



# Nontrivial solutions for critical potential elliptic systems

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## ABSTRACT

We consider potential elliptic systems involving  $p$ -Laplace operators, critical nonlinearities and lower-order perturbations. Suitable necessary and sufficient conditions for existence of nontrivial solutions are presented. In particular, a number of results on Brezis–Nirenberg type problems are extended in a unified framework.

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## 1. Introduction and main results

In the 80s decade, Brezis and Nirenberg investigated, in the celebrated paper [6], the existence of a nontrivial solution  $u$  for the critical problem

$$\begin{cases} -\Delta u = |u|^{\frac{4}{n-2}} u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

on a bounded domain with smooth boundary  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , where  $\lambda$  is a real parameter. Since then a lot of attention has been devoted to various questions and extensions related to (1). We refer for instance to Chapter 3 of the Struwe's book [25] and references therein for an overview on the so called Brezis–Nirenberg problem.

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A possible extension has been addressed to the quasilinear problem

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2)$$

where  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$  stands for the  $p$ -Laplace operator,  $1 < p < n$  and  $p^* = np/(n-p)$  denotes the critical Sobolev exponent for the embedding of  $W_0^{1,p}(\Omega)$  into  $L^q(\Omega)$ . As is well known, the existence of a positive solution  $u$  for (2) relies strongly on the value of the parameter  $\lambda$  compared with the first eigenvalue  $\lambda_1$  corresponding to the  $p$ -Laplace operator on  $\Omega$  under Dirichlet boundary condition. In particular, let us recall that when  $p = 2$ , according to the main result of [6], problem (1) admits a positive solution  $u$  if, and only if,  $\lambda \in ]0, \lambda_1[$ , provided that  $n \geq 4$ . The result has been extended by Egnell [14], Garcia Azorero and Peral Alonso [15] and Guedda and Veron [19], who have proved that problem (2) admits a positive solution  $u$  if, and only if,  $\lambda \in ]0, \lambda_1[$ , provided that  $p > 1$  and  $n \geq p^2$ . When  $\lambda \in ]0, \lambda_1[$ , such a solution  $u$  can for instance be obtained as a least energy solution through the minimization of the functional

$$\phi(u) = \int_{\Omega} |\nabla u|^p dx + \lambda \int_{\Omega} |u|^p dx$$

constrained to the Nehari manifold  $\{u \in W_0^{1,p}(\Omega) : \int_{\Omega} |u|^{p^*} dx = 1\}$  and the fact that  $|u|$  also minimizes  $\phi$ . Moreover, it is well known that, if  $\Omega$  is star-shaped, then problem (2) has no nontrivial solution  $u$  for any  $\lambda < 0$ , see [6,14,19]. However, when  $\lambda \geq \lambda_1$ , some results on existence of a nontrivial solution  $u$  have been established in the literature, see [7,16] for  $p = 2$  and [2] and the recent work [13] for  $p \neq 2$ .

Another possible extension concerns the semilinear system

$$\begin{cases} -\Delta u = f(u) + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

where  $u = (u_1, \dots, u_k)$ ,  $\Delta u = (\Delta u_1, \dots, \Delta u_k)$ ,  $k \geq 1$ ,  $f(u) = \frac{1}{2^*} \nabla F(u)$  and  $g(u) = \frac{1}{2} \nabla G(u)$ , where  $F, G : \mathbb{R}^k \rightarrow \mathbb{R}$  are  $C^1$  functions with  $F$  positively homogeneous of degree  $2^*$  and  $G$  homogeneous of degree 2. We recall that a function  $H : \mathbb{R}^k \rightarrow \mathbb{R}$  is homogeneous of degree  $q$  when  $H(\rho t) = \rho^q H(t)$  for any  $\rho > 0$  and  $t \in \mathbb{R}^k$ . Problem (1) corresponds exactly to the choice  $k = 1$ ,  $F(t) = |t|^{2^*}$  and  $G(t) = \lambda t^2$ . More generally, problems (2) and (3) can be simultaneously recovered from the quasilinear system

$$\begin{cases} -\Delta_p u = f(u) + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

where  $u = (u_1, \dots, u_k)$ ,  $\Delta_p u = (\Delta_p u_1, \dots, \Delta_p u_k)$ ,  $f(u) = \frac{1}{p^*} \nabla F(u)$  and  $g(u) = \frac{1}{p} \nabla G(u)$ , where  $F, G : \mathbb{R}^k \rightarrow \mathbb{R}$  are  $C^1$  functions with  $F$  positively homogeneous of degree  $p^*$  and  $G$  homogeneous of degree  $p$ . Such systems are known in the literature as potential systems (or gradient systems) and  $F$  and  $G$  are called potential functions. When  $k = 1$ , problem (4) takes the form (2), since  $F$  and  $G$  become  $F(t) = |t|^{p^*}$  and  $G(t) = \lambda |t|^p$ , modulo constant factors. For  $k \geq 2$ , there are many homogeneous potential functions. Some typical examples are:

- (i)  $F(t) = |t|_q^{p^*}$ ,  $F(t) = (\pi_l(t))^{p^*/l}$ ;
- (ii)  $G(t) = |t|_q^p$ ,  $G(t) = (\pi_l(t))^{p/l}$ ,  $G(t) = |\langle At, t \rangle|^{(p-2)/2} \langle At, t \rangle$ ,

where  $|t|_q := (\sum_{i=1}^n |t_i|^q)^{1/q}$  is the Euclidean  $q$ -norm with  $q \geq 1$ ,  $\pi_l$  is the  $l$ th elementary symmetric polynomial,  $l = 1, \dots, k$ ,  $\langle \cdot, \cdot \rangle$  denotes the usual Euclidean inner product and  $A = (a_{ij})$  is a real

$k \times k$  matrix. Necessary and sufficient conditions for the existence of a nontrivial solution  $u$  of the system (3) have been discussed by various authors. In particular, Amster et al. [1] and Bartsch and Guo [3] focused on the existence problem in the situation, closely related to (1), when  $G(t) = \langle At, t \rangle$ , where  $A$  is a symmetric  $k \times k$  matrix, and, more recently, Montenegro [22] addressed the problem for general potential functions  $G$ . Dealing with system (4), sufficient conditions for the existence of a positive weak solution  $u$  (i.e. a weak solution whose coordinates are nonnegative nonzero functions) have been established in the case  $k = 2$  by De Moraes Filho and Souto [12]. A goal common to all these works is the search for non-resonance conditions on the potential function  $G$ , which extend the condition  $\lambda < \lambda_1$  in the corresponding scalar context, and some advances have been obtained in this matter. Moreover, the existence of a nontrivial solution  $u$  has been achieved in the same variational spirit as that previously described. In a precise way, one considers the minimization problem of the functional

$$\Phi_G(u) = \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} G(u) dx$$

constrained to the manifold  $\{u \in W_0^{1,p}(\Omega, \mathbb{R}^k) : \int_{\Omega} F(u) dx = 1\}$ , where  $W_0^{1,p}(\Omega, \mathbb{R}^k)$  denotes the vector Sobolev space  $W_0^{1,p}(\Omega) \times \cdots \times W_0^{1,p}(\Omega)$ . The case  $k \geq 2$  contrasts to the scalar one in the sense that the functional  $\Phi_G$  and the related manifold are, in most examples, not invariant by the map  $(u_1, \dots, u_k) \mapsto (|u_1|, \dots, |u_k|)$ . In particular, when  $k \geq 2$ , it is not clear that the existence of a minimizer for  $\Phi_G$  leads readily to a nonnegative minimizer.

