c_5' |\bar{V}|_{L^2}^{1/2} ||\bar{V}||^{1/2} |\bar{V}|_{H^m}^{1/2} |\bar{V}|_{H^{m+1}}^{3/2},
\end{aligned}
$$

where we wrote

$$
q_1 = \sum_{j \in \mathbb{Z}^2} |\bar{u}_j||j'|^m e^{i(x \cdot j')}, \quad q_2 = \sum_{j \in \mathbb{Z}^2} |\bar{n}_j| e^{i(x \cdot j)}, \quad q_3 = \sum_{j \in \mathbb{Z}^2} |\bar{n}_j||j'|^{m+1} e^{i(x \cdot j')},
$$

$$
q_4 = \sum_{j \in \mathbb{Z}^2} |\bar{n}_j||j'|^m e^{i(x \cdot j')}, \quad q_5 = \sum_{j \in \mathbb{Z}^2} |\bar{u}_j| e^{i(x \cdot j')}.
$$

Estimating similarly the other terms, we finally obtain

Lemma 3.6. *There exists a constant $c_3 > 0$ depending only on L_1 and L_3 such that, for all $\bar{V}$ in $\tilde{V} \cap (H_{\mathrm{per}}^{2m}(\mathscr{M}))^3$,*

$$
\tilde{b}_{\mathrm{r}}(\bar{V}, \bar{V}, \Delta^m \bar{V}) \leqslant c_3 |\bar{V}|^{1/2} ||\bar{V}||^{1/2} |\bar{V}|_{H^m}^{1/2} |\bar{V}|_{H^{m+1}}^{3/2}. \tag{3.41}
$$

Returning to (3.39) and using Young's inequality, we find:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\bar{V}|_{H^m}^2 + c_1|\bar{V}|_{H^{m+1}}^2 &\leqslant c_3|\bar{V}|^{1/2}||\bar{V}||^{1/2}|\bar{V}|_{H^m}^{1/2}|\bar{V}|_{H^{m+1}}^{3/2} + |(\tilde{S}_{\mathrm{r}}, \Delta^m \bar{V})_{L^2}| \\
&\leqslant \frac{c_1}{2}|\bar{V}|_{H^{m+1}}^2 + c_1'|\tilde{S}_{\mathrm{r}}|_{H^{m-1}}^2 + c_2'|\bar{V}|_{L^2}^2|\bar{V}|_{H^1}^2|\bar{V}|_{H^m}^2
\end{aligned}
$$

or

$$
\frac{d}{dt}|\bar{V}|_{H^m}^2 + c_1|\bar{V}|_{H^{m+1}}^2 \leqslant 2c_1'|\tilde{S}_{\mathrm{r}}|_{H^{m-1}}^2 + 2c_2'|\bar{V}|_{L^2}^2|\bar{V}|_{H^1}^2|\bar{V}|_{H^m}^2. \tag{3.42}
$$

Applying the classical Gronwall lemma to (3.42) we obtain estimates in $L^\infty(0, t_\star; H^m)$ for all $t_\star > 0$, with the bounds depending on the initial data.

Bounds uniform in time, $\bar{V} \in L^\infty(\mathbb{R}_+; H^m)$, can be obtained by using the induction hypothesis and applying the uniform Gronwall lemma to (3.42). The bound thus obtained is independent of $|U_0|_m$ when $t \geqslant t_m(U_0)$ but the bound of $\bar{V}$ in $L^\infty(0, t_m(U_0); H^m)$ depends of course on $|U_0|_m$.

Applying classical methods (see, e.g., [11,12,20]) to the above a priori estimates, we find:

Theorem 3.7. *For any* $m \in \mathbb{N}$, $m \geqslant 2$, *given* $\bar{V}_0 \in (H^m_{\text{per}}(\mathcal{M}))^3 \cap \tilde{\boldsymbol{V}}$ *and* $\tilde{S}_{\text{r}} \in (H^{m-1}_{\text{per}}(\mathcal{M}))^3 \cap \tilde{\boldsymbol{V}}$, *there exists a unique solution* $\bar{V}$ *of* (3.27) *in* $L^\infty(\mathbb{R}_+; (H^m_{\text{per}}(\mathcal{M}))^3)$.

4. First-order error estimates

We introduce as in Section 2 the first-order approximate solution $V^1(s)$

$$V^1(s) = e^{-sL}[\bar{V}(s) + \varepsilon G_{\text{np}}(\bar{V}, s)]. \tag{4.1}$$

Here $\bar{V}(s)$ is the solution of the renormalized equation,

$$\frac{d\bar{V}}{ds} = \varepsilon G_{\text{r}}(\bar{V}),$$

$$\bar{V}(0) = V_0. \tag{4.2}$$

Our aim in this section is to compare the approximate solution $V^1(s)$ to the exact solution $V(s)$, which satisfies

$$\frac{dV}{ds} + \tilde{L}V = \varepsilon \mathcal{G}(V),$$

$$V(0) = V_0. \tag{4.3}$$

The notations we have used are as follows:

$$\mathcal{G}(V) := -\tilde{A}V - \tilde{B}(V, V) + \tilde{S},$$

$$G(s, V) := e^{\tilde{L}s}\mathcal{G}(e^{-\tilde{L}s}V).$$

The resonant and non-resonant parts of $G(s, V)$ are defined as in (2.21),

$$G(s, V) = G_{\text{r}}(V) + G_{\text{n}}(s, V), \tag{4.4}$$

and the primitive $G_{\text{np}}(s, V)$ of $G_{\text{n}}(s, V)$ is defined as in (2.22).

Denoting the error by

$$W(s) = V^1(s) - V(s) = e^{-sL}[\bar{V}(s) + \varepsilon G_{\text{np}}(\bar{V}(s), s)] - V(s), \tag{4.5}$$

we find after straightforward computations that it satisfies:

$$\frac{dW}{ds} + \tilde{L}W + \varepsilon\tilde{A}W + \varepsilon\tilde{B}(W, W) + \varepsilon\tilde{B}(V^1, W) + \varepsilon\tilde{B}(W, V^1) = \varepsilon^2 R_\varepsilon,$$

$$W(0) = 0, \tag{4.6}$$

where

$$\begin{aligned} R_\varepsilon = & -\tilde{A}e^{-s\tilde{L}}G_{\mathrm{np}}(s, \bar{V}) - \tilde{B}(e^{-s\tilde{L}}\bar{V}, e^{-s\tilde{L}}G_{\mathrm{np}}(s, \bar{V})) \\ & - \tilde{B}(e^{-s\tilde{L}}G_{\mathrm{np}}(s, \bar{V}), e^{-s\tilde{L}}\bar{V}) - \varepsilon\tilde{B}(e^{-s\tilde{L}}G_{\mathrm{np}}(s, \bar{V}), e^{-s\tilde{L}}G_{\mathrm{np}}(s, \bar{V})) \\ & - e^{-s\tilde{L}}\nabla_{\bar{V}}G_{\mathrm{np}}(s, \bar{V}) \cdot G_{\mathrm{r}}(\bar{V}). \end{aligned} \tag{4.7}$$

We take the scalar product of (4.6) with W in $(L^2(\mathscr{M}))^3$ and, using the coercivity and orthogonality properties, we obtain,

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}|W|_{L^2}^2 + \varepsilon c_1|W|_{H^1}^2 & \leqslant \varepsilon|\tilde{b}(W, V^1, W)| + \varepsilon^2|(R_\varepsilon, W)_{L^2}| \\ & \leqslant \varepsilon|\tilde{b}(W, V^1, W)| + \varepsilon^2 c_0|R_\varepsilon|_{L^2}|W|_{H^1}. \end{aligned} \tag{4.8}$$

The first term on the r.h.s. is bounded using Lemma 3.1,

$$|\tilde{b}(W, V^1, W)| \leqslant c|W|_{L^2}^{1/2}|V^1|_{H^1}^{1/2}|V^1|_{H^2}^{1/2}|W|_{H^1}^{3/2}; \tag{4.9}$$

applying Young's inequality to this and to $|R_\varepsilon|_{L^2}|W|_{H^1}$, we find

$$\frac{d}{ds}|W|_{L^2}^2 + \varepsilon c_1|W|_{H^1}^2 \leqslant \varepsilon^2 c'|R_\varepsilon|_{L^2}^2 + \varepsilon c|W|_{L^2}^2|V^1|_{H^1}^2|V^1|_{H^2}^2. \tag{4.10}$$

It remains to estimate R_ε and V^1.

Estimates for R_ε: We start with

$$\begin{aligned} |R_\varepsilon|_{L^2} \leqslant & \, c|e^{-\tilde{L}s}G_{\mathrm{np}}(\bar{V}, s)|_{H^2} + |e^{-\tilde{L}s}\nabla_{\bar{V}}G_{\mathrm{np}}(\bar{V}, s) \cdot G_{\mathrm{r}}(\bar{V})|_{L^2} \\ & + |\tilde{B}(e^{-\tilde{L}s}\bar{V}, e^{-\tilde{L}s}G_{\mathrm{np}}(\bar{V}, s))|_{L^2} + |\tilde{B}(e^{-\tilde{L}s}G_{\mathrm{np}}(\bar{V}, s), e^{-\tilde{L}s}\bar{V})|_{L^2} \\ & + \varepsilon|\tilde{B}(e^{-\tilde{L}s}G_{\mathrm{np}}(\bar{V}, s), e^{-\tilde{L}s}G_{\mathrm{np}}(\bar{V}, s))|_{L^2}. \end{aligned} \tag{4.11}$$

Note that since the eigenvalues of the matrix $\tilde{L}_k$ are purely imaginary for all $k \in \mathbb{Z}^2$,

$$|e^{-\tilde{L}s}V| \leqslant |V|, \tag{4.12}$$

where here $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^3$.

By arguments similar to those used in the proof of Lemmas 3.1 and 3.6, one can show that, for all $p \in \mathbb{N}$,

$$|\tilde{B}(V, V^\flat)|_{H^p} \leqslant c_4|V|_{H^{p+2}}|V^\flat|_{H^{p+2}}, \quad \forall V, V^\flat \in \tilde{V} \cap (H_{\mathrm{per}}^{p+2}(\mathscr{M}))^3. \tag{4.13}$$

Using this and (4.12) in (4.11), we have

$$|R_\varepsilon|_{L^2} \leqslant c|G_{\mathrm{np}}(\bar{V},s)|_{H^2} + 2c_4|\bar{V}|_{H^2}|G_{\mathrm{np}}(\bar{V},s)|_{H^2} + \varepsilon c_4|G_{\mathrm{np}}(\bar{V},s)|^2_{H^2}$$
$$+ c|\nabla_{\bar{V}}G_{\mathrm{np}}(\bar{V},s)\cdot G_{\mathrm{r}}(\bar{V})|_{L^2}. \tag{4.14}$$

To continue we need to estimate $|G_{\mathrm{np}}(s,\bar{V})|_{H^2}$ and $|\nabla_{\bar{V}}G_{\mathrm{np}}(s,\bar{V})\cdot G_{\mathrm{r}}(\bar{V})|_{L^2}$.

