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Initial layers and zero-relaxation limits of Euler–Maxwell equations

Mohamed-Lasmer Hajjej [a,b], Yue-Jun Peng [a,b,*]

[a] Clermont Université, Université Blaise Pascal, 63000 Clermont-Ferrand, France
[b] Laboratoire de Mathématiques, CNRS-UMR 6620, Complexe Scientifique Les Cézeaux, BP 80026, 63171 Aubière cedex, France

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ABSTRACT

In this paper we consider zero-relaxation limits for periodic smooth solutions of Euler–Maxwell systems. For well-prepared initial data, we propose an approximate solution based on a new asymptotic expansion up to any order. For ill-prepared initial data, we construct initial layer corrections in an explicit way. In both cases, the asymptotic expansions are valid in time intervals independent of the relaxation time and their convergence is justified by establishing uniform energy estimates.

1. Introduction

Euler–Maxwell equations appear in the modeling of plasmas under conditions on the frequency collision of particles. One example is the modeling of ionospheric plasmas. For a magnetized plasma composed of electrons and ions, let n_e and u_e (respectively, n_i and u_i) be the density and velocity vector of the electrons (respectively, ions), E and B' be respectively the electric field and magnetic field. Theses variables are functions of a three-dimensional position vector $x \in \mathbb{R}^3$ and of the time $t > 0$. The fields E and B' are coupled to the electron density through the Maxwell equations and act on electrons via the Lorentz force. In this paper, we consider the periodic case in a torus $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^3$.

In vacuum, variables (n_ν, u_ν, E, B') satisfy a two-fluid Euler–Maxwell equations (see [3,6,24]):

* Corresponding author at: Laboratoire de Mathématiques, CNRS-UMR 6620, Complexe Scientifique Les Cézeaux, BP 80026, 63171 Aubière cedex, France.

E-mail addresses: hajjej@math.univ-bpclermont.fr (M.-L. Hajjej), peng@math.univ-bpclermont.fr (Y.-J. Peng).

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$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ m_\nu \partial_t (n_\nu u_\nu) + m_\nu \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = q_\nu n_\nu \big(E + u_\nu \times B'\big) - \dfrac{m_\nu n_\nu u_\nu}{\tau_\nu}, \\ \varepsilon_0 \partial_t E - \mu_0^{-1} \nabla \times B' = -(q_e n_e u_e + q_i n_i u_i), \quad \varepsilon_0 \operatorname{div} E = q_e n_e + q_i n_i, \\ \partial_t B' + \nabla \times E = 0, \quad \operatorname{div} B' = 0, \end{cases} \tag{1.1}$$

for $\nu = e, i$ and $(t, x) \in (0, \infty) \times \mathbb{T}$, where $\otimes$ stands for the tensor product and $p_\nu = p_\nu(n_\nu)$ is the pressure function which is sufficiently smooth and strictly increasing for $n_\nu > 0$. In (1.1) the physical parameters are the charges of the electron $q_e = -q$ and of the ion $q_i = q > 0$, the electron mass $m_e > 0$ and the ion mass $m_i > 0$, the momentum relaxation times $\tau_e > 0$ and $\tau_i > 0$, and the vacuum permittivity $\varepsilon_0 > 0$ and the vacuum permeability $\mu_0 > 0$. Recall that the speed of light c and the Debye length λ' are defined by

$$c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}, \qquad \lambda' = \left(\frac{\varepsilon_0 K_B T_e}{n_0 q^2}\right)^{1/2},$$

where $K_B > 0$ is the Boltzmann constant, $T_e > 0$ is the temperature of the electron, and $n_0 > 0$ is the mean density of the plasma [6, p. 350]. Let us define

$$\lambda = \varepsilon_0^{1/2}, \qquad \gamma = \frac{1}{\varepsilon_0^{1/2} c}.$$

Then $\lambda > 0$ is a scaled Debye length since it is proportional to λ'. Remark that $\gamma \to 0$ as $c \to \infty$.

Let us introduce a scaling for the magnetic field $B' = \gamma B$. Then the scaled two-fluid Euler–Maxwell equations are written as:

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ m_\nu \partial_t (n_\nu u_\nu) + m_\nu \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = q_\nu n_\nu (E + \gamma u_\nu \times B) - \dfrac{m_\nu n_\nu u_\nu}{\tau_\nu}, \\ \gamma \lambda^2 \partial_t E - \nabla \times B = -\gamma (q_e n_e u_e + q_i n_i u_i), \quad \lambda^2 \operatorname{div} E = q_e n_e + q_i n_i, \\ \gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \quad \nu = e, i. \end{cases} \tag{1.2}$$

For smooth solutions with $n_\nu > 0$, the second equation of (1.2) is equivalent to

$$m_\nu \partial_t u_\nu + m_\nu (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = q_\nu (E + \gamma u_\nu \times B) - \frac{m_\nu u_\nu}{\tau_\nu}, \tag{1.3}$$

where $\cdot$ denotes the inner product of $\mathbb{R}^3$ and the enthalpy function h_ν is defined by

$$h_\nu(n_\nu) = \int_1^{n_\nu} \frac{p'_\nu(s)}{s} \, ds, \quad \nu = e, i. \tag{1.4}$$

Since p_ν is sufficiently smooth and strictly increasing on $(0, +\infty)$, so is h_ν.

In the plasma when the ions are non-moving and become a uniform background with a given stationary density, by letting $n_i = b$, $u_i = 0$ and deleting the Euler equations for ions, a one-fluid Euler–Maxwell model is formally derived. For simplifying the discussion, in the sequel we take $q = 1$. Replacing (n_e, u_e) by (n, u), m_e by m and τ_e by τ, the one-fluid Euler–Maxwell equations read:

$$
\begin{cases}
\partial_t n + \operatorname{div}(nu) = 0, \\
m\partial_t u + m(u \cdot \nabla)u + \nabla h(n) = -E - \gamma u \times B - \dfrac{mu}{\tau}, \\
\gamma\lambda^2 \partial_t E - \nabla \times B = \gamma nu, \quad \lambda^2 \operatorname{div} E = b - n, \\
\gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0,
\end{cases}
\tag{1.5}
$$

for $(t, x) \in (0, \infty) \times \mathbb{T}$. It is complemented by periodic initial conditions:

$$
t = 0: \quad (n, u, E, B) = \left(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau\right). \tag{1.6}
$$

The given function b depends only on x. This is compatible with system (1.5). Indeed, we have

$$
\partial_t n = -\operatorname{div}(nu) = -\partial_t\left(\lambda^2 \operatorname{div} E\right) = \partial_t n - \partial_t b,
$$

which implies that $\partial_t b = 0$. Moreover, since we consider periodic smooth solutions, b is supposed to be sufficiently smooth and periodic.

In (1.2) the physical parameters m_ν, τ_ν, γ and λ^2 can be chosen independently of each other according to physical situations. They are very small compared to the physical size of the other quantities. Therefore, it is important to study the limits of system (1.2) or (1.5) as these parameters go to zero. The formal asymptotic limits of the two-fluid Euler–Maxwell equations (1.2) have been investigated in [22]. In the one-fluid Euler–Maxwell equations (1.5), the non-relativistic limit $\gamma \to 0$, the quasi-neutral limit $\lambda \to 0$ and the limit of their combination $\gamma = \lambda \to 0$ have been rigorously justified in [19], [20] and [21], respectively. The results show that these limits of (1.5) are respectively a compressible Euler–Poisson system, an electron magnetohydrodynamics system and incompressible Euler equations. The justifications are valid for smooth periodic solutions in time intervals independent of the parameters γ and λ. We mention also that in the two-fluid Euler–Maxwell equations the non-relativistic limit can be justified in a similar way [25], however, the justifications of the quasi-neutral limit and the combined limit $\gamma = \lambda \to 0$ are still open problems. For the kinetic version of the above limits in Vlasov–Maxwell equations, we refer to [5] for the combined limit and to [4] for the non-relativistic limit.

Another interesting problems rely on the limit of the mass ratio between electrons and ions in system (1.2). Since the electron mass is much smaller than the ion mass, for fixing the idea we let $m_i = 1$. Then we may consider the zero-electron mass limit $m_e \to 0$ and the combined limit $m_e \to 0$ with $\tau_e, \tau_i \to 0$ in (1.2). The formal equations of limits can be easily derived (see Appendix A), however, the mathematical justification of these limits is a quite open problem. We leave these problems for a future investigation. On this topic, we refer to [2] for a rigorous justification of the electron mass limit in Euler–Poisson equations.

In what follows, we only consider the one-fluid Euler–Maxwell system (1.5), which is symmetrizable hyperbolic in the sense of Friedrichs [8]. Its local existence of smooth solutions is a well-known result due to Kato [12]. The global existence and the long-time stability of smooth solutions have been recently obtained in [23] when the solutions are close to a constant equilibrium. In a simplified one-dimensional Euler–Maxwell system, the global existence of entropy solutions has been studied in [7] by the compensated compactness method.

In this paper, we are interested in the zero-relaxation limit $\tau \to 0$ of system (1.5) under the conditions $m = O(1)$, $\gamma = O(1)$ and $\lambda = O(1)$. We assume, throughout this paper, that $m = \gamma = \lambda = 1$. The usual time scaling for studying the limit $\tau \to 0$ is $t' = \tau t$. Since $t = 0$ if and only if $t' = 0$, this change of scaling does not affect the initial condition (1.6). Rewriting still t' by t, system (1.5) becomes (see [22,23])

$$\begin{cases} \partial_t n + \dfrac{1}{\tau}\operatorname{div}(nu) = 0, \\ \partial_t u + \dfrac{1}{\tau}(u \cdot \nabla)u + \dfrac{1}{\tau}\nabla h(n) = -\dfrac{E}{\tau} - \dfrac{u \times B}{\tau} - \dfrac{u}{\tau^2}, \\ \partial_t E - \dfrac{1}{\tau}\nabla \times B = \dfrac{nu}{\tau}, \quad \operatorname{div} E = b - n, \\ \partial_t B + \dfrac{1}{\tau}\nabla \times E = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{1.7}$$

Remark that the time scaling $t' = \tau t$ may reveal the long-time asymptotic behavior of solutions. Indeed, $t = t'\tau^{-1} = O(\tau^{-1})$ for fixed $t' > 0$. Then for a fixed time $T_1 > 0$, a local-in-time convergence for system (1.7) on the time interval $[0, T_1]$ means the convergence for system (1.5) on a long-time interval $[0, T_1\tau^{-1}]$. On the other hand, as $\tau \to 0$, a convergence error $O(\tau^r)$ with $r > 0$ implies a rate $O(t^{-r})$ of the long-time asymptotics (see (4.8) in Remark 4.2).

For $m \geqslant 1$, the authors of [23] proposed an asymptotic expansion to (1.7) of the form:

$$\left(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m\right) = \sum_{j=0}^{m} \tau^j \left(n^j, \tau u^j, E^j, B^j\right).$$

They established the convergence in Sobolev spaces of the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ of (1.7) to $(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m)$ with order $O(\tau^m)$ when the initial data are well-prepared and the initial error has the same order. Here the well-prepared initial data mean that compatibility conditions hold. Unfortunately, this result cannot deal with the case of ill-prepared data and the case $m = 0$ of the well-prepared initial data in which the error disappears. The goal of this paper is to improve the above result in two directions.

