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The Stefan problem for the Fisher–KPP equation ☆

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We study the Fisher–KPP equation with a free boundary governed by a one-phase Stefan condition. Such a problem arises in the modeling of the propagation of a new or invasive species, with the free boundary representing the propagation front. In one space dimension this problem was investigated in Du and Lin (2010) [11], and the radially symmetric case in higher space dimensions was studied in Du and Guo (2011) [10]. In both cases a spreading-vanishing dichotomy was established, namely the species either successfully spreads to all the new environment and stabilizes at a positive equilibrium state, or fails to establish and dies out in the long run; moreover, in the case of spreading, the asymptotic spreading speed was determined. In this paper, we consider the non-radially symmetric case. In such a situation, similar to the classical Stefan problem, smooth solutions need not exist even if the initial data are smooth. We thus introduce and study the "weak solution" for a class of free boundary problems that include the Fisher–KPP as a special case. We establish the existence and uniqueness of the weak solution, and through suitable comparison arguments, we extend some of the results obtained earlier in Du and Lin (2010) [11] and Du and Guo (2011) [10] to this general case. We also show that the classical Aronson–Weinberger result on the spreading speed obtained through the traveling wave solution approach is a limiting case of our free boundary problem here.

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1. Introduction

In this paper, we investigate the following free boundary problem

$$\begin{cases} u_t - d\Delta u = a(x)u - b(x)u^2 & \text{for } x \in \Omega(t), t > 0, \\ u = 0 \quad \text{and} \quad u_t = \mu|\nabla_x u|^2 & \text{for } x \in \Gamma(t), t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega_0, \end{cases}$$

where $\Omega(t) \subset \mathbb{R}^N$ $(N \geqslant 2)$ is bounded by the free boundary $\Gamma(t)$, with $\Omega(0) = \Omega_0$, μ and d are given positive constants, a, b are positive functions in $C(\mathbb{R}^N)$, and $u_0 > 0$ in Ω_0. This is an analogue of the classical one-phase Stefan problem but with a logistic type nonlinear source term on the right side of the differential equation. Such a diffusive equation is often called a Fisher–KPP equation, and has been widely used in the study of traveling wave solutions and propagation problems. In most of the paper, we actually consider a more general nonlinear term $g(x,u)$ which includes $a(x)u - b(x)u^2$ as a special case.

Similar to the classical Stefan problem, smooth solutions to the above free boundary problem need not exist even if the initial data are smooth (see [15] and explanations below). We will thus introduce and study the weak solutions.

The above free boundary problem arises from our efforts to better understand the nature of the spreading of invasive species. Since this is rather different from the traditional applications of free boundary problems, to motivate this research, we give some detailed accounts of the background below.

In invasion ecology, ample empirical evidences suggest that a great number of successful invasive species spread to their new environments with a constant speed after a short starting period. A classical example is the 1951 observation of Skellam [27] on the spreading of muskrat in Europe in the early 1900s: He calculated the area of the muskrat range from a map obtained from field data, took the square root (which gives the spreading radius) and plotted it against years, and found that the data points lay on a straight line. We refer to [26] for more empirical examples and for discussions of relevant mathematical models.

One of the most successful theories for the mathematical description of the propagation of species is based on the "traveling wave solutions". In the pioneering work of Fisher in 1937 [21], he made use of the equation

$$u_t - du_{xx} = au - bu^2, \quad t > 0,\ x \in \mathbb{R}^1 \tag{1.1}$$

to study the propagation of advantageous genes, where the function $u = u(t,x)$ stands for the population density at time t and location x of a spreading species that carries the advantageous genes, with diffusion rate d, intrinsic growth rate a, in a habitat with carrying capacity a/b. Fisher showed that for any constant c satisfying $c \geqslant c^* := 2\sqrt{ad}$, there exists a solution of the form $u = W(x - ct)$ with the property that

$$cW' + dW'' + aW - bW^2 = 0, \qquad W(-\infty) = a/b, \qquad W(+\infty) = 0; \tag{1.2}$$

no such solution exists if $c < c^*$. Such a solution is called a *traveling wave solution*, and the number c^* is called the minimal speed of the traveling waves. In another well-known paper by Kolmogorov et al. [23], the same result was proved for a more general class of equations whose nonlinearity has a similar behavior, now called Fisher–KPP type, or *monostable* type. Fisher [21] claims that c^* is the "spreading speed" for the advantageous gene in his research, and used a probabilistic argument to support his claim.

In 1975, Aronson and Weinberger [1] established a rather general theory based on the traveling wave solutions, which contains a rigorous proof of the 1937 claim of Fisher: For a spreading population $u(t,x)$ governed by the above equation (1.1) with initial distribution $u(0,x)$ confined to a bounded set of x (i.e., $u(0,x) = 0$ outside a bounded set), it was proved in [1] that

$$\lim_{t\to\infty,\,|x|\leqslant(c^*-\epsilon)t} u(t,x)=a/b, \qquad \lim_{t\to\infty,\,|x|\geqslant(c^*+\epsilon)t} u(t,x)=0 \tag{1.3}$$

for any small $\epsilon>0$. This means that if an observer travels in the direction of propagation at a speed c which is below c^*, then he would find that the population is close to the positive steady-state level a/b, while if his speed is above c^*, he would observe that the population is nearly 0. Therefore the transition phase of the solution (namely the level set $\{u=\lambda\}$ with $0<\lambda<a/b$), which is used to represent the propagation front here, propagates linearly in t at the speed c^* (for large time).

These mathematical results have been extended to higher dimensions in [2], and extensive further development on traveling wave solutions and the spreading speed has been achieved in several directions (e.g., [3–7,22,24,28–30]).

This approach for the propagation problem is a remarkable achievement. Nevertheless, it has some shortcomings. For example, it does not give a precise location of the spreading front. As the solution $u(t,x)$ is positive for all x once $t>0$, the front can only be described as the "transition phase" of the solution, which is a collection of level sets $\{u=\lambda\}$ with λ varying in a certain range. Moreover, when the logistic reaction term is used as in (1.1), this approach predicts persistent propagation (or spreading) regardless of the initial size of the species; namely, starting with any nontrivial (i.e., not identically zero) initial population $u(0,x)$, one has $u(t,x)\to a/b$ as $t\to\infty$ for all fixed x, that is, as time grows, the new population will spread to the entire available space and establish itself. This is in sharp contrast to numerous empirical evidences which indicate that successful spreading depends on the initial size [26,25].

The phenomenon that a species starting with small initial size may fail to establish is often explained by the "Allee effect" (populations may shrink at very low densities). Such an effect is usually incorporated in the model by replacing the logistic reaction term $au-bu^2$ by a bistable reaction term such as $f(u)=au(1-u)(u-\theta)$, $\theta\in(0,1/2)$. It is well known that when the logistic reaction term is replaced by such a bistable reaction term, as time $t\to\infty$, the solution of (1.1) with a nonnegative initial function u_0 whose supporting set is nonempty and compact may go to 0, or converge to a positive steady-state, depending on the size of u_0 (see e.g., [1,18,14]).

Recently, Du and Lin [11] used the one space dimension version of the above free boundary problem to study the spreading of species, and demonstrated that, even with the logistic reaction term, this free boundary model can predict both spreading and vanishing, according to the initial size of the population. The results of [11] have been extended in [10] to the situation of higher space dimensions in the radially symmetric case. In such a case the solution can be written as $u(t,r)$, $r=|x|$, $x\in\mathbb{R}^N$ $(N\geqslant 2)$, and it satisfies

$$\begin{cases} u_t-d\Delta u=u\big(\alpha(r)-\beta(r)u\big), & t>0,\ 0<r<h(t),\\ u_r(t,0)=0, \quad u\big(t,h(t)\big)=0, & t>0,\\ h'(t)=-\mu u_r\big(t,h(t)\big), & t>0,\\ h(0)=h_0, \quad u(0,r)=u_0(r), & 0\leqslant r\leqslant h_0, \end{cases} \tag{1.4}$$

where due to the radial symmetry, $\Delta u=u_{rr}+\frac{N-1}{r}u_r$, $r=h(t)$ is the moving boundary to be determined, h_0, μ and d are given positive constants. It was assumed that $\alpha,\beta\in C^{\nu_0}([0,\infty))$ for some $\nu_0\in(0,1)$, and there are positive constants $\kappa_1\leqslant\kappa_2$ such that

$$\kappa_1\leqslant\alpha(r)\leqslant\kappa_2, \qquad \kappa_1\leqslant\beta(r)\leqslant\kappa_2 \quad \text{for } r\in[0,\infty). \tag{1.5}$$

The initial function $u_0(r)$ satisfies

$$u_0\in C^2\big([0,h_0]\big), \qquad u_0'(0)=u_0(h_0)=0, \qquad u_0>0 \quad \text{in } [0,h_0). \tag{1.6}$$

Thus problem (1.4) describes the spreading of a new or invasive species with population density $u(t,|x|)$ over an N-dimensional habitat, which is radially symmetric but heterogeneous. The initial function $u_0(|x|)$ stands for the population in the very early stage of its introduction, which occupies an initial region B_{h_0}. Here and in what follows we use B_R to stand for the ball with center at 0 and radius R. The spreading front is represented by the free boundary $|x| = h(t)$, which is the $(N-1)$-dimensional sphere $\partial B_{h(t)}$ whose radius $h(t)$ grows at a rate that is proportional to the population gradient at the front: $h'(t) = -\mu u_r(t, h(t))$. The coefficient function $\alpha(|x|)$ represents the intrinsic growth rate of the species, $\beta(|x|)$ measures its intra-specific competition, and d is the diffusion rate.

The free boundary model gives a precise prediction of the location of the spreading front for any future time $t > 0$, which is an advantage over the Cauchy problem model, where the spreading front is approximated by a continuum of level set of the solution, $\{u = \lambda\}$, with λ varying in a certain range. As is typical with ecological models, a thorough justification of the free boundary condition $h'(t) = -\mu u_r(t, h(t))$ is difficult to supply, due partly to the lack of first principles for such ecological problems. Nevertheless, this free boundary condition can be deduced from the following consideration based on the population pressure at the front. In the process of spreading, the front of the population range expands under the pressure of diffusion (random walk of the species). On the other hand, since the population density is close to 0 near the front, to counter the Allee effect, the random movement of the individuals of the species at the front is affected by a tendency to stay close to the population range instead of moving away from it (for example, driven by the desire to find a mating partner), which generates a viscosity-like force at the front. It is natural to assume that this viscosity-like force at the front is a constant for a given species. Therefore the front propagates in a way that keeps the diffusion pressure at the front at a certain constant level k, determined by the viscosity-like force there. One can then use Fick's law to deduce the free boundary condition with $\mu = d/k$, where d is the diffusion rate in (1.4) (see [8] for details). It will follow from a general result of this paper that the corresponding Cauchy problem of (1.4) is the limiting problem of this free boundary problem as $\mu \to \infty$, that is, the free boundary problem reduces to the Cauchy problem when the diffusion pressure (or equivalently the viscosity-like force) at the front is decreased to 0. On the other extreme end $\mu = 0$, clearly the free boundary problem reduces to a fixed boundary problem with Dirichlet boundary conditions.

It was shown in [10] that (1.4) has a unique solution $(u(t,r), h(t))$ defined for all $t > 0$, with $u(t,r) > 0$ and $h'(t) > 0$ for $t > 0$ and $0 \leqslant r < h(t)$. Moreover, a spreading-vanishing dichotomy holds for (1.4), namely, as time $t \to \infty$, the population $u(t,r)$ either successfully establishes itself in the new environment (called spreading), in the sense that $h(t) \to \infty$ and $u(t,r) \to \hat{U}(r)$, where $\hat{U}(r)$ is the unique positive solution of the problem

$$-d\Delta U = U\big(\alpha\big(|x|\big) - \beta\big(|x|\big)U\big) \quad \text{in } \mathbb{R}^N,$$

or the population fails to establish and vanishes eventually (called vanishing), namely $h(t) \to h_\infty \leqslant R^*$ and $u(t,r) \to 0$, where $R^* > 0$ is determined by an eigenvalue problem, independent of the initial data. Furthermore, when spreading occurs, and when $\lim_{r\to\infty} \alpha(r)$ and $\lim_{r\to\infty} \beta(r)$ exist, for large time, the spreading speed approaches a positive constant k_0, i.e., $h(t) = [k_0 + o(1)]t$ as $t \to \infty$. The asymptotic spreading speed k_0 is uniquely determined by an auxiliary elliptic problem induced from (1.4), and is independent of the initial population size u_0. Moreover, if $\lim_{r\to\infty} \alpha(r) = a$ and $\lim_{r\to\infty} \beta(r) = b$, we have the following result (see Corollary 3.7 and Proposition 3.1 in [10]):

$$\lim_{\frac{a\mu}{bd}\to\infty} \frac{k_0}{\sqrt{ad}} = 2, \qquad \lim_{\frac{a\mu}{bd}\to 0} \frac{k_0}{\sqrt{ad}} \frac{bd}{a\mu} = 1/\sqrt{3}.$$

Hence when the quantity $\frac{a\mu}{bd}$ is large, the spreading speed k_0 is well approximated by the formula

$$k_0 \approx 2\sqrt{ad},$$

while when this quantity is small, k_0 can be approximated by the formula

$$k_0 \approx \frac{a\mu}{bd}\frac{\sqrt{ad}}{\sqrt{3}}.$$

The main purpose of this paper is to extend the results of [10] to the non-radially symmetric case, hence showing the phenomena revealed in the special cases in [11] and [10] are robust. In the general case, the free boundary condition can be described as follows: The velocity of the movement of a point x on the free boundary $\Gamma(t) \subset \mathbb{R}^N$ is in the direction of the outward normal ν_x at x, with magnitude proportional to the directional derivative of u at x in the direction ν_x. If $\Gamma(t)$ is expressed by

$$\Gamma(t) = \{x \in \mathbb{R}^N\colon\ \Phi(t,x) = 0\}$$

with

$$\nabla_x \Phi \neq 0 \quad \text{on } \Gamma(t), \qquad \Phi(t,x) < 0 \quad \text{in } \Omega(t), \tag{1.7}$$

where $\Omega(t)$ denotes the region in $\mathbb{R}^N$ bounded by $\Gamma(t)$, then

$$\nu_x = \frac{\nabla_x \Phi}{|\nabla_x \Phi|},$$

where we use the notation

$$\nabla \Phi(x,t) = (\Phi_t, \nabla_x \Phi) = (\Phi_t, \Phi_{x_1}, \ldots, \Phi_{x_N}).$$

Hence the condition governing the free boundary can be expressed by

$$\frac{\Phi_t}{|\nabla_x \Phi|} = \mu \frac{\partial u}{\partial \nu_x} \quad \text{on } \Gamma(t), \tag{1.8}$$

or

$$\Phi_t = \mu \nabla_x u \cdot \nabla_x \Phi \quad \text{on } \Gamma(t),$$

where μ is a positive constant.

Thus in the non-radially symmetric case, the corresponding free boundary problem is given by

$$\begin{cases} u_t - d\Delta u = a(x)u - b(x)u^2 & \text{for } x \in \Omega(t), t > 0, \\ u = 0 & \text{for } x \in \Gamma(t), t > 0, \\ \Phi_t = \mu \nabla_x u \cdot \nabla_x \Phi & \text{for } x \in \Gamma(t), t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega_0, \end{cases} \tag{1.9}$$

where μ and d are given positive constants. We assume that $a, b \in C(\mathbb{R}^N)$, and there are positive constants $\kappa_1 \leqslant \kappa_2$ such that

$$\kappa_1 \leqslant a(x) \leqslant \kappa_2, \qquad \kappa_1 \leqslant b(x) \leqslant \kappa_2 \quad \text{for } x \in \mathbb{R}^N. \tag{1.10}$$

The initial function $u_0(x)$ satisfies

$$u_0 \in C(\overline{\Omega}_0) \cap H^1(\Omega_0), \qquad u_0 > 0 \quad \text{in } \Omega_0, \qquad u_0 = 0 \quad \text{on } \partial\Omega_0. \tag{1.11}$$

This is a variant of the classical one-phase Stefan problem. By a *classical solution* of the problem (1.9) for $0 < t < T$, we mean a pair of functions (u, Φ) such that $\Phi \in C^1(\overline{\bigcup_{0<t<T} \Omega(t)})$ satisfies (1.7) and u, $\nabla_x u$ are continuous in $\bigcup_{0\leqslant t<T} \overline{\Omega(t)}$ and $\nabla_x^2 u$, u_t are continuous in $\bigcup_{0<t<T} \Omega(t)$. Moreover, (u, Φ) satisfies all the identities in (1.9).