In [12], the authors proved that the system (4) admits a nonnegative nontrivial solution  $u$ , provided that  $k = 2$ ,  $p > 1$ ,  $n \geq p^2$  and  $F$  and  $G$  are positive homogeneous functions on  $[0, +\infty) \times [0, +\infty) \setminus \{(0, 0)\}$  satisfying

$$(t_1, t_2) \mapsto F(t_1^{1/p^*}, t_2^{1/p^*}) \text{ is concave,} \quad (5)$$

$$D_{t_1} F(0, 1) = 0, \quad D_{t_2} F(1, 0) = 0, \quad (6)$$

$$D_{t_1} G(0, 1) > 0, \quad D_{t_2} G(1, 0) > 0, \quad (7)$$

$$\max_{|(t_1, t_2)|_p=1} G(t_1, t_2) < \lambda_1. \quad (8)$$

On the other hand, it is still meaningful to look for a nontrivial solution  $u$  of the system (4) under less stringent requirements, since some simple examples of potential functions  $F$  and  $G$  can be easily built such that one or more of the above conditions do not hold. To this end, a more careful examination of the hypotheses above must be done. First, the positivity condition on  $F$  is natural if we focus on least energy solutions, since the functional  $\Phi_G$  is to be bounded below for any  $p$ -homogeneous function  $G$ . Dealing with the function  $G$ , we can write  $G(t) = \lambda|t|^p$  whenever  $k = 1$ . In this case, the positivity of  $G$  is equivalent to  $\lambda > 0$  and the assumption (8) is equivalent to  $\lambda < \lambda_1$ , so that one gets  $\lambda \in ]0, \lambda_1[$ , regardless of the remaining assumptions. A question that arises then concerns the possibility of eliminating the conditions (5)–(7). Another important point here is that in general the function  $G$  can change sign when  $k \geq 2$  and so the positivity of  $G$  significantly restricts the available examples of  $p$ -homogeneous functions. For this reason, another question is if the positivity condition can be weakened to allow signed functions  $G$ .

In the present paper, we answer affirmatively to two of the questions raised above. In particular, we produce necessary or sufficient conditions for existence of a nontrivial solution  $u$  of the system (4), namely:

- (a) The nonnegativity of  $G$  somewhere in  $\mathbb{R}^k$  is a necessary condition for star-shaped domains  $\Omega$  as can easily be seen from Pohozaev type variational identities;

(b) The positivity of  $G$  at a certain suitable point in  $\mathbb{R}^k$  and the inequality

$$M_G := \max_{|t|_p=1} G(t) < \lambda_1$$

are sufficient conditions for arbitrary domains  $\Omega$ ;

(c) The inequality  $M_G < \lambda_1$  is a necessary condition in the context of positive solutions for a class of potential functions  $F$  and  $G$  and arbitrary domains  $\Omega$ .

The assertions (a)–(c) reveal that the number  $M_G$  plays, for  $k \geq 2$  and in some specific models of potential functions  $F$  and  $G$ , a similar role to that one taken by the parameter  $\lambda$  in problem (2) (see Section 5). For other results concerning critical variational systems involving  $p$ -Laplace operators, we refer to [10,11,20] for  $k=2$  and  $p=2$ , [4] for  $k=2$  and  $p \neq 2$ , among others.

Before we go further and state our main theorems, a little notation should be introduced. Denote by  $E_k$  the vector Sobolev space  $W_0^{1,p}(\Omega, \mathbb{R}^k) := W_0^{1,p}(\Omega) \times \cdots \times W_0^{1,p}(\Omega)$  endowed with the norm

$$\|u\|_{E_k} = \left( \int_{\Omega} |\nabla u|^p dx \right)^{1/p} := \left( \sum_{i=1}^k \|u_i\|^p \right)^{1/p}$$

for  $u = (u_1, \dots, u_k)$ , where

$$\|u_i\| = \left( \int_{\Omega} |\nabla u_i|^p dx \right)^{1/p}.$$

We also denote the usual norm on the Lebesgue space  $L^q(\Omega)$  by  $|u|_q$  and on the corresponding vector Lebesgue space  $L^q(\Omega, \mathbb{R}^k)$  by

$$\|u\|_q := \left( \sum_{i=1}^k |u_i|_q^q \right)^{1/q}.$$

For a weak solution of the system (4) we mean a map  $u \in E_k$  such that

$$\int_{\Omega} \mathcal{A}(\nabla u) \cdot \nabla \psi dx = \int_{\Omega} (f(u) + g(u)) \cdot \psi dx, \quad (9)$$

for any map  $\psi \in C_0^1(\Omega, \mathbb{R}^k) := C_0^1(\Omega) \times \cdots \times C_0^1(\Omega)$ , where  $\mathcal{A} : \mathbb{R}^{k^2} \rightarrow \mathbb{R}^{k^2}$  denotes the map

$$\mathcal{A}(t_1, \dots, t_k) := (|t_1|^{p-2}t_1, \dots, |t_k|^{p-2}t_k).$$

Assume that  $\Omega$  has smooth boundary, where for smoothness of boundary we mean of  $C^2$  class. In this case, the regularity theory for critical equations involving  $p$ -Laplace operators guarantees that every weak solution  $u \in W_0^{1,p}(\Omega)$  of the problem (2) belongs to  $C^{1,\alpha}(\overline{\Omega})$ , see [19]. An analogous claim also holds for any  $k \geq 2$ . Precisely, every weak solution  $u \in E_k$  of the system (4) belongs to  $C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^k) := C^{1,\alpha}(\overline{\Omega}) \times \cdots \times C^{1,\alpha}(\overline{\Omega})$ . For completeness, we include a proof of this fact in the next section.

We next present two non-existence theorems. A first necessary condition for the existence of a nontrivial solution  $u$  of the system (4) comes naturally from a Pohozaev type identity due to Pucci and Serrin [23] and Guedda and Véron [19] applied to star-shaped domains.

**Theorem 1.1.** Assume that  $\Omega$  has smooth boundary. Let  $F, G : \mathbb{R}^k \rightarrow \mathbb{R}$  be  $C^1$  functions with  $F$  positively homogeneous of degree  $p^*$  and  $G$  homogeneous of degree  $p$ . If  $G$  is negative on  $\mathbb{R}^k \setminus \{0\}$  and  $\Omega$  is star-shaped, then the system (4) has no nontrivial weak solution  $u$ .

Assume now that  $G$  is positive somewhere in  $\mathbb{R}^k$ , that is  $M_G > 0$ . In this case, the inequality  $M_G \geq \lambda_1$  is a necessary condition for certain potential functions  $F$  and  $G$  as shown in the following result:

**Theorem 1.2.** Assume that  $\Omega$  has smooth boundary. Let  $F, G : \mathbb{R}^k \rightarrow \mathbb{R}$  be  $C^1$  functions with  $F$  positively homogeneous of degree  $p^*$  and  $G$  even and homogeneous of degree  $p$ . Suppose moreover that  $D_{t_i} F$  is positive on  $\{t \in \mathbb{R}^k : t_j > 0, j = 1, \dots, k\}$  and  $D_{t_i} G$  is non-decreasing on  $\mathbb{R}^k$ , where for a non-decreasing function  $D_{t_i} G$  we mean that  $D_{t_i} G(s) \leq D_{t_i} G(t)$  for any  $s, t \in \mathbb{R}^k$  satisfying  $s_i \leq t_i$  for all  $i = 1, \dots, k$ . If  $M_G \geq \lambda_1$ , then the system (4) has no positive weak solution  $u$ , i.e. weak solution whose coordinates are nonnegative nonzero functions.

