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On the spectral theory of Gesztesy–Šeba realizations of 1-D Dirac operators with point interactions on a discrete set

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Dedicated with deep respect to Fritz Gesztesy on the occasion of his 60th birthday

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A B S T R A C T

We investigate spectral properties of Gesztesy–Šeba realizations $D_{X,\alpha}$ and $D_{X,\beta}$ of the 1-D Dirac differential expression D with point interactions on a discrete set $X = \{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$. Here $\alpha := \{\alpha_n\}_{n=1}^{\infty}$ and $\beta := \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R}$. The Gesztesy–Šeba realizations $D_{X,\alpha}$ and $D_{X,\beta}$ are the relativistic counterparts of the corresponding Schrödinger operators $H_{X,\alpha}$ and $H_{X,\beta}$ with δ- and δ'-interactions, respectively. We define the minimal operator D_X as the direct sum of the minimal Dirac operators on the intervals (x_{n-1}, x_n). Then using the regularization procedure for direct sum of boundary triplets we construct an appropriate boundary triplet for the maximal operator D_X^* in the case $d_*(X) := \inf\{|x_i - x_j|,\ i \neq j\} = 0$. It turns out that the boundary operators $B_{X,\alpha}$ and $B_{X,\beta}$ parameterizing the realizations $D_{X,\alpha}$ and $D_{X,\beta}$ are Jacobi matrices. These matrices substantially differ from the ones appearing in spectral theory of Schrödinger operators with point interactions. We show that certain spectral properties of the operators $D_{X,\alpha}$ and $D_{X,\beta}$ correlate with the corresponding spectral properties of the Jacobi matrices $B_{X,\alpha}$ and $B_{X,\beta}$, respectively. Using this connection we investigate spectral properties (self-adjointness, discreteness, absolutely continuous and singular spectra) of Gesztesy–Šeba realizations. Moreover, we investigate the non-relativistic limit as the velocity of light $c \to \infty$. Most of our results are new even in the case $d_*(X) > 0$.

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Contents

1. Introduction

It is well known that many exactly solvable models describing complicated physical phenomena are expressed in terms of operators with point interactions (see [3,5,28,45] and comprehensive lists of references therein). In the 1-D case, the most known models are the Schrödinger operators $H_{X,\alpha}$ and $H_{X,\beta}$ associated with the formal differential expressions

$$\ell_{X,\alpha} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{x_n \in X} \alpha_n \delta(x - x_n), \qquad \ell_{X,\beta} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{x_n \in X} \beta_n \delta'(x - x_n), \tag{1.1}$$

where $\delta(\cdot)$ is a Dirac delta-function. These operators describe δ- and δ'-interactions, respectively, on a discrete set $X = \{x_n\}_{n \in I} \subset \mathcal{I} = (a, b) \subseteq \mathbb{R}$, and the coefficients $\alpha_n, \beta_n \in \mathbb{R}$ are called the strengths of the interaction at the point $x = x_n$.

Investigation of these models was originated by the "Kronig–Penney model" [50], a simple model for a non-relativistic electron moving in a fixed crystal lattice ($X = \mathbb{Z}$, $\mathcal{I} = \mathbb{R}$, $\alpha_n \equiv \alpha$). For a more mathematically rigorous approach to this model see for instance, [33] and [29] and the monograph [3].

Let $\alpha := \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R} \cup \{+\infty\}$ and $\beta := \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R} \cup \{+\infty\}$. There are several ways to associate well-defined linear operators with $\ell_{X,\alpha}$ and $\ell_{X,\beta}$ (see [3,15,61]). In $L^2(\mathcal{I})$, the minimal symmetric operators $H_{X,\alpha}$ and $H_{X,\beta}$ are naturally associated with (1.1). Namely, assuming that $\mathcal{I} = (a, +\infty)$ and $I = \mathbb{N}$ one defines the operators $H^0_{X,\alpha}$ and $H^0_{X,\beta}$ by the differential expression $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the domains, respectively,

$$\begin{aligned}\operatorname{dom}\bigl(H^0_{X,\alpha}\bigr) = \bigl\{ f \in W^{2,2}_{\mathrm{comp}}(\mathcal{I}\setminus X):\ & f'(a+) = 0,\ f(x_n+) = f(x_n-),\\ & f'(x_n+) - f'(x_n-) = \alpha_n f(x_n),\ n\in\mathbb{N}\bigr\},\end{aligned}\tag{1.2}$$

$$\begin{aligned}\operatorname{dom}\bigl(H^0_{X,\beta}\bigr) = \bigl\{ f \in W^{2,2}_{\mathrm{comp}}(\mathcal{I}\setminus X):\ & f'(a+) = 0,\ f'(x_n+) = f'(x_n-),\\ & f(x_n+) - f(x_n-) = \beta_n f'(x_n),\ n\in\mathbb{N}\bigr\}.\end{aligned}\tag{1.3}$$

Let $H_{X,\alpha}$ and $H_{X,\beta}$ be the closures of $H^0_{X,\alpha}$ and $H^0_{X,\beta}$, respectively. In general, the operators $H_{X,\alpha}$ and $H_{X,\beta}$ are symmetric but not automatically self-adjoint. Then one is interested in finding self-adjointness criteria and in the spectral analysis of such self-adjoint realizations.

In this paper we investigate two families of operators with point interactions which are the relativistic counterparts of $\ell_{X,\alpha}$ and $\ell_{X,\beta}$. Namely, we consider the cases where the differential expression $-\frac{d^2}{dx^2}$ in (1.1) is replaced by the Dirac differential expression

$$D \equiv D^c := -ic\frac{d}{dx}\otimes\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{c^2}{2}\otimes\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \equiv \begin{pmatrix} c^2/2 & -ic\frac{d}{dx}\\ -ic\frac{d}{dx} & -c^2/2\end{pmatrix}.\tag{1.4}$$

Here $c > 0$ denotes the velocity of light. Relativistic operators with point interactions have received a lot of attention recently (see e.g. [3,6–9,13,17,23,26,25,24,27,30,32,36,37,39,51,69,73] and references therein).

Assume that $\mathcal{I} = (a,b)$ with $-\infty < a < b \leqslant \infty$ and $I = \mathbb{N}$. Following [30] (see also [3, Appendix J][1]), we define the operators $D_{X,\alpha}$ and $D_{X,\beta}$ (realizations of D) to be the closures in $L^2(\mathcal{I})\otimes\mathbb{C}^2$ of the operators

$$\begin{aligned} D^0_{X,\alpha} &= D,\\ \operatorname{dom}\bigl(D^0_{X,\alpha}\bigr) &= \Bigl\{ f\in W^{1,2}_{\mathrm{comp}}(\mathcal{I}\backslash X)\otimes\mathbb{C}^2:\ f_1\in AC_{\mathrm{loc}}(\mathcal{I}),\ f_2\in AC_{\mathrm{loc}}(\mathcal{I}\backslash X);\\ &\qquad f_2(a+) = 0,\ f_2(x_n+) - f_2(x_n-) = -\frac{i\alpha_n}{c} f_1(x_n),\ n\in\mathbb{N}\Bigr\},\end{aligned}\tag{1.5}$$

and

$$\begin{aligned} D^0_{X,\beta} &= D,\\ \operatorname{dom}\bigl(D^0_{X,\beta}\bigr) &= \bigl\{ f\in W^{1,2}_{\mathrm{comp}}(\mathcal{I}\backslash X)\otimes\mathbb{C}^2:\ f_1\in AC_{\mathrm{loc}}(\mathcal{I}\backslash X),\ f_2\in AC_{\mathrm{loc}}(\mathcal{I});\\ &\qquad f_2(a+) = 0,\ f_1(x_n+) - f_1(x_n-) = i\beta_n c f_2(x_n),\ n\in\mathbb{N}\bigr\},\end{aligned}\tag{1.6}$$

respectively, i.e., $D_{X,\alpha} = \overline{D^0_{X,\alpha}}$ and $D_{X,\beta} = \overline{D^0_{X,\beta}}$. It is easily seen that both operators $D_{X,\alpha}$ and $D_{X,\beta}$ are symmetric. The domains of the adjoint operators $D^*_{X,\alpha}$ and $D^*_{X,\beta}$ are described explicitly: $\operatorname{dom}(D^*_{X,\alpha})$ and $\operatorname{dom}(D^*_{X,\beta})$ are given by formulae (1.5) and (1.6), respectively, with $W^{1,2}(\mathcal{I}\backslash X)$ in place of $W^{1,2}_{\mathrm{comp}}(\mathcal{I}\backslash X)$ (see Theorem 5.9(i)). The important feature of realizations $D_{X,\alpha}$ and $D_{X,\beta}$ is that *they are always self-adjoint, $D_{X,\alpha} = D^*_{X,\alpha}$ and $D_{X,\beta} = D^*_{X,\beta}$, provided that the interval $\mathcal{I}$ is infinite* (see Proposition 5.5 and Theorem 5.9(ii)).

The realizations $D_{X,\alpha}$ and $D_{X,\beta}$ have originally been introduced by Gesztesy and Šeba [30] (see also [3, Appendix J]) in the case of $\mathcal{I} = \mathbb{R}$ and $I = \mathbb{Z}$, i.e. $X = \{x_n\}_{n\in\mathbb{Z}}$. In what follows we will call

[1] There are typos in the definition of $D_{X,\beta}$ given in [3, Appendix J]: in formulae (J.17) and (J.23) there should be a sign $+$ instead of $-$.

these operators *Gesztesy–Šeba realizations* (in short, GS-realizations). These realizations turn out to be closely related to their non-relativistic counterparts $H_{X,\alpha}$ and $H_{X,\beta}$ associated with the differential expression (1.1).

Gesztesy and Šeba [30] investigated the realizations $D_{X,\alpha}$ and $D_{X,\beta}$ in the framework of extension theory of symmetric operators and treating the operators $D_{X,\alpha}$ and $D_{X,\beta}$ as extensions of the minimal operator

$$D_X := \bigoplus_{n\in\mathbb{Z}} D_n, \quad D_n = D, \qquad \operatorname{dom}(D_n) = W_0^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2. \tag{1.7}$$

In fact, they assumed in addition that $d_*(X) > 0$ where

$$d_*(X) := \inf_n d_n, \qquad d^*(X) := \sup_n d_n \quad \text{and} \quad d_n := x_n - x_{n-1}. \tag{1.8}$$

Clearly, D_n is a symmetric operator with deficiency indices $n_\pm(D_n) = 2$. These authors also computed the resolvent differences $(D_{X,\alpha} - z)^{-1} - (D_{\text{free}} - z)^{-1}$ and $(D_{X,\beta} - z)^{-1} - (D_{\text{free}} - z)^{-1}$, where D_{free} is the free Dirac operator D. In the periodic case ($X = \mathbb{Z}$, $\alpha_k = \alpha_0$, $\beta_k = \beta_0$, $k \in \mathbb{Z}$) they proved that the spectra $\sigma(D_{X,\alpha})$ and $\sigma(D_{X,\beta})$ have a band-zone structure.

Moreover, assuming $d_*(X) > 0$ they proved the following non-relativistic limit

$$s - \lim_{c\to+\infty} \left(D^c_{X,\alpha} - \left(z + c^2/2\right)\right)^{-1} = (H_{X,\alpha} - z)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.9}$$

In the present paper we study the spectral properties of the GS-realizations $D_{X,\alpha}$ and $D_{X,\beta}$ for arbitrary $d_*(X) \geqslant 0$. Moreover, we investigate the GS-realizations $D_{X,\alpha}(Q) := D_{X,\alpha} + Q$ and $D_{X,\beta}(Q) := D_{X,\beta} + Q$ of a general Dirac operator $D + Q$ with 2×2 matrix potential $Q = Q^*$.

Spectral analysis of the GS-operators $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$ consists (at least partially) of the following problems:

(a) Finding self-adjointness criteria for $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$.
(b) Discreteness of the spectra of the operators $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$.
(c) Characterization of continuous, absolutely continuous, and singular parts of the spectra of the operators $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$.
(d) Resolvent comparability of the operators $D_{X,\alpha^{(1)}}(Q)$ and $D_{X,\alpha^{(2)}}(Q)$ with $\alpha^{(1)} \neq \alpha^{(2)}$ i.e. finding conditions for the inclusion $(D_{X,\alpha^{(1)}}(Q) - i)^{-1} - (D_{X,\alpha^{(2)}}(Q) - i)^{-1} \in \mathfrak{S}_p(\mathfrak{H})$ to be valid. Here $\mathfrak{S}_p(\mathfrak{H})$ denotes the Neumann–Schatten ideal.

We investigate spectral properties of these operators by applying the technique of boundary triplets and the corresponding Weyl functions (see Section 2 for the precise definitions). This new approach to extension theory of symmetric operators has appeared and was intensively elaborated during the last three decades (see [31,20,22,14,67], [18, Chapter 9] and references therein).

The main ingredient of this approach is the following abstract version of the Green formula for the adjoint A^* of a symmetric operator A:

$$\left(A^* f, g\right)_{\mathfrak{H}} - \left(f, A^* g\right)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}\left(A^*\right). \tag{1.10}$$

Here $\mathcal{H}$ is an auxiliary Hilbert space and the mapping $\Gamma := \binom{\Gamma_0}{\Gamma_1} : \operatorname{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is required to be surjective. The mapping Γ leads to a natural parametrization of self-adjoint extensions of A by means of self-adjoint linear relations (multi-valued operators) in $\mathcal{H}$, see [31,20]. For instance, any extension $\widetilde{A} = \widetilde{A}^*$ disjoint with $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ admits a representation

$$\widetilde{A} = A_B := A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0) \quad \text{with } B = B^* \in \mathcal{C}(\mathcal{H}), \tag{1.11}$$

where the graph of the "boundary" operator B in $\mathcal{H}$ is $\Gamma\,\mathrm{dom}(\widetilde{A}) := \{\{\Gamma_0 f, \Gamma_1 f\}\colon\ f \in \mathrm{dom}(\widetilde{A})\}$. As distinguished from the von Neumann approach, parametrization (1.11) yields a natural description of all proper (in particular, self-adjoint) extensions in terms of (abstract) boundary conditions.

In particular, this approach was successfully applied to boundary value problems for smooth elliptic operators on bounded or unbounded domains with a smooth compact boundary (see [34,12,54] and the monograph [35]), to the maximal Sturm–Liouville operator $-d^2/dx^2 + T$ in $\mathfrak{H} = L^2([0,1];\mathcal{H})$ with an unbounded operator potential $T = T^* \geqslant aI$, $T \in \mathcal{C}(\mathcal{H})$ ([31], see also [20] and [57] for the case of $\mathfrak{H} = L^2(\mathbb{R}_+;\mathcal{H})$), as well as to 3-D and 2-D Schrödinger operators with infinitely many δ-interactions (see [58] and references therein).

The most relevant to our paper is the article [44] where this approach was applied to 1-D Schrödinger operators in the case $d_*(X) = 0$ (for the case $d_*(X) > 0$ see works [43,59]). Namely, confining ourselves to the case of $\mathcal{I} \subset \mathbb{R}_+$ we treat the GS-operators $D_{X,\alpha}$ and $D_{X,\beta}$ as extensions of the minimal operator D_X given by (1.7) with $I = \mathbb{N}$ in place of $I = \mathbb{Z}$.

A boundary triplet for the operator A^* always exists whenever $n_+(A) = n_-(A)$, though it is not unique. Its role in extension theory is similar to that of a coordinate system in analytic geometry. It enables us to describe all self-adjoint extensions in terms of (abstract) boundary conditions in place of the second von Neumann formula, although this description is simple and adequate only under a suitable choice of a boundary triplet. Note that in the case $n_\pm(A) = \infty$ a construction of a suitable boundary triplet is a rather difficult problem.