Note that if (u, Φ) is a classical solution of (1.9), then by the maximum principle and Hopf boundary lemma, we find that $u > 0$ in $\Omega(t)$ and $\nabla_x u \neq 0$ on $\Gamma(t)$. Thus we may take a suitable extension of $-u$ over $\bigcup_{0\leqslant t<T} \overline{\Omega(t)}$ as Φ, and then the condition

$$\Phi_t = \mu \nabla_x u \cdot \nabla_x \Phi \quad \text{for } x \in \Gamma(t),\ 0 < t < T$$

is reduced to

$$u_t = \mu |\nabla_x u|^2 \quad \text{for } x \in \Gamma(t),\ 0 < t < T.$$

As mentioned earlier, a smooth solution to (1.9) does not exist in general (even for smooth initial data u_0). For example, if Ω_0 is an annulus, $u_0(x)$ is smooth and radially symmetric, and a, b are positive constants, so that the free boundary problem has a radially symmetric solution $(u(t,|x|), \Gamma(t))$, then it is easy to show that for small $t > 0$, the free boundary $\Gamma(t)$ consists of two spheres that enclose an annulus. As t increases, the part of the free boundary that consists of the small sphere shrinks while the big sphere expands. If $\mu > 0$ is large, then one can show that as t passes across a certain finite t_0, the small sphere shrinks to a point and then disappears, so for $t > t_0$ the free boundary $\Gamma(t)$ consists of only the big expanding sphere. Thus the solution forms a singularity at $t = t_0$. As in [19] and [15], we shall transform the problem (1.9) into a “generalized” one, and look for weak solutions.

In Section 2, we introduce the notion of weak solution for a class of problems that include (1.9) as a special case. More precisely we give a weak formulation for the problem

$$\begin{cases} u_t - d\Delta u = g(x,u) & \text{for } x \in \Omega(t), 0 < t < T, \\ u = 0 & \text{for } x \in \Gamma(t), 0 \leqslant t < T, \\ \Phi_t = \mu \nabla_x u \cdot \nabla_x \Phi & \text{for } x \in \Gamma(t), 0 < t < T, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega_0, \end{cases} \tag{1.12}$$

where $g(x,u)$ has the following properties:

$$\begin{aligned} &\text{(i)} && g(x,u) \text{ is continuous for } (x,u) \in \mathbb{R}^N \times [0,\infty), \\ &\text{(ii)} && g(x,u) \text{ is locally Lipschitz in } u \text{ uniformly for } x \in \mathbb{R}^N, \\ &\text{(iii)} && g(x,0) \equiv 0, \\ &\text{(iv)} && \text{there exists } c^* > 0 \text{ such that } g(x,u) \leqslant c^* u \text{ for all } x \in \mathbb{R}^N \text{ and } u \geqslant 0. \end{aligned} \tag{1.13}$$

Our definition of weak solutions follows the approach of Friedman [15] for the classical Stefan problem.

In Section 3, we show that the weak solution defined in Section 2 exists (Theorem 3.1) and is monotone with respect to μ, which implies its uniqueness (Theorem 3.5).

In Section 4, we prove that the weak solution depends continuously with the initial function u_0 (Theorem 4.1) and enjoys the usual comparison principle (Theorem 4.3).

In Section 5, we examine the case that $\mu \to \infty$. We show that in the limit, the solution of the free boundary problem (1.12) converges to the solution $U(t,x)$ of the corresponding Cauchy problem, with initial function u_0 (extended to 0 outside Ω_0); see Theorem 5.4.

In Section 6, we consider the long-time dynamical behavior of the Fisher–KPP problem (1.9). We make use of suitable comparison arguments and the results established for the radially symmetric case in [10] to obtain sufficient conditions for vanishing and spreading of the weak solution (Theorem 6.2), and also obtain rather sharp estimates on the spreading speed of the generalized free boundary (Theorem 6.4). For large μ, we show that the Aronson–Weinberger property (1.3) is recovered in the free boundary model (Theorem 6.6). However, without knowing the regularity of the free boundary, we are unable to establish a sharp spreading-vanishing dichotomy as in the simpler cases treated in [11] and [10].

A sharp spreading-vanishing dichotomy will be established in a forthcoming paper, where the regularity of the free boundary is considered. The proof for this sharp dichotomy relies on the regularity of the free boundary and on the results of this paper.

2. Weak formulation of the free boundary problem

Let Ω_0 be a bounded smooth domain in $\mathbb{R}^N$ ($N \geqslant 2$) and $\Gamma(t)$, $\Phi(t,x)$ and $\Omega(t)$ be as in Section 1, with $\Omega(0) = \Omega_0$. Instead of considering (1.12) for all $t > 0$, it is convenient to start by considering $0 < t < T$ for some arbitrarily given $T \in (0,\infty)$.

For a classical solution we require u_0 to be smooth and positive in Ω_0, and take the value 0 on $\partial\Omega_0$. But only (1.11) is required for weak solutions to be defined below.

To formulate a weak version of (1.12), as in [15], we extend the solution u to a bigger region $[0,T] \times G$, for some bounded domain $G \subset \mathbb{R}^N$, by defining $u(t,x) = 0$ for $x \in G \setminus \Omega(t)$. To choose G, we use an auxiliary radially symmetric free boundary problem which will guarantee that $\Omega(t)$ stays inside G for $0 \leqslant t \leqslant T$. This will be proved after the weak solution is defined and comparison results for weak solutions established. Hence the definition of the weak solution will turn out to be independent of the choice of G.

To find such a domain G, we choose a ball $B_{R_0}(x_0) \supset \Omega_0$, and a radial function $\bar{u}_0 \in C^2(\overline{B}_{R_0}(x_0))$ such that $\bar{u}_0 > 0$ in $B_{R_0}(x_0)$, $\bar{u}_0(R_0) = 0$ and

$$u_0(x) \leqslant \bar{u}_0\big(|x - x_0|\big) \quad \text{for } x \in \Omega_0.$$

By the properties of $g(x,u)$ stated in (1.13), $g(x,u) \leqslant c^* u$ for all $x \in \mathbb{R}^N$ and $u \geqslant 0$. We now choose $M^* > 0$ such that

$$\frac{\pi}{4}\frac{M^*\mu}{c^*} \geqslant \max\left\{R_0, \frac{2}{c^*}\right\}, \qquad M^*\cos\left(\frac{\pi}{2R_0} r\right) \geqslant \bar{u}_0(r), \quad \forall r \in [0, R_0).$$

With these choices we can show that

$$v^*(t,r) := M^* e^{c^* t}\cos\left(\frac{\pi}{2h^*(t)} r\right), \quad h^*(t) := \frac{\pi}{4}\frac{M^*\mu}{c^*} e^{c^* t}$$

form an upper solution to the problem

$$\begin{cases} v_t - d\Delta v = c^* v, & t > 0,\ 0 < r < h(t), \\ v_r(t,0) = 0, \quad v\big(t,h(t)\big) = 0, & t > 0, \\ h'(t) = -\mu v_r\big(t,h(t)\big), & t > 0, \\ h(0) = R_0, \quad v(0,r) = \bar{u}_0(r), & 0 \leqslant r \leqslant R_0, \end{cases} \tag{2.1}$$

which would guarantee, by suitable comparison arguments to be established later, that $\Omega(t)$ in (1.12) is contained in G for $t \in [0,T]$ provided that G is a smooth domain such that $G \supset B_\Lambda(x_0)$ with $\Lambda = h^*(T)$. We fix such a G. Clearly $\Omega_0 \Subset G$. We denote $G_T = (0,T) \times G$, $S = \bigcup_{0 \leqslant t < T} \Gamma(t)$, and

$$\alpha(w) = \begin{cases} w & \text{if } w > 0, \\ w - d\mu^{-1} & \text{if } w \leqslant 0, \end{cases}$$

$$\tilde{u}_0(x) = \begin{cases} u_0(x) & \text{for } x \in \Omega_0, \\ 0 & \text{for } x \in \mathbb{R}^N \backslash \Omega_0. \end{cases}$$

Definition 2.1. Suppose that Ω_0 is smooth, u_0 satisfies (1.11), g satisfies (1.13), and G is chosen as above. A nonnegative function $u \in H^1(G_T) \cap L^\infty(G_T)$ is called a *weak* solution of (1.12) over G_T if

$$\int_0^T \int_G \left[d\nabla_x u \cdot \nabla_x \phi - \alpha(u)\phi_t\right] dx\,dt - \int_G \alpha(\tilde{u}_0)\phi(0,x)\,dx = \int_0^T \int_G g(x,u)\phi\,dx\,dt \tag{2.2}$$

for every function $\phi \in C(\overline{G}_T) \cap H^1(G_T)$ such that $\phi = 0$ on $(\{T\} \times G) \cup ([0,T] \times \partial G)$.

As in [15], for each weak solution $u(t,x)$, the function $\alpha(u(t,x))$ is defined as $u(t,x)$ if $u(t,x) > 0$; at points where $u(t,x) = 0$ the function $\alpha(u(t,x))$ is only required to satisfy $-d\mu^{-1} \leqslant \alpha(u(t,x)) \leqslant 0$ and to be such that it is altogether a measurable function. However, if $v(x)$ is continuous and positive in Ω_0 and identically zero in $G \setminus \Omega_0$, with $\partial\Omega_0$ smooth (say Lipschitz), then we understand that $\alpha(v) = -d\mu^{-1}$ on $G \setminus \Omega_0$.

Remark 2.2. The choice of the test functions in (2.2) implies that if u is a weak solution over G_T, and $\tilde{G}$ is a subdomain of G with smooth boundary that contains $B_\Lambda(x_0)$, and $\sigma \in (0,T]$, then the restriction of u on $\tilde{G}_\sigma$ is a weak solution of (1.12) over $\tilde{G}_\sigma$.

Theorem 2.3. (a) *Assume that* (u,Φ) *is a classical solution of* (1.12)*. Then*

$$w(t,x) := \begin{cases} u(t,x) & \text{for } x \in \Omega(t),\ 0 < t < T, \\ 0 & \text{for } x \in G\backslash\Omega(t),\ 0 < t < T \end{cases}$$

is a weak solution of (1.12) *in* G_T.

(b) *Let* w *be a weak solution of* (1.12) *in* G_T*. Assume that there exists a* C^1 *function* Φ *in* $\overline{G_T}$ *satisfying*

$$\Omega(t) \equiv \left\{x \in G\colon\ w(t,x) > 0\right\} = \left\{x \in G\colon\ \Phi(t,x) < 0\right\}$$

with $\Omega(0) = \Omega_0$*, and*

$$\nabla_x \Phi \neq 0 \quad \text{on } \Gamma(t) \equiv \partial\Omega(t), \qquad \Phi < 0 \quad \text{in } \Omega(t), \qquad \Phi > 0 \quad \text{in } G\backslash\overline{\Omega(t)}.$$

Setting $u = w$ *in* $\bigcup_{0<t<T} \overline{\Omega(t)}$*, and assume that* u*,* $\nabla_x u$ *are continuous in* $\bigcup_{0\leqslant t<T} \overline{\Omega(t)}$ *and that* $\nabla_x^2 u$*,* u_t *are continuous in* $\bigcup_{0<t<T} \Omega(t)$*. Then* (u,Φ) *is a classical solution of* (1.12)*.*

Proof. To prove (a), we first use the divergence theorem over $G_* := \bigcup_{0<t<T}(G\backslash\overline{\Omega(t)})$ for

$$\int_{G_*} div\,\Psi\,dV$$

with $\Psi(t,x) = (\phi(t,x), 0, \ldots, 0) \in \mathbb{R}^{N+1}$, to obtain

$$\int_0^T \int_{G\setminus\overline{\Omega(t)}} \phi_t\,dx\,dt = -\int_S \phi \frac{\Phi_t}{|\nabla\Phi|}\,d\sigma - \int_{G\setminus\overline{\Omega}_0} \phi(0,x)\,dx$$
$$= -\int_0^T \int_{\Gamma(t)} \phi \frac{\Phi_t}{|\nabla_x\Phi|}\,dS_x\,dt - \int_{G\setminus\overline{\Omega}_0} \phi(0,x)\,dx. \tag{2.3}$$

Then we multiply both sides of the first equation in (1.12) by ϕ and integrate over $\bigcup_{0<t<T}\Omega(t)$. Since $u=0$ on $\Gamma(t)$, we obtain, by the divergence theorem and integration by parts,

$$\int_0^T \int_{\Omega(t)} \left(u\frac{\partial\phi}{\partial t} - d\nabla_x u\cdot\nabla_x\phi\right)dx\,dt + d\int_0^T \int_{\Gamma(t)} \phi\frac{\partial u}{\partial\nu_x}\,dS_x\,dt + \int_0^T \int_{\Omega(t)} g(x,u)\phi\,dx\,dt$$
$$= -\int_{\Omega_0} u_0\phi(0,x)\,dx. \tag{2.4}$$

Eq. (2.2) now follows from (1.8), (2.3) and (2.4).

Suppose, conversely, that w is a weak solution of (1.12) satisfying the assumption in statement (b). Since now $w\equiv 0$ in $\bigcup_{0\leqslant t\leqslant T}[G\setminus\Omega(t)]$, (2.2) reduces to

$$\int_0^T \int_{\Omega(t)} (-d\nabla_x w\cdot\nabla_x\phi + w\phi_t)\,dx\,dt + \int_{\Omega_0} u_0\phi(0,x)\,dx + \int_0^T \int_{\Omega(t)} g(x,w)\phi\,dx\,dt$$
$$+\int_0^T \int_{G\setminus\Omega(t)} \alpha(w)\phi_t\,dx\,dt + \int_{G\setminus\Omega_0}\left(-\frac{d}{\mu}\right)\phi(0,x)\,dx = 0. \tag{2.5}$$

Taking ϕ with support in $\bigcup_{0<t<T}\Omega(t)$, we find that (2.5) is reduced to

$$\int_0^T \int_{\Omega(t)} (-d\nabla_x w\cdot\nabla_x\phi + w\phi_t)\,dx\,dt + \int_0^T \int_{\Omega(t)} g(x,w)\phi\,dx\,dt = 0,$$

which gives, after integration by parts,

$$\int_0^T \int_{\Omega(t)} \big[w_t - d\Delta w - g(x,w)\big]\phi\,dx\,dt = 0.$$

Hence w satisfies the differential equation in (1.12) in the classical sense.