The key tools in the proof of the preceding theorem consist of strong maximum and comparison principles and Hopf's lemma related to  $p$ -Laplace operators.

We also have the following two main existence theorems, the first for the case  $n \geq p^2$  and second when  $n < p^2$ .

**Theorem 1.3.** Let  $p > 1, n \geq p^2, F, G : \mathbb{R}^k \rightarrow \mathbb{R}$  be  $C^1$  functions with  $F$  positively homogeneous of degree  $p^*$  and  $G$  homogeneous of degree  $p$ . If  $M_G < \lambda_1$  and  $G(t_0) > 0$  for some maximum point  $t_0$  of  $F$  on  $\mathbb{S}_p^{k-1} := \{t \in \mathbb{R}^k : |t|_p = 1\}$ , then the system (4) has a nontrivial weak solution  $u$ .

**Remark 1.1.** The conditions  $M_G < \lambda_1$  and the  $G(t_0) > 0$  assumed above correspond, in the case  $k = 1$ , to  $\lambda < \lambda_1$  and  $\lambda > 0$ , respectively. In other words, Theorem 1.3 extends completely the main existence results of [6,15].

In lower dimensions, we need the best Sobolev constant

$$K(n, p)^p := \sup \left\{ |u|_{p^*}^p : u \in W_0^{1,p}(\Omega), \int_{\Omega} |\nabla u|^p dx = 1 \right\}$$

for the embedding of  $W_0^{1,p}(\Omega)$  into  $L^{p^*}(\Omega)$ .

**Theorem 1.4.** Let  $1 < p < n, n < p^2, F, G : \mathbb{R}^n \rightarrow \mathbb{R}$  be  $C^1$  functions with  $F$  positively homogeneous of degree  $p^*$  and  $G$  homogeneous of degree  $p$ . Let  $\varphi_1 \in W_0^{1,p}(\Omega)$  be a positive eigenfunction of  $-\Delta_p$  corresponding to  $\lambda_1$  and normalized by  $|\varphi_1|_{p^*} = 1$ , and set

$$\bar{\lambda} := \lambda_1 - \left( K(n, p)^p \int_{\Omega} \varphi_1^p dx \right)^{-1}.$$

If  $M_G < \lambda_1$  and  $G(t_0) > \bar{\lambda}$  for some maximum point  $t_0$  of  $F$  on  $\mathbb{S}_p^{k-1}$ , then the system (4) has a nontrivial weak solution  $u$ .

The paper is organized into four sections. In Section 2, we exhibit the exact set of extremal maps associated to sharp Sobolev  $F$ -inequalities on to the space  $E_k$ , characterize the coercivity of the functional  $\Phi_G$  on  $E_k$  in terms of  $M_G$  and  $\lambda_1$  and, finally, state some basic results on minimization of  $\Phi_G$  and on  $C^{1,\alpha}$  regularity up to the boundary of  $\Omega$  for weak solutions of the system (4). Sections 3

and 4 are devoted to the proofs of our theorems. In Section 5, we present two typical examples which illustrate the sharpness of our assumptions.

## 2. The key vector tools

Throughout this section we assume that  $F, G : \mathbb{R}^k \rightarrow \mathbb{R}$  are  $C^1$  functions with  $F$  positively homogeneous of degree  $p^*$  and  $G$  homogeneous of degree  $p$  and that  $f(u) = \frac{1}{p^*} \nabla F(u)$  and  $g(u) = \frac{1}{p} \nabla G(u)$ .

### 2.1. The extremal set

Let  $E_k := W_0^{1,p}(\Omega, \mathbb{R}^k)$  as in the introduction. Define

$$K_F(\Omega) := \sup \left\{ \left( \int_{\Omega} F(u) dx \right)^{1/p^*} : u \in E_k, \|u\|_{E_k} = 1 \right\}.$$

By a usual scaling argument, it follows that  $K_F(\Omega) = K_F(\mathbb{R}^n)$ , where

$$K_F(\mathbb{R}^n) := \sup \left\{ \left( \int_{\mathbb{R}^n} F(u) dx \right)^{1/p^*} : u \in \mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k), \|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)} = 1 \right\}$$

with

$$\mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k) := \mathcal{D}^{1,p}(\mathbb{R}^n) \times \dots \times \mathcal{D}^{1,p}(\mathbb{R}^n)$$

and

$$\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)} := \left( \sum_{i=1}^k \int_{\mathbb{R}^n} |\nabla u_i|^p dx \right)^{1/p}.$$

In order to estimate the least energy level associated to (4), we need to find maps that realize the optimal constant  $K_F(\mathbb{R}^n)$ . The lemma below classifies these maps.

**Lemma 2.1.** *We have:*

- (a)  $K_F(\mathbb{R}^n) = M_F^{1/p^*} K(n, p)$ , where  $M_F$  is the maximum value of  $F$  on  $\mathbb{S}_p^{k-1}$ ,
- (b)  $K_F(\mathbb{R}^n)$  is achieved exactly by maps of the form  $u = t_0 v_0$ , where  $t_0 \in \mathbb{S}_p^{k-1}$  is a maximum point of  $F$  and  $v_0$  is an extremal function for  $K(n, p)$ .

Recall from Theorem 2.1 of [26] that  $K(n, p)$  is achieved by the functions

$$v_{\varepsilon}(x) = c_{p,n} \varepsilon^{(n-p)/p(p-1)} \left( \varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}} \right)^{(n-p)/p}$$

for any  $\varepsilon > 0$ ; here  $c_{p,n}$  is normalized so that  $|v_{\varepsilon}|_{p^*} = 1$ . In particular, the part (b) of the preceding lemma provides explicit extremal maps for  $K_F(\mathbb{R}^n)$ .

**Proof of Lemma 2.1.** The  $p^*$ -homogeneity of  $F$  yields

$$F(t) \leq M_F \left( \sum_{i=1}^k |t_i|^p \right)^{p^*/p}$$

for any  $t \in \mathbb{R}^k$ . Recall that  $p < p^*$  since  $1 < p < n$ . So, by Minkowski's inequality,

$$\begin{aligned} \left( \int_{\mathbb{R}^n} F(u) dx \right)^{p/p^*} &\leq M_F^{p/p^*} \left( \int_{\mathbb{R}^n} \left( \sum_{i=1}^k |u_i|^p \right)^{p^*/p} dx \right)^{p/p^*} \\ &\leq M_F^{p/p^*} \sum_{i=1}^k \left( \int_{\mathbb{R}^n} |u_i|^{p^*} dx \right)^{p/p^*} \\ &\leq M_F^{p/p^*} K(n, p)^p \sum_{i=1}^k \|u_i\|^p \\ &= M_F^{p/p^*} K(n, p)^p \|u\|_{E_k}^p, \end{aligned} \quad (10)$$

so that

$$K_F(\mathbb{R}^n) \leq M_F^{1/p^*} K(n, p).$$

Consider now the map  $u_0 = t_0 v_0 \in E_k$  where  $t_0 \in \mathbb{S}_p^{k-1}$  satisfies  $M_F = F(t_0)$  and  $v_0 \in W_0^{1,p}(\Omega)$  is an extremal function for  $K(n, p)$ . Then,

$$\begin{aligned} \left( \int_{\mathbb{R}^n} F(u_0) dx \right)^{p/p^*} &= M_F^{p/p^*} \left( \int_{\mathbb{R}^n} |v_0|^{p^*} dx \right)^{p/p^*} \\ &= M_F^{p/p^*} K(n, p)^p \|u_0\|^p \\ &= M_F^{p/p^*} K(n, p)^p \|t_0 v_0\|_{E_k}^p \\ &= M_F^{p/p^*} K(n, p)^p \|u_0\|_{E_k}^p, \end{aligned}$$

so that

$$K_F(\mathbb{R}^n) = M_F^{1/p^*} K(n, p).$$

In particular,  $K_F(\mathbb{R}^n)$  is achieved by the map  $u_0 = t_0 v_0$ . It remain to show that an arbitrary extremal map  $u$  for  $K_F(\mathbb{R}^n)$  can always be placed in this form. In fact,  $u$  satisfies (10) with equality in all steps. Moreover, the second equality corresponds exactly to Minkowski's inequality, so that there exist  $t_1 \in \mathbb{S}_p^{k-1}$  and a function  $v_1 \in W_0^{1,p}(\Omega)$  such that  $u = t_1 v_1$ . Finally, from the first equality, one deduces that  $F(t_1) = M_F$ , and, from the third one, that  $v_1$  is an extremal function for  $K(n, p)$ .  $\square$