First, we propose a different asymptotic expansion to (1.7) of the form:

$$\left(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m\right) = \sum_{j=0}^{m} \tau^{2j} \left(n^j, \tau u^j, E^j, \tau B^j\right), \tag{1.8}$$

where the first order profile (n^0, u^0, E^0) satisfies a drift-diffusion system, as shown in [23]. The motivation of this expansion is the following consideration. If we replace u by τu and B by τB, then system (1.7) becomes

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \tau^2\left(\partial_t u + (u \cdot \nabla)u\right) + \nabla h(n) = -E - \tau^2 u \times B - u, \\ \partial_t E - \nabla \times B = nu, \quad \operatorname{div} E = b - n, \\ \tau^2 \partial_t B - \nabla \times E = 0, \quad \operatorname{div} B = 0, \end{cases}$$

in which the only small parameter is τ^2. With expansion (1.8), for $m \geqslant 0$ we prove the convergence of the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ of (1.7) to $(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m)$ with a higher order $O(\tau^{2(m+1)})$ when the initial data are well-prepared and the initial error has the same order. This includes the case $m = 0$. In the proof of the result, we have to treat the order of the remainder $R_B^{\tau,m}$ for variable B. Indeed, there is a loss of one order for $R_B^{\tau,m}$ in comparison with those for variables n, u and E. This is overcome by introducing a correction term into E_τ^m so that the new remainder for B becomes zero without changing the order of the other remainders.

Second, for ill-prepared initial data, the above convergence result is not valid because the approximate solution cannot satisfy the prescribed initial conditions. In this case, we construct initial layer corrections with exponential decay to zero and prove the convergence of the first order asymptotic expansion. The analysis shows that there are no first order initial layers on variables n, E and B.

However, we have to consider the second order initial layer corrections to obtain the desired order of remainders.

The zero-relaxation limit $\tau \to 0$ in the Euler–Poisson system was extensively studied by many authors. See [17,10,11,9,15,14,1,26] and the references therein. Remark that the Euler–Maxwell system and the Euler–Poisson system are essentially different due to the coupling terms and to the difference between Poisson equation and Maxwell equations. Finally, assuming $\tau_e = \tau_i = \tau$, so that the change of scaling of time is possible, the zero-relaxation limit $\tau \to 0$ in the two-fluid Euler–Maxwell system can be carried out in a similar way. Indeed, here the essential point in the proof is that the equations for u_ν are dissipative. Then we may treat the energy estimates of the Euler equations for both $\nu = e, i$ in a similar way to the one-fluid case. In Appendix A, we give a formal description of the zero-relaxation limit in the two-fluid Euler–Maxwell equations and write down the limit equations. For avoiding the tedious calculations, the rigorous justification of the limit is omitted.

For later use in this paper, we recall some results on Moser-type calculus inequalities in Sobolev spaces and the local existence of smooth solutions for symmetrizable hyperbolic equations. For any $s > 0$, we denote by $\|\cdot\|_s$ the norm of the usual Sobolev space $H^s(\mathbb{T})$, and by $\|\cdot\|$ and $\|\cdot\|_\infty$ the norms of $L^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$, respectively. In addition, we denote by $C([0,T), X)$ (respectively, $C^1([0,T), X)$) the space of continuous (respectively, continuously differentiable) functions on $[0,T)$ with values on a Banach space X. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we denote by

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \quad \text{with } |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

Lemma 1.1 *(Moser-type calculus inequalities). (See [13,16].) Let $s \geqslant 1$ be an integer. Suppose $u \in H^s(\mathbb{T})$, $\nabla u \in L^\infty(\mathbb{T})$ and $v \in H^{s-1}(\mathbb{T}) \cap L^\infty(\mathbb{T})$. Then for all multi-indexes $\alpha \in \mathbb{N}^3$ with $1 \leqslant |\alpha| \leqslant s$, we have $\partial_x^\alpha(uv) - u\partial_x^\alpha v \in L^2(\mathbb{T})$ and*

$$\left\|\partial_x^\alpha(uv) - u\partial_x^\alpha v\right\| \leqslant C_s\left(\|\nabla u\|_\infty \left\|D^{|\alpha|-1}v\right\| + \left\|D^{|\alpha|}u\right\| \|v\|_\infty\right),$$

where

$$\left\|D^{s'}u\right\| = \sum_{|\alpha|=s'} \left\|\partial_x^\alpha u\right\|, \quad \forall s' \in \mathbb{N}.$$

Moreover, if $s \geqslant 3$, then the embedding $H^{s-1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ is continuous and we have

$$\|uv\|_{s-1} \leqslant C_s\|u\|_{s-1}\|v\|_{s-1}, \qquad \left\|\partial_x^\alpha(uv) - u\partial_x^\alpha v\right\| \leqslant C_s\|u\|_s\|v\|_{s-1}, \quad \forall |\alpha| \leqslant s.$$

Lemma 1.2. *(See [19].) Let $s \geqslant 0$ be an integer and $f \in H^s(\mathbb{T})$ and $g \in H^s(\mathbb{T})$. Then problem*

$$\nabla \times B = f, \qquad \operatorname{div} B = g, \qquad \operatorname{div} f = 0, \qquad m(g) = 0 \tag{1.9}$$

has a unique solution $B \in H^{s+1}(\mathbb{T})$ in the class $m(B) = 0$, where

$$m(B) = \int_{\mathbb{T}} B\,dx.$$

Proposition 1.1 *(Local existence of smooth solutions). (See [12,16].) Let $s \geqslant 3$ be an integer and $(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau) \in H^s(\mathbb{T})$ with $n_0^\tau \geqslant \kappa$ for some given constant $\kappa > 0$, independent of τ. Then there exist $T_1^\tau > 0$ and a unique smooth solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem* (1.5)–(1.6) *defined in the time interval $[0, T_1^\tau]$, with $(n^\tau, u^\tau, E^\tau, B^\tau) \in C^1([0, T_1^\tau]; H^{s-1}(\mathbb{T})) \cap C([0, T_1^\tau]; H^s(\mathbb{T}))$.*

This paper is organized as follows. In the next section, we derive asymptotic expansions of solutions and state the convergence result to problem (1.6)–(1.7) in the case of well-prepared initial data. In particular, we add a correction term to derive desired error estimates. In Section 3 we consider the asymptotic expansions in the case of ill-prepared initial data by constructing initial layer corrections which exponentially decay to zero. The justification of the both two asymptotic expansions is given in the last section. For this purpose, we prove a more general convergence theorem which implies the convergence of the both expansions. Finally, in Appendix A, we consider the formal derivation of the combined zero-electron mass and zero-relaxation limits.

2. Case of well-prepared initial data

2.1. Formal asymptotic expansions

In this section we consider the zero-relaxation limit $\tau \to 0$ in problem (1.6)–(1.7) with well-prepared initial data. Based on the discussion on the asymptotic expansion, we make the following ansatz for both the approximate solution and its initial data:

$$(n_\tau, u_\tau, E_\tau, B_\tau)(0,x) = \sum_{j\geqslant 0} \tau^{2j}(n_j, \tau u_j, E_j, \tau B_j)(x), \quad x \in \mathbb{T}, \tag{2.1}$$

$$(n_\tau, u_\tau, E_\tau, B_\tau)(t,x) = \sum_{j\geqslant 0} \tau^{2j}\big(n^j, \tau u^j, E^j, \tau B^j\big)(t,x), \quad t > 0,\ x \in \mathbb{T}, \tag{2.2}$$

where $(n_j, u_j, E_j, B_j)_{j\geqslant 0}$ are given sufficiently smooth data with $n_0 \geqslant \text{constant} > 0$ in $\mathbb{T}$. The validity of expansions (2.1)–(2.2) is discussed in Section 4 (see Theorem 4.1).

Now let us determine the profiles (n^j, u^j, E^j, B^j) for all $j \geqslant 0$. Putting expression (2.2) into system (1.7) and identifying the coefficients in powers of τ, we see that $(n^j, u^j, E^j, B^j)_{j\geqslant 0}$ are solutions of the following systems:

$$\begin{cases} \partial_t n^0 + \operatorname{div}\big(n^0 u^0\big) = 0, \\ \nabla h\big(n^0\big) = -\big(E^0 + u^0\big), \\ \nabla \times E^0 = 0, \quad \operatorname{div} E^0 = b - n^0, \\ \nabla \times B^0 = \partial_t E^0 - n^0 u^0, \quad \operatorname{div} B^0 = 0, \end{cases} \tag{2.3}$$

and for $j \geqslant 1$,

$$\begin{cases} \partial_t n^j + \displaystyle\sum_{k=0}^{j} \operatorname{div}\big(n^k u^{j-k}\big) = 0, \\ \partial_t u^{j-1} + \displaystyle\sum_{k=0}^{j-1} \big(u^k \cdot \nabla\big) u^{j-1-k} + \nabla\big(h'\big(n^0\big)n^j + h^{j-1}\big(\big(n^k\big)_{k\leqslant j-1}\big)\big) \\ \quad = -E^j - \displaystyle\sum_{k=0}^{j-1} u^k \times B^{j-1-k} - u^j, \\ \nabla \times E^j = -\partial_t B^{j-1}, \quad \operatorname{div} E^j = -n^j, \\ \nabla \times B^j = \partial_t E^j - \displaystyle\sum_{k=0}^{j} n^k u^{j-k}, \quad \operatorname{div} B^j = 0, \end{cases} \tag{2.4}$$

where $h^0 = 0$ and h^{j-1} is a function depending only on $(n^k)_{0\leqslant k\leqslant j-1}$ and is defined for $j \geqslant 2$ by

$$h\left(\sum_{j\geqslant 0}\tau^{2j}n^j\right) = \sum_{j\geqslant 0} c_j\tau^{2j},$$

with

$$c_0 = h(n^0), \qquad c_1 = h'(n^0)n^1, \qquad c_j = h'(n^0)n^j + h^{j-1}\big((n^k)_{k\leqslant j-1}\big), \quad \forall j \geqslant 2.$$

In (2.3), equation $\nabla \times E^0 = 0$ implies the existence of a potential ϕ^0 such that $E^0 = -\nabla\phi^0$. Then (n^0, ϕ^0) solves a classical system drift-diffusion equations:

$$\begin{cases} \partial_t n^0 - \operatorname{div}\big(n^0\nabla\big(h(n^0) - \phi^0\big)\big) = 0, \\ -\Delta\phi^0 = b - n^0, \end{cases} \qquad t > 0,\ x \in \mathbb{T}, \tag{2.5}$$

with the initial condition:

$$n^0(0, x) = n_0, \quad x \in \mathbb{T}. \tag{2.6}$$

The existence of smooth solutions to problem (2.5)–(2.6) can be easily established, at least locally in time. The solution ϕ^0 is unique in the class $m(\phi^0) = 0$. See for instance [18]. Then (u^0, E^0) are given by

$$u^0 = -\nabla\big(h(n^0) - \phi^0\big), \qquad E^0 = -\nabla\phi^0. \tag{2.7}$$