Applying (2.3) in (2.5) we deduce

$$\int_0^T \int_{\Omega(t)} (-d\nabla_x w\cdot\nabla_x\phi + w\phi_t)\,dx\,dt + \int_{\Omega_0} u_0\phi(0,x)\,dx + \int_0^T \int_{\Omega(t)} g(x,w)\phi\,dx\,dt$$

$$+\int_0^T\int_{G\setminus\Omega(t)}\left[\alpha(w)+\frac{d}{\mu}\right]\phi_t\,dx\,dt+\frac{d}{\mu}\int_0^T\int_{\Gamma(t)}\phi\frac{\Phi_t}{|\nabla_x\Phi|}\,dS_x\,dt=0. \tag{2.6}$$

Taking ϕ with support in $\bigcup_{0<t<T}[G\setminus\overline{\Omega}(t)]$ we deduce from (2.6) that

$$\int_0^T\int_{G\setminus\overline{\Omega}(t)}\left[\alpha(w)+\frac{d}{\mu}\right]\phi_t\,dx\,dt=0,$$

which implies that $\alpha(w)=-d/\mu$ a.e. in $\bigcup_{0<t<T}[G\setminus\overline{\Omega}(t)]$. Hence (2.6) can be simplified to

$$\int_0^T\int_{\Omega(t)}(-d\nabla_x w\cdot\nabla_x\phi+w\phi_t)\,dx\,dt+\int_{\Omega_0}u_0\phi(0,x)\,dx+\int_0^T\int_{\Omega(t)}g(x,w)\phi\,dx\,dt$$
$$+\frac{d}{\mu}\int_0^T\int_{\Gamma(t)}\phi\frac{\Phi_t}{|\nabla_x\Phi|}\,dS_x\,dt=0. \tag{2.7}$$

By the smoothness assumption on w, for any $\delta\in(0,T)$, we can use the divergence theorem and the proved differential identity to deduce

$$\begin{aligned}&\int_\delta^T\int_{\Omega(t)}(-d\nabla_x w\cdot\nabla_x\phi+w\phi_t)\,dx\,dt\\
&=-\int_\delta^T\int_{\Omega(t)}[w_t-d\Delta w]\phi\,dx\,dt-d\int_\delta^T\int_{\Gamma(t)}\phi\frac{\partial w}{\partial\nu_x}\,dS_x\,dt-\int_{\Omega(\delta)}w(\delta,x)\phi(\delta,x)\,dx\\
&=-\int_\delta^T\int_{\Omega(t)}g(x,w)\phi\,dx\,dt-d\int_\delta^T\int_{\Gamma(t)}\phi\frac{\partial w}{\partial\nu_x}\,dS_x\,dt-\int_{\Omega(\delta)}w(\delta,x)\phi(\delta,x)\,dx.\end{aligned}$$

Letting $\delta\to0$ in the first term and the last three terms, we obtain

$$\begin{aligned}&\int_0^T\int_{\Omega(t)}(-d\nabla_x w\cdot\nabla_x\phi+w\phi_t)\,dx\,dt\\
&\quad=-\int_0^T\int_{\Omega(t)}g(x,w)\phi\,dx\,dt-d\int_0^T\int_{\Gamma(t)}\phi\frac{\partial w}{\partial\nu_x}\,dS_x\,dt-\int_{\Omega_0}w(0,x)\phi(0,x)\,dx.\end{aligned}$$

Substituting this into (2.7) we obtain

$$\int_{\Omega_0}\left[u_0-w(0,x)\right]\phi(0,x)\,dx-\frac{d}{\mu}\int_0^T\int_{\Gamma(t)}\left(\frac{\Phi_t}{|\nabla_x\Phi|}+\mu\frac{\partial w}{\partial\nu_x}\right)\phi\,dS_x\,dt=0. \tag{2.8}$$

Taking ϕ with support in $\bigcup_{0\leqslant t<T}\Omega(t)$, we see from (2.8) that $w(0,x)=u_0(x)$ in Ω_0. Thus the initial condition in (1.12) is satisfied. Moreover, (2.8) is reduced to

$$\int_0^T\int_{\Gamma(t)}\left(\frac{\Phi_t}{|\nabla_x\Phi|}-\mu\frac{\partial w}{\partial\nu_x}\right)\phi\,dS_x\,dt=0,$$

which implies that

$$\Phi_t=\mu\nabla_x w\cdot\nabla_x\Phi\quad\text{for } x\in\Gamma(t),\ 0<t<T.$$

Thus all the identities in (1.12) are satisfied in the classical sense. This completes the proof of Theorem 2.3. □

3. Existence and uniqueness of weak solutions

In this section we will prove the existence of a weak solution of (1.12), and then prove a comparison result which implies the uniqueness of the weak solution. The existence proof is adapted from that of [15], but we correct the mistake there differently to [16] and with considerable simplifications (see our argument below for proving $J_m^1\to 0$). Our uniqueness proof is different from that of [15], though the idea of constructing test functions is from [15], which followed [19].

Theorem 3.1. *There exists a weak solution w of* (1.12) *over G_T.*

For the proof of this theorem, we will adapt the approximation arguments of [15]. Some preparations are needed before the proof. Let $\{\alpha_m(w)\}$ be a sequence of smooth functions such that $\alpha_m(w)\to\alpha(w)$ uniformly in any compact subset of $\mathbb{R}^1\setminus\{0\}$, and $\alpha_m(0)\to -d\mu^{-1}$, $w-d\mu^{-1}\leqslant\alpha_m(w)\leqslant w$ for all $w\in\mathbb{R}^1$. We may choose the $\alpha_m(w)$ in such a way that

$$\alpha_m'(u)\geqslant 1.\tag{3.1}$$

We now consider the following sequence of approximating problems:

$$\begin{cases}\dfrac{\partial\alpha_m(w_m)}{\partial t}-d\Delta w_m=g(x,w_m) & \text{in } G_T,\\ w_m=0 & \text{on } (0,T)\times\partial G,\\ w_m(0,x)=\tilde u_0(x) & \text{in } G.\end{cases}\tag{3.2}$$

It is well known that (see, for example [20,17]) (3.2) admits a unique solution w_m, and $w_m\geqslant 0$.

We will need the following comparison result.

Lemma 3.2. *Suppose that $\tilde\alpha(u)$ is a smooth function of $u\in\mathbb{R}^1$ such that $\tilde\alpha'(u)\geqslant 1$ for all u, and $f(t,x,u)$ is a continuous function which is locally Lipschitz continuous in u. If $u(t,x)$ and $v(t,x)$ satisfy* (*in the classical sense*)

$$\begin{aligned}&\frac{\partial}{\partial t}\tilde\alpha(u)-d\Delta u\geqslant f(t,x,u)\quad \text{in } G_T,\\ &\frac{\partial}{\partial t}\tilde\alpha(v)-d\Delta v\leqslant f(t,x,v)\quad \text{in } G_T,\\ &u\geqslant v\quad \text{on } (0,T)\times\partial G,\\ &u(0,x)\geqslant v(0,x)\quad \text{in } G,\end{aligned}$$

then

$$u(t,x) \geqslant v(t,x) \quad \text{in } G_T.$$

Proof. This follows from the standard maximum principle. Write $w = u - v$. Then w satisfies

$$\begin{aligned}\tilde{\alpha}'(u)w_t - d\Delta w &\geqslant f(t,x,u) - f(t,x,v) - \big[\tilde{\alpha}'(u) - \tilde{\alpha}'(v)\big]v_t \\ &= \big[c_1(t,x) + c_2(t,x)v_t\big]w \quad \text{in } G_T.\end{aligned}$$

Since $w \geqslant 0$ on $(0,T) \times \partial G$ and $w(0,x) \geqslant 0$ in G, and $\tilde{\alpha}'(u) \geqslant 1$, we can apply the standard comparison principle to deduce that $w \geqslant 0$ in G_T. □

Using (1.13) and Lemma 3.2, we find that $0 \leqslant w_m(t,x) \leqslant \overline{w}(t)$ for $t \in (0,T)$ and $x \in G$, where

$$\overline{w}(t) := \|\tilde{u}_0\|_\infty e^{c^* t}.$$

Therefore,

$$\max_{\overline{G}_T} |w_m| \leqslant C_1 := \|\tilde{u}_0\|_\infty e^{c^* T} \tag{3.3}$$

for $m \geqslant 1$. It follows that

$$\iint_{G_T} |w_m|^2 \, dx\, dt \leqslant C_2 := C_1^2 |G_T|. \tag{3.4}$$

Lemma 3.3. *There is a positive constant C_3, independent of m, such that*

$$\iint_{G_T} \left|\frac{\partial w_m}{\partial t}\right|^2 dx\, dt \leqslant C_3$$

and

$$\int_G \big|\nabla_x w_m(t,x)\big|^2 dx \leqslant C_3, \quad \forall t \in [0,T].$$

Proof. For any $\sigma \in (0,T]$, we multiply the first equation in (3.2) by $\partial w_m/\partial t$ and integrate the resulting equation over G_σ to obtain

$$\begin{aligned}&\iint_{G_\sigma} \alpha_m'(w_m)\left(\frac{\partial w_m}{\partial t}\right)^2 dx\, dt + d\iint_{G_\sigma} \nabla_x w_m \cdot \nabla_x \frac{\partial w_m}{\partial t}\, dx\, dt \\ &\quad = d\int_0^\sigma \int_{\partial G} \frac{\partial w_m}{\partial t}\frac{\partial w_m}{\partial \nu}\, dS_x\, dt + \iint_{G_\sigma} g(x, w_m)\frac{\partial w_m}{\partial t}\, dx\, dt.\end{aligned}$$

Using (3.1), (3.2), (3.3) and (3.4), we see that

$$\begin{aligned}
&\frac{1}{2}\iint\limits_{G_\sigma}\left(\frac{\partial w_m}{\partial t}\right)^2 dx\,dt+\frac{d}{2}\int\limits_G\left|\nabla_x w_m(\sigma,x)\right|^2 dx\\
&\quad\leqslant \frac{d}{2}\int\limits_G|\nabla_x\tilde{u}_0|^2\,dx+A\iint\limits_{G_\sigma}|w_m|\left|\frac{\partial w_m}{\partial t}\right|dx\,dt\\
&\quad\leqslant \frac{d}{2}\int\limits_G|\nabla_x\tilde{u}_0|^2\,dx+A^2\iint\limits_{G_\sigma}(w_m)^2\,dx\,dt+\frac{1}{4}\iint\limits_{G_\sigma}\left(\frac{\partial w_m}{\partial t}\right)^2 dx\,dt\\
&\quad\leqslant \frac{d}{2}\int\limits_G|\nabla_x\tilde{u}_0|^2\,dx+A^2C_2+\frac{1}{4}\iint\limits_{G_\sigma}\left(\frac{\partial w_m}{\partial t}\right)^2 dx\,dt
\end{aligned}\tag{3.5}$$

where $A=A(g,C_1)$ is independent of m. It follows from (3.5) that

$$\iint\limits_{G_\sigma}\left(\frac{\partial w_m}{\partial t}\right)^2 dx\,dt+2d\int\limits_G\left|\nabla_x w_m(\sigma,x)\right|^2 dx\leqslant C_3(d,T,g,C_2).\tag{3.6}$$

This completes the proof. □

The next lemma is a trivial consequence of (3.4) and Lemma 3.3.

Lemma 3.4. *The sequence $\{w_m\}$ of approximating functions is bounded in $H^1(G_T)$:*

$$\|w_m\|_{H^1(G_T)}\leqslant C_4\quad \textit{with } C_4 \textit{ independent of } m.$$

Proof of Theorem 3.1. In what follows we shall select various subsequences from $\{w_m\}$ and, to avoid inundation by subscripts, always denote the subsequence again by $\{w_m\}$. Lemma 3.4 implies, by Rellich's Lemma, that there is a subsequence $\{w_m\}$ and a function $w\in H^1(G_T)$ such that, as $m\to\infty$,

$$w_m\to w\quad \text{weakly in } H^1(G_T) \text{ and strongly in } L^2(G_T).\tag{3.7}$$

In particular, $w_m\to w$ and $w\geqslant 0$ almost everywhere in G_T. Moreover, in view of (3.3),

$$0\leqslant w\leqslant C_1\quad \text{in } G_T.$$

Furthermore, using Lemma 3.3, we deduce

$$\int\limits_G|\nabla_x w|^2\,dx\leqslant C_3\quad \text{for a.e. } t\in[0,T],\tag{3.8}$$

since the set

$$\left\{v\in H^1(G_T)\colon \int\limits_G\left|\nabla_x v(t,x)\right|^2 dx\leqslant C_3 \text{ for a.e. } t\in[0,T]\right\}$$

is closed and convex in $H^1(G_T)$, and such sets are closed under the weak limit.

In order to complete the proof of Theorem 3.1 it remains to show that w is a weak solution. Let ϕ be a test function as in Definition 2.1 and consider a w_m from the sequence $\{w_m\}$. Since w_m is a classical solution of (3.2), it is also a weak solution; that is,

$$\iint\limits_{G_T}\left[\alpha_m(w_m)\frac{\partial\phi}{\partial t}-d\nabla_x w_m\cdot\nabla_x\phi\right]dx\,dt+\int\limits_G\alpha_m(\tilde{u}_0)\phi(0,x)\,dx+\iint\limits_{G_T}g(x,w_m)\phi\,dx\,dt=0.$$

Therefore

$$\begin{aligned}&\iint\limits_{G_T}\left[\alpha(w)\frac{\partial\phi}{\partial t}-d\nabla_x w\cdot\nabla_x\phi\right]dx\,dt+\int\limits_G\alpha(\tilde{u}_0)\phi(0,x)\,dx+\iint\limits_{G_T}g(x,w)\phi\\&\quad=\iint\limits_{G_T}\big[\alpha(w)-\alpha_m(w_m)\big]\frac{\partial\phi}{\partial t}\,dx\,dt+\iint\limits_{G_T}d\nabla_x(w_m-w)\cdot\nabla_x\phi\,dx\,dt\\&\qquad+\int\limits_G\big[\alpha(\tilde{u}_0)-\alpha_m(\tilde{u}_0)\big]\phi(0,x)\,dx+\iint\limits_{G_T}\big[g(x,w)-g(x,w_m)\big]\phi\,dx\,dt\\&\quad\equiv J_m^1+J_m^2+J_m^3+J_m^4.\end{aligned}$$

Thus it will suffice to prove that

$$\lim_{m\to\infty}J_m^k=0\quad(k=1,2,3,4).$$

The fact that $J_m^k\to 0$ for $k=2,3,4$ is apparent. It remains to show that $J_m^1\to 0$. Since $\{\alpha_m(w_n)\}$ is a bounded sequence in G_T, by passing to a subsequence we can find $W\in L^2(G_T)$ such that $\alpha_m(w_m)\to W$ weakly in $L^2(G_T)$. On the set $G_T^+:=\{(t,x)\in G_T\colon\ w(t,x)>0\}$, since $w_m\to w$ a.e., we have $\alpha_m(w_m)\to\alpha(w)=w$ a.e. It follows that $W(t,x)=w(t,x)$ a.e. in G_T^+.

On the set $G_T^0:=G_T\setminus G_T^+$, we have $w_m\to 0$ a.e. Hence it follows from $w_m-d\mu^{-1}\leqslant\alpha_m(w_m)\leqslant w_m$ that

$$-d\mu^{-1}\leqslant W(t,x)\leqslant 0\quad\text{a.e. in } G_T^0.$$

Thus upon defining

$$\alpha(w)=\begin{cases}w(t,x)&\text{for }(t,x)\in G_T^+,\\W(t,x)&\text{for }(t,x)\in G_T^0,\end{cases}$$

we have $\alpha_m(w_m)\to\alpha(w)$ weakly in $L^2(G_T)$, and hence $J_m^1\to 0$ as $m\to\infty$. This completes the proof. □

Next we prove a comparison result which implies that the weak solution obtained in Theorem 3.1 is unique.

Theorem 3.5. *Suppose that $\mu_1\geqslant\mu_2>0$ and u_1 and u_2 are weak solutions of* (1.12) *with $\mu=\mu_1$ and μ_2, respectively. Then $u_1\geqslant u_2$ a.e. in G_T. In particular, the weak solution to* (1.12) *is unique.*

The proof of Theorem 3.5 is also based on an approximation argument. We now introduce the approximation functions and some estimates to be used in the proof.

