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On the compatibility condition for linear systems and a factorization formula for wave functions

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ABSTRACT

The well-known compatibility condition for linear systems $w_x = Gw$ and $w_t = Fw$ is considered and new results are obtained. In this way, a factorization formula for wave functions, which is basic in the inverse spectral transform approach to initial–boundary value problems, is proved in greater generality than before. Some applications follow.

1. Introduction

Zero curvature representation of the integrable nonlinear equations is a well-known approach (see [1,11,21,38] and references in [11]), which was developed soon after seminal Lax pairs appeared in [19]. Namely, many integrable nonlinear equations admit the representation (zero curvature representation)

$$G_t(x,t,z) - F_x(x,t,z) + \big[G(x,t,z), F(x,t,z)\big] = 0,$$
$$G_t := \frac{\partial}{\partial t}G, \quad [G,F] := GF - FG, \tag{1.1}$$

which is the compatibility condition of the auxiliary linear systems

$$\frac{\partial}{\partial x}w(x,t,z) = G(x,t,z)w(x,t,z), \qquad \frac{\partial}{\partial t}w(x,t,z) = F(x,t,z)w(x,t,z). \tag{1.2}$$

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Here G and F are $m \times m$ matrix functions, and z is the spectral parameter, which will be omitted sometimes in our notations.

Solution of the integrable nonlinear equations is closely related to the Lax pairs and zero curvature representations, which have been mentioned above, and has been a great breakthrough in the second half of the 20th century. An active study of the cases, which are close to integrable in a certain sense, followed (see, e.g., some references in [3,16]). Initial–boundary value problems for the integrable nonlinear equations can be considered as an important example, where integrability is "spoiled" by the boundary conditions. These problems are of great current interest, and the inverse spectral transform (ISpT) method [4,5,14,29,30] is one of the fruitful approaches in this domain. In particular, some further developments of the results and methods from [29,30] are given in [22,23,25,26,31,32].

Below we assume that x, t belong to a semi-strip

$$\mathcal{D} = \big\{(x,t)\colon\ 0 \leqslant x < \infty,\ 0 \leqslant t < a\big\}. \tag{1.3}$$

We normalize fundamental solutions of the auxiliary systems by the initial conditions

$$\frac{d}{dx} W(x,t,z) = G(x,t,z)W(x,t,z), \quad W(0,t,z) = I_m; \tag{1.4}$$

$$\frac{d}{dt} R(x,t,z) = F(x,t,z)R(x,t,z), \quad R(x,0,z) = I_m, \tag{1.5}$$

where I_m is the identity matrix of order m. If condition (1.1) holds, the fundamental solution of (1.4) admits factorization

$$W(x,t,z) = R(x,t,z)W(x,0,z)R(t,z)^{-1}, \quad R(t,z) := R(0,t,z). \tag{1.6}$$

Formula (1.6) is one of the basic and actively used formulas in the inverse spectral transform method (see [22,23,25,26,29–32] and references therein). It was derived in [29,30] under some smoothness conditions (continuous differentiability of G and F, in particular): see formulas (1.6) in [29, p. 22] and in [30, p. 39].

Here we prove (1.6) under weaker conditions and in much greater detail, which is important for applications. Namely, we prove the following theorem.

Theorem 1.1. *Let $m \times m$ matrix functions G and F and their derivatives G_t and F_x exist on the semi-strip $\mathcal{D}$, let G, G_t, and F be continuous with respect to x and t on $\mathcal{D}$, and let* (1.1) *hold. Then the equality*

$$W(x,t,z)R(t,z) = R(x,t,z)W(x,0,z), \quad R(t,z) := R(0,t,z), \tag{1.7}$$

holds.

Constructions similar to (1.6) appear also in the theory of Knizhnik–Zamolodchikov equation (see Theorem 3.1 in [34] and see also [33]).

We note that the solvability of system (1.2) in the domain $\mathcal{D}$ is of independent interest as one of the well-posedness and compatibility problems in domains with a boundary. The well-posedness of initial and initial–boundary value problems is a difficult area, which is actively studied (see, e.g., recent works [6,7,18] and references therein). Formula (1.7) implies that the condition (1.1) is, indeed, the compatibility condition and $w(x,t,z) = W(x,t,z)R(t,z) = R(x,t,z)W(x,0,z)$ satisfies (1.2).