Since (n^0, u^0, E^0) are known, B^0 solves the linear system of curl-div equations of type (1.9) in the class $m(B^0) = 0$. More precisely, using $\nabla \times E^0 = 0$ and formula

$$\nabla \times \nabla \times B^0 = \nabla \operatorname{div} B^0 - \Delta B^0,$$

we obtain

$$\Delta B^0 = \nabla \times \big(n^0u^0\big) \quad \text{in } \mathbb{T} \quad \text{and} \quad m(B_0) = 0.$$

From (2.7) and the fourth equation of system (2.3) we get the first order compatibility conditions:

$$u_0 = -\nabla\big(h(n_0) - \phi_0\big), \qquad E_0 = -\nabla\phi_0, \qquad B_0 = B^0(0, .), \tag{2.8}$$

where ϕ_0 is determined by

$$-\Delta\phi_0 = b - n_0 \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_0) = 0. \tag{2.9}$$

For $j \geqslant 1$, the profiles (n^j, u^j, E^j, B^j) are obtained by induction in j. Assume that $(n^k, u^k, E^k, B^k)_{0\leqslant k\leqslant j-1}$ are smooth and have already been determined in previous steps. Equations for B^j are of curl-div type and determine a unique smooth B^j in the class $m(B^j) = 0$. Moreover, from $\operatorname{div} B^j = 0$, we deduce the existence of a given vector ψ^j such that $B^j = -\nabla \times \psi^j$. Then, equation $\nabla \times E^j = -\partial_t B^{j-1}$ in (2.4) becomes $\nabla \times (E^j - \partial_t\psi^{j-1}) = 0$. It follows that there is a potential function ϕ^j such that

$$E^j = \partial_t \psi^{j-1} - \nabla \phi^j. \tag{2.10}$$

From (2.4) we also get

$$\begin{aligned} u^j = {} & \nabla\big(\phi^j - h'\big(n^0\big)n^j - h^{j-1}\big(\big(n^k\big)_{k\leqslant j-1}\big)\big) \\ & - \left(\partial_t u^{j-1} + \partial_t \psi^{j-1} + \sum_{k=0}^{j-1}\big(\big(u^k\cdot\nabla\big)u^{j-1-k} + u^k \times B^{j-1-k}\big)\right). \end{aligned} \tag{2.11}$$

Therefore, in the class $m(\phi^j) = 0$, (n^j, ϕ^j) solves a linearized system of drift-diffusion equations:

$$\begin{cases} \partial_t n^j - \operatorname{div}\big(n^0\nabla\big(h'\big(n^0\big)n^j - \phi^j\big)\big) + \operatorname{div}\big(n^j u^0\big) \\ \quad = f^j\big(\big(V^k, \partial_t V^k, \partial_x V^k, \partial_t\partial_x V^k, \partial_x^2 V^k\big)_{0\leqslant k\leqslant j-1}\big) + \operatorname{div}\big(n^0 \partial_t \psi^{j-1}\big), \quad t > 0,\ x \in \mathbb{T}, \\ \Delta \phi^j = n^j + \partial_t\big(\operatorname{div}\psi^{j-1}\big), \end{cases} \tag{2.12}$$

with the initial condition:

$$n^j(0, x) = n_j(x), \quad x \in \mathbb{T}, \tag{2.13}$$

where f^j is a given smooth function and $V^k = (n^k, u^k, \psi^k)$. Problem (2.12)–(2.13) is linear. It admits a unique global smooth solution. Then (u^j, E^j) are given by (2.10)–(2.11). Thus, we get the high-order compatibility conditions for $j \geqslant 1$:

$$\begin{aligned} u_j = {} & \nabla\big(\phi_j - h'(n_0)n_j - h^{j-1}\big((n_k)_{k\leqslant j-1}\big)\big) \\ & - \left(\partial_t u^{j-1}\big|_{t=0} + \psi^{j-1}\big|_{t=0} + \sum_{k=0}^{j-1}\big((u_k\cdot\nabla)u_{j-1-k} + u_k \times B^{j-1-k}(0,.)\big)\right), \end{aligned} \tag{2.14}$$

$$E_j = \partial_t \psi^{j-1}(0,.) - \nabla\phi_j, \qquad B_j = B^j(0,.), \tag{2.15}$$

where ϕ_j is determined by

$$\Delta\phi_j = n_j + \partial_t\big(\operatorname{div}\psi^{j-1}\big)\big|_{t=0} \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_j) = 0. \tag{2.16}$$

We conclude the above discussion with the following result.

Proposition 2.1. *Let $s \geqslant 3$ be an integer. Assume $(n_j, u_j, E_j, B_j) \in H^{s+1}(\mathbb{T})$ for $j \geqslant 0$, with $n_0 \geqslant$ constant > 0 in $\mathbb{T}$, and satisfy the compatibility conditions (2.8)–(2.9) and (2.14)–(2.16) for $j \geqslant 1$. Then there exists a unique asymptotic expansion up to any order of the form (2.2), i.e. there exist $T_1 > 0$ and a unique smooth solution $(n^j, u^j, E^j, B^j)_{j\geqslant 0}$ in the time interval $[0, T_1]$ of problems (2.5)–(2.7) and (2.10)–(2.13) for $j \geqslant 1$. Moreover, $n^0 \geqslant$ constant > 0 in $[0, T_1] \times \mathbb{T}$ and*

$$\big(n^j, u^j, E^j, B^j\big) \in C^1\big(0, T_1; H^s(\mathbb{T})\big) \cap C\big(0, T_1; H^{s+1}(\mathbb{T})\big), \quad \forall j \geqslant 0.$$

In particular, the formal zero-relaxation limit $\tau \to 0$ of the Euler–Maxwell system (1.7) is the classical drift-diffusion system (2.5) and (2.7).

2.2. Convergence results

Let $m \geqslant 0$ be a fixed integer. We denote by

$$\left(n_\tau^m, u_\tau^m, \tilde{E}_\tau^m, B_\tau^m\right) = \sum_{j=0}^{m} \tau^{2j}\left(n^j, \tau u^j, E^j, \tau B^j\right) \tag{2.17}$$

an approximate solution of order m, where $(n^j, u^j, E^j, B^j)_{0 \leqslant j \leqslant m}$ are constructed in the previous subsection. From the construction of the approximate solution, for $(t,x) \in [0, T_1] \times \mathbb{T}$ we have

$$\operatorname{div} \tilde{E}_\tau^m = b - n_\tau^m, \qquad \operatorname{div} B_\tau^m = 0. \tag{2.18}$$

We define the remainders $R_n^{\tau,m}$, $R_u^{\tau,m}$, $R_E^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$ by

$$\begin{cases} \partial_t n_\tau^m + \dfrac{1}{\tau} \operatorname{div}\left(n_\tau^m u_\tau^m\right) = R_n^{\tau,m}, \\ \partial_t u_\tau^m + \dfrac{1}{\tau}\left(u_\tau^m \cdot \nabla\right) u_\tau^m + \dfrac{1}{\tau} \nabla h\left(n_\tau^m\right) = -\dfrac{\tilde{E}_\tau^m}{\tau} - \dfrac{u_\tau^m}{\tau^2} - \dfrac{u_\tau^m \times B_\tau^m}{\tau} + R_u^{\tau,m}, \\ \partial_t \tilde{E}_\tau^m - \dfrac{1}{\tau} \nabla \times B_\tau^m = \dfrac{n_\tau^m u_\tau^m}{\tau} + R_E^{\tau,m}, \\ \partial_t B_\tau^m + \dfrac{1}{\tau} \nabla \times \tilde{E}_\tau^m = \tilde{R}_B^{\tau,m}. \end{cases} \tag{2.19}$$

It is clear that the convergence rate depends strongly on the order of the remainders with respect to τ. Since the last equation in (2.19) is linear, for sufficiently smooth profiles $(n^j, u^j, E^j, B^j)_{j \geqslant 0}$, it is easy to see that

$$\tilde{R}_B^{\tau,m} = \tau^{2m+1} \partial_t B^m. \tag{2.20}$$

Moreover, a further computation gives

$$R_n^{\tau,m} = O\left(\tau^{2(m+1)}\right), \qquad R_E^{\tau,m} = O\left(\tau^{2(m+1)}\right), \qquad R_u^{\tau,m} = O\left(\tau^{2m+1}\right). \tag{2.21}$$

In (2.20)–(2.21), there is a loss of one order for the remainders $R_u^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$. For $R_u^{\tau,m}$ this loss will be recovered in the error estimate of convergence due to the dissipation term for u. However, the situation is different for $\tilde{R}_B^{\tau,m}$ since the equation for B is not dissipative. A simple way to remedy this is to introduce a correction term into $\tilde{E}_\tau^m$ so that

$$E_\tau^m = \tilde{E}_\tau^m + \tau^{2(m+1)} E_c^{m+1} = \sum_{j=0}^{m} \tau^{2j} E^j + \tau^{2(m+1)} E_c^{m+1}. \tag{2.22}$$

In view of (2.18)–(2.20), E_c^{m+1} should be defined by

$$\nabla \times E_c^{m+1} = -\partial_t B^m, \qquad \operatorname{div} E_c^{m+1} = 0, \qquad m\left(E_c^{m+1}\right) = 0 \tag{2.23}$$

so that the new remainder $R_B^{\tau,m}$ of B satisfies

$$R_B^{\tau,m} \overset{def}{=} \partial_t B_\tau^m + \frac{1}{\tau}\nabla \times E_\tau^m = 0 \tag{2.24}$$

and we still have

$$\operatorname{div} E_\tau^m = b - n_\tau^m, \qquad \operatorname{div} B_\tau^m = 0. \tag{2.25}$$

Since the correction term is of order $O(\tau^{2(m+1)})$, the orders of the remainders $R_n^{\tau,m}$, $R_u^{\tau,m}$ and $R_E^{\tau,m}$ are not changed. Moreover, the correction term does not affect assumption (2.28) below.

We conclude the above discussion with the following result.

Proposition 2.2. *Let the assumptions of Proposition* 2.1 *hold. For all integers $m \geqslant 0$ and $s \geqslant 3$ the remainders $R_n^{\tau,m}$, $R_u^{\tau,m}$, $R_E^{\tau,m}$ and $R_B^{\tau,m}$ satisfy* (2.24) *and*

$$\sup_{0 \leqslant t \leqslant T_1} \left\| \left(R_n^{\tau,m}, R_E^{\tau,m}\right)(t,\cdot)\right\|_s \leqslant C_m \tau^{2(m+1)}, \qquad \sup_{0 \leqslant t \leqslant T_1} \left\| R_u^{\tau,m}(t,\cdot)\right\|_s \leqslant C_m \tau^{2m+1}, \tag{2.26}$$

where $C_m > 0$ is a constant independent of τ.

The convergence result of this section is stated as follows of which the proof is given in Section 4.