With u_1 and u_2 as given in the statement of Theorem 3.5, we have

$$\iint_{G_T} \big[\alpha_2(u_2) - \alpha_1(u_1)\big](\phi_t + de\Delta\phi + e\ell\phi)\,dx\,dt = d\big(\mu_2^{-1} - \mu_1^{-1}\big) \int_{G\setminus\overline{\Omega}_0} \phi(0,x)\,dx \tag{3.9}$$

for every $\phi \in C^2(\overline{G}_T)$ that vanishes on $(\{T\}\times G)\cup([0,T]\times\partial G)$, where

$$\ell(t,x) = \begin{cases} \frac{g(x,u_2(t,x))-g(x,u_1(t,x))}{u_2(t,x)-u_1(t,x)} & \text{if } u_2(t,x)\neq u_1(t,x), \\ 0 & \text{if } u_2(t,x)=u_1(t,x), \end{cases}$$

and for $i=1,2$, $\alpha_i(u)$ denotes $\alpha(u)$ with $\mu=\mu_i$, and

$$e(t,x) = \begin{cases} \frac{u_2(t,x)-u_1(t,x)}{\alpha_2(u_2(t,x))-\alpha_1(u_1(t,x))} & \text{if } u_2(t,x)\neq u_1(t,x), \\ 0 & \text{if } u_2(t,x)=u_1(t,x). \end{cases}$$

It is easily checked that if we write

$$\alpha_2\big(u_2(t,x)\big) - \alpha_1\big(u_1(t,x)\big) = \overline{\alpha}(t,x)\big[u_2(t,x)-u_1(t,x)\big]$$

when $u_1(t,x)\neq u_2(t,x)$, then

$$\overline{\alpha}(t,x) \geqslant 1 \quad \text{a.e. in } G_T.$$

Therefore, there is $0<\tilde{C}_1\leqslant 1$ such that

$$0 \leqslant e(t,x) \leqslant \tilde{C}_1 \quad \text{for almost all } (t,x)\in G_T.$$

We approximate e in $L^2(G_T)$ by a sequence of smooth functions $e_m \in C^\infty(\overline{G}_T)$ with

$$\frac{1}{m} \leqslant e_m(t,x) \leqslant \tilde{C}_2, \quad (t,x)\in\overline{G_T}$$

for some $\tilde{C}_2$ independent of m. We also approximate ℓ, u_1, u_2 by smooth ℓ_m, u_m^1, u_m^2 such that

$$\|\ell_m - \ell\|_{L^2(G_T)} \to 0, \qquad \big\|u_m^1 - u_1\big\|_{L^2(G_T)} \to 0, \qquad \big\|u_m^2 - u_2\big\|_{L^2(G_T)} \to 0.$$

By (1.13) and the fact that $u_1, u_2 \in L^\infty(G_T)$, we may require that

$$\|\ell_m\|_\infty \leqslant \tilde{C}_3, \qquad \big\|u_m^1\big\|_\infty \leqslant \tilde{C}_3, \qquad \big\|u_m^2\big\|_\infty \leqslant \tilde{C}_3,$$

for some $\tilde{C}_3>0$ independent of m.

For any f in $C^\infty(G_T)$ with compact support, we solve

$$\begin{cases} \dfrac{\partial \phi_m}{\partial t} + de_m \Delta \phi_m + e_m \ell_m \phi_m = f & \text{in } G_T, \\ \phi_m = 0 & \text{on } \{T\} \times G, \\ \phi_m = 0 & \text{on } [0, T] \times \partial G. \end{cases} \tag{3.10}$$

The existence of smooth solutions ϕ_m to (3.10) follows from [17] and [20].

Lemma 3.6. *There is a positive constant $C_5 = C_5(T, f)$, independent of m, such that*

$$\max_{\overline{G}_T} |\phi_m| \leqslant C_5.$$

Proof. Choose large positive constants A and B so that

$$A > \tilde{C}_2 \tilde{C}_3 + 2\tilde{C}_2 \tilde{C}_3^2 + \|f\|_\infty \geqslant |e_m \ell_m| + |f|,$$

and

$$B > e^{AT}.$$

Then set

$$y(t) = Be^{-At} - 1 \quad \text{and} \quad z^\pm = y \pm \phi_m.$$

In G_T we have

$$\frac{\partial z^\pm}{\partial t} + de_m \Delta z^\pm + e_m \ell_m z^\pm = -A \pm f + (-A + e_m \ell_m) y < 0. \tag{3.11}$$

On $(\{T\} \times G) \cup ([0, T] \times \partial G)$, $z^\pm = y > 0$. It follows from the maximum principle (applied to $z^\pm(T - t, x)$) that

$$z^\pm \geqslant 0 \quad \text{in } G_T.$$

This implies that $y \geqslant \pm\phi_m$ in G_T and therefore

$$\max_{\overline{G}_T} |\phi_m| \leqslant \max_{t \in [0,T]} y(t) = C_5 := B - 1.$$

This completes the proof. $\square$

Lemma 3.7. *There is a positive constant $C_6 = C_6(T, f)$, independent of m, such that*

$$\left\| e_m^{1/2} \Delta \phi_m \right\|_{L^2(G_T)} \leqslant C_6.$$

Proof. Multiplying (3.10) by $\Delta\phi_m$ and integrating over G_T, we obtain

$$\iint\limits_{G_T} \left(\frac{\partial \phi_m}{\partial t} \Delta\phi_m + d e_m |\Delta\phi_m|^2 + e_m \ell_m \phi_m \Delta\phi_m \right) dx\,dt = \iint\limits_{G_T} f \Delta\phi_m \, dx\,dt.$$

Moreover,

$$\begin{aligned}
\iint\limits_{G_T} \frac{\partial \phi_m}{\partial t} \Delta\phi_m \, dx\,dt &= \int_0^T \int\limits_{\partial G} \frac{\partial \phi_m}{\partial t} \frac{\partial \phi_m}{\partial \nu} \, dS_x \, dt - \int_0^T \int\limits_G \nabla_x \left(\frac{\partial \phi_m}{\partial t} \right) \cdot \nabla_x \phi_m \, dx\,dt \\
&= -\frac{1}{2} \int_0^T \int\limits_G \frac{\partial}{\partial t} |\nabla_x \phi_m|^2 \, dx\,dt \\
&= \frac{1}{2} \int\limits_G \left| \nabla_x \phi_m(0,x) \right|^2 dx.
\end{aligned}$$

Since $\|e_m\|_{L^\infty(G_T)} \leqslant \tilde{C}_2$ and $\|\ell_m \phi_m\|_{L^\infty(\overline{G_T})} \leqslant \tilde{C}_4$, where $\tilde{C}_4 > 0$ is independent of m, we also have, for any $0 < \epsilon < d/2$,

$$\begin{aligned}
\left| \iint\limits_{G_T} e_m \ell_m \phi_m \Delta\phi_m \, dx\,dt \right| &\leqslant \epsilon \iint\limits_{G_T} e_m |\Delta\phi_m|^2 \, dx\,dt + \epsilon^{-1} \iint\limits_{G_T} e_m \ell_m^2 \phi_m^2 \, dx\,dt \\
&\leqslant \epsilon \iint\limits_{G_T} e_m |\Delta\phi_m|^2 \, dx\,dt + C(\epsilon, T),
\end{aligned}$$

where

$$C(\epsilon, T) := \epsilon^{-1} \tilde{C}_2 \tilde{C}_4^2 |G_T|.$$

Furthermore, since $f \in C_0^\infty(G_T)$,

$$\iint\limits_{G_T} f \Delta\phi_m \, dx\,dt = \iint\limits_{G_T} \phi_m \Delta f \, dx\,dt.$$

Hence

$$\frac{1}{2} \int\limits_G \left| \nabla_x \phi_m(0,x) \right|^2 dx + (d - \epsilon) \iint\limits_{G_T} e_m |\Delta\phi_m|^2 \, dx\,dt \leqslant C_7(\epsilon, T, f). \tag{3.12}$$

It follows from (3.12) that

$$\left[\iint\limits_{G_T} e_m |\Delta\phi_m|^2 \, dx\,dt \right]^{1/2} \leqslant C_8(T, f),$$

which completes the proof. □

Let $e = e(t, x)$ be as the above. We now use a special choice of e_m as in [9]. By convolving e with appropriate mollification kernels, one can find a sequence of functions $\bar{e}_m \in C^\infty(\overline{G_T})$ such that

$$0 \leqslant \bar{e}_m(x, t) \leqslant \sup_{G_T} e$$

in $\overline{G_T}$ and

$$\|e - \bar{e}_m\|_{L^2(G_T)} < \frac{1}{m} \tag{3.13}$$

for all $m \geqslant 1$. Set

$$e_m = \bar{e}_m + \frac{1}{m}. \tag{3.14}$$

Then by Lemma 5 of [9] there is a positive constant C_9, independent of m, such that

$$\left\| \frac{e}{e_m} \right\|_{L^2(G_T)} \leqslant C_9. \tag{3.15}$$

Proof of Theorem 3.5. Take $f \in C_0^\infty(G_T)$ nonnegative and let e_m be chosen as above. For ϕ_m determined by (3.10), by the maximum principle (applied to $\phi_m(T-t, x)$) we deduce that $\phi_m \leqslant 0$. We shall establish that for any such f,

$$\iint_{G_T} \big[\alpha_2(u_2) - \alpha_1(u_1)\big] f \, dx \, dt \leqslant 0, \tag{3.16}$$

from which it follows that $\alpha_2(u_2) \leqslant \alpha_1(u_1)$, and hence in the a.e. sense, $u_1(t, x) > 0$ whenever $u_2(t, x) > 0$, which implies $u_1 \geqslant u_2$ a.e. in G_T, as both u_1 and u_2 are nonnegative by definition.

Taking the smooth function ϕ_m as a test function in (3.9), we obtain, due to $\mu_1 \geqslant \mu_2 > 0$ and $\phi_m \leqslant 0$,

$$\iint_{G_T} \big[\alpha_2(u_2) - \alpha_1(u_1)\big] \left\{ \frac{\partial \phi_m}{\partial t} + de\Delta\phi_m + e\ell\phi_m \right\} dx \, dt \leqslant 0.$$

Hence

$$\begin{aligned}
&\iint_{G_T} \big[\alpha_2(u_2) - \alpha_1(u_1)\big] f \, dx \, dt \\
&\quad = \iint_{G_T} \big[\alpha_2(u_2) - \alpha_1(u_1)\big] \left\{ \frac{\partial \phi_m}{\partial t} + de_m\Delta\phi_m + e_m\ell_m\phi_m \right\} dx \, dt \\
&\quad \leqslant \iint_{G_T} \big[\alpha_2(u_2) - \alpha_1(u_1)\big] \big\{ d(e_m - e)\Delta\phi_m + (e_m\ell_m - e\ell)\phi_m \big\} \, dx \, dt
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left| \iint_{G_T} \big[\alpha_2(u_2) - \alpha_1(u_1)\big] d(e_m - e)\Delta\phi_m \,dx\,dt \right| \\
&\quad + \left| \iint_{G_T} \big[\alpha_2(u_2) - \alpha_1(u_1)\big](e_m\ell_m - e\ell)\phi_m \,dx\,dt \right| \\
&\leqslant M_1 \iint_{G_T} |e_m - e||\Delta\phi_m| \,dx\,dt + M_2 \iint_{G_T} |e_m\ell_m - e\ell| \,dx\,dt \\
&\equiv M_1 I_m^1 + M_2 I_m^2,
\end{aligned}
$$

for some M_1 and M_2 independent of m. To obtain (3.16), it suffices to show that

$$\lim_{m\to\infty} I_m^1 = 0, \qquad \lim_{m\to\infty} I_m^2 = 0. \tag{3.17}$$

The first limit follows from the arguments in the proof of (3.11) in [9], by making use of (3.15) and Lemma 3.7 above. The second limit follows directly from

$$\|e - e_m\|_{L^2(G_T)} \to 0, \qquad \|\ell - \ell_m\|_{L^2(G_T)} \to 0.$$

Thus (3.16) holds, and the proof of Theorem 3.5 is complete. □

4. Basic properties of the weak solution

In this section we obtain some basic properties for the weak solution $u(t,x)$ of (1.12). Our first result implies that the weak solution $u(t,x)$ of (1.12) is stable with respect to the initial function. Let u_1, u_2 be two weak solutions of (1.12) in G_T corresponding, respectively, to the initial functions u_0^1, u_0^2. Set

$$L = \max\big\{ \|u_0^1\|_\infty, \|u_0^2\|_\infty \big\}.$$

Theorem 4.1. *There is a constant $C = C(T,L)$ such that*

$$\|u_1 - u_2\|_{L^2(G_T)} \leqslant C\sqrt{\|u_0^1 - u_0^2\|_{L^2(\Omega_0)}}. \tag{4.1}$$

Proof. Let $f \in C_0^\infty(G_T)$ and consider the solution ϕ_m of (3.10). Using the definition of weak solutions and the notation for ℓ, ℓ_m used in the proof of (3.16), we obtain

$$
\begin{aligned}
&\iint_{G_T} \big[\alpha(u_1) - \alpha(u_2)\big]\big\{ f + d(e - e_m)\Delta\phi_m + (e\ell - e_m\ell_m)\phi_m \big\} \,dx\,dt \\
&\quad = \int_{\Omega_0} \big[u_0^2 - u_0^1\big]\phi_m(0,x) \,dx.
\end{aligned}
$$

Therefore

$$\left| \iint_{G_T} \big[\alpha(u_1) - \alpha(u_2)\big] f \,dx\,dt \right| \leqslant \left| \iint_{G_T} \big[\alpha(u_1) - \alpha(u_2)\big] d(e_m - e)\Delta\phi_m \,dx\,dt \right|$$

$$+\left|\iint\limits_{G_T}\big[\alpha(u_1)-\alpha(u_2)\big](e\ell-e_m\ell_m)\phi_m\,dx\,dt\right| + \int\limits_{\Omega_0}\big|u_0^2-u_0^1\big|\big|\phi_m(0,x)\big|\,dx.$$

As $m\to\infty$ the first and second terms on the right side of the above inequality tend to zero by precisely the same argument used in Section 3 to demonstrate (3.16). From the proof of Lemma 3.6 we see that

$$\max_{\overline{G_T}}|\phi_m|\leqslant C_5\big(T,\|f\|_\infty\big)$$

for all $m\geqslant 1$. Hence

$$\left|\iint\limits_{G_T}\big[\alpha(u_1)-\alpha(u_2)\big]f\,dx\,dt\right|\leqslant C_5\big(T,\|f\|_\infty\big)\int\limits_{\Omega_0}\big|u_0^2-u_0^1\big|\,dx. \tag{4.2}$$

Now $[\alpha(u_1)-\alpha(u_2)]$ is a bounded measurable function in G_T. It can be approximated in $L^2(G_T)$ by a sequence $f_i\in C_0^\infty(G_T)$ such that $\{f_i\}$ is bounded in $L^\infty(G_T)$ by a bound determined by the $L^\infty(G_T)$ norm of $[\alpha(u_1)-\alpha(u_2)]$ which, in turn, can be estimated in terms of L by (3.3). Thus we may replace f by f_i in (4.2) and let $i\to\infty$ to obtain

$$\iint\limits_{G_T}\big[\alpha(u_1)-\alpha(u_2)\big]^2\,dx\,dt\leqslant C_{10}\int\limits_{\Omega_0}\big|u_0^2-u_0^1\big|\,dx,$$

with C_{10} depending only on T and L. Since

$$\big|\alpha(u_1)-\alpha(u_2)\big|\geqslant|u_1-u_2|\quad\text{in } G_T,$$

Schwarz's inequality then yields

$$\iint\limits_{G_T}|u_1-u_2|^2\,dx\,dt\leqslant C\sqrt{\int\limits_{\Omega_0}\big|u_0^1-u_0^2\big|^2\,dx}$$

for some constant $C=C(T,L)$. This completes the proof. □

Remark 4.2. It is possible to replace the stability inequality (4.1) with a linear one:

$$\|u_1-u_2\|_{L^1(G_T)}\leqslant C\big\|u_0^1-u_0^2\big\|_{L^1(\Omega_0)}. \tag{4.3}$$

To obtain (4.3), we choose $f_i\in C_0^\infty(G_T)$ such that $\{f_i\}$ is bounded in $L^\infty(G_T)$ and converges to $sgn(\alpha(u_1)-\alpha(u_2))$ in $L^2(G_T)$, where $sgn(u)=1,0$ or -1 according to whether $u>0, u=0$ or $u<0$. Then replace f by f_i in (4.2) and let $i\to\infty$.

We next prove a comparison result for weak solutions. Suppose that g and $\hat{g}$ both satisfy (1.13), Ω_0 and $\hat{\Omega}_0$ are bounded smooth domains in $\mathbb{R}^N$, u_0 satisfies (1.11) and $\hat{u}_0$ satisfies (1.11) with Ω_0 replaced by $\hat{\Omega}_0$. Let u and $\hat{u}$ be the weak solution of (1.12) corresponding to (Ω_0, u_0, g) and $(\hat{\Omega}_0, \hat{u}_0, \hat{g})$, over G_T and $\hat{G}_T$, respectively.