For the adjoint operator D_X^* of D_X given by (1.7) it is natural to search for boundary triplets constructed as a direct sum of triplets Π_n for operators D_n^*, that is, $\Pi_D := \{\mathcal{H}, \Gamma_0, \Gamma_1\} := \bigoplus_{n=1}^\infty \Pi_n$, where Π_n is a boundary triplet for D_n^*, $n \in I$, and

$$\mathcal{H} := \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n, \qquad \Gamma_0 := \bigoplus_{n\in\mathbb{N}} \Gamma_0^{(n)}, \qquad \Gamma_1 := \bigoplus_{n\in\mathbb{N}} \Gamma_1^{(n)}. \tag{1.12}$$

If $d_*(X) > 0$, then it is easily seen that the triplet (1.12) is a boundary triplet for D_X^* if one chooses $\Pi_n = \{\mathcal{H}, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ in the standard way with $\Gamma_j^{(n)}$, $j \in \{0,1\}$, given by

$$\begin{aligned}
\Gamma_0^{(n)} f &= -i\sqrt{\frac{c}{2}}\begin{pmatrix} f_2(x_{n-1}+) - f_2(x_n-) \\ f_1(x_{n-1}+) - f_1(x_n-) \end{pmatrix}, \\
\Gamma_1^{(n)} f &= \sqrt{\frac{c}{2}}\begin{pmatrix} f_1(x_{n-1}+) + f_1(x_n-) \\ f_2(x_{n-1}+) + f_2(x_n-) \end{pmatrix}, \quad n \in \mathbb{N},
\end{aligned} \tag{1.13}$$

where $f = \binom{f_1}{f_2}$ (see [21, formula (66)]). However, this direct sum is no longer a boundary triplet for D_X^* whenever $d_*(X) = 0$ (see Proposition 3.8). To construct a boundary triplet we use a regularization procedure elaborated in [57] and [44]. This procedure was already applied in [44] to 1-D Schrödinger operators with point interactions. However, in comparison with the Schrödinger case, one meets an additional difficulty of an algebraic character. Namely, we are searching for a boundary triplet such that the corresponding boundary operator (cf. (1.11)) is a Jacobi (tri-diagonal) matrix and for this purpose we need to construct an appropriate boundary triplet Π_n for the Dirac operator D_n^* on the interval $[x_{n-1}, x_n]$. Let us emphasize that only a sequence of boundary triplets $\widetilde{\Pi}^{(n)} = \{\mathcal{H}, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ given by $\mathcal{H} = \mathbb{C}^2$,

$$\widetilde{\Gamma}_0^{(n)} f = \begin{pmatrix} f_1(x_{n-1}+) \\ icf_2(x_n-) \end{pmatrix}, \qquad \widetilde{\Gamma}_1^{(n)} f = \begin{pmatrix} icf_2(x_{n-1}+) \\ f_1(x_n-) \end{pmatrix}, \quad n \in \mathbb{N}, \tag{1.14}$$

(see (3.8)) leads after an appropriate regularization to a new sequence of triplets Π_n for D_n^* having desirable properties (see Theorem 3.10). Namely, only *in the triplet* $\Pi_D = \bigoplus_1^\infty \Pi_n$ given by (3.54), (3.55), *the parametrization of GS-realizations is given by means of Jacobi matrices*. Let us also mention that a boundary triplet $\widetilde{\Pi}^{(n)}$ for D_n^* of the form (1.14) differs from (1.13) and the other ones known in the literature (see, e.g., [11,21]).

Recall that one of the main results in [44] states that certain spectral properties of $H_{X,\alpha}$ (self-adjointness, discreteness, etc.) correlate with the corresponding spectral properties of the Jacobi matrix

$$B_{X,\alpha}(H) := \begin{pmatrix} 0 & -d_1^{-2} & 0 & 0 & 0 & \dots \\ -d_1^{-2} & -d_1^{-2} & d_1^{-3/2}d_2^{-1/2} & 0 & 0 & \dots \\ 0 & d_1^{-3/2}d_2^{-1/2} & \alpha_1 d_2^{-1} & -d_2^{-2} & 0 & \dots \\ 0 & 0 & -d_2^{-2} & -d_2^{-2} & d_2^{-3/2}d_3^{-1/2} & \dots \\ 0 & 0 & 0 & d_2^{-3/2}d_3^{-1/2} & \alpha_2 d_3^{-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{1.15}$$

As usual we identify the Jacobi matrix $B_{X,\alpha}(H)$ with (the closed) minimal symmetric operator associated with it and denote it by the same letter. We emphasize that the Jacobi operator $B_{X,\alpha}(H)$ is a boundary operator for $H_{X,\alpha}$ in the triplet $\Pi_H = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ in the sense of (1.11), that is

$$\begin{aligned} H_{X,\alpha} &= H_{\min}^* \restriction \operatorname{dom}(H_{B_{X,\alpha}(H)}), \\ \operatorname{dom}\bigl(H_{B_{X,\alpha}}(H)\bigr) &= \bigl\{f \in W^{2,2}(\mathcal{I} \setminus X)\colon\ \Gamma_1 f = B_{X,\alpha}(H)\Gamma_0 f\bigr\}. \end{aligned} \tag{1.16}$$

In the present paper we establish similar results for GS-realizations $D_{X,\alpha}$ and $D_{X,\beta}$. For instance, we show (see Proposition 5.4 and Theorem 5.26) that self-adjointness and discreteness of the spectrum of $D_{X,\alpha}$ correlate with the corresponding properties of the following Jacobi matrix

$$B_{X,\alpha} = \begin{pmatrix} 0 & -\frac{\nu(d_1)}{d_1^2} & 0 & 0 & 0 & \dots \\ -\frac{\nu(d_1)}{d_1^2} & -\frac{\nu(d_1)}{d_1^2} & \frac{\nu(d_1)}{d_1^{3/2}d_2^{1/2}} & 0 & 0 & \dots \\ 0 & \frac{\nu(d_1)}{d_1^{3/2}d_2^{1/2}} & \frac{\alpha_1}{d_2} & -\frac{\nu(d_2)}{d_2^2} & 0 & \dots \\ 0 & 0 & -\frac{\nu(d_2)}{d_2^2} & -\frac{\nu(d_2)}{d_2^2} & \frac{\nu(d_2)}{d_2^{3/2}d_3^{1/2}} & \dots \\ 0 & 0 & 0 & \frac{\nu(d_2)}{d_2^{3/2}d_3^{1/2}} & \frac{\alpha_2}{d_3} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{1.17}$$

where $\nu(x) := (1 + (c^2x^2)^{-1})^{-1/2}$. Emphasize that similar to (1.16), the Jacobi operator $B_{X,\alpha}$ in (1.17) is just a boundary operator for the GS-realization $D_{X,\alpha}$ in the triplet $\Pi_D = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, that is

$$\begin{aligned} D_{X,\alpha} &= D_{B_{X,\alpha}} = D_X^* \restriction \operatorname{dom}(D_{B_{X,\alpha}}), \\ \operatorname{dom}(D_{B_{X,\alpha}}) &= \bigl\{f \in W^{1,2}(\mathcal{I} \setminus X) \otimes \mathbb{C}^2\colon\ \Gamma_1 f = B_{X,\alpha}\Gamma_0 f\bigr\}. \end{aligned} \tag{1.18}$$

Representation (1.18) plays a crucial role in the paper: it allows us to solve the problems (a)–(d) regarding the operator $D_{X,\alpha}$ by combining known results on Jacobi matrices with the technique elaborated in [20,22].

For instance, applying the Carleman test (see e.g. [1], and [10, Chapter VII.1.2]) to the Jacobi matrix $B_{X,\alpha}$ we get that $B_{X,\alpha} = B_{X,\alpha}^*$, and hence $D_{X,\alpha}$ is always self-adjoint whenever $\mathcal{I} = \mathbb{R}_+$. It is not the case for GS-realizations $D_{X,\alpha}$ on a finite interval $\mathcal{I}$: under certain conditions on the sequences

α and X it might happen that either $D_{X,\alpha}$ has the non-trivial deficiency indices $n_\pm(D_{X,\alpha}) = 1$ (see Theorem 5.11) or $n_\pm(D_{X,\alpha}) = 0$, i.e. it is self-adjoint. More precisely, applying the Dennis–Wall test (see e.g. [1, Problem 2, p. 25]) to the matrix $B_{X,\alpha}$ we show in Proposition 5.7 that the GS-operator $D_{X,\alpha}$ on a finite interval $\mathcal{I} = (a,b)$ is self-adjoint provided that

$$\sum_{n\in\mathbb{N}} \sqrt{d_n d_{n+1}}|\alpha_n| = +\infty. \tag{1.19}$$

Next, applying known results on discreteness spectra of Jacobi matrices [16] to the matrix $B_{X,\alpha}$, we obtain (see Proposition 5.30) that the GS-operator $D^c_{X,\alpha}$ on the half-line $\mathbb{R}_+$ has discrete spectrum provided that $\lim_{n\to\infty} d_n = 0$ and

$$\lim_{n\to\infty} \frac{|\alpha_n|}{d_n} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{c}{\alpha_n} > -\frac{1}{4}. \tag{1.20}$$

Note, that condition $\lim_{n\to\infty} d_n = 0$ is necessary for the minimal Dirac operator D_X on $\mathbb{R}_+$ to have extensions (realizations) with discrete spectrum. It is worth to mention that conditions (1.20) provide the discreteness property of the GS-operators $D_{X,\alpha}(Q) := D_{X,\alpha} + Q$ with certain unbounded potentials Q (see Proposition 5.30 and Example 5.32).

Using parametrization (1.18), (1.17), we express the inclusion $(D_{X,\alpha} - z)^{-1} - (D_N - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H})$ in terms of $\alpha = \{\alpha_n\}_1^\infty$ and $\{d_n\}_1^\infty$. Here D_N is the Neumann realization of D, $\mathrm{dom}(D_N) = \{f \in W^{1,2}(\mathbb{R}_+) \otimes \mathbb{C}^2\colon\ f_2(+0) = 0\}$. Based on this result we prove (see Theorem 5.21) that

$$\sigma_{\mathrm{ess}}(D_{X,\alpha}) = \sigma_{\mathrm{ess}}(D_N) = \mathbb{R} \setminus \bigl(-c^2/2, c^2/2\bigr), \quad \text{whenever } \lim_{n\to\infty} \alpha_n/d_n = 0, \tag{1.21}$$

and

$$\sigma_{ac}(D_{X,\alpha}) = \sigma_{ac}(D_N) = \mathbb{R} \setminus \bigl(-c^2/2, c^2/2\bigr) \quad \text{if } \{\alpha_n/d_n\}_1^\infty \in l^1(\mathbb{N}). \tag{1.22}$$

We also find conditions guarantying that the spectrum $\sigma(D_{X,\alpha})$ is purely singular.

Finally, assuming that the operators $D^c_{X,\alpha} := D_{X,\alpha}$ and $H_{X,\alpha}$ are self-adjoint and using parameterizations (1.18) and (1.16) we prove (see Theorem 5.43) the non-relativistic limit (1.9) in the case $d_*(X) \geqslant 0$. In particular, (1.9) holds whenever $\mathcal{I} = \mathbb{R}_+$ and $H_{X,\alpha} = H^*_{X,\alpha}$. The latter happens if, for instance, $H_{X,\alpha}$ is lower semibounded (see [4]).

Similar results are also valid for the GS-realizations $D_{X,\beta}$. The simplest way to prove that is to extract them from the corresponding properties of the operators $D_{X,\alpha}$. This can be done by noticing that $D_{X,\beta}$ is unitarily equivalent to $-\widehat{D}_{X,\alpha}$ where $\alpha = \beta c^2$ and that the resolvent difference $(\widehat{D}_{X,\alpha} + z)^{-1} - (D_{X,\alpha} + z)^{-1}$ is a rank-one operator (see Proposition 5.39).

The paper is organized as follows.

Section 2 is preparatory. It contains necessary definitions and statements on the theory of boundary triplets of symmetric operators, Weyl functions, γ-fields, etc. We also consider a family of symmetric operators $\{S_n\}_{n\in\mathbb{N}}$ and a family of boundary triplets Π_n for S^*_n, $n \in \mathbb{N}$. Following [57] and [44] we discuss conditions guarantying that the direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ of boundary triplets Π_n is either a B-generalized or an ordinary boundary triplet. We also discuss and complete regularization procedure for Π_n such that a direct sum of regularized boundary triplets forms already a boundary triplet for the operator $A^* = \bigoplus_{n=1}^\infty S^*_n$ (see Theorem 2.12).

In Section 3 we construct boundary triplets for maximal Dirac operators on finite intervals and half-lines and compute the corresponding Weyl functions. Using the explicit form of the Weyl functions and applying the regularization procedure described in Section 2, we construct a boundary triplet Π_D for the maximal operator D^*_X. We also describe trace properties of functions from the space $W^{1,2}(\mathbb{R}_+ \setminus X)$ and show that the direct sum $\bigoplus_{n=1}^\infty \Pi_n$ is an ordinary boundary triplet if and only if $0 < d^*(X) < \infty$ and it is a generalized boundary triplet (in the sense of [22]) whenever $d^*(X) < \infty$.

In Section 4 we apply boundary triplets technique to prove the non-relativistic limit for any m-dissipative (m-accumulative) realization of the expression D_X. To this end we compute the corresponding limits of the Weyl function and γ-field.

In Section 5 we investigate spectral properties of GS-realizations $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$ and solve problems (a)–(d). Moreover, we show (see Remark 5.10) that the operators $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$ on the line are self-adjointness for any continuous (not necessarily bounded) 2×2 potential matrix $Q(\cdot)=Q(\cdot)^*$. We also find certain sufficient conditions for the operator $D_{X,\alpha}$ on a finite interval either to be self-adjoint or to have deficiency indices $n_\pm(D_{X,\alpha})=1$ (see Proposition 5.7 and Theorem 5.11). Comparison of these results shows that roughly speaking $D_{X,\alpha}$ is self-adjoint on a finite interval whenever the sequence $\{\alpha_n\}_1^\infty$ grows faster than the sequence $\{d_n\}_1^\infty$ decays.

Moreover, using parameterizations (1.18) and (1.16) and the general result on non-relativistic limits (see Theorem 4.8) we prove relation (1.9) as well as similar relation for $D^c_{X,\beta}$.

Notations. Throughout the paper $\mathfrak{H}$, $\mathcal{H}$ denote separable Hilbert spaces. $[\mathfrak{H},\mathcal{H}]$ denotes the set of bounded operators from $\mathfrak{H}$ to $\mathcal{H}$; $[\mathfrak{H}]:=[\mathfrak{H},\mathfrak{H}]$. $\mathcal{C}(\mathfrak{H})$ and $\widetilde{\mathcal{C}}(\mathfrak{H})$ are the sets of closed operators and linear relations in $\mathfrak{H}$, respectively. By $\mathfrak{S}_p$, $p\in(0,\infty)$, we denote the Neumann–Schatten ideals. Let T be a linear operator in a Hilbert space $\mathfrak{H}$. In what follows, $\operatorname{dom}(T)$, $\ker(T)$, $\operatorname{ran}(T)$ are the domain, the kernel, the range of T, respectively; $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_{ac}(T)$, and $\sigma_s(T)$, denote the spectrum, point spectrum, continuous, absolutely continuous and singular spectrum of $T=T^*$, respectively; $\rho(T)$ and $\widehat{\rho}(T)$ denote the resolvent set, and the set of regular type points of T, respectively; $R_T(\lambda):=(T-\lambda I)^{-1}$, $\lambda\in\rho(T)$, is the resolvent of T.

Let X be a discrete subset of $\mathcal{I}\subseteq\mathbb{R}$. We define the following Sobolev spaces

$$\begin{aligned}
W^{1,2}(\mathcal{I}\setminus X) &:= \bigl\{f\in L^2(\mathcal{I}):\ f\in AC_{\mathrm{loc}}(\mathcal{I}\setminus X),\ f'\in L^2(\mathcal{I})\bigr\},\\
W^{2,2}(\mathcal{I}\setminus X) &:= \bigl\{f\in L^2(\mathcal{I}):\ f,f'\in AC_{\mathrm{loc}}(\mathcal{I}\setminus X),\ f''\in L^2(\mathcal{I})\bigr\},\\
W^{1,2}_0(\mathcal{I}\setminus X) &:= \bigl\{f\in W^{1,2}(\mathcal{I}):\ f(x_k)=0 \text{ for all } x_k\in X\bigr\},\\
W^{2,2}_0(\mathcal{I}\setminus X) &:= \bigl\{f\in W^{2,2}(\mathcal{I}):\ f(x_k)=f'(x_k)=0 \text{ for all } x_k\in X\bigr\},\\
W^{k,2}_{\mathrm{comp}}(\mathcal{I}\setminus X) &:= \bigl\{f\in W^{k,2}(\mathcal{I}\setminus X):\ \operatorname{supp} f \text{ is compact in } \mathcal{I}\bigr\} = W^{k,2}(\mathcal{I}\setminus X)\cap L^2_{\mathrm{comp}}(\mathcal{I}).
\end{aligned}$$

Let I be a subset of $\mathbb{Z}$, $I\subseteq\mathbb{Z}$. For any non-negative sequence $\{c_n\}_{n\in I}$ we denote by $l^2(I;\{c_n\},\mathcal{H}):=l^2(I;\{c_n\})\otimes\mathcal{H}$ the weighted Hilbert space of $\mathcal{H}$-valued sequences, i.e. $f=\{f_n\}_{n\in I}\in l^2(I;\{c_n\},\mathcal{H})$ if $\|f\|^2=\sum_{n\in I}c_n\|f_n\|^2_{\mathcal{H}}<\infty$; $l^2_0(I,\mathcal{H})$ is a subset of finite sequences in $l^2(I;\{c_n\},\mathcal{H})$, i.e. the sequences with compact supports; we also abbreviate $l^2(\mathbb{N};\{c_n\}):=l^2(\mathbb{N};\{c_n\},\mathbb{C})$, $l^2_0(\mathbb{N};\{c_n\}):=l^2_0(\mathbb{N};\{c_n\},\mathbb{C})$. As usual $l^p(\mathbb{N})$, $p\in[1,\infty)$, denotes the space of p-summable complex-valued sequences $f=\{f_n\}_{n\in\mathbb{N}}$; $l^\infty(\mathbb{N})$ denotes the space of bounded complex-valued sequences and $c_0(\mathbb{N})$ is a subspace of $l^\infty(\mathbb{N})$ consisting of sequences $f=\{f_n\}_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty}f_n=0$. $\chi_\pm(\cdot)$ denotes the indicator function of $\mathbb{R}_\pm$.

2. Preliminaries

2.1. Boundary triplets and Weyl functions

In this section we briefly recall the basic facts of the theory of boundary triplets and the corresponding Weyl functions (we refer to [20,22,31] for a detailed exposition of boundary triplets). Besides, we discuss a regularization procedure for direct sum of boundary triplets following [44] and [57]. Moreover, we slightly complete [44, Theorem 3.13] (see Theorem 2.12).