Theorem 2.1. *Let $m \geqslant 0$ and $s \geqslant 3$ be any fixed integers. Let the assumptions of Proposition* 2.1 *hold. Suppose*

$$\operatorname{div} E_0^\tau = b - n_0^\tau, \qquad \operatorname{div} B_0^\tau = 0 \quad \text{in } \mathbb{T} \tag{2.27}$$

and

$$\left\| \left(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau\right) - \sum_{j=0}^m \tau^{2j}(n_j, \tau u_j, E_j, \tau B_j) \right\|_s \leqslant C_1 \tau^{2(m+1)}, \tag{2.28}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geqslant T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (1.6)–(1.7) *satisfies*

$$\left\| \left(n^\tau, u^\tau, E^\tau, B^\tau\right)(t) - \left(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m\right)(t) \right\|_s \leqslant C_2 \tau^{2(m+1)}, \quad \forall t \in [0, T_1].$$

Moreover,

$$\left\| u^\tau - u_\tau^m \right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leqslant C_2 \tau^{2m+3}.$$

3. Case of ill-prepared initial data

3.1. Initial layer corrections

In Theorem 2.1, compatibility conditions are made on the initial data. These conditions mean that the initial profiles $(u^j, E^j, B^j)(0,\cdot)$ are determined through the resolution of the problems (2.3)–(2.4) for (n^j, u^j, E^j, B^j). Then $(u_0^\tau, E_0^\tau, B_0^\tau)$ cannot be given explicitly. If these conditions are not satisfied, the phenomenon of initial layers occurs. In this section, we consider this situation for so-called ill-prepared initial data. We seek a simplest possible form of an asymptotic expansion with initial layer corrections such that its remainders are at least of order $O(\tau)$ for variable u.

Let the initial data of an approximate solution $(n_\tau, u_\tau, E_\tau, B_\tau)$ have an asymptotic expansion of the form:

$$(n_\tau, u_\tau, E_\tau, B_\tau)|_{t=0} = (n_0, \tau u_0, E_0, \tau B_0) + O\big(\tau^2\big), \tag{3.1}$$

where (n_0, u_0, E_0, B_0) are given smooth functions. Taking into account the expansion in the case of well-prepared initial data, a simplest form of an asymptotic expansion including initial layer corrections is

$$\begin{aligned}(n_\tau, u_\tau, E_\tau, B_\tau)(t,x) &= \big(n^0, \tau u^0, E^0 + \tau^2 E_c^1, \tau B^0\big)(t,x) \\ &\quad + \big(\big(n_I^0, \tau u_I^0, E_I^0, \tau B_I^0\big) + \tau^2\big(n_I^1, \tau u_I^1, E_I^1, \tau B_I^1\big)\big)(z,x) + O\big(\tau^2\big),\end{aligned} \tag{3.2}$$

where $z = t/\tau^2 \in \mathbb{R}$ is the fast variable, the subscript I stands for the initial layer variables and E_c^1 is the correction term defined by (2.23) with $m = 0$. As we will see below, this expansion is enough to give the remainders at least of order $O(\tau)$ for variable u, which is the case of well-prepared initial data for $m = 0$.

Obviously, (n^0, u^0, E^0, B^0) still satisfies the drift-diffusion system (2.3). It remains to determine the initial layer profiles $(n_I^0, u_I^0, E_I^0, B_I^0)$ and $(n_I^1, u_I^1, E_I^1, B_I^1)$. Putting expression (3.2) into system (1.7) and using (2.3), we obtain

$$\partial_z n_I^0 = 0, \qquad \partial_z E_I^0 = 0, \qquad \partial_z B_I^0 + \nabla \times E_I^0 = 0 \tag{3.3}$$

and

$$\partial_z u_I^0 + u_I^0 = 0. \tag{3.4}$$

Eqs. (3.3) imply that there are no first order initial layers for variables n, E and B. Therefore, up to a constant for variable B, we may take

$$n^0(0,x) = n_0(x), \qquad E^0(0,x) = E_0(x) \quad \text{and} \quad B^0(0,x) = B_0(x). \tag{3.5}$$

Moreover, expressions (3.1) and (3.2) for u imply that

$$u^0(0,x) + u_I^0(0,x) = u_0(x), \tag{3.6}$$

which determines the initial value of u_I^0, where $u^0(0,\cdot)$ is given by (2.8)–(2.9). Together with (3.4), we obtain

$$u_I^0(z,x) = u_I^0(0,x)e^{-z} = \big(u_0(x) - u^0(0,x)\big)e^{-z}. \tag{3.7}$$

Similarly, the second order initial layers n_I^1 and E_I^1 satisfy

$$u_I^1 = 0, \tag{3.8}$$

$$\partial_z n_I^1(z,x) + \operatorname{div}\big(n^0(0,x)u_I^0(z,x)\big) = 0, \tag{3.9}$$

$$\partial_z E_I^1(z,x) = n^0(0,x)u_I^0(z,x) \tag{3.10}$$

and

$$\partial_z B_I^1(z,x) + \nabla \times E_I^1(z,x) = 0. \tag{3.11}$$

Let (n_1, E_1, B_1) be smooth functions such that

$$E_1(x) = -n^0(0,x)\big(u_0(x) - u^0(0,x)\big) \tag{3.12}$$

and

$$n_1 = \operatorname{div} E_1, \qquad \operatorname{div} B_1 = 0. \tag{3.13}$$

Set

$$\big(n_I^1, E_I^1, B_I^1\big)(0,x) = (n_1, E_1, B_1)(x).$$

Together with (3.7) and (3.9)–(3.12), it is easy to obtain

$$n_I^1(z,x) = n_1(x) - \operatorname{div}\big(n^0(0,x)\big(u_0(x) - u^0(0,x)\big)\big)\big(1 - e^{-z}\big), \tag{3.14}$$

$$E_I^1(z,x) = -n^0(0,x)\big(u_0(x) - u^0(0,x)\big)e^{-z} \tag{3.15}$$

and

$$B_I^1(z,x) = B_1(x) + \nabla \times \big[n^0(0,x)\big(u_0(x) - u^0(0,x)\big)\big]\big(1 - e^{-z}\big). \tag{3.16}$$

Finally, from (3.13) we have

$$\operatorname{div} E_I^1 + n_I^1 = 0, \qquad \operatorname{div} B_I^1 = 0. \tag{3.17}$$

Thus, the initial layer profiles $(n_I^0, u_I^0, E_I^0, B_I^0)$ and $(n_I^1, u_I^1, E_I^1, B_I^1)$ are completely determined by (3.3), (3.7)–(3.8) and (3.14)–(3.16). They are smooth functions of (z,x) and bounded with respect to z.

3.2. Convergence results

According to the asymptotic expansions above, set

$$\begin{cases} n_{\tau,I}(t,x) = n^0(t,x) + \tau^2 n_I^1\big(t/\tau^2, x\big), \\ u_{\tau,I}(t,x) = \tau\big(u^0(t,x) + u_I^0\big(t/\tau^2, x\big)\big), \\ E_{\tau,I}(t,x) = E^0(t,x) + \tau^2\big(E_c^1(t,x) + E_I^1\big(t/\tau^2, x\big)\big), \\ B_{\tau,I}(t,x) = \tau\big(B^0(t,x) + \tau^2 B_I^1\big(t/\tau^2, x\big)\big). \end{cases} \tag{3.18}$$

Then we have

$$t = 0: \quad (n_{\tau,I}, u_{\tau,I}, E_{\tau,I}, B_{\tau,I}) = (n_0, \tau u_0, E_0, \tau B_0) + \tau^2\big(n_1, 0, E_1 + E_c^1(0,\cdot), \tau B_1\big). \tag{3.19}$$

Moreover, Eqs. (2.3), (2.25) and (3.17) imply that

$$\operatorname{div} E_{\tau,I} = b - n_{\tau,I}, \qquad \operatorname{div} B_{\tau,I} = 0. \tag{3.20}$$

Define the remainders $R_n^{\tau,I}$, $R_u^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ by

$$\begin{cases} \partial_t n_{\tau,I} + \dfrac{1}{\tau}\operatorname{div}(n_{\tau,I}u_{\tau,I}) = R_n^{\tau,I}, \\ \partial_t u_{\tau,I} + \dfrac{1}{\tau}(u_{\tau,I}\cdot\nabla)u_{\tau,I} + \dfrac{1}{\tau}\nabla h(n_{\tau,I}) = -\dfrac{E_{\tau,I}}{\tau} - \dfrac{u_{\tau,I}}{\tau^2} - \dfrac{u_{\tau,I}\times B_{\tau,I}}{\tau} + R_u^{\tau,I}, \\ \partial_t E_{\tau,I} - \dfrac{1}{\tau}\nabla\times B_{\tau,I} = \dfrac{n_{\tau,I}u_{\tau,I}}{\tau} + R_E^{\tau,I}, \\ \partial_t B_{\tau,I} + \dfrac{1}{\tau}\nabla\times E_{\tau,I} = R_B^{\tau,I}. \end{cases} \tag{3.21}$$

Using Eqs. (2.3), (2.23) for $(n^0, u^0, E^0, B^0, E_c^1)$ and (3.4), (3.9)–(3.11) for $(u_I^0, n_I^1, E_I^1, B_I^1)$, we obtain

$$\begin{aligned} R_n^{\tau,I} &= \partial_t\big(n^0+\tau^2 n_I^1\big) + \operatorname{div}\big(\big(n^0+\tau^2 n_I^1\big)\big(u^0+u_I^0\big)\big) \\ &= \partial_z n_I^1 + \operatorname{div}\big(n^0 u_I^0\big) + \tau^2\operatorname{div}\big(n_I^1\big(u^0+u_I^0\big)\big) \\ &= \operatorname{div}\big(\big(n^0(t,x)-n^0(0,x)\big)u_I^0(z,x)\big) + \tau^2\operatorname{div}\big(n_I^1\big(u^0+u_I^0\big)\big), \end{aligned}$$

$$\begin{aligned} R_u^{\tau,I} &= \tau\big(\partial_t\big(u^0+u_I^0\big) + \big(u^0+u_I^0\big)\cdot\nabla\big(u^0+u_I^0\big) + \big(u^0+u_I^0\big)\times\big(B^0+\tau^2 B_I^1\big)\big) \\ &\quad + \frac{1}{\tau}\big(\nabla h\big(n^0+\tau^2 n_I^1\big) + \big(E^0+\tau^2 E_c^1+\tau^2 E_I^1\big) + \big(u^0+u_I^0\big)\big) \\ &= \frac{1}{\tau}\big(\nabla h\big(n^0\big) + E^0 + u^0\big) + \frac{1}{\tau}\big(\partial_z u_I^0 + u_I^0\big) + \frac{1}{\tau}\nabla\big(h\big(n^0+\tau^2 n_I^1\big) - h\big(n^0\big)\big) \\ &\quad + \tau\big(\partial_t u^0 + \big(u^0+u_I^0\big)\cdot\nabla\big(u^0+u_I^0\big) + \big(u^0+u_I^0\big)\times B^0\big) \\ &\quad + \tau\big(E_I^1+E_c^1\big) + \tau^3\big(n^0+n_I^1\big)\times B_I^1 \\ &= \tau\big(\partial_t u^0 + \big(u^0+u_I^0\big)\cdot\nabla\big(u^0+u_I^0\big) + \big(u^0+u_I^0\big)\times B^0\big) \\ &\quad + \tau\big(E_I^1+E_c^1\big) + \frac{1}{\tau}\nabla\big(h\big(n^0+\tau^2 n_I^1\big) - h\big(n^0\big)\big) + \tau^3\big(n^0+n_I^1\big)\times B_I^1, \end{aligned}$$

$$\begin{aligned} R_E^{\tau,I} &= \partial_t\big(E^0+\tau^2 E_c^1+\tau^2 E_I^1\big) - \nabla\times\big(B^0+\tau^2 B_I^1\big) - \big(n^0+\tau^2 n_I^1\big)\big(u^0+u_I^0\big) \\ &= \big(\partial_t E^0 - \nabla\times B^0 - n^0 u^0\big) + \partial_z E_I^1 - n^0 u_I^0 + \tau^2\big(n_I^1\big(u^0+u_I^0\big) + \partial_t E_c^1 - \nabla\times B_I^1\big) \\ &= \big(n^0(0,x)-n^0(t,x)\big)u_I^0(z,x) + \tau^2\big(n_I^1\big(u^0+u_I^0\big) + \partial_t E_c^1 - \nabla\times B_I^1\big) \end{aligned}$$

and

$$\begin{aligned} R_B^{\tau,I} &= \tau\,\partial_t\big(B^0+\tau^2 B_I^1\big) + \frac{1}{\tau}\nabla\times\big(E^0+\tau^2 E_c^1+\tau^2 E_I^1\big) \\ &= \frac{1}{\tau}\nabla\times E^0 + \tau\big(\partial_t B^0 + \nabla\times E_c^1\big) + \tau\big(\partial_z B_I^1 + \nabla\times E_I^1\big) \\ &= 0. \end{aligned}$$