Theorem 4.3. *Suppose that $\Omega_0 \subset \hat{\Omega}_0$, $u_0 \leqslant \hat{u}_0$ and $g \leqslant \hat{g}$. Then $u \leqslant \hat{u}$ in $G_T \cap \hat{G}_T$.*

Proof. By Remark 2.2, we may choose G large enough so that both u and $\hat{u}$ are defined over the same G_T. So we assume from now on that $G_T = \hat{G}_T$.

Let w_m and $\hat{w}_m$ be determined by (3.2) with reaction term g and $\hat{g}$ respectively, and the initial functions are obtained by extending u_0 and $\hat{u}_0$, respectively. By the comparison principle we clearly have $w_m \leqslant \hat{w}_m$ in G_T. By the proof of Theorem 3.1, we have $w_m \to u$ and $\hat{w}_m \to \hat{u}$ in $L^2(G_T)$. It follows that

$$u \leqslant \hat{u} \quad \text{in } G_T.$$

This completes the proof. $\square$

Define

$$\Omega(t) := \left\{x \in \mathbb{R}^N:\ u(t,x) > 0\right\}$$

and

$$\Omega^*(t) := \left\{x \in \mathbb{R}^N:\ |x - x_0| < h^*(t)\right\}.$$

We show next that

$$\Omega(t) \subset \Omega^*(t) \subset G, \quad \forall t \in [0, T]. \tag{4.4}$$

Clearly this would justify our claim on the choice of G before Definition 2.1. To this end, we first establish a global existence result by using minor modifications of the arguments in [10].

Consider the radially symmetric free boundary problem

$$\begin{cases} v_t - d\Delta v = g^*(r, v), & t > 0,\ 0 < r < h(t), \\ v_r(t, 0) = 0, \quad v\big(t, h(t)\big) = 0, & t > 0, \\ h'(t) = -\mu v_r\big(t, h(t)\big), & t > 0, \\ h(0) = R, \quad v(0, r) = v_0(r), & 0 \leqslant r \leqslant R, \end{cases} \tag{4.5}$$

where $g^*(r, v)$ is Hölder continuous, locally Lipschitz in v uniformly for $r \in [0, \infty)$, and there exists $C > 0$ such that

$$g^*(r, v) \leqslant Cv \quad \text{for all } r \geqslant 0,\ v \geqslant 0;$$

$v_0 \in C^2([0, R])$ and $v_0(r) > 0$ in $[0, R)$, $v_0(R) = 0$.

Proposition 4.4. *Problem* (4.5) *has a unique classical solution defined for all $t > 0$.*

Proof. The local existence and uniqueness can be proved by exactly the same argument used in the proof of Theorem 4.1 in [10], as the special nonlinearity in [10] was not needed in the proof there.

We may then proceed as in the proof of Theorem 4.3 in [10], but with the following modifications of Lemma 4.2 there:

Let (v,h) be a solution of (4.5) defined for $t \in (0,T_0)$ for some $T_0 \in (0,\infty)$. Then for any given $T = T_0 + \sigma$, $\sigma > 0$, there exist constants C_1 and C_2 depending on T but independent of T_0 such that

$$0 < v(t,r) \leqslant C_1, \qquad 0 < h'(t) \leqslant C_2 \quad \text{for } 0 < t < T_0,\ 0 \leqslant r < h(t). \tag{4.6}$$

To find C_1, we use $g^*(r,v) \leqslant Cv$ and the comparison principle to obtain

$$v(t,r) \leqslant \bar{v}(t) := \|v_0\|_\infty e^{Ct},$$

and hence we may take $C_1 := \|v_0\|_\infty e^{CT}$.

To find C_2, we may use the same construction as in the proof of Lemma 4.2 in [10], with some obvious modifications. □

By Proposition 4.4, we know that (2.1) has a unique classical solution (v,h) defined for all $t > 0$. It is easily checked that (v^*,h^*) is an upper solution of (2.1), and hence by the comparison principal (see [10]), we have

$$h(t) \leqslant h^*(t), \qquad v(t,r) \leqslant v^*(t,r) \quad \text{for } t > 0,\ 0 \leqslant r \leqslant h(t).$$

Denote

$$\mathcal{G}(t) = \big\{x\colon\ |x - x_0| < h(t)\big\},$$
$$\Phi(t,x) = |x - x_0| - h(t), \qquad V(t,x) = v\big(t, |x - x_0|\big).$$

We also extend $V(t,\cdot)$ to be zero outside $\mathcal{G}(t)$. Clearly (V,Φ) is a classical solution of the following problem:

$$\begin{cases} u_t - d\Delta u = c^* u & \text{for } x \in \mathcal{G}(t),\ 0 < t < T, \\ u = 0 & \text{for } x \in \Gamma(t),\ 0 < t < T, \\ \Phi_t = \mu \nabla_x u \cdot \nabla_x \Phi & \text{for } x \in \Gamma(t),\ 0 < t < T, \\ u(0,x) = \bar{u}_0\big(|x - x_0|\big) & \text{for } x \in \mathcal{G}(0), \end{cases} \tag{4.7}$$

where $\Gamma(t) = \partial\mathcal{G}(t) = \{x\colon\ |x - x_0| = h(t)\}$. By Theorems 2.3 and 3.5, V is the unique weak solution of (4.7) over G_T.

Let w_m be determined by (3.2), and let v_m be given by

$$\begin{cases} \dfrac{\partial \alpha_m(v_m)}{\partial t} - d\Delta v_m = c^* v_m & \text{in } G_T, \\ v_m = 0 & \text{on } (0,T) \times \partial G, \\ v_m(0,x) = \bar{u}_0\big(|x - x_0|\big) & \text{in } G, \end{cases}$$

where $\bar{u}_0$ is extended by 0 for $|x - x_0| > R$. By the comparison principle we clearly have $w_m \leqslant v_m$ in G_T. By the proof of Theorem 3.1, we have $w_m \to u$ and $v_m \to V$ in $L^2(G_T)$. It follows that

$$u(t,x) \leqslant V\big(t, |x|\big) \quad \text{in } G_T.$$

Therefore $u(t,x)=0$ for $|x-x_0|\geqslant h(t)$, which implies that $\Omega(t)\subset\Omega^*(t)$, since $h(t)\leqslant h^*(t)$. This proves (4.4).

Let us now look at the global nature of the weak solution of (1.12). We claim that the weak solution over G_T can be uniquely extended to all $t>T$. Firstly we observe that Remark 2.2 implies that the week solution does not depend on the particular choice of G and we may just take $G=B_{h^*(T)}(x_0)$. With G taken this way, for any $t>T$, we can fix $\hat{T}>t$ and then choose $\hat{G}=B_{h^*(\hat{T})}(x_0)$ and use Theorems 3.1 and 3.5 to conclude that (1.12) with G_T replaced by $\hat{G}_{\hat{T}}$ has a unique weak solution $\hat{u}$. By Remark 2.2, the restriction of $\hat{u}$ over G_T agrees with u, and so this is the unique extension of u to $T\leqslant t<\hat{T}$. Thus the weak solution can be regarded as uniquely defined for all $t>0$, or (1.12) has a unique weak solution with $T=\infty$. In particular, (1.9) has a unique weak solution defined for all $t>0$. Note however that this unique global weak solution satisfies (2.2) in the following sense: For any $T>0$ one can find G such that (2.2) is satisfied over G_T.

5. Asymptotic limit of the weak solution as $\mu\to\infty$

In this section, we study the asymptotic behavior of the weak solution of (1.12) as $\mu\to\infty$. To emphasize its dependence on μ, we denote the unique weak solution of (1.12) by u_μ, and denote $\Omega_\mu(t)=\{x\colon u_\mu(t,x)>0\}$. Let us note that the domain G in the definition of weak solutions depends on μ, and so we will also write G^μ instead of G, and G^μ_T instead of G_T.

Firstly we derive some bounds for u_μ that is independent of μ. From (3.3) we see that for any given $\sigma>0$,

$$0\leqslant u_\mu(t,x)\leqslant\|\tilde{u}_0\|_\infty e^{c^*\sigma},\quad\forall x\in\Omega_\mu(t),\ 0\leqslant t\leqslant\sigma,\ \mu>0.\tag{5.1}$$

Further bounds for u_μ are given in the following two lemmas.

Lemma 5.1. *Given any $\sigma>0$ and any ball $B_R(z_0)$ of radius R such that $B_R(z_0)\subset G^\mu$ for all large μ, say $\mu\geqslant\mu_0$, there exists $C=C(\sigma,R)>0$ so that*

$$\int_0^\sigma\int_{B_{R/2}(z_0)}|\nabla_x u_\mu|^2\,dx\,dt\leqslant C,\quad\forall\mu\geqslant\mu_0.\tag{5.2}$$

Proof. Fix $T>\sigma$ and let w_m be the approximate solutions given by (3.2). Clearly it suffices to show that (5.2) holds for w_m with every $m\geqslant1$ and all $\mu\geqslant\mu_0$. Note that G_T now becomes G^μ_T. To simplify notations, in the following, we will abuse the notation a little by writing B_R for $B_R(z_0)$, etc. We will also write ∇ instead of ∇_x.

Let $\eta(x)$ be a smooth function satisfying

$$0\leqslant\eta\leqslant1,\qquad\eta\equiv1\quad\text{in }B_{R/2},\qquad\eta\equiv0\quad\text{in }B_R^c:=\mathbb{R}^N\setminus B_R.$$

We now multiply (3.2) by $\alpha_m(w_m)\eta^2$ and integrate the resulting equation over $[0,\sigma]\times B_R$. After suitable integration by parts, we obtain

$$\int_0^\sigma\int_{B_R}\alpha_m(w_m)\eta^2\frac{\partial}{\partial t}\alpha_m(w_m)\,dx\,dt+d\int_0^\sigma\int_{B_R}\nabla w_m\cdot\nabla\big[\alpha_m(w_m)\eta^2\big]\,dx\,dt$$
$$=\int_0^\sigma\int_{B_R}g(x,w_m)\alpha_m(w_m)\eta^2\,dx\,dt.\tag{5.3}$$

We have

$$\int_0^\sigma \int_{B_R} \alpha_m(w_m)\eta^2 \frac{\partial}{\partial t}\alpha_m(w_m)\,dx\,dt = \frac{1}{2}\int_{B_R} \alpha_m^2\big(w_m(\sigma,x)\big)\eta^2(x)\,dx - \frac{1}{2}\int_{B_R} \alpha_m^2\big(\tilde{u}_0(x)\big)\eta^2(x)\,dx$$
$$\geqslant -C_1(R), \quad \forall m \geqslant 1,\ \forall \mu \geqslant \mu_0,$$

and by (3.3),

$$\begin{aligned}
&\int_0^\sigma \int_{B_R} \nabla w_m \cdot \nabla\big[\alpha_m(w_m)\eta^2\big]\,dx\,dt \\
&\quad = \int_0^\sigma \int_{B_R} \big\{|\nabla w_m|^2\alpha_m'(w_m)\eta^2 + [\nabla w_m \cdot \nabla\eta]\alpha_m(w_m)2\eta\big\}\,dx\,dt \\
&\quad \geqslant \int_0^\sigma \int_{B_R} |\nabla w_m|^2\eta^2\,dx\,dt - \frac{1}{2}\int_0^\sigma \int_{B_R} |\nabla w_m|^2\eta^2\,dx\,dt - C(\sigma)\int_0^\sigma \int_{B_R} |\nabla\eta|^2\,dx\,dt \\
&\quad \geqslant \frac{1}{2}\int_0^\sigma \int_{B_{R/2}} |\nabla w_m|^2\,dx\,dt - C_2(\sigma,R), \quad \forall m \geqslant 1,\ \forall \mu \geqslant \mu_0.
\end{aligned}$$

Clearly we also have

$$\int_0^\sigma \int_{B_R} g(x, w_m)\alpha_m(w_m)\eta^2\,dx\,dt \leqslant C_3(\sigma,R), \quad \forall m \geqslant 1,\ \forall \mu \geqslant \mu_0.$$

Substituting these estimates into (5.3), we find that w_m satisfies

$$\int_0^\sigma \int_{B_{R/2}} |\nabla_x w_m|^2\,dx\,dt \leqslant C(\sigma,R), \quad \forall m \geqslant 1,\ \forall \mu \geqslant \mu_0, \tag{5.4}$$

as we wanted. □

Lemma 5.2. *Given any $\sigma > 0$ and any ball B_R of radius R such that $B_R \subset G^\mu$ for all large μ, say $\mu \geqslant \mu_0$, there exists $C = C(\sigma,R) > 0$ so that*

$$\|u_\mu\|_{H^1([0,\sigma]\times B_{R/4})} \leqslant C, \quad \forall \mu \geqslant \mu_0. \tag{5.5}$$

Proof. As in the proof of Lemma 5.1, we fix $T > \sigma$ and let w_m be the approximate solutions given by (3.2). In view of (5.1) and Lemma 5.1, it suffices to show that

$$\int_0^\sigma \int_{B_{R/4}} \left|\frac{\partial w_m}{\partial t}\right|^2 dx\,dt \leqslant C(\sigma,R), \quad \forall m \geqslant 1,\ \forall \mu \geqslant \mu_0. \tag{5.6}$$

Let $\xi(x)$ be a smooth function such that

$$0 \leqslant \xi \leqslant 1, \qquad \xi \equiv 1 \quad \text{in } B_{R/4}, \qquad \xi \equiv 0 \quad \text{in } B^c_{R/2}.$$

We multiply (3.2) by $\frac{\partial w_m}{\partial t}\xi^2$ and integrate the resulting equation over $[0,\sigma] \times B_{R/2}$. After suitable integration by parts, we obtain

$$\int_0^\sigma \int_{B_{R/2}} \alpha_m'(w_m)\left(\frac{\partial w_m}{\partial t}\right)^2 \xi^2 \, dx\, dt + d\int_0^\sigma \int_{B_{R/2}} \nabla w_m \cdot \nabla\left[\frac{\partial w_m}{\partial t}\xi^2\right] dx\, dt$$
$$= \int_0^\sigma \int_{B_{R/2}} g(x, w_m)\frac{\partial w_m}{\partial t}\xi^2 \, dx\, dt. \tag{5.7}$$

Making use of (5.4) we obtain

$$\int_0^\sigma \int_{B_{R/2}} \nabla w_m \cdot \nabla\left[\frac{w_m}{\partial t}\xi^2\right] dx\, dt$$
$$= \int_0^\sigma \int_{B_{R/2}} \xi^2 \nabla w_m \cdot \nabla\left(\frac{\partial w_m}{\partial t}\right) dx\, dt + \int_0^\sigma \int_{B_{R/2}} 2\xi \frac{\partial w_m}{\partial t} \nabla w_m \cdot \nabla\xi \, dx\, dt$$
$$= \frac{1}{2}\int_{B_{R/2}} \left|\nabla w_m(\sigma, x)\right|^2 \xi^2(x)\, dx - \frac{1}{2}\int_{B_{R/2}} \left|\nabla \tilde{u}_0(x)\right|^2 \xi^2(x)\, dx + \int_0^\sigma \int_{B_{R/2}} 2\xi \frac{\partial w_m}{\partial t} \nabla w_m \cdot \nabla\xi \, dx\, dt$$
$$\geqslant -\frac{1}{2}\int_{B_{R/2}} \left|\nabla \tilde{u}_0(x)\right|^2 \xi^2(x)\, dx - \frac{1}{2d}\int_0^\sigma \int_{B_{R/2}} \left(\frac{\partial w_m}{\partial t}\right)^2 \xi^2 \, dx\, dt - 2d\int_0^\sigma \int_{B_{R/2}} |\nabla w_m|^2 |\nabla \xi|^2 \, dx\, dt$$
$$\geqslant -\frac{1}{2d}\int_0^\sigma \int_{B_{R/2}} \left(\frac{\partial w_m}{\partial t}\right)^2 \xi^2 \, dx\, dt - C_4(\sigma, R), \quad \forall m \geqslant 1,\ \forall \mu \geqslant \mu_0.$$

By (3.3), we also have, for all $m \geqslant 1$ and $\mu \geqslant \mu_0$,

$$\int_0^\sigma \int_{B_{R/2}} g(x, w_m)\frac{\partial w_m}{\partial t}\xi^2 \, dx\, dt \leqslant \frac{1}{4}\int_0^\sigma \int_{B_{R/2}} \left(\frac{\partial w_m}{\partial t}\right)^2 \xi^2 \, dx\, dt + C_5(\sigma, R).$$

Substituting the above estimates into (5.7), and recalling $\alpha_m'(w_m) \geqslant 1$, we deduce

$$\int_0^\sigma \int_{B_{R/4}} \left(\frac{\partial w_m}{\partial t}\right)^2 dx\, dt \leqslant \int_0^\sigma \int_{B_{R/2}} \left(\frac{\partial w_m}{\partial t}\right)^2 \xi^2 \, dx\, dt \leqslant C_6(\sigma, R), \quad \forall m \geqslant 1,\ \forall \mu \geqslant \mu_0.$$

Hence (5.6) holds and the proof is complete. □

Next we estimate $\Omega_\mu(t)$ for large μ.

