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On weighted-norm estimates for nonstationary incompressible Navier–Stokes flows in a 3D exterior domain

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ABSTRACT

We study the time-decay of weighted norms of weak and strong solutions to the Navier–Stokes equations in a 3D exterior domain. Moment estimates for weak solutions and weighted L^q-estimates for strong solutions are deduced, both of which seem to be optimal. The relation is discussed between the space–time decay and the vanishing of the total net force exerted by the fluid to the body. A class of initial data is given so that the total net force associated to the corresponding fluid flows does not vanish.

1. Introduction

In an exterior domain $\Omega \subset \mathbb{R}^n$ $(n = 3)$ with smooth boundary $\partial\Omega$, we study the space–time decay properties of solutions to the Navier–Stokes initial value problem

$$\begin{aligned}
&\partial_t u + u \cdot \nabla u = \Delta u - \nabla p \quad (x \in \Omega,\ t > 0),\\
&\nabla \cdot u = 0 \quad (x \in \Omega,\ t \geqslant 0),\\
&u|_{\partial\Omega} = 0, \qquad u \to 0 \quad \big(|x| \to \infty\big),\\
&u|_{t=0} = a,
\end{aligned} \tag{1.1}$$

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for unknown velocity $u = u(x,t) = (u_1, \dots, u_n)$, unknown pressure $p = p(x,t)$ and a prescribed initial velocity $a = a(x)$. The kinematic viscosity is normalized to be one.

There is an extensive literature dealing with decay properties of weak and strong solutions to (1.1) (see, e.g., [6–8,17,23,24,27,28,30–33,36]). For weak solutions, L^2 decay properties have been studied and the algebraic decay rates, similar to those for solutions of the heat equation, are obtained. The results show that for each $a \in L^2_\sigma(\Omega)$, the space of the L^2 solenoidal vector fields, there is a weak solution u defined for all $t \geqslant 0$ such that $\|u(t)\|_2 \to 0$ as $t \to \infty$. Hereafter, $\|\cdot\|_r$ denotes the norm of $L^r(\Omega)$. If $a \in L^2_\sigma(\Omega) \cap L^r(\Omega)$ for some $1 \leqslant r < 2$, then the weak solution satisfies

$$\big\|u(t)\big\|_2 \leqslant c(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}. \tag{1.2}$$

See [6,7] and [11]. For strong solutions, L^q-theory was developed by Iwashita [24] and Chen [11] for $n \geqslant 3$, and by Dan and Shibata [12] for $n = 2$ (see also [1] and [16]). They proved the estimates

$$\big\|u_0(t)\big\|_q \leqslant ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_p \quad (1 < p \leqslant q < \infty,\ 1 \leqslant p < q \leqslant \infty), \tag{1.3}$$

$$\big\|\nabla u_0(t)\big\|_q \leqslant ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_p \quad (1 < p \leqslant q \leqslant n,\ 1 \leqslant p < q \leqslant n), \tag{1.4}$$

on solutions u_0 of the Stokes problem, i.e., the linearized version of (1.1). These estimates were applied by [8,11] and [24] to extend results of Kato [25] for the Cauchy problem to the case of (1.1), and we know that if $n \geqslant 3$, if a is in the space $L^n_\sigma(\Omega)$ of L^n solenoidal vector fields and if $\|a\|_n$ is sufficiently small, then (1.1) admits a unique strong solution u defined for all $t \geqslant 0$. Moreover, if $a \in L^r(\Omega) \cap L^n_\sigma(\Omega)$ for some $1 < r \leqslant n$, then

$$t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}u \in BC\big([0,\infty); L^q(\Omega)\big) \quad (r \leqslant q \leqslant \infty), \tag{1.5}$$

$$t^{\frac{1}{2}+\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\nabla u \in BC\big([0,\infty); L^q(\Omega)\big) \quad (r \leqslant q \leqslant n). \tag{1.6}$$

In [18] we extended (1.5) and (1.6) to the case where $r = 1 < q$.

In this paper we first discuss estimates for the L^2-moments of weak solutions of the form:

$$\text{(M)} \qquad \int_\Omega |x|^{2\alpha}\big|u(x,t)\big|^2\,dx + \int_0^t\int_\Omega |x|^{2\alpha}\big|\nabla u(x,\tau)\big|^2\,dx\,d\tau \leqslant c.$$

For the Cauchy problem, the following are known: M.E. Schonbek and T.P. Schonbek [37] proved (M) with $\alpha = 3/2$ for smooth solutions on $\mathbb{R}^3$ (see also [15]). He and Xin [22] proved (M) for weak solutions, with $\alpha = 3/2$, assuming $a \in L^1(\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$ and $|x|^{3/2}a \in L^2(\mathbb{R}^3)$. Bae and Jin [3] proved (M) for weak solutions, with $1 < \alpha < 5/2$, assuming $a \in L^2_\sigma(\mathbb{R}^3)$, $(1+|x|)a \in L^1(\mathbb{R}^3)$ and $|x|^\alpha a \in L^2(\mathbb{R}^3)$. Brandolese [9] found a local smooth solution $u \in C([0,T); \mathbf{Z}_\alpha)$, with some $T > 0$, assuming $a \in \mathbf{Z}_\alpha$ for $3/2 < \alpha < 9/2$ ($\alpha \neq 5/3, 7/2$). Here, $f \in \mathbf{Z}_\alpha$ means that

$$\big(1+|x|^2\big)^{\alpha-2} f \in L^2\big(\mathbb{R}^3\big), \qquad \big(1+|x|^2\big)^{\alpha-1}\nabla f \in L^2\big(\mathbb{R}^3\big), \qquad \big(1+|x|^2\big)^{\alpha}\Delta f \in L^2\big(\mathbb{R}^3\big).$$

For problem (1.1), the corresponding results are still incomplete. Farwig and Sohr [14] found a class of weak solutions u with associated pressures p such that

$$|x|^\alpha\partial_t u,\ |x|^\alpha\partial_x^2 u,\ |x|^\alpha\nabla p \in L^s\big(0,\infty; L^q(\Omega)\big) \quad (n = 3),$$

for $1 < q < 3/2$ and $1 < s < 2$ with $3/q + 2/s - 4 \leqslant \alpha < \min\{1/2, 3 - 3/q\}$. Farwig [13] then gave another class of weak solutions such that

$$\big\| |x|^{\frac{\alpha}{2}} u(t)\big\|_2^2 + c_\alpha \int_s^t \big\| |x|^{\frac{\alpha}{2}} \nabla u\big\|_2^2 \, d\tau \leqslant \big\| |x|^{\frac{\alpha}{2}} u(s)\big\|_2^2 \quad (0 < \alpha < 1),$$

$$\big\| |x|^{\frac{1}{2}} u(t)\big\|_2^2 + 2 \int_s^t \big\| |x|^{\frac{1}{2}} \nabla u\big\|_2^2 \, d\tau \leqslant \big\| |x|^{\frac{1}{2}} u(s)\big\|_2^2 + c_{a,\delta} |t-s|^\delta,$$

for $s = 0$, a.e. $s > 0$ and all $t \geqslant s$, where $\delta > 0$ is arbitrary. Recently, Bae and Jin have studied decay rates of L^2-moments. When $n = 2$, they prove in [4] that there is a weak solution u satisfying

$$\big\| |x|^\alpha u(t)\big\|_p = O\big(t^{-\frac{1}{2}+\frac{1}{p}+\frac{\alpha}{2}+\delta}\big) \quad \text{for large } t,$$

for all $\delta > 0$ and $0 < \alpha \leqslant 1$, if $a \in L^r(\Omega) \cap L^2(\Omega)$ and $|x|a \in L^{\frac{2r}{2-r}}(\Omega)$ with $1 < r \leqslant 2p/(p+2) < 2 \leqslant p < \infty$. Moreover, in case $n = 3$ they prove in [5] that there is a weak solution such that

$$\big\| |x| u(t)\big\|_2 \leqslant c_\delta (1+t)^{\frac{5}{4}-\frac{3}{2r}+\delta},$$

for all $\delta > 0$, if $a \in L^r(\Omega) \cap L^2(\Omega)$ for some $1 < r < 6/5$, $|x|a \in L^{6/5}(\Omega)$ and $|x|^2 a \in L^2(\Omega)$.

This paper improves the above results on L^2-moments and gives weak solutions satisfying

$$\big\| |x|^\alpha u(t)\big\|_2^2 + \int_0^t \big\| |x|^\alpha \nabla u(\tau)\big\|_2^2 \, d\tau \leqslant c \quad (1 < \alpha < n/2),$$

$$\big\| |x|^\beta u(t)\big\|_2 \leqslant c(1+t)^{-\frac{n(\alpha-\beta)}{4\alpha}} \quad (0 \leqslant \beta \leqslant \alpha < n/2),$$

for all $t \geqslant 0$. The restriction $\alpha < n/2$ comes from our estimates on pressures. But, this condition on α is optimal in the following sense: in Theorem 2.5 (see Section 2), we will show that strong solutions behave in general as $|u(x,t)| \approx |x|^{-n}$ for large $|x|$. So $|x|^\alpha u$ is in $L^2(\Omega)$ only when $\alpha < n/2$. In a special case, however, this restriction on α is relaxed. Indeed, we show that one can take $\alpha < 1 + n/2$ if the associated pressure p satisfies

$$G(t) = \int_{\partial\Omega} (y \partial_\nu p - p\nu)(y,t) \, dS_y = 0, \quad t \in (0,\infty), \tag{1.7}$$

where ν is the unit outward normal to $\partial\Omega$.

We next discuss the behavior of weighted L^q-norms of strong solutions. For the Cauchy problem, the estimates $t^{\frac{\beta}{2}} \||x|^\alpha u\|_q + t^{\frac{1+\beta}{2}} \||x|^\alpha \nabla u\|_q \leqslant c$ are known to be valid if $\alpha \geqslant 0$, $\beta \geqslant 0$ and

$$\alpha + 2\beta = n - n/q \quad \text{or} \quad \alpha + 2\beta = n + 1 - n/q, \quad n < q \leqslant \infty. \tag{1.8}$$

See [2,3,15,22,34] and [35] for the details. See also [19] for solutions with some symmetries. The balance relation (1.8) agrees with that for solutions of the linear heat equation on $\mathbb{R}^n$.

On the other hand, for (1.1) with $n = 3$, He and Xin [21] gave strong solutions such that $\||x|^\alpha u(t)\|_q \leqslant c$ for $\alpha = 3/7 - 3/q$, $7 < q \leqslant \infty$. Recently, Bae and Jin have adapted the ideas of [21] and proved

$$\big\| |x|^2 u(t)\big\|_p \leqslant c_\delta t^{1-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})+\delta} \quad \text{for large } t > 0,$$

with an arbitrary $\delta > 0$, assuming that $a \in L^r(\Omega) \cap L^3(\Omega)$ for some $1 < r < 6/5$, and

$$|x|a, |x|^2 a \in L^r(\Omega), \qquad |x|a \in L^{6/5}(\Omega), \qquad |x|^2 a \in L^2(\Omega).$$

However, these results are not optimal. In this paper we deduce the optimal decay rates in space and time and establish a balance relation between these two kinds of decays which is similar to that of solutions to the Cauchy problem.

It should be noticed that for (1.1), the spatial decay property of a solution is closely connected with the vanishing of the total net force exerted by the fluid to the body $\mathbb{R}^n \setminus \Omega$. Indeed, it is shown in [18] that the following three statements are equivalent:

(a) The total net force vanishes, i.e., we have

$$\mathcal{F}(t) = \int_{\partial\Omega} \big(T[u, p] \cdot \nu\big)(y, t)\, dS_y = 0, \tag{1.9}$$

where $T[u, p] = (T_{jk}[u, p])_{j,k=1}^n = (\partial_j u_k + \partial_k u_j - \delta_{jk} p)$ is the stress tensor.

(b) The solution u is in $C([0, T); L^1(\Omega))$.

(c) Assertion (1.7) holds, i.e., $G(t) = 0$.

In this paper we further show that if $|x|^{n(1-\frac{1}{r})} a \in L^r(\Omega)$ for some $1 \leqslant r < \infty$, then in general we have $t^{\frac{n}{2}(1-\frac{1}{r})} |x|^{n(1-\frac{1}{r})} u \in L^\infty_{\mathrm{loc}}([0, \infty); L^r_w(\Omega))$, where L^r_w is the weak L^r-space, and that

$$(1.9) \text{ holds} \quad \text{if and only if} \quad t^{\frac{n}{2}(1-\frac{1}{r})} |x|^{n(1-\frac{1}{r})} u \in L^\infty_{\mathrm{loc}}\big([0, \infty); L^r(\Omega)\big).$$

See Theorem 2.5 in Section 2.

Finally, we give a class of initial data a such that the corresponding strong solutions satisfy $\mathcal{F} \neq 0$ (or, equivalently, $G \neq 0$). For such data, our moment estimates (Theorem 2.2) and the time-decay rates (Theorem 2.4) are optimal. But, we do not know if our class is vacuous or not.

Throughout the paper we assume $n = 3$; but we use the notation n to denote the space dimension. Indeed, our results on strong solutions are valid for all dimensions $n \geqslant 3$ and, moreover, our notation (of using n) would be convenient for the reader to understand the nature of assumptions in our main results (Theorems 2.1–2.6 below).