Estimates for $G_{\mathrm{np}}(\bar{V},s)$: We recall from Appendix A that $G_{\mathrm{n}} = \tilde{A}_{\mathrm{n}} + \tilde{B}_{\mathrm{n}} + \tilde{S}_{\mathrm{n}}$, with $\tilde{A}_{\mathrm{n}}$, $\tilde{B}_{\mathrm{n}}$ and $\tilde{S}_{\mathrm{n}}$ being defined in (A.17), (A.23), (A.25) and (A.26). To estimate

$$G_{\mathrm{np}}(s,\bar{V}) = \int_0^s G_{\mathrm{n}}(s,\bar{V})\,ds,$$

we shall need to bound terms of the forms:

$$I_1(j) = \frac{e^{s\alpha\beta_j}-1}{\alpha\beta_j}, \tag{4.15}$$

$$I_2(j,l) = \frac{e^{s(\alpha_1\beta_j+\alpha_2\beta_l)}-1}{\alpha_1\beta_j+\alpha_2\beta_l}, \quad \text{where } \beta_j - \beta_l \neq 0, \tag{4.16}$$

$$I_3(j,l,k) = \frac{e^{s(\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k)}-1}{\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k}, \tag{4.17}$$

In these expressions, the αs can take on the values of $\pm i$ and the β's are real and not less than 1 [cf. (3.11)].

We now obtain bounds for the denominators in (4.16) and (4.17). It turns out that, provided that the Burgers number N does not lie in a certain set of measure zero, $\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k \neq 0$. Similarly, it can also be shown (cf. Appendix B.2) that, when N lies outside a small set, the denominators can be bounded from below.

$I_1(j)$ is easily estimated:

$$|I_1(j)| = \left|\frac{e^{\alpha s\beta_j}-1}{\alpha\beta_j}\right| = \frac{\sqrt{2(1-\cos s\beta_j)}}{\beta_j} \leqslant 2. \tag{4.18}$$

To estimate $I_2(j,l)$, $\beta_j \neq \beta_l$, we distinguish two cases:

(i) For $\alpha_1 = \alpha_2$, we obtain $|I_2(j,l)| = 2/|\beta_j+\beta_l| \leqslant 1$.

(ii) For $\alpha_1 = -\alpha_2$, we need a bound for $2/|\beta_j-\beta_l|$. We assume without loss of generality that $\beta_j > \beta_l$; writing $N' = N^2(L_3/L_1)^2$ we find,

$$|I_2(j,l)| = \frac{2}{\beta_j-\beta_l} = \frac{2(\beta_j+\beta_l)}{\beta_j^2-\beta_l^2} = \frac{2(\beta_j+\beta_l)}{N'(j_1/j_3)^2 - N'(l_1/l_3)^2} = \frac{2}{N'}\frac{\beta_j+\beta_l}{j_1^2 l_3^2 - j_3^2 l_1^2} j_3^2 l_3^2$$

$$\leqslant \frac{2}{N'}(\beta_j+\beta_l)j_3^2 l_3^2 \leqslant \frac{2}{N'}\left(\sqrt{1+N'(j_1/j_3)^2}+\sqrt{1+N'(l_1/l_3)^2}\right)j_3^2 l_3^2$$
$$\leqslant c(N')|j|^2|l|^2.$$

To estimate $I_3(j,k,l)$ we also consider two cases:
(i) All α_i have the same sign, which immediately leads to $|I_3|\leqslant 2/3$.
(ii) $\alpha_1=\alpha_2=-\alpha_3$, for which we compute

$$\begin{aligned}|I_3| &\leqslant \frac{2}{|\beta_j+\beta_l-\beta_k|}\\ &= \frac{2|(\beta_j+\beta_l+\beta_k)(-\beta_j+\beta_l+\beta_k)(-\beta_l+\beta_j+\beta_k)|}{|(\beta_j+\beta_l+\beta_k)(\beta_j+\beta_l-\beta_k)(-\beta_j+\beta_l+\beta_k)(\beta_j-\beta_l+\beta_k)|}\\ &\leqslant \frac{|J_1|}{|J_2|},\end{aligned}$$

where

$$J_1 = 2(\beta_j+\beta_l+\beta_k)(-\beta_j+\beta_l+\beta_k)(-\beta_l+\beta_j+\beta_k)j_3^4 l_3^4 k_3^4,$$

$$\begin{aligned}J_2 &= 3j_3^4 l_3^4 k_3^4 + 2N'(j_1^2 l_3^4 k_3^4 j_3^2 + l_1^2 l_3^2 j_3^4 k_3^4 + k_1^2 k_3^2 j_3^4 l_3^4)\\ &\quad + N'^2(2j_1^2 j_3^2 l_1^2 l_3^2 k_3^4 + 2j_1^2 j_3^2 k_1^2 k_3^2 l_3^4 + 2l_1^2 l_3^2 k_1^2 k_3^2 j_3^4 - j_1^4 l_3^4 k_3^4 - l_1^4 j_3^4 k_3^4 - k_1^4 j_3^4 l_3^4).\end{aligned}$$

Setting

$$\sigma_1 = 2j_1^2 j_3^2 l_1^2 l_3^2 k_3^4 + 2j_1^2 j_3^2 k_1^2 k_3^2 l_3^4 + 2l_1^2 l_3^2 k_1^2 k_3^2 j_3^4 - j_1^4 l_3^4 k_3^4 - k_1^4 j_3^4 k_3^4 - k_1^4 j_3^4 l_3^4,$$

$$\sigma_2 = 2(j_1^2 l_3^4 k_3^4 j_3^2 + l_1^2 l_3^2 j_3^4 k_3^4 + k_1^2 k_3^2 j_3^4 l_3^4),$$

$$\sigma_3 = 3j_3^4 l_3^4 k_3^4,$$

we need to estimate $1/|N'^2\sigma_1 + N'\sigma_2 + \sigma_3|$. For this we recall from [19]:[2]

For any $\delta>0$ and for almost all $v\in\mathbb{R}$, there exists a constant K depending on v and δ such that

$$|v^2 q + vp + r| > K(v,\delta)(|q|+|p|+|r|)^{-(2+\delta)}, \quad \forall p,q,r\in\mathbb{Z}. \tag{4.19}$$

For the convenience of the reader, we provide in Appendix B.1 an elementary proof of a weaker result in which the power $2+\delta$ is replaced by $3+\delta$.

Choosing N' such that (4.19) holds (almost all real numbers satisfy this property), we estimate I_3 as:

$$|I_3| \leqslant J_1 K(N',\delta)(|\sigma_1|+|\sigma_2|+|\sigma_3|)^{2+\delta} \leqslant K(N',\delta)|j|^{12+4\delta}|l|^{12+4\delta}|k|^{12+4\delta}. \tag{4.20}$$

[2] Pointed out to us by Yann Bugeaud (personal communication).

We note that this result implies that the denominator $\alpha_1\beta_j + \alpha_2\beta_l + \alpha_3\beta_k$ in (4.17) is never zero for almost all $N' \in \mathbb{R}$.

We are now ready to estimate $|G_{\rm np}(s, \bar{V})|_{H^2}$: Taking into account (A.14), (A.16) and (A.17), we see that $\tilde{A}_{\rm np}(\bar{V}, s)$ only contains terms of type I_1 and we have

$$
\begin{aligned}
|\tilde{A}_{\rm np}(s, \bar{V})|_{H^2} &\leqslant c'\left[\sum_k (|I_1(k)|\,|k'|^2|\bar{m}_k|)^2(1+|k'|^2)^2\right]^{1/2} \\
&\quad + c''\left[\sum_k (|I_1(k)|\,|k'|^2(|\bar{u}_k|+|\bar{n}_k|))^2(1+|k'|^2)^2\right]^{1/2} \\
&\leqslant c|\bar{V}|_{H^4}.
\end{aligned}
\tag{4.21}
$$

Next, we estimate $\tilde{B}_{\rm np}(s, \bar{V})$. From (A.23) and (A.25), the most problematic terms (imposing the highest regularity on $\bar{V}$) are those which, after integration, are of type I_3. We only estimate the typical term $M_{1,2,{\rm np}}$ (see the appendix for details on $M_{1,2,{\rm n}}$), which we bound using (4.20):

$$
\begin{aligned}
&\left| -\frac{i}{8} \sum_{j_3 l_3 \neq 0}^{\alpha,k} I_3(j,l,k)(l_1' - l_3'\delta_j) X_{\alpha_2,j}(\bar{V}) X_{\alpha_3,l}(\bar{V}) e^{i(k_1'x+k_3'z)} \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \right|_{H^2} \\
&\leqslant c(N',\delta)\left[\sum_{j_3 l_3 \neq 0}^{\alpha,k} |j'|^{13+4\delta}|l'|^{13+4\delta}|k'|^{12+4\delta}(|\bar{u}_j|+|\bar{n}_j|)(|\bar{u}_l|+|\bar{n}_l|)(1+|k'|^2)^2\right]^{1/2} \\
&\leqslant c(N',\delta)|q_1^2|_{H^{14+4\delta}} \leqslant c(N',\delta)|q_1|^2_{H^{14+4\delta}} \leqslant c(N',\delta)|\bar{V}|^2_{H^{27+8\delta}},
\end{aligned}
$$

where $q_1 := \sum_j |j'|^{13+4\delta}(|u_j|+|n_j|)e^{i(j_1'x+j_3'z)}$, and we set $X_{\alpha,k}(\bar{V}) = \bar{u}_k - \alpha\bar{n}_k$; we have used $|l_1' - l_3'(j_1'/j_3')| \leqslant |j'|\,|l'|$. The sums $\sum^\alpha$ and $\sum^k$ are defined in the Appendix, after (A.5) and (A.7).