2.1.1. Linear relations, boundary triplets, and self-adjoint extensions

1. The set $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$ is the set of closed linear subspaces of $\mathcal{H}\oplus\mathcal{H}$. Recall that $\operatorname{dom}(\Theta)=\{f:\ \{f,f'\}\in\Theta\}$, $\operatorname{ran}(\Theta)=\{f':\ \{f,f'\}\in\Theta\}$, and $\operatorname{mul}(\Theta)=\{f':\ \{0,f'\}\in\Theta\}$ are the domain, the range, and the multi-valued part of Θ. A closed linear operator A in $\mathcal{H}$ is identified with its graph $\mathrm{gr}(A)$, so that the set $\mathcal{C}(\mathcal{H})$ of closed linear operators in $\mathcal{H}$ is viewed as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$. In particular, a linear relation Θ is an operator if and only if $\mathrm{mul}(\Theta)$ is trivial. We recall that the adjoint relation $\Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ of $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ is defined by

$$\Theta^* = \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} : \ (f', h)_{\mathcal{H}} = (f, h')_{\mathcal{H}} \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is said to be *symmetric* if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$.

For a symmetric linear relation $\Theta \subseteq \Theta^*$ in $\mathcal{H}$ the multi-valued part $\mathrm{mul}(\Theta)$ is the orthogonal complement of $\mathrm{dom}(\Theta)$ in $\mathcal{H}$. Therefore setting $\mathcal{H}_{\mathrm{op}} := \overline{\mathrm{dom}(\Theta)}$ and $\mathcal{H}_\infty = \mathrm{mul}(\Theta)$, one arrives at the orthogonal decomposition $\Theta = \Theta_{\mathrm{op}} \oplus \Theta_\infty$ where Θ_{op} is a symmetric operator in $\mathcal{H}_{\mathrm{op}}$, the operator part of Θ, and $\Theta_\infty = \{\binom{0}{f'}:\ f' \in \mathrm{mul}(\Theta)\}$, a "pure" linear relation in $\mathcal{H}_\infty$.

2. Let A be a densely defined closed symmetric operator in a separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $\mathrm{n}_\pm(A) = \dim \mathfrak{N}_{\pm \mathrm{i}} \leqslant \infty$, where $\mathfrak{N}_z := \ker(A^* - z)$ is the defect subspace.

Definition 2.1. (See [31].) A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called an (ordinary) boundary triplet for the adjoint operator A^* if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}(A^*) \to \mathcal{H}$ are linear mappings such that the second abstract Green identity

$$\left(A^* f, g\right)_{\mathfrak{H}} - \left(f, A^* g\right)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}\left(A^*\right), \tag{2.1}$$

holds and the mapping $\Gamma := \binom{\Gamma_0}{\Gamma_1} : \mathrm{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

First, note that a boundary triplet for A^* exists whenever the deficiency indices of A are equal, $n_+(A) = n_-(A)$. Moreover, $n_\pm(A) = \dim \mathcal{H}$ and $\ker(\Gamma) = \ker(\Gamma_0) \cap \ker(\Gamma_1) = \mathrm{dom}(A)$. Note also that Γ is a bounded mapping from $\mathfrak{H}_+ = \mathrm{dom}(A^*)$ equipped with the graph norm to $\mathcal{H} \oplus \mathcal{H}$.

A boundary triplet for A^* is not unique. Moreover, for any self-adjoint extension $\widetilde{A} := \widetilde{A}^*$ of A there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that $\ker(\Gamma_0) = \mathrm{dom}(\widetilde{A})$.

Definition 2.2.

(i) A closed extension A' of A is called a *proper extension*, if $A \subset A' \subset A^*$. The set of all proper extensions of A completed by the (non-proper) extensions A and A^* is denoted by Ext_A.
(ii) Two proper extensions A', A'', of A are called disjoint if $\mathrm{dom}(A') \cap \mathrm{dom}(A'') = \mathrm{dom}(A)$ and transversal if in addition $\mathrm{dom}(A') + \mathrm{dom}(A'') = \mathrm{dom}(A^*)$.

Recall that an operator $T \in \mathcal{C}(\mathcal{H})$ is called dissipative if $\mathrm{Im}(Tf, f) \geqslant 0$ for $f \in \mathrm{dom}(T)$. It is called m-dissipative if it has no proper dissipative extensions. It is known (and easily seen) that dissipative T is m-dissipative if and only if $\mathbb{C}_- \subset \rho(T)$.

The operator T is called accumulative (m-accumulative) if $-T$ is dissipative (m-dissipative).

Any dissipative (accumulative) extension $\widetilde{A}$ of A is necessarily a proper extension, $\widetilde{A} \in \mathrm{Ext}_A$. Moreover, if A' and A'' are disjoint and self-adjoint, then $\mathrm{dom}(A') + \mathrm{dom}(A'')$ is dense in $\mathrm{dom}(A^*)$.

Fixing a boundary triplet Π one can parameterize the set Ext_A in the following way.

Proposition 2.3. *(See [22].) Let A be as above and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$\mathrm{Ext}_A \ni \widetilde{A} \to \Gamma \,\mathrm{dom}(\widetilde{A}) = \left\{\{\Gamma_0 f, \Gamma_1 f\}:\ f \in \mathrm{dom}(\widetilde{A})\right\} =: \Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \tag{2.2}$$

establishes a bijective correspondence between the sets Ext_A and $\widetilde{\mathcal{C}}(\mathcal{H})$. We put $A_\Theta := \widetilde{A}$ where Θ is defined by (2.2), i.e. $A_\Theta := A^* \upharpoonright \Gamma^{-1}\Theta = A^* \upharpoonright \{f \in \mathrm{dom}(A^*):\ \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}$. Then:

(i) *A_Θ is m-dissipative (m-accumulative) if and only if so is Θ.*
(ii) *A_Θ is symmetric (self-adjoint) if and only if so is Θ. Moreover,* $\mathrm{n}_\pm(A_\Theta) = \mathrm{n}_\pm(\Theta)$.
(iii) *The extensions A_Θ and A_0 are disjoint (transversal) if and only if Θ is an operator. In this case A_Θ admits a representation*

$$A_\Theta = A^* \restriction \ker(\Gamma_1 - \Theta\Gamma_0). \tag{2.3}$$

Moreover, the extensions A_Θ and A_0 are transversal if and only if $\Theta \in [\mathcal{H}]$.

The linear relation Θ (the operator B) in the correspondence (2.2) (resp. (2.3)) is called *the boundary relation* (*the boundary operator*).

We emphasize that *in the case of differential operators opposed to the von Neumann parametrization the parametrization* (2.2)–(2.3) *describes the set of proper extensions directly in terms of boundary conditions*.

It follows immediately from Proposition 2.3 that the extensions

$$A_0 := A^* \restriction \ker(\Gamma_0) \quad \text{and} \quad A_1 := A^* \restriction \ker(\Gamma_1)$$

are self-adjoint. Clearly, $A_j = A_{\Theta_j}$, $j \in \{0, 1\}$, where the subspaces $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\}$ are self-adjoint relations in $\mathcal{H}$. Note that Θ_0 is a "pure" linear relation.

2.1.2. Weyl functions, γ-fields, and Krein type formula for resolvents

1. In [20,22] the concept of the classical Weyl–Titchmarsh m-function from the theory of Sturm–Liouville operators was generalized to the case of symmetric operators with equal deficiency indices. The role of abstract Weyl functions in the extension theory is similar to that of the classical Weyl–Titchmarsh m-function in the spectral theory of singular Sturm–Liouville operators.

Definition 2.4. (See [20].) Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator-valued functions $\gamma(\cdot) : \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M(\cdot) : \rho(A_0) \to [\mathcal{H}]$ defined by

$$\gamma(z) := (\Gamma_0 \restriction \mathfrak{N}_z)^{-1} \quad \text{and} \quad M(z) := \Gamma_1\gamma(z), \quad z \in \rho(A_0), \tag{2.4}$$

are called the *γ-field* and the *Weyl function*, respectively, corresponding to the boundary triplet Π.

The γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.4) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$ and the following relations hold (see [20])

$$\gamma(z) = \big(I + (z - \zeta)(A_0 - z)^{-1}\big)\gamma(\zeta), \quad z, \zeta \in \rho(A_0), \tag{2.5}$$

$$M(z) - M(\zeta)^* = (z - \bar{\zeta})\gamma(\zeta)^*\gamma(z), \quad z, \zeta \in \rho(A_0). \tag{2.6}$$

Identities (2.5) and (2.6) mean that $\gamma(\cdot)$ and $M(\cdot)$ are the γ-field and the Q-function of the operator A_0, respectively, in the sense of M.G. Krein (see [49]). It follows from (2.6) that $M(\cdot)$ is an $R[\mathcal{H}]$-function (or *Nevanlinna function*), i.e., $M(\cdot)$ is an ($[\mathcal{H}]$-valued) holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ satisfying

$$\operatorname{Im} z \cdot \operatorname{Im} M(z) \geqslant 0, \qquad M\big(z^*\big) = M(\bar{z}), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.7}$$

Moreover, due to (2.6) $M(\cdot) \in R^u[\mathcal{H}]$, i.e. it satisfies $0 \in \rho(\operatorname{Im} M(i))$.

It is well known that $M(\cdot)$ admits an integral representation (see, for instance, [1,2])

$$M(z) = C_0 + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma_M(t), \quad z \in \rho(A_0), \tag{2.8}$$

where $\Sigma_M(\cdot)$ is an operator-valued Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\Sigma_M(t) \in [\mathcal{H}]$ and $C_0 = C_0^* \in [\mathcal{H}]$. The integral in (2.8) is understood in the strong sense. Note that the spectral measure $E_{A_0}(\cdot)$ of the extension $A_0 = A_0^*$ and the measure $\Sigma_M(\cdot)$ from the integral representation (2.8) are equivalent (see [11]). Moreover, these operator measures are spectrally equivalent in the sense of [55]. Note also that a linear term $C_1 z$ is missing in (2.8) since A is densely defined (see [20]).

2. Recall that a symmetric operator A in $\mathfrak{H}$ is said to be *simple* if there is no non-trivial subspace which reduces it to a self-adjoint operator. In other words, A is simple if it does not admit an (orthogonal) decomposition $A = A' \oplus S$ where A' is a symmetric operator and S is a self-adjoint operator acting on a non-trivial Hilbert space.

It is easily seen (and well known) that A is simple if and only if $\operatorname{span}\{\mathfrak{N}_z(A)\colon z \in \mathbb{C} \setminus \mathbb{R}\} = \mathfrak{H}$.

If A is simple, then the Weyl function $M(\cdot)$ determines the boundary triplet Π uniquely up to the unitary equivalence (see [20]). In particular, $M(\cdot)$ contains the full information about the spectral properties of A_0. Moreover, the spectrum of a proper (not necessarily self-adjoint) extension $A_\Theta \in \operatorname{Ext}_A$ can be described by means of $M(\cdot)$ and the boundary relation Θ.

Proposition 2.5. *(See [20].) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and the γ-field. Then for any $\widetilde{A} = A_\Theta \in \operatorname{Ext}_A$ with $\rho(A_\Theta) \neq \emptyset$ the following Krein type formula holds*

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)\big(\Theta - M(z)\big)^{-1}\gamma^*(\bar{z}), \quad z \in \rho(A_0) \cap \rho(A_\Theta). \tag{2.9}$$

Moreover, if $\widetilde{A}$ is simple, then for any $z \in \rho(A_0)$ the following equivalence holds

$$z \in \sigma_i(A_\Theta) \quad \Leftrightarrow \quad 0 \in \sigma_i\big(\Theta - M(z)\big), \quad i \in \{\mathrm{p}, \mathrm{c}, \mathrm{r}\}.$$

Formula (2.9) is a generalization of the classical Krein formula for canonical resolvents (cf. [2,49]). It establishes a one-to-one correspondence between the set of proper extensions $\widetilde{A} = A_\Theta$ with non-empty resolvent set and the set of the corresponding linear relations Θ in $\mathcal{H}$. Note also that all objects in (2.9) are expressed in terms of the boundary triplet Π (see formulae (2.3) and (2.4)) (cf. [20,22]).

We emphasize that *precisely two parameterizations* (2.2)–(2.3) *and* (2.9) *of the set* Ext_A *make it possible application of Krein's type formula for resolvents to boundary value problems.*

The following result is deduced from (2.9).

Proposition 2.6. *(See [20].) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $\Theta_1, \Theta_2 \in \widetilde{\mathcal{C}}(\mathcal{H})$ and let $\mathfrak{S}_p(\mathfrak{H})$, $p \in (0, \infty]$, be the Neumann–Schatten ideal in $[\mathfrak{H}]$. Then:*

(i) *For any $z \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2})$ and $\zeta \in \rho(\Theta_1) \cap \rho(\Theta_2)$ the following equivalence holds*

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \quad \Leftrightarrow \quad (\Theta_1 - \zeta)^{-1} - (\Theta_2 - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H}). \tag{2.10}$$

In particular, $(A_{\Theta_1} - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Leftrightarrow (\Theta_1 - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H})$.

(ii) *If, in addition, $\Theta_1, \Theta_2 \in \mathcal{C}(\mathcal{H})$ and $\operatorname{dom}(\Theta_1) = \operatorname{dom}(\Theta_2)$, then the following implication holds*

$$\overline{\Theta_1 - \Theta_2} \in \mathfrak{S}_p(\mathcal{H}) \quad \Rightarrow \quad (A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}). \tag{2.11}$$

(iii) *Moreover, if $\Theta_1, \Theta_2 \in [\mathcal{H}]$, then implication (2.11) becomes the equivalence.*

2.1.3. Generalized boundary triplets of bounded type

In many applications the notion of a boundary triplet is too restrictive because of the assumption $\operatorname{dom}(\Gamma_j)=\mathfrak{H}_+$, $j\in\{0,1\}$. Inspiring by possible applications as well as certain theoretical reasons this concept was relaxed in [22, Section 6].

Definition 2.7. (See [22].) Let A be a closed densely defined symmetric operator in $\mathfrak{H}$ with equal deficiency indices. Let $A_*\supseteq A$ be a not necessarily closed extension of A such that $(A_*)^*=A$. A triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is called a *generalized boundary triplet of bounded type* (in short, *B-generalized boundary triplet*) for A^* if $\mathcal{H}$ is a Hilbert space and $\Gamma_j:\operatorname{dom}(\Gamma):=\operatorname{dom}(\Gamma_0)\cap\operatorname{dom}(\Gamma_1)=\operatorname{dom}(A_*)\to\mathcal{H}$, $j\in\{0,1\}$, are linear mappings such that

$(B1)$ Γ_0 is surjective,
$(B2)$ $A_{*0}:=A_*\restriction\ker(\Gamma_0)$ is a self-adjoint operator,
$(B3)$ the Green's identity holds

$$(A_*f,g)_{\mathfrak{H}}-(f,A_*g)_{\mathfrak{H}}=(\Gamma_1 f,\Gamma_0 g)_{\mathcal{H}}-(\Gamma_0 f,\Gamma_1 g)_{\mathcal{H}},\qquad f,g\in\operatorname{dom}(A_*)=\operatorname{dom}(\Gamma). \tag{2.12}$$

Note that one always has $A\subseteq A_*\subseteq A^*=\overline{A_*}$.

For any B-generalized boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ we set $A_{*j}:=A^*\lceil\ker(\Gamma_j)$, $j\in\{0,1\}$. Note that the extensions A_{*0} and A_{*1} are always disjoint but not necessarily transversal.

Starting with Definition 2.7 of a B-generalized boundary triplet Π, one can introduce concepts of the (generalized) γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ corresponding to Π in much the same way as in Definition 2.4 for (ordinary) boundary triplet (for detail see [22]). Let us mention only the following result (cf. [22, Proposition 6.2] and [19, Proposition 5.9]).