2. Notation and main results

We always assume that $n = 3$ and that the origin of $\mathbb{R}^n$ is in $\mathbb{R}^n \setminus \overline{\Omega}$. $L^q(\Omega)$, $1 \leqslant q \leqslant \infty$, denotes the Lebesgue space of real-valued functions as well as that of vector functions, with norm $\|\cdot\|_q$, and $C^\infty_{0,\sigma}(\Omega)$ the set of smooth solenoidal vector fields with compact support in Ω. $L^q_\sigma(\Omega)$, $1 < q < \infty$, is the closure of $C^\infty_{0,\sigma}(\Omega)$ in the norm $\|\cdot\|_q$. Let $\mathcal{H}^1(\mathbb{R}^n)$ be the Hardy space defined in [33,39]. Given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0, T; X)$, $1 \leqslant p \leqslant \infty$, the set of strongly measurable functions $f : (0, T) \to X$ such that $\int_0^T \|f(t)\|_X^p\, dt < \infty$ (obvious modification when $p = \infty$). $P : L^q(\Omega) \to L^q_\sigma(\Omega)$ is the bounded projection as defined in [32], and the Stokes operator $A = -P\Delta$ is the closed linear operator in $L^q_\sigma(\Omega)$, with (dense) domain $D(A) = D(A_q) = H^{2,q}(\Omega) \cap H^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$. We know that $-A_q$ generates in $L^q_\sigma(\Omega)$ a bounded analytic semigroup $\{e^{-tA}\}_{t \geqslant 0}$. Using this we define

$$D_q^{1-1/s,s} = \left\{ v \in L^q_\sigma(\Omega)\colon\ \|v\|_{D_q^{1-1/s,s}} = \|v\|_q + \left(\int_0^\infty \big\|t^{\frac{1}{s}} A e^{-tA} v\big\|_q^s\, dt/t \right)^{\frac{1}{s}} < +\infty \right\},$$

with $1 < s < \infty$. We need these spaces for specifying our initial data.

Definition 2.1. Let $a \in L^2_\sigma(\Omega)$. A vector function u on $\Omega \times [0,\infty)$ is called a *weak solution* to problem (1.1) if:

(1) $u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^{1,2}_0(\Omega))$ for all $T > 0$.
(2) For every $\phi \in C_0([0,\infty); H^{1,2}_0(\Omega)) \cap C^1_0([0,\infty); L^2_\sigma(\Omega))$, we have

$$\int_0^\infty \int_\Omega \big(-u \cdot \partial_\tau \phi + \nabla u \cdot \nabla \phi + (u \cdot \nabla) u \cdot \phi\big)\, dx\, d\tau = \int_\Omega \phi(x,0) \cdot a(x)\, dx.$$

(3) u satisfies $\nabla \cdot u = 0$ in Ω in the sense of distributions.

Definition 2.2. Let $a \in L^n_\sigma(\Omega)$. A vector function u is called a *strong solution* to problem (1.1) if $u \in BC([0,\infty); L^n_\sigma(\Omega))$ and if (2) and (3) in Definition 2.1 hold for u.

Our main results are as follows. The first result deals with the existence and estimates of weak solutions in weighted L^2-spaces.

Theorem 2.1. *For each $a \in L^2_\sigma(\Omega)$, there exists a weak solution u such that*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u\|_2^2\, d\tau \leqslant \|u(s)\|_2^2 \quad \text{for } s = 0, \text{ a.e. } s > 0, \text{ and all } t \geqslant s. \tag{2.1}$$

Moreover, if $a \in L^1(\Omega) \cap L^2_\sigma(\Omega) \cap D^{1-1/s,s}_p$, $n+1 = 2/s + n/p$, and $6/5 \leqslant p < n/(n-1)$ and if $|x|^\alpha a \in L^2(\Omega)$ for some $1 < \alpha < n/2$, the weak solution given above satisfies

$$\big\||x|^\alpha u(t)\big\|_2^2 + \int_0^t \big\||x|^\alpha \nabla u\big\|_2^2\, d\tau \leqslant c, \qquad \big\||x|^\beta u(t)\big\|_2 \leqslant c(1+t)^{-\frac{n(\alpha-\beta)}{4\alpha}} \quad (0 \leqslant \beta \leqslant \alpha), \tag{2.2}$$

for all $t \geqslant 0$, with c depending only on α, $\|a\|_1$, $\|a\|_{D^{1-1/s,s}_q}$ and $\||x|^\alpha a\|_2$.

We note that $6/5 = (2^*)' = 2n/(n+2)$ with $2^* = 2n/(n-2)$ and $n = 3$, according to the Sobolev embedding theorem. As will be seen from the proof, the restriction $\alpha < n/2$ comes from our estimates on the pressures. But, condition $\alpha < n/2$ is optimal, as mentioned in Introduction, since our weak solutions behave like $|x|^{-n}$ as $|x| \to \infty$. On the other hand, if p satisfies $G = 0$, where G is the function defined in (1.7), then u will behave like $|x|^{-n-1}$. We now discuss the validity of this conjecture. However, it is now known that condition $G = 0$ is closely connected with some symmetry conditions on $\{u, p\}$; so we state our result in the following form.

Theorem 2.2. *Suppose Ω is invariant under the reflection $x \mapsto -x$. Let $a \in L^1(\Omega) \cap L^2_\sigma(\Omega) \cap D^{1-1/s,s}_p$, $n+1 = 2/s + n/p$, and $6/5 \leqslant p < n/(n-1)$. If $a(-x) = -a(x)$ and $|x|^\alpha a \in L^2(\Omega)$ for some $1 < \alpha < 1 + n/2$, then a weak solution u exists, satisfying $G = 0$ and*

$$\big\||x|^\alpha u(t)\big\|_2^2 + \int_0^t \big\||x|^\alpha \nabla u\big\|_2^2\, d\tau \leqslant c, \qquad \big\||x|^\beta u(t)\big\|_2 \leqslant c(1+t)^{-\frac{n(\alpha-\beta)}{4\alpha}} \quad (0 \leqslant \beta \leqslant \alpha). \tag{2.3}$$

As shown in [18], $G = 0$ is equivalent to (1.9). The result above is the same as those given in [3] and [9] for solutions to the Cauchy problem.

We next deal with strong solutions and prove the existence of those solutions which decay more rapidly than those treated, e.g., in [7,8,11] and [24].

Theorem 2.3. *Let $a \in L^1(\Omega) \cap L^n_\sigma(\Omega) \cap D_p^{1-1/s,s}$, $2/s + n/p = n + 1$, and $6/5 \leqslant p < n/(n-1)$. There is a constant $\lambda > 0$ so that $\|a\|_n \leqslant \lambda$ implies the existence of a strong solution u defined for all $t \geqslant 0$ such that*

$$\|u\|_r \leqslant ct^{-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \quad \big(1 \leqslant \ell \leqslant \min\{n,r\},\ 1 < r \leqslant \infty\big),$$
$$\|\nabla u\|_r \leqslant ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \quad (1 \leqslant \ell \leqslant r \leqslant n),$$
$$\big\|\partial^2 u\big\|_r + \|\partial_t u\|_r + \|\nabla p\|_r \leqslant ct^{-1-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \quad (1 \leqslant \ell \leqslant r \leqslant n/2,\ r > 1) \tag{2.4}$$

and

$$\big\|\nabla u(t)\big\|_{L^r(\Omega_\delta)} \leqslant ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})} + c_\delta t^{-\frac{n}{2}} \quad (n < r < \infty), \tag{2.5}$$

for all $t > 0$, where $\Omega_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\}$, $\delta > 0$.

(2.4) is given in [18] (see Theorem 1), and (2.5) will be proved in Section 5. The last term in (2.5) comes from a boundary integral in the representation formula of u, which does not appear in the case of the Cauchy problem.

The result below deals with the time-decay of weighted norms of strong solutions.

Theorem 2.4. *Let $a \in L^1(\Omega) \cap L^n_\sigma(\Omega) \cap D_p^{1-1/s,s}$, $2/s + n/p = n + 1$, and $6/5 \leqslant p < n/(n-1)$. Suppose $|x|^\alpha a \in L^r(\Omega)$ with $\alpha = n(1-1/r)$ for some $1 \leqslant r < \infty$. Then, there is a number $\lambda_1 > 0$ so that $\|a\|_n \leqslant \lambda_1$ ensures the existence of a strong solution u satisfying*

$$\big\||x|^\alpha u(t)\big\|_q \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \quad \big(\max\big\{r, n/(n-1)\big\} < q \leqslant \infty\big) \text{ for all } t > 0. \tag{2.6}$$

For the Cauchy problem, there are strong solutions u satisfying $t^\beta \||x|^\alpha u\|_q \leqslant c$, $n < q \leqslant \infty$, with $\alpha = n(1-1/r)$, $\beta = (n/2)(1/r - 1/q)$ and $1 < r \leqslant q \leqslant \infty$. See [22]. Our result above is similar to that of [22] and improves that of [5]. The relation between the space and time decays given above agrees with that of the Cauchy problem.

We finally discuss the relation between the decay properties of solutions u and the vanishing of the associated total net force, i.e., the validity of (1.9). Define $V(x,t) = (V_{jk}(x,t))$ by

$$V_{jk}(x,t) = E_t(x)\delta_{jk} + \partial_j\partial_k(\mathcal{N} * E_t)(x), \tag{2.7}$$

where $\mathcal{N} = c_n|x|^{2-n}$ is the Newtonian potential and $E_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Moreover, recall the function $\mathcal{F}(t) = (\mathcal{F}_j(t))_{j=1}^n$ defined in (1.9). We shall prove

Theorem 2.5. *Let $a \in L^1(\Omega) \cap L^n_\sigma(\Omega) \cap D_p^{1-1/s,s}$, $2/s + n/p = n + 1$, and $6/5 \leqslant p < n/(n-1)$. Suppose $|x|^\alpha a \in L^r(\Omega)$ with $\alpha = n(1-1/r)$ for some $1 \leqslant r < \infty$. If $\|a\|_n \leqslant \lambda_1$, the strong solution obtained in Theorem 2.4 satisfies*

$$\left\||x|^\alpha\left(u(t) - V(\,\cdot\,,t)\cdot\int_0^t \mathcal{F}\,d\tau\right)\right\|_r \leqslant c\big(1 + t^{-\frac{n}{2}(1-\frac{1}{r})}\big) \quad \text{for all } t > 0. \tag{2.8}$$

This implies that $t^{\frac{\alpha}{2}}|x|^{\alpha}u \in L^{\infty}_{\mathrm{loc}}([0,\infty); L^r_w(\Omega))$ *and that* $t^{\frac{\alpha}{2}}|x|^{\alpha}u \in L^{\infty}_{\mathrm{loc}}([0,\infty); L^r(\Omega))$ *if and only if* (1.9) *holds.*

To see that (2.8) is in general optimal, we need to construct a velocity field a for which the corresponding solution does not satisfy (1.9). To this end, the following result would be useful. Let $\nu = (\nu_1, \nu_2, \nu_3)$ be the unit outward normal to $\partial\Omega$ and consider the functions h_k, $k = 1, 2, 3$, satisfying

$$\Delta h_k = 0, \qquad \partial h_k/\partial\nu|_{\partial\Omega} = -\nu_k, \qquad \big|h_k(x)\big| = O\big(|x|^{-1}\big) \quad \big(|x| \to \infty\big).$$

Now, we know [29] that if $a \in L^1(\Omega) \cap D(A_2)$, a (unique) strong solution u exists at least locally in time, satisfying $\|u(t) - a\|_{H^{2,2}(\Omega)} \to 0$ as $t \to 0$. In this situation we prove

Theorem 2.6. *A strong solution u and the associated pressure p satisfy* (1.9) *if and only if*

$$\int\limits_{\partial\Omega} (\partial_\nu u_k + \partial_\nu u_i \cdot \partial_i h_k)\, dS_x + \int\limits_{\Omega} (u_i u_j)\partial^2_{ij} h_k \, dx = 0 \quad \textit{for all } k \in \{1, 2, 3\}.$$

Therefore, if $a \in L^1(\Omega) \cap D(A_2)$ satisfies

$$\int\limits_{\partial\Omega} (\partial_\nu a_k + \partial_\nu a_i \cdot \partial_i h_k)\, dS_x + \int\limits_{\Omega} (a_i a_j)\partial^2_{ij} h_k \, dx \neq 0 \quad \textit{for some } k \in \{1, 2, 3\},$$

then the corresponding $\{u, p\}$ does not satisfy (1.9)*. In particular, if $a \in C^{\infty}_{0,\sigma}(\Omega)$ and*

$$\int\limits_{\Omega} (a_i a_j)\partial^2_{ij} h_k \, dx \neq 0 \quad \textit{for some } k \in \{1, 2, 3\}, \tag{2.9}$$

then $\{u, p\}$ does not satisfy (1.9)*.*

Let Ω be the exterior to the unit ball, and so $h = -c\nabla|x|^{-1}$. If $a(-x) = -a(x)$, the corresponding $\{u, p\}$ satisfies $u(-x, t) = -u(x, t)$, $p(-x, t) = p(x, t)$. Direct calculation then gives

$$\int\limits_{\partial\Omega} (\partial_\nu u_k + \partial_\nu u_i \cdot \partial_i h_k)\, dS_x + \int\limits_{\Omega} (u_i u_j)\partial^2_{ij} h_k \, dx = 0 \quad \text{for all } k \in \{1, 2, 3\}.$$

Hence, (1.9) holds by Theorem 2.6. However, by now we have no examples of a satisfying (2.9).

We prove Theorems 2.1–2.2 by establishing necessary estimates for approximate solutions which are uniform in approximation parameter and then invoking the fact that our weak solutions become strong after a finite time. Construction of the approximate solutions will be described in Section 3. Theorems 2.3–2.6 are obtained by directly estimating the strong solutions whose existence is now well known. In dealing with strong solutions, we freely make use of the results obtained in our previous paper [18].