Theorem 1.1 is proved in Section 2. Section 3 is dedicated to applications to initial–boundary value problems, and Theorem 3.2 on the evolution of the Weyl function for the "focusing" modified Korteweg–de Vries (mKdV) equation is proved there as an example.

As usual, by $\mathbb{N}$ we denote the set of positive integers, by $\mathbb{C}$ we denote the complex plane, and by $\mathbb{C}^m$ is denoted the m-dimensional coordinate space over $\mathbb{C}$. By $\Im z$ is denoted the imaginary part of $z \in \mathbb{C}$, and $\arg z$ is the argument of z. By $C^k(\mathcal{D})$ we denote the functions and matrix functions, which are k times continuously differentiable on $\mathcal{D}$.

2. Proof of Theorem 1.1

The spectral parameter z is non-essential for the formulation of Theorem 1.1 and for its proof and we shall omit it in this section. We shall need the proposition below.

Proposition 2.1. *Let the $m \times m$ matrix function W be given on the semi-strip $\mathcal{D}$ by Eq.* (1.4)*, where $G(x,t)$ and $G_t(x,t)$ are continuous matrix functions in x and t.*

(i) *Then the derivative W_t exists and matrix functions W and W_t are continuous with respect to x and t on the semi-strip $\mathcal{D}$.*

(ii) *Moreover, the mixed derivative W_{tx} exists and the equality $W_{tx} = W_{xt}$ holds on $\mathcal{D}$.*

Proof. Consider system

$$\frac{d}{dx}y = \widehat{G}(x,y)y, \quad \widehat{G}(x,y) = \widehat{G}(x,y_{m+1}) := \begin{bmatrix} G(x,y_{m+1}) & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.1}$$

where $\widehat{G}$ is an $(m+1) \times (m+1)$ matrix function and y_{m+1} is the last entry of the column vector $y \in \mathbb{C}^{m+1}$. Denote by W_j and e_j the j-th columns of W and I_m, respectively ($1 \leqslant j \leqslant m$). It easily follows from (1.4) that the solution of (2.1) with the initial condition

$$y(0) = g = \begin{bmatrix} e_j \\ t \end{bmatrix} \tag{2.2}$$

has the form

$$y(x,g) = \begin{bmatrix} W_j(x,t) \\ t \end{bmatrix}. \tag{2.3}$$

Putting $G(x,t) = G(0,t)$ for $-\varepsilon \leqslant x \leqslant 0$ whereas $t \geqslant 0$, and putting $G(x,t) = G(x,0) + tG_t(x,0)$ for $-\varepsilon \leqslant t \leqslant 0$ ($\varepsilon > 0$) we extend G so that G and G_t remain continuous on the rectangles

$$\mathcal{D}(a_1,a_2) = \big\{(x,t)\colon\ -\varepsilon \leqslant x \leqslant a_1,\ -\varepsilon \leqslant t \leqslant a_2 < a\big\}, \quad a_1, a_2 \in \mathbb{R}_+. \tag{2.4}$$

Hence, it follows from the definition of $\widehat{G}$ in (2.1) that $\widehat{G}(x,y)$ and, as a consequence, the vector function $\widehat{G}(x,y)y$ are continuous on $\mathcal{D}(a_1,a_2)$ together with their derivatives with respect to the entries of y. Thus, according to the classical theory of ordinary differential equations (see, for instance, theorem on pp. 305–306 in [36]) the partial first derivatives of $y(x,g)$ with respect to the entries of g exist in the interior $\mathcal{D}_i(a_1,a_2)$ of $\mathcal{D}(a_1,a_2)$. Moreover, y and its partial derivatives with respect to the entries of g are continuous. In particular, since by (2.2) we have $g_{m+1} = t$, the functions y and y_t are continuous in all rectangles $\mathcal{D}_i(a_1,a_2)$. Taking into account (2.3), we see that W and W_t are continuous in the rectangles $\mathcal{D}_i(a_1,a_2)$, and the statement (i) is proved.