Now we establish error estimates for $(R_n^{\tau,I}, R_u^{\tau,I}, R_E^{\tau,I}, R_B^{\tau,I})$. For $R_n^{\tau,I}$ and $R_E^{\tau,I}$, there is $\eta\in[0,t]\subset[0,T_1]$ such that

$$n^0(t,x) - n^0(0,x) = t\partial_t n^0(\eta,x) = \tau^2 z\partial_t n^0(\eta,x).$$

Since function $z\mapsto ze^{-z}$ is bounded for $z\geqslant 0$ and

$$\partial_t n^0 = -\operatorname{div}(n^0 u^0)$$

it follows from (3.7) that

$$\left(n^0(t,x) - n^0(0,x)\right)u_I^0(z,x) = O\left(\tau^2\right).$$

Then

$$R_n^{\tau,I} = O\left(\tau^2\right) \quad \text{and} \quad R_E^{\tau,I} = O\left(\tau^2\right).$$

Finally, for $R_u^{\tau,I}$, we have

$$h\left(n^0 + \tau^2 n_I^1\right) - h\left(n^0\right) = O\left(\tau^2\right).$$

Thus

$$R_u^{\tau,I} = O(\tau).$$

From the previous discussions on the remainders, we obtain the following error estimates.

Proposition 3.1. *Let $s \geqslant 3$ be an integer. For given smooth data, the remainders $R_n^{\tau,I}$, $R_u^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ satisfy*

$$\sup_{0\leqslant t\leqslant T_1} \left\| \left(R_n^{\tau,I}, R_E^{\tau,I}\right)(t,\cdot)\right\|_s \leqslant C\tau^2, \qquad \sup_{0\leqslant t\leqslant T_1} \left\| R_u^{\tau,I}(t,\cdot)\right\|_s \leqslant C\tau, \qquad R_B^{\tau,I} = 0, \tag{3.22}$$

where $C > 0$ is a constant independent of τ.

The convergence result with initial layers can be stated as follows.

Theorem 3.1. *Let $s \geqslant 3$ be a fixed integer and $(n_0, u_0, E_0, B_0) \in H^{s+1}(\mathbb{T})$ with $n_0 \geqslant$ constant > 0 in $\mathbb{T}$. Suppose*

$$\operatorname{div} E_0^\tau = b - n_0^\tau, \qquad \operatorname{div} B_0^\tau = 0 \quad \text{in } \mathbb{T} \tag{3.23}$$

and

$$\left\| \left(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau\right) - (n_0, \tau u_0, E_0, \tau B_0)\right\|_s \leqslant C_1\tau^2, \tag{3.24}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geqslant T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (1.6)–(1.7) satisfies

$$\left\| \left(n^\tau, u^\tau, E^\tau, B^\tau\right) - \left(n^0, u_{\tau,I}, E^0, B^0\right)(t)\right\|_s \leqslant C_2\tau^2, \quad \forall t \in [0, T_1].$$

Moreover,

$$\left\| u^\tau - u_{\tau,I}\right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leqslant C_2\tau^3.$$

4. Justification of asymptotic expansions

4.1. Statement of the main result

In this section, we justify rigorously the asymptotic expansions of solutions $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (1.6)–(1.7) developed in Sections 2–3. We prove a more general convergence result which implies both Theorems 2.1 and 3.1. As a consequence, we obtain the existence of exact solutions $(n^\tau, u^\tau, E^\tau, B^\tau)$ in a time interval independent of τ. To justify rigorously the asymptotic expansions (2.2) and (3.18), it suffices to obtain the uniform estimates of the smooth solutions to (1.7) with respect to the parameter τ.

Let $(n^\tau, u^\tau, E^\tau, B^\tau)$ be the exact solution to (1.7) with initial data $(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau)$ and $(n_\tau, u_\tau, E_\tau, B_\tau)$ be an approximate periodic solution defined on $[0, T_1]$, with

$$(n_\tau, u_\tau, E_\tau, B_\tau) \in C\big([0, T_1], H^{s+1}(\mathbb{T})\big) \cap C^1\big([0, T_1], H^s(\mathbb{T})\big).$$

We define the remainders of the approximate solution by

$$\begin{cases} R_n^\tau = \partial_t n_\tau + \dfrac{1}{\tau}\operatorname{div}(n_\tau u_\tau), \\ R_u^\tau = \partial_t u_\tau + \dfrac{1}{\tau}(u_\tau \cdot \nabla)u_\tau + \dfrac{1}{\tau}\nabla h(n_\tau) + \dfrac{E_\tau}{\tau} + \dfrac{u_\tau \times B_\tau}{\tau} + \dfrac{u_\tau}{\tau^2}, \\ R_E^\tau = \partial_t E_\tau - \dfrac{1}{\tau}\nabla \times B_\tau - \dfrac{n_\tau u_\tau}{\tau}, \\ R_B^\tau = \partial_t B_\tau + \dfrac{1}{\tau}\nabla \times E_\tau. \end{cases} \tag{4.1}$$

Suppose

$$\operatorname{div} E_\tau = b - n_\tau, \qquad \operatorname{div} B_\tau = 0, \tag{4.2}$$

$$\sup_{0 \leqslant t \leqslant T_1} \big\|(n_\tau, E_\tau, B_\tau)(t, \cdot)\big\|_s \leqslant C_1, \qquad \sup_{0 \leqslant t \leqslant T_1} \big\|u_\tau(t, \cdot)\big\|_s \leqslant C_1 \tau, \tag{4.3}$$

$$\big\|\big(n_0^\tau - n_\tau(0, \cdot), u_0^\tau - u_\tau(0, \cdot), E_0^\tau - E_\tau(0, \cdot), B_0^\tau - B_\tau(0, \cdot)\big)\big\|_s \leqslant C_1 \tau^{\lambda+1}, \tag{4.4}$$

$$\sup_{0 \leqslant t \leqslant T_1} \big\|\big(R_n^\tau, R_E^\tau\big)(t, \cdot)\big\|_s \leqslant C_1 \tau^{\lambda+1}, \qquad \sup_{0 \leqslant t \leqslant T_1} \big\|R_u^\tau(t, \cdot)\big\|_s \leqslant C_1 \tau^\lambda, \qquad R_B^\tau = 0, \tag{4.5}$$

where $\lambda \geqslant 0$ and $C_1 > 0$ are constants independent of τ.

Theorem 4.1. *Let $s \geqslant 3$ be an integer and $\lambda \geqslant 0$. Under the above assumptions, there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geqslant T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ of the periodic problem* (1.6)–(1.7) *satisfies*

$$\big\|\big(n^\tau, u^\tau, E^\tau, B^\tau\big)(t) - (n_\tau, u_\tau, E_\tau, B_\tau)(t)\big\|_s \leqslant C_2 \tau^{\lambda+1}, \quad \forall t \in [0, T_1]. \tag{4.6}$$

Moreover,

$$\big\|u^\tau - u_\tau\big\|_{L^2(0,T_1; H^s(\mathbb{T}))} \leqslant C_2 \tau^{\lambda+2}. \tag{4.7}$$

Remark 4.1. It is clear that Theorem 4.1 implies Theorems 2.1 and 3.1. In particular, $\lambda = 2m+1$ with $m \geqslant 0$ in Section 2 and $\lambda = 1$ in Section 3, since

$$\left\|(n_{\tau,I}, E_{\tau,I}, B_{\tau,I})(t) - \left(n^0, E^0, B^0\right)(t)\right\|_s = O\left(\tau^2\right),$$

uniformly with respect to t.

Remark 4.2. Theorem 4.1 holds in the scaled time variable $t' = \tau t$. It can be written down in the normal time variable $t = t'/\tau$. Precisely, since Theorem 4.1 is valid for $t' \in [0, T_1]$, it is valid for $t \in [0, T_1\tau^{-1}]$, so that the periodic problem (1.5)–(1.6) admits a unique solution $(\tilde{n}^\tau, \tilde{u}^\tau, \tilde{E}^\tau, \tilde{B}^\tau)$ on a long-time interval $[0, T_1\tau^{-1}]$ as $\tau \to 0$. This solution satisfies

$$\left(\tilde{n}^\tau, \tilde{u}^\tau, \tilde{E}^\tau, \tilde{B}^\tau\right) \in C\left(\left[0, T_1\tau^{-1}\right], H^s(\mathbb{T})\right) \cap C^1\left(\left[0, T_1\tau^{-1}\right], H^{s-1}(\mathbb{T})\right).$$

With the notations of Theorem 4.1, we have

$$\left(\tilde{n}^\tau, \tilde{u}^\tau, \tilde{E}^\tau, \tilde{B}^\tau\right)(t,x) = \left(n^\tau, u^\tau, E^\tau, B^\tau\right)(\tau t, x).$$

Then estimate (4.6) becomes

$$\left\|\left(\tilde{n}^\tau, \tilde{u}^\tau, \tilde{E}^\tau, \tilde{B}^\tau\right)(t) - (n_\tau, u_\tau, E_\tau, B_\tau)(\tau t)\right\|_s \leqslant C_2\tau^{\lambda+1}, \quad \forall t \in \left[0, T_1\tau^{-1}\right]. \tag{4.8}$$

On the other hand, the change of variable $t = t'/\tau$ gives

$$\int_0^{T_1} \left\|u^\tau\left(t'\right) - u_\tau\left(t'\right)\right\|_s^2 dt' = \tau \int_0^{T_1\tau^{-1}} \left\|\tilde{u}^\tau(t) - u_\tau(\tau t)\right\|_s^2 dt.$$