3. Preliminaries

Let Ω be a smooth exterior domain in $\mathbb{R}^n$, $n = 3$. We construct approximate solutions u^{ε}, $\varepsilon > 0$, by solving

$$\begin{gathered}\partial_t u^\varepsilon - \Delta u^\varepsilon + \left(u^\varepsilon * \phi_\varepsilon\right)\cdot\nabla u^\varepsilon = -\nabla p^\varepsilon \quad \text{in } \Omega\times(0,\infty),\\ \nabla\cdot u^\varepsilon = 0 \quad \text{in } \Omega\times[0,\infty),\\ u^\varepsilon|_{\partial\Omega} = 0, \qquad u^\varepsilon|_{t=0} = a^\varepsilon = e^{-\varepsilon A}a. \end{gathered} \tag{3.1}$$

Here, $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$ is the standard mollifier on $\mathbb{R}^n$ and $\phi_\varepsilon * u^\varepsilon$ is the convolution of ϕ_ε and the extension of u^ε to $\mathbb{R}^n$ defined as $u^\varepsilon = 0$ outside Ω. As is well known (see [36]), the function u^ε is obtained by solving the integral equation

$$u^\varepsilon(t) = e^{-tA}a^\varepsilon - \int_0^t e^{-(t-\tau)A}P\left(\left(\phi_\varepsilon * u^\varepsilon\right)\cdot\nabla u^\varepsilon\right)(\tau)\,d\tau. \tag{3.1$'$}$$

Indeed, we know that if $a \in L^2_\sigma(\Omega)$ and $\varepsilon > 0$, then a unique solution u^ε to (3.1$'$) exists for all $t \geqslant 0$, satisfying $u^\varepsilon \in L^2(0,T;D(A_2)) \cap H^{1,2}(0,T;L^2_\sigma(\Omega))$ for each fixed $0 < T < \infty$. The standard energy method yields

$$\left\|u^\varepsilon(t)\right\|_2^2 + 2\int_0^t \left\|\nabla u^\varepsilon\right\|_2^2\,d\tau = \left\|a^\varepsilon\right\|_2^2 \leqslant \|a\|_2^2 \quad \text{for all } t \geqslant 0. \tag{3.2}$$

Moreover, we know by [6] and [11] that if $a \in L^1(\Omega)\cap L^2_\sigma(\Omega)$, then

$$\left\|u^\varepsilon(t)\right\|_r \leqslant c(1+t)^{-\frac{n}{2}(1-\frac{1}{r})} \quad (1 < r \leqslant 2) \tag{3.3}$$

with $c > 0$ independent of ε and $t > 0$. The result below is due to [1] and [16].

Lemma 3.1. *Let $a \in L^2_\sigma(\Omega)\cap D_p^{1-1/s,s}$ with $n+1 = 2/s + n/p$, $1 < p < n/(n-1)$, $1 < s < 2$. Then, there exists a number $c > 0$ independent of ε such that*

$$\int_0^\infty \left(\left\|\partial_t u^\varepsilon\right\|_p^s + \left\|\partial_x^2 u^\varepsilon\right\|_p^s + \left\|\nabla p^\varepsilon\right\|_p^s\right)dt \leqslant c\left(\|a\|_2^2 + \|a\|_{D_p^{1-1/s,s}}\right)^s. \tag{3.4}$$

The following is proved in [36].

Lemma 3.2. *Let $a \in L^2_\sigma(\Omega)$ and let u^ε satisfy* (3.1)*. Then, as $\varepsilon \to 0$, a subsequence of u^ε converges to a weak solution u of* (1.1) *such that*

$$\left\|u(t)\right\|_2^2 + 2\int_s^t \|\nabla u\|_2^2\,d\tau \leqslant \left\|u(s)\right\|_2^2 \quad \textit{for } s = 0\textit{, a.e. } s > 0\textit{, and all } t \geqslant s. \tag{3.5}$$

Furthermore, there is $t_0 > 0$ so that u becomes a strong solution of (1.1) *for $t \geqslant t_0$.*

The function u given above is a strong solution for $t \geqslant t_0$; so the proof of Theorem 1 in [18] applies with minor change to show the following

Lemma 3.3. *Let $a \in L^2_\sigma(\Omega) \cap L^1(\Omega)$ and let u be the corresponding weak solution satisfying* (3.5)*. Then there exists $t_0 > 0$ such that*

$$\|\partial_t u\|_r + \left\|\partial_x^2 u\right\|_r + \|\nabla p\|_r \leqslant c(t-t_0)^{-1-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \quad (1 < \ell < r < n/2) \text{ for all } t > t_0. \tag{3.6}$$

Lemmas 3.1 and 3.3 together imply

Lemma 3.4. *Let $a \in L^2_\sigma(\Omega) \cap L^1(\Omega) \cap D_p^{1-1/s,s}$, with $n+1 = 2/s + n/p$ and $1 < p < n/(n-1)$. Let u be the weak solution satisfying* (3.5)*. Then there is a constant c' depending on u, so that*

$$\int_0^\infty \left(\|\partial_t u\|_p + \left\|\partial_x^2 u\right\|_p + \|\nabla p\|_p\right) dt \leqslant c\left(\|a\|_2^2 + \|a\|_{D_p^{1-1/s,s}}\right) + c'.$$

We conclude this section with the following, which is needed in the next section in order to deduce our assertion by applying Gronwall's inequality to approximate solutions u^ε.

Lemma 3.5. *Let $a^t = e^{-tA}a$, with $a \in L^2_\sigma(\Omega) \cap D_p^{1-1/s,s}$, $n+1 = n/p + 2/s$ and $1 < p < n/(n-1)$. If $|x|^\alpha a \in L^2(\Omega)$ for some $1 < \alpha < n/2$, then*

$$\lim_{t\to 0} \left\| |x|^\alpha \left(a^t - a\right)\right\|_2 = 0. \tag{3.7}$$

Furthermore, let p^t be the pressure associated to the Stokes flow a^t and suppose that

$$\int_{\partial\Omega} \left(T\left[a^t, p^t\right]\cdot\nu\right)(y,t)\,dS_y = 0 \quad \text{for a.e. } t > 0. \tag{3.8}$$

Then (3.7) *holds for $1 < \alpha < 1 + n/2$, provided that $|x|^\alpha a \in L^2(\Omega)$.*

Proof. We invoke the representation

$$a^t(x) = (E_t * \tilde{a})(x) + \int_0^t \int_{\partial\Omega} V(x-y, t-\tau)\cdot\left(T\left[a^t, p^t\right]\cdot\nu\right)(y,\tau)\,dS_y\,d\tau \equiv (E_t * \tilde{a})(x) + b^t,$$

where $V = (V_i)$ is defined in (2.7), and $\tilde{a} = a$ in Ω and $\tilde{a} = 0$ outside Ω. First we show that

$$\left\| |x|^\alpha (E_t * \tilde{a} - \tilde{a})\right\|_2 \to 0 \quad (t \to 0). \tag{3.9}$$

For simplicity we write $\tilde{a} = a$. Direct calculation gives

$$\begin{aligned} |x|^\alpha |E_t * a - a| &= |x|^\alpha \left| \int E_t(y)\left[a(x-y) - a(x)\right] dy \right| \\ &\leqslant c \int |y|^\alpha \left|E_t(y)\left[a(x-y) - a(x)\right]\right| dy \\ &\quad + c \int E_t(y)|x-y|^\alpha \left|a(x-y) - a(x)\right| dy \end{aligned}$$

$$= ct^{\alpha/2} \int |y/\sqrt{t}|^{\alpha} t^{-n/2} E_1(y/\sqrt{t}) \left| a(x-y) - a(x) \right| dy$$
$$+ ct^{-n/2} \int E_1(y/\sqrt{t}) |x-y|^{\alpha} \left| a(x-y) - a(x) \right| dy.$$

Integrating in x via the change of variables $z = y/\sqrt{t}$ gives

$$\left\| |x|^{\alpha} (E_t * a - a) \right\|_2 \leqslant ct^{\alpha/2} \int |z|^{\alpha} E_1(z) \left\| a(\cdot - \sqrt{t}z) - a \right\|_2 dz$$
$$+ c \int E_1(z) \left\| |\cdot - \sqrt{t}z|^{\alpha} \left(a(\cdot - \sqrt{t}z) - a \right) \right\|_2 dz.$$

The first term tends to 0 as $t \to 0$. On the other hand,

$$|x - \sqrt{t}z|^{\alpha} \left(a(x - \sqrt{t}z) - a(x) \right) = |x - \sqrt{t}z|^{\alpha} a(x - \sqrt{t}z) - |x|^{\alpha} a(x) + a(x) \left[|x|^{\alpha} - |x - \sqrt{t}z|^{\alpha} \right].$$

Hence,

$$\left\| |\cdot - \sqrt{t}z|^{\alpha} \left(a(\cdot - \sqrt{t}z) - a(\cdot) \right) \right\|_2$$
$$\leqslant \left\| |\cdot - \sqrt{t}z|^{\alpha} a(\cdot - \sqrt{t}z) - |\cdot|^{\alpha} a(\cdot) \right\|_2 + \left\| a(\cdot) \left(|\cdot|^{\alpha} - |\cdot - \sqrt{t}z|^{\alpha} \right) \right\|_2.$$

The dominated convergence theorem gives

$$\lim_{t \to 0} \int E_1(z) \left\| |\cdot - \sqrt{t}z|^{\alpha} a(\cdot - \sqrt{t}z) - |\cdot|^{\alpha} a(\cdot) \right\|_2 dz = 0.$$

Furthermore,

$$|x - \sqrt{t}z|^{\alpha} - |x|^{\alpha} = \int_0^1 \frac{d}{d\theta} |x - \theta\sqrt{t}z|^{\alpha} d\theta = -\alpha \int_0^1 |x - \theta\sqrt{t}z|^{\alpha-2} (x - \theta\sqrt{z}) \cdot \sqrt{t}z \, d\theta$$

and so

$$\left| |x - \sqrt{t}z|^{\alpha} - |x|^{\alpha} \right| \leqslant \alpha |z| \sqrt{t} \int_0^1 |x - \theta\sqrt{t}z|^{\alpha-1} d\theta \leqslant c_{\alpha} |z| \sqrt{t} |x|^{\alpha-1} + c_{\alpha} \left(\sqrt{t}|z| \right)^{\alpha}.$$

Thus,

$$\left\| a\left(|\cdot - \sqrt{t}z|^{\alpha} - |\cdot|^{\alpha} \right) \right\|_2 \leqslant c_{\alpha} \sqrt{t} |z| \left\| |x|^{\alpha-1} a \right\|_2 + c_{\alpha} \left(\sqrt{t}|z| \right)^{\alpha} \|a\|_2$$
$$\leqslant c_{\alpha} \sqrt{t} |z| \left\| |x|^{\alpha} a \right\|_2^{\frac{\alpha-1}{\alpha}} \|a\|_2^{\frac{1}{\alpha}} + c_{\alpha} \left(\sqrt{t}|z| \right)^{\alpha} \|a\|_2.$$

We conclude that

$$\int E_1(z) \left\| a(\cdot) \left(|\cdot - \sqrt{z}|^{\alpha} - |\cdot|^{\alpha} \right) \right\|_2 dz$$
$$\leqslant c_{\alpha} \sqrt{t} \left\| |x|^{\alpha} a \right\|_2^{1-1/\alpha} \|a\|_2^{1/\alpha} \int |z| E_1(z) \, dz + c_{\alpha} t^{\alpha/2} \|a\|_2 \int |z|^{\alpha} E_1(z) \, dz \to 0$$

as $t \to 0$. This proves (3.9).

Now let $\Omega_1 = \Omega \cap \{|x| > R\}$. We may assume $|x-y| \geqslant 1$ whenever $x \in \Omega_1$ and $y \in \partial\Omega$. Therefore, if $1 < \alpha < n/2$, then

$$\begin{aligned}
\big\| |x|^\alpha b^t \big\|_{L^2(\Omega_1)} &\leqslant \int_0^t \int_{\partial\Omega} \big\| |x|^\alpha V(t-\tau) \big\|_{L^2(|x|\geqslant 1)} \big| T\big[a^t, p^t\big] \big| \, dS_y \, d\tau \\
&\quad + \int_0^t \int_{\partial\Omega} \big\| V(t-\tau) \big\|_{L^2(|x|\geqslant 1)} |y|^\alpha \big| T\big[a^t, p^t\big] \big| \, dS_y \, d\tau \\
&\leqslant c \int_0^t (t-\tau+1)^{\frac{\alpha}{2}-\frac{n}{4}} \big(\big\| \partial_x^2 a^\tau \big\|_p + \big\| \nabla p^\tau \big\|_p \big)(\tau) \, d\tau \\
&\quad + c \int_0^t (t-\tau+1)^{-\frac{n}{4}} \big(\big\| \partial_x^2 a^\tau \big\|_p + \big\| \nabla p^\tau \big\|_p \big) \, d\tau \\
&\leqslant c \int_0^t \big(\big\| \partial_x^2 a^\tau \big\|_p + \big\| \nabla p^\tau \big\|_p \big) \, d\tau \to 0 \quad \text{as } t \to 0.
\end{aligned}$$

Here we have used estimate (4.5). This, together with (3.9), gives $\lim_{t\to 0} \| |x|^\alpha (e^{tA} a - a) \|_{L^2(\Omega_1)} = 0$ if $1 < \alpha < n/2$. On the other hand,

$$\limsup_{t\to 0} \big\| |x|^\alpha \big(e^{-tA} a - a\big) \big\|_{L^2(\Omega\setminus\Omega_1)} \leqslant c \lim_{t\to 0} \big\| e^{-tA} a - a \big\|_2 = 0.$$

So we have proved (3.7) for $1 < \alpha < n/2$. The case $1 < \alpha < 1 + n/2$ is treated similarly, by using

$$b^t = -\int_0^1 \int_0^t \int_{\partial\Omega} (y \cdot \nabla_x V)(x - y\theta, t-\tau) \cdot \big(T\big[a^t, p^t\big] \cdot \nu \big)(y,\tau) \, dS_y \, d\tau \, d\theta,$$

which follows from (3.8). We have

$$\begin{aligned}
\big\| |x|^\alpha b^t \big\|_{L^2(\Omega_1)} &\leqslant c \int_0^t (t-\tau+1)^{\frac{\alpha}{2}-\frac{n+2}{4}} \big(\big\| \partial_x^2 a^\tau \big\|_p + c \big\| \nabla p^\tau \big\|_p \big) \, d\tau \\
&\quad + c \int_0^t (t-\tau+1)^{-\frac{n+2}{4}} \big(\big\| \partial_x^2 a^\tau \big\|_p + \big\| \nabla p^\tau \big\|_p \big) \, d\tau \\
&\leqslant c \int_0^t \big(\big\| \partial_x^2 a^\tau \big\|_p + \big\| \nabla p^\tau \big\|_p \big) \, d\tau \to 0 \quad \text{as } t \to 0.
\end{aligned}$$

The estimate in $L^2(\Omega \setminus \Omega_1)$ is the same as in the previous case. $\square$

4. Moment estimates for weak solutions

This section establishes the estimates (Proposition 4.1 and Lemma 4.2 below) for L^2-moments of weak solutions, thereby proving Theorems 2.1 and 2.2.