Proposition 2.8. *Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a B-generalized boundary triplet for A^*, $A_*=A^*\lceil\operatorname{dom}(\Gamma)$, and let $M(\cdot)$ be the corresponding Weyl function. Then:*

(i) *$M(\cdot)$ is an $[\mathcal{H}]$-valued Nevanlinna function satisfying* $\ker(\operatorname{Im}M(z))=\{0\}$, $z\in\mathbb{C}_+$.
(ii) *Π is an ordinary boundary triplet if and only if* $0\in\rho(\operatorname{Im}M(i))$.
(iii) *Moreover, if $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is a generalized boundary triplet for A^* (a boundary relation in the sense of [19]) and $M(\cdot)$ is an $R[\mathcal{H}]$-function satisfying* $\ker(\operatorname{Im}M(i))=\{0\}$, *then $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is a B-generalized boundary triplet for A^*, i.e. $(B1)$ and $(B2)$ are satisfied.*

2.2. Direct sums of boundary triplets

Let S_n be a densely defined symmetric operator in a Hilbert space $\mathfrak{H}_n$ with $\mathrm{n}_+(S_n)=\mathrm{n}_-(S_n)\leqslant\infty$, $n\in\mathbb{N}$. Consider the operator $A:=\bigoplus_{n=1}^\infty S_n$ acting in $\mathfrak{H}:=\bigoplus_{n=1}^\infty\mathfrak{H}_n$, the Hilbert direct sum of Hilbert spaces $\mathfrak{H}_n$. By definition, $\mathfrak{H}=\{f=\bigoplus_{n=1}^\infty f_n:\ f_n\in\mathfrak{H}_n,\ \sum_{n=1}^\infty\|f_n\|^2<\infty\}$. Clearly,

$$A^*=\bigoplus_{n=1}^\infty S_n^*,$$
$$\operatorname{dom}\bigl(A^*\bigr)=\Biggl\{f=\bigoplus_{n=1}^\infty f_n\in\mathfrak{H}:\ f_n\in\operatorname{dom}\bigl(S_n^*\bigr),\ \sum_{n\in\mathbb{N}}\bigl\|S_n^*f_n\bigr\|^2<\infty\Biggr\}. \tag{2.13}$$

We equip the domains $\operatorname{dom}(S_n^*)=:\mathfrak{H}_{n+}$ and $\operatorname{dom}(A^*)=:\mathfrak{H}_+$ with the graph norms $\|f_n\|^2_{\mathfrak{H}_{n+}}:=\|f_n\|^2+\|S_n^*f_n\|^2$ and $\|f\|^2_{\mathfrak{H}_+}:=\|f\|^2+\|A^*f\|^2=\sum_n\|f_n\|^2_{\mathfrak{H}_{n+}}$, respectively.

Further, let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$. By $\|\Gamma_j^{(n)}\|$ we denote the norm of the linear mapping $\Gamma_j^{(n)} \in [\mathfrak{H}_{n+}, \mathcal{H}_n]$, $j \in \{0,1\}$, $n \in \mathbb{N}$. Let also $\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ be a Hilbert direct sum of $\mathcal{H}_n$. Define mappings Γ_0 and Γ_1 by setting

$$\Gamma_j := \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}, \quad \operatorname{dom}(\Gamma_j) = \left\{ f = \bigoplus_{n=1}^{\infty} f_n \in \operatorname{dom}\bigl(A^*\bigr) \colon \sum_{n \in \mathbb{N}} \bigl\| \Gamma_j^{(n)} f_n \bigr\|_{\mathcal{H}_n}^2 < \infty \right\}. \tag{2.14}$$

Clearly, $\operatorname{dom}(\Gamma) := \operatorname{dom}(\Gamma_1) \cap \operatorname{dom}(\Gamma_0)$ is dense in $\mathfrak{H}_+$. Define the operators $S_{nj} := S_n^* \upharpoonright \ker \Gamma_j^{(n)}$ and $A_j := \bigoplus_{n=1}^{\infty} S_{nj}$, $j \in \{0,1\}$. Then A_0 and A_1 are self-adjoint extensions of A. Note that A_0 and A_1 are disjoint but not necessarily transversal. Finally, we set

$$A_* = A^* \upharpoonright \operatorname{dom}(\Gamma) \quad \text{and} \quad A_{*j} := A_* \upharpoonright \ker(\Gamma_j), \quad j \in \{0,1\}. \tag{2.15}$$

Clearly, A_{*j} is symmetric (not necessarily self-adjoint or even closed!) extension of A, $A_{*j} \subset A_j$, $j \in \{0,1\}$, and

$$\operatorname{dom}(A_{*j}) = \left\{ f = \bigoplus_{n=1}^{\infty} f_n \in \mathfrak{H} \colon\ f_n \in \ker \Gamma_j^{(n)},\ \sum_{n \in \mathbb{N}} \bigl(\bigl\| S_n^* f_n \bigr\|^2 + \bigl\| \Gamma_{j'}^{(n)} f_n \bigr\|^2 \bigr) < \infty \right\},$$
$$\bigl(0' := 1,\ 1' := 0\bigr).$$

Definition 2.9. Let Γ_j be defined by (2.14) and $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$. A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ will be called a direct sum of boundary triplets and will be assigned as $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n$.

It easily follows from (2.13)–(2.15) and Definition 2.9, that for $f = \bigoplus_{n=1}^{\infty} f_n$, $g = \bigoplus_{n=1}^{\infty} g_n \in \operatorname{dom}(A_*) = \operatorname{dom}(\Gamma)$ Green's identity (2.12) holds

$$\begin{aligned}
(A_* f, g)_{\mathfrak{H}} - (f, A_* g)_{\mathfrak{H}} &= \sum_{n \in \mathbb{N}} \bigl[\bigl(S_n^* f_n, g_n \bigr)_{\mathfrak{H}_n} - \bigl(f_n, S_n^* g_n \bigr)_{\mathfrak{H}_n} \bigr] \\
&= \sum_{n \in \mathbb{N}} \bigl[\bigl(\Gamma_1^{(n)} f_n, \Gamma_0^{(n)} g_n \bigr)_{\mathcal{H}_n} - \bigl(\Gamma_0^{(n)} f_n, \Gamma_1^{(n)} g_n \bigr)_{\mathcal{H}_n} \bigr] \\
&= (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}.
\end{aligned} \tag{2.16}$$

The series in the above equality converge due to (2.13) and (2.14). However the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ *is not a boundary triplet and even a B-generalized boundary triplet for A^* without additional restrictions*. This fact was discovered in [42] (in this connection see also simple examples in [44,57] and Proposition 3.8 below). At the same time, according to [44, Theorem 3.2] $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is always a boundary relation in the sense of [19].

The following criterion has been obtained in [57,44].

Theorem 2.10. *Let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* and $M_n(\cdot)$ the corresponding Weyl function, $n \in \mathbb{N}$. A direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an ordinary boundary triplet for the operator $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if*

$$C_1 = \sup_n \bigl\| M_n(i) \bigr\|_{\mathcal{H}_n} < \infty \quad \text{and} \quad C_2 = \sup_n \bigl\| \bigl(\operatorname{Im} M_n(i) \bigr)^{-1} \bigr\|_{\mathcal{H}_n} < \infty. \tag{2.17}$$

Theorem 2.10 makes it possible to construct a boundary triplet by regularizing each summand in a direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of arbitrary boundary triplets. The corresponding result was obtained in [57, Theorem 5.3] (see also [56] and [44, Theorems 3.10, 3.11]).

Theorem 2.11. *(See [57,56].) Let S_n be a symmetric operator in $\mathfrak{H}_n$ with deficiency indices $\mathrm{n}_\pm(S_k) = \mathrm{n}_n \leqslant \infty$ and $S_{n0} = S_{n0}^* \in \operatorname{Ext} S_n$, $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ there exists a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for S_n^* such that $\ker \Gamma_0^{(n)} = \operatorname{dom}(S_{n0})$ and $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an ordinary boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ satisfying $\ker \Gamma_0 = \operatorname{dom}(\widetilde{A}_0) := \bigoplus_{n=1}^{\infty} \operatorname{dom}(S_{n0})$.*

Next we assume that the operator $A = \bigoplus_{n=1}^{\infty} S_n$ has a regular real point, i.e., there exists $a = \bar{a} \in \widehat{\rho}(A)$. The latter is equivalent to the existence of $\varepsilon > 0$ such that

$$(a - \varepsilon, a + \varepsilon) \subset \bigcap_{n=1}^{\infty} \widehat{\rho}(S_n). \tag{2.18}$$

Emphasize that condition $a \in \bigcap_{n=1}^{\infty} \widehat{\rho}(S_n)$ is not enough for the inclusion $a \in \widehat{\rho}(A)$.

It is known [48] (see also [20]) that under condition (2.18) for every $k \in \mathbb{N}$ there exists a self-adjoint extension $\widetilde{S}_k = \widetilde{S}_k^*$ of S_k preserving the gap $(a - \varepsilon, a + \varepsilon)$. Moreover, the Weyl function of the pair $\{S_k, \widetilde{S}_k\}$ is regular within the gap $(a - \varepsilon, a + \varepsilon)$. Assuming condition (2.18) to be satisfied, one can simplify conditions (2.17) of Theorem 2.10 (cf. [44, Theorem 3.13]). In the following theorem we slightly complete [44, Theorem 3.13].

Theorem 2.12. *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of symmetric operators satisfying (2.18). Let also $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a - \varepsilon, a + \varepsilon) \subset \rho(S_{n0})$ and let $M_n(\cdot)$ be the corresponding Weyl function. Then:*

(i) $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms a B-generalized boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if

$$C_3 := \sup_{n \in \mathbb{N}} \bigl\| M_n(a) \bigr\|_{\mathcal{H}_n} < \infty \quad \text{and} \quad C_4 := \sup_{n \in \mathbb{N}} \bigl\| M_n'(a) \bigr\|_{\mathcal{H}_n} < \infty, \tag{2.19}$$

where $M_n'(a) := (dM_n(z)/dz)|_{z=a}$.

(ii) $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is an ordinary boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if in addition to (2.19) the following condition is satisfied

$$C_5 := \sup_{n \in \mathbb{N}} \bigl\| \bigl(M_n'(a)\bigr)^{-1} \bigr\|_{\mathcal{H}_n} < \infty. \tag{2.20}$$

Proof. (i) According to (2.8) each $M_n(\cdot)$, $n \in \mathbb{N}$, admits a representation

$$M_n(z) = C_{0,n} + \int\limits_{\mathbb{R} \setminus G_\varepsilon} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\Sigma_n(t), \qquad \int\limits_{\mathbb{R}} \frac{1}{1 + t^2} \, d\Sigma_n(t) \in [\mathcal{H}_n], \tag{2.21}$$

where $C_{0,n} = C_{0,n}^* \in [\mathcal{H}_n]$ and $G_\varepsilon := (a - \varepsilon, a + \varepsilon)$. Hence

$$M_n(a) = C_{0,n} + \int\limits_{\mathbb{R} \setminus G_\varepsilon} \frac{1 + at}{(t - a)(1 + t^2)} \, d\Sigma_n(t) \quad \text{and} \quad M_n'(a) = \int\limits_{\mathbb{R} \setminus G_\varepsilon} \frac{1}{(t - a)^2} \, d\Sigma_n(t). \tag{2.22}$$

Noting that with some $k > 0$

$$\left|(1+at)(t-a)\left(1+t^2\right)^{-1}\right| \leqslant k, \quad t \in \mathbb{R}, \tag{2.23}$$

we get from (2.22) that the second condition in (2.19) implies

$$\sup_n \left\| M_n(a) - C_{0,n} \right\|_{\mathcal{H}_n} \leqslant k \sup_n \left\| M_n'(a) \right\|_{\mathcal{H}_n} < \infty.$$

Combining this estimate with the first condition in (2.19) yields $\sup_n \|C_{0,n}\|_{\mathcal{H}_n} < \infty$.

Further, it follows from (2.21) that

$$M_n(i) = C_{0,n} + i \int\limits_{\mathbb{R}\setminus G_\varepsilon} \frac{1}{1+t^2}\, d\Sigma_n(t) \in [\mathcal{H}_n]. \tag{2.24}$$

It is easily seen that there exist constants $k_1, k_2 > 0$ such that

$$0 < k_1 < \left(1+t^2\right)(t-a)^{-2} < k_2, \quad t \in \mathbb{R} \setminus (a-\varepsilon, a+\varepsilon). \tag{2.25}$$

Taking this inequality into account and combining (2.24) with (2.22) we get that the second condition in (2.19) is equivalent to $\sup_n \|M_n(i) - C_{0,n}\|_{\mathcal{H}_n} < \infty$. Combining this estimate with $\sup_n \|C_{0,n}\|_{\mathcal{H}_n} < \infty$ yields $\sup_n \|M_n(i)\|_{\mathcal{H}_n} < \infty$, i.e. $M(i) \in [\mathcal{H}]$. The latter means that $M(\cdot) \in R[\mathcal{H}]$. Since $\ker \operatorname{Im} M(i) = 0$, it remains to apply Proposition 2.8(iii).

(ii) Using representation (2.22) for $M_n'(a)$ we rewrite condition (2.20) as

$$\int\limits_{\mathbb{R}\setminus G_\varepsilon} \frac{1}{(t-a)^2}\, d\Sigma_n(t) = M_n'(a) \geqslant C_5^{-1}, \quad n \in \mathbb{N}. \tag{2.26}$$

Combining these inequalities with representation (2.24) and taking into account inequality (2.25) we obtain

$$\begin{aligned} \operatorname{Im} M_n(i) &= \int\limits_{\mathbb{R}\setminus G_\varepsilon} \frac{1}{1+t^2}\, d\Sigma_n(t) = \int\limits_{\mathbb{R}\setminus G_\varepsilon} \frac{1}{(t-a)^2} \cdot \frac{(t-a)^2}{1+t^2}\, d\Sigma_n(t) \\ &\geqslant k_2^{-1} \int\limits_{\mathbb{R}\setminus G_\varepsilon} \frac{1}{(t-a)^2}\, d\Sigma_n(t) \geqslant k_2^{-1} C_5^{-1}, \quad n \in \mathbb{N}. \end{aligned} \tag{2.27}$$

This is amount to saying that $\sup_n \|(\operatorname{Im} M_n(\mathrm{i}))^{-1}\|_{\mathcal{H}_n} \leqslant k_2 C_5$. To complete the proof it remains to apply Theorem 2.10. □

For operators $A = \bigoplus_{n=1}^\infty S_n$ satisfying (2.18) we complete Theorem 2.12 by presenting a regularization procedure for $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ leading to a boundary triplet. In applications to symmetric operators with a gap this regularization is substantially simpler than the one described in Theorem 2.11.

Corollary 2.13. *Let $\{S_n\}_{n=1}^\infty$ be a sequence of symmetric operators satisfying (2.18). Let also $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a-\varepsilon, a+\varepsilon) \subset \rho(S_{n0})$, $S_{n0} = S_n^* \upharpoonright \ker(\widetilde{\Gamma}_0^{(n)})$, and $\widetilde{M}_n(\cdot)$ the corresponding Weyl function. Assume also that for some operators R_n such that $R_n, R_n^{-1} \in [\mathcal{H}_n]$, the following conditions are satisfied*

$$\sup_n \left\| R_n^{-1}\left(\widetilde{M}_n'(a)\right)\left(R_n^{-1}\right)^* \right\|_{\mathcal{H}_n} < \infty \quad \text{and} \quad \sup_n \left\| R_n^*\left(\widetilde{M}_n'(a)\right)^{-1} R_n \right\|_{\mathcal{H}_n} < \infty, \quad n \in \mathbb{N}. \tag{2.28}$$

Then the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets

$$\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\} \quad \text{with } \Gamma_0^{(n)} := R_n \widetilde{\Gamma}_0^{(n)}, \ \Gamma_1^{(n)} := (R_n^{-1})^* (\widetilde{\Gamma}_1^{(n)} - \widetilde{M}_n(a) \widetilde{\Gamma}_0^{(n)}), \tag{2.29}$$

forms a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$.

Example 2.14.

(i) Let $F(z) = Bz$ where $B \in [\mathcal{H}]$, $B = B^* > 0$ and $0 \in \sigma(B) \setminus \sigma_p(B)$. Then $0 (\in \sigma_c(B))$ is an accumulation point for $\sigma(B)$ and the operator B admits a decomposition $B = \bigoplus_{n=1}^{\infty} B_n$ with $B_n = B_n^* \in [\mathcal{H}_n]$ and $0 \in \rho(B_n)$, $n \in \mathbb{N}$. Clearly, $F(\cdot) = \bigoplus_{n=1}^{\infty} F_n(\cdot) \in R[\mathcal{H}]$, where $F_n(z) = B_n z$ and $F_n(\cdot) \in R^u[\mathcal{H}_n]$, i.e. $0 \in \rho(\operatorname{Im} F_n(i))$, $n \in \mathbb{N}$. However, $-F^{-1}(\cdot) \in R(\mathcal{H}) \setminus R[\mathcal{H}]$, i.e. $-F^{-1}(\cdot)$ is a Nevanlinna function with (unbounded) values in $\mathcal{C}(\mathcal{H})$. Clearly,

$$F_n(0) = 0, \qquad F_n'(0) = B_n \in [\mathcal{H}_n] \quad \text{and} \quad \sup_{n \in \mathbb{N}} \| F_n'(0) \| = \|B\| < \infty, \tag{2.30}$$

and conditions (2.19) are satisfied. At the same time, $\sup_{n\in\mathbb{N}} \|(F_n'(0))^{-1}\| = \sup_{n\in\mathbb{N}} \|B_n^{-1}\| = \infty$ and condition (2.20) is violated. Thus, condition (2.20) is not implied by conditions (2.19).

(ii) Let $F(z) = Bz$ where $B = B^* \in \mathcal{C}(\mathcal{H}) \setminus [\mathcal{H}]$ is unbounded positively definite operator, $0 \in \rho(B)$. Clearly, $B = \bigoplus_{n=1}^{\infty} B_n$ where $B_n \in [\mathcal{H}]$, $n \in \mathbb{N}$ and $F(\cdot) = \bigoplus_{n=1}^{\infty} F_n(\cdot)$ with $F_n = B_n z$. It is easily seen that

$$F_n(0) = 0, \qquad (F_n'(0))^{-1} = B_n^{-1} \quad \text{and}$$
$$\sup_{n\in\mathbb{N}} \| (F_n'(0))^{-1} \| = \sup_{n\in\mathbb{N}} \| B_n^{-1} \| = \| B^{-1} \| < \infty. \tag{2.31}$$

On the other hand, $\sup_{n\in\mathbb{N}} \|F_n'(0)\| = \sup_{n\in\mathbb{N}} \|B_n\| = \infty$ and the second condition in (2.19) is violated. This example shows that the second condition in (2.19) does not follow from the first one and condition (2.20).