Hence, (4.7) implies that

$$\int_0^{T_1\tau^{-1}} \left\|\tilde{u}^\tau(t) - u_\tau(\tau t)\right\|_s^2 dt \leqslant C_2\tau^{2\lambda+3}. \tag{4.9}$$

4.2. Proof of the main result

By Proposition 1.1, the exact solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ is defined in a time interval $[0, T_1^\tau]$ with $T_1^\tau > 0$. Since $n^\tau \in C([0, T_1^\tau], H^s(\mathbb{T}))$ and the embedding from $H^s(\mathbb{T})$ to $C(\mathbb{T})$ is continuous, we have $n^\tau \in C([0, T_1^\tau] \times \mathbb{T})$. From (4.3)–(4.4) and assumption $n_0^\tau \geqslant \kappa > 0$, we deduce that there exist $T_2^\tau \in (0, T_1^\tau]$ and a constant $C_0 > 0$, independent of τ, such that

$$\frac{\kappa}{2} \leqslant n^\tau(t,x) \leqslant C_0 \quad \forall (t,x) \in \left[0, T_2^\tau\right] \times \mathbb{T}.$$

Similarly, the function $t \mapsto \|(n^\tau(t,\cdot), u^\tau(t,\cdot), E^\tau(t,\cdot), B^\tau(t,\cdot))\|_s$ is continuous in $C([0, T_2^\tau])$. From (4.3), the sequence $(\|(n^\tau(0,\cdot), u^\tau(0,\cdot), E^\tau(0,\cdot), B^\tau(0,\cdot))\|_s)_{\tau>0}$ is bounded. Then there exist $T_3^\tau \in (0, T_2^\tau]$ and a constant, still denoted by C_0, such that

$$\left\|\left(n^\tau(t,\cdot), u^\tau(t,\cdot), E^\tau(t,\cdot), B^\tau(t,\cdot)\right)\right\|_s \leqslant C_0, \quad \forall t \in \left(0, T_3^\tau\right].$$

Then we define $T^\tau = \min(T_1, T_3^\tau) > 0$ so that the exact solution and the approximate solution are both defined in the time interval $[0, T^\tau]$. In this time interval, we denote by

$$\left(N^\tau, U^\tau, F^\tau, G^\tau\right) = \left(n^\tau - n_\tau, u^\tau - u_\tau, E^\tau - E_\tau, B^\tau - B_\tau\right). \tag{4.10}$$

Obviously, $(N^\tau, U^\tau, F^\tau, G^\tau)$ satisfies the following problem:

$$\begin{cases} \partial_t N^\tau + \dfrac{1}{\tau}\left(\left(U^\tau + u_\tau\right)\cdot\nabla\right)N^\tau + \dfrac{1}{\tau}\left(N^\tau + n_\tau\right)\operatorname{div} U^\tau \\ \quad = -\dfrac{1}{\tau}\left(N^\tau \operatorname{div} u_\tau + \left(U^\tau\cdot\nabla\right)n_\tau\right) - R_n^\tau, \\ \partial_t U^\tau + \dfrac{1}{\tau}\left(\left(U^\tau + u_\tau\right)\cdot\nabla\right)U^\tau + \dfrac{1}{\tau}h'\left(N^\tau + n_\tau\right)\nabla N^\tau \\ \quad = -\dfrac{1}{\tau}\left[\left(U^\tau\cdot\nabla\right)u_\tau + \left(h'\left(N^\tau + n_\tau\right) - h'(n_\tau)\right)\nabla n_\tau\right] - \dfrac{U^\tau}{\tau^2} \\ \qquad - \dfrac{1}{\tau}\left[F^\tau + \left(U^\tau + u_\tau\right)\times G^\tau + U^\tau\times B_\tau\right] - R_u^\tau, \\ \partial_t F^\tau - \dfrac{1}{\tau}\nabla\times G^\tau = \dfrac{1}{\tau}\left(N^\tau U^\tau + N^\tau u_\tau + n_\tau U^\tau\right) - R_E^\tau, \quad \operatorname{div} F^\tau = -N^\tau, \\ \partial_t G^\tau + \dfrac{1}{\tau}\nabla\times F^\tau = 0, \quad \operatorname{div} G^\tau = 0, \\ t = 0\colon \quad \left(N^\tau, U^\tau, F^\tau, G^\tau\right) = \left(n_0^\tau - n_\tau(0,\cdot), u_0^\tau - u_\tau(0,\cdot), E_0^\tau - E_\tau(0,\cdot), B_0^\tau - B_\tau(0,\cdot)\right). \end{cases} \tag{4.11}$$

Set

$$W_I^\tau = \begin{pmatrix} N^\tau \\ U^\tau \end{pmatrix}, \qquad W_{II}^\tau = \begin{pmatrix} F^\tau \\ G^\tau \end{pmatrix}, \qquad W^\tau = \begin{pmatrix} W_I^\tau \\ W_{II}^\tau \end{pmatrix},$$

$$A_i^I\left(n^\tau, u^\tau\right) = \begin{pmatrix} u_i^\tau & n^\tau e_i^t \\ h'(n^\tau)e_i & u_i^\tau I_3 \end{pmatrix}, \quad i = 1, 2, 3,$$

$$H_1\left(W_I^\tau\right) = \begin{pmatrix} -(U^\tau\cdot\nabla)n_\tau - N^\tau \operatorname{div} u_\tau \\ -(U^\tau\cdot\nabla)u_\tau - (h'(N^\tau + n_\tau) - h'(n_\tau))\nabla n_\tau \end{pmatrix}, \qquad H_2\left(W_I^\tau\right) = \begin{pmatrix} 0 \\ -U^\tau \end{pmatrix},$$

$$H_3\left(W^\tau\right) = \begin{pmatrix} 0 \\ -F^\tau - (U^\tau + u_\tau)\times G^\tau - U^\tau\times B_\tau \end{pmatrix}, \qquad R^\tau = \begin{pmatrix} R_n^\tau \\ R_u^\tau \end{pmatrix},$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, y_i denotes the ith component of $y \in \mathbb{R}^3$ and I_3 is the 3×3 unit matrix. Then system (4.11) for unknown W_I^τ can be rewritten as

$$\partial_t W_I^\tau + \frac{1}{\tau}\sum_{i=1}^{3} A_i^I\left(n^\tau, u^\tau\right)\partial_{x_i} W_I^\tau = \frac{1}{\tau}\left(H_1\left(W_I^\tau\right) + H_3\left(W^\tau\right)\right) + \frac{1}{\tau^2}H_2\left(W_I^\tau\right) - R^\tau. \tag{4.12}$$

It is symmetrizable hyperbolic with symmetrizer

$$A_0^I\left(n^\tau\right) = \begin{pmatrix} (n^\tau)^{-1} & 0 \\ 0 & (h'(n^\tau))^{-1} I_3 \end{pmatrix},$$

which is a positive definite matrix when $0 < \frac{\kappa}{2} \leqslant n^\tau = N^\tau + n_\tau \leqslant C_0$. Moreover,

$$\tilde{A}_i^I(n^\tau, u^\tau) = A_0^I(n^\tau) A_i^I(n^\tau, u^\tau) = u_i^\tau A_0^I(n^\tau) + D_i \tag{4.13}$$

is symmetric for all $1 \leqslant i \leqslant 3$, where each D_i is a constant matrix

$$D_i = \begin{pmatrix} 0 & e_i^t \\ e_i & 0 \end{pmatrix}.$$

The existence and uniqueness of smooth solutions to (1.6)–(1.7) is equivalent to that of (4.11). Thus, in order to prove Theorem 4.1, it suffices to establish uniform estimates of W^τ with respect to τ. In what follows, we denote by $C > 0$ various constants independent of τ and for $\alpha \in \mathbb{N}^3$, $(W_{I\alpha}^\tau, W_{II\alpha}^\tau) = \partial_x^\alpha (W_I^\tau, W_{II}^\tau)$, etc. The main estimates are contained in the following two lemmas for W_I^τ and W_{II}^τ, respectively. We first consider the estimate for W_I^τ.

Lemma 4.1. *Under the assumptions of Theorem* 4.1, *for all $t \in (0, T^\tau]$, as $\tau \to 0$ we have*

$$\|W_I^\tau(t)\|_s^2 + \frac{1}{\tau^2} \int_0^t \|U^\tau(\xi)\|_s^2 \, d\xi \leqslant C \int_0^t \left(\|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi)\|_s^4\right) d\xi + C\tau^{2(\lambda+1)}. \tag{4.14}$$

Proof. For $\alpha \in \mathbb{N}^3$ with $|\alpha| \leqslant s$, differentiating Eqs. (4.12) with respect to x yields

$$\begin{aligned} &\partial_t W_{I\alpha}^\tau + \frac{1}{\tau} \sum_{i=1}^3 A_i^I(n^\tau, u^\tau) \partial_{x_i} W_{I\alpha}^\tau \\ &\quad = \frac{1}{\tau}\left[\partial_x^\alpha H_1(W_I^\tau) + \partial_x^\alpha H_3(W^\tau)\right] + \frac{1}{\tau^2} \partial_x^\alpha H_2(W_I^\tau) - \partial_x^\alpha R^\tau \\ &\qquad + \frac{1}{\tau} \sum_{i=1}^3 \left[A_i^I(n^\tau, u^\tau) \partial_{x_i} W_{I\alpha}^\tau - \partial_x^\alpha\left(A_i^I(n^\tau, u^\tau) \partial_{x_i} W_I^\tau\right)\right]. \end{aligned} \tag{4.15}$$

Multiplying (4.15) by $A_0^I(n^\tau)$ and taking the inner product of the resulting equations with $W_{I\alpha}^\tau$, by employing the classical energy estimate for symmetrizable hyperbolic equations, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(A_0^I(n^\tau) W_{I\alpha}^\tau, W_{I\alpha}^\tau\right) - \frac{2}{\tau^2}\left(A_0^I(n^\tau) \partial_x^\alpha H_2(W_I^\tau), W_{I\alpha}^\tau\right) \\ &\quad = \frac{2}{\tau}\left(A_0^I(n^\tau)\left[\partial_x^\alpha H_1(W_I^\tau) + \partial_x^\alpha H_3(W^\tau)\right], W_{I\alpha}^\tau\right) + \frac{2}{\tau}\left(J_\alpha^\tau, W_{I\alpha}^\tau\right) \\ &\qquad + \left(\operatorname{div} A_\tau^I(n^\tau, u^\tau) W_{I\alpha}^\tau, W_{I\alpha}^\tau\right) - 2\left(A_0^I(n^\tau) \partial_x^\alpha R^\tau, W_{I\alpha}^\tau\right), \end{aligned} \tag{4.16}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\mathbb{T})$,

$$J_\alpha^\tau = -\sum_{i=1}^3 A_0^I(n^\tau)\left[\partial_x^\alpha\left(A_i^I(n^\tau, u^\tau) \partial_{x_i} W_I^\tau\right) - A_i^I(n^\tau, u^\tau) \partial_x^\alpha\left(\partial_{x_i} W_I^\tau\right)\right]$$

and

$$\operatorname{div} A_\tau^I(n^\tau, u^\tau) = \partial_t A_0^I(n^\tau) + \frac{1}{\tau} \sum_{i=1}^3 \partial_{x_i} \tilde{A}_i^I(n^\tau, u^\tau). \tag{4.17}$$