Proposition 4.1. *Let $a \in L^1(\Omega) \cap L^2_\sigma(\Omega) \cap D_p^{1-1/s,s}$, $n+1 = 2/s + n/p$, and $6/5 \leqslant p < n/(n-1)$. Let u be the weak solution satisfying* (3.5). *If $|x|^\alpha a \in L^2(\Omega)$ and $1 < \alpha < n/2$, then for all $t \geqslant 0$,*

$$\big\||x|^\alpha u(t)\big\|_2^2 + \int_0^t \big\||x|^\alpha \nabla u\big\|_2^2\, d\tau \leqslant c, \tag{4.1}$$

$$\big\||x|^\beta u(t)\big\|_2 \leqslant c(1+t)^{-\frac{n(\alpha-\beta)}{4\alpha}} \quad (0 \leqslant \beta \leqslant \alpha). \tag{4.2}$$

Remark. As seen from the proof given below, the restriction $\alpha < n/2$ comes from estimates on $\||x|^{\alpha-1} p^\varepsilon\|_2$. Indeed, we have to require $2(\alpha - 1 - n + 1) + n < 0$ in estimating I_1 and I_2 in (4.4). However, the condition $\alpha < n/2$ is in general optimal in the following sense: In Section 6 we show that in general our solutions u behave like $|u(x,t)| \approx c|x|^{-n}$ for large $|x|$. So $|x|^\alpha u(x,t)$ is in $L^2(\Omega)$ only if $\alpha < n/2$.

Proof of Proposition 4.1. For brevity we write $v = u^\varepsilon$, $b = u^\varepsilon * \phi_\varepsilon$ and $p = p^\varepsilon$. We multiply (3.1) by $2|x|^{2\alpha} v$ and integrate by parts to get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |x|^{2\alpha}|v|^2\,dx + 2\int_\Omega |x|^{2\alpha}|\nabla v|^2\,dx \\
&\quad\leqslant c\int_\Omega |v|^2\big|\Delta|x|^{2\alpha}\big|\,dx + c\int_\Omega |v||\nabla v||x|^{2\alpha-1}\,dx + c\int_\Omega |v|^2|b||x|^{2\alpha-1}\,dx + c\int_\Omega |p||x|^{2\alpha-1}|v|\,dx \\
&\quad\equiv K_1 + K_2 + K_3 + K_4.
\end{aligned} \tag{4.3}$$

We estimate each term on the right-hand side of (4.3). Let $2^* = 2n/(n-2)$. By the Hölder and Sobolev inequalities, we have

$$K_1 \leqslant c\big\||x|^{\alpha-1}v\big\|_2^2 \leqslant c\big\||x|^\alpha v\big\|_2^{\frac{2(\alpha-1)}{\alpha}}\|v\|_2^{\frac{2}{\alpha}}, \qquad K_4 \leqslant c\big\||x|^\alpha v\big\|_2\big\||x|^{\alpha-1}p\big\|_2,$$

$$K_2 \leqslant c\big\||x|^{\alpha-1}v\big\|_2\big\||x|^\alpha\nabla v\big\|_2 \leqslant c\big\||x|^\alpha v\big\|_2^{\frac{\alpha-1}{\alpha}}\|v\|_2^{\frac{1}{\alpha}}\big\||x|^\alpha\nabla v\big\|_2 \leqslant \frac12\big\||x|^\alpha\nabla v\big\|_2^2 + c\big\||x|^\alpha v\big\|_2^{\frac{2(\alpha-1)}{\alpha}}\|v\|_2^{\frac{2}{\alpha}},$$

and

$$\begin{aligned}
K_3 &\leqslant c\big\||x|^\alpha v\big\|_2\big\||x|^\alpha v\big\|_{2^*}^{\frac{\alpha-1}{\alpha}}\|v\|_{2^*}^{\frac{1}{\alpha}}\|b\|_n \\
&\leqslant c\big\||x|^\alpha v\big\|_2\big\||x|^\alpha\nabla v\big\|_2^{\frac{\alpha-1}{\alpha}}\|\nabla v\|_2^{\frac{1}{\alpha}}\|b\|_2^{\frac{4-n}{2}}\|\nabla b\|_2^{\frac{n-2}{2}} \\
&\leqslant \frac12\big\||x|^\alpha\nabla v\big\|_2^2 + c\big\||x|^\alpha v\big\|_2^{\frac{2\alpha}{\alpha+1}}\|\nabla v\|_2^{\frac{2}{\alpha+1}}\|b\|_2^{\frac{\alpha(4-n)}{\alpha+1}}\|\nabla b\|_2^{\frac{\alpha(n-2)}{\alpha+1}}.
\end{aligned}$$

Here, we have applied the estimate (see [10]) $\||x|^{\alpha-1}f\|_2 \leqslant c\||x|^\alpha\nabla f\|_2$ for $\alpha - 1 > -n/2$, to get

$$\big\||x|^\alpha v\big\|_{2^*} \leqslant c\big(\big\||x|^\alpha\nabla v\big\|_2 + \big\||x|^{\alpha-1}v\big\|_2\big) \leqslant c\big\||x|^\alpha\nabla v\big\|_2.$$

In what follows we estimate $\||x|^{\alpha-1}p\|_2$, invoking the representation (see Section 6 in [18]):

$$\begin{aligned} p(x,t) &= -\int_0^1 \int_{\partial\Omega} (y\cdot\nabla\mathcal{N})(x-y\theta)\partial_\nu p\,dS_y\,d\theta \\ &\quad - \int_{\partial\Omega} \partial_\nu\mathcal{N}(x-y)p(y,t)\,dS_y + \nabla^2 \int_\Omega \mathcal{N}(x-y):(b\otimes v)\,dy \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{4.4}$$

Here, $\mathcal{N} = c_n|x|^{2-n}$ is the Newtonian potential. The term I_1 is deduced from the standard single-layer potential by the fact that $\int_{\partial\Omega}\partial_\nu p\,dS = \int_\Omega \Delta p\,dx = -\int_\Omega \partial_j b_k \partial_k v_j\,dx = 0$, which is obtained via Lemma 4.2 below.

Lemma 4.2. (i) *If $\nabla p \in L^q(\Omega)$, $1 < q < n' = n/(n-1)$, and if $\Delta p \in L^1(\Omega)$, then the normal derivative $\partial_\nu p = \nu\cdot\nabla p|_{\partial\Omega}$ makes sense in $W^{-1/q,q}(\partial\Omega)$, satisfying*

$$\langle\partial_\nu p, f|_{\partial\Omega}\rangle = \int_\Omega \nabla p\cdot\nabla f\,dx + \int_\Omega (\Delta p) f\,dx \quad \textit{for all } f\in H^{1,q'}(\Omega),$$

$$\|\partial_\nu p\|_{W^{-1/q,q}(\partial\Omega)} \leqslant c\big(\|\nabla p\|_q + \|\Delta p\|_1\big).$$

(ii) *If $\nabla p \in L^{n'}(\Omega)$ and $\Delta p = g|_\Omega$ for some $g\in\mathcal{H}^1(\mathbb{R}^n)$, then $\partial_\nu p \in W^{-1/n',n'}(\partial\Omega)$ is well defined and satisfies*

$$\langle\partial_\nu p, f|_{\partial\Omega}\rangle = \int_\Omega \nabla p\cdot\nabla f\,dx + \int_\Omega (\Delta p) f\,dx \quad \textit{for all } f\in H^{1,n}(\Omega),$$

$$\|\partial_\nu p\|_{W^{-1/n',n'}(\partial\Omega)} \leqslant c\big(\|\nabla p\|_{n'} + \inf\big\{\|g\|_{\mathcal{H}^1(\mathbb{R}^n)}\colon g|_\Omega = \Delta p\big\}\big).$$

See Proposition 2.2 and Corollary 2.3 in [20] for the proof of Lemma 4.2. We continue the proof of Proposition 4.1. Since $\alpha < n/2$, we have $|x|^{\alpha-1}\nabla\mathcal{N}\in L^2(\Omega)$. Thus, Lemma 4.2(i) implies

$$\big\||x|^{\alpha-1}I_1\big\|_{L^2(\Omega_\delta)} \leqslant c\|\partial_\nu p\|_{W^{-1/p,p}(\partial\Omega)} \leqslant c\big(\|\nabla p\|_p + \|\Delta p\|_1\big) \leqslant c\big(\|\nabla p\|_p + \|\nabla b\|_2\|\nabla v\|_2\big),$$

where $\Omega_\delta = \{x\in\Omega\colon \operatorname{dist}(x,\partial\Omega) > \delta\}$. Furthermore, we know that

$$\|f\|_{L^p(\partial\Omega)} \leqslant c\|\nabla f\|_p \quad \text{for all } f\in H^{1,p}(\Omega) \text{ and } 1<p<n. \tag{4.5}$$

See Lemma 4.1 in [18] and Lemma 2.1 in [20]. By (4.5), I_2 is estimated as

$$\big\||x|^{\alpha-1}I_2\big\|_{L^2(\Omega_\delta)} \leqslant c\|p\|_{L^p(\partial\Omega)} \leqslant c\|\nabla p\|_p.$$

Since $-n/2 < \alpha - 1 < n/2$, from the weighted estimates on singular integrals [38–40] we get $\||x|^{\alpha-1}I_3\|_2 \leqslant c\||x|^{\alpha-1}bv\|_2$. Therefore,

$$\big\||x|^{\alpha-1}p\big\|_{L^2(\Omega_\delta)} \leqslant c\big(\|\nabla p\|_p + \|\nabla b\|_2\|\nabla v\|_2 + \big\||x|^{\alpha-1}bv\big\|_2\big).$$

On the other hand, since $6/5 \leqslant p$, we have $p^* \geqslant 2$. Thus, Hölder and Sobolev inequalities yield $\||x|^{\alpha-1}p\|_{L^2(\Omega\setminus\Omega_\delta)} \leqslant c\|p\|_{p^*} \leqslant c\|\nabla p\|_p$. Hence, $\||x|^{\alpha-1}p\|_2 \leqslant c(\|\nabla p\|_p + \|\nabla b\|_2\|\nabla v\|_2 + \||x|^{\alpha-1}bv\|_2)$, and so

$$K_4 \leqslant c\big\||x|^\alpha v\big\|_2\big\||x|^{\alpha-1}bv\big\|_2 + c\big\||x|^\alpha v\big\|_2\big(\|\nabla p\|_p + \|\nabla b\|_2\|\nabla v\|_2\big).$$

The first term on the right-hand side is estimated as

$$\begin{aligned}\big\||x|^\alpha v\big\|_2\big\||x|^{\alpha-1}bv\big\|_2 &\leqslant \big\||x|^\alpha v\big\|_2\big\||x|^{\alpha-1}v\big\|_{2^*}\|b\|_n\\ &\leqslant c\big\||x|^\alpha v\big\|_2\big\||x|^\alpha v\big\|_{2^*}^{\frac{\alpha-1}{\alpha}}\|v\|_{2^*}^{\frac{1}{\alpha}}\|b\|_2^{\frac{4-n}{2}}\|\nabla b\|^{\frac{n-2}{2}}\\ &\leqslant c\big\||x|^\alpha v\big\|_2\big\||x|^\alpha \nabla v\big\|_2^{\frac{\alpha-1}{\alpha}}\|\nabla v\|_2^{\frac{1}{\alpha}}\|b\|_2^{\frac{4-n}{2}}\|\nabla b\|_2^{\frac{n-2}{2}}\\ &\leqslant \frac{1}{2}\big\||x|^\alpha\nabla v\big\|_2^2 + c\big\||x|^\alpha v\big\|_2^{\frac{2\alpha}{\alpha+1}}\|\nabla v\|_2^{\frac{2}{\alpha+1}}\|b\|_2^{\frac{\alpha(4-n)}{\alpha+1}}\|\nabla b\|_2^{\frac{\alpha(n-2)}{\alpha+1}}.\end{aligned}$$

We thus obtain

$$\begin{aligned}K_4 \leqslant{}& \frac{1}{2}\big\||x|^\alpha\nabla v\big\|_2^2 + c\big\||x|^\alpha v\big\|_2^{\frac{2\alpha}{\alpha+1}}\|\nabla v\|_2^{\frac{2}{\alpha+1}}\|b\|_2^{\frac{\alpha(4-n)}{\alpha+1}}\|\nabla b\|_2^{\frac{\alpha(n-2)}{\alpha+1}}\\ &+ c\big\||x|^\alpha v\big\|_2\big(\|\nabla p\|_p + \|\nabla b\|_2\|\nabla v\|_2\big).\end{aligned}$$

Inserting the above estimates in (4.3) gives

$$\begin{aligned}&\frac{d}{dt}\big\||x|^\alpha v\big\|_2^2 + \big\||x|^\alpha\nabla v\big\|_2^2\\ &\quad\leqslant c\big\||x|^\alpha v\big\|_2^{\frac{2\alpha}{\alpha+1}}\|\nabla v\|_2^{\frac{2}{\alpha+1}}\|b\|_2^{\frac{\alpha(4-n)}{\alpha+1}}\|\nabla b\|_2^{\frac{\alpha(n-2)}{\alpha+1}}\\ &\qquad + c\Big(\big\||x|^\alpha v\big\|_2^{\frac{2(\alpha-1)}{\alpha}}\|v\|_2^{\frac{2}{\alpha}} + \big\||x|^\alpha v\big\|_2\big(\|\nabla p\|_p + \|\nabla b\|_2\|\nabla v\|_2\big)\Big).\end{aligned}\tag{4.6}$$

Here, we define

$$\begin{aligned}F_\varepsilon(t) &\equiv \|\nabla v\|_2^{\frac{2}{\alpha+1}}\|b\|_2^{\frac{\alpha(4-n)}{\alpha+1}}\|\nabla b\|_2^{\frac{\alpha(n-2)}{\alpha+1}} + \|v\|_2^{\frac{2}{\alpha}} + \|\nabla p\|_p + \|\nabla b\|_2\|\nabla v\|_2,\\ Y_\varepsilon(t) &\equiv 1 + \big\||x|^\alpha u^\varepsilon(t)\big\|_2^2 = 1 + \big\||x|^\alpha v(t)\big\|_2^2.\end{aligned}$$