Remark 2.15.

(i) In [44, Theorem 3.13] it is incorrectly stated that the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an ordinary boundary triplet for A^* whenever both the first condition in (2.19) and condition (2.20) are satisfied. However the proof in [44, Theorem 3.13] can easily be fixed by posing the second condition in (2.19) and using formula (2.6) connecting $M(i)$ and $M(a)$. In Theorem 2.12(ii) we presented another proof of this fact that seems to be simpler.

(ii) Note also that the first inequality in (2.28) was occasionally missed in [44, Corollary 3.15]. We mention also a misprint in formula (59) of [44]: there should be R_n in place of R_n^{-1}.

3. Dirac operators with point interactions on the line

Let D be the differential expression

$$D = -ic\frac{d}{dx} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3 = \begin{pmatrix} c^2/2 & -ic\frac{d}{dx} \\ -ic\frac{d}{dx} & -c^2/2 \end{pmatrix} \tag{3.1}$$

acting on $\mathbb{C}^2$-valued functions of a real variable. Here

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.2}$$

are the Pauli matrices in $\mathbb{C}^2$ and $c > 0$ denotes the velocity of light.

We set

$$k(z) := c^{-1}\sqrt{z^2 - \left(c^2/2\right)^2}, \quad z \in \mathbb{C}, \tag{3.3}$$

where the branch of the multifunction $\sqrt{\cdot}$ is selected such that $k(x) > 0$ for $x > c^2/2$. It is easily seen that $k(\cdot)$ is holomorphic in $\mathbb{C}$ with two cuts along the half-lines $(-\infty, -c^2/2]$ and $[c^2/2, \infty)$ and $k(\bar{z}) = -\overline{k(z)}$.

Thus, $k(\cdot)$ itself is not R-function (Nevanlinna function), although the extension

$$\widetilde{k}(z) = \begin{cases} k(z), & z \in \mathbb{C}_+, \\ -k(\bar{z}), & z \in \mathbb{C}_- \end{cases} \tag{3.4}$$

is already an R-function, i.e. a holomorphic function in $\mathbb{C}\backslash\mathbb{R}$, that maps $\mathbb{C}_+$ into $\mathbb{C}_+$ and satisfies $f(\bar{z}) = \overline{f(z)}$. Next we put

$$k_1(z) := \frac{ck(z)}{z + c^2/2} = \sqrt{\frac{z - c^2/2}{z + c^2/2}}, \quad z \in \mathbb{C}. \tag{3.5}$$

We can independently define the right-hand side in $\mathbb{C}\backslash\{(-\infty, -c^2/2] \cup [c^2/2, \infty)\}$ by selecting the branch of the corresponding multifunction in such a way that $\sqrt{\frac{x-c^2/2}{x+c^2/2}} > 0$ for $x > c^2/2$.

Next we construct boundary triplets for D_X^* using the technique elaborated in [44] and [57].

3.1. *Boundary triplets for Dirac building blocks*

We begin with a construction of a boundary triplet for the maximal Dirac operator on an interval.

3.1.1. *The case of a finite interval*

Let D_n be the minimal operator generated in $L^2[x_{n-1}, x_n] \otimes \mathbb{C}^2$ by the differential expression (3.1)

$$D_n = D \restriction \mathrm{dom}(D_n), \quad \mathrm{dom}(D_n) = W_0^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2. \tag{3.6}$$

We also put $d_n := x_n - x_{n-1} > 0$.

Lemma 3.1. *D_n is a symmetric operator with deficiency indices $n_\pm(D_n) = 2$. Its adjoint D_n^* is given by*

$$D_n^* = D \restriction \mathrm{dom}\left(D_n^*\right), \quad \mathrm{dom}\left(D_n^*\right) = W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2.$$

The defect subspace $\mathfrak{N}_z := \ker(D_n^* - z)$ is spanned by the vector functions $f_n^\pm(\cdot, z)$,

$$f_n^\pm(x, z) := \begin{pmatrix} e^{\pm ik(z)x} \\ \pm k_1(z) e^{\pm ik(z)x} \end{pmatrix}. \tag{3.7}$$

Moreover, the following is true:

(i) *The triplet* $\widetilde{\Pi}^{(n)} = \{\mathbb{C}^2, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$, *where*

$$\widetilde{\Gamma}_0^{(n)} f := \widetilde{\Gamma}_0^{(n)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1(x_{n-1}+) \\ icf_2(x_n-) \end{pmatrix} \quad \text{and}$$
$$\widetilde{\Gamma}_1^{(n)} f := \widetilde{\Gamma}_1^{(n)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} icf_2(x_{n-1}+) \\ f_1(x_n-) \end{pmatrix}, \tag{3.8}$$

forms a boundary triplet for D_n^*.

(ii) *The spectrum of the operator* $D_{n,0} := D_n^* \upharpoonright \ker \widetilde{\Gamma}_0^{(n)}$, *where*

$$\operatorname{dom}(D_{n,0}) = \bigl\{\{f_1, f_2\}^\tau \in W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2\colon\ f_1(x_{n-1}+) = f_2(x_n-) = 0\bigr\}, \tag{3.9}$$

is discrete,

$$\sigma(D_{n,0}) = \sigma_d(D_{n,0}) = \left\{\pm\sqrt{\frac{c^2\pi^2}{d_n^2}\left(j+\frac{1}{2}\right)^2 + \left(\frac{c^2}{2}\right)^2},\ j = 0, 1, \ldots\right\}. \tag{3.10}$$

(iii) *The* γ*-field* $\widetilde{\gamma}_n(\cdot) : \mathbb{C}^2 \to L^2[x_{n-1}, x_n] \otimes \mathbb{C}^2$, *corresponding to the triplet* $\widetilde{\Pi}^{(n)}$ *is given in the standard basis in* $\mathbb{C}^2$ *by*

$$\widetilde{\gamma}_n(z)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\cos(d_n k(z))}\begin{pmatrix} \cos(k(z)(x_n - x)) & -(ck_1(z))^{-1}\sin(k(z)(x_{n-1} - x)) \\ -ik_1(z)\sin(k(z)(x_n - x)) & -ic^{-1}\cos(k(z)(x_{n-1} - x)) \end{pmatrix}$$
$$\times \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad z \in \rho(D_{n,0}). \tag{3.11}$$

(iv) *The Weyl function* $\widetilde{M}_n(\cdot)$ *corresponding to the triplet* $\widetilde{\Pi}^{(n)}$ *is*

$$\widetilde{M}_n(z) = \frac{1}{\cos(d_n k(z))}\begin{pmatrix} ck_1(z)\sin(d_n k(z)) & 1 \\ 1 & (ck_1(z))^{-1}\sin(d_n k(z)) \end{pmatrix},$$
$$z \in \rho(D_{n,0}). \tag{3.12}$$

Proof. (i) and (ii) are straightforward.

(iii) Since f_n^- and f_n^+ form a basis in the defect subspace $\mathfrak{N}_z$, we get from the definition of the γ-field,

$$\widetilde{\gamma}_n(z)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = w_1(z) f_n^-(x, z) + w_2(z) f_n^+(x, z).$$

Applying to this identity the mapping $\widetilde{\Gamma}_0^{(n)}$ and using (3.7), (3.8) and definition (2.4) we get

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^{-ik(z)x_{n-1}} & e^{ik(z)x_{n-1}} \\ -ick_1(z)e^{-ik(z)x_n} & ick_1(z)e^{ik(z)x_n} \end{pmatrix}\begin{pmatrix} w_1(z) \\ w_2(z) \end{pmatrix} =: \Lambda(z)\begin{pmatrix} w_1(z) \\ w_2(z) \end{pmatrix}. \tag{3.13}$$

Hence $\binom{w_1(z)}{w_2(z)} = \Lambda^{-1}(z)\binom{v_1}{v_2}$. Setting $\Delta(z) := \det \Lambda(z) = 2ick_1(z)\cos(d_n k(z))$ we find

$$\widetilde{\gamma}(z)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\Delta(z)}\big(ick_1(z)e^{ik(z)x_n}v_1 - e^{ik(z)x_{n-1}}v_2\big)\begin{pmatrix} e^{-ik(z)x} \\ -k_1(z)e^{-ik(z)x} \end{pmatrix} + \frac{1}{\Delta(z)}\big(ick_1(z)e^{-ik(z)x_n}v_1 + e^{-ik(z)x_{n-1}}v_2\big)\begin{pmatrix} e^{ik(z)x} \\ k_1(z)e^{ik(z)x} \end{pmatrix}$$
$$= \frac{1}{\cos(k(z)d_n)}\begin{pmatrix} \cos(k(z)(x_n - x)) & -(ck_1(z))^{-1}\sin(k(z)(x_{n-1} - x)) \\ -ik_1(z)\sin(k(z)(x_n - x)) & (ic)^{-1}\cos(k(z)(x_{n-1} - x)) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This proves (3.11).

(iv) This statement is immediate from (3.11), (3.8) and the identity $\widetilde{M}_n(z) = \widetilde{\Gamma}_1^{(n)}\widetilde{\gamma}_n(z)$. □

3.1.2. The case of a half-line

In this section we construct boundary triplets for the Dirac operator D on half-lines $\mathbb{R}_a^- := (-\infty, a)$ and $\mathbb{R}_b^+ := (b, +\infty)$.

Denote by D_{a-} the minimal Dirac operator generated by differential expression (3.1) in $L^2(\mathbb{R}_a^-) \otimes \mathbb{C}^2$, i.e.

$$D_{a-} = D \restriction \operatorname{dom}(D_{a-}), \quad \operatorname{dom}(D_{a-}) = W_0^{1,2}\big(\mathbb{R}_a^-\big) \otimes \mathbb{C}^2. \tag{3.14}$$

Lemma 3.2. *D_{a-} is a closed symmetric operator with deficiency indices $n_\pm(D_{a-}) = 1$. Its adjoint D_{a-}^* is given by*

$$D_{a-}^* = D \restriction \operatorname{dom}\big(D_{a-}^*\big), \quad \operatorname{dom}\big(D_{a-}^*\big) = W^{1,2}\big(\mathbb{R}_a^-\big) \otimes \mathbb{C}^2.$$

The defect subspace $\ker(D_{a-}^* - z)$ is spanned by the vector function

$$f_a^-(x,z) := \begin{pmatrix} e^{-ik(z)x} \\ -k_1(z)e^{-ik(z)x} \end{pmatrix}. \tag{3.15}$$

Moreover, the following hold:

(i) *The triplet $\Pi^{(a-)} = \{\mathbb{C}, \Gamma_0^{(a-)}, \Gamma_1^{(a-)}\}$ where*

$$\Gamma_0^{(a-)} f := \Gamma_0^{(a-)}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = icf_2(a-) \quad \text{and} \quad \Gamma_1^{(a-)} f := \Gamma_1^{(a-)}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f_1(a-), \tag{3.16}$$

forms a boundary triplet for D_{a-}^*.

(ii) *The spectrum of the operator $D_{a-,0} := D_{a-}^* \restriction \ker \widetilde{\Gamma}_0^{(a-)}$ is absolutely continuous, of the multiplicity one,*

$$\sigma(D_{a-,0}) = \sigma_{ac}(D_{a-,0}) = \big(-\infty, c^2/2\big] \cup \big[c^2/2, +\infty\big). \tag{3.17}$$

(iii) *The corresponding γ-field $\gamma_{a-}(\cdot) : \mathbb{C} \to L^2(\mathbb{R}_a^-) \otimes \mathbb{C}^2$, is given by*

$$\gamma_{a-}(z)w = w\frac{ie^{ik(z)a}}{ck_1(z)} f_a^-(z), \quad z \in \rho(D_{a_-,0}). \tag{3.18}$$

(iv) *The Weyl function $M_{a-}(\cdot)$, corresponding to the triplet $\Pi^{(a-)}$ is*

$$M_{a-}(z) = \frac{i}{ck_1(z)}, \quad z \in \rho(D_{a_-,0}). \tag{3.19}$$

Next we denote by D_{b+} the minimal Dirac operator generated by differential expression (3.1) in $L^2(\mathbb{R}_b^+) \otimes \mathbb{C}^2$, i.e.

$$D_{b+} = D \upharpoonright \operatorname{dom}(D_{b+}), \quad \operatorname{dom}(D_{b+}) = W_0^{1,2}\big(\mathbb{R}_b^+\big) \otimes \mathbb{C}^2. \tag{3.20}$$

Lemma 3.3. *D_{b+} is a symmetric operator with the deficiency indices $n_\pm(D_{b+}) = 1$. The adjoint operator D_{b+}^* is given by*

$$D_{b+}^* = D \upharpoonright \operatorname{dom}\big(D_{b+}^*\big), \quad \operatorname{dom}\big(D_{b+}^*\big) = W^{1,2}\big(\mathbb{R}_b^+\big) \otimes \mathbb{C}^2, \tag{3.21}$$

and the defect subspace $\mathfrak{N}_z = \ker(D_{b+}^* - z)$ is spanned by the vector function $f_b^+(\cdot, z)$,

$$f_b^+(x,z) := \begin{pmatrix} e^{ik(z)x} \\ k_1(z)e^{ik(z)x} \end{pmatrix}. \tag{3.22}$$

Moreover, the following is true:

(i) *The triplet $\Pi^{(b+)} = \{\mathbb{C}, \Gamma_0^{(b+)}, \Gamma_1^{(b+)}\}$ where*

$$\Gamma_0^{(b+)} f := \Gamma_0^{(b+)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f_1(b+) \quad \textit{and} \quad \Gamma_1^{(b+)} f := \Gamma_1^{(b+)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = icf_2(b+), \tag{3.23}$$

forms a boundary triplet for D_{b+}^*.

(ii) *The spectrum of the operator $D_{b+,0} := D_{b+}^* \upharpoonright \ker \Gamma_0^{(b+)} = D_{b+,0}^*$ is absolutely continuous of the multiplicity one,*

$$\sigma(D_{b+,0}) = \sigma_{ac}(D_{b+,0}) = \big(-\infty, c^2/2\big] \cup \big[c^2/2, +\infty\big). \tag{3.24}$$

(iii) *The corresponding γ-field $\gamma_{b+}(\cdot) : \mathbb{C} \to L^2(\mathbb{R}_b^+) \otimes \mathbb{C}^2$, is*

$$\gamma_{b+}(z)w = we^{-ik(z)b} f_b^+(z), \quad z \in \rho(D_{b+,0}). \tag{3.25}$$

(iv) *The Weyl function $M_{b+}(\cdot)$, corresponding to the triplet $\Pi^{(b+)}$ is*

$$M_{b+}(z) = ick_1(z), \quad z \in \rho(D_{b+,0}). \tag{3.26}$$

The proofs of Lemmas 3.2 and 3.3 are straightforward.

3.2. *Trace properties of functions from the Sobolev space $W^{1,2}(\mathbb{R}_+ \setminus X)$*

Let $X = \{x_n\}_{n=1}^\infty$ be a discrete subset of the interval $\mathcal{I} = (a,b)$, $x_{n-1} < x_n$, $n \in \mathbb{N}$, with the accumulation point b, i.e. such that $b := \sup X \equiv \lim_{n\to\infty} x_n$. We set $x_0 := a$ and

$$d_n := x_n - x_{n-1}, \qquad d_*(X) := \inf_n d_n, \qquad d^*(X) := \sup_n d_n. \tag{3.27}$$

In what follows we always assume for convenience that $a := x_0 = 0$. Then we define the minimal operator D_X on $\mathfrak{H} = L^2(\mathcal{I}) \otimes \mathbb{C}^2$ by setting

$$D_X := \bigoplus_{n\in\mathbb{N}} D_n, \tag{3.28}$$

where D_n, $n \in \mathbb{N}$, is given by (3.6). Clearly,

$$\operatorname{dom}(D_X) = W_0^{1,2}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2 = \bigoplus_{n\in\mathbb{N}} W_0^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2. \tag{3.29}$$

Investigating non-relativistic limit in the case $b < \infty$ we will also consider operators $D_X \oplus D_{b+}$.