## 2.2. Nonlinear eigenvalues and coercivity

Assume that  $G$  is positive somewhere and so then  $M_G > 0$ . Define

$$\lambda_{1,G} = \inf_{u \in E_G} \|u\|_{E_k}^p,$$

where  $E_G := \{u \in E_k : \int_{\Omega} G(u) dx = 1\}$ .

Note that  $\lambda_{1,G}$  is a positive eigenvalue of the system

$$\begin{cases} -\Delta_p u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (11)$$

Moreover,  $\lambda_{1,G}$  is the smallest positive eigenvalue of (11) as can easily be checked by multiplying the  $i$ th equation by  $u_i$ , integrating by parts and using the relation  $g(u) \cdot u = G(u)$ .

The next lemma expresses the value of  $\lambda_{1,G}$  in terms of  $M_G$  and  $\lambda_1$ .

**Lemma 2.2.** *We have*

$$\lambda_{1,G} = \frac{\lambda_1}{M_G}.$$

**Proof.** First,

$$\begin{aligned} \lambda_{1,G} &= \inf_{u \in E_G} \|u\|_{E_k}^p \geq \frac{1}{M_G} \inf_{u \in E_G} \frac{\|u\|_{E_k}^p}{\|u\|_p^p} \geq \frac{1}{M_G} \inf_{u \in E_k \setminus \{0\}} \frac{\|u\|_{E_k}^p}{\|u\|_p^p} \\ &\geq \frac{1}{M_G} \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_p^p}{|u|_p^p} = \frac{\lambda_1}{M_G}. \end{aligned}$$

Choose now  $t_0 \in \mathbb{S}_p^{k-1}$  such that  $G(t_0) = M_G$  and  $\varphi_1 \in W_0^{1,p}(\Omega)$  an eigenfunction of  $-\Delta_p$  corresponding to  $\lambda_1$ . Then,  $\theta_1 = t_0 \varphi_1 \in E_k$  and

$$\lambda_{1,G} \leq \frac{\|\theta_1\|_{E_k}^p}{\int_{\Omega} G(\theta_1) dx} = \frac{1}{G(t_0)} \frac{\|\varphi_1\|_p^p}{|\varphi_1|_p^p} = \frac{\lambda_1}{M_G},$$

so that the desired conclusion follows.  $\square$

**Remark 2.1.** From this proof, one concludes readily that maps of the form  $\theta = t_1 \varphi_1$ , where  $t_1$  is a maximum point of  $G$  on  $\mathbb{S}_p^{k-1}$  and  $\varphi_1 \in W_0^{1,p}(\Omega)$  is a principal eigenfunction of  $-\Delta_p$ , are eigenfunctions of (11) associated to  $\lambda_{1,G}$ .

Consider the functional on  $E_k$ ,

$$\Phi_G(u) = \|u\|_{E_k}^p - \int_{\Omega} G(u) dx.$$

We now characterize the coercivity of  $\Phi_G$  on  $E_k$  in terms of  $G$  and  $\lambda_1$ .

**Lemma 2.3.** *The functional  $\Phi_G$  is coercive on  $E_k$  if, and only if,  $M_G < \lambda_1$ .*



**Proof.** By Lemma 2.2, it suffices to show that coercivity of  $\Phi_G$  on  $E_k$  is equivalent to  $\lambda_{1,G} > 1$ . Assume first that  $\Phi_G$  is coercive on  $E_k$ , so that there exists a constant  $0 < a < 1$  such that

$$\Phi_G(u) = \|u\|_{E_k}^p - 1 \geq a\|u\|_{E_k}^p$$

for any  $u \in E_G$ . In other words,

$$\|u\|_{E_k}^p \geq \frac{1}{1-a}$$

for any  $u \in E_G$ , so that

$$\lambda_{1,G} \geq \frac{1}{1-a} > 1.$$

Let now  $\lambda_{1,G} > 1$  and fix  $u \in E_k$ . If

$$\int_{\Omega} G(u) dx \leq 0,$$

then

$$\Phi_G(u) \geq a\|u\|_{E_k}^p$$

for any constant  $0 < a < 1$ . Otherwise, by definition of  $\lambda_{1,G}$ , we have

$$\frac{\|u\|_{E_k}^p}{\int_{\Omega} G(u) dx} \geq \lambda_{1,G},$$

so that

$$\Phi_G(u) = \|u\|_{E_k}^p - \int_{\Omega} G(u) dx \geq \|u\|_{E_k}^p - \frac{1}{\lambda_{1,G}} \|u\|_{E_k}^p = \frac{\lambda_{1,G} - 1}{\lambda_{1,G}} \|u\|_{E_k}^p.$$

This ends the proof.  $\square$

The next subsections are rather standard and we only include them for the reader's convenience.

### 2.3. Least energy solutions

As usual, nontrivial solutions of (4) correspond, up to a scaling, to critical points of  $\Phi_G(u)$  constrained to the Nehari manifold  $E_F := \{u \in E_k : \int_{\Omega} F(u) dx = 1\}$ . A special kind of solution is the so-called least energy solution which is characterized as a minimum point of the functional  $\Phi_G$  constrained to  $E_F$  and its existence relies on the value of the least energy level of  $\Phi_G$ .

**Lemma 2.4.** *If*

$$c := \inf_{u \in E_F} \Phi_G(u) < K_F(\mathbb{R}^n)^{-p},$$

*then the constrained functional  $\Phi_G|_{E_F}$  admits a minimum point.*

**Proof.** Let  $(u_m) \subset E_F$  be a minimizing sequence for  $c$ . It is clear that  $(u_m)$  is bounded in  $E_k$ . Thus, up to a subsequence,  $u_m \rightharpoonup u$  in  $E_k$ ,  $u_m \rightarrow u$  in  $L^p(\Omega, \mathbb{R}^k)$ ,  $u_m \rightarrow u$  in  $L^{p^*}(\Omega, \mathbb{R}^k)$  and  $u_m \rightarrow u$  almost everywhere in  $\Omega$ . The weak convergence in  $E_k$  produces

$$\int_{\Omega} |\nabla u_m|^p dx = \int_{\Omega} |\nabla(u_m - u)|^p dx + \int_{\Omega} |\nabla u|^p dx + o(1) \quad (12)$$

where  $o(1) \rightarrow 0$  as  $m \rightarrow \infty$ . By a vector version of the Brézis–Lieb lemma (see [5,3]), it follows that

$$\int_{\Omega} F(u_m) dx = \int_{\Omega} F(u_m - u) dx + \int_{\Omega} F(u) dx + o(1). \quad (13)$$