Since $\||x|^\alpha v\|_2^{\frac{2\alpha}{\alpha+1}} + \||x|^\alpha v\|_2^{\frac{2(\alpha-1)}{\alpha}} + \||x|^\alpha v\|_2 \leqslant cY_\varepsilon(t)$ with $c > 0$ independent of ε, (4.6) gives

$$\frac{d}{dt}Y_\varepsilon(t) + \big\||x|^\alpha\nabla u^\varepsilon(t)\big\|_2^2 \leqslant cF_\varepsilon(t)Y_\varepsilon(t).$$

Lemma 3.1, (3.2), (3.3) together imply $F_\varepsilon \in L^1(0,T)$ for each fixed $0 < T < \infty$ and

$$\int_0^T F_\varepsilon\, d\tau \leqslant c_T$$

with $c_T > 0$ independent of ε. Via Gronwall's lemma and Lemma 3.5, we conclude that

$$\big\||x|^{\alpha}v(t)\big\|_2^2+\int_0^t\big\||x|^{\alpha}\nabla v\big\|_2^2\,d\tau\leqslant c_T\quad\big(t\in(0,T]\big),$$

with $c_T>0$ independent of ε. Passing to the limit as $\varepsilon\to 0$ yields

$$\big\||x|^{\alpha}u(t)\big\|_2^2+\int_0^t\big\||x|^{\alpha}\nabla u\big\|_2^2\,d\tau\leqslant c_T\quad\big(t\in(0,T]\big).\tag{4.7}$$

We next invoke Lemma 3.3 to see that $\int_{2t_0}^{\infty}\|\nabla p\|_p\,d\tau<\infty$. Since u is a strong solution for $t>t_0$, defining $Y(t)=1+\||x|^{\alpha}u(t)\|_2^2$ and

$$F(t)=\|\nabla u\|_2^{\frac{2+\alpha(n-2)}{\alpha+1}}\|u\|_2^{\frac{\alpha(4-n)}{\alpha+1}}+\|u\|_2^{\frac{2}{\alpha}}+\|\nabla p\|_p+\|\nabla u\|_2^2,\quad t>t_0,$$

we obtain $dY/dt\leqslant cFY$ and $\int_{2t_0}^{\infty}F\,d\tau<\infty$. Hence, the foregoing argument applies to obtain

$$\big\||x|^{\alpha}u(t)\big\|_2^2+\int_{2t_0}^t\big\||x|^{\alpha}\nabla u\big\|_2^2\,d\tau\leqslant c\quad\text{for all } t>2t_0.$$

This, together with (4.7) for $t=T=2t_0$, gives (4.1). On the other hand, since

$$\big\||x|^{\beta}u(t)\big\|_2\leqslant\big\||x|^{\alpha}u(t)\big\|_2^{\frac{\beta}{\alpha}}\big\|u(t)\big\|_2^{\frac{\alpha-\beta}{\alpha}},$$

we see that (4.2) follows from (3.3). This completes the proof of Proposition 4.1. □

In the proof of (4.1), the restriction $\alpha<n/2$ results from the estimates on I_1 and I_2 in (4.4). To estimate I_3, we need only require $\alpha-1<n(1-1/2)$, i.e., $\alpha<1+n/2$. So we can reasonably expect the improvement of the order of the moments, assuming that the total net force vanishes. In the result below, we assume that Ω is invariant under the reflection $x\mapsto -x$.

Proposition 4.3. *Let $a\in L^1(\Omega)\cap L^2_\sigma(\Omega)\cap D_p^{1-1/s,s}$, $n+1=2/s+n/p$, and $6/5\leqslant p<n/(n-1)$. Suppose further that $a(-x)=-a(x)$ and $|x|^{\alpha}a\in L^2(\Omega)$ for some $1<\alpha<1+n/2$. Then the weak solution u given in Proposition* 4.1 *satisfies*

$$\big\||x|^{\alpha}u(t)\big\|_2^2+\int_0^t\big\||x|^{\alpha}\nabla u\big\|_2^2\,d\tau\leqslant c,\qquad\big\||x|^{\beta}u(t)\big\|_2\leqslant c(1+t)^{-\frac{n(\alpha-\beta)}{4\alpha}}\quad(0\leqslant\beta\leqslant\alpha),\tag{4.8}$$

for all $t\geqslant 0$.

Proof. In the proof of Proposition 4.1, we have only to replace the estimates on I_1 and I_2 by new ones. The assumption $a(-x)=-a(x)$ implies $u^{\varepsilon}(-x,t)=-u^{\varepsilon}(x,t)$ and $p^{\varepsilon}(-x,t)=p^{\varepsilon}(x,t)$. See [20] for the details. Thus, for each $\varepsilon>0$, we have

$$G_\varepsilon(t)\equiv\int_{\partial\Omega}\big(y\partial_\nu p^{\varepsilon}-p^{\varepsilon}\nu\big)(y,t)\,dS_y=0,$$

and this implies that, with $p = p^{\varepsilon}$,

$$I_1 + I_2 = \int_0^1 \int_0^1 \int_{\partial\Omega} \theta(\partial_\nu p)(y,t) y_k y_\ell (\partial_k \partial_\ell \mathcal{N})(x - y\theta\rho)\, dS_y\, d\theta\, d\rho$$
$$- \int_0^1 \int_{\partial\Omega} p(y,t) \nu_k y_\ell (\partial_k \partial_\ell \mathcal{N})(x - y\theta)\, dS_y\, d\theta.$$

Since $1 < \alpha < 1 + n/2$, we have $|x|^{\alpha-1}\partial_k\partial_\ell \mathcal{N}(x) \in L^2(\Omega)$. So, splitting Ω as in the proof of Proposition 4.1 and applying Lemma 4.2, we get

$$\big\| |x|^{\alpha-1}(I_1 + I_2)\big\|_2 \leqslant c\big(\|\nabla p\|_p + \|\Delta p\|_1\big) \leqslant c\big(\|\nabla p\|_p + \|\nabla b\|_2 \|\nabla v\|_2\big).$$

The estimate on I_3 is the same as before. We thus see that if $1 < \alpha < 1 + n/2$, then

$$\big\| |x|^{\alpha-1} p\big\|_2 \leqslant c\big(\|\nabla p\|_p + \|\nabla b\|_2 \|\nabla v\|_2 + \big\| |x|^{\alpha-1} b v\big\|_2\big).$$

Using this, we can prove (4.8). This completes the proof of Proposition 4.3. □

5. L^q-estimates for strong solutions

We prove the desired L^q-estimates for strong solutions u which imply Theorem 2.3. Let $V = (V_{jk})$ be the functions defined in (2.7), i.e.,

$$V_{jk}(x,t) = E_t(x)\delta_{jk} + \partial_j \partial_k (\mathcal{N} * E_t)(x), \quad E_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \tag{5.1}$$

where $\mathcal{N}$ is the Newtonian potential. Furthermore, let $\nu = (\nu_1, \ldots, \nu_n)$ be the unit outward normal to $\partial\Omega$ and let

$$T[u,p] = (\partial_j u_i + \partial_i u_j - \delta_{ij} p)_{i,j=1}^n$$

denote the stress tensor. We denote by dS the surface element on $\partial\Omega$. By (4.5) and Hölder's inequality, we have

$$\int_{\partial\Omega} \big|T[u,p](y,t)\big|\, dS_y \leqslant c\big(\big\|\partial^2 u\big\|_r + \|\nabla p\|_r\big) \quad (1 < r < n). \tag{5.2}$$

By (3.6) and (5.2), the boundary integral

$$\int_0^t \int_{\partial\Omega} V(x-y, t-\tau) \cdot \big(T[u,p] \cdot \nu\big)(y,\tau)\, dS_y\, d\tau$$

is well defined. Therefore, the function $u = (u_1, \ldots, u_n)$ is represented as

$$\begin{aligned} u(x,t) &= \int_\Omega E_t(x-y)a(y)\,dy \\ &\quad + \int_0^t \int_{\partial\Omega} V(x-y,t-\tau)\cdot\big(T[u,p]\cdot\nu\big)(y,\tau)\,dS_y\,d\tau \\ &\quad - \left(\int_0^{t/2} + \int_{t/2}^t\right)\int_\Omega V(x-y,t-\tau)\cdot(u\cdot\nabla u)(y,\tau)\,dy\,d\tau \\ &\equiv I_1+I_2+I_3+I_4. \end{aligned} \tag{5.3}$$

The result below improves Lemma 3.3.

Lemma 5.1. *Let $a \in L^1(\Omega)\cap L^n_\sigma(\Omega)\cap D_p^{1-1/s,s}$ with $2/s+n/p=n+1$ for some $1<p<n/(n-1)$. There is $\lambda>0$ so that if $\|a\|_n \leqslant \lambda$, then*

$$\begin{gathered} \|u\|_r \leqslant ct^{-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \quad \big(1\leqslant \ell \leqslant \min\{n,r\}\leqslant\infty,\ r>1\big), \\ \|\nabla u\|_r \leqslant ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \quad (1\leqslant \ell\leqslant r\leqslant n), \\ \big\|\partial_x^2 u\big\|_r + \|\partial_t u\|_r + \|\nabla p\|_r \leqslant ct^{-1-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \quad (1\leqslant \ell\leqslant r\leqslant n/2,\ r>1), \end{gathered} \tag{5.4}$$

for all $t>0$.

The proof of Lemma 5.1 is similar to that of Theorem 1 in [18], so omitted here.

Proposition 5.2. *Under the assumptions of Lemma* 5.1, *we have*

$$\big\|\nabla u(t)\big\|_{L^r(\Omega_\delta)} \leqslant ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})} + c_\delta t^{-\frac{n}{2}} \quad (n<r<\infty), \tag{5.5}$$

for all $t>0$.

Proof. First we observe that

$$\big|\partial_x^m V(x,t)\big| \leqslant c\big(|x|^2+t\big)^{-\frac{m+n}{2}} \quad (m=0,1,2,\ldots), \tag{5.6}$$

with $c>0$ depending only on m. To prove (5.6), note that (5.1) can be rewritten in the form

$$V_{jk}(x,t) = E_t(x)\delta_{jk} + \int_0^\infty \partial_j\partial_k E_{s+t}(x)\,ds$$

as seen via the Fourier transform. It is easy to see that $|\partial_x^m E_t(x)| \leqslant c_m(|x|^2+t)^{-\frac{n+m}{2}}$; so we need only prove

$$\left|\int_0^\infty \partial_x^{m+2} E_{t+s}(x)\,ds\right| \leqslant c_m\big(|x|^2+t\big)^{-\frac{n+m}{2}}.$$

Direct calculation using the change of variable $s=\sigma(t+|x|^2)$ gives

$$\left|\int_0^\infty \partial_x^{m+2} E_{s+t}(x)\,ds\right| \leqslant c_m \int_0^\infty \left(|x|^2+(s+t)\right)^{-\frac{n+m+2}{2}}\,ds = c_m'\left(|x|^2+t\right)^{-\frac{n+m}{2}}$$

and this implies (5.6).

Now we can prove (5.5), using (5.3) and (5.6). It is easy to see that

$$\|\nabla I_1\|_r \leqslant c\|a\|_1 t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})} \quad (1<r\leqslant\infty) \text{ for all } t>0.$$

We write

$$I_2 = \left(\int_0^{t/2} + \int_{t/2}^t\right)\int_{\partial\Omega} V(x-y,t-\tau)\cdot\left(T[u,p]\cdot\nu\right)(y,\tau)\,dS_y\,d\tau \equiv I_{21}+I_{22}. \tag{5.7}$$

By (5.2), (5.4) and Lemma 3.4, we deduce

$$\begin{aligned}\|\nabla I_{21}\|_r &\leqslant c\int_0^{t/2}(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\left\|T[u,p]\right\|_{L^p(\partial\Omega)}\,d\tau\\ &\leqslant c\left(\int_0^\infty\left(\left\|\partial^2 u\right\|_p+\|\nabla p\|_p\right)d\tau\right)t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}.\end{aligned}$$

Applying (5.6) and Lemma 5.1, we get

$$\begin{aligned}\|\nabla I_{22}\|_{L^r(\Omega_\delta)} &\leqslant c\left\|\int_{t/2}^t\int_{\partial\Omega}\left|\nabla V(x-y,t-\tau)\right|\left|T[u,p](y,\tau)\right|dS_y\,d\tau\right\|_{L^r(\Omega_\delta)}\\ &\leqslant c\int_{t/2}^t\left(t-\tau+\delta^2\right)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\left(\left\|\partial^2 u\right\|_{n/2}+\|\nabla p\|_{n/2}\right)d\tau\\ &\leqslant ct^{-\frac{n}{2}}\int_{t/2}^t\left(t-\tau+\delta^2\right)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\,d\tau \leqslant c_\delta t^{-\frac{n}{2}}.\end{aligned}$$

On the other hand, since $\tilde{u}\cdot\nabla\tilde{u}\in\mathcal{H}^1(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\|\nabla I_3\|_r &\leqslant c\int_0^{t/2}(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\left\|\tilde{u}\cdot\nabla\tilde{u}(\tau)\right\|_{\mathcal{H}^1(\mathbb{R}^n)}\,d\tau\\ &\leqslant ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\int_0^t\left\|u(\tau)\right\|_2\left\|\nabla u(\tau)\right\|_2\,d\tau\\ &\leqslant ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}.\end{aligned}$$

Furthermore, since $a\in L^n_\sigma(\Omega)$, we have $\|\nabla u(\tau)\|_n\leqslant c\tau^{-\frac{1}{2}}$. Hence, if $n<r<\infty$, then

$$\|\nabla I_4\|_r \leqslant c\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\big\|u(\tau)\big\|_\infty\big\|\nabla u(\tau)\big\|_n d\tau$$

$$\leqslant c\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\tau^{-\frac{1}{2}-\frac{n}{2}}\,d\tau \leqslant ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}.$$

Collecting terms gives (5.5). The proof is complete. □

6. Weighted L^q-estimates for strong solutions

This section establishes the weighted L^q-estimates for our strong solutions and proves Theorem 2.4. We begin with

Lemma 6.1. *Let $1 \leqslant r < \infty$, $\alpha = n(1-1/r)$ and let $V = (V_{jk})$ be defined by (5.1). Then,*

$$\big\||\cdot|^\alpha \nabla^j V(\cdot,t)\big\|_q \leqslant ct^{\frac{\alpha}{2}-\frac{j}{2}-\frac{n}{2}(1-\frac{1}{q})} = ct^{-\frac{j}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \quad \textit{if } \frac{n}{q}-\frac{n}{r} < j,\ j=0,1.$$

The same estimates hold also in case $r = q = \infty$.