Here we construct (ordinary) boundary triplets for Dirac operators with point interactions on the half-line as well as on the line. It is natural to define a boundary triplet for $D_X^* = \bigoplus_{n=1}^{\infty} D_n^*$ as the direct sum $\widetilde{\Pi} = \bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n$ of boundary triplets $\widetilde{\Pi}^{(n)} = \{\mathbb{C}^2, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ defined in Lemma 3.1. However, $\widetilde{\Pi}$ is not an ordinary boundary triplet, in general. First we find necessary and sufficient conditions for a discrete set $X = \{x_n\}_{n=0}^{\infty}$ which guarantee this property for the direct sum $\bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n$. This problem is closely related to the property of trace mapping defined on the Sobolev space $W^{1,2}(\mathbb{R}_+)$ by

$$\pi : W^{1,2}(\mathbb{R}_+) \to l_2(\mathbb{N}), \quad \pi(f) = \{f(x_n)\}_{n=1}^{\infty}. \tag{3.30}$$

Proposition 3.4. *Let $X = \{x_n\}_{n=1}^{\infty}$ be as above. Then the mapping π is surjective if and only if $d_*(X) > 0$.*

Proof. *Sufficiency.* Let $d_*(X) > 0$. Denote by $u_0(\cdot) \in C_0^{\infty}(\mathbb{R})$ a function with compact support $\operatorname{supp} u_0 \subset (-d_*(X)/2, d_*(X)/2)$ and satisfying $u_0(0) = 1$. Next we put $u_n(x) := u_0(x - x_{n-1} + d_n/2)$, $n \in \mathbb{N}$, and note that $\operatorname{supp} u_n \subset [x_{n-1}, x_n]$ and $\|u_n\|_{W^{1,2}} = \|u_0\|_{W^{1,2}}$, $n \in \mathbb{N}$. Since $\operatorname{supp} u_k \cap \operatorname{supp} u_j = \emptyset$ for $j \neq k$, for any sequence $\{a_k\}_1^{\infty} \in l^2(\mathbb{N})$ the following series converges in $W^{1,2}(\mathbb{R}_+)$,

$$f := \sum_{k=1}^{\infty} a_k u_k \in W^{1,2}(\mathbb{R}_+) \quad \text{and} \quad \|f\|^2_{W^{1,2}(\mathbb{R}_+)} = \|u_0\|^2_{W^{1,2}(\mathbb{R}_+)} \cdot \sum_{k=1}^{\infty} a_k^2.$$

Clearly, $f(x_k) = a_k$, $k \in \mathbb{N}$, i.e. $\pi(f) = \{a_k\}_1^{\infty}$ and the mapping π is surjective.

Necessity. Assume that π is surjective. Choose any sequence $\{u_n\}_1^{\infty} \in W^{1,2}(\mathbb{R}_+)$ satisfying

$$u_n(x_n) = 1, \qquad u_n(x_{n-1}) = 0, \quad n \in \mathbb{N}. \tag{3.31}$$

Then, with the above notation $d_n = x_n - x_{n-1}$ we get

$$d_n^{-1} = d_n^{-1}\left(\int_{x_{n-1}}^{x_n} u_n'(t)\,dt\right)^2 \leqslant \int_{x_{n-1}}^{x_n} \left|u_n'(t)\right|^2 dt \leqslant \|u_n\|^2_{W^{1,2}(\mathbb{R}_+)}, \quad n \in \mathbb{N}. \tag{3.32}$$

If π is surjective, then, by closed graph theorem, there exists a bounded “inverse”, i.e. a surjective mapping $\pi^{(-1)}$ such that

$$\pi^{(-1)} : l^2(\mathbb{N}) \to W_1 \subset W^{1,2}(\mathbb{R}_+), \quad \pi\pi^{(-1)} = I_{l^2}, \tag{3.33}$$

where W_1 is a (closed) subspace of $W^{1,2}(\mathbb{R}_+)$. Hence there exists a bounded in $W^{1,2}(\mathbb{R}_+)$ sequence $\{v_n\}_1^{\infty} \subset W^{1,2}(\mathbb{R}_+)$ covering the coordinate basis $e_n := \{\delta_{mn}\}_{m=1}^{\infty}$, $n \in \mathbb{N}$, in $l^2(\mathbb{N})$, i.e. satisfying

$\pi(v_n) = e_n$, $n \in \mathbb{N}$. Substituting the sequence $\{v_n\}_1^\infty$ in (3.32) in place of $\{u_n\}_1^\infty$, we conclude that the sequence $\{d_n^{-1}\}_1^\infty$ is bounded, i.e. $d_*(X) > 0$. □

Next we give a complete trace characteristic of the space $W^{1,2}(\mathbb{R}_a^+ \setminus X)$ assuming for convenience that $a = 0$. Due to the embedding theorem, the trace mappings

$$\pi_\pm : W^{1,2}(\mathbb{R}_+ \setminus X) \to l^2(\mathbb{N}), \quad \pi_+(f) = \big\{f(x_{n-1}+)\big\}_1^\infty, \qquad \pi_-(f) = \big\{f(x_n-)\big\}_1^\infty, \tag{3.34}$$

are well defined for functions with compact supports, i.e. for $f \in \bigoplus_1^N W^{1,2}[x_{n-1}, x_n]$, $N \in \mathbb{N}$. We assume $\pi_\pm$ to be defined on its maximal domain $\operatorname{dom}(\pi_\pm) := \{f \in W^{1,2}(\mathbb{R}_+ \setminus X)\colon \pi_\pm f \in l^2(\mathbb{N})\}$. Clearly, $\operatorname{dom}(\pi_\pm)$ is dense in $W^{1,2}(\mathbb{R}_+ \setminus X)$ although, in general, $\operatorname{dom}(\pi_\pm) \neq W^{1,2}(\mathbb{R}_+ \setminus X)$.

Proposition 3.5. *Let $X = \{x_n\}_{n=1}^\infty$ be as above with $x_0 = 0$ and $X \subset \overline{\mathbb{R}}_+$. Then:*

(i) *For any pair of sequences $a^\pm = \{a_n^\pm\}_1^\infty$ satisfying*

$$a^\pm = \big\{a_n^\pm\big\}_1^\infty \in l^2\big(\mathbb{N}; \{d_n\}\big) \quad \text{and} \quad \big\{a_n^+ - a_n^-\big\}_1^\infty \in l^2\big(\mathbb{N}; \big\{d_n^{-1}\big\}\big), \tag{3.35}$$

there exists a (non-unique) function $f \in W^{1,2}(\mathbb{R}_+ \setminus X)$ such that $\pi_\pm(f) = a^\pm$. Moreover, the mapping $\pi_+ - \pi_- : W^{1,2}(\mathbb{R}_+ \setminus X) \to l^2(\mathbb{N}; \{d_n^{-1}\})$ is surjective and contractive, i.e.

$$\sum_{n\in\mathbb{N}} d_n^{-1} \big|f(x_n-) - f(x_{n-1}+)\big|^2 \leqslant \|f\|^2_{W^{1,2}(\mathbb{R}_+\setminus X)}, \quad f \in W^{1,2}(\mathbb{R}_+ \setminus X). \tag{3.36}$$

(ii) *Assume in addition, that $d^*(X) < \infty$. Then the mapping $\pi_\pm$ can be extended to a bounded surjective mapping from $W^{1,2}(\mathbb{R}_+ \setminus X)$ onto $l^2(\mathbb{N}; \{d_n\})$. More precisely, the following estimate holds*

$$\begin{aligned}\sum_{n\in\mathbb{N}} d_n\big(\big|f(x_{n-1}+)\big|^2 + \big|f(x_n-)\big|^2\big) &\leqslant 4\big(d^*(X)^2 \|f'\|^2_{L^2(\mathbb{R}_+)} + \|f\|^2_{L^2(\mathbb{R}_+)}\big) \\ &\leqslant C_1 \|f\|^2_{W^{1,2}(\mathbb{R}_+\setminus X)}, \quad f \in W^{1,2}(\mathbb{R}_+ \setminus X),\end{aligned} \tag{3.37}$$

where $C_1 := 4\max\{d^*(X)^2, 1\}$. Besides, the traces $a^\pm := \pi_\pm(f)$ of each $f \in W^{1,2}(\mathbb{R}_+ \setminus X)$ satisfy conditions (3.35). Moreover, the assumption $d^*(X) < \infty$ is necessary for the inequality (3.37) to hold with some $C_1 > 0$.

(iii) *The trace mapping $\pi_\pm : \operatorname{dom}(\pi_\pm) \to l^2(\mathbb{N})$ is closed. Moreover, it is surjective, $\operatorname{ran}(\pi_\pm) \supset l^2(\mathbb{N}\}$, if and only if $d^*(X) < \infty$.*

(iv) *The mapping $\pi_\pm$ is bounded, i.e. $\operatorname{dom}(\pi_\pm) = W^{1,2}(\mathbb{R}_+ \setminus X)$, if and only if*

$$0 < d_*(X) < d^*(X) < \infty. \tag{3.38}$$

Proof. (i) Let conditions (3.35) be satisfied. Define a function g_n by setting

$$g_n(x) = a_n^+ + d_n^{-1}(x - x_{n-1})\big(a_n^- - a_n^+\big), \quad x \in [x_{n-1}, x_n],\ n \in \mathbb{N}. \tag{3.39}$$

Let us check that the piecewise linear function $g = \bigoplus_{n\in\mathbb{N}} g_n$ has the required properties. Clearly, $g_n(x_{n-1}+) = a_n^+$ and $g_n(x_n-) = a_n^-$. Moreover, $g_n'(x) = d_n^{-1}(a_n^- - a_n^+)$ and

$$\|g'\|^2_{L^2(\mathbb{R}_+)} = \sum_n \|g_n'\|^2_{L^2[x_{n-1},x_n]} = \sum_n d_n^{-1}\big|a_n^- - a_n^+\big|^2 < \infty. \tag{3.40}$$

In other words, $g' \in L^2(\mathbb{R}_+)$ if and only if $\{a_n^+ - a_n^-\}_1^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\})$.

Next, it is easily seen that

$$\|g_n\|^2_{L^2[x_{n-1},x_n]} = d_n\big(\mathrm{Re}\big(a_n^+\overline{a_n^-}\big) + 3^{-1}\big|a_n^- - a_n^+\big|^2\big)$$
$$= 3^{-1}d_n\big(\big|a_n^+\big|^2 + \big|a_n^-\big|^2 + \mathrm{Re}\big(a_n^+\overline{a_n^-}\big)\big), \quad n \in \mathbb{N}. \tag{3.41}$$

On the other hand, by the Cauchy–Schwartz inequality

$$6^{-1}\big(|z_1|^2 + |z_2|^2\big) \leqslant 3^{-1}\big(|z_1|^2 + |z_2|^2 + \mathrm{Re}(z_1\bar{z}_2)\big) \leqslant 2^{-1}\big(|z_1|^2 + |z_2|^2\big). \tag{3.42}$$

Combining (3.41) with (3.42) we arrive at the following two-sided estimate for $g = \bigoplus_{n\in\mathbb{N}} g_n$

$$6^{-1}\sum_{n=1}^{\infty} d_n\big(\big|a_n^+\big|^2 + \big|a_n^-\big|^2\big) \leqslant \|g\|^2_{L^2(\mathbb{R}_+)} = \sum_{n=1}^{\infty}\|g_n\|^2_{L^2[x_{n-1},x_n]}$$
$$\leqslant 2^{-1}\sum_{n=1}^{\infty} d_n\big(\big|a_n^+\big|^2 + \big|a_n^-\big|^2\big). \tag{3.43}$$

In other words, $g \in L^2(\mathbb{R}_+)$ if and only if $a^\pm \in l^2(\mathbb{N}; \{d_n\})$. Thus, it follows from (3.43) and (3.40) that $g = \bigoplus_{n\in\mathbb{N}} g_n \in W^{1,2}(\mathbb{R}_+ \setminus X)$ if and only if both assumptions in (3.35) are satisfied.

To prove surjectivity of the mapping $\pi_+ : W^{1,2}(\mathbb{R}_+ \setminus X) \to l^2(\mathbb{N}; \{d_n\})$ we choose any $a^+ = \{a_n^+\} \in l^2(\mathbb{N}; \{d_n\})$ and put $a^- := \{a_n^-\} = a^+$. Clearly, $\{a_n^+ - a_n^-\}_1^\infty = \{0\}_1^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\})$ and both conditions (3.35) are satisfied. Thus, the step function $g = \bigoplus_{n\in\mathbb{N}} g_n$ with $g_n := a_n \in W^{1,2}[x_{n-1}, x_n]$, $n \in \mathbb{N}$, belongs to $W^{1,2}(\mathbb{R}_+ \setminus X)$ and satisfies $\pi_\pm(g) = a^\pm$.

Further, for any $f_n \in W^{1,2}[x_{n-1}, x_n]$, $n \in \mathbb{N}$, one easily gets

$$d_n^{-1}\big|f_n(x_n-) - f_n(x_{n-1}+)\big|^2 = d_n^{-1}\left(\int_{x_{n-1}}^{x_n} f_n'(t)\,dt\right)^2 \leqslant \int_{x_{n-1}}^{x_n}\big|f_n'(t)\big|^2\,dt$$
$$\leqslant \|f_n\|^2_{W^{1,2}[x_{n-1},x_n]}. \tag{3.44}$$

Taking a sum one arrives at the inequality (3.36).

(ii) Next, let $d^*(X) < \infty$. By the Sobolev embedding theorem, for any $f \in W^{1,2}(\mathbb{R}_+ \setminus X)$

$$\max\big\{d_n\big|f(x_{n-1}+)\big|^2, d_n\big|f(x_n-)\big|^2\big\}$$
$$\leqslant 2\big(d_n^2\|f'\|^2_{L^2[x_{n-1},x_n]} + \|f\|^2_{L^2[x_{n-1},x_n]}\big), \quad n \in \mathbb{N}. \tag{3.45}$$

Taking a sum and noting that $d^*(X) < \infty$ we arrive at (3.37). It follows that the mapping $\pi_\pm$ (see (3.34)) originally defined on functions with compact supports can be extended to bounded surjective mappings from $W^{1,2}(\mathbb{R}_+ \setminus X)$ onto $l^2(\mathbb{N}; \{d_n\})$.

Further, let $f \in W^{1,2}(\mathbb{R}_+ \setminus X)$ and let $a_n^+ := f(x_{n-1}+)$, $a_n^- := f(x_n-)$. Since $d^*(X) < \infty$, it follows from (3.37) that the sequences $a^\pm = \{a_n^\pm\}_1^\infty$ satisfy the first condition in (3.35). The second condition in (3.35) is implied by (3.36).

It remains to prove the necessity of the assumption $d^*(X) < \infty$ for the validity of inequality (3.37). Choose $f_0 \in W^{1,2}[0, 1]$ such that $\|f_0\|^2_{L^2[0,1]} = \frac{1}{C_1}$ and $f_0(0) = f_0(1) = 1$ and put

$$f_n(x) := f_0\big((x - x_{n-1})d_n^{-1}\big), \quad n \in \mathbb{N}. \tag{3.46}$$

Clearly, $f_n(x_{n-1}+) = f_n(x_n-) = 1$ and

$$\|f_n\|^2_{W^{1,2}[x_{n-1},x_n]} = \frac{1}{d_n}\int_0^1 \left|f_0'(t)\right|^2 dt + d_n \int_0^1 \left|f_0(t)\right|^2 dt, \quad n \in \mathbb{N}. \tag{3.47}$$

Substituting f_n in (3.37) with account of (3.47) we arrive at the estimate

$$2d_n \leqslant C_1\bigl(d_n^{-1}\|f_0'\|^2_{L^2[0,1]} + d_n\|f_0\|^2_{L^2[0,1]}\bigr) = C_1\bigl(d_n^{-1}\|f_0'\|^2_{L^2[0,1]} + C_1^{-1}d_n\bigr), \quad n \in \mathbb{N}.$$

This estimate is equivalent to

$$d_n \leqslant C_1 d_n^{-1}\|f_0'\|^2_{L^2[0,1]}, \quad n \in \mathbb{N}.$$

In turn, the latter is equivalent to $d_n^2 \leqslant C_1\|f_0'\|^2_{L^2[0,1]}$ for $n \in \mathbb{N}$, which implies $d^*(X) < \infty$.

(iii) Let $\lim_{n\to\infty} f_n = f$ in $W^{1,2}(\mathbb{R}_+ \setminus X)$ and $\lim_{n\to\infty} \pi_\pm(f_n) = h_\pm$ in $l^2(\mathbb{N})$. By the embedding theorem, $\pi_\pm(f_n)$ weakly converges to $\pi_\pm(f)$ as $n \to \infty$. Thus, $f \in \operatorname{dom}(\pi_\pm)$, $\pi_\pm(f) = h_\pm$ and the mapping $\pi_\pm$ is closed.

Further, let $d^*(X) < \infty$. Then the space $l^2(\mathbb{N})$ is continuously embedded in $l^2(\mathbb{N}; \{d_n\})$. Therefore the surjectivity of the mapping $\pi_\pm$ is implied by the statement (i).

Conversely, let the trace mapping $\pi_+ : \operatorname{dom}(\pi_+) \to l^2(\mathbb{N})$ be surjective, i.e. $\operatorname{ran}(\pi_+) = l^2(\mathbb{N})$. Then, by (i), $l^2(\mathbb{N})$ is a subset of $l^2(\mathbb{N}; \{d_n\})$. It is easily seen that the identical embedding $i_+ : l^2(\mathbb{N}) \hookrightarrow l^2(\mathbb{N}; \{d_n\})$ is closed. By the closed graph theorem, i_+ is continuous, i.e. there exists a constant $C > 0$ such that $\sum_n d_n|a_n|^2 \leqslant C\sum_n |a_n|^2$. It follows that $d^*(X) = \sup_n d_n \leqslant C$.