We can now write

$$
|\tilde{B}_{\rm np}(\bar{V}, s)|_{H^2} \leqslant c(N',\delta)|\bar{V}|^2_{H^{27+8\delta}}. \tag{4.22}
$$

Finally, noting that,

$$
|\tilde{S}_{\rm np}|_{H^2} \leqslant |\tilde{S}|_{H^2},
$$

we obtain the following estimate:

$$
|G_{\rm np}(s, \bar{V})|_{H^2} \leqslant c_1(N',\delta)|\bar{V}|_{H^4} + c_2(N',\delta)|\bar{V}|^2_{H^{27+8\delta}} + c_3(N',\delta)|\tilde{S}|_{H^2}, \tag{4.23}
$$

valid, as (4.19) tells us, for almost every $N' \in \mathbb{R}$.

Estimates for $|\nabla_{\bar{V}} G_{\mathrm{np}}(s,\bar{V})\cdot G_{\mathrm{r}}(\bar{V})|_{L^2}$: We consider the bilinear form

$$\tilde{B}_{\mathrm{np}}(s,\bar{V},V^{\sharp}) = \begin{pmatrix} \tilde{B}^{(1)}_{\mathrm{np}}(s,\bar{V},V^{\sharp}) \\ \tilde{B}^{(2)}_{\mathrm{np}}(s,\bar{V},V^{\sharp}) \\ \tilde{B}^{(3)}_{\mathrm{np}}(s,\bar{V},V^{\sharp}) \end{pmatrix},$$

whose Fourier components are:

For $k_3=0$, $\tilde{B}^{(1)}_{\mathrm{np},k}(s,\bar{V},V^{\sharp})=0$, $\tilde{B}^{(3)}_{\mathrm{np},k}(s,\bar{V},V^{\sharp})=0$, and

$$\begin{aligned}\tilde{B}^{(2)}_{\mathrm{np},k}(s,\bar{V},V^{\sharp}) ={}& \frac{ik_1'}{2}\sum_{j_3 l_3\neq 0}^{\alpha,k} I_1(j)X_{\alpha_1,j}(\bar{V})m_l^{\sharp}\vec{\gamma}_l\cdot\vec{\phi}_k \\ &+\frac{ik_1'}{4}\sum_{\substack{j_3l_3\neq 0\\ \alpha_1\beta_j+\alpha_2\beta_l\neq 0}}^{\alpha,k} I_2(j,l)\alpha_2 X_{\alpha_1,j}(\bar{V})X_{\alpha_2,l}(V^{\sharp})\vec{\phi}_l\cdot\vec{\phi}_k,\end{aligned}$$

For $k_3\neq 0$:

$$\tilde{B}_{\mathrm{np},k}(s,\bar{V},V^{\sharp}) = \begin{pmatrix} M^k_{1,2,\mathrm{np}}(s,\bar{V},V^{\sharp}) \\ M^k_{3,\mathrm{np}}(s,\bar{V},V^{\sharp}) \end{pmatrix},$$

where

$$\begin{aligned}M^k_{1,2,\mathrm{np}}(s,\bar{V},V^{\sharp}) ={}& \frac{i}{8}\sum_{j_3l_3\neq 0}^{\alpha,k} I_3(j,l,k)(l_1'-l_3'\delta_j)X_{\alpha_2,j}(\bar{V})X_{\alpha_3,l}(V^{\sharp})\begin{pmatrix}1\\-\alpha_1\end{pmatrix} \\ &+\frac{i}{4}\sum_{\substack{j_3l_3\neq 0\\ \alpha_1\beta_k+\alpha_2\beta_j\neq 0}}^{\alpha,k} I_2(j,k)(l_1'-l_3'\delta_j)\alpha_1 X_{\alpha_2,j}(\bar{V})m_l^{\sharp}\vec{\phi}_k\cdot\vec{\gamma}_l\begin{pmatrix}1\\-\alpha_1\end{pmatrix} \\ &+\frac{i}{8}\sum_{j_3l_3\neq 0}^{\alpha,k} I_3(j,l,k)(l_1'-l_3'\delta_j)\alpha_1\alpha_3 X_{\alpha_2,j}(\bar{V})X_{\alpha_3,l}(V^{\sharp})\vec{\phi}_k\cdot\vec{\phi}_l\begin{pmatrix}1\\-\alpha_1\end{pmatrix} \\ &+\frac{i}{4}\sum_{\substack{l_3=0\\ \alpha_1\beta_k+\alpha_2\beta_j\neq 0}}^{\alpha,k} I_2(j,k)l_1'\alpha_1 X_{\alpha_2,j}(\bar{V})n_l^{\sharp}\vec{\phi}_k\cdot\vec{\phi}_l\begin{pmatrix}1\\-\alpha_1\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}M^k_{3,\mathrm{np}}(s,\bar{V},V^{\sharp}) ={}& \frac{i}{2}\sum_{l_3=0}^{\alpha,k} I_1(j)l_1' X_{\alpha_1,j}(\bar{V})n_l^{\sharp}\vec{\phi}_l\cdot\vec{\gamma}_k \\ &+\frac{i}{2}\sum_{j_3l_3\neq 0}^{\alpha,k}(l_1'-l_3'\delta_j)I_1(j)X_{\alpha_1,j}(\bar{V})m_l^{\sharp}\vec{\gamma}_l\cdot\vec{\gamma}_k \\ &+\frac{i}{4}\sum_{\substack{j_3l_3\neq 0\\ \alpha_1\beta_j+\alpha_2\beta_l\neq 0}}^{\alpha,k} I_2(j,l)(l_1'-l_3'\delta_j)\alpha_2 X_{\alpha_1,j}(\bar{V})X_{\alpha_2,l}(V^{\sharp})\vec{\phi}_l\cdot\vec{\gamma}_k.\end{aligned}$$

Since $G_{\mathrm{np}}(s, \bar{V}) = \tilde{A}_{\mathrm{np}}(s, \bar{V}) + \tilde{B}_{\mathrm{np}}(s, \bar{V}) + \tilde{S}_{\mathrm{np}}$, we have

$$\nabla_{\bar{V}} G_{\mathrm{np}}(\bar{V}, s) \cdot G_{\mathrm{r}}(\bar{V}) = \nabla_{\bar{V}} \tilde{A}_{\mathrm{np}}(\bar{V}, s) \cdot G_{\mathrm{r}}(\bar{V}) + \nabla_{\bar{V}} \tilde{B}_{\mathrm{np}}(\bar{V}, s) \cdot G_{\mathrm{r}}(\bar{V})$$

$$= \tilde{A}_{\mathrm{np}}(G_{\mathrm{r}}(\bar{V}), s) + \tilde{B}_{\mathrm{np}}(\bar{V}, G_{\mathrm{r}}(\bar{V}), s) + \tilde{B}_{\mathrm{np}}(G_{\mathrm{r}}(\bar{V}), \bar{V}, s).$$

Using the same type of argument as before, we have the estimates:

$$|\tilde{A}_{\mathrm{np}}(G_{\mathrm{r}}(\bar{V}), s)|_{L^2} \leqslant c |G_{\mathrm{r}}(\bar{V})|_{H^2},$$

$$|\tilde{B}_{\mathrm{np}}(\bar{V}, G_{\mathrm{r}}(\bar{V}), s)|_{L^2} \leqslant K(N', \delta) |\bar{V}|_{H^{27+8\delta}} |G_{\mathrm{r}}(\bar{V})|_{H^{27+8\delta}},$$

$$|\tilde{B}_{\mathrm{np}}(G_{\mathrm{r}}(\bar{V}), \bar{V}, s)|_{L^2} \leqslant K(N', \delta) |\bar{V}|_{H^{27+8\delta}} |G_{\mathrm{r}}(\bar{V})|_{H^{27+8\delta}}.$$

We bound $G_{\mathrm{r}}(\bar{V}) = -\tilde{A}_{\mathrm{r}}(\bar{V}) - \tilde{B}_{\mathrm{r}}(\bar{V}, \bar{V}) + \tilde{S}_{\mathrm{r}}$ using

$$|\tilde{S}_{\mathrm{r}}|_{H^m} \leqslant |\tilde{S}|_{H^m},$$

$$|\tilde{B}_{\mathrm{r}}(\bar{V}, \bar{V})|_{H^m} \leqslant c |\bar{V}|^2_{H^{m+2}},$$

$$|\tilde{A}_{\mathrm{r}}(\bar{V})|_{H^m} \leqslant c |\bar{V}|_{H^{m+2}},$$

for all $m \in \mathbb{N}$. Finally, we find:

$$|\nabla_{\bar{V}} G_{\mathrm{np}}(\bar{V}, s) \cdot G_{\mathrm{r}}(\bar{V})|_{L^2} \leqslant K(N', \delta, |\bar{V}|_{H^{29+8\delta}}, |\tilde{S}|_{H^{27+8\delta}}). \tag{4.24}$$

Putting the estimates we have just derived into (4.14), we have

$$|R_\varepsilon|_{L^2} \leqslant K(N', \delta, |\bar{V}|_{H^{29+8\delta}}, |\tilde{S}|_{H^{27+8\delta}}). \tag{4.25}$$

Using Theorem 3.7, we can write this in terms of the initial conditions:

$$|R_\varepsilon|_{L^2} \leqslant K(N', \delta, |V_0|_{H^{29+8\delta}}, |\tilde{S}|_{H^{28+8\delta}}). \tag{4.26}$$

Estimates for $W(s)$: Note that $V^1(s) = e^{s\tilde{L}}[\bar{V}(s) + \varepsilon G_{\mathrm{np}}(s, \bar{V}(s))]$ has been bounded by (4.23),

$$|V^1(s)|_{H^2} \leqslant K(N', \delta, |\bar{V}|_{H^{27+8\delta}}, |\tilde{S}|_{H^2}), \quad \forall s > 0, \tag{4.27}$$

or, using Theorem 3.7 again,

$$|V^1(s)|_{H^2} \leqslant K(N', \delta, |V_0|_{H^{27+8\delta}}, |\tilde{S}|_{H^{26+8\delta}}), \quad \forall s > 0. \tag{4.28}$$

Putting this into (4.10), we have

$$\frac{d}{ds} |W|^2_{L^2} + \varepsilon c_1 |W|^2_{H^1} \leqslant \varepsilon^2 \kappa_1 + \varepsilon \kappa_2 |W|^2_{L^2}, \tag{4.29}$$

where κ_1 and κ_2 are constants depending on N', δ, $|V_0|_{H^{29+8\delta}}$ and $|\tilde{S}|_{H^{28+8\delta}}$. The desired bound on $W(s)$ follows from this using the classical Gronwall lemma:

$$|W(s)|^2_{L^2} \leqslant \varepsilon^2 \frac{\kappa_1}{\kappa_2} e^{\varepsilon \kappa_2 s}, \quad \forall s \geqslant 0. \tag{4.30}$$

Taking $\delta = 1/8$ and collecting the results in this section, we have the following:

Theorem 4.1. *For any L_1 and L_3, and for almost all Burgers numbers $N \in \mathbb{R}$, given $V_0 \in (H^{30}_{\mathrm{per}}(\mathcal{M}))^3 \cap \tilde{V}$, and $\tilde{S} \in (H^{29}_{\mathrm{per}}(\mathcal{M}))^3 \cap \tilde{V}$, the difference between the solution V of the original system* (3.20) *and the approximate solution V^1 given by* (4.1) *satisfies*

$$|V^1(t) - V(t)|^2_{L^2} \leqslant \varepsilon^2 \kappa' e^{\kappa'' t}, \quad \forall t \geqslant 0, \tag{4.31}$$

where κ' and κ'' are constants depending on N, L_1, L_3, V_0 and $\tilde{S}$.