Proof. The result follows immediately from (5.6), and the details are omitted. □

The main result in this section is the following

Proposition 6.2. *Let $a \in L^1(\Omega)\cap L^n_\sigma(\Omega)\cap D_p^{1-1/s,s}$ with $2/s+n/p = n+1$ and $1 < p < n/(n-1)$. Suppose $|x|^\alpha a \in L^r(\Omega)$ with $\alpha = n(1-1/r)$ for some $1 \leqslant r < \infty$. Then there is a constant $\lambda_1 > 0$ such that if $\|a\|_n \leqslant \lambda_1$, then there is $c > 0$ so that*

$$\big\||x|^\alpha u(t)\big\|_q \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \quad \big(\max\{r, n/(n-1)\} < q \leqslant \infty\big)\ \textit{for all } t > 0. \tag{6.1}$$

Proof. First let $\|a\|_n \leqslant \lambda$ so that we can apply Lemma 5.1 and Proposition 5.2. Obviously,

$$|x|^\alpha \leqslant 2^{\frac{\alpha}{2}}\big(|y|^\alpha + |x-y|^\alpha\big), \quad \alpha \geqslant 0. \tag{6.2}$$

Hereafter I_k $(k = 1,2,3,4)$ denotes the terms given in (5.3). If $q > \max\{r, n/(n-1)\}$, then

$$\begin{aligned}\big\||x|^\alpha I_1\big\|_q &\leqslant c\bigg\|\int_\Omega E_t(x-y)|y|^\alpha a(y)\,dy\bigg\|_q + c\bigg\|\int_\Omega |x-y|^\alpha E_t(x-y)a(y)\,dy\bigg\|_q \\ &\leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\big\||y|^\alpha a\big\|_r + ct^{-\frac{n}{2}(1-\frac{1}{q})+\frac{\alpha}{2}}\|a\|_1 \\ &= c\big(\big\||y|^\alpha a\big\|_r + \|a\|_1\big)t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}.\end{aligned} \tag{6.3}$$

Let I_{21} and I_{22} be defined in (5.7). Since we may assume $|x-y| \geqslant 1$ whenever $x \in \Omega_\delta$ and $y \in \partial\Omega$, applying Lemma 3.4, (6.2) and (5.6) with $m = 0$ gives

$$\begin{aligned}\big\||x|^\alpha I_{21}\big\|_{L^q(\Omega_\delta)} &\leqslant c\int_0^{t/2}(t-\tau+1)^{-\frac{n}{2}(1-\frac{1}{q})}\big\|T[u,p]\big\|_{L^1(\partial\Omega)}(\tau)\,d\tau\\ &\quad + c\int_0^{t/2}(t-\tau+1)^{\frac{\alpha}{2}-\frac{n}{2}(1-\frac{1}{q})}\big\|T[u,p]\big\|_{L^1(\partial\Omega)}(\tau)\,d\tau\\ &\leqslant ct^{\frac{\alpha}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_0^t\big(\big\|\partial^2 u\big\|_p+\|\nabla p\|_p\big)(\tau)\,d\tau \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\end{aligned}$$

for all $t>0$. We next apply Lemma 5.1 and (6.2) to estimate I_{22} as

$$\begin{aligned}\big\||x|^\alpha I_{22}\big\|_{L^q(\Omega_\delta)} &\leqslant c\int_{t/2}^{t}(t-\tau+1)^{-\frac{n}{2}(1-\frac{1}{q})}\big\|T[u,p]\big\|_{L^1(\partial\Omega)}(\tau)\,d\tau\\ &\quad + c\int_{t/2}^{t}(t-\tau+1)^{\frac{\alpha}{2}-\frac{n}{2}(1-\frac{1}{q})}\big\|T[u,p]\big\|_{L^1(\partial\Omega)}(\tau)\,d\tau\\ &\leqslant c\int_{t/2}^{t}(t-\tau+1)^{-\frac{n}{2}(1-\frac{1}{q})}\big(\big\|\partial^2 u\big\|_{n/2}+\|\nabla p\|_{n/2}\big)(\tau)\,d\tau\\ &\quad + c\int_{t/2}^{t}(t-\tau+1)^{\frac{\alpha}{2}-\frac{n}{2}(1-\frac{1}{q})}\big(\big\|\partial^2 u\big\|_{n/2}+\|\nabla p\|_{n/2}\big)(\tau)\,d\tau\\ &\leqslant c\int_{t/2}^{t}(t-\tau+1)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\big(\big\|\partial^2 u\big\|_{n/2}+\|\nabla p\|_{n/2}\big)(\tau)\,d\tau\\ &\leqslant ct^{-\frac{n}{2}}\int_{t/2}^{t}(t-\tau+1)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\,d\tau.\end{aligned}$$

Suppose first $-\beta-1+n/2\leqslant 0$ and $\beta<1$, where $\beta=(n/2)(1/r-1/q)$. Since $n/2>1$, we have

$$\begin{aligned}\int_{t/2}^{t}(t-\tau+1)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\,d\tau &\leqslant \int_{t/2}^{t}(t-\tau+1)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})+\frac{n}{2}-1}\,d\tau\\ &\leqslant \int_{t/2}^{t}(t-\tau)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-1+\frac{n}{2}}\,d\tau \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})+\frac{n}{2}}.\end{aligned}$$

Thus, in this case we get $\||x|^\alpha I_{22}\|_{L^q(\Omega_\delta)}\leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}$ for all $t>0$.

If $\beta>1$, then $\int_{t/2}^t(t-\tau+1)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\,d\tau\leqslant c$. Therefore, replacing $\|\partial^2 u\|_{n/2}+\|\nabla p\|_{n/2}$ by $\|\partial^2 u\|_\ell+\|\nabla p\|_\ell$, with a suitable $\ell\in(1,n/2]$, gives $\||x|^\alpha I_{22}\|_{L^q(\Omega_\delta)}\leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}$ for all $t>0$.

If $\beta=1$, the foregoing estimate gives $\||x|^\alpha I_{22}\|_{L^q(\Omega_\delta)}\leqslant ct^{-\frac{n}{2}}\log(e+t)\leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}$.

If $-\beta - 1 + n/2 > 0$, then $\beta < -1 + n/2 \leqslant 1$. Using this, we get

$$\begin{aligned}\big\||x|^\alpha I_{22}\big\|_{L^q(\Omega_\delta)} &\leqslant c\int_{t/2}^{t}(t-\tau+1)^{-\frac{n}{2}(1-\frac{1}{q})}\big(\big\|\partial^2 u^\varepsilon\big\|_\ell + \big\|\nabla p^\varepsilon\big\|_\ell\big)(\tau)\,d\tau \\ &\quad + c\int_{t/2}^{t}(t-\tau+1)^{\frac{\alpha}{2}-\frac{n}{2}(1-\frac{1}{q})}\big(\big\|\partial^2 u^\varepsilon\big\|_\ell + \big\|\nabla p^\varepsilon\big\|_\ell\big)(\tau)\,d\tau \\ &\leqslant c\int_{t/2}^{t}(t-\tau+1)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\big(\big\|\partial^2 u^\varepsilon\big\|_\ell + \big\|\nabla p^\varepsilon\big\|_\ell\big)(\tau)\,d\tau \\ &\leqslant ct^{-1}\int_{t/2}^{t}(t-\tau)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\,d\tau \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\end{aligned}$$

for some $\ell \in (1, n/2]$. Here, we have used $\|\partial^2 u(\tau)\|_\ell + \|\nabla p(\tau)\|_\ell \leqslant c\tau^{-1}$, which follows from (5.4) by taking $\ell = r \leqslant n/2$. We thus obtain, for all possible choices of r and q,

$$\big\||x|^\alpha I_{22}\big\|_{L^q(\Omega_\delta)} \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \quad \text{for all } t > 0.$$

Now, integration by parts gives

$$\begin{aligned}I_4 &= -\int_{t/2}^{t}\int_\Omega V(x-y, t-\tau)\cdot(u\cdot\nabla u)(y,\tau)\,dy\,d\tau \\ &= -\int_{t/2}^{t}\int_\Omega (\nabla_x V)(x-y, t-\tau) : (u\otimes u)(y,\tau)\,dy\,d\tau.\end{aligned}$$

Since $a \in L^n_\sigma(\Omega)$, (5.4) with $\ell = n$ and $r = \infty$ shows $\|u(\tau)\|_\infty \leqslant c\|a\|_n\tau^{-1/2}$. Hence,

$$\begin{aligned}\big\||x|^\alpha I_4\big\|_q &\leqslant c\int_{t/2}^{t}\big\|\nabla V(t-\tau)\big\|_1\big\||y|^\alpha u(\tau)\big\|_q\big\|u(\tau)\big\|_\infty\,d\tau \\ &\quad + c\int_{t/2}^{t}\big\||\cdot|^\alpha\nabla V(\cdot, t-\tau)\big\|_r\big\|u(\tau)\big\|_\infty\big\|u(\tau)\big\|_{qr'/(q+r')}\,d\tau \\ &\leqslant c\|a\|_n\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\big\||y|^\alpha u(\tau)\big\|_q\,d\tau \\ &\quad + c\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\,d\tau\end{aligned}$$

$$\leqslant c\|a\|_n \int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \big\| |y|^\alpha u(\tau) \big\|_q \, d\tau + ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}.$$

Here, we have used $1/q + 1/r' < 1$, which follows from $q > r$. Finally, we estimate

$$\begin{aligned} I_3 &= -\int_0^{t/2} \int_\Omega V(x-y, t-\tau) \cdot (u \cdot \nabla u)(y,\tau)\, dy\, d\tau \\ &= -\int_0^{t/2} \int_\Omega (\nabla_x V)(x-y, t-\tau) : (u \otimes u)(y,\tau)\, dy\, d\tau. \end{aligned}$$

We write

$$\begin{aligned} \big\| |x|^\alpha I_3 \big\|_q &\leqslant c \left\| \int_0^{t/2} \int_\Omega (\nabla_x V)(x-y, t-\tau) : |y|^\alpha (u \otimes u)(y,\tau)\, dy\, d\tau \right\|_q \\ &\quad + c \left\| \int_0^{t/2} \int_\Omega |x-y|^\alpha (\nabla_x V)(x-y, t-\tau) : (u \otimes u)(y,\tau)\, dy\, d\tau \right\|_q \\ &\equiv J_1 + J_2. \end{aligned}$$

By (6.2) and (3.3) with $r = 2$, we have

$$\begin{aligned} J_2 &\leqslant c \int_0^{t/2} \big\| |\cdot|^\alpha \nabla_x V(t-\tau) \big\|_q \big\| (u \otimes u)(\tau) \big\|_1 \, d\tau \\ &\leqslant c \int_0^{t/2} (t-\tau)^{\frac{\alpha}{2} - \frac{1}{2} - \frac{n}{2}(1-\frac{1}{q})} \big\| u(\tau) \big\|_2 \big\| u(\tau) \big\|_2 \, d\tau \\ &\leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \int_0^{t/2} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{n}{2}} \, d\tau \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}. \end{aligned}$$

To estimate J_1, suppose first $\beta = (n/2)(1/r - 1/q) < 1$. We apply $\|u(\tau)\|_n \leqslant c\|a\|_n$, as well as Young's inequality for convolution, to get

$$\begin{aligned} J_1 &\leqslant c \int_0^{t/2} \big\| \nabla V(t-\tau) \big\|_{n/(n-1)} \big\| u(\tau) \big\|_n \big\| |y|^\alpha u(\tau) \big\|_q \, d\tau \\ &\leqslant c\|a\|_n \int_0^{t/2} (t-\tau)^{-1} \big\| |y|^\alpha u(\tau) \big\|_q \, d\tau \leqslant c\|a\|_n t^{-1} \int_0^{t/2} \big\| |y|^\alpha u(\tau) \big\|_q \, d\tau. \end{aligned}$$

Here, condition $q > n/(n-1)$ is used to apply Young's inequality. If $r = 1$, then $\alpha = 0$, and so

$$J_1 \leqslant c \int_0^{t/2} (t-\tau)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \|u(\tau)\|_2 \|u(\tau)\|_2 \, d\tau \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}.$$

If $\beta \geqslant 1$ and $r > 1$, we take $q_1 > \max\{r, n/(n-1)\}$ so that $\beta_1 = (n/2)(1/r - 1/q_1) < 1$. Then

$$\begin{aligned} J_1 &\leqslant c \int_0^{t/2} (t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{n}+\frac{1}{q_1}-\frac{1}{q})} \|u(\tau)\|_n \| |y|^\alpha u(\tau)\|_{q_1} \, d\tau \\ &\leqslant c\|a\|_n t^{-1-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q})} \int_0^t \| |y|^\alpha u(\tau)\|_{q_1} \, d\tau \end{aligned}$$

since $\|u(\tau)\|_n \leqslant c\|a\|_n$. We thus obtain

$$\| |x|^\alpha I_3 \|_q \leqslant \begin{cases} c\|a\|_n t^{-1-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q})} \int_0^t \| |y|^\alpha u(\tau)\|_{q_1} \, d\tau + ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} & (\beta \geqslant 1,\ r > 1), \\ ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} & (r = 1), \\ c\|a\|_n t^{-1} \int_0^{t/2} \| |y|^\alpha u(\tau)\|_q \, d\tau + ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} & (\beta < 1), \end{cases} \tag{6.4}$$

with $q_1 > \max\{r, n/(n-1)\}$ such that $\beta_1 = (n/2)(1/r - 1/q_1) < 1$.