In view of (1.4) and the considerations above the derivatives W_x, W_{xt}, and W_t exist and are continuous in the rectangles $\mathcal{D}_i(a_1,a_2)$. Hence, by a stronger formulation (see, e.g., the notes [2,35] or the book [20, p. 201]) of the well-known theorem on mixed derivatives, W_{tx} exists in $\mathcal{D}_i(a_1,a_2)$ and $W_{tx} = W_{xt}$. Thus, the statement (ii) is valid. □

Now, we can follow the scheme from [30, Chapter 3] (see also [32, Chapter 12]).

Proof of Theorem 1.1. According to statement (i) in Proposition 2.1 the matrix function W_t exists and is continuous. Introduce $U(x,t)$ by the equality

$$U := W_t - FW. \tag{2.5}$$

By (1.4), (2.5), and statement (ii) in Proposition 2.1 we have

$$U_x = W_{tx} - F_x W - F W_x = W_{xt} - F_x W - FGW. \tag{2.6}$$

It is immediate also from (1.4) that

$$W_{xt} = (GW)_t = G_t W + G W_t. \tag{2.7}$$

Formulas (2.6) and (2.7) imply

$$U_x = G_t W + G W_t - F_x W - FGW = (G_t - F_x + GF - FG)W + GW_t - GFW. \tag{2.8}$$

From (1.1), (2.8), and definition (2.5) we see that $U_x = GU$, that is, U and W satisfy the same equation. Taking into account $W(0,t) = I_m$, we derive $W_t(0,t) = 0$, and so by (2.5) we have $U(0,t) = -F(0,t)$. Finally, as

$$U_x = GU, \qquad W_x = GW, \qquad U(0,t) = -F(0,t), \qquad W(0,t) = I_m,$$

we have $U(x,t) = -W(x,t)F(0,t)$ or, equivalently,

$$W_t(x,t) - F(x,t)W(x,t) = -W(x,t)F(0,t). \tag{2.9}$$

Put

$$Y(x,t) = W(x,t)R(t), \qquad Z(x,t) = R(x,t)W(x,0). \tag{2.10}$$

Recall that $R(t) = R(0,t)$. Therefore (1.5), (2.9), and (2.10) imply that

$$\begin{aligned} Y_t(x,t) &= \big(F(x,t)W(x,t) - W(x,t)F(0,t)\big)R(t) + W(x,t)F(0,t)R(t) \\ &= F(x,t)Y(x,t), \quad Y(x,0) = W(x,0). \end{aligned} \tag{2.11}$$

Formulas (1.5) and (2.10) imply that

$$Z_t(x,t) = F(x,t)Z(x,t), \qquad Z(x,0) = W(x,0). \tag{2.12}$$

By (2.11) and (2.12) $Y = Z$, that is, (1.7) holds. The theorem is proved. □

Remark 2.2. Though the case of continuous F is more convenient for applications, it is immediate from the proof that the statement of Theorem 1.1 is true, when F is differentiable with respect to x, and measurable and summable with respect to t on all finite intervals from $\mathbb{R}_+$.

According to the proof of Theorem 1.1 the following remark is also true.

Remark 2.3. Theorem 1.1 holds on the domains more general than $\mathcal{D}$. In particular, it holds if we consider $(x,t) \in \mathcal{I}_1 \times \mathcal{I}_2$, where $\mathcal{I}_k$ $(k = 1,2)$ is the interval $[0, b_k)$ $(0 < b_k \leqslant \infty)$.

Another interesting case of matrix factorizations related to boundary value problems is treated in [8,15].

3. Some applications

The *matrix "focusing" mKdV equation* has the form

$$4v_t = v_{xxx} + 3\big(v_x v^* v + v v^* v_x\big), \tag{3.1}$$

where $v(x,t)$ is a $p \times p$ matrix function. Eq. (3.1) is equivalent (see [9,11,37] and references therein) to zero curvature equation (1.1), where the $m \times m$ $(m = 2p)$ matrix functions $G(x,t,z)$ and $F(x,t,z)$ are given by the formulas

$$G = izj + V, \quad j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ -v^* & 0 \end{bmatrix}, \tag{3.2}$$

$$F = -iz^3 j - z^2 V - \frac{iz}{2}\big(V^2 + V_x j\big) + \frac{1}{4}\big(V_{xx} - 2V^3 - V_x V + V V_x\big). \tag{3.3}$$

At first we omit the variable t in V and v. The Weyl theory of the skew-self-adjoint Dirac system (also called Zakharov–Shabat or AKNS system)

$$\frac{d}{dx} w(x,z) = \big(izj + V(x)\big) w(x,z), \quad x \geqslant 0 \tag{3.4}$$

was treated in [10,12,22,23] (see also preliminaries in [28]).