Remark 4.2. We can redo the above estimates, using the bounds on I_3 given in Appendix 3 instead, to arrive at the following:

Theorem 4.3. *Let $\mu > 0$, L_1 and L_3 be fixed. Take $V_0 \in (H^{11}_{\mathrm{per}}(\mathcal{M}))^3 \cap \tilde{V}$ and $\tilde{S} \in (H^{10}_{\mathrm{per}}(\mathcal{M}))^3 \cap \tilde{V}$. Then there exists a set $\Theta^{\mu}_3(L_1, L_3)$ having a Lebesque measure* $\mathrm{mes}\, \Theta^{\mu}_3(L_1, L_3) \leqslant \mu$ *such that, for all Burgers numbers $N \notin \Theta^{\mu}_3(L_1, L_3)$, the difference between the solution V of the original system* (3.20) *and the approximate solution V^1 given by* (4.1) *satisfies,*

$$|V^1(t) - V(t)|^2_{L^2} \leqslant \varepsilon^2 \kappa' e^{\kappa'' t}, \quad \forall t \geqslant 0, \tag{4.32}$$

where κ' and κ'' are constants depending on N, L_1, L_3, μ, V_0 and $\tilde{S}$.

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Appendix A. Derivation of the renormalized equation

Following the algorithm briefly explained in Section 2.2, we start by solving the linear system obtained from (3.20) by dropping all order-ε terms (zeroth order approximation).

For $k_3 = 0$ we find

$$u_{(k_1,0)} = 0, \quad m_{(k_1,0)} = 0, \tag{A.1}$$

and $n'_{(k_1,0)} = 0$ which implies that $n_{(k_1,0)}(s) = n_{(k_1,0)}(0)$.

For $k_3 \neq 0$ we find, as we already saw, the system (3.15):

$$u_k' - \beta_k n_k = 0,$$

$$n_k' + \beta_k u_k = 0,$$

$$m_k' = 0. \tag{A.2}$$

Setting $V_k = (u_k, n_k, m_k)$, this system of ordinary differential equations can be written as

$$V_k' + \tilde{L}_k V_k = 0, \quad \text{where } \tilde{L}_k = \begin{pmatrix} 0 & -\beta_k & 0 \\ \beta_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.3}$$

Its solution is $V_k(s) = e^{-s\tilde{L}_k} V_k(0)$; with

$$e^{-s\tilde{L}_k} = \begin{pmatrix} \frac{1}{2}\sum^{\alpha} e^{s\alpha\beta_k} R_\alpha & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where } R_\alpha = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$$

and $\alpha = \pm i$, we have explicitly,

$$V_k(s) = \begin{pmatrix} \frac{1}{2}\sum^{\alpha} e^{s\alpha\beta_k}(u_k(0) - \alpha n_k(0)) \\ \frac{1}{2}\sum^{\alpha} \alpha e^{s\alpha\beta_k}(u_k(0) - \alpha n_k(0)) \\ m_k(0) \end{pmatrix}. \tag{A.4}$$

Denoting $X_{\alpha,k}(V) \coloneqq u_k - \alpha n_k$, (A.4) reads

$$u_k(s) = \frac{1}{2}\sum^{\alpha} e^{s\alpha\beta_k} X_{\alpha,k}(V_0),$$

$$n_k(s) = \frac{1}{2}\sum^{\alpha} \alpha e^{s\alpha\beta_k} X_{\alpha,k}(V_0),$$

$$m_k(s) = m_k(0). \tag{A.5}$$

Here and throughout this paper, $\sum^{\alpha}$ always range over $\alpha = \pm i$; similarly for α_j.

For the $\mathcal{O}(\varepsilon)$ approximation, we need to separate the r.h.s. $G(s, V)$ into its resonant and non-resonant parts,

$$G(s, V) = e^{s\tilde{L}}\mathcal{G}(e^{-s\tilde{L}}V) = G_{\mathrm{r}}(V) + G_{\mathrm{n}}(s, V), \tag{A.6}$$

and then compute the primitive G_{np} of G_{n}. As usual, we analyse separately the cases $k_3 = 0$ and $k_3 \neq 0$.

The case $k_3 = 0$: In this case, the equations of motion (3.3) read

$$u_k = 0,$$

$$n'_k = -\varepsilon\nu_{\mathbf{v}}|k'|^2 n_k - \varepsilon i \sum\nolimits^k k'_1 u_j (m_l \vec{\gamma}_l + n_l \vec{\phi}_l) \cdot \vec{\phi}_k + \varepsilon S_{n,k},$$

$$m_k = 0, \tag{A.7}$$

where the superscript k in $\sum^k$ means that it is taken over $j + l = k$ with k fixed. Since here the fast linear operator vanishes, $\tilde{L}_{(k_1,0)} = 0$, we have

$$e^{s\tilde{L}_k}\tilde{S}_k = \tilde{S}_k,$$

$$\left\{e^{s\tilde{L}_k}\tilde{A}e^{-s\tilde{L}_k}\right\}_k = \tilde{A}_k = \tilde{A}_{\mathrm{r},k},$$

$$\left\{e^{s\tilde{L}_k}\tilde{B}(e^{-s\tilde{L}}V, e^{-s\tilde{L}}V)\right\}_k = \tilde{B}_k\left(e^{-s\tilde{L}}V, e^{-s\tilde{L}}V\right). \tag{A.8}$$

The u and m components of $\tilde{B}_k$ vanish, so we only need to compute

$$\begin{aligned}\tilde{B}_k^{(n)} &= i\sum\nolimits^k k'_1 u_j (m_l\vec{\gamma}_l + n_l\vec{\phi}_l)\cdot\vec{\phi}_k \\ &= ik'_1\sum\nolimits^k\left[m_l(0)\vec{\gamma}_l + \frac{1}{2}\sum\nolimits^{\alpha_2} X_{\alpha_2,l}(V_0)\alpha_2 e^{s\alpha_2\beta_l}\vec{\phi}_l\right]\cdot\vec{\phi}_k\left[\frac{1}{2}\sum\nolimits^{\alpha_1} X_{\alpha_1,j}(V_0)e^{s\alpha_1\beta_j}\right].\end{aligned} \tag{A.9}$$

The resonant part (i.e. the s-independent part) of this expression obtains when $\alpha_1\beta_j + \alpha_2\beta_l = 0$, which only happens when $\alpha_1 = -\alpha_2$ and $\beta_j = \beta_l$; this gives us

$$\begin{aligned}\tilde{B}_{\mathrm{r},k}^{(n)}(V,V) &= \frac{ik'_1}{4}\sum_{\substack{j_3 l_3\neq 0\\ \beta_j=\beta_l}}^{\alpha,k} \alpha X_{-\alpha,j}(V)X_{\alpha,l}(V)\vec{\phi}_l\cdot\vec{\phi}_k \\ &= \frac{ik'_1}{2}\sum_{\substack{j_3 l_3\neq 0\\ \beta_j=\beta_l}}^{k}(n_l u_j - n_j u_l)\vec{\phi}_l\cdot\vec{\phi}_k.\end{aligned} \tag{A.10}$$

The non-resonant part of $\tilde{B}_k^{(n)}$ is

$$\begin{aligned}\tilde{B}_{\mathrm{n},k}^{(n)}(s,V,V) &= \frac{ik'_1}{2}\sum_{j_3 l_3\neq 0}^{\alpha,k} e^{s\alpha_1\beta_j} m_l X_{\alpha_1,j}(V)\vec{\gamma}_l\cdot\vec{\phi}_k \\ &\quad + \frac{ik'_1}{4}\sum_{\substack{j_3 l_3\neq 0\\ \alpha_1\beta_j+\alpha_2\beta_l\neq 0}}^{\alpha,k} e^{s(\alpha_1\beta_j+\alpha_2\beta_l)}\alpha_2 X_{\alpha_1,j}(V)X_{\alpha_2,l}(V)\vec{\phi}_l\cdot\vec{\phi}_k.\end{aligned} \tag{A.11}$$

The case $k_3 \neq 0$: We begin with the linear operator A_k [cf. (3.20)],

$$\tilde{A}_k V_k = \begin{pmatrix} |k'|^2 v_{\boldsymbol{v}} u_k \\ |k'|^2 v_{\boldsymbol{v}} n_k + (v_\rho - v_{\boldsymbol{v}})|k'|^2 (N\delta_k/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k \\ |k'|^2 v_{\boldsymbol{v}} m_k + |k'|^2 (v_\rho - v_{\boldsymbol{v}})(1/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k \end{pmatrix}, \tag{A.12}$$

which we split into its diagonal and off-diagonal parts,

$$\tilde{A}_{1,k} V_k = v_{\boldsymbol{v}} |k'|^2 V_k,$$

$$\tilde{A}_{2,k} V_k = \begin{pmatrix} 0 \\ |k'|^2 (v_\rho - v_{\boldsymbol{v}})(N\delta_k/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k \\ |k'|^2 (v_\rho - v_{\boldsymbol{v}})(1/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k \end{pmatrix}. \tag{A.13}$$

Since $\tilde{A}_{1,k}$ is diagonal, it is completely resonant. To find the resonant part of $\tilde{A}_{2,k}$, we compute, using $V_k = e^{s\tilde{L}_k} V_0$,

$$\begin{aligned} e^{s\tilde{L}_k} \tilde{A}_{2,k} V_k &= \begin{pmatrix} \frac{1}{2}\sum^\alpha e^{s\alpha\beta_k} R_{-\alpha} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 \\ |k'|^2 (v_\rho - v_{\boldsymbol{v}})(N\delta_k/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k \\ |k'|^2 (v_\rho - v_{\boldsymbol{v}})(1/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{v_\rho - v_{\boldsymbol{v}}}{2} \sum^\alpha \alpha |k'|^2 (N\delta_k/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k e^{s\alpha\beta_k} \\ \dfrac{v_\rho - v_{\boldsymbol{v}}}{2} \sum^\alpha |k'|^2 (N\delta_k/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k e^{s\alpha\beta_k} \\ |k'|^2 (v_\rho - v_{\boldsymbol{v}})(1/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k \end{pmatrix}. \end{aligned} \tag{A.14}$$