By Lemma 2.1, we have

$$\left( \int_{\Omega} F(u_m - u) dx \right)^{\frac{p}{p^*}} \leq K_F(\mathbb{R}^n)^p \int_{\Omega} |\nabla(u_m - u)|^p dx. \quad (14)$$

From (12), (13) and (14), we get

$$\begin{aligned} \left( 1 - \int_{\Omega} F(u) dx \right)^{\frac{p}{p^*}} &= \left( \int_{\Omega} F(u_m - u) dx \right)^{\frac{p}{p^*}} + o(1) \\ &\leq K_F(\mathbb{R}^n)^p \int_{\Omega} |\nabla(u_m - u)|^p dx + o(1) \\ &= K_F(\mathbb{R}^n)^p \left( \int_{\Omega} |\nabla u_m|^p dx - \int_{\Omega} |\nabla u|^p dx \right) + o(1). \end{aligned}$$

Note that

$$\Phi_G(u_m) = c + o(1).$$

Since

$$\int_{\Omega} G(u_m) dx = \int_{\Omega} G(u) dx + o(1),$$

we also have

$$\int_{\Omega} |\nabla u_m|^p dx - \int_{\Omega} |\nabla u|^p dx = \Phi_G(u_m) - \Phi_G(u) + o(1).$$

Thus,

$$K_F(\mathbb{R}^n)^p \left( \int_{\Omega} |\nabla u_m|^p dx - \int_{\Omega} |\nabla u|^p dx \right) = K_F(\mathbb{R}^n)^p (c - \Phi_G(u)) + o(1),$$

so that

$$\left(1 - \int_{\Omega} F(u) dx\right)^{\frac{p}{p^*}} \leq K_F (\mathbb{R}^n)^p (c - \Phi_G(u)).$$

Using that

$$\Phi_G(u) \geq c \left( \int_{\Omega} F(u) dx \right)^{\frac{p}{p^*}},$$

we arrive at

$$\left(1 - \int_{\Omega} F(u) dx\right)^{\frac{p}{p^*}} \leq K_F (\mathbb{R}^n)^p c \left(1 - \left( \int_{\Omega} F(u) dx \right)^{\frac{p}{p^*}}\right).$$

By Fatou's lemma, we have

$$\int_{\Omega} F(u) dx \leq \liminf \int_{\Omega} F(u_m) dx = 1.$$

Using now the assumption

$$K_F (\mathbb{R}^n)^p c < 1,$$

we discover that

$$\int_{\Omega} F(u) dx = 1.$$

On the other hand,

$$\begin{aligned} \int_{\Omega} |\nabla u_m|^p dx - \int_{\Omega} |\nabla u|^p dx &= \Phi_G(u_m) - \Phi_G(u) + o(1) \\ &= c - \Phi_G(u) + o(1) \\ &\leq c - c \left( \int_{\Omega} F(u) dx \right)^{\frac{p}{p^*}} + o(1) = o(1), \end{aligned}$$

so that, by (12), we obtain

$$\int_{\Omega} |\nabla(u_m - u)|^p dx = o(1).$$

Therefore,  $u_m \rightarrow u$  in  $E_k$  and  $\Phi_G(u) = c$ .  $\square$

## 2.4. Regularity of weak solutions

The lemma below is an extension to potential systems of a classical regularity result for critical quasilinear equations of the type (2). Its proof bases on a slight adaptation to the vector setting of the proof in the scalar case due to Trudinger [29].

**Lemma 2.5.** *Let  $u \in E_k$  be a weak solution of the system (4). If  $\Omega$  has smooth boundary, then  $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^k)$ .*

**Proof.** Given real numbers  $l > 0$  and  $\beta > 1$ , consider the functions  $\tau$  and  $\sigma$  given by

$$\tau(s) = \begin{cases} s^\beta & \text{if } 0 \leq s \leq l, \\ \beta l^{\beta-1}(s-l) + l^\beta & \text{if } s > l, \end{cases}$$

$$\sigma(s) = \begin{cases} s^{(\beta-1)p+1} & \text{if } 0 \leq s \leq l, \\ ((\beta-1)p+1)l^{(\beta-1)p}(s-l) + l^{(\beta-1)p+1} & \text{if } s > l. \end{cases}$$

Let  $u = (u_1, \dots, u_k)$ . In what follows, we assume that  $u_i \geq 0$ , otherwise we perform the argument for the positive and negative parts of  $u_i$ . Also, it is convenient to choose  $\beta > 1$  such that  $\beta p \leq p^*$ . Since  $\tau$  and  $\sigma$  are Lipschitz functions, taking  $\psi = (\sigma(u_1), \dots, \sigma(u_k)) \in E_k$  as a test function in (9), we get

$$\begin{aligned} \sum_{i=1}^k \int_{\Omega} |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla \sigma(u_i) dx &= \int_{\Omega} \mathcal{A}(\nabla u) \cdot \nabla \psi dx \\ &= \int_{\Omega} h(u) \cdot \psi dx = \sum_{i=1}^k \int_{\Omega} h_i(u) \sigma(u_i) dx, \end{aligned}$$

where  $h(t) = f(t) + g(t)$ . We remark now that there exist constants  $c_0, c_1 > 0$ , independent of  $l$ , such that

$$c_0 \tau'(s)^p \leq \sigma'(s)$$

for all  $s \geq 0$  and

$$|h(t)| \leq c_1(|t|^{p^*-1} + 1)$$

for all  $t \in \mathbb{R}^k$ . So, we obtain a constant  $c_2 > 0$ , independent of  $l$ , such that

$$\begin{aligned} \int_{\Omega} |\nabla(\tau(u_i))|^p dx &= \int_{\Omega} |\nabla u_i|^p \tau'(u_i)^p dx \leq c_2 \int_{\Omega} |u|^{p^*-1} \sigma(u_i) dx + c_2 \\ &\leq c_2 \int_{\Omega} |u|^{p^*-1} \sigma(|u|) dx + c_2. \end{aligned}$$

On the other hand, there exists a constant  $c_3 > 0$ , independent of  $l$ , such that

$$s^{p-1} \sigma(s) \leq c_3 \tau(s)^p$$

for all  $s \geq 0$ . Thus, we arrive at

$$\int_{\Omega} |\nabla(\tau(u_i))|^p dx \leq c_3 c_2 \int_{\Omega} |u|^{p^*-p} \tau(|u|)^p dx + c_2.$$

Applying the classical Sobolev inequality, we derive constants  $c_4, c_5 > 0$ , independent of  $l$ , such that

$$\begin{aligned} \left( \int_{\Omega} \tau(u_i)^{p^*} dx \right)^{\frac{p}{p^*}} &\leq c_4 \int_{\Omega} |\nabla(\tau(u_i))|^p dx \\ &\leq c_5 \int_{\Omega} |u|^{p^*-p} \tau(|u|)^p dx + c_5. \end{aligned}$$

Moreover, it follows easily from the definition of  $\tau$  that there is a constant  $c_6 > 0$ , independent of  $l$ , such that

$$\tau(|u|) \leq c_6 \sum_{i=1}^k \tau(u_i).$$

Using Hölder and Young inequalities, we find a constant  $c_7 > 0$ , independent of  $l$ , such that

$$\left( \int_{\Omega} \tau(|u|)^{p^*} dx \right)^{\frac{p}{p^*}} \leq c_7 \int_{\Omega} \tau(|u|)^p dx + c_7.$$

Taking now the limit  $l \rightarrow +\infty$  in the inequality above, we find

$$\left( \int_{\Omega} |u|^{\beta p^*} dx \right)^{\frac{p}{p^*}} \leq c_7 \int_{\Omega} |u|^{\beta p} dx + c_7.$$

Therefore, we get  $|u| \in L^q(M)$  with  $q = \beta p^* > p^*$ . Standard arguments based on the well-known Moser's iterative scheme imply that  $|u| \in L^{q_0}(\Omega)$  for any  $q_0 \geq 1$  (for details we refer the reader to [17,19]) and, since  $\Omega$  has smooth boundary, the desired conclusion follows from the classical elliptic regularity theory applied to each equation of (4) (see Lieberman [21] and Tolksdorf [28]).  $\square$

### 3. Proof of Theorems 1.1 and 1.2

For the proof of Theorem 1.1 we need the following generalized Pohozaev identity due to Pucci and Serrin [23] for classical solutions and, by means of an approximation scheme, to Guedda and Véron [19] for weak solutions in  $C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^k)$ .