For the case of measurable matrix function v such that

$$\sup_{0<x<\infty} \big\| v(x) \big\| \leqslant M, \tag{3.5}$$

the Weyl matrix function φ of system (3.4) is uniquely defined in the semi-plane $\Im z < -M$ by the inequality

$$\int_0^\infty \begin{bmatrix} \varphi(z)^* & I_p \end{bmatrix} W(x,z)^* W(x,z) \begin{bmatrix} \varphi(z) \\ I_p \end{bmatrix} dx < \infty, \quad \Im z < -M < 0, \tag{3.6}$$

where W is the fundamental solution of (3.4) normalized by $W(0,z) = I_m$. Weyl functions are constructed using pairs of meromorphic $p \times p$ matrix functions $P_1(z)$, $P_2(z)$, which are nonsingular and have property-j, that is,

$$\mathcal{P}(z)^* \mathcal{P}(z) > 0, \qquad \mathcal{P}(z)^* j \mathcal{P}(z) \leqslant 0, \quad \mathcal{P} := \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}. \tag{3.7}$$

Theorem 3.1. *(See [22].) The above system* (3.4) *with v satisfying* (3.5) *has a unique Weyl function. This Weyl function is holomorphic in the semi-plane $\Im z < -M$. It is given by the equality*

$$\varphi(z) = \lim_{r \to \infty} \big(\mathcal{A}_{11}(r,z) P_1(r,z) + \mathcal{A}_{12}(r,z) P_2(r,z)\big)$$
$$\times \big(\mathcal{A}_{21}(r,z) P_1(r,z) + \mathcal{A}_{22}(r,z) P_2(r,z)\big)^{-1} \quad (\Im z < -M), \tag{3.8}$$

$$\mathcal{A}(r,z) = \big\{\mathcal{A}_{kp}(r,z)\big\}_{k,p=1}^2 := W(r,\bar{z})^*, \tag{3.9}$$

where the pairs $\{P_1, P_2\}$ are arbitrary pairs satisfying (3.7) *and W is the fundamental solution of* (3.4) *normalized by $W(0,z) = I_m$.*

Our next theorem on the evolution of the Weyl function in the case of the focusing mKdV follows from Theorems 1.1 and 3.1. The case of the defocusing mKdV was earlier treated in [29,30,32].

Theorem 3.2. *Let a $p \times p$ matrix function $v \in C^1(\mathcal{D})$ have a continuous partial second derivative v_{xx}, and let v_{xxx} exist. Assume that v satisfies mKdV* (3.1) *and that the inequalities*

$$\sup_{(x,t)\in\mathcal{D}} \left\| v(x,t) \right\| \leqslant M, \qquad \sup_{(x,t)\in\mathcal{D}} \left(\left\| v_x(x,t) \right\| + \left\| v_{xx}(x,t) \right\| \right) < \infty \tag{3.10}$$

hold.

Then the evolution $\varphi(t,z)$ of the Weyl function of the skew-self-adjoint Dirac system (1.4)*, where G has the form* (3.2)*, is given by the equality*

$$\varphi(t,z) = \left(R_{11}(t,z)\varphi(0,z) + R_{12}(t,z)\right)\left(R_{21}(t,z)\varphi(0,z) + R_{22}(t,z)\right)^{-1} \tag{3.11}$$

in the semi-plane $\Im z < -M < 0$. Here the block matrix function

$$R(t,z) = \left\{R_{kn}(t,z)\right\}_{k,n=1}^{2} = R(0,t,z) \tag{3.12}$$

is defined by the boundary values $v(0,t)$, $v_x(0,t)$, and $v_{xx}(0,t)$ via formulas (1.5) *and* (3.3).