Let us estimate each term of Eqs. (4.16). First, a direct computation gives

$$-\big(A_0^I(n^\tau)\partial_x^\alpha H_2(W_I^\tau), W_{I\alpha}^\tau\big) = \big((h'(n^\tau))^{-1}U_\alpha^\tau, U_\alpha^\tau\big) \geqslant C^{-1}\|U_\alpha^\tau\|^2. \tag{4.18}$$

On the other hand, since $A_0^I(n^\tau)$ is a positive definite matrix, we get:

$$\big(A_0^I(n^\tau)W_{I\alpha}^\tau, W_{I\alpha}^\tau\big) \geqslant C^{-1}\|W_{I\alpha}^\tau\|^2. \tag{4.19}$$

Moreover, we use the expression of $H_1(W_I^\tau)$ to compute:

$$\begin{aligned}\big(A_0^I(n^\tau)\partial_x^\alpha H_1(W_I^\tau), W_{I\alpha}^\tau\big) &= -(n^\tau)^{-1}N_\alpha^\tau\partial_x^\alpha\big[(U^\tau\cdot\nabla)n_\tau + N^\tau \operatorname{div} u_\tau\big]\\ &\quad - (h'(n^\tau))^{-1}U_\alpha^\tau\cdot\partial_x^\alpha\big[(U^\tau\cdot\nabla)u_\tau + \big(h'(N^\tau+n_\tau)-h'(n_\tau)\big)\nabla n_\tau\big].\end{aligned}$$

By Lemma 1.1 and $u_\tau = O(\tau)$, we get

$$\begin{aligned}\frac{2}{\tau}\big(A_0^I(n^\tau)\partial_x^\alpha H_1(W_I^\tau), W_{I\alpha}^\tau\big) &\leqslant \frac{C}{\tau}\big(\|N^\tau\|_s\|U^\tau\|_s + \tau\|N^\tau\|_s^2 + \tau\|U^\tau\|_s^2\big)\\ &\leqslant \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\|W^\tau\|_s^2.\end{aligned} \tag{4.20}$$

Here and hereafter, ε denotes a small constant independent of τ and $C_\varepsilon > 0$ denotes a constant depending only on ε.

For the term containing H_3, we have

$$\begin{aligned}\frac{2}{\tau}\big(A_0^I(n^\tau)\partial_x^\alpha H_3(W^\tau), W_{I\alpha}^\tau\big) &= -\frac{2}{\tau}\big((h'(n^\tau))^{-1}U_\alpha^\tau, \partial_x^\alpha\big[F^\tau + (U^\tau+u_\tau)\times G^\tau + U^\tau\times B_\tau\big]\big)\\ &\leqslant \frac{\varepsilon}{\tau^2}\|U_\alpha^\tau\|^2 + C_\varepsilon\int_{\mathbb{T}}\big|\partial_x^\alpha\big[F^\tau + (U^\tau+u_\tau)\times G^\tau + U^\tau\times B_\tau\big]\big|^2\,dx\\ &\leqslant \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\big(\|W^\tau\|_s^2 + \|U^\tau\|_s^2\|G^\tau\|_s^2\big).\end{aligned}$$

Therefore,

$$\frac{2}{\tau}\big(A_0^I(n^\tau)\partial_x^\alpha H_3(W^\tau), W_{I\alpha}^\tau\big) \leqslant \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\big(\|W^\tau\|_s^2 + \|W^\tau\|_s^4\big). \tag{4.21}$$

Now we consider the estimate for the term containing J_α^τ. Let us first point out that a direct application of Lemma 1.1 to J_α^τ does not yield the desired result. We have to develop the terms in the summation of J_α^τ to see the appearance of terms U^τ or $U^\tau + u_\tau$. By the definition of $A_i^I(n^\tau, u^\tau)$, we have

$$\begin{aligned}&\partial_x^\alpha\big(A_i^I(n^\tau,u^\tau)\partial_{x_i}W_I^\tau\big) - A_i^I(n^\tau,u^\tau)\partial_x^\alpha\big(\partial_{x_i}W_I^\tau\big)\\ &\quad= \begin{pmatrix}\partial_x^\alpha((U^\tau+u_\tau)_i\partial_{x_i}N^\tau) - (U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}N^\tau\\ \partial_x^\alpha(h'(N^\tau+n_\tau)\partial_{x_i}N^\tau e_i) - h'(N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}N^\tau e_i\end{pmatrix}\\ &\qquad+ \begin{pmatrix}\partial_x^\alpha((N^\tau+n_\tau)_i\partial_{x_i}U^\tau\cdot e_i^t) - (N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}U^\tau\cdot e_i^t\\ \partial_x^\alpha((U^\tau+u_\tau)_i\partial_{x_i}U^\tau) - (U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}U^\tau\end{pmatrix}.\end{aligned}$$

Then,

$$\begin{aligned}
&\big(A_0^I(n^\tau)\big(\partial_x^\alpha\big(A_i^I(n^\tau,u^\tau)\partial_{x_i}W_I^\tau\big)-A_i^I(n^\tau,u^\tau)\partial_x^\alpha\big(\partial_{x_i}W_I^\tau\big)\big),W_\alpha^\tau\big)\\
&\quad=\big(n^\tau\big)^{-1}\big[\partial_x^\alpha\big(\big(U^\tau+u_\tau\big)_i\partial_{x_i}N^\tau\big)-\big(U^\tau+u_\tau\big)_i\partial_x^\alpha\partial_{x_i}N^\tau\big]N_\alpha^\tau\\
&\qquad+\big(h'(n^\tau)\big)^{-1}\big[\partial_x^\alpha\big(\big(U^\tau+u_\tau\big)_i\partial_{x_i}U^\tau\big)-\big(U^\tau+u_\tau\big)_i\partial_x^\alpha\partial_{x_i}U^\tau\big]U_\alpha^\tau\\
&\qquad+\big(n^\tau\big)^{-1}\big(\partial_x^\alpha\big(\big(N^\tau+n_\tau\big)_i\partial_{x_i}U^\tau\cdot e_i^t\big)-\big(N^\tau+n_\tau\big)\partial_x^\alpha\partial_{x_i}U^\tau\cdot e_i^t\big)N_\alpha^\tau\\
&\qquad+\big(h'(n^\tau)\big)^{-1}\big[\partial_x^\alpha\big(h'\big(N^\tau+n_\tau\big)\partial_{x_i}N^\tau e_i\big)-h'\big(N^\tau+n_\tau\big)\partial_x^\alpha\partial_{x_i}N^\tau e_i\big]U_\alpha^\tau\\
&\quad=J_{i1}+J_{i2}+J_{i3}+J_{i4}.
\end{aligned}$$

Noting (4.3) for u_τ and applying Lemma 1.1 to each term on the right-hand side of the above equation gives

$$\begin{aligned}
|J_{i1}+J_{i2}|&\leqslant C\big(\tau+\|U^\tau\|_s\big)\|W_I^\tau\|_s^2\leqslant\frac{\varepsilon}{\tau}\|U^\tau\|_s^2+C_\varepsilon\tau\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big),\\
|J_{i3}+J_{i4}|&\leqslant C\big(1+\|N^\tau\|_s\big)\|N^\tau\|_s\|U^\tau\|_s\leqslant\frac{\varepsilon}{\tau}\|U^\tau\|_s^2+C_\varepsilon\tau\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big),
\end{aligned}$$

which imply that

$$\frac{2}{\tau}\big(J_\alpha^\tau,W_\alpha^\tau\big)\leqslant\frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2+C_\varepsilon\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big). \tag{4.22}$$

Using the expression of $A_0^I(n^\tau)$, we have obviously

$$-2\big(A_0^I(n^\tau)\partial_x^\alpha R^\tau,W_{I\alpha}^\tau\big)=-2\big(\big(n^\tau\big)^{-1}N_\alpha^\tau,\partial_x^\alpha R_n^\tau\big)-2\big(\big(h'(n^\tau)\big)^{-1}U_\alpha^\tau,\partial_x^\alpha R_u^\tau\big).$$

Together with (4.5) this yields

$$-2\big(A_0^I(n^\tau)\partial_x^\alpha R^\tau,W_{I\alpha}^\tau\big)\leqslant C\|W^\tau\|_s^2+\frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2+C_\varepsilon\tau^{2(\lambda+1)}. \tag{4.23}$$

Finally, for $i=1,2,3$, it follows from (4.13) and (4.17) that

$$\begin{aligned}
\operatorname{div}A_\tau^I\big(n^\tau,u^\tau\big)&=\big(A_0^I\big)'\big(n^\tau\big)\partial_t n^\tau+\frac{1}{\tau}\sum_{i=1}^{3}\partial_{x_i}\big[u_i^\tau A_0^I\big(n^\tau\big)\big]\\
&=\big(A_0^I\big)'\big(n^\tau\big)\left(\partial_t n^\tau+\frac{1}{\tau}\nabla n^\tau\cdot u^\tau\right)+\frac{1}{\tau}\operatorname{div}u^\tau A_0^I\big(n^\tau\big).
\end{aligned}$$

Using the first equation of (1.7), we deduce that

$$\operatorname{div}A_\tau^I\big(n^\tau,u^\tau\big)=\frac{\operatorname{div}u^\tau}{\tau}\big[A_0^I\big(n^\tau\big)-n^\tau\big(A_0^I\big)'\big(n^\tau\big)\big].$$

Noting

$$\frac{\kappa}{2}\leqslant n^\tau=N^\tau+n_\tau\leqslant C_0,\qquad u^\tau=U^\tau+u_\tau,\qquad u_\tau=O(\tau),$$

we obtain

$$\left\|\operatorname{div} u^\tau\right\|_\infty \leqslant C\left\|\operatorname{div}\left(U^\tau+u_\tau\right)\right\|_{s-1} \leqslant C\left(\left\|U^\tau\right\|_s+\tau\right).$$

Here in the last inequality we have used the continuous embedding $H^{s-1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. Therefore,

$$\left\|\operatorname{div} A_\tau^I\left(n^\tau, u^\tau\right)\right\|_\infty \leqslant C\left(1+\frac{1}{\tau}\left\|U^\tau\right\|_s\right).$$

We conclude that

$$\left(\operatorname{div} A_\tau^I\left(n^\tau, u^\tau\right) W_{I\alpha}^\tau, W_{I\alpha}^\tau\right) \leqslant \frac{\varepsilon}{\tau^2}\left\|U^\tau\right\|_s^2+C_\varepsilon\left(\left\|W^\tau\right\|_s^2+\left\|W^\tau\right\|_s^4\right). \tag{4.24}$$

Thus, together with (4.16) and (4.19)–(4.24), we obtain, for all $|\alpha| \leqslant s$,

$$\frac{d}{dt}\left(A_0^I\left(n^\tau\right) W_{I\alpha}^\tau, W_{I\alpha}^\tau\right)+\frac{\kappa}{\tau^2}\left\|U_\alpha^\tau\right\|^2 \leqslant \frac{C\varepsilon}{\tau^2}\left\|U^\tau\right\|_s^2+C_\varepsilon\left(\left\|W^\tau\right\|_s^2+\left\|W^\tau\right\|_s^4\right)+C_\varepsilon \tau^{2(\lambda+1)}.$$

Integrating this equation over $(0,t)$ with $t \in (0, T^\tau) \subset (0, T_1)$ and summing up over all $|\alpha| \leqslant s$, taking $\varepsilon > 0$ sufficiently small such that the term including $\frac{C\varepsilon}{\tau^2}\|U^\tau\|_s^2$ can be controlled by the left-hand side, together with condition (4.4) for the initial data and noting (4.19), we get (4.14). $\square$

Now, let us establish the estimate for W_{II}^τ.