Lemma 5.3. *For any given $\epsilon \in (0,1)$ and $R > 1$, there exists $\hat{\mu} = \hat{\mu}(\epsilon, R) > 0$ such that*

$$\overline{B}_R(0) \subset \Omega_\mu(t), \quad \forall t \in \left[\epsilon, \epsilon^{-1}\right],\ \forall \mu \geqslant \hat{\mu}.$$

Proof. By (3.3), we have

$$0 \leqslant w_m \leqslant C_1(\epsilon), \qquad 0 \leqslant u_\mu \leqslant C_1(\epsilon)$$

for all $m \geqslant 1$, $\mu > 0$ and $t \in [0, \epsilon^{-1}]$.

By our assumption on $g(x,u)$, we can find $c > 0$ such that

$$g(x,u) \geqslant -cu \quad \text{for } x \in \mathbb{R}^N,\ u \in \left[0, C_1(\epsilon)\right].$$

We now choose a ball $B_{r_0}(y_0) \subset \Omega_0$ and a C^1 radial function $\underline{u}_0(r)$ $(r = |x - y_0|)$ satisfying $\underline{u}_0(|x - y_0|) < u_0(x)$ for $|x - y_0| < r_0$ and

$$\underline{u}_0(r) > 0 \quad \text{for } r \in [0, r_0), \qquad \underline{u}_0(r_0) = 0, \qquad \underline{u}_0'(r_0) < 0.$$

We then consider the auxiliary radially symmetric free boundary problem

$$\begin{cases} v_t - d\Delta v = -cv, & t > 0,\ 0 < r < k(t), \\ v_r(t,0) = 0, \quad v\big(t, k(t)\big) = 0, & t > 0, \\ k'(t) = -\mu v_r\big(t, k(t)\big), & t > 0, \\ k(0) = r_0, \quad v(0,r) = \underline{u}_0(r), & 0 \leqslant r \leqslant r_0. \end{cases} \tag{5.8}$$

By Proposition 4.4, (5.8) has a unique solution (v_μ, k_μ) defined for all $t \geqslant 0$, and $k_\mu'(t) > 0$ due to the Hopf boundary lemma. We extend $v_\mu(t,r)$ to $r > k_\mu(t)$ by the value 0 and still use v_μ to denote the extended function. Much as before we can apply a comparison argument to show that

$$w_m \geqslant v_\mu \quad \text{and} \quad u_\mu \geqslant v_\mu \quad \text{for } 0 \leqslant t \leqslant \epsilon^{-1}.$$

It follows that

$$\Omega_\mu(t) \supset B_{k_\mu(t)}(y_0), \quad \forall t \in \left[0, \epsilon^{-1}\right]. \tag{5.9}$$

Applying Theorem 3.5, we find that $k_\mu(t)$ is non-decreasing in μ. Thus $\lim_{\mu\to\infty} k_\mu(t) = k_\infty(t) \in (0, \infty]$ always exists. Since $k_\mu(t) \geqslant k_\mu(\epsilon)$ for $t \in [\epsilon, \epsilon^{-1}]$, to complete the proof of the lemma, it suffices to show that $k_\infty(\epsilon) = \infty$.

For this purpose, we choose a smooth increasing function $\tilde{k}(t)$ for $t \in [0, \epsilon)$ satisfying $\tilde{k}(0) = r_0$, $\tilde{k}(\epsilon) = +\infty$. Then we consider the initial–boundary value problem

$$\begin{cases} v_t - d\Delta v = -cv, & t \in (0, \epsilon),\ 0 < r < \tilde{k}(t), \\ v_r(t,0) = 0, \quad v\big(t, \tilde{k}(t)\big) = 0, & t \in (0, \epsilon), \\ v(0,r) = \underline{u}_0(r), & 0 \leqslant r \leqslant r_0. \end{cases} \tag{5.10}$$

By standard theory on parabolic equations (5.10) has a unique positive solution $\tilde{v}(t,r)$. Moreover, for any $\delta \in (0,\epsilon)$, there exists $M_\delta > 0$ such that

$$\frac{\tilde{k}'(t)}{-\tilde{v}_r(t,\tilde{k}(t))} \leqslant M_\delta \quad \text{for } t \in (0,\epsilon-\delta]. \tag{5.11}$$

For $\mu > M_\delta$, $(\tilde{v},\tilde{k})$ is easily seen to be a lower solution to (5.8) in the range $0 \leqslant t < \epsilon - \delta$. By the comparison principle (see [10]), we obtain that

$$k_\mu(t) \geqslant \tilde{k}(t), \qquad v_\mu(t,r) \geqslant \tilde{v}(t,r) \quad \text{for } t \in (0,\epsilon-\delta] \text{ and } 0 < r < \tilde{k}(t).$$

It follows that

$$k_\infty(t) := \lim_{\mu\to+\infty} k_\mu(t) \geqslant \tilde{k}(t) \quad \text{for } t \in (0,\epsilon-\delta].$$

Thus, in view of $k'_\mu(t) > 0$,

$$k_\infty(\epsilon) \geqslant k_\infty(\epsilon-\delta) \geqslant \tilde{k}(\epsilon-\delta).$$

Letting $\delta \to 0$ we obtain

$$k_\infty(\epsilon) = +\infty,$$

as desired. □

We are now ready to prove the main result of this section.

Theorem 5.4. *Let u_μ be the unique solution to problem* (1.12)*. Then*

$$\lim_{\mu\to\infty} \Omega_\mu(t) = \mathbb{R}^N, \quad \forall t > 0, \tag{5.12}$$

and

$$u_\mu \to U \quad \text{in } C_{\text{loc}}^{\frac{1+\theta}{2},1+\theta}\big((0,\infty)\times\mathbb{R}^N\big) \text{ as } \mu \to \infty, \tag{5.13}$$

where θ can be any number in $(0,1)$ and $U(t,x)$ is the unique solution of the Cauchy problem

$$\begin{cases} U_t - d\Delta U = g(x,U) & \text{in } (0,\infty)\times\mathbb{R}^N, \\ U(0,x) = \tilde{u}_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{5.14}$$

Proof. Let μ_n be an arbitrary increasing sequence of positive numbers converging to ∞ as $n \to \infty$. We write $u_n := u_{\mu_n}$ and $\Omega_n(t) := \Omega_{\mu_n}(t)$. Clearly it suffices to prove the desired limit along μ_n. The limit for $\Omega_n(t)$ follows trivially from Lemma 5.3. It remains to prove the limit for u_n.

From Theorem 3.5 and (5.1), we find that, for any given $\epsilon \in (0,1)$,

$$0 \leqslant u_n(t,x) \leqslant u_{n+1}(t,x) \leqslant C(\epsilon) \quad \text{for } x \in \Omega_n(t),\ 0 \leqslant t \leqslant \epsilon^{-1}.$$

In view of Lemma 5.3, there exists a measurable function $U(t,x)$ defined on $(0,\infty)\times\mathbb{R}^N$, such that for any given bounded domain $D\subset\mathbb{R}^N$,

$$\lim_{n\to\infty} u_n(t,x) = U(t,x) \leqslant C(\epsilon) \quad \text{a.e. in } \left[\epsilon, \epsilon^{-1}\right]\times D. \tag{5.15}$$

On the other hand, from Lemmas 5.3 and 5.2, for all large n, $\|u_n\|_{H^1([0,\epsilon^{-1}]\times D)}$ has a bound independent of n. Therefore, by passing to a subsequence, u_n converges weakly in $H^1([0,\epsilon^{-1}]\times D)$ and strongly in $L^2([0,\epsilon^{-1}]\times D)$ to some function $\tilde{U}\in H^1([0,\epsilon^{-1}]\times D)$. Due to (5.15), which holds for any $\epsilon\in(0,1)$, we necessarily have $\tilde{U}=U$. This also implies that the entire original sequence u_n converges to U weakly in $H^1([0,\epsilon^{-1}]\times D)$ and strongly in $L^2([0,\epsilon^{-1}]\times D)$.

We now consider (2.2) for u_n with $T=\epsilon^{-1}$ and test function ϕ chosen such that $\phi=0$ if $t\geqslant\epsilon^{-1}$ or $x\in D^c=\mathbb{R}^N\setminus D$. We have

$$\int_0^T\int_D \left[d\nabla u_n\cdot\nabla\phi - \alpha_n(u_n)\phi_t\right]dx\,dt - \int_D \alpha_n(\tilde{u}_0)\phi(0,x)\,dx = \int_0^T\int_D g(x,u_n)\phi\,dx\,dt, \tag{5.16}$$

where

$$u_n - d/\mu_n \leqslant \alpha_n(u_n)\leqslant u_n, \qquad \tilde{u}_0 - d/\mu_n\leqslant\alpha_n(\tilde{u}_0)\leqslant\tilde{u}_0.$$

Hence

$$\alpha_n(u_n)-u_n\to 0, \qquad \alpha_n(\tilde{u}_0)-\tilde{u}_0\to 0$$

as $n\to\infty$ in the L^∞ norm. Letting $n\to\infty$ in (5.16), we deduce

$$\int_0^T\int_D [d\nabla U\cdot\nabla\phi - U\phi_t]\,dx\,dt - \int_D \tilde{u}_0(x)\phi(0,x)\,dx = \int_0^T\int_D g(x,U)\phi\,dx\,dt.$$

Since D and $T=\epsilon^{-1}$ are arbitrary, this implies that U satisfies (5.14) in the weak sense. By standard parabolic regularity, we find that $U\in C_{\mathrm{loc}}^{\frac{1+\theta}{2},1+\theta}((0,\infty)\times\mathbb{R}^N)$, and $u_n\to U$ in $C_{\mathrm{loc}}^{\frac{1+\theta}{2},1+\theta}((0,\infty)\times\mathbb{R}^N)$, $\forall\theta\in(0,1)$. □

Corollary 5.5. *Let u_μ and U be given as in Theorem* 5.4; *then $u_\mu(t,x)\leqslant U(t,x)$ for all $t>0$ and $x\in\Omega_\mu(t)$.*

Proof. From the proof of Theorem 5.4, we see that u_μ increases to U as $\mu\to\infty$. Therefore $u_\mu\leqslant U$ for every $\mu>0$. □

6. Dynamical behavior of the Fisher–KPP equation

We now make use of comparison arguments and the results on radially symmetric problems in [10] to investigate the dynamical behavior of the weak solution to (1.9), which is (1.12) with $T=\infty$ and with $g(x,u)$ taking the Fisher–KPP form:

$$g(x,u)=a(x)u-b(x)u^2.$$

Let $B_{R_0}(x_0)$ and $\bar{u}_0$ be given as in (2.1). We choose $\bar{a}, \bar{b} \in C^{\nu_0}([0,\infty))$ such that

$$\kappa_2 \geqslant \bar{a}\big(|x-x_0|\big) \geqslant a(x), \qquad \kappa_1 \leqslant \bar{b}\big(|x-x_0|\big) \leqslant b(x), \quad \forall x \in \mathbb{R}^N.$$

Then consider the problem

$$\begin{cases} v_t - d\Delta v = v\big(\bar{a}(r) - \bar{b}(r)v\big), & t>0,\ 0<r<h(t), \\ v_r(t,0)=0, \quad v\big(t,h(t)\big)=0, & t>0, \\ h'(t) = -\mu v_r\big(t,h(t)\big), & t>0, \\ h(0)=R_0, \quad v(0,r)=\bar{u}_0(r), & 0 \leqslant r \leqslant R_0. \end{cases} \tag{6.1}$$

It follows from [10] that (6.1) possesses a (unique) classical solution $(v(t,r), h(t))$ such that

$$h'(t)>0, \qquad v(t,r)>0 \quad \text{for } 0 \leqslant r < h(t),\ t>0.$$

If we define $\bar{u}(t,x) = v(t,|x-x_0|)$ for $|x-x_0| \leqslant h(t)$ and extend it to be zero for $|x-x_0|>h(t)$ $(t>0)$, then $\bar{u}$ is the weak solution of the free boundary problem induced from (6.1) over G_T, for any $T>0$. Now the argument used in the proof of Theorem 4.3 shows that

$$u \leqslant \bar{u} \quad \text{in } G_T,\ \forall T>0. \tag{6.2}$$

Thus, if we define

$$\mathcal{G}(t) = \big\{x\colon |x-x_0|<h(t)\big\}, \qquad \Omega(t) = \big\{x\colon u(t,x)>0\big\},$$

then

$$\Omega(t) \subset \mathcal{G}(t), \quad \forall t>0.$$

To obtain a lower bound for u and $\Omega(t)$, we choose r_0 and $\underline{u}_0$ as in (5.8), and also choose $\underline{a}, \underline{b} \in C^{\nu_0}([0,\infty))$ such that

$$\kappa_1 \leqslant \underline{a}\big(|x-y_0|\big) \leqslant a(x), \qquad \kappa_2 \geqslant \underline{b}\big(|x-y_0|\big) \geqslant b(x), \quad \forall x \in \mathbb{R}^N.$$

Then consider the problem

$$\begin{cases} w_t - d\Delta w = w\big(\underline{a}(r) - \underline{b}(r)w\big), & t>0,\ 0<r<k(t), \\ w_r(t,0)=0, \quad w\big(t,k(t)\big)=0, & t>0, \\ k'(t) = -\mu w_r\big(t,k(t)\big), & t>0, \\ k(0)=r_0, \quad w(0,r)=\underline{u}_0(r) & 0 \leqslant r \leqslant r_0. \end{cases} \tag{6.3}$$

It follows from [10] that (6.3) possesses a (unique) classical solution $(w(t,r), k(t))$ such that

$$k'(t)>0, \qquad w(t,r)>0 \quad \text{for } 0 \leqslant r < k(t),\ t>0.$$

If we define $\underline{u}(t,x) = w(t, |x - y_0|)$ for $|x - y_0| \leqslant k(t)$ and extend it to be zero for $|x - y_0| > k(t)$ ($t > 0$), then $\underline{u}$ is the weak solution of the free boundary problem induced from (6.3), and we can similarly use the argument in the proof of Theorem 4.3 to conclude that

$$u \geqslant \underline{u} \quad \text{in } G_T,\ \forall T > 0. \tag{6.4}$$

Therefore, if we define

$$\mathcal{O}(t) := \{x \in \mathbb{R}^N\colon\ |x - y_0| < k(t)\},$$

then it follows from (6.2) and (6.4) that

$$\mathcal{O}(t) \subset \Omega(t) \subset \mathcal{G}(t), \quad \forall t \in (0, T). \tag{6.5}$$

Let us now look at the regularity of the weak solution inside $\bigcup_{t>0} \Omega(t)$. By Definition 2.1, for any open set $O \Subset \bigcup_{t>0} \Omega(t)$, $u(t,x)$ satisfies

$$u_t - \Delta u = a(x)u - b(x)u^2$$

in the usual weak sense for parabolic equations. Moreover, it follows from (6.2) that u is uniformly bounded from above. Hence it follows from standard parabolic regularity that $u \in C_{loc}^{\frac{\theta+1}{2}, 1+\theta}(O)$ for each $\theta \in (0,1)$. In particular, this is true for

$$O = \bigcup_{t>0} \mathcal{O}(t) = \{(t,x)\colon\ |x - y_0| < k(t),\ t > 0\}.$$

Summarizing the above discussion, we have the following result.