Proof. As $V^* = -V$ and $(V_x j)^* = V_x j$, it is immediate from (3.3) that $F(x,t,\bar{z})^* + F(x,t,z) = 0$. Hence, it follows from (1.5) that

$$\frac{\partial}{\partial t}\left(R(x,t,\bar{z})^* R(x,t,z)\right) = 0.$$

Therefore, using equalities $R(x,0,z) = I_m$ and (3.12), we get

$$R(x,t,\bar{z})^* R(x,t,z) = I_m, \qquad R(t,\bar{z})^* R(t,z) = I_m,$$

or, equivalently,

$$R(x,t,\bar{z})^* = R(x,t,z)^{-1}, \qquad R(t,\bar{z})^* = R(t,z)^{-1}. \tag{3.13}$$

Because of the smoothness conditions on v, we see that G and F given by (3.2) and (3.3), respectively, satisfy the requirements of Theorem 1.1, that is, (1.7) holds. In view of (3.13) rewrite (1.7) in the form

$$\mathcal{A}(x,t,z)R(x,t,z) = R(t,z)\mathcal{A}(x,0,z), \tag{3.14}$$

where $\mathcal{A}(x,t,z) := W(x,t,\bar{z})^*$ (compare with (3.9)). Let $\mathcal{P}(x,z)$ satisfy (3.7) and put

$$\widetilde{\mathcal{P}}(x,t,z) = \begin{bmatrix} \widetilde{P}_1(x,t,z) \\ \widetilde{P}_2(x,t,z) \end{bmatrix} := R(x,t,z)\mathcal{P}(x,z). \tag{3.15}$$

By (3.14) and (3.15) we have

$$\mathcal{A}(x,t,z)\widetilde{\mathcal{P}}(x,t,z) = R(t,z)\mathcal{A}(x,0,z)\mathcal{P}(x,z). \tag{3.16}$$

Now, taking into account that $\mathcal{P}(x,z)$ is a nonsingular pair with property-j, we show that $\widetilde{\mathcal{P}}(x,t,z)$ is a nonsingular pair with property-j too. According to (1.5), (3.3), and (3.10) we get

$$\frac{\partial}{\partial t}\left(R(x,t,z)^* jR(x,t,z)\right) = R(x,t,z)^*\left(i\left(\bar{z}^3 - z^3\right)I_m + O\left(z^2\right)\right)R(x,t,z) \tag{3.17}$$

for $z \to \infty$. Formula (3.17) implies that for some

$$M_1 > M > 0 \quad \Big(M \geqslant \sup_{(x,t)\in\mathcal{D}} \big\|v(x,t)\big\|\Big), \tag{3.18}$$

and for all z from the domain

$$\mathcal{D}_1 = \{z\colon\ z \in \mathbb{C},\ \Im z < -M_1,\ 0 > \arg z > -\pi/4\} \tag{3.19}$$

we have

$$\frac{\partial}{\partial t}\big(R(x,t,z)^* jR(x,t,z)\big) \leqslant 0,$$

and so

$$R(x,t,z)^* jR(x,t,z) \leqslant j. \tag{3.20}$$

Relations (3.7), (3.15), and (3.20) imply that

$$\widetilde{\mathcal{P}}(x,t,z)^* \widetilde{\mathcal{P}}(x,t,z) > 0, \qquad \widetilde{\mathcal{P}}(x,t,z)^* j\widetilde{\mathcal{P}}(x,t,z) \leqslant 0 \quad (z \in \mathcal{D}_1). \tag{3.21}$$

Clearly, it suffices to prove (3.11) for values of z from $\mathcal{D}_1$. (According to (3.18) and (3.19) the domain $\mathcal{D}_1$ belongs to the semi-plane $\Im z < -M$.)