**Lemma 3.1.** Assume that  $\Omega$  has smooth boundary. Let  $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^k)$  be a weak solution of the system

$$\begin{cases} -\Delta_p u = h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where  $h(u) = \nabla H(u)$  for some  $H \in C^1(\mathbb{R}^k)$  with  $H(0) = 0$ . Then,

$$\int_{\partial\Omega} |\nabla u|^p x \cdot \nu(x) \, ds = np \int_{\Omega} H(u) \, dx - (n-p) \int_{\Omega} h(u) \cdot u \, dx.$$

**Proof of Theorem 1.1.** Let

$$H(u) = \frac{1}{p^*} F(u) + \frac{1}{p} G(u).$$

Since  $F$  and  $G$  are homogeneous of degree  $p^*$  and  $p$ , respectively, it follows that

$$h(u) \cdot u = F(u) + G(u).$$

Let  $u \in E_k$  be a weak solution of the system (4). By Lemma 2.5, we have  $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^k)$ . Applying now Lemma 3.1 with the function  $H$  chosen above, we derive

$$\begin{aligned} \int_{\partial\Omega} |\nabla u|^p x \cdot \nu(x) \, ds &= \left[ \frac{np}{p^*} - (n-p) \right] \int_{\Omega} F(u) \, dx + p \int_{\Omega} G(u) \, dx \\ &= p \int_{\Omega} G(u) \, dx. \end{aligned} \quad (15)$$

Note that  $x \cdot \nu(x) > 0$  on  $\partial\Omega$ , since  $\Omega$  is star-shaped. In particular, the left-hand side of (15) is always nonnegative. Thus, if  $G$  is negative on  $\mathbb{R}^k \setminus \{0\}$ , then the right-hand side of (15) is negative unless  $u = 0$  on  $\Omega$ . This concludes the proof of theorem.  $\square$

The proof of Theorem 1.2 requires strong maximum and comparison principles and Hopf's lemma related to  $p$ -Laplace operators on smooth domains.

**Proof of Theorem 1.2.** Let  $u \in E_k$  be a nonnegative nontrivial weak solution of (4). By Lemma 2.5, we have  $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^k)$ . Thanks to the assumptions of theorem, Hopf's lemma and the strong maximum principle, it follows that  $u_i > 0$  in  $\Omega$  and  $\frac{\partial}{\partial \nu} u < 0$  on  $\partial\Omega$ ,  $i = 1, \dots, k$ , see [24] or [30]. Here, it is important to emphasize that Hopf's lemma requires regularity of the boundary of  $\Omega$  and that the principal operator be uniformly elliptic which is true since each component  $u_i$  belongs to  $C^1(\overline{\Omega})$ . Let  $\theta_1 \in E_k$  be a smooth eigenfunction of (11) corresponding to  $\lambda_{1,G}$ . It is clear that  $-\theta_1$  is also an eigenfunction, since  $G$  is an even function. Thus, we can assume that, at least, one component of  $\theta_1$  is positive somewhere. In particular, the set

$$S := \{s > 0: u > s\theta_1 \text{ in } \Omega\}$$

is upper bounded and nonempty. Define

$$s_* := \sup S > 0.$$

Note that  $u \geq s_* \theta_1$  in  $\Omega$ . Since

$$0 < \lambda_{1,G} = \frac{\lambda_1}{M_G} \leq 1,$$

we discover that

$$\begin{aligned}
 -\Delta_p u + \Delta_p(s_*\theta_1) &= -\Delta_p u - s_*^{p-1}(-\Delta_p \theta_1) \\
 &= f(u) + g(u) - s_*^{p-1}\lambda_{1,G}g(\theta_1) \\
 &> g(u) - \lambda_{1,G}g(s_*\theta_1) \\
 &\geq g(u) - \lambda_{1,G}g(u) \\
 &\geq 0.
 \end{aligned}$$

Since  $|\nabla u| \neq 0$  on  $\partial\Omega$ , we can evoke the strong comparison principle (see for instance [27,9,8]), so that  $u - s_*\theta_1 > 0$  in  $\Omega$  and  $\frac{\partial}{\partial\nu}(u - s_*\theta_1) < 0$  on  $\partial\Omega$ . Therefore,  $u - (s_* + \varepsilon)\theta_1 > 0$  in  $\Omega$  for  $\varepsilon > 0$  small enough and this contradicts the definition of  $s_*$ .  $\square$

#### 4. Proof of Theorems 1.3 and 1.4

**Proof of Theorem 1.3.** Let

$$c = \inf_{u \in E_F} \Phi_G(u).$$

We first show that  $c < K_F(\mathbb{R}^n)^{-p}$ . Let  $\eta \in C_0^\infty(\Omega)$  be a cutoff function with  $\eta = 1$  in a neighborhood of 0 and  $v_\varepsilon$  be the extremal function for  $K(n, p)$  presented in Section 2. Then,  $w_\varepsilon = \eta v_\varepsilon \in W_0^{1,p}(\Omega)$  and

$$\begin{aligned}
 \|w_\varepsilon\|^p &= K(n, p)^{-p} + O(\varepsilon^{p+\frac{n-p^2}{p-1}}), \\
 |w_\varepsilon|_{p^*}^p &= 1 + O(\varepsilon^n), \\
 |w_\varepsilon|_p^p &= \begin{cases} a\varepsilon^p + O(\varepsilon^{p+\frac{n-p^2}{p-1}}), & n > p^2, \\ a\varepsilon^p |\log \varepsilon| + O(\varepsilon^p), & n = p^2, \end{cases}
 \end{aligned}$$

for some constant  $a > 0$ , see [15].

Let now  $u_\varepsilon = t_0 w_\varepsilon \in E_k$ , where  $t_0 \in \mathbb{S}_p^{k-1}$  satisfies  $F(t_0) = M_F$  and  $G(t_0) > 0$ , by hypothesis. Since  $n > p$ , for  $n > p^2$ , we have

$$\begin{aligned}
 c &\leq \frac{\Phi_G(u_\varepsilon)}{(\int_\Omega F(u_\varepsilon) dx)^{p/p^*}} \\
 &= \frac{\|u_\varepsilon\|_{E_k}^p - \int_\Omega G(u_\varepsilon) dx}{(\int_\Omega F(u_\varepsilon) dx)^{p/p^*}} \\
 &= \frac{\|w_\varepsilon\|^p - G(t_0)|w_\varepsilon|_p^p}{M_F^{p/p^*}|w_\varepsilon|_{p^*}^p} \\
 &= \frac{K(n, p)^{-p} - aG(t_0)\varepsilon^p + O(\varepsilon^{p+\frac{n-p^2}{p-1}})}{M_F^{p/p^*} + O(\varepsilon^n)} \\
 &< K(n, p)^{-p} M_F^{-p/p^*} = K_F(\mathbb{R}^n)^{-p}
 \end{aligned}$$

provided  $\varepsilon > 0$  is small enough.