Theorem 6.1. *Problem* (1.9) *has a unique weak solution $u(t,x)$, which is defined for all $t > 0$. Moreover,*

$$\mathcal{O}(t) \subset \Omega(t) \subset \mathcal{G}(t), \quad \forall t \geqslant 0, \tag{6.6}$$

$$\underline{u}(t,x) \leqslant u(t,x) \leqslant \bar{u}(t,x) \quad \textit{for a.e. } (t,x) \in [0,\infty) \times \mathbb{R}^N, \tag{6.7}$$

and $u \in C_{loc}^{\frac{\theta+1}{2}, 1+\theta}(O)$ for each $\theta \in (0,1)$, with

$$O = \bigcup_{t>0} \mathcal{O}(t) = \{(t,x)\colon\ |x - y_0| < k(t),\ t > 0\}.$$

We are now ready to make use of Theorem 6.1 and the results in [10] to study the long-time asymptotic behavior of the weak solution $u(t,x)$ of (1.9). We will obtain sufficient conditions for spreading and vanishing respectively. Moreover, when spreading occurs, we will give estimates on the spreading speed.

Let us first recall the threshold criteria for spreading and vanishing of the radially symmetric free boundary problem (6.1) given in [10]. For a radially symmetric positive continuous function α, if $\lambda_1(d, \alpha, R)$ denotes the principal eigenvalue of the problem

$$-d\Delta\psi = \lambda\alpha\psi \quad \text{in } B_R; \qquad \psi = 0 \quad \text{on } \partial B_R, \tag{6.8}$$

then there is a unique $R_\alpha > 0$ such that

$$\lambda_1(d,\alpha,R_\alpha)=1 \tag{6.9}$$

and

$$1>\lambda_1(d,\alpha,R) \quad \text{for } R>R_\alpha; \qquad 1<\lambda_1(d,\alpha,R) \quad \text{for } R<R_\alpha.$$

Moreover, $R_{\alpha_1}\leqslant R_{\alpha_2}$ if $\alpha_1\geqslant\alpha_2$. We now set

$$R_*:=R_{\bar{a}}, \qquad R^*:=R_{\underline{a}}.$$

Then from [10] we find that spreading for (6.1) happens if $R_0\geqslant R_*$, or if $R_0<R_*$ and $\mu>\mu^*$, where $\mu^*>0$ depends on $\bar{u}_0$; and vanishing happens for (6.1) if $R_0<R_*$ and $\mu\leqslant\mu^*$.

Similarly spreading happens for (6.3) if $r_0\geqslant R^*$, or if $r_0<R^*$ and $\mu>\mu_*$, where $\mu_*>0$ depends on $\underline{u}_0$; and vanishing happens for (6.3) if $r_0<R^*$ and $\mu\leqslant\mu_*$.

Let

$$h_\infty=\lim_{t\to+\infty}h(t), \qquad k_\infty=\lim_{t\to+\infty}k(t).$$

Then from [10] we find that the statement that vanishing happens to (6.1) is equivalent to $h_\infty\leqslant R_*$; similarly vanishing happens to (6.3) is equivalent to $k_\infty\leqslant R^*$.

Theorem 6.2. (a) *If $h_\infty\leqslant R_*$, then the weak solution $u(x,t)$ of* (1.9) *vanishes, i.e.,*

$$B_{k_\infty}(x_0)\subset\lim_{t\to+\infty}\Omega(t)\subset B_{h_\infty}(x_0)$$

and

$$\lim_{t\to+\infty}\left\|u(t,\cdot)\right\|_{L^\infty(\Omega(t))}=0.$$

(b) *If $k_\infty>R^*$ (and hence $k_\infty=\infty$), then the weak solution $u(x,t)$ of* (1.9) *spreads, i.e.,*

$$\lim_{t\to+\infty}\Omega(t)=\mathbb{R}^N$$

and

$$\lim_{t\to+\infty}u(t,x)=\hat{U}(x) \quad \textit{locally uniformly for } x\in\mathbb{R}^N \tag{6.10}$$

where $\hat{U}$ is the unique positive solution of the equation

$$-d\Delta\hat{U}=\hat{U}\big[a(x)-b(x)\hat{U}\big] \quad \textit{for } x\in\mathbb{R}^N. \tag{6.11}$$

Proof. Part (a) follows directly from (6.6) and (6.7). It remains to prove part (b). The existence and uniqueness of a positive solution of (6.11) follows from Theorem 2.3 of [12] (by choosing both γ and τ there to be 0).

Since $k_\infty>R^*$, from [10] we have $k_\infty=+\infty$, and by (6.6),

$$\lim_{t\to+\infty}\Omega(t)=\mathbb{R}^N.$$

To show (6.10), we use a squeezing argument introduced in [13]. We first consider the Dirichlet problem

$$-d\Delta v = v\big[a(x) - b(x)v\big] \quad \text{in } B_R(x_0), \qquad v = 0 \quad \text{on } \partial B_R(x_0)$$

and the boundary blow-up problem

$$-d\Delta z = z\big[a(x) - b(x)z\big] \quad \text{in } B_R, \qquad z = +\infty \quad \text{on } \partial B_R(x_0).$$

When R is large, it is well known that these problems have positive solutions v_R and z_R, respectively. By the comparison principle given in [13], as $R \to +\infty$, v_R increases to the unique positive solution $\hat{U}$ of (6.11) and z_R decreases to $\hat{U}$.

Choose an increasing sequence of positive number R_n such that $R_n \to +\infty$ as $n \to \infty$, and v_{R_n} exists for all $n \geqslant 1$. Then, as $n \to \infty$, both v_{R_n} and z_{R_n} converge to $\hat{U}$. For each n, we can find $T_n > 0$ such that

$$\overline{B_{R_n}(x_0)} \subset \mathcal{O}(t) \subset \Omega(t) \quad \text{for all } t \geqslant T_n.$$

Thus

$$u(t,x) > 0 \quad \text{for } (t,x) \in [T_n, \infty) \times \overline{B_{R_n}(x_0)}$$

and is smooth in this range. Hence it satisfies

$$u_t - d\Delta u = u\big(a(x) - b(x)u\big) \quad \text{for } (t,x) \in [T_n, \infty) \times \overline{B_{R_n}(x_0)} \tag{6.12}$$

in the usual sense.

We now choose a positive function $\xi_n \in C^2(B_{R_n}(x_0))$ with $\xi_n = 0$ on $\partial B_{R_n}(x_0)$ and $\xi_n(x) \leqslant u(x, T_n)$ for $x \in B_{R_n}(x_0)$ and consider the problem

$$\begin{cases} v_t - d\Delta v = v\big(a(x) - b(x)v\big), & (t,x) \in [T_n, \infty) \times B_{R_n}(x_0), \\ v(x,t) = 0, & (t,x) \in [T_n, \infty) \times \partial B_{R_n}(x_0), \\ v(x, T_n) = \xi_n(x), & x \in B_{R_n}(x_0). \end{cases} \tag{6.13}$$

By standard theory on parabolic logistic equations we see that (6.13) admits a unique positive solution $\underline{v}_n$ and

$$\underline{v}_n(\cdot, t) \to v_{R_n} \quad \text{uniformly for } x \in B_{R_n}(x_0) \text{ as } t \to +\infty. \tag{6.14}$$

By the comparison principle, we have

$$\underline{v}_n(t,x) \leqslant u(t,x) \quad \text{for } (t,x) \in [T_n, \infty) \times B_{R_n}(x_0).$$

Therefore,

$$\varliminf_{t \to +\infty} u(t,x) \geqslant v_{R_n} \quad \text{uniformly in } B_{R_n}(x_0).$$

Sending $n \to +\infty$, we obtain

$$\varliminf_{t\to+\infty} u(t,x) \geqslant \hat{U} \quad \text{locally uniformly in } \mathbb{R}^N. \tag{6.15}$$

Analogously, by arguments similar to those in the proof of Theorem 4.1 of [13], we see that

$$\varlimsup_{t\to+\infty} u(t,x) \leqslant z_{R_n}(x) \quad \text{uniformly for } x \in B_{R_n}(x_0),$$

which implies (by sending $n \to \infty$)

$$\varlimsup_{t\to+\infty} u(t,x) \leqslant \hat{U}(x) \quad \text{locally uniformly for } x \in \mathbb{R}^N. \tag{6.16}$$

From (6.15) and (6.16) we see that (6.10) holds. □

We now consider the propagation speed of the "generalized" free boundary $\partial\Omega(t)$ by making use of (6.6). We need the following result of [11] after corrections (the corrections will appear in [8]).

Proposition 6.3. *For any given constants $a > 0$, $b > 0$, $d > 0$ and $k \in [0, 2\sqrt{ad})$, the problem*

$$-dZ'' + kZ' = aZ - bZ^2 \quad \text{in } (0,\infty), \qquad Z(0) = 0 \tag{6.17}$$

admits a unique positive solution $Z = Z_k$, and it satisfies $Z_k(r) \to \frac{a}{b}$ as $r \to +\infty$. Moreover, $Z_k'(r) > 0$ for $r \geqslant 0$, $Z_{k_1}'(0) > Z_{k_2}'(0)$, $Z_{k_1}(r) > Z_{k_2}(r)$ for $r > 0$ and $k_1 < k_2$, and for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu, a, b, d) \in (0, 2\sqrt{ad})$ such that $\mu Z_{k_0}'(0) = k_0$. Furthermore, $k_0(\mu, a, b, d)$ depends continuously on its arguments, is increasing in μ and $\lim_{\mu\to+\infty} k_0(\mu, a, b, d) = 2\sqrt{ad}$.

In our discussion below, since d is always fixed, we often write $k_0(\mu, a, b)$ instead of $k_0(\mu, a, b, d)$. When $k_\infty = \infty$, by (6.6), we necessarily have $h_\infty = \infty$. Hence we can apply Theorem 3.6 of [10] to conclude that

$$\varlimsup_{t\to\infty} \frac{h(t)}{t} \leqslant k_0\big(\mu, \bar{a}^\infty, \bar{b}_\infty\big), \qquad \varliminf_{t\to\infty} \frac{h(t)}{t} \geqslant k_0\big(\mu, \bar{a}_\infty, \bar{b}^\infty\big),$$

and

$$\varlimsup_{t\to\infty} \frac{k(t)}{t} \leqslant k_0\big(\mu, \underline{a}^\infty, \underline{b}_\infty\big), \qquad \varliminf_{t\to\infty} \frac{k(t)}{t} \geqslant k_0\big(\mu, \underline{a}_\infty, \underline{b}^\infty\big),$$

where we have used the notation

$$\alpha^\infty := \varlimsup_{r\to\infty} \alpha(r), \qquad \alpha_\infty := \varliminf_{r\to\infty} \alpha(r)$$

for a function $\alpha(r)$.

If we denote

$$c_*(\mu) := k_0\big(\mu, \underline{a}_\infty, \underline{b}^\infty\big), \qquad c^*(\mu) := k_0\big(\mu, \bar{a}^\infty, \bar{b}_\infty\big),$$

then from (6.6) we find that, for any given small $\epsilon > 0$, there exists $T_\epsilon > 0$ such that

$$\{x\colon |x| \leqslant [c_*(\mu) - \epsilon]t\} \subset \Omega(t) \subset \{x\colon |x| \leqslant [c^*(\mu) + \epsilon]t\}, \quad \forall t \geqslant T_\epsilon. \tag{6.18}$$

Thus we may regard $c_*(\mu)$ and $c^*(\mu)$ as a lower and an upper bound, respectively, for the propagation speed of the free boundary $\partial\Omega(t)$.

In particular, if we assume that

$$\lim_{|x|\to\infty} a(x) = a_\infty > 0, \qquad \lim_{|x|\to\infty} b(x) = b_\infty > 0, \tag{6.19}$$

then it is possible to choose $\bar{a}$, $\underline{a}$ and $\bar{b}$, $\underline{b}$ such that

$$\lim_{r\to\infty} \bar{a}(r) = \lim_{r\to\infty} \underline{a}(r) = a_\infty, \qquad \lim_{r\to\infty} \bar{b}(r) = \lim_{r\to\infty} \underline{b}(r) = b_\infty. \tag{6.20}$$

In such a case, we obtain

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{k(t)}{t} = k_0(\mu, a_\infty, b_\infty),$$

and hence $c_*(\mu) = c^*(\mu) = k_0(\mu, a_\infty, b_\infty)$. So when (6.19) holds we may regarded $k_0(\mu, a_\infty, b_\infty)$ as the asymptotic propagation speed of $\partial\Omega(t)$.

Our next result describes the large time behavior of the weak solution to (1.9) inside the ball $\{x\colon |x| < c_*(\mu)t\}$, which considerably improves the conclusion in (6.10).

Theorem 6.4. *Suppose that* (6.19) *holds, $k_\infty > R^*$, and $u(t,x)$, $\hat{U}(x)$ are as in part* (b) *of Theorem* 6.2. *Then*

$$\lim_{t\to+\infty} \max_{|x|\leqslant[c_*(\mu)-\epsilon]t} |u(t,x) - \hat{U}(x)| = 0 \tag{6.21}$$

for every small $\epsilon > 0$, where $c_*(\mu) = k_0(\mu, a_\infty, b_\infty)$.

Remark 6.5. Eq. (6.21) gives almost the best possible estimate for $u(t,x)$, since $\Omega(t) \subset \{x\colon |x| \leqslant [c_*(\mu) + \epsilon]t\}$ for every $\epsilon > 0$ and all large t, due to (6.6) and $\lim_{t\to\infty} h(t)/t = c_*(\mu)$.

Before giving the proof of Theorem 6.4, let us observe that for each $k \in [0, 2\sqrt{ad})$, if we define $z_k(t,x) = Z_k(kt - x)$, then z_k satisfies

$$(z_k)_t - d(z_k)_{xx} = az_k - bz_k^2 \quad \text{for } t \in \mathbb{R}^1,\ x \in (-\infty, kt); \qquad z_k(t, kt) = 0 \quad \text{for } t \in \mathbb{R}^1.$$

Thus as t increases, $z_k(t,x)$ behaves like a wave that travels to the right at the constant speed k, with the wave front at $x = kt$. For $k = k_0 = k_0(\mu, a, b, d)$, we have the extra property that

$$k_0 = \mu \frac{\partial z_{k_0}}{\partial x}(t, k_0 t), \quad \forall t \in \mathbb{R}^1.$$

Analogously $\tilde{z}_k(t,x) := Z_k(kt + x)$ defines a wave that travels at the constant speed k to the left.

In comparison with the classical traveling waves generated by W given in (1.2), by analogy, we may call the above waves generated by Z_k "semi-waves". The proof of Theorem 6.4 will rely crucially on these semi-waves $Z_k(x - kt)$.