On the other hand, since $\Omega_1 \equiv \{x \in \Omega\colon\ \mathrm{dist}(x, \partial\Omega) \leqslant \delta\}$ is bounded, we get

$$\| |x|^\alpha u(t)\|_{L^q(\Omega_1)} \leqslant c\|u(t)\|_q. \tag{6.5}$$

Since we are assuming $0 \notin \overline{\Omega}$, there is a constant c_0 such that $|x| \geqslant c_0$ for all $x \in \Omega$. Then $a \in L^r(\Omega)$ because $|x|^\alpha a \in L^r(\Omega)$. If $1 \leqslant r \leqslant n$, (5.4) with $r = q$ and $\ell = r$ yields

$$\|u(t)\|_q \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \quad \text{for all } t > 0,$$

with c independent of ε. Consider next the case $r > n$. Then (5.4) with $r = \ell = n$ yields $\|\nabla u(\tau)\|_n \leqslant c\|a\|_n \tau^{-1/2}$ and $\|u(\tau)\|_n \leqslant c\|a\|_n$. Furthermore, recall that u solves the integral equation

$$\begin{aligned} u(t) &= e^{-tA}a - \int_0^t e^{-(t-\tau)A} P(u \cdot \nabla u)(\tau) \, d\tau \\ &= e^{-tA}a - \left(\int_0^{t/2} + \int_{t/2}^t \right) e^{-(t-\tau)A} P(u \cdot \nabla u)(\tau) \, d\tau. \end{aligned}$$

By (1.3) and (1.4), we get, for $q > n/(n-1)$,

$$\begin{aligned} \|u(t)\|_q \leqslant{}& ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \|a\|_r + c \int_{t/2}^t (t-\tau)^{-\frac{1}{2}} \|u(\tau)\|_q \|\nabla u(\tau)\|_n \, d\tau \\ &+ c \int_0^{t/2} (t-\tau)^{-1+\frac{n}{2q}} \|u(\tau)\|_n \|\nabla u(\tau)\|_n \, d\tau \end{aligned}$$

$$\leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|a\|_r + c\|a\|_n \int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\big\|u(\tau)\big\|_q\,d\tau$$

$$+ ct^{-1+\frac{n}{2q}} \int_0^t (1+\tau)^{-\frac{1}{2}}\,d\tau$$

$$\leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} + c\|a\|_n \int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\big\|u(\tau)\big\|_q\,d\tau.$$

Here, we have used $\int_0^t (1+\tau)^{-\frac{1}{2}}\,d\tau \leqslant c\int_0^t \tau^{-\frac{n}{2r}}\,d\tau \leqslant ct^{1-\frac{n}{2r}}$ for $r > n$. By a standard argument, we find a constant $\lambda_2 > 0$ so that if $\|a\|_n \leqslant \lambda_2$, then $\|u(t)\|_q \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}$ for all $t > 0$. This, together with (6.5), gives $\||x|^\alpha u(t)\|_{L^q(\Omega_1)} \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}$. Collecting the above estimates gives

$$\big\||x|^\alpha u(t)\big\|_q \leqslant ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} + c\|a\|_n \int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\big\||y|^\alpha u(\tau)\big\|_q\,d\tau$$
$$+ \begin{cases} c\|a\|_n t^{-1-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q})} \int_0^t \||y|^\alpha u(\tau)\|_{q_1}\,d\tau + ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} & (\beta \geqslant 1,\ r > 1), \\ ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} & (r = 1), \\ c\|a\|_n t^{-1} \int_0^{t/2} \||y|^\alpha u(\tau)\|_q\,d\tau + ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} & (\beta < 1), \end{cases} \tag{6.6}$$

for some $q_1 > \max\{r, n/(n-1)\}$ such that $\beta_1 = (n/2)(1/r - 1/q_1) < 1$. Now we deduce (6.1) as in [25] by taking $\|a\|_n$ small. Define

$$M_q(t) = t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\big\||x|^\alpha u(t)\big\|_q,$$

so that (6.6) gives

$$M_q(t) \leqslant c + c\|a\|_n \int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}} M_q(\tau)\,d\tau$$
$$+ \begin{cases} c\|a\|_n t^{-1+\frac{n}{2}(\frac{1}{r}-\frac{1}{q_1})} \int_0^t \tau^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q_1})} M_{q_1}(\tau)\,d\tau & (\beta \geqslant 1,\ r > 1), \\ c & (r = 1), \\ c\|a\|_n t^{-1+\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \int_0^t \tau^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} M_q(\tau)\,d\tau & (\beta < 1). \end{cases} \tag{6.7}$$

Suppose first $r = 1$. We note that $\sup_{0 \leqslant \tau \leqslant t} M_q(\tau)$ is finite for any $t > 0$, which can be justified rigorously following the arguments given in [22] in the case of the Cauchy problem. By (6.7) we see that the function

$$g_q(t) = \sup_{0 \leqslant \tau \leqslant t} M_q(\tau)$$

satisfies

$$g_q(t) \leqslant c_1 + c_2\|a\|_n B(1/2, 1/2) g_q(t),$$

where $B(p,q)$ is the beta function. Taking $\|a\|_n$ small, we obtain $g_q(t) \leqslant c_3$ for all $t > 0$ and this proves (6.1) in case $r = 1$. When $\beta < 1$, (6.7) gives

$$g_q(t) \leqslant c_1 + c_2\|a\|_n\big(B(1/2, 1/2) + 1\big)g_q(t),$$

and so $g_q(t) \leqslant c_3$ for all $t > 0$ if $\|a\|_n$ is taken small. This implies (6.1). When $\beta \geqslant 1$ and $r > 1$, we already know that $g_{q_1}(t) \leqslant c$ for all $t > 0$ if $\|a\|_n$ is small. So (6.7) gives

$$g_q(t) \leqslant c_1 + c_2\|a\|_n B(1/2, 1/2)g_q(t) + c_3.$$

Hence, $g_q(t) \leqslant c_4$ for all $t > 0$ if we take $\|a\|_n$ small. The proof is complete. □

7. L^1-summability of flows and weighted estimates

This section proves Theorem 2.5, i.e., we shall prove

Proposition 7.1. *Let $a \in L^1(\Omega) \cap L^n_\sigma(\Omega) \cap D_p^{1-1/s,s}$ with $2/s + n/p = n + 1$ for some $1 < p < n/(n-1)$. Suppose $|x|^\alpha a \in L^r(\Omega)$ with $\alpha = n(1 - 1/r)$ for some $1 < r < \infty$. If $\|a\|_n \leqslant \lambda_1$, the strong solution given in Proposition 6.2 satisfies*

$$\left\| |x|^\alpha \left(u(t) - V(\cdot\,,t) \cdot \int_0^t \mathcal{F}\,d\tau \right) \right\|_r \leqslant c\big(1 + t^{-\frac{n}{2}(1-\frac{1}{r})}\big) \quad \text{for all } t > 0. \tag{7.1}$$

This implies that $t^{\frac{\alpha}{2}}|x|^\alpha u \in L^\infty_{\mathrm{loc}}([0,\infty); L^r_w(\Omega))$ and that

$$t^{\frac{\alpha}{2}}|x|^\alpha u \in L^\infty_{\mathrm{loc}}\big([0,\infty); L^r(\Omega)\big) \quad \text{if and only if} \quad \mathcal{F}(t) = 0 \quad \text{for almost all } t > 0.$$

Here, L^r_w denotes the weak L^r-space.

Proof. The strong solution obtained in Proposition 6.2 is written as

$$\begin{aligned} u(x,t) &= \int_\Omega E_t(x-y)a(y)\,dy + \int_0^t \int_{\partial\Omega} V(x-y, t-\tau) \cdot \big(T[u,p] \cdot \nu\big)(y,\tau)\,dS_y\,d\tau \\ &\quad - \int_0^t \int_\Omega (\nabla V)(x-y, t-\tau) : (u \otimes u)(y,\tau)\,dy\,d\tau \\ &\equiv K_1 + K_2 + K_3. \end{aligned}$$

In the same way as in the proof of (6.4), we have

$$\big\| |x|^\alpha K_1 \big\|_r \leqslant c\big(\|a\|_1 + \big\| |y|^\alpha a \big\|_r\big) \qquad \text{for all } t > 0.$$

We estimate K_3 as

$$
\begin{aligned}
\big\| |x|^\alpha K_3 \big\|_r &\leqslant c \left\| \int_0^t \int_\Omega (\nabla V)(x-y,t-\tau) : |y|^\alpha (u \otimes u)(y,\tau)\, dy\, d\tau \right\|_r \\
&\quad + c \left\| \int_0^t \int_\Omega |x-y|^\alpha (\nabla V)(x-y,t-\tau) : (u \otimes u)(y,\tau)\, dy\, d\tau \right\|_r \\
&\leqslant c \int_0^t \big\| \nabla V(\,\cdot\,,t-\tau) \big\|_1 \big\| u(\tau) \big\|_{2r} \big\| |y|^\alpha u(\tau) \big\|_{2r}\, d\tau \\
&\quad + c \int_0^t \big\| |\cdot|^\alpha \nabla V(\,\cdot\,,t-\tau) \big\|_r \big\| u(\tau) \big\|_2^2\, d\tau.
\end{aligned}
$$

To estimate the last two terms, we invoke Lemma 6.1 and $\|u(t)\|_2 \leqslant c(1+t)^{-\frac{n}{4}}$. The first term is estimated by using

$$
\big\| u(t) \big\|_{2r} \leqslant c(1+t)^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{2r})} \quad (2r \geqslant n), \tag{7.2}
$$

which is obtained from (5.4), and

$$
\big\| u(t) \big\|_{2r} \leqslant c(1+t)^{-\frac{n}{2}(1-\frac{1}{2r})} \quad (2 < 2r < n). \tag{7.3}
$$

Using (7.2), (7.3) and the estimate (6.1) for $\| |x|^\alpha u(t) \|_{2r}$, we get

$$
\big\| |x|^\alpha K_3 \big\|_r \leqslant c \quad \text{for all } t > 0.
$$

For K_2, we write

$$
\begin{aligned}
K_2 - V(x,t) \cdot \int_0^t \mathcal{F}\, d\tau &= - \int_0^t \int_{\partial\Omega} \int_0^1 (y \cdot \nabla_x V)(x - y\theta, t-\tau) \big(T[u,p] \cdot \nu \big)(y,\tau)\, d\theta\, dS_y\, d\tau \\
&\quad - \int_0^t \int_{\partial\Omega} \int_0^1 (\tau \cdot \partial_t V)(x, t - \tau\theta) \big(T[u,p] \cdot \nu \big)(y,\tau)\, d\theta\, dS_y\, d\tau \\
&\equiv K_{21} + K_{22}.
\end{aligned}
$$

We fix δ so large that $|\nabla_x V(x - y\theta, t-\tau)| \leqslant c(|x| + \sqrt{t-\tau})^{-n-1} \leqslant c|x|^{-n-1}$, whenever $x \in \Omega_\delta$, $y \in \partial\Omega$ and $\theta \in [0,1]$. We may further assume that $|x| \geqslant 1$, whenever $x \in \Omega_\delta$. It follows that

$$
\big\| |x|^\alpha K_{21} \big\|_{L^r(\Omega_\delta)} \leqslant c \int_0^t \int_{\partial\Omega} \big| T[u,p] \big|\, dS_y\, d\tau \leqslant c \int_0^t \big(\big\| \partial_x^2 u \big\|_p + \|\nabla p\|_p \big)\, d\tau \leqslant c,
$$

by Lemma 3.4. Next, we estimate K_{22} as

$$\begin{aligned}
&\left\| |x|^{\alpha} K_{22} \right\|_{L^r(\Omega_\delta)} \\
&\quad \leqslant c \left\| \int_0^t \int_{\partial\Omega} \int_0^1 \left|(\tau \cdot \partial_\tau V)(x, t-\tau\theta)\right| |y\theta|^{\alpha} \left|T[u,p]\cdot \nu\right|(y,\tau)\, d\theta\, dS_y\, d\tau \right\|_{L^r(\Omega_\delta)} \\
&\qquad + c \left\| \int_0^t \int_{\partial\Omega} \int_0^1 |x-y\theta|^{\alpha} |\tau \cdot \partial_\tau V|(x, t-\tau\theta) \left|T[u,p]\cdot \nu\right|(y,\tau)\, d\theta\, dS_y\, d\tau \right\|_{L^r(\Omega_\delta)}.
\end{aligned}$$

Note that $|\partial_\tau V(x, t-\tau\theta)| \leqslant c(|x|^2 + t - \tau)^{-\frac{n+2}{2}}$. So

$$\left|(\tau \cdot \partial_\tau V)(x, t-\tau\theta)\right| |y\theta|^{\alpha} \leqslant C\tau \left(|x|^2 + \sqrt{t-\tau}\right)^{-\frac{n+2}{2}} |y\theta|^{\alpha} \leqslant C\left(1+\sqrt{t-\tau}\right)^{-2}$$

when $y \in \partial\Omega$, $\tau, \theta \in (0,1)$ and $|x| \geqslant \delta$ as $x \in \Omega_\delta$. We also note that $|x - y\theta| \leqslant 2|x|$ as $y \in \partial\Omega$ and $\theta \in (0,1)$. So, when $x \in \Omega_\delta$, we have

$$\begin{aligned}
|x-y\theta|^{\alpha} |\tau \cdot \partial_\tau V|(x, t-\tau\theta) &\leqslant C|x|^{\alpha} \left(|x|^2 + \sqrt{t-\tau}\right)^{-\frac{n+2}{2}} \\
&\leqslant C|x|^{-\frac{n}{r}} \left(1+\sqrt{t-\tau}\right)^{-2} \leqslant C\left(1+\sqrt{t-\tau}\right)^{-2}.
\end{aligned}$$