For  $n = p^2$ , we also have

$$\begin{aligned}
 c &\leq \frac{\Phi_G(u_\varepsilon)}{(\int_\Omega F(u_\varepsilon) dx)^{p/p^*}} \\
 &= \frac{\|u_\varepsilon\|_{E_k}^p - \int_\Omega G(u_\varepsilon) dx}{(\int_\Omega F(u_\varepsilon) dx)^{p/p^*}} \\
 &= \frac{\|w_\varepsilon\|^p - G(t_0)|w_\varepsilon|_p^p}{M_F^{p/p^*}|w_\varepsilon|_p^p} \\
 &= \frac{K(n, p)^{-p} - aG(t_0)|\log \varepsilon| \varepsilon^p + O(\varepsilon^p)}{M_F^{p/p^*} + O(\varepsilon^n)} \\
 &< K_F(\mathbb{R}^n)^{-p}
 \end{aligned}$$

for  $\varepsilon > 0$  small.

By Lemma 2.4, the constrained functional  $\Phi_G|_{E_F}$  admits a minimizer  $u \in E_F$ . In particular, there exists a Lagrange multiplier  $\mu \in \mathbb{R}$  such that

$$-\Delta_p u = g(u) + \mu f(u) \quad \text{in } \Omega.$$

Moreover, one has

$$\mu = \Phi_G(u),$$

since

$$\int_\Omega f(u) \cdot u dx = \int_\Omega F(u) dx = 1.$$

Using now the assumption  $M_G < \lambda_1$ , from Lemma 2.3, we derive  $\mu > 0$ . Finally, one easily checks that  $\mu^{1-p^*} u$  is a nontrivial weak solution of (4).  $\square$

**Proof of Theorem 1.4.** Let  $\varphi_1 \in W_0^{1,p}(\Omega)$  be an eigenfunction of  $-\Delta_p$  corresponding to  $\lambda_1$  and normalized by  $|\varphi_1|_{p^*} = 1$ . Set  $u_0 = t_0 \varphi_1 \in E_F$ , where  $t_0 \in \mathbb{S}_p^{k-1}$  satisfies  $F(t_0) = M_F$  and  $G(t_0) > \bar{\lambda}$ , as assumed in the statement of theorem. Then,

$$\begin{aligned}
 c &\leq \frac{\Phi_G(u_0)}{(\int_\Omega F(u_0) dx)^{p/p^*}} = \frac{\|u_0\|_{E_k}^p - \int_\Omega G(u_0) dx}{M_F^{p/p^*}} \\
 &= \frac{\|\varphi_1\|^p - G(t_0)|\varphi_1|_p^p}{M_F^{p/p^*}} \\
 &= \frac{(\lambda_1 - G(t_0))|\varphi_1|_p^p}{M_F^{p/p^*}} \\
 &< K(n, p)^{-p} M_F^{-p/p^*} = K_F(\mathbb{R}^n)^{-p}.
 \end{aligned}$$

The proof of Theorem 1.4 proceeds now as that one of Theorem 1.3.  $\square$



## 5. Illustrative examples

For some kinds of potential functions  $F$  and  $G$  and domains  $\Omega$ , the condition  $M_G < \lambda_1$  is necessary and sufficient for the existence of either a nonnegative nontrivial solution or a positive solution  $u$  of the system (4) as illustrates the following model:

$$F(t) = |t|_r^{p^*}, \quad G(t) = \lambda |t|_s^p$$

with  $r, s \geq 1$ ,  $p > 1$ ,  $n \geq p^2$  and  $\lambda \in \mathbb{R}$ . Obviously, nontrivial weak solutions  $u$  do not exist if  $\lambda < 0$ , provided that  $\Omega$  is a star-shaped domain with smooth boundary. Moreover, the non-existence of nonnegative nontrivial solutions extends readily to  $\lambda = 0$ . In fact, in this case, the identity (15) reduces to

$$\int_{\partial\Omega} |\nabla u|^{p_X} \cdot \nu(x) \, ds = 0,$$

so that  $\nabla u = 0$  on  $\partial\Omega$  and so, the strong maximum principle and Hopf's lemma, both applied to each equation of (4), lead to  $u = 0$  in  $\Omega$ .

Assume now that  $\lambda > 0$ . By Theorem 1.3, one easily deduces that a nonnegative nontrivial weak solution  $u$  of (4) exists if

$$\lambda < \begin{cases} k^{1-p/s}\lambda_1 & \text{if } 1 \leq s < p, \\ \lambda_1 & \text{if } s \geq p. \end{cases} \quad (16)$$

The nonnegativity of  $u$  comes from the invariance of  $F$  and  $G$  under the map  $(t_1, \dots, t_k) \mapsto (|t_1|, \dots, |t_k|)$ . A fact that deserves mention in this specific example is that a solution  $u$  of the system (4) can be presented into an explicit form. Precisely, the positive map  $u = \mu(v, \dots, v)$  solves (4) for  $1 \leq s < p$ , where  $\mu > 0$  is such that

$$\mu^{p^*-p} k^{-1+p^*/r} = 1$$

and  $v \in C^1(\overline{\Omega})$  is a positive solution of

$$\begin{cases} -\Delta_p v = |v|^{p^*-2} v + k^{-1+p/s} \lambda |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

and nonnegative nonzero maps of the type  $u = (0, \dots, w, \dots, 0)$  solve (4) for  $s \geq p$ , where  $w \in C^1(\overline{\Omega})$  is a positive solution of

$$\begin{cases} -\Delta_p w = |w|^{p^*-2} w + \lambda |w|^{p-2} w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 1.2, we then conclude that the system (4) admits a positive solution  $u$  if, and only if,  $\lambda < k^{1-p/s}\lambda_1$ , provided that  $1 \leq s < p$ . On the other hand, the necessity of (16) is not as straightforward for existence of nonnegative nontrivial solutions when  $s \geq p$ . In fact, let  $u \in E_k$  be a nonnegative nontrivial weak solution of (4). By Lemma 2.5, we have  $u = (u_1, \dots, u_k) \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^k)$ . By the strong maximum principle, it follows that either  $u_i > 0$  in  $\Omega$  or  $u_i = 0$  in  $\Omega$ . Module a rearrangement, we can assume that  $u_i > 0$  in  $\Omega$ ,  $i = 1, \dots, m$ , and  $u_i = 0$  in  $\Omega$ ,  $i = m+1, \dots, k$ , for some  $m \in \{1, \dots, k\}$ . Let  $\tilde{u} = (u_1, \dots, u_m)$ . Define now the function  $\tilde{G} : \mathbb{R}^m \rightarrow \mathbb{R}$  by  $\tilde{G}(t) = G(t, 0)$ . Since  $s \geq p$ , one easily

checks that

$$0 < \lambda_{1,\tilde{G}} = \lambda_{1,G} = \frac{\lambda_1}{\lambda} \leq 1.$$