(iv) Let conditions (3.38) be satisfied and $f = \bigoplus_1^\infty f_n \in W^{1,2}(\mathbb{R}_+ \setminus X)$. Since $d_*(X) > 0$, we get from (3.37)

$$\begin{aligned}\bigl\|\pi_+(f)\bigr\|_{l^2(\mathbb{N})} &= \sum_{n\in\mathbb{N}} \bigl|f_n(x_{n-1}+)\bigr|^2 \leqslant d_*(X)^{-1}\sum_{n\in\mathbb{N}} d_n\bigl|f_n(x_{n-1}+)\bigr|^2 \\ &\leqslant C_3\|f\|^2_{W^{1,2}(\mathbb{R}_+\setminus X)},\end{aligned} \tag{3.48}$$

where $C_3 = C_1 d_*(X)^{-1}$. Similarly we get $\|\pi_-(f)\|_{l^2(\mathbb{N})} = \sum_{n\in\mathbb{N}} |f_n(x_n-)|^2 \leqslant C_3\|f\|^2_{W^{1,2}(\mathbb{R}_+\setminus X)}$.

Conversely, let $\operatorname{dom}(\pi_+) = W^{1,2}(\mathbb{R}_+ \setminus X)$. Then $l^2(\mathbb{N})$ is isomorphic algebraically and topologically to the quotient space $W^{1,2}(\mathbb{R}_+ \setminus X)/\ker\pi_+$. Combining this fact with the statement (i), we get that $l^2(\mathbb{N})$ is a subset of the weighted l^2-space $l^2(\mathbb{N}; \{d_n\})$. Hence, as it is proved in the step (iii), $d^*(X) < \infty$. In turn, by the statement (ii), we get that $l^2(\mathbb{N})$ coincides algebraically and topologically with $l^2(\mathbb{N}; \{d_n\})$. This immediately yields conditions (3.38). □

Remark 3.6. Let $d^*(X) < \infty$. Starting with $f \in W^{1,2}(\mathbb{R}_+ \setminus X)$ we set $a_n^+ := f(x_{n-1}+)$, $a_n^- := f(x_n-)$, $n \in \mathbb{N}$, and define the function $g = \bigoplus_1^\infty g_n$ where g_n, $n \in \mathbb{N}$, is given by (3.39). It is proved in statement (ii) that the sequences $a^\pm = \{a_n^\pm\}_1^\infty$ satisfy conditions (3.37) and $g \in W^{1,2}(\mathbb{R}_+ \setminus X)$. Therefore f admits the unique decomposition

$$f = g + f_0, \quad \text{where } f_0 := f - g \in W_0^{1,2}(\mathbb{R}_+ \setminus X).$$

In the case $d^*(X) = \infty$ this decomposition fails since $g \notin W_0^{1,2}(\mathbb{R}_+ \setminus X)$, in general.

Remark 3.7. Assume that $d^*(X) < \infty$. Then using the continuity and surjectivity of the trace mapping $\tau := (\pi_-, \pi_+)$ furnished in Proposition 3.5, and following the approach from [63] one can obtain a description of the set of self-adjoint extensions of the operator D_X by means of the Krein type formula for resolvents. It is a way alternative to that discussed in the next section.

3.3. Boundary triplets for Dirac operators with point interactions

Here we construct a boundary triplet for the operator $D_X^* := \bigoplus_{n=1}^{\infty} D_n^*$. First we show that without additional restriction on X the direct sum $\bigoplus_{n=1}^{\infty} \widetilde{\Pi}^{(n)}$ of boundary triplets $\widetilde{\Pi}^{(n)}$ given by (3.8) forms only *a B-generalized boundary triplet* for D_X^*.

Proposition 3.8. *Let X be as above, $d^*(X) < \infty$, and let $\widetilde{\Pi}^{(n)} = \{\mathbb{C}^2, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ be the boundary triplet for the operator D_n^* defined in Lemma 3.1. Let also $A := D_X := \bigoplus_1^{\infty} D_n$, $\mathcal{H} = l^2(\mathbb{N}) \otimes \mathbb{C}^2$, and $\widetilde{\Gamma}_j = \bigoplus_{n=1}^{\infty} \widetilde{\Gamma}_j^{(n)}$, $j \in \{0, 1\}$, i.e.*

$$\widetilde{\Gamma}_0 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \left\{ \begin{pmatrix} f_1(x_{n-1}+) \\ icf_2(x_n-) \end{pmatrix} \right\}_{n\in\mathbb{N}} \quad \text{and}$$

$$\widetilde{\Gamma}_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \left\{ \begin{pmatrix} icf_2(x_{n-1}+) \\ f_1(x_n-) \end{pmatrix} \right\}_{n\in\mathbb{N}}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \operatorname{dom}\bigl(D_X^*\bigr). \tag{3.49}$$

Then:

(i) *The mappings $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ are densely defined and closed. Moreover, the operator $A_* := A^* \restriction \operatorname{dom}(A_*)$ satisfies*

$$\begin{aligned} \operatorname{dom}(A_*) &:= \operatorname{dom}(\widetilde{\Gamma}_0) \cap \operatorname{dom}(\widetilde{\Gamma}_1) = \operatorname{dom}(\widetilde{\Gamma}_0) = \operatorname{dom}(\widetilde{\Gamma}_1) \\ &= \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in W^{1,2}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2 : \right. \\ &\qquad \left. \bigl\{f_j(x_{n-1}+)\bigr\}_1^{\infty}, \bigl\{f_j(x_n-)\bigr\}_1^{\infty} \in l^2(\mathbb{N}),\ j \in \{1, 2\} \right\}. \end{aligned} \tag{3.50}$$

(ii) *The direct sum $\widetilde{\Pi} := \bigoplus_{n=1}^{\infty} \widetilde{\Pi}^{(n)} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ forms a B-generalized boundary triplet for D_X^* in the sense of Definition 2.7. In particular, $\operatorname{ran}(\widetilde{\Gamma}_0) = \mathcal{H}$.*

(iii) *The transposed triplet $\widetilde{\Pi}^{\top} = \{\mathcal{H}, \widetilde{\Gamma}_0^{\top}, \widetilde{\Gamma}_1^{\top}\} := \{\mathcal{H}, \widetilde{\Gamma}_1, -\widetilde{\Gamma}_0\}$ also forms a B-generalized boundary triplet for D_X^*. In particular, $\operatorname{ran}(\widetilde{\Gamma}_1) = \mathcal{H}$.*

(iv) *The triplet $\widetilde{\Pi}$ is an (ordinary) boundary triplet for the operator $D_X^* = \bigoplus_{n=1}^{\infty} D_n^*$ if and only if $0 < d_*(X) < d^*(X) < \infty$.*

Proof. (i) By Definition 2.9 and formula (3.8), the domain of $\widetilde{\Gamma}_0$ is given by

$$\begin{aligned} \operatorname{dom}(\widetilde{\Gamma}_0) = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in W^{1,2}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2 : \right. \\ \left. \bigl\{f_1(x_{n-1}+)\bigr\}_1^{\infty}, \bigl\{f_2(x_n-)\bigr\}_1^{\infty} \in l^2(\mathbb{N}) \right\}. \end{aligned} \tag{3.51}$$

Since $d^*(X) < \infty$, it follows from (3.36) that

$$\sum_{n\in\mathbb{N}}\left|f_1(x_n-)-f_1(x_{n-1}+)\right|^2 \leqslant d^*(X)\sum_{n\in\mathbb{N}} d_n^{-1}\left|f_1(x_n-)-f_1(x_{n-1}+)\right|^2$$
$$\leqslant d^*(X)\|f\|^2_{W^{1,2}(\mathbb{R}_+\setminus X)}<\infty. \tag{3.52}$$

Combining this inequality with (3.51), yields $\{f_1(x_n-)\}_1^\infty \in l^2(\mathbb{N})$. The inclusion $\{f_2(x_{n-1}+)\}_1^\infty \in l^2(\mathbb{N})$ is proved similarly. Hence $\mathrm{dom}(\widetilde{\Gamma}_0)=\mathrm{dom}(\widetilde{\Gamma}_0)\cap\mathrm{dom}(\widetilde{\Gamma}_1)$. The equality $\mathrm{dom}(\widetilde{\Gamma}_1)=\mathrm{dom}(\widetilde{\Gamma}_0)\cap\mathrm{dom}(\widetilde{\Gamma}_1)$ is proved in much the same way.

(ii) Due to (i) $\ker\widetilde{\Gamma}_0\subset\mathrm{dom}(\widetilde{\Gamma}_1)=\mathrm{dom}(A_*)$. Hence

$$A_{*0}:=A_*\lceil\ker\widetilde{\Gamma}_0=A^*\lceil\ker\widetilde{\Gamma}_0=\bigoplus_{n=1}^{\infty}D_{n,0}=A_0=A_0^*, \tag{3.53}$$

i.e. $A_{*0}=A_0$ is self-adjoint. The Green's identity (2.16) is obviously satisfied for $f,g\in\mathrm{dom}(A_*)$ (see (2.16)). It remains to show that $\mathrm{ran}(\Gamma_0)=\mathcal{H}=l^2(\mathbb{N})\otimes\mathbb{C}^2$. This fact is immediate from (3.51) and Proposition 3.5(iii).

(iii) The proof is similar to the proof of (ii).

(iv) Let conditions (3.38) be satisfied. Then, by Proposition 3.5(iii), $\mathrm{dom}(\pi_\pm)=W^{1,2}(\mathbb{R}_+\setminus X)$. Combining this fact with (3.50) we get $\mathrm{dom}(\widetilde{\Gamma}_j)=W^{1,2}(\mathbb{R}_+\setminus X)\otimes\mathbb{C}^2$, $j\in\{0,1\}$. Hence Green's formula (2.12) holds for all $f,g\in\mathrm{dom}(A^*)$.

Next let us prove the surjectivity of the mapping $\widetilde{\Gamma}=\{\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$. Let $a_k=\{a_{kn}\}_{n=1}^\infty\in l^2(\mathbb{N})$, $k\in\{1,\dots,4\}$. By Proposition 3.5(iii), there exist $f_1,f_2\in W^{1,2}(\mathbb{R}_+\setminus X)$ such that

$$\pi_+(f_1)=\left\{f_1(x_{n-1}+)\right\}_{n=1}^\infty=\{a_{1n}\}_{n=1}^\infty,\qquad \pi_-(f_1)=\left\{f_1(x_n-)\right\}_{n=1}^\infty=\{a_{4n}\}_{n=1}^\infty,$$
$$\pi_+(f_2)=\left\{icf_2(x_{n-1}+)\right\}_{n=1}^\infty=\{a_{3n}\}_{n=1}^\infty,\qquad \pi_-(f_2)=\left\{icf_2(x_n-)\right\}_{n=1}^\infty=\{a_{2n}\}_{n=1}^\infty.$$

Combining these relations with (3.49), yields the surjectivity of the mapping $\widetilde{\Gamma}$.

Conversely let $\mathrm{dom}(\widetilde{\Gamma}_j)=W^{1,2}(\mathbb{R}_+\setminus X)\otimes\mathbb{C}^2$, $j\in\{0,1\}$. Then, by (3.49), $\mathrm{dom}(\pi_\pm)=W^{1,2}(\mathbb{R}_+\setminus X)$. Now Proposition 3.5(iii) yields the condition $d_*(X)>0$. $\square$

Remark 3.9.

(i) We emphasize the difference between the trace mappings $\pi_\pm: W^{1,2}(\mathbb{R}_+\setminus X)\to l^2(\mathbb{N})$ and $\pi: W^{1,2}(\mathbb{R}_+)\to l^2(\mathbb{N})$ (see (3.30)). According to Proposition 3.5(iii) the first one is surjective if and only if $d^*(X)<\infty$. At the same time, by Proposition 3.4, the second one is surjective if and only if $0<d_*(X)<d^*(X)<\infty$.

(ii) We emphasize that the mapping $\widetilde{\Gamma}_j$, $j\in\{0,1\}$, is bounded, $\widetilde{\Gamma}_j\in[\mathfrak{H}_+,\mathcal{H}]$, if and only if $d_*(X)>0$. Indeed, it follows from (3.1) and (3.8) that the estimate

$$\|\widetilde{\Gamma}_0^{(n)}f_n\|_{\mathbb{C}^2}\leqslant\widetilde{C}_{0n}\left(\|D_n^*f_n\|^2_{L^2[x_{n-1},x_n]\otimes\mathbb{C}^2}+\|f_n\|^2_{L^2[x_{n-1},x_n]\otimes\mathbb{C}^2}\right),\quad f_n\in\mathrm{dom}\left(D_n^*\right),\ n\in\mathbb{N},$$

yields (in fact, is equivalent to) the estimate

$$\left|f_{1n}(x_{n-1}+)\right|^2+\left|f_{2n}(x_n-)\right|^2$$
$$\leqslant k_n\left(\|f_{1n}\|^2_{W^{1,2}[x_{n-1},x_n]}+\|f_{2n}\|^2_{W^{1,2}[x_{n-1},x_n]}\right),\quad f_{1n},f_{2n}\in W^{1,2}[x_{n-1},x_n].$$

Thus, the mapping $\widetilde{\Gamma}_0=\bigoplus_{n\in\mathbb{N}}\widetilde{\Gamma}_0^{(n)}$ is bounded, $\widetilde{\Gamma}_0\in[\mathfrak{H}_+,\mathcal{H}]$, if and only if $\sup_n k_n<\infty$. In turn, according to the Sobolev embedding theorem, the latter is amount to saying that $d_*(X)>0$.

This fact is similar to that for Schrödinger operator (cf. [44, Corollary 4.9]). It also shows that the condition $\sup_n \|\widetilde{\Gamma}_0^{(n)}\| < \infty$ is only sufficient for a triplet $\widetilde{\Pi} = \bigoplus_{n\in\mathbb{N}} \widetilde{\Pi}_n$ to form a B-generalized boundary triplet (cf. [44, Proposition 3.6]).

To obtain an appropriate boundary triplet for the operator $D_X^* = \bigoplus_{n=1}^\infty D_n^*$ in the case $d_*(X) = 0$ we regularize the boundary triplets $\widetilde{\Pi}_n$ for D_n^*, $n \in \mathbb{N}$, given by (3.8). To this end we apply the regularization procedure proposed in Corollary 2.13 (cf. formula (2.29)).

Theorem 3.10. *Let $X = \{x_n\}_{n=1}^\infty$ be as above and $d^*(X) < +\infty$. Define the mappings*

$$\Gamma_j^{(n)} : W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2 \to \mathbb{C}^2, \quad n \in \mathbb{N},\ j \in \{0, 1\},$$

by setting

$$\Gamma_0^{(n)} f := \begin{pmatrix} d_n^{1/2} f_1(x_{n-1}+) \\ icd_n^{3/2}\sqrt{1 + \frac{1}{c^2 d_n^2}}\, f_2(x_n-) \end{pmatrix}, \tag{3.54}$$

$$\Gamma_1^{(n)} f := \begin{pmatrix} icd_n^{-1/2}(f_2(x_{n-1}+) - f_2(x_n-)) \\ d_n^{-3/2}(1 + \frac{1}{c^2 d_n^2})^{-1/2}(f_1(x_n-) - f_1(x_{n-1}+) - icd_n f_2(x_n-)) \end{pmatrix}. \tag{3.55}$$

Then:

(i) *For any $n \in \mathbb{N}$, $\Pi^{(n)} = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ is a boundary triplet for D_n^*.*
(ii) *The direct sum $\Pi := \bigoplus_{n=1}^\infty \Pi^{(n)} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with $\mathcal{H} = l^2(\mathbb{N}, \mathbb{C}^2)$ and $\Gamma_j = \bigoplus_{n=1}^\infty \Gamma_j^{(n)}$, $j \in \{0, 1\}$, is a boundary triplet for the operator $D_X^* = \bigoplus_{n=1}^\infty D_n^*$.*

Proof. According to Lemma 3.1(ii) each operator $D_{n,0} = D_{n,0}^*$ has a gap $(-\alpha_n, \alpha_n) \supset (-c^2/2, c^2/2)$. Hence the symmetric operator $D_X = \bigoplus_{n=1}^\infty D_n$ has a gap $(-\alpha, \alpha) := \bigcap_n^\infty (-\alpha_n, \alpha_n)$. Since $d^*(X) < \infty$, it follows from (3.10) that $\alpha > c^2/2$, i.e. $(-\alpha, \alpha) \supset (-c^2/2, c^2/2)$. Moreover, by (3.12),

$$\widetilde{M}_n\left(\frac{c^2}{2}\right) = \begin{pmatrix} 0 & 1 \\ 1 & d_n \end{pmatrix} \quad \text{and} \quad \widetilde{M}_n'\left(\frac{c^2}{2}\right) = \begin{pmatrix} d_n & d_n^2/2 \\ d_n^2/2 & d_n/c^2 + d_n^3/3 \end{pmatrix}. \tag{3.56}$$

Since $c^2/2 \in (0, \alpha)$, we can apply Corollary 2.13 to regularize the sequence of boundary triplets $\widetilde{\Pi}^{(n)} = \{\mathbb{C}^2, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ for D_n^*, $n \in \mathbb{N}$, defined by (3.8). Starting with (3.56) we define the matrices $R_n = R_n^*$ and $Q_n = Q_n^*$ by setting

$$R_n := \begin{pmatrix} d_n^{1/2} & 0 \\ 0 & d_n^{3/2}\sqrt{1 + \frac{1}{c^2 d_n^2}} \end{pmatrix}, \qquad Q_n := \widetilde{M}_n\left(\frac{c^2}{2}\right) = \begin{pmatrix} 0 & 1 \\ 1 & d_n \end{pmatrix}, \quad n \in \mathbb{N}. \tag{3.57}$$