Proof of Theorem 6.4. Let (v,h) and (w,k) be the solution of (6.1) and (6.3), respectively, and assume that $\bar{a},\bar{b}$ and $\underline{a},\underline{b}$ are chosen so that (6.20) holds. By Theorem 6.1, we have

$$w\big(t,|x-y_0|\big)\leqslant u(t,x)\leqslant v\big(t,|x-x_0|\big). \tag{6.22}$$

By [13],

$$\lim_{|x|\to\infty}\hat{U}(x)=\frac{a_\infty}{b_\infty}.$$

Hence for any given small $\epsilon>0$, there exists $R_\epsilon>0$ such that

$$\left|\hat{U}(x)-\frac{a_\infty}{b_\infty}\right|<\epsilon \quad \text{for } |x|\geqslant R_\epsilon. \tag{6.23}$$

We next make use of the estimates for $w(t,r)$ and $v(t,r)$ given in the proof of Theorem 3.6 in [10]. It was shown there that for any given small $\delta>0$, there exist positive numbers T^δ, R_1^δ and R_2^δ such that

$$v\big(t+T^\delta,r+R_1^\delta\big)\leqslant(1-\delta)^{-2}U_\delta\big(\xi(t)-r\big) \quad \text{for } t\geqslant 0,\ 0\leqslant r\leqslant\xi(t), \tag{6.24}$$

where

$$\xi(t)=(1-\delta)^{-2}k^\delta t+R_2^\delta,$$

and $U_\delta(r)$ stands for $Z_{k_0}(r)$ with $a=a_\infty+\delta$, $b=b_\infty-\delta$ and

$$k_0=k_0(\mu,a_\infty+\delta,b_\infty-\delta,d)\equiv k^\delta.$$

And similarly, there exist positive numbers $\tilde{T}^\delta$, $\tilde{R}_1^\delta$ and $\tilde{R}_2^\delta$ such that

$$w\big(t+\tilde{T}^\delta,r+\tilde{R}_1^\delta\big)\geqslant(1-\delta)^2V_\delta\big(\eta(t)-r\big) \quad \text{for } t\geqslant 0,\ 0\leqslant r\leqslant\eta(t), \tag{6.25}$$

where

$$\eta(t)=(1-\delta)^2k_\delta t+\tilde{R}_2^\delta,$$

and $V_\delta(r)$ stands for $Z_{k_0}(r)$ with $a=a_\infty-\delta$, $b=b_\infty+\delta$ and

$$k_0=k_0(\mu,a_\infty-\delta,b_\infty+\delta,d)\equiv k_\delta.$$

Since

$$\lim_{\delta\to 0}(1-\delta)^2k_\delta=\lim_{\delta\to 0}(1-\delta)^{-2}k^\delta=k_0(\mu,a_\infty,b_\infty,d)=c_*(\mu),$$

we can find $\delta_\epsilon\in(0,\epsilon)$ sufficiently small so that

$$\big|(1-\delta_\epsilon)^2k_{\delta_\epsilon}-c_*(\mu)\big|<\epsilon/2, \qquad \big|(1-\delta_\epsilon)^{-2}k^{\delta_\epsilon}-c_*(\mu)\big|<\epsilon/2.$$

We now fix $\delta = \delta_\epsilon$ in U_δ, ξ, V_δ and η. Then clearly

$$\xi(t) - r \geqslant \big[c_*(\mu) - \epsilon\big]t - r + R_2^{\delta_\epsilon} + \frac{\epsilon}{2}t,$$

and

$$\eta(t) - r \geqslant \big[c_*(\mu) - \epsilon\big]t - r + \tilde{R}_2^{\delta_\epsilon} + \frac{\epsilon}{2}t.$$

By Proposition 6.3, we have

$$\lim_{r\to\infty} U_{\delta_\epsilon}(r) = \frac{a_\infty + \delta_\epsilon}{b_\infty - \delta_\epsilon} < \frac{a_\infty + \epsilon}{b_\infty - \epsilon},$$

and

$$\lim_{r\to\infty} V_{\delta_\epsilon}(r) = \frac{a_\infty - \delta_\epsilon}{b_\infty + \delta_\epsilon} > \frac{a_\infty - \epsilon}{b_\infty + \epsilon}.$$

Thus we can find $T_1^\epsilon > 0$ such that for $r \geqslant T_1^\epsilon$,

$$U_{\delta_\epsilon}(r) \leqslant \frac{a_\infty + \epsilon}{b_\infty - \epsilon}, \qquad V_{\delta_\epsilon}(r) \geqslant \frac{a_\infty - \epsilon}{b_\infty + \epsilon}.$$

It follows that, if

$$0 \leqslant r \leqslant \big[c_*(\mu) - 2\epsilon/3\big]t \quad \text{and} \quad t \geqslant (\epsilon/6)^{-1}T_1^\epsilon,$$

then

$$v\big(t + T^{\delta_\epsilon}, r + R_1^{\delta_\epsilon}\big) \leqslant (1-\delta_\epsilon)^{-2} U_{\delta_\epsilon}\big(\xi(t) - r\big) \leqslant (1-\epsilon)^{-2}\frac{a_\infty + \epsilon}{b_\infty - \epsilon},$$

and

$$w\big(t + \tilde{T}^{\delta_\epsilon}, r + \tilde{R}_1^{\delta_\epsilon}\big) \geqslant (1-\delta_\epsilon)^{2} V_{\delta_\epsilon}\big(\eta(t) - r\big) \geqslant (1-\epsilon)^{2}\frac{a_\infty - \epsilon}{b_\infty + \epsilon}.$$

Combining these with (6.22), we obtain

$$(1-\epsilon)^2\frac{a_\infty - \epsilon}{b_\infty + \epsilon} \leqslant u(t,x) \leqslant (1-\epsilon)^{-2}\frac{a_\infty + \epsilon}{b_\infty - \epsilon} \tag{6.26}$$

provided that

$$t \geqslant \frac{6}{\epsilon}T_1^\epsilon + \max\big\{T^{\delta_\epsilon}, \tilde{T}^{\delta_\epsilon}\big\}$$

and

$$0 \leqslant |x - x_0| - R_1^{\delta_\epsilon} \leqslant \big[c_*(\mu) - 2\epsilon/3\big]t, \qquad 0 \leqslant |x - y_0| - \tilde{R}_1^{\delta_\epsilon} \leqslant \big[c_*(\mu) - 2\epsilon/3\big]t.$$

We now take

$$T_2^\epsilon := \frac{6}{\epsilon}\max\{T_1^\epsilon, |x_0|, |y_0|\} + \max\{T^{\delta_\epsilon}, \tilde{T}^{\delta_\epsilon}\}, \qquad \tilde{R}_\epsilon := \max\{R_\epsilon, |x_0| + R_1^{\delta_\epsilon}, |y_0| + \tilde{R}_1^{\delta_\epsilon}\},$$

and find that (6.26) holds if

$$t \geqslant T_2^\epsilon \quad \text{and} \quad \tilde{R}_\epsilon \leqslant |x| \leqslant [c_*(\mu) - \epsilon]t.$$

In view of (6.23), this implies that, for such t and x,

$$|u(t,x) - \hat{U}(x)| \leqslant I(\epsilon),$$

where

$$I(\epsilon) = \epsilon + \max\left\{\left[(1-\epsilon)^{-2}\frac{a_\infty + \epsilon}{b_\infty - \epsilon} - \frac{a_\infty}{b_\infty}\right], \left[\frac{a_\infty}{b_\infty} - (1-\epsilon)^2\frac{a_\infty - \epsilon}{b_\infty + \epsilon}\right]\right\}.$$

By (6.10),

$$\lim_{t\to\infty} u(t,x) = \hat{U}(x) \quad \text{uniformly for } |x| \leqslant \tilde{R}_\epsilon.$$

Hence we can find $T_3^\epsilon \geqslant T_2^\epsilon$ such that

$$|u(t,x) - \hat{U}(x)| \leqslant I(\epsilon) \quad \text{for } t \geqslant T_3^\epsilon \text{ and } |x| \leqslant \tilde{R}_\epsilon.$$

So finally we find that for all $t \geqslant T_3^\epsilon$ and $|x| \leqslant [c_*(\mu) - \epsilon]t$, we have

$$|u(t,x) - \hat{U}(x)| \leqslant I(\epsilon).$$

Since $I(\epsilon) \to 0$ as $\epsilon \to 0$, this implies that (6.21) holds. The proof is now complete. □

Theorem 6.6. *Suppose that* (6.19) *holds. Then for any given small* $\epsilon > 0$, *there exists a large* $\mu_\epsilon > 0$ *such that*

$$\lim_{t\to\infty} \max_{|x| \leqslant (2\sqrt{a_\infty d} - \epsilon)t} |u_\mu(t,x) - \hat{U}(x)| = 0 \quad \textit{uniformly for } \mu \geqslant \mu_\epsilon. \tag{6.27}$$

Proof. Recall that $k_\infty = \infty$ for every $\mu > 0$ if $r_0 \geqslant R^*$, and for the case $0 < r_0 < R^*$, there exists $\mu_* > 0$ such that $k_\infty = \infty$ if and only if $\mu \geqslant \mu_*$. Hence we can always find some $\mu_0 > 0$ such that $k_\infty = \infty$ for $\mu \geqslant \mu_0$.

For any given small $\epsilon > 0$, since $\lim_{\mu\to\infty} k_0(\mu, a_\infty, b_\infty, d) = 2\sqrt{a_\infty d}$, we can find $\mu_\epsilon \geqslant \mu_0$ such that

$$k_0(\mu, a_\infty, b_\infty, d) > 2\sqrt{a_\infty d} - \epsilon, \quad \forall \mu \geqslant \mu_\epsilon.$$

We may now apply Theorem 6.4 to conclude that

$$\lim_{t\to\infty} \max_{|x| \leqslant (2\sqrt{a_\infty d} - \epsilon)t} |u_{\mu_\epsilon}(t,x) - \hat{U}(x)| = 0. \tag{6.28}$$

By Corollary 5.5, we have $u_\mu(t,x) \leqslant U(t,x)$, where $U(t,x)$ is the solution of the Cauchy problem (5.14) with $g(x,u) = a(x)u - b(x)u^2$. Thus, due to Theorem 3.5,

$$u_{\mu_\epsilon}(t,x) \leqslant u_\mu(t,x) \leqslant U(t,x) \quad \text{for } \mu \geqslant \mu_\epsilon.$$

Hence to prove (6.27), in view of (6.28), it suffices to show that

$$\lim_{t\to\infty} \max_{|x|\leqslant(2\sqrt{a_\infty d}-\epsilon)t} \left|U(t,x) - \hat{U}(x)\right| = 0. \tag{6.29}$$

We now set to prove (6.29). For any given small $\delta > 0$, since $\lim_{|x|\to\infty} \hat{U}(x) = \frac{a_\infty}{b_\infty}$ by [13], and $\lim_{|x|\to\infty} a(x) = a_\infty$, $\lim_{|x|\to\infty} b(x) = b_\infty$ by assumption, we can find $R_1^\delta > 0$ such that

$$\frac{a_\infty - \delta}{b_\infty + \delta} \leqslant \hat{U}(x) \leqslant \frac{a_\infty + \delta}{b_\infty - \delta}, \qquad a(x) \leqslant a_\infty + \delta, \qquad b(x) \geqslant b_\infty - \delta \quad \text{for } |x| \geqslant R_1^\delta. \tag{6.30}$$

On the other hand, by [13], we also have

$$\lim_{t\to\infty} U(t,x) = \hat{U}(x) \quad \text{locally uniformly in } \mathbb{R}^N. \tag{6.31}$$

Therefore, in view of (6.30), we can find $T_1^\delta > 0$ such that

$$U(t,x) \leqslant \frac{a_\infty + 2\delta}{b_\infty - 2\delta} \quad \text{for } t \geqslant T_1^\delta \text{ and } |x| = R_1^\delta.$$

We now consider the auxiliary problem

$$\begin{cases} v_t - d\Delta v = (a_\infty + 2\delta)v - (b_\infty - 2\delta)v^2, & t \geqslant T_1^\delta,\ |x| \geqslant R_1^\delta, \\ v = \dfrac{a_\infty + 2\delta}{b_\infty - 2\delta}, & t \geqslant T_1^\delta,\ |x| = R_1^\delta, \\ v = m_0, & t = T_1^\delta,\ |x| \geqslant R_1^\delta, \end{cases} \tag{6.32}$$

where m_0 is a large positive constant satisfying $m_0 \geqslant U(T_1^\delta, x)$ for all $|x| \geqslant R_1^\delta$. Such an m_0 exists because $U(t,x) \leqslant \overline{U}(t)$, where $\overline{U}(t)$ is the unique solution of the following ODE problem:

$$u' = \kappa_2 u - \kappa_1 u^2 \quad \text{for } t > 0; \qquad u(0) = \|\tilde{u}_0\|_\infty,$$

which satisfies $\overline{U}(t) \to \kappa_2/\kappa_1$ as $t \to \infty$.

It is easily seen that $U(t,x)$ is a lower solution to (6.32), while the unique solution of the ODE problem

$$V' = (a_\infty + 2\delta)V - (b_\infty - 2\delta)V^2 \quad \text{for } t > T_1^\delta; \qquad V\left(T_1^\delta\right) = V_0 := \max\left\{m_0, \frac{a_\infty + 2\delta}{b_\infty - 2\delta}\right\}$$

is an upper solution to (6.32). It follows that

$$U(t,x) \leqslant V(t) \quad \text{for } t \geqslant T_1^\delta \text{ and } |x| \geqslant R_1^\delta.$$

Since $V(t) \to \frac{a_\infty+2\delta}{b_\infty-2\delta}$ as $t \to \infty$, we can find $T_2^\delta \geqslant T_1^\delta$ such that

$$V(t) \leqslant \frac{a_\infty + 3\delta}{b_\infty - 3\delta} \quad \text{for } t \geqslant T_2^\delta.$$

It follows that

$$U(t,x) \leqslant \frac{a_\infty + 3\delta}{b_\infty - 3\delta} \quad \text{for } t \geqslant T_2^\delta \text{ and } |x| \geqslant R_1^\delta.$$

Combining this with (6.30), we obtain

$$U(t,x) - \hat{U}(x) \leqslant J(\delta) := \frac{a_\infty + 3\delta}{b_\infty - 3\delta} - \frac{a_\infty - \delta}{b_\infty + \delta} \quad \text{for } t \geqslant T_2^\delta \text{ and } |x| \geqslant R_1^\delta.$$

By (6.31), we can find $T_3^\delta \geqslant T_2^\delta$ such that

$$U(t,x) - \hat{U}(x) \leqslant J(\delta) \quad \text{for } t \geqslant T_3^\delta \text{ and } |x| \leqslant R_1^\delta.$$

Thus

$$U(t,x) - \hat{U}(x) \leqslant J(\delta), \quad \forall t \geqslant T_3^\delta,\ \forall x \in \mathbb{R}^N. \tag{6.33}$$

By (6.28), we can find $T_4^\delta \geqslant T_3^\delta$ such that

$$u_{\mu_\epsilon}(t,x) - \hat{U}(x) \geqslant -J(\delta) \quad \text{for all } t \geqslant T_4^\delta \text{ and } |x| \leqslant (2\sqrt{a_\infty d} - \epsilon)t.$$

Combining this with (6.33) and $U(t,x) \geqslant u_{\mu_\epsilon}(t,x)$, we obtain

$$\left|U(t,x) - \hat{U}(x)\right| \leqslant J(\delta) \quad \text{for all } t \geqslant T_4^\delta \text{ and } |x| \leqslant (2\sqrt{a_\infty d} - \epsilon)t.$$

Since $J(\delta) \to 0$ as $\delta \to 0$, this is equivalent to say that (6.29) holds. The proof of the theorem is now complete. □

Remark 6.7. If $a(x) \equiv a$ and $b(x) \equiv b$ are constants, then necessarily $\hat{U}(x) \equiv \frac{a}{b}$, and hence (6.27) reduces to

$$\lim_{t\to\infty} \max_{|x|\leqslant(2\sqrt{ad}-\epsilon)t} \left|u_\mu(t,x) - \frac{a}{b}\right| = 0 \quad \text{uniformly for } \mu \geqslant \mu_\epsilon.$$

Due to $k_0(\mu,a,b,d) < 2\sqrt{ad}$ and (6.6), we have $u_\mu(t,x) \equiv 0$ for $|x| \geqslant (2\sqrt{ad} + \epsilon)t$ for every $\epsilon > 0$, every $\mu > 0$ and all large t. Thus u_μ exhibits the Aronson–Weinberger property (1.3) for all $\mu \geqslant \mu_\epsilon$.

Remark 6.8. Under the assumptions of Theorem 6.6, apart from (6.29) for the Cauchy problem solution $U(t,x)$, one can also modify the arguments in [2] to show that

$$\lim_{t\to\infty} \max_{|x|\geqslant(2\sqrt{a_\infty d}+\epsilon)t} U(t,x) = 0 \tag{6.34}$$

for every $\epsilon > 0$. Clearly (6.29) and (6.34) together gives an extension of the classical Aronson–Weinberger result (1.3) to the situation that the environment is only asymptotically homogeneous at infinity.

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