Continuing the computations for $e^{s\tilde{L}_k} \tilde{A}_{2,k} V_k$, we obtain

$$\begin{pmatrix} \dfrac{v_\rho - v_{\boldsymbol{v}}}{2} \sum^{\alpha_1} \alpha_1 |k'|^2 \dfrac{N\delta_k}{\beta_k} \left[m_k(0) \dfrac{1}{\beta_k} + \dfrac{N\delta_k}{2\beta_k} \sum^{\alpha_2} \alpha_2 X_{\alpha_2,k}(V_0) e^{s\alpha_2\beta_k} \right] e^{s\alpha_1\beta_k} \\ \dfrac{v_\rho - v_{\boldsymbol{v}}}{2} \sum^{\alpha_1} |k'|^2 \dfrac{N\delta_k}{\beta_k} \left[m_k(0) \dfrac{1}{\beta_k} + \dfrac{N\delta_k}{2\beta_k} \sum^{\alpha_2} \alpha_2 X_{\alpha_2,k}(V_0) e^{s\alpha_2\beta_k} \right] e^{s\alpha_1\beta_k} \\ (v_\rho - v_{\boldsymbol{v}}) |k'|^2 \dfrac{1}{\beta_k} \left[m_k(0) \dfrac{1}{\beta_k} + \dfrac{N\delta_k}{2\beta_k} \sum^\alpha \alpha X_{\alpha,k}(V_0) e^{s\alpha\beta_k} \right] \end{pmatrix}, \tag{A.15}$$

where (A.5) has been used for the last equation. Using the fact that $\sum^\alpha X_{\alpha,k}(V_0) = 2u_k(0)$ and $\sum^\alpha \alpha X_{\alpha,k}(V_0) = 2n_k(0)$, we obtain from the last expression the resonant part of $\tilde{A}_k$:

$$\{\tilde{A}_{2,\mathrm{r}} V\}_k = \begin{pmatrix} \frac{\nu_\rho - \nu_{\mathbf{v}}}{2} |k'|^2 (N\delta_k/\beta_k)^2 u_k \\ \frac{\nu_\rho - \nu_{\mathbf{v}}}{2} |k'|^2 (N\delta_k/\beta_k)^2 n_k \\ (\nu_\rho - \nu_{\mathbf{v}}) |k'|^2 (1/\beta_k)^2 m_k \end{pmatrix}. \tag{A.16}$$

The non-resonant part of $\tilde{A}_2$ is then

$$\tilde{A}_{2,\mathrm{n}} = \tilde{A}_2 - \tilde{A}_{2,\mathrm{r}}. \tag{A.17}$$

Next, we treat the bilinear from $\tilde{B}$:

$$\begin{aligned} e^{s\tilde{L}_k} \tilde{B}_k(e^{-s\tilde{L}} V_0, e^{-s\tilde{L}} V_0) &= \begin{pmatrix} \frac{1}{2}\sum^\alpha e^{s\alpha\beta_k} R_{-\alpha} & 0 \\ 0 & I \end{pmatrix} \cdot \tilde{B}_k(e^{-s\tilde{L}} V_0, e^{-s\tilde{L}} V_0) \\ &= \begin{pmatrix} M_{1,2}^k \\ M_3^k \end{pmatrix}, \end{aligned} \tag{A.18}$$

where we denoted by $M_{1,2}^k$ the u and n components of the resulting column and by M_3^k the m component. We have

$$\begin{aligned} M_{1,2}^k = &\frac{i}{2} \sum^\alpha e^{s\alpha\beta_k} \sum^k (l_1' - l_3'\delta_j) u_j u_l \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} \\ &+ \frac{i}{2} \sum^\alpha \alpha e^{s\alpha\beta_k} \sum^k (l_1' - l_3'\delta_j) u_j (m_l \vec{\gamma}_l + n_l \vec{\phi}_l) \cdot \vec{\phi}_k \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}, \end{aligned} \tag{A.19}$$

or, using (A.5),

$$\begin{aligned} M_{1,2}^k = &\frac{i}{8} \sum_{j_3 l_3 \neq 0}^{\alpha,k} e^{s(\alpha_1\beta_k + \alpha_2\beta_j + \alpha_3\beta_l)} (l_1' - l_3'\delta_j) X_{\alpha_2,j}(V_0) X_{\alpha_3,l}(V_0) \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ &+ \frac{i}{4} \sum_{j_3 l_3 \neq 0}^{\alpha,k} e^{s(\alpha_1\beta_k + \alpha_2\beta_j)} (l_1' - l_3'\delta_j) \alpha_1 X_{\alpha_2,j}(V_0) m_l(0) \vec{\gamma}_l \cdot \vec{\phi}_k \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ &+ \frac{i}{8} \sum_{j_3 l_3 \neq 0}^{\alpha,k} e^{s(\alpha_1\beta_k + \alpha_2\beta_j + \alpha_3\beta_l)} (l_1' - l_3'\delta_j) \alpha_1 \alpha_3 X_{\alpha_2,j}(V_0) X_{\alpha_3,l}(V_0) \vec{\phi}_k \cdot \vec{\phi}_l \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ &+ \frac{i}{4} \sum_{l_3 = 0}^{\alpha,k} e^{s(\alpha_1\beta_k + \alpha_2\beta_j)} l_1' \alpha_1 X_{\alpha_2,j}(V_0) n_{(l_1,0)}(0) \vec{\phi}_l \cdot \vec{\phi}_k \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix}. \end{aligned}$$

The resonant part of this expression obtains when $\alpha_1\beta_k + \alpha_2\beta_j = 0$ (implying that $\alpha_1 = -\alpha_2$ and $\beta_k = \beta_j$), or when $\alpha_1\beta_k + \alpha_2\beta_j + \alpha_3\beta_l = 0$. As shown in Section 4, the latter scenario does not happen if the Burgers number N lies outside a set of measure zero. Assuming the generic situation, the resonant part of $M^k_{1,2}$ is

$$M^k_{1,2,\mathrm{r}} = \frac{i}{4} \sum_{\substack{j_3 l_3 \neq 0 \\ \beta_k = \beta_j}}^{\alpha,k} (l'_1 - l'_3 \delta_j) \alpha X_{-\alpha,j}(V) m_l \vec{\phi}_k \cdot \vec{\gamma}_l \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} + \frac{i}{4} \sum_{\substack{l_3 = 0 \\ \beta_k = \beta_j}}^{\alpha,k} l'_1 \alpha X_{-\alpha,j}(V) n_l \vec{\phi}_k \cdot \vec{\phi}_l \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}. \tag{A.20}$$

After some elementary computations we obtain

$$M^k_{1,\mathrm{r}} = -\frac{i}{2} \sum_{\substack{j_3 l_3 \neq 0 \\ \beta_k = \beta_j}}^{k} (l'_1 - l'_3 \delta_j) n_j m_l \vec{\phi}_k \cdot \vec{\gamma}_l - \frac{i}{2} \sum_{\substack{l_3 = 0 \\ \beta_k = \beta_j}}^{k} l'_1 n_j n_l \vec{\phi}_k \cdot \vec{\phi}_l, \tag{A.21}$$

$$M^k_{2,\mathrm{r}} = \frac{i}{2} \sum_{\substack{j_3 l_3 \neq 0 \\ \beta_k = \beta_j}}^{k} (l'_1 - l'_3 \delta_j) u_j m_l \vec{\phi}_k \cdot \vec{\gamma}_l + \frac{i}{2} \sum_{\substack{l_3 = 0 \\ \beta_k = \beta_j}}^{k} l'_1 u_j n_l \vec{\phi}_k \cdot \vec{\phi}_l. \tag{A.22}$$

Similarly, the non-resonant part of $M^k_{1,2}$ is

$$\begin{aligned} M^k_{1,2,\mathrm{n}} = & \frac{i}{8} \sum_{j_3 l_3 \neq 0}^{\alpha,k} e^{s(\alpha_1 \beta_k + \alpha_2 \beta_j + \alpha_3 \beta_l)} (l'_1 - l'_3 \delta_j) X_{\alpha_2,j}(V) X_{\alpha_3,l}(V) \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ & + \frac{i}{4} \sum_{\substack{j_3 l_3 \neq 0 \\ \alpha_1 \beta_k + \alpha_2 \beta_j \neq 0}}^{\alpha,k} e^{s(\alpha_1 \beta_k + \alpha_2 \beta_j)} (l'_1 - l'_3 \delta_j) \alpha_1 X_{\alpha_2,j}(V) m_l \vec{\phi}_k \cdot \vec{\gamma}_l \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ & + \frac{i}{8} \sum_{j_3 l_3 \neq 0}^{\alpha,k} e^{s(\alpha_1 \beta_k + \alpha_2 \beta_j + \alpha_3 \beta_l)} (l'_1 - l'_3 \delta_j) \alpha_1 \alpha_3 X_{\alpha_2,j}(V) X_{\alpha_3,l}(V) \vec{\phi}_k \cdot \vec{\phi}_l \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ & + \frac{i}{4} \sum_{\substack{l_3 = 0 \\ \alpha_1 \beta_k + \alpha_2 \beta_j \neq 0}}^{\alpha,k} e^{s(\alpha_1 \beta_k + \alpha_2 \beta_j)} l'_1 \alpha_1 X_{\alpha_2,j}(V) n_l \vec{\phi}_k \cdot \vec{\phi}_l \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix}. \end{aligned} \tag{A.23}$$