Then, applying for (5.2), the two terms at the right-hand side of the estimate of $\| |x|^{\alpha} K_{22} \|_{L^r(\Omega_\delta)}$ are estimated as

$$\begin{aligned}
&\leqslant c \int_0^t \left(1+\sqrt{t-\tau}\right)^{-2} \left(\left\| \partial_x^2 u \right\|_p + \|\nabla p\|_p \right) \tau\, d\tau \\
&\leqslant c \int_0^1 \left(\left\| \partial_x^2 u \right\|_p + \|\nabla p\|_p \right) d\tau + c \int_1^t \left(1+\sqrt{t-\tau}\right)^{-2} \left(\left\| \partial_x^2 u \right\|_{n/2} + \|\nabla p\|_{n/2} \right) \tau\, d\tau \\
&= c + c \int_1^t \left(1+\sqrt{t-\tau}\right)^{-2} \left(\left\| \partial_x^2 u \right\|_{n/2} + \|\nabla p\|_{n/2} \right) \tau\, d\tau,
\end{aligned}$$

by Lemma 3.4. Take $\ell = 1$ and $r = n/2$ in (5.4) to have $\|\partial_x^2 u\|_{n/2} + \|\nabla p\|_{n/2} \leqslant c\tau^{-\frac{n}{2}}$ for all $\tau > 0$. Since $1 + \sqrt{t-\tau} \geqslant 2(t-\tau)^{1/4}$ and since $\tau^{1-\frac{n}{2}} \leqslant \tau^{-1/2}$ if $\tau \geqslant 1$ and $n \geqslant 3$, we obtain

$$\int_1^t \left(1+\sqrt{t-\tau}\right)^{-2} \left(\left\| \partial_x^2 u \right\|_{n/2} + \|\nabla p\|_{n/2} \right) \tau\, d\tau \leqslant c \int_1^t (t-\tau)^{-1/2} \tau^{-1/2}\, d\tau \leqslant c.$$

Hence, $\| |x|^{\alpha} K_{22} \|_{L^r(\Omega_\delta)} \leqslant c$, and so $\| |x|^{\alpha} K_2 \|_{L^r(\Omega_\delta)} \leqslant c$ for all $t > 0$. Collecting terms gives

$$\left\| |x|^{\alpha} \left(u(t) - V(\cdot, t) \cdot \int_0^t \mathcal{F}\, d\tau \right) \right\|_{L^r(\Omega_\delta)} \leqslant c \quad (1 < r < \infty) \text{ for all } t > 0.$$

Furthermore, since $\Omega' = \Omega \setminus \Omega_\delta$ is bounded, Lemmas 3.4 and 5.1 together imply

$$\left\| |x|^{\alpha}\left(u(t) - V(\cdot,t)\cdot\int_0^t \mathcal{F}\,d\tau\right)\right\|_{L^r(\Omega')} \leqslant c\left(\|u(t)\|_r + \|V(\cdot,t)\|_r\int_0^t \left(\|\partial_x^2 u\|_p + \|\nabla p\|_p\right)d\tau\right)$$
$$\leqslant ct^{-\frac{n}{2}(1-\frac{1}{r})}$$

for $1 < r < \infty$. This proves (7.1).

To complete the proof of Proposition 7.1, we invoke the following, which will be proved below:

$$|x|^{\alpha}V(x,t) \in L^r_w(\Omega)\setminus L^r(\Omega) \quad \big(\alpha = n(1-1/r)\big). \tag{7.4}$$

By (7.1), (7.4) and the Fourier transform of V, we see that $|x|^{\alpha}u \in L^{\infty}_{\mathrm{loc}}([0,\infty); L^r(\Omega))$ if and only if $\int_0^t \mathcal{F}\,d\tau = 0$ for a.e. $t > 0$, which is equivalent to $\mathcal{F}(t) = 0$ for a.e. $t > 0$. This proves Proposition 7.1. □

Proof of (7.4). From (5.6) we easily see that $|x|^{\alpha}V \in L^r_w(\Omega)$; so we need only show $|x|^{\alpha}V \notin L^r(\Omega)$. Recall (5.1):

$$V_{jk}(x,t) = E_t(x)\delta_{jk} + \partial^2_{jk}(\mathcal{N} * E_t)(x),$$

where $\mathcal{N} = c|x|^{2-n}$ is the Newtonian potential on $\mathbb{R}^n$. It thus suffices to show that the function $W = \nabla^2(\mathcal{N} * E_t)$, which is smooth on $\mathbb{R}^n$, satisfies

$$|x|^{\alpha}W \notin L^r\big(\mathbb{R}^n\big). \tag{7.5}$$

To prove (7.5), suppose that $\||x|^{\alpha}W\|_{r,\mathbb{R}^n} < \infty$. Then the result of [10] gives

$$\big\||x|^{\alpha-2}(\mathcal{N} * E_t)\big\|_{r,\mathbb{R}^n} \leqslant c\big\||x|^{\alpha}W\big\|_{r,\mathbb{R}^n} < \infty, \tag{7.6}$$

since $\alpha - 2 + n/r = n - 2 > 0$. Now, $\mathcal{N}$ and E_t are nonnegative; so we have

$$|x|^{\alpha-2}(\mathcal{N} * E_t)(x) \geqslant c|x|^{\alpha-2}\int_{|y|\leqslant R}\mathcal{N}(x-y)E_t(y)\,dy \quad \text{for any fixed } R > 0.$$

Here we choose x so that $|x| > 2R$. Then $|y| < |x|/2$ if $|y| \leqslant R$, and so $|x-y| \leqslant c|x|$, which gives $|x-y|^{2-n} \geqslant c|x|^{2-n}$. Hence, $|x|^{\alpha-2}\int_{|y|\leqslant R}\mathcal{N}(x-y)E_t(y)\,dy \geqslant c|x|^{-\frac{n}{r}}$ whenever $|x| > 2R$, and so

$$\big\||x|^{\alpha-2}(\mathcal{N} * E_t)\big\|_{r,\mathbb{R}^n} \geqslant \big\||x|^{\alpha-2}(\mathcal{N} * E_t)\big\|_{r,\{|x|>2R\}} \geqslant c\left(\int_{|x|>2R}|x|^{-n}\,dx\right)^{1/r} = \infty,$$

contradicting (7.6). So we get (7.5); and the proof of (7.4) is complete. □

8. On initial data for flows with non-vanishing net force

This section proves Theorem 2.6. Namely, we give a class of smooth initial data $a \in L^1(\Omega) \cap L^n_{\sigma}(\Omega)$ for which the corresponding (local) strong solutions do not satisfy (1.9).

Let h_k $(k = 1,2,3)$ be the solution to the exterior Neumann problem

$$\Delta h_k = 0, \qquad \partial_{\nu}h_k|_{\partial\Omega} = -\nu_k, \qquad \big|h_k(x)\big| = O\big(|x|^{-1}\big) \quad \big(|x| \to \infty\big). \tag{8.1}$$

We first prove

Proposition 8.1. *Let u be a strong solution with the associated pressure p. Then*

$$\int_{\partial\Omega} \partial_\nu p(y_k + h_k)\, dS_y = \int_{\partial\Omega} (y_k \partial_\nu p - p\nu_k)\, dS_y - \int_{\Omega} u_i u_j \partial_{ij}^2 h_k\, dx \quad (k = 1, 2, 3). \tag{8.2}$$

Proof. Observe first that

$$-\Delta p = \partial_i u_j \partial_j u_i = \partial_i (u_j \partial_j u_i) = \partial_{ij}^2 (u_i u_j) \quad \text{in } \Omega. \tag{8.3}$$

Choose $\phi_N(x) = \phi(x/N) \in C_0^\infty(\mathbb{R}^n)$ so that $\phi_N = 1$ for $|x| \leqslant N$ and $\phi_N = 0$ for $|x| \geqslant 2N$. Integrating by parts gives

$$\begin{aligned}
\int_{\partial\Omega} \partial_\nu p(y_k + h_k)\, dS_y &= \int_{\partial\Omega} \partial_\nu p(y_k + h_k)\phi_N\, dS_y \\
&= \int_{\Omega} \big((x_k + h_k)\phi_N \Delta p - p\Delta\big[(x_k + h_k)\phi_N\big]\big)\, dx
\end{aligned}$$

for large N; and this last integral is computed, with the aid of (8.3), as follows:

$$\begin{aligned}
&= \int_{\Omega} (x_k\phi_N \Delta p - 2p\partial_k\phi_N - x_k p \Delta\phi_N)\, dx - \int_{\Omega} (h_k\phi_N \partial_i u_j \partial_j u_i + h_k p \Delta\phi_N + 2p\nabla h_k \cdot \nabla\phi_N)\, dx \\
&= \int_{\Omega} (x_k\phi_N \Delta p - 2p\partial_k\phi_N - x_k p \Delta\phi_N)\, dx - \int_{\Omega} \big(u_i u_j \partial_{ij}^2 (h_k\phi_N) + h_k p \Delta\phi_N + 2p\nabla h_k \cdot \nabla\phi_N\big)\, dx.
\end{aligned}$$

But, since $\partial_\nu \phi_N = 0$ for large N, we get

$$\begin{aligned}
\int_{\Omega} (-2p\partial_k\phi_N - x_k p\Delta\phi_N)\, dx &= \int_{\Omega} (-p\partial_k\phi_N + x_k \nabla p \cdot \nabla\phi_N)\, dx \\
&= \int_{\partial\Omega} (y_k \partial_\nu p - p\nu_k)\, dS_y - \int_{\Omega} x_k\phi_N \Delta p\, dx,
\end{aligned}$$

and so

$$\begin{aligned}
\int_{\partial\Omega} \partial_\nu p(y_k + h_k)\, dS_y &= \int_{\partial\Omega} (y_k \partial_\nu p - p\nu_k)\, dS_y \\
&\quad - \int_{\Omega} \big(h_k p \Delta\phi_N + u_i u_j \partial_{ij}^2 (h_k\phi_N) + 2p\nabla h_k \cdot \nabla\phi_N\big)\, dx.
\end{aligned} \tag{8.4}$$

Since $\int_{\partial\Omega} \nu_k\, dS_y = 0$, a classical result in potential theory [26] shows that

$$\big|\nabla^j h_k(x)\big| \leqslant c|x|^{-2-j} \quad (j = 0, 1, 2, \ldots) \text{ for large } |x|. \tag{8.5}$$

So, letting $N \to \infty$ in (8.4) gives (8.2). The proof is complete. $\square$

Proposition 8.2. *Let u be a strong solution given in Theorem* 2.3*, with associated pressure p. Then $u \in C([0,T); L^1(\Omega))$ for all $T > 0$, if and only if*

$$\int_{\partial\Omega} \partial_\nu p(y_k + h_k)\, dS_y + \int_{\Omega} u_i u_j \partial_{ij}^2 h_k\, dx = 0 \quad (k = 1, 2, 3). \tag{8.6}$$

Proof. We know by Theorem 3 of [18] that u is in $C([0,T); L^1(\Omega))$ if and only if

$$G(t) \equiv \int_{\partial\Omega} (y_k \partial_\nu p - p\nu_k)\, dS_y = 0 \quad \text{for } k = 1, 2, 3 \text{ and all } t > 0,$$

which is equivalent to (1.9). Thus, (8.6) follows from (8.2). The proof is complete. □

Lemma 8.3. *If $\Delta u \in L^1(\Omega)$ and $\nabla u \in L^q(\Omega)$ for some $q \in [1, 3/2]$, then*

$$\int_{\partial\Omega} \partial_\nu p(y_k + h_k)\, dS_y = \int_{\partial\Omega} (\partial_\nu u_k + \partial_\nu u_i \cdot \partial_i h_k)\, dS_y \quad (k = 1, 2, 3). \tag{8.7}$$

Proof. Let ϕ_N be the cut-off functions employed above. From (1.1) we get $\partial_\nu p = \Delta u \cdot \nu$ on $\partial\Omega$. Applying the divergence theorem yields

$$\int_{\partial\Omega} \partial_\nu p(y_k + h_k)\, dS_y = \int_{\partial\Omega} \Delta u \cdot \nu(y_k + h_k)\phi_N\, dS_y = \int_{\Omega} \nabla \cdot \big[(\Delta u)(x_k + h_k)\phi_N\big]\, dx.$$

Since $\nabla \cdot (\Delta u) = \Delta(\nabla \cdot u) = 0$, the last term is computed as

$$\begin{aligned}
&= \int_{\Omega} (\Delta u_k \cdot \phi_N + x_k \Delta u_i \cdot \partial_i \phi_N + \phi_N \Delta u_i \partial_i h_k + h_k \Delta u_i \cdot \partial_i \phi_N)\, dx \\
&= \int_{\partial\Omega} (\partial_\nu u_k + \partial_\nu u_i \cdot \partial_i h_k)\, dS - \int_{\Omega} (\nabla u_k \cdot \nabla \phi_N + \partial_k u_i \cdot \partial_i \phi_N + x_k \nabla u_i \cdot \nabla \partial_i \phi_N)\, dx \\
&\quad - \int_{\Omega} (\phi_N \nabla u_i \cdot \nabla \partial_i h_k + \nabla u_i \cdot \nabla \phi_N \cdot \partial_i h_k)\, dx + \int_{\Omega} h_k \Delta u_i \cdot \partial_i \phi_N\, dx.
\end{aligned}$$

Using the assumptions on u and (8.5), we get (8.7) by letting $N \to \infty$. This proves Lemma 8.3. □

From Proposition 8.2 and Lemma 8.3, we obtain

Proposition 8.4. *Let u be a strong solution. Then $u \in C([0,T); L^1(\Omega))$ if and only if*

$$\int_{\partial\Omega} (\partial_\nu u_k + \partial_\nu u_i \cdot \partial_i h_k)\, dS_y + \int_{\Omega} u_i u_j \partial_{ij}^2 h_k\, dx = 0 \quad (k = 1, 2, 3) \text{ for a.e. } t > 0.$$

Now, if $a \in L^1(\Omega) \cap D(A_2)$, there exists a unique strong solution u defined on some $[0, T)$ such that $\|u(t) - a\|_{H^{2,2}(\Omega)} \to 0$ as $t \to 0$; see [29]. Thus, if

$$\int_{\partial\Omega} (\partial_\nu a_k + \partial_\nu a_i \cdot \partial_i h_k)\, dS_y + \int_{\Omega} (a_i a_j)\partial^2_{ij} h_k\, dx \neq 0 \quad \text{for some } k \in \{1,2,3\},$$

the strong solution u is not in $C([0,T); L^1(\Omega))$. For example, suppose $a \in C^\infty_{0,\sigma}(\Omega)$ satisfies

$$\int_{\Omega} (a_i a_j)\partial^2_{ij} h_k\, dx \neq 0 \quad \text{for some } k \in \{1,2,3\}. \tag{8.8}$$

Then, the corresponding strong solution is not in $C([0,T); L^1(\Omega))$. However, we should note that we do not know if the class of initial data $a \in C^\infty_{0,\sigma}(\Omega)$ satisfying (8.8) is vacuous or not.

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