In a way similar to the proofs of (3.13) and (3.20) we derive

$$\mathcal{A}(x,t,z) = W(x,t,z)^{-1}, \qquad W(x,t,z)^* jW(x,t,z) \geqslant j \quad (\Im z < -M). \tag{3.22}$$

It is immediate from (3.22) that

$$\mathcal{A}(x,t,z)^* j\mathcal{A}(x,t,z) \leqslant j \qquad (\Im z < -M). \tag{3.23}$$

Hence, inequalities (3.7) and (3.21) imply

$$\det\big(\mathcal{A}_{21}(x,0,z)P_1(x,z) + \mathcal{A}_{22}(x,0,z)P_2(x,z)\big) \neq 0 \quad (\Im z < -M), \tag{3.24}$$

$$\det\big(\mathcal{A}_{21}(x,t,z)\widetilde{P}_1(x,t,z) + \mathcal{A}_{22}(x,t,z)\widetilde{P}_2(x,t,z)\big) \neq 0 \quad (z \in \mathcal{D}_1). \tag{3.25}$$

In view of (3.24) rewrite (3.16) as

$$\mathcal{A}(x,t,z)\widetilde{\mathcal{P}}(x,t,z) = R(t,z)\begin{bmatrix} \phi(x,0,z) \\ I_p \end{bmatrix}\big(\mathcal{A}_{21}(x,0,z)P_1(x,z) + \mathcal{A}_{22}(x,0,z)P_2(x,z)\big), \tag{3.26}$$

$$\begin{aligned}\phi(x,0,z) := {}&\big(\mathcal{A}_{11}(x,0,z)P_1(x,z) + \mathcal{A}_{12}(x,0,z)P_2(x,z)\big) \\ &\times \big(\mathcal{A}_{21}(x,0,z)P_1(x,z) + \mathcal{A}_{22}(x,0,z)P_2(x,z)\big)^{-1}.\end{aligned} \tag{3.27}$$

According to (3.24)–(3.26) we get

$$\begin{aligned}&\big(\mathcal{A}_{11}(x,t,z)\widetilde{P}_1(x,t,z) + \mathcal{A}_{12}(x,t,z)\widetilde{P}_2(x,t,z)\big) \\ &\quad\times \big(\mathcal{A}_{21}(x,t,z)\widetilde{P}_1(x,t,z) + \mathcal{A}_{22}(x,t,z)\widetilde{P}_2(x,t,z)\big)^{-1} \\ &= \big(R_{11}(t,z)\phi(x,0,z) + R_{12}(t,z)\big)\big(R_{21}(t,z)\phi(x,0,z) + R_{22}(t,z)\big)^{-1}.\end{aligned} \tag{3.28}$$

As $\widetilde{\mathcal{P}}(x,t,z)$ satisfies (3.21) for $z \in \mathcal{D}_1$, using (3.8) we derive

$$\varphi(t,z) = \lim_{x\to\infty}\big(\mathcal{A}_{11}(x,t,z)\widetilde{P}_1(x,t,z) + \mathcal{A}_{12}(x,t,z)\widetilde{P}_2(x,t,z)\big) \\ \times \big(\mathcal{A}_{21}(x,t,z)\widetilde{P}_1(x,t,z) + \mathcal{A}_{22}(x,t,z)\widetilde{P}_2(x,t,z)\big)^{-1} \quad (z \in \mathcal{D}_1). \tag{3.29}$$

In a similar way we derive from (3.8) and (3.27) that

$$\varphi(0,z) = \lim_{x\to\infty}\phi(x,0,z) \quad (\Im z < -M). \tag{3.30}$$

Let us show that

$$\det\big(R_{21}(t,z)\varphi(0,z) + R_{22}(t,z)\big) \neq 0 \quad (z \in \mathcal{D}_1). \tag{3.31}$$

Indeed, it follows from (3.7), (3.23), and (3.27) that

$$\begin{bmatrix} \phi(x,0,z)^* & I_p \end{bmatrix} j \begin{bmatrix} \phi(x,0,z) \\ I_p \end{bmatrix} \leqslant 0. \tag{3.32}$$

By (3.30) and (3.32) the inequality

$$\begin{bmatrix} \varphi(0,z)^* & I_p \end{bmatrix} j \begin{bmatrix} \varphi(0,z) \\ I_p \end{bmatrix} \leqslant 0 \tag{3.33}$$

is valid. Finally, inequalities (3.20) and (3.33) imply

$$\begin{bmatrix} \varphi(0,z)^* & I_p \end{bmatrix} R(t,z)^* j R(t,z) \begin{bmatrix} \varphi(0,z) \\ I_p \end{bmatrix} \leqslant 0 \quad (z \in \mathcal{D}_1). \tag{3.34}$$