We turn now to the m component of M,

$$\begin{aligned} M^k_3 = & \, i \sum_{j_3 \neq 0}^{k} (l'_1 - l'_3 \delta_j) u_j (m_l \vec{\gamma}_l + n_l \vec{\phi}_l) \cdot \vec{\gamma}_k \\ = & \frac{i}{2} \sum_{l_3 = 0}^{\alpha,k} e^{\alpha_1 s \beta_j} l'_1 X_{\alpha_1,j}(V_0) n_l(0) \vec{\phi}_l \cdot \vec{\gamma}_k \\ & + \frac{i}{2} \sum_{j_3 l_3 \neq 0}^{k} e^{\alpha_1 s \beta_j} (l'_1 - l'_3 \delta_j) X_{\alpha_1,j}(V_0) m_l(0) \vec{\gamma}_l \cdot \vec{\gamma}_k \\ & + \frac{i}{4} \sum_{j_3 l_3 \neq 0}^{\alpha,k} e^{s(\alpha_1 \beta_j + \alpha_2 \beta_l)} (l'_1 - l'_3 \delta_j) \alpha_2 X_{\alpha_1,j}(V_0) X_{\alpha_2,l}(V_0) \vec{\phi}_l \cdot \vec{\gamma}_k, \end{aligned}$$

where we have use (A.5) for the last equality. Its resonant part is

$$\begin{aligned} M^k_{3,\mathrm{r}} &= \frac{i}{4} \sum_{\substack{j_3 l_3 \neq 0 \\ \beta_j = \beta_l}}^{\alpha,k} (l_1' - l_3' \delta_j) \alpha X_{-\alpha,j}(V) X_{\alpha,l}(V) \vec{\phi}_l \cdot \vec{\gamma}_k \\ &= \frac{i}{2} \sum_{\substack{j_3 l_3 \neq 0 \\ \beta_j = \beta_l}}^{k} (l_1' - l_3' \delta_j)(u_j n_l - n_j u_l) \vec{\phi}_l \cdot \vec{\gamma}_k, \end{aligned} \tag{A.24}$$

while its non-resonant part is

$$\begin{aligned} M^k_{3,\mathrm{n}} = &\frac{i}{2} \sum_{l_3 = 0}^{\alpha,k} e^{\alpha_1 s \beta_j} l_1' X_{\alpha_1,j}(V) n_{(l_1,0)} \vec{\phi}_l \cdot \vec{\gamma}_k \\ &+ \frac{i}{2} \sum_{j_3 l_3 \neq 0}^{\alpha,k} e^{\alpha_1 s \beta_j} (l_1' - l_3' \delta_j) X_{\alpha_1,j}(V) m_l \vec{\gamma}_l \cdot \vec{\gamma}_k \\ &+ \frac{i}{4} \sum_{\substack{j_3 l_3 \neq 0 \\ \alpha_1 \beta_j + \alpha_2 \beta_l \neq 0}}^{\alpha,k} e^{s(\alpha_1 \beta_j + \alpha_2 \beta_l)} \alpha_2 (l_1' - l_3' \delta_j) X_{\alpha_1,j}(V) X_{\alpha_2,l}(V) \vec{\phi}_l \cdot \vec{\gamma}_k. \end{aligned} \tag{A.25}$$

Finally, we compute

$$\begin{aligned} \{e^{\tilde{L}s} \tilde{S}_k\}_k &= \begin{pmatrix} \frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} R_{-\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_{u,k} \\ S_{n,k} \\ S_{m,k} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} (S_{u,k} + \alpha S_{n,k}) \\ -\frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} \alpha (S_{u,k} + \alpha S_{n,k}) \\ S_{m,k} \end{pmatrix}, \end{aligned}$$

whence we find

$$\tilde{S}_{\mathrm{r},k} = \begin{pmatrix} 0 \\ 0 \\ S_{m,k} \end{pmatrix} \quad \text{and} \quad \tilde{S}_{\mathrm{n},k} = \begin{pmatrix} \frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} (S_{u,k} + \alpha S_{n,k}) \\ -\frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} \alpha (S_{u,k} + \alpha S_{n,k}) \\ 0 \end{pmatrix}. \tag{A.26}$$

The renormalized system: We have now computed all the terms in the renormalized system,

$$\frac{dV}{dt} + \tilde{A}_{\mathrm{r}} V + \tilde{B}_{\mathrm{r}}(V, V) = \tilde{S}_{\mathrm{r}}, \tag{A.27}$$

written here in the slow time t. Explicitly, we have in Fourier modes for $k=(k_1,0)$:

$$\frac{du_k}{dt}=0,$$

$$\frac{dm_k}{dt}=0,$$

$$\frac{dn_k}{dt}=-\nu_{\mathbf{v}}(k_1')^2 n_{(k_1,0)}-\frac{i}{2}\sum^{k}_{\substack{j_3 l_3\neq 0\\ \beta_j=\beta_l}} k_1'(n_l u_j-n_j u_l)\vec{\phi}_l\cdot\vec{\phi}_k+S_{n,k}. \tag{A.28}$$

For $k_3\neq 0$, we have

$$\frac{du_k}{dt}=-\nu_{\mathbf{v}}|k'|^2u_k-\frac{\nu_\rho-\nu_{\mathbf{v}}}{2}|k'|^2\frac{N^2\delta_k^2}{\beta_k^2}u_k+\frac{i}{2}\sum^{k}_{\substack{j_3 l_3\neq 0\\ \beta_k=\beta_j}}(l_1'-l_3'\delta_j)n_j m_l\vec{\gamma}_l\cdot\vec{\phi}_k$$
$$+\frac{i}{2}\sum^{k}_{\substack{l_3=0\\ \beta_k=\beta_j}} l_1' n_j n_l\vec{\phi}_k\cdot\vec{\phi}_l,$$

$$\frac{dn_k}{dt}=-\nu_{\mathbf{v}}|k'|^2n_k-\frac{\nu_\rho-\nu_{\mathbf{v}}}{2}|k'|^2\frac{N^2\delta_k^2}{\beta_k^2}n_k-\frac{i}{2}\sum^{k}_{\substack{j_3 l_3\neq 0\\ \beta_k=\beta_j}}(l_1'-l_3'\delta_j)u_j m_l\vec{\phi}_k\cdot\vec{\gamma}_l$$
$$-\frac{i}{2}\sum^{k}_{\substack{l_3=0\\ \beta_k=\beta_j}} l_1' u_j n_l\vec{\phi}_k\cdot\vec{\phi}_l,$$

$$\frac{dm_k}{dt}=-\nu_{\mathbf{v}}|k'|^2m_k-(\nu_\rho-\nu_{\mathbf{v}})|k'|^2\frac{1}{\beta_k^2}m_k$$
$$-\frac{i}{2}\sum^{k}_{\substack{j_3 l_3\neq 0\\ \beta_j=\beta_l}}(l_1'-l_3'\delta_j)(u_j n_l-n_j u_l)\vec{\phi}_l\cdot\vec{\gamma}_k+S_{m,k}.$$

Properties of the renormalized system: As mentioned in the Introduction, the renormalized linear operator $\tilde{A}_{\mathrm{r}}$ and bilinear operator $\tilde{B}_{\mathrm{r}}$ in (A.27) enjoy some properties of their original counterparts, as we now show:

$$\tilde{a}_{\mathrm{r}}(V,V)=\langle\tilde{A}_{\mathrm{r}}V,V\rangle_{\tilde{\mathbf{V}}',\tilde{\mathbf{V}}}$$
$$=\nu_{\mathbf{v}}\sum_k|k'|^2|n_k|^2+\nu_{\mathbf{v}}\sum_k|k'|^2|u_k|^2$$
$$+\frac{\nu_\rho-\nu_{\mathbf{v}}}{2}\sum_k|k'|^2\frac{N^2\delta_k^2}{\beta_k^2}|u_k|^2+\frac{\nu_\rho-\nu_{\mathbf{v}}}{2}\sum_k|k'|^2\frac{N^2\delta_k^2}{\beta_k^2}|n_k|^2$$
$$+\nu_{\mathbf{v}}\sum_k|k'|^2|m_k|^2+(\nu_\rho-\nu_{\mathbf{v}})\sum_k|k'|^2\frac{1}{\beta_k^2}|m_k|^2.$$

After some elementary computations we have

$$\tilde{a}_{\mathrm{r}}(V,V) \geqslant \min(\nu_{\boldsymbol{v}},\nu_\rho)(||u||^2+||n||^2+||m||^2),$$

thus proving the coercivity of a_{r} in $\tilde{\boldsymbol{V}}$.

We turn now to the trilinear form $\tilde{b_{\mathrm{r}}}(V,V^\flat,V^\sharp) = \langle \tilde{B}_{\mathrm{r}}(V,V^\flat), V^\sharp \rangle_{\tilde{\boldsymbol{V}}',\tilde{\boldsymbol{V}}}$,

$$\begin{aligned}
\tilde{b_{\mathrm{r}}}(V,V^\flat,V^\sharp) = &-\frac{i}{2}\sum^{\mathrm{c}}_{\substack{j_3\neq 0,k_3=0\\ \beta_j=\beta_l}} k_1'(n_l^\flat u_j - u_l^\flat n_j)n_k^\sharp \vec{\phi}_l\cdot\vec{\phi}_k\\
&-\frac{i}{2}\sum^{\mathrm{c}}_{\substack{j_3l_3k_3\neq 0\\ \beta_j=\beta_k}}(l_1'-l_3'\delta_j)u_k^\sharp n_j m_l^\flat \vec{\phi}_k\cdot\vec{\gamma}_l\\
&-\frac{i}{2}\sum^{\mathrm{c}}_{\substack{j_3\neq 0,l_3=0\\ \beta_j=\beta_k}} l_1' n_j n_l^\flat u_k^\sharp \vec{\phi}_l\cdot\vec{\phi}_k\\
&+\frac{i}{2}\sum^{\mathrm{c}}_{\substack{j_3l_3k_3\neq 0\\ \beta_j=\beta_k}}(l_1'-l_3'\delta_j)u_j m_l^\flat n_k^\sharp \vec{\phi}_k\cdot\vec{\gamma}_l\\
&+\frac{i}{2}\sum^{\mathrm{c}}_{\substack{j_3\neq 0,l_3=0\\ \beta_j=\beta_k}} l_1' u_j n_k^\sharp n_l^\flat \vec{\phi}_l\cdot\vec{\phi}_k\\
&+\frac{i}{2}\sum^{\mathrm{c}}_{\substack{j_3l_3k_3\neq 0\\ \beta_j=\beta_l}}(l_1'-l_3'\delta_j)(u_j n_l^\flat - u_l^\flat n_j)m_k^\sharp \vec{\gamma}_k\cdot\vec{\phi}_l. \qquad \text{(A.29)}
\end{aligned}$$