Let  $\tilde{\theta}_1 \in E_m$  be a smooth eigenfunction of (11) corresponding to  $\lambda_{1,\tilde{G}}$ . Without loss of generality, we can assume that  $\tilde{\theta}_1$  is a nonnegative nontrivial map. In particular, the set

$$\tilde{S} := \{s > 0: \tilde{u} > s\tilde{\theta}_1 \text{ in } \Omega\}$$

is upper bounded and nonempty. Define

$$\tilde{s}_* := \sup \tilde{S} > 0.$$

Note that  $\tilde{u} \geq \tilde{s}_*\tilde{\theta}_1$  in  $\Omega$ . Arguing now as in the proof of Theorem 1.2, we have

$$\begin{aligned} -\Delta_p \tilde{u} + \Delta_p(\tilde{s}_*\tilde{\theta}_1) &= -\Delta_p \tilde{u} - \tilde{s}_*^{p-1}(-\Delta_p \tilde{\theta}_1) \\ &= f(\tilde{u}, 0) + g(\tilde{u}, 0) - \tilde{s}_*^{p-1}\lambda_{1,\tilde{G}}g(\tilde{\theta}_1, 0) \\ &> g(\tilde{u}, 0) - \lambda_{1,\tilde{G}}g(\tilde{s}_*\tilde{\theta}_1, 0) \\ &\geq g(\tilde{u}, 0) - \lambda_{1,\tilde{G}}g(\tilde{u}, 0) \\ &\geq 0, \end{aligned}$$

and this inequality contradicts the definition of  $\tilde{s}_*$ . In conclusion, the system (4) admits a nonnegative nontrivial solution  $u$  if, and only if,  $\lambda < \lambda_1$ , provided that  $s \geq p$ .

Another canonical model, closely related to the scalar case, is given by

$$G(t) = |\langle At, t \rangle|^{(p-2)/2} \langle At, t \rangle,$$

where  $A = (a_{ij})$  a symmetric  $k \times k$  matrix. A direct inequality is

$$m_p M_G \leq \|A\|^{p/2},$$

where

$$\|A\| := \max\{|\mu_i|: \mu_i \text{ is an eigenvalue of } A\}$$

and

$$m_p := \min_{|t|_p=1} |t|_2^{-p} = \begin{cases} 1 & \text{if } 1 < p < 2, \\ k^{-p/2+1} & \text{if } p \geq 2. \end{cases}$$

In fact,

$$\begin{aligned} \|A\|^{p/2} &= \max_{|t|_2=1} |\langle At, t \rangle|^{p/2} = \max_{|t|_2=1} |G(t)| \\ &= \max_{t \neq 0} \frac{|G(t)|}{|t|_2^p} = \max_{t \neq 0} \frac{|t|_p^p |G(t)|}{|t|_2^p |t|_p^p} \geq m_p M_G. \end{aligned}$$

For a moment, let us recall the statement of the Perron–Frobenius theorem for matrices with nonnegative entries. Let  $A = (a_{ij})$  be a  $k \times k$  matrix with nonnegative entries. Then, the following statement holds (see a proof in [18]): “There exists a nonnegative number  $r$ , called the Perron root or the Perron–Frobenius eigenvalue, such that  $r$  is an eigenvalue of  $A$  and any other eigenvalue  $\lambda$  (possibly complex) is strictly smaller or equal in absolute value than  $r$ ,  $|\lambda| \leq r$ . Moreover,  $r$  admits an eigenvector with nonnegative entries”. In particular, the spectral radius of  $A$ , denoted by  $\rho(A)$ , is equal to  $r$ . If, furthermore,  $A$  is a symmetric matrix, then  $\rho(A) = \|A\|$ . In a nutshell, using the Perron–Frobenius theorem and the symmetry of  $A$ , we deduce that  $\|A\|$  is the largest eigenvalue of  $A$  whenever  $a_{ij} \geq 0$  for all  $i, j = 1, \dots, k$ . In this case, the equality  $m_p M_G = \|A\|^{p/2}$  happens provided that some eigenvector corresponding to  $\|A\|$  is a minimum point of  $m_p$ . The precise description of the minimizers of  $m_p$  is either points of the type  $(0, \dots, \pm 1, \dots, 0)$ , if  $1 < p < 2$ , or points of the type  $(\pm 1/k^{1/p}, \dots, \pm 1/k^{1/p})$ , if  $p \geq 2$ . However, in most examples the inequality becomes strict, so that the condition  $\|A\|^{p/2} < m_p \lambda_1$  is in general stronger than the one  $M_G < \lambda_1$ . In addition, one easily constructs examples of functions  $F$  positively homogeneous of degree  $p^*$  and symmetric matrices  $A$  such that  $M_G < \lambda_1$ ,  $\|A\|^{p/2} > m_p \lambda_1$ , and  $G(t_0) > 0$  in higher dimensions or  $G(t_0) > \bar{\lambda}$  in lower dimensions for some maximum point  $t_0$  of  $F$  on  $\mathbb{S}_p^{k-1}$ . As consequence, Theorems 1.3 and 1.4 can be evoked for an interesting class of functions  $F$  and not necessarily positive matrices  $A$ .

Finally, consider a symmetric  $k \times k$  matrix  $A = (a_{ij})$  with positive entries and with an eigenvector  $t_1 \in \mathbb{S}_p^{k-1}$  corresponding to  $\|A\|$  which minimizes  $m_p$ . Let

$$G(t) = \langle At, t \rangle^{p/2}$$

and  $F : \mathbb{R}^k \rightarrow \mathbb{R}$  be a  $C^1$  function positively homogeneous of degree  $p^*$  with a nonnegative maximizer  $t_0$  on  $\mathbb{S}_p^{k-1}$  and such that  $D_{t_i} F$  is positive on  $\{t \in \mathbb{R}^k : t_j > 0, j = 1, \dots, k\}$ .

Under these conditions,  $p \geq 2$  and  $n \geq p^2$ , Theorems 1.2 and 1.3 reveal that the system (4) admits a positive solution  $u$  if, and only if,  $\|A\|^{p/2} < m_p \lambda_1$ . Indeed, the assumption  $p \geq 2$  allows to guarantee that  $D_{t_i} G$  is non-decreasing on  $\mathbb{R}^k$  for  $k \geq 2$ , which, unlike the case  $k = 1$ , fails in general for  $1 < p < 2$ . The existence of a positive solution  $u$  is then attained by applying Lemma 2.4 to a slightly modified functional. Precisely, it suffices to consider

$$\Phi_G^+(u) = \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} G(u^+) dx$$

on  $E_F^+ = \{u \in E : \int_{\Omega} F(u^+) dx = 1\}$ , where  $u^+ = (u_1^+, \dots, u_k^+)$  and  $u_i^+$  denotes the positive part of  $u_i$ . First, one easily checks that Lemma 2.4 holds for  $\Phi_G^+$ . Mimicking the proof of Theorem 1.3, we conclude that

$$c^+ := \inf_{u \in E_F^+} \Phi_G^+(u) < K_F(\mathbb{R}^n)^{-p}$$

which leads readily to the existence of a solution  $u \in E_F^+$  of the system

$$-\Delta_p u = g(u^+) + \mu f(u^+) \quad \text{in } \Omega$$

for some  $\mu \in \mathbb{R}$ . By the weak maximum principle, it is clear that  $u^+$  is nontrivial. By taking then  $u^+ \in E$  as a test function in the preceding system, one deduces that  $\mu > 0$ , since  $u^+ \in E_F^+$ ,  $\Phi_G^+(u^+) = \Phi_G(u^+)$  and  $\Phi_G$  is coercive on  $E$ . Finally, the strong maximum principle implies the positivity of  $u$ , since  $a_{ij} > 0$  for every  $i, j = 1, \dots, k$ .

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