It is immediate from (3.34) that (3.31) holds. In view of analyticity of both parts of (3.11), relations (3.28)–(3.31) imply (3.11) in the whole semi-plane $\Im z < -M$. □

In a way similar to [17] and the more general constructions for self-adjoint systems in [30,32] (see also some references therein), one can use structured operators to solve inverse problem for the skew-self-adjoint system (3.4) too. Namely, to recover v, which satisfies condition (3.5), from the Weyl function φ we use operators S_l (acting in $L_p^2(0,l)$, $0 < l < \infty$) of the form

$$S_l f = f(x) + \frac{1}{2}\int_0^l \int_{|x-r|}^{x+r} s'\left(\frac{\zeta + x - r}{2}\right) s'\left(\frac{\zeta + r - x}{2}\right)^* d\zeta\, f(r)\, dr. \tag{3.35}$$

Here $s' := \frac{d}{dx}s$. Below we give the procedure from [22] modified in accordance with [12,23].

First, we recover a $p \times p$ matrix function $s(x)$ with the entries from $L^2(0,l)$ (i.e., $s(x) \in L^2_{p\times p}(0,l)$) via the Fourier transform. That is, we put

$$s(x) = \frac{i}{2\pi}e^{-\eta x}\,\text{l.i.m.}_{a\to\infty}\int_{-a}^{a} e^{i\xi x} z^{-1}\varphi(z/2)\, d\xi \quad (z = \xi + i\eta,\ \eta < -2M), \tag{3.36}$$

the limit l.i.m. being the limit in $L^2_{p\times p}(0,l)$. Formula (3.36) has sense for any $l < \infty$, and so the matrix function $s(x)$ is defined on the non-negative real semi-axis $x \geqslant 0$. Moreover, s is absolutely continuous, it does not depend on the choice of $\eta < -2M$, s' is bounded on any finite interval, and $s(0) = 0$. To define the operator S_l we substitute $s'(x)$ into (3.35).

Next, we denote the $p \times 2p$ block rows of W by ω_1 and ω_2:

$$\omega_1(x) = [\, I_p \quad 0\,]\, W(x,0), \qquad \omega_2(x) = [\, 0 \quad I_p\,]\, W(x,0). \tag{3.37}$$

It follows from (3.2) and (3.4) that $W(x,0)^* W(x,0) = I_m$. Hence, by (3.2), (3.4), and (3.37) we have

$$v(x) = \omega_1'(x)\omega_2(x)^*, \tag{3.38}$$

and ω_1, ω_2 satisfy the equalities

$$\omega_1(0) = [\, I_p \quad 0\,], \qquad \omega_1\omega_1^* \equiv I_p, \qquad \omega_1'\omega_1^* \equiv 0, \qquad \omega_1\omega_2^* \equiv 0. \tag{3.39}$$

It is immediate that ω_1 is uniquely recovered from ω_2 using (3.39).

Finally, we obtain ω_2 via the formula

$$\omega_2(l) = [\, 0 \quad I_p\,] - \int_0^l \big(S_l^{-1}s'\big)(x)^* \big[I_p \quad s(x)\big]\, dx \quad (0 < l < \infty), \tag{3.40}$$

where S_l^{-1} is applied to s' columnwise.

Theorem 3.3. *Assume that φ is the Weyl function of system* (3.4)*, where j and V have the form* (3.2) *and v satisfies* (3.5)*. Then v is recovered from φ via formulas* (3.38)–(3.40)*, where s and S_l are given by equalities* (3.35) *and* (3.36)*. All the mentioned above relations are well defined and the inequalities $S_l \geqslant I$ hold.*

Another inverse problem, where condition (3.5) on v is substituted by a condition on φ, is also solved in [12,23,28] using the same procedure.

Remark 3.4. One can apply Theorems 3.2 and 3.3 to recover solutions of mKdV. Theorems on the evolution of the Weyl functions constitute also the first step in proofs of uniqueness and existence of the solutions of nonlinear equations via ISpT method (see, for instance, [28]).

The application of Theorem 1.1 to the GBDT version (see [13,24,27] and references therein) of the Bäcklund–Darboux transformation is of interest and shall be discussed in the next paper.

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