Interchanging k with l and using the elementary relation

$$l_1' + k_1' - (k_3'+l_3')(j_1'/j_3') = -j_1' + j_3'(j_1'/j_3') = 0 \quad (\text{since } j+l+k=0),$$

we now compute

$$\begin{aligned}
\tilde{b_{\mathrm{r}}}(V,V^\flat,V^\flat) = &-\frac{i}{2}\sum^{\mathrm{c}}_{\substack{j_3\neq 0,l_3=0\\ \beta_j=\beta_k}} l_1'(n_k^\flat u_j - u_k^\flat n_j)n_l^\flat \vec{\phi}_l\cdot\vec{\phi}_k\\
&-\frac{i}{2}\sum^{\mathrm{c}}_{\substack{j_3l_3k_3\neq 0\\ \beta_j=\beta_k}}(l_1'-l_3'\delta_j)u_k^\flat n_j m_l^\flat \vec{\gamma}_l\cdot\vec{\phi}_k\\
&-\frac{i}{2}\sum^{\mathrm{c}}_{\substack{j_3\neq 0,l_3=0\\ \beta_j=\beta_k}} l_1' n_j u_k^\flat n_l^\flat \vec{\phi}_l\cdot\vec{\phi}_k
\end{aligned}$$

$$
\begin{aligned}
&+\frac{i}{2}\sum^{c}_{\substack{j_3l_3k_3\neq 0\\ \beta_j=\beta_k}}(l_1'-l_3'\delta_j)u_j m_l^\flat n_k^\flat \vec{\phi}_k\cdot\vec{\gamma}_l\\
&+\frac{i}{2}\sum^{c}_{\substack{j_3\neq 0,l_3=0\\ \beta_j=\beta_k}} l_1' u_j n_k^\flat n_l^\flat \vec{\phi}_l\cdot\vec{\phi}_k\\
&+\frac{i}{2}\sum^{c}_{\substack{j_3l_3k_3\neq 0\\ \beta_j=\beta_l}}(l_1'-l_3'\delta_j)(n_l^\flat u_j-n_j u_l^\flat)m_k^\flat\vec{\phi}_l\cdot\vec{\gamma}_k\\
=&-\frac{i}{2}\sum^{c}_{\substack{j_3l_3k_3\neq 0\\ \beta_j=\beta_k}}\left(l_1'+k_1'-l_3'\frac{j_1'}{j_3'}-k_3'\frac{j_1'}{j_3'}\right)\\
&\times(u_l^\flat n_j-u_j n_l^\flat)m_k^\flat\vec{\phi}_l\cdot\vec{\gamma}_k=0. \qquad\text{(A.30)}
\end{aligned}
$$

We have thus proved that the orthogonality of $b(V,V^\sharp,V^\flat)$ is preserved in the renormalized system.

Lemma A.1. *There exists a constant $c_2>0$ such that for all $V=(u,n,m)$, $V^\flat=(u^\flat,n^\flat,m^\flat)$, $V^\sharp=(v^\sharp,n^\sharp,m^\sharp)$, with $V^\sharp\in\tilde{\boldsymbol{V}}$ and V, $V^\flat\in\tilde{\boldsymbol{V}}_2$, we have*

$$
\begin{aligned}
|\tilde{b}_{\mathrm{r}}(V,V^\flat,V^\sharp)|\leqslant\ & c_2\|V\|^{1/2}|\Delta V|_{L^2}^{1/2}|V^\flat|_{L^2}^{1/2}\|V^\flat\|^{1/2}|V^\sharp|_{L^2}\\
&+c_2|V|_{L^2}^{1/2}\|V\|^{1/2}\|V^\flat\|^{1/2}|\Delta V^\flat|_{L^2}^{1/2}|V^\sharp|_{L^2}, \qquad\text{(A.31)}
\end{aligned}
$$

$$
|\tilde{b}_{\mathrm{r}}(V,V^\flat,V^\sharp)|\leqslant c_2\|V\|\|V^\flat\|\|V^\sharp\|. \qquad\text{(A.32)}
$$

Proof. We need to estimate each term of $\tilde{b}_{\mathrm{r}}(V,V^\flat,V^\sharp)$. In order to facilitate the computations we write:

$$
u_1=\sum_{j=(j_1,j_3)\in\mathbb{Z}^2}|u_j|e^{i(xj_1'+zj_3')},\quad u_2=\sum_{j=(j_1,j_3)\in\mathbb{Z}^2}|j'|\,|u_j|e^{i(xj_1'+zj_3')},
$$

and similarly for n and m. We estimate $|l_1'-l_3'(j_1'/j_3')|$ taking into account the summation conditions $\beta_j=\beta_k\Leftrightarrow|j_1'/j_3'|=|k_1'/k_3'|$: When $j_1'/j_3'=k_1'/k_3'$, we have from $j+l+k=0$ that $|l_1'-l_3'(j_1'/j_3')|=0$. When $j_1'/j_3'=-k_1'/k_3'$, we write $j_1'=-sk_1'$, $j_3'=sk_3'$, and using $j+l+k=0$ again we have $|l_1'-l_3'(j_1'/j_3')|=2|k_1'|\leqslant 2(|j'|+|l'|)$. We also have $|k_1'-k_3'(j_1'/j_3')|=2|k_1'|\leqslant 2(|j'|+|l'|)$.

We can now proceed and estimate $|\tilde{b_{\rm r}}(V, V^\flat, V^\sharp)|$:

$$\begin{aligned}
&\left|\frac{i}{2}\sum^{\rm c}_{\substack{j_3\neq 0,k_3=0\\ \beta_j=\beta_l}} k_1'(n_l^\flat u_j - u_l^\flat n_j)n^\sharp_{(k_1,0)}\vec{\phi}_l\cdot\vec{\phi}_k|\right|\\
&\leqslant c\sum^{\rm c}_{\substack{j_3\neq 0,k_3=0\\ \beta_j=\beta_l}} (|u_j|\,|n_l^\flat|\,|n_k^\sharp| + |n_j|\,|u_l^\flat|\,|n_k^\sharp|)(|j'|+|l'|)\\
&\leqslant c\int_{\mathscr{M}} u_2 n_1^\flat n_1^\sharp\, d\mathscr{M} + c\int_{\mathscr{M}} u_1^\flat n_2 n_1^\sharp\, d\mathscr{M} + c\int_{\mathscr{M}} u_1 n_1^\sharp n_2^\flat\, d\mathscr{M} + c\int_{\mathscr{M}} u_2^\flat n_1 n_1^\sharp\, d\mathscr{M}\\
&\leqslant c|u_2|_{L^4(\mathscr{M})}|n_1^\sharp|_{L^4(\mathscr{M})}|n_1^\flat|_{L^2(\mathscr{M})} + c|u_1^\flat|_{L^4(\mathscr{M})}|n_2|_{L^4(\mathscr{M})}|n_1^\sharp|_{L^2(\mathscr{M})}\\
&\quad + c|n_2^\flat|_{L^4(\mathscr{M})}|u_1|_{L^4(\mathscr{M})}|n_1^\sharp|_{L^2(\mathscr{M})} + c|u_2^\flat|_{L^4(\mathscr{M})}|n_1|_{L^4(\mathscr{M})}|n_1^\sharp|_{L^2(\mathscr{M})}.
\end{aligned}$$

Using the fact that $|u|_{L^4(\mathscr{M})}\leqslant c|u|_{H^{1/2}(\mathscr{M})}$ in space dimension two, we find

$$\begin{aligned}
&\left|\frac{i}{2}\sum^{\rm c}_{\substack{j_3\neq 0,k_3=0\\ \beta_j=\beta_l}} k_1'(n_l^\flat u_j - u_l^\flat n_j)n^\sharp_{(k_1,0)}\vec{\phi}_l\cdot\vec{\phi}_k\right|\\
&\leqslant c||V||^{1/2}|\Delta V|^{1/2}_{L^2}|V^\flat|^{1/2}_{L^2}||V^\flat||^{1/2}|V^\sharp|_{L^2} + c|V|^{1/2}_{L^2}||V||^{1/2}||V^\flat||^{1/2}|\Delta V^\flat|^{1/2}_{L^2}|V^\sharp|_{L^2}.
\end{aligned}$$

All the other terms can be estimated in the same manner, giving us (A.31). The proof of (A.32) follows using the same type of argument. □

Appendix B. Auxiliary results

B.1. *A result in number theory*

In this section we prove for interested readers a (weaker) analogue of the small denominator estimate (4.19) used in Section 4.

Lemma B.1. *For any $\delta > 3$ and for almost every $\xi\in(0,R)$, where R is an arbitrarily natural number, there exists a constant $\gamma>0$ such that $|p+q\xi+r\xi^2|>\gamma|p^2+q^2+r^2|^{-\delta/2}$ for all $(p,q,r)\in\mathbb{Z}^3\backslash\{\mathbf{0}\}$.*

Proof. We need to show that the set

$$\Omega = \{\xi\in(0,R)\colon\ \forall\gamma>0\exists(p,q,r)\in\mathbb{Z}^3\backslash\{\mathbf{0}\}\ \text{with}\ |p+q\xi+r\xi^2|\leqslant\gamma|p^2+q^2+r^2|^{-\delta/2}\}$$

has measure zero.

We first split $\mathbb{Z}^3\backslash\{\mathbf{0}\}$ into $Z_1 + Z_2 + Z_3 + Z_4$, where

$$Z_1 = \{(p,q,r)\colon\ r\xi^2 + q\xi + p = 0 \text{ has no solution in } \mathbb{R}\},$$

$$Z_2 = \{(p,q,r)\colon\ r\xi^2 + q\xi + p = 0 \text{ has a double root } |\xi_*| \leqslant 2R\},$$

$$Z_3 = \{(p,q,r)\colon\ r\xi^2 + q\xi + p = 0 \text{ has two simple roots}\},$$

and Z_4 covers the other cases which do not concern us. Noting that

$$\Omega = \bigcap_{\gamma} \bigcup_{p,q,r} \Omega_\gamma(p,q,r),$$

we fix γ and (p,q,r), and compute the measure of the set

$$\Omega_\gamma(p,q,r) = \{\xi \in (0,R)\colon\ |p + q\xi + r\xi^2| \leqslant \gamma|p^2 + q^2 + r^2|^{-\delta/2}\}. \tag{B.1}$$

We now consider Z_1, Z_2 and Z_3 in turn.

$(p,q,r) \in Z_1$: mes $\Omega_\gamma(p,q,r) = 0$ for $\gamma < 1/4$, because

$$\min_{\xi \in \mathbb{R}} |r\xi^2 + q\xi + p| = \frac{|q^2 - 4pr|}{4|r|} \geqslant \gamma|p^2 + q^2 + r^2|^{-1}$$

and $|q^2 - 4pr| \geqslant 1$ in this case.