Lemma 4.2. *Under the assumptions of Theorem 4.1, for all $t \in (0, T^\tau]$, as $\tau \to 0$ we have*

$$\left\|W_{II}^\tau(t)\right\|_s^2 \leqslant \int_0^t\left(\frac{1}{2\tau^2}\left\|U^\tau(\xi)\right\|_s^2+C\left\|W^\tau(\xi)\right\|_s^2+C\left\|W^\tau(\xi)\right\|_s^4\right) d\xi+C\tau^{2(\lambda+1)}. \tag{4.25}$$

Proof. For a multi-index $\alpha \in \mathbb{N}^3$ with $|\alpha| \leqslant s$, differentiating the third and fourth equations of (4.11) with respect to x, we have

$$\begin{cases} \partial_t F_\alpha^\tau-\dfrac{1}{\tau}\nabla \times G_\alpha^\tau=\dfrac{1}{\tau}\partial_x^\alpha\left(N^\tau U^\tau+N^\tau u_\tau+n_\tau U^\tau\right)-\partial_x^\alpha R_E^\tau, \\ \partial_t G_\alpha^\tau+\dfrac{1}{\tau}\nabla \times F_\alpha^\tau=0, \end{cases} \tag{4.26}$$

where $W_{II\alpha}^\tau=(F_\alpha^\tau, G_\alpha^\tau)=\partial_x^\alpha(F^\tau, G^\tau)$.

By the vector analysis formula:

$$\operatorname{div}(f \times g)=(\nabla \times f)\cdot g-(\nabla \times g)\cdot f,$$

the singular term appearing in Sobolev energy estimates vanishes, i.e.,

$$\int_{\mathbb{T}}\left(-\frac{1}{\tau}\nabla \times G_\alpha^\tau \cdot F_\alpha^\tau+\frac{1}{\tau}\nabla \times F_\alpha^\tau \cdot G_\alpha^\tau\right) dx=\frac{1}{\tau}\int_{\mathbb{T}}\operatorname{div}\left(F_\alpha^\tau \times G_\alpha^\tau\right) dx=0.$$

Hence, using $u_\tau = O(\tau)$ and (4.5), a standard energy estimate for (4.26) yields

$$\begin{aligned}\frac{d}{dt}\|W_{II\alpha}^\tau\|^2 &\leqslant \frac{2}{\tau}\int_{\mathbb{T}}\big(\big|\partial_x^\alpha\big(N^\tau U^\tau\big)\big|+\big|\partial_x^\alpha\big(N^\tau u_\tau\big)\big|+\big|\partial_x^\alpha\big(n_\tau U^\tau\big)\big|\big)\big|F_\alpha^\tau\big|\,dx+\int_{\mathbb{T}}\big|\partial_x^\alpha R_E^\tau\big|\big|F_\alpha^\tau\big|\,dx\\ &\leqslant \frac{1}{2\tau^2}\|U^\tau\|_s^2+C\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big)+C\tau^{2(\lambda+1)}. \qquad (4.27)\end{aligned}$$

Integrating (4.27) over $(0,t)$, with $t\in(0,T^\tau)$, summing up over α satisfying $|\alpha|\leqslant s$ and using (4.4) we obtain the lemma. □

Proof of Theorem 4.1. Let $\tau\to 0$ and $\varepsilon>0$ be sufficiently small. By Lemmas 4.1 and 4.2, for $t\in(0,T^\tau]$ we have

$$\|W^\tau(t)\|_s^2+\frac{1}{\tau^2}\int_0^t\|U^\tau(\xi)\|_s^2\,d\xi\leqslant C\int_0^t\big(\|W^\tau(\xi)\|_s^2+\|W^\tau(\xi)\|_s^4\big)\,d\xi+C\tau^{2(\lambda+1)}. \qquad (4.28)$$

Let

$$y(t)=C\int_0^t\big(\|W^\tau(\xi)\|_s^2+\|W^\tau(\xi)\|_s^4\big)\,d\xi+C\tau^{2(\lambda+1)}.$$

Then it follows from (4.28) that

$$\|W^\tau(t)\|_s^2\leqslant y(t),\qquad \frac{1}{\tau^2}\int_0^t\|U^\tau(\xi)\|_s^2\,d\xi\leqslant y(t),\quad \forall t\in\big(0,T^\tau\big], \qquad (4.29)$$

and

$$y'(t)=C\big(\|W^\tau(t)\|_s^2+\|W^\tau(t)\|_s^4\big)\leqslant C\big(y(t)+y^2(t)\big),$$

with

$$y(0)=C\tau^{2(\lambda+1)}.$$

A straightforward computation yields

$$y(t)\leqslant C\tau^{2(\lambda+1)}e^{Ct}\leqslant C\tau^{2(\lambda+1)}e^{CT_1},\quad \forall t\in\big[0,T^\tau\big].$$

Therefore, from (4.29) we obtain

$$\|W^\tau(t)\|_s\leqslant\sqrt{y(t)}\leqslant C\tau^{\lambda+1},\qquad \int_0^t\|U^\tau(\xi)\|_s^2\leqslant\tau^2y(t)\leqslant C\tau^{2(\lambda+2)},\quad \forall t\in\big[0,T^\tau\big].$$

In particular, this implies that W^τ is bounded in $L^\infty(0,T^\tau;H^s(\mathbb{T}))$, so is $(n^\tau,u^\tau,E^\tau,B^\tau)$. By a standard argument on the time extension of smooth solutions, we obtain $T_3^\tau\geqslant T_1$, i.e. $T^\tau=T_1$. This finishes the proof of Theorem 4.1. □

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Appendix A. Formal derivation of combined limits

We give a formal derivation of the combined zero-relaxation and zero-electron mass limits in system (1.2). For simplicity, let $\tau_i=\tau_e=\tau$, $q_i=-q_e=1$, $m_i=1$ and $\lambda=\gamma=1$. Then there are only two parameters τ and m_e in (1.2). Similarly to the one-fluid equations, we make the time scaling by replacing t by t/τ. We replace also u by τu. With this simplification, system (1.2) is written as:

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \\ m_\nu \tau^2 \big(\partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu\big) + \nabla h_\nu(n_\nu) = q_\nu(E + \tau u_\nu \times B) - m_\nu u_\nu, \\ \partial_t E - \dfrac{1}{\tau} \nabla \times B = n_e u_e - n_i u_i, \quad \operatorname{div} E = n_i - n_e, \\ \partial_t B + \dfrac{1}{\tau} \nabla \times E = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{A.1}$$

There are several situations of limits in view of the two parameters. First, the formal limit equations in the zero-electron mass limit $m_e \to 0$ of (A.1) are

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \\ \nabla h_e(n_e) = -(E + \tau u_e \times B), \\ \tau^2 \big(\partial_t u_i + (u_i \cdot \nabla) u_i\big) + \nabla h_i(n_i) = E + \tau u_i \times B - u_i, \\ \partial_t E - \dfrac{1}{\tau} \nabla \times B = n_e u_e - n_i u_i, \quad \operatorname{div} E = n_i - n_e, \\ \partial_t B + \dfrac{1}{\tau} \nabla \times E = 0, \quad \operatorname{div} B = 0, \end{cases} \tag{A.2}$$

in which only the momentum equations for electrons are changed. Up to our knowledge this limit system has not been analyzed mathematically. Furthermore, replacing B by τB and letting $\tau \to 0$ in (A.2), we obtain the limit equations

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \\ \nabla h_e(n_e) = \nabla \phi = -E, \\ \nabla h_i(n_i) = -u_i - \nabla \phi, \\ -\Delta \phi = n_i - n_e, \\ \nabla \times B = n_i u_i - n_e u_e - \partial_t \nabla \phi, \quad \operatorname{div} B = 0, \end{cases} \tag{A.3}$$

from which we deduce relations

$$n_e = h_e^{-1}(\phi), \qquad u_i = -\nabla\big(h_i(n_i) + \phi\big), \qquad E = -\nabla \phi \tag{A.4}$$

and a drift-diffusion type system

$$\begin{cases} \partial_t n_i - \operatorname{div}\big(n_i \nabla \big(h_i(n_i) + \phi\big)\big) = 0, \\ -\Delta \phi = n_i - h_e^{-1}(\phi), \end{cases} \tag{A.5}$$

where h_e^{-1} is the inverse function of h_e. Hence, we may determine (n_i, ϕ) by (A.5) and (n_e, u_i, E) by (A.4). Note that when n_e is given, the first equation in (A.3) for u_e is an incompressibility-type condition, which is not sufficient to determine u_e. The determination of u_e requires also another equations which can be derived from a high order asymptotic expansion.

Second, in the zero-relaxation limit $\tau \to 0$ of (A.1), still replacing B by τB, we obtain the classical drift-diffusion equations:

$$\begin{cases} m_\nu \partial_t n_\nu - \operatorname{div}\big(n_\nu \nabla\big(h_\nu(n_\nu) + q_\nu \phi\big)\big) = 0, \quad \nu = e, i, \\ -\Delta\phi = n_i - n_e, \end{cases} \tag{A.6}$$

together with

$$\begin{cases} m_\nu u_\nu = -\big(\nabla h_\nu(n_\nu) + q_\nu \nabla\phi\big), \quad \nu = e, i, \quad E = -\nabla\phi, \\ \nabla \times B = n_i u_i - n_e u_e - \partial_t \nabla\phi, \quad \operatorname{div} B = 0. \end{cases} \tag{A.7}$$

Taking the zero-electron mass limit $m_e \to 0$ in (A.6)–(A.7) we still obtain (A.3), which is also the combined limit system (A.1) as $(m_e, \tau) \to 0$. Therefore, the three limits $m_e \to 0$ then $\tau \to 0$, $\tau \to 0$ then $m_e \to 0$ and $(m_e, \tau) \to 0$ are formally commutative.

Another interesting limit is $m_e \to 0$ and $m_e \tau^2 = 1$. Hence, $\tau \to +\infty$. Replace B by B/τ, the limit $m_e \to 0$ and $m_e \tau^2 = 1$ in (A.1) gives equations

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \\ \partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = q_\nu (E + \tau u_\nu \times B), \\ \partial_t E = n_e u_e - n_i u_i, \quad \operatorname{div} E = n_i - n_e, \\ \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{A.8}$$

This is still a symmetrizable hyperbolic system for (n_ν, u_ν), $\nu = e, i$, coupled to a degenerate Maxwell system. Therefore, it is hopeful to prove a result on the local-in-time smooth solutions to (A.8). However, the justification of the limit is also an unsolved problem.

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