$(p,q,r) \in Z_2$: in this case $|r| \geqslant 1$ and $q^2 - 4pr = 0$, which implies $pr \geqslant 0$. We then have,

$$\text{mes}\, \Omega_\gamma(p,q,r) \leqslant \sqrt{\gamma/|r|}\,|p^2 + q^2 + r^2|^{-\delta/4}. \tag{B.2}$$

Since the root $|\xi_*| \leqslant 2R$, $q^2 \leqslant 8r^2$ and (using $4pr = q^2$) also $p^2 \leqslant 4r^2R^4$. Therefore $\sqrt{|r|} \geqslant C(R)|p^2 + q^2 + r^2|^{1/4}$ and

$$\text{mes}\, \Omega_\gamma(p,q,r) \leqslant \sqrt{\gamma}\, C(R)|p^2 + q^2 + r^2|^{-(\delta+1)/4}. \tag{B.3}$$

Since $q^2 = 4pr$, this is equivalent to (allowing us to sum over $\mathbb{Z}^2\backslash\{\mathbf{0}\}$ in (B.6) below)

$$\text{mes}\, \Omega_\gamma(p,q,r) \leqslant \sqrt{\gamma}\, C(R)|p^2 + r^2|^{-(\delta+1)/4}. \tag{B.4}$$

$(p,q,r) \in Z_3$: as before, we assume that $r \geqslant 1$; the case $r \leqslant -1$ is similar, and the "linear" case $r = 0$ is easy. We denote $\eta = \gamma|p^2 + q^2 + r^2|^{-\delta/2}$, $\Delta = q^2 - 4pr$, $\Delta_- = q^2 - 4p(r - \eta)$ and $\Delta_+ = q^2 - 4p(r + \eta)$. Considering the neighborhood of a root, and noting that $\Delta_- > 0$ whenever $\gamma < 1/4$, we have

$$\text{mes}\{\xi\colon\ |r\xi^2 + q\xi + p| \leqslant \eta\} \leqslant \frac{\sqrt{\Delta_+} - \sqrt{\Delta_-}}{2r} = \frac{8\eta}{\sqrt{\Delta_+} + \sqrt{\Delta_-}} \leqslant \frac{8\eta}{\sqrt{\Delta}} \leqslant 8\eta.$$

Regardless of where the roots lie, we thus have

$$\Omega_\gamma(p,q,r) \leqslant 16\gamma|p^2+q^2+r^2|^{-\delta/2}. \tag{B.5}$$

Putting together the results of the three cases, we have

$$\operatorname{mes}\Omega_\gamma \leqslant 16\gamma \sum\nolimits_{p,q,r} |p^2+q^2+r^2|^{-\delta/2} + \sqrt{\gamma}C(R)\sum\nolimits_{p,r} |p^2+r^2|^{-(\delta+1)/4} \tag{B.6}$$

where the first sum is taken over $\mathbb{Z}^3\backslash\{\mathbf{0}\}$ and the second over $\mathbb{Z}^2\backslash\{\mathbf{0}\}$. Both sums converge when $\delta > 3$, giving us

$$\operatorname{mes}\Omega_\gamma \leqslant \sqrt{\gamma}C(\delta,R), \tag{B.7}$$

valid for $\gamma < 1/4$, whence it follows that $\operatorname{mes}\Omega = 0$. $\square$

B.2. *Another estimate for small denominators*

In this section, following an alternate approach due to Babin et al. [3], we present another way of estimating the three-wave resonances. In a sense the method is an improvement of that used in Section 4 because we require less regularity on the initial data. On the other hand, it is weaker because it is valid only for Burgers numbers belonging to a certain quasi-resonant set.

Recall that $\beta_k = [1+N^2(k_1'/k_3')^2]^{1/2}$. As in Section 4, we need to estimate the term

$$I_3 = \frac{e^{s(\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k)}-1}{\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k}, \tag{B.8}$$

where $\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k \neq 0$, α_1, α_2, $\alpha_3 = \pm i$ and $j+l+k=0$.

The problem is non-trivial only when the α_i are not of the same sign; with no loss of generality, we suppose that $\alpha_1=\alpha_2=-\alpha_3$. In estimating $|\beta_j+\beta_l-\beta_k|^{-1}$, we have two cases:

Case 1: If $|\beta_l-\beta_k| \leqslant \beta_j/2$, then $|\beta_j+\beta_l-\beta_k|^{-1} \leqslant 2/\beta_j \leqslant 2$ and we are done.

Case 2: If $|\beta_l-\beta_k| \geqslant \beta_j/2$, some work is needed. We estimate

$$\begin{aligned}|I_3| &\leqslant \frac{2}{|\beta_j+\beta_l-\beta_k|}\\ &= \frac{2|(\beta_j+\beta_l+\beta_k)(-\beta_j+\beta_l+\beta_k)(-\beta_l+\beta_j+\beta_k)|}{|(\beta_j+\beta_l+\beta_k)(\beta_j+\beta_l-\beta_k)(-\beta_j+\beta_l+\beta_k)(\beta_j-\beta_l+\beta_k)|}\\ &=: 2I_3'.\end{aligned} \tag{B.9}$$

Denoting $\lambda = N^2$ and $\chi_k = (k_1'/k_3')^2$, we have

$$I_3' = \frac{|(\beta_j + \beta_l + \beta_k)(-\beta_j + \beta_l + \beta_k)(-\beta_l + \beta_j + \beta_k)|}{|P(\lambda)|}, \tag{B.10}$$

where

$$P(\lambda) = \lambda^2(\chi_k^2 + \chi_j^2 + \chi_l^2 - 2\chi_k\chi_j - 2\chi_j\chi_l - 2\chi_k\chi_l) - 2\lambda(\chi_j + \chi_l + \chi_k) - 3. \tag{B.11}$$

The discriminant of this quadratic polynomial is

$$\Delta = 2[(\chi_j - \chi_l)^2 + (\chi_l - \chi_k)^2 + (\chi_k - \chi_j)^2] \geqslant 0. \tag{B.12}$$

Since $P(\lambda) = 0$ has no more than two solutions for each fixed (j, l), the set of Burgers numbers N for which $\beta_j + \beta_l - \beta_k = 0$ is at most countable. We denote the solutions of $P(\lambda) = 0$ by $\lambda_\pm(j, l)$.

We define the three-wave quasi-resonant set $\Theta_3^\mu(L_1, L_3)$:

Given $\mu > 0$ and a sequence of positive numbers $\{\xi_{(j,l)}\}$ with $\sum_{(j,l)} \xi_{(j,l)} \leqslant 1$, we define the three-wave quasi-resonant set $\Theta_3^\mu(L_1, L_3)$ as:

$$\Theta_3^\mu(L_1, L_3) = \bigcup_{(j,l)\in\mathbb{Z}^2} \{N\colon\ 2|N - N^\star(j, l, L_1, L_3)| \leqslant \mu\xi_{(j,l)}\}, \tag{B.13}$$

where $N^\star(j, l, L_1, L_3) \coloneqq \sqrt{\lambda_\pm(j, l, L_1, L_3)}$. It is obvious that the Lebesque measure mes $\Theta_3^\mu(L_1, L_3) \leqslant \mu$ for all L_1 and L_3.

For j, l, L_1 and L_3 given, the set $\{N\colon\ 2|N - N^\star(j, l, L_1, L_3)| \leqslant \mu\xi_{(j,l)}\}$ can be defined approximately by $|P(\lambda)| \leqslant \delta$. For δ small, we have

$$\begin{aligned} \delta &\simeq \left|\frac{d\lambda}{d\delta}(0)\right|^{-1} |\lambda(\delta) - \lambda_\pm(j, l, L_1, L_3)| \\ &\simeq 2N_\pm(j, l, L_1, L_3)|N - N_\pm(j, l, L_1, L_3)| \left|\frac{d\lambda}{d\delta}(0)\right|^{-1}, \end{aligned} \tag{B.14}$$

where

$$\left|\frac{d\lambda}{d\delta}(0)\right| = \frac{1}{\sqrt{\Delta}} = \frac{1}{\sqrt{2[(\chi_j - \chi_l)^2 + (\chi_l - \chi_k)^2 + (\chi_k - \chi_j)^2]}}. \tag{B.15}$$

or, using $\beta_j - \beta_k = N^2(\chi_j - \chi_k)$,

$$\left|\frac{d\lambda}{d\delta}(0)\right| = \frac{N^2}{\sqrt{2[(\beta_j^2 - \beta_l^2)^2 + (\beta_l^2 - \beta_k^2)^2 + (\beta_k^2 - \beta_j^2)^2]}} \leqslant \frac{N^2}{2\sqrt{2}}. \tag{B.16}$$

Since $\beta_k \leqslant \max(1, N^2)|k'|$, for $N \notin \Theta_3^\mu(L_1, L_3)$, we have, using (B.14) that

$$\frac{1}{|\beta_j + \beta_l - \beta_k|} \leqslant C(N) \frac{(|k'| + |l'| + |j'|)^3}{|P(\lambda)|} \leqslant C(N, L_1, L_3) \frac{(|k'| + |j'| + |l'|)^3}{\mu \xi_{(j,l)}}. \tag{B.17}$$

We now choose $\xi_{(j,l)}$: For any $\eta > 0$ we can take

$$\xi_{(j,l)} = c(\eta) |j'|^{-2-\eta} |l'|^{-2-\eta}, \tag{B.18}$$

where $c(\eta) = \left(\sum_{j,l \in \mathbb{Z}^2} |j'|^{-2-\eta} |l'|^{-2-\eta}\right)^{-1}$. Substituting this into (B.17), we obtain the following bound:

$$\frac{1}{|\beta_j + \beta_l - \beta_k|} \leqslant C(N, L_1, L_3, \eta) \frac{(|k'| + |j'| + |l'|)^3}{\mu} |l'|^{2+\eta} |j'|^{2+\eta}, \quad N \notin \Theta_3^\mu(L_1, L_3).$$

We can now conclude with the following result:

Lemma B.2. *Let $\eta > 0$ and $\mu > 0$; then for every $L_1, L_3 \in \mathbb{R}$ and $N \notin \Theta_3^\mu(L_1, L_3)$ we have $\beta_j + \beta_l - \beta_k \neq 0$ for all j, l, k with $j + l + k = 0$, and*

$$\frac{1}{|\beta_j + \beta_l - \beta_k|} \leqslant \max\left(2, C(N, L_1, L_3, \eta) \frac{(|k'| + |j'| + |l'|)^3}{\mu} |l'|^{2+\eta} |j'|^{2+\eta}\right). \tag{B.19}$$

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