Next we define a new sequence of boundary triplets $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ by formulas (2.29),

$$\Gamma_0^{(n)} := R_n \widetilde{\Gamma}_0^{(n)}, \qquad \Gamma_1^{(n)} := R_n^{-1}\big(\widetilde{\Gamma}_1^{(n)} - Q_n \widetilde{\Gamma}_0^{(n)}\big), \quad n \in \mathbb{N}. \tag{3.58}$$

Clearly, the corresponding Weyl function is

$$M_n(z) = R_n^{-1}\big(\widetilde{M}_n(z) - Q_n\big) R_n^{-1}, \quad n \in \mathbb{N}. \tag{3.59}$$

Let us check that the family $\{M_n(\cdot)\}_{n=1}^{\infty}$ of the Weyl functions satisfies conditions (2.28) of Corollary 2.13. Indeed, by (3.57), $M_n(c^2/2)=0$. Moreover, combining (3.59) with (3.57) and (3.56) we get

$$M_n'\left(\frac{c^2}{2}\right) = R_n^{-1}\widetilde{M}_n'\left(\frac{c^2}{2}\right)R_n^{-1} = R_n^{-1}\begin{pmatrix} d_n & d_n^2/2 \\ d_n^2/2 & d_n/c^2 + d_n^3/3 \end{pmatrix} R_n^{-1}$$
$$= \begin{pmatrix} 1 & \frac{1}{2}(1+\frac{1}{c^2 d_n^2})^{-1/2} \\ \frac{1}{2}(1+\frac{1}{c^2 d_n^2})^{-1/2} & \frac{1}{3}\frac{3+c^2d_n^2}{1+c^2d_n^2} \end{pmatrix}, \quad n\in\mathbb{N}. \tag{3.60}$$

Hence $\sup_{n\in\mathbb{N}}\|M_n'(c^2/2)\|<\infty$ and the first condition in (2.28) is satisfied. Further,

$$\left(M_n'(c^2/2)\right)^{-1} = R_n\widetilde{M}_n'(c^2/2)^{-1}R_n$$
$$= \frac{1}{\Delta(c^2/2)}\begin{pmatrix} \frac{1}{3}\frac{3+c^2d_n^2}{1+c^2d_n^2} & -\frac{1}{2}(1+\frac{1}{c^2d_n^2})^{-1/2} \\ -\frac{1}{2}(1+\frac{1}{c^2d_n^2})^{-1/2} & 1 \end{pmatrix}, \tag{3.61}$$

where $\Delta(c^2/2) = 12(1+c^2d_n^2)(12+c^2d_n^2)^{-1}$. Hence $\sup_{n\in\mathbb{N}}\|(M_n'(c^2/2))^{-1}\|<\infty$ and the second condition in (2.28) is satisfied too. Thus, by Corollary 2.13, the direct sum $\bigoplus_{n=1}^{\infty}\Pi^{(n)}$ is an ordinary boundary triplet for the operator D_X^*.

To complete the proof it remains to note that the mappings $\Gamma_0^{(n)}$ and $\Gamma_1^{(n)}$, $n\in\mathbb{N}$, defined by (3.58) coincide with the mappings given by (3.54), (3.55). □

Remark 3.11. It follows from (3.56) that both conditions

$$\sup_{n\in\mathbb{N}}\left\|\widetilde{M}_n(c^2/2)\right\|<\infty \quad \text{and} \quad \sup_{n\in\mathbb{N}}\left\|\widetilde{M}_n'(c^2/2)\right\|<\infty$$

are satisfied for any discrete sequence $X=\{x_n\}_{n=1}^{\infty}$ whenever $d^*(X)<\infty$. Applying Theorem 2.12(i) we obtain an alternative proof of Proposition 3.8(ii).

Further, it is easily seen that $\widetilde{M}_n^{-1}(\frac{c^2}{2}) = \begin{pmatrix} -d_n & 1 \\ 1 & 0\end{pmatrix}$ and

$$(\widetilde{M}_n^{-1})'\left(\frac{c^2}{2}\right) = \widetilde{M}_n^{-1}\left(\frac{c^2}{2}\right)\widetilde{M}_n'\left(\frac{c^2}{2}\right)\widetilde{M}_n^{-1}\left(\frac{c^2}{2}\right) = \begin{pmatrix} d_n^3/3 + d_n/c^2 & -d_n^2/2 \\ -d_n^2/2 & d_n \end{pmatrix}, \quad n\in\mathbb{N}.$$

Thus, the sequence $\{-\widetilde{M}_n^{-1}(\cdot)\}_{n\in\mathbb{N}}$ satisfies conditions (2.19) at the point $a=c^2/2$ provided that $d^*(X)<\infty$. Again, by Theorem 2.12(i), the direct sum $\widetilde{\Pi} := \bigoplus_{n=1}^{\infty}\widetilde{\Pi}^{(n)}$ forms a B-generalized boundary triplet for D_X^* in the sense of Definition 2.7. These reasonings give an alternative proof of Proposition 3.8(iii).

At the same time, condition $\sup_{n\in\mathbb{N}}\|(\widetilde{M}_n'(c^2/2))^{-1}\|<\infty$ is satisfied if and only if $d_*(X)>0$. Hence Theorem 2.12(ii) gives an alternative proof of Proposition 3.8(iv). On the other hand, this example shows that condition (2.20) of Theorem 2.12 is not implied by conditions (2.19) (cf. Example 2.14(i)).

Corollary 3.12. *Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for the operator D_X^* defined in Theorem 3.10, i.e. $\Pi=\bigoplus_{n=1}^{\infty}\Pi^{(n)}$. Then:*

(i) *The set of closed proper extensions of* D_X *is parameterized as follows*

$$\widetilde{D}_X = D_{X,\Theta} := D_X^* \upharpoonright \operatorname{dom}(D_{X,\Theta}),$$
$$\operatorname{dom}(D_{X,\Theta}) = \{ f \in W^{1,2}(\mathcal{I} \setminus X) \otimes \mathbb{C}^2 \colon \{\Gamma_0 f, \Gamma_1 f\} \in \Theta \}, \tag{3.62}$$

where $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \setminus \{\{0\} \cup \mathcal{H} \oplus \mathcal{H}\}$.

(ii) $D_{X,\Theta}$ *is symmetric (self-adjoint) if and only if so is* Θ. *Moreover,* $n_\pm(D_{X,\Theta}) = n_\pm(\Theta)$.

(iii) $D_{X,\Theta}$ *is m-dissipative (m-accumulative) if and only if so is* Θ.

(iv) $\widetilde{D}_X = D_{X,\Theta}$ *is disjoint with* $D_{X,0} := D_X^* \upharpoonright \ker \Gamma_0$ *if and only if* Θ *is a closed operator. In this case relation (3.62) takes the form*

$$\widetilde{D}_X = D_{X,\Theta} := D^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0). \tag{3.63}$$

Moreover, $\widetilde{D}_X = D_{X,\Theta}$ *and* $D_{X,0}$ *are transversal if and only if (3.63) holds with* $\Theta \in [\mathcal{H}]$.

Proof. According to (3.28) $D_X^* = \bigoplus_{n=1}^\infty D_n^*$, hence

$$\operatorname{dom}(D_X^*) = \bigoplus_{n=1}^\infty \operatorname{dom}(D_n^*) = \bigoplus_{n=1}^\infty W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2 = W^{1,2}(\mathcal{I} \setminus X) \otimes \mathbb{C}^2.$$

One completes the proof by applying Proposition 2.3. □

Remark 3.13. Consider a Dirac operator with point interactions supported on the set $X = \{x_k\}_{k \in I} \subset \mathcal{I} = (a,b)$, $x_{k-1} < x_k$, $k \in I$, where either $I = \mathbb{N}$ or $I = \mathbb{Z}$. Moreover, we assume as usual that $\lim_{k\to\infty} x_k = b \leqslant \infty$ and $\lim_{k\to-\infty} x_k = a \geqslant -\infty$ in the second case. Now in place of (3.28) the minimal operator is $D_X := \bigoplus_{n\in I} D_n$.

To investigate the non-relativistic limit on the line we also will consider the operators

$$D_{a-} \oplus D_X \oplus D_{b+}, \tag{3.64}$$

where the first (resp. the third) summand is missing whenever $a = -\infty$ (resp. $b = \infty$). The corresponding maximal operators are given by $D_X^* = \bigoplus_{n\in I} D_n^*$ and $D_{a-}^* \oplus D_X^* \oplus D_{b+}^*$, respectively.

The appropriate boundary triplets for the maximal operators are of the form $\bigoplus_{n\in I} \Pi^{(n)}$ and $\Pi^{(a-)} \oplus (\bigoplus_{n\in I}^\infty \Pi^{(n)}) \oplus \Pi^{(b+)}$, respectively. Here $\Pi^{(a-)}$ and $\Pi^{(b+)}$ are the boundary triplets defined in Lemmas 3.2 and 3.3, respectively. Using these boundary triplets one parameterizes the set of proper extensions of the operators D_X and (3.64) in just the same way as in Corollary 3.12.

4. Non-relativistic limit

Here we investigate the non-relativistic resolvent limit of maximal dissipative (accumulative) extension $D_{X,\Theta}^c$ defined by (3.62). To this end we consider the operator $-d^2/dx^2$ with point interactions on a discrete set and following [44] describe the corresponding boundary triplets, Weyl functions, etc.

4.1. Boundary triplets for Schrödinger building blocks

First we present a boundary triplet for the maximal operator $-\frac{d^2}{dx^2}$ on a finite interval. Let H_n denote the minimal operator associated with the differential expression $-\frac{d^2}{dx^2}$ in $L^2[x_{n-1}, x_n]$ by

$$H_n := -\frac{d^2}{dx^2} \upharpoonright \operatorname{dom}(H_n), \quad \operatorname{dom}(H_n) = W_0^{2,2}[x_{n-1}, x_n], \ n \in \mathbb{N}. \tag{4.1}$$

Lemma 4.1. *The operator H_n is symmetric in $L^2[x_{n-1}, x_n]$ with the deficiency indices $n_\pm(H_n) = 2$. Its adjoint H_n^* is given by*

$$H_n^* = -\frac{d^2}{dx^2} \restriction \operatorname{dom}\bigl(H_n^*\bigr), \quad \operatorname{dom}\bigl(H_n^*\bigr) = W^{2,2}[x_{n-1}, x_n],$$

and the defect subspace $\mathfrak{N}_z = \ker(H_n^* - z)$ is spanned by the functions $f_{n,H}^\pm(\cdot)$,

$$f_{n,H}^\pm(z)(x) := e^{\pm i\sqrt{z}x}, \qquad \operatorname{Im}(\sqrt{z}\,) \geqslant 0. \tag{4.2}$$

Moreover, the following hold:

(i) *The triplet $\widetilde{\Pi}_H^{(n)} = \{\mathbb{C}^2, \widetilde{\Gamma}_{0,H}^{(n)}, \widetilde{\Gamma}_{1,H}^{(n)}\}$ where*

$$\widetilde{\Gamma}_{0,H}^{(n)} f := \begin{pmatrix} f(x_{n-1}+) \\ f'(x_n-) \end{pmatrix}, \qquad \widetilde{\Gamma}_{1,H}^{(n)} f := \begin{pmatrix} f'(x_{n-1}+) \\ f(x_n-) \end{pmatrix}, \tag{4.3}$$

forms a boundary triplet for H_n^*.

(ii) *The spectrum of the operator $H_{n,0} := H_n^* \restriction \ker \widetilde{\Gamma}_{0,H}^{(n)}$ is discrete,*

$$\sigma(H_{n,0}) = \sigma_d(H_{n,0}) = \left\{ \frac{\pi^2}{d_n^2}\left(j + \frac{1}{2}\right)^2, \ j \in \{0\} \cup \mathbb{N} \right\}. \tag{4.4}$$

(iii) *The γ-field $\widetilde{\gamma}_n(\cdot) : \mathbb{C}^2 \to L^2[x_{n-1}, x_n]$, corresponding to the triplet $\widetilde{\Pi}^{(n)}$ is given by*

$$\begin{aligned} \widetilde{\gamma}_n(z)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \frac{1}{\cos(d_n\sqrt{z}\,)}\left(\cos\bigl(\sqrt{z}(x_n - x)\bigr), \ -\frac{\sin(\sqrt{z}(x_{n-1} - x))}{\sqrt{z}}\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \frac{1}{\cos(d_n\sqrt{z}\,)}\left(w_1\cos\bigl(\sqrt{z}(x_n - x)\bigr) - w_2\frac{\sin(\sqrt{z}(x_{n-1} - x))}{\sqrt{z}}\right), \\ z \in \rho(H_{n,0}). \end{aligned} \tag{4.5}$$

(iv) *The Weyl function $\widetilde{M}_{n,H}(\cdot)$, corresponding to the triplet $\widetilde{\Pi}_H^{(n)}$ is*

$$\widetilde{M}_{n,H}(z) = \frac{1}{\cos(d_n\sqrt{z}\,)}\begin{pmatrix} \sqrt{z}\sin(d_n\sqrt{z}\,) & 1 \\ 1 & z^{-1/2}\sin(d_n\sqrt{z}\,) \end{pmatrix}, \quad z \in \rho(H_{n,0}). \tag{4.6}$$

Next we present a boundary triplet for the operator $-\frac{d^2}{dx^2}$ on the half-line. Denote by H_{a-} the minimal operator associated with the differential expression $-\frac{d^2}{dx^2}$ on $L^2(\mathbb{R}_a^-)$ by

$$H_{a-} = -\frac{d^2}{dx^2} \restriction \operatorname{dom}(H_{a-}), \quad \operatorname{dom}(H_{a-}) = W_0^{2,2}\bigl(\mathbb{R}_a^-\bigr). \tag{4.7}$$

Similarly, H_{b+} denotes the minimal operator generated by the expression $-d^2/dx^2$ on $L^2(\mathbb{R}_b^+)$,

$$H_{b+} = -\frac{d^2}{dx^2} \restriction \operatorname{dom}(H_{b+}), \quad \operatorname{dom}(H_{b+}) = W_0^{2,2}\bigl(\mathbb{R}_b^+\bigr). \tag{4.8}$$

Lemma 4.2. *The operator H_{a-} is symmetric with the deficiency indices $n_\pm(H_{a-}) = 1$. Its adjoint H^*_{a-} is given by*

$$H^*_{a-} := (H_{a-})^* = -\frac{d^2}{dx^2} \upharpoonright \operatorname{dom}\big(H^*_{a-}\big), \quad \operatorname{dom}\big(H^*_{a-}\big) = W^{2,2}\big(\mathbb{R}^-_a\big),$$

and the defect subspace $\mathfrak{N}_z = \ker(H^*_{a-} - z)$ is spanned by the vector function

$$f^-_a(x,z) := e^{-i\sqrt{z}x}, \qquad \operatorname{Im}(\sqrt{z}) > 0. \tag{4.9}$$

Moreover, the following hold:

(i) *The triplet $\Pi^{(a-)}_H = \{\mathbb{C}^2, \Gamma^{(a-)}_{0,H}, \Gamma^{(a-)}_{1,H}\}$, where*

$$\Gamma^{(a-)}_{0,H} f := f'(a-), \qquad \Gamma^{(a-)}_{1,H} f := f(a-), \tag{4.10}$$

forms a boundary triplet for the operator H^*_{a-}.

(ii) *The spectrum of the operator $H_{a-,0} := H^*_{a-} \upharpoonright \ker \Gamma^{(a-)}_{0,H} = (H_{a-,0})^*$ is absolutely continuous,*

$$\sigma(H_{a-,0}) = \sigma_{ac}(H_{a-,0}) = [0,+\infty). \tag{4.11}$$

(iii) *The corresponding γ-field $\gamma_{a-,H}(\cdot) : \mathbb{C} \to L^2(\mathbb{R}^-_a)$, is*

$$\gamma_{a-,H}(z)w = w\frac{i}{\sqrt{z}} e^{i\sqrt{z}a} f^-_{a,H}(\cdot,z), \quad w \in \mathbb{C}_+,\ z \in \rho(H_{a-,0}). \tag{4.12}$$

(iv) *The Weyl function $M_{a-,H}(\cdot)$ corresponding to the triplet $\Pi^{(a-)}_H$ is*

$$M_{a-,H}(z) = \frac{i}{\sqrt{z}}, \quad z \in \rho(H_{a-,0}). \tag{4.13}$$

Lemma 4.3. *The operator H_{b+} is symmetric with the deficiency indices $n_\pm(H_{b+}) = 1$. Its adjoint H^*_{b+} is given by*

$$H^*_{b+} := (H_{b+})^* = -\frac{d^2}{dx^2} \upharpoonright \operatorname{dom}\big(H^*_{b+}\big), \quad \operatorname{dom}\big(H^*_{b+}\big) = W^{2,2}\big(\mathbb{R}^+_b\big),$$

and the defect subspace $\mathfrak{N}_z = \ker(H^*_{b+} - z)$ is spanned by the vector function

$$f^+_{b,H}(x,z) := e^{i\sqrt{z}x}, \qquad \operatorname{Im}(\sqrt{z}) > 0. \tag{4.14}$$

Moreover, the following hold:

(i) *The triplet $\Pi^{(b+)}_H = \{\mathbb{C}^2, \Gamma^{(b+)}_{0,H}, \Gamma^{(b+)}_{1,H}\}$, where*

$$\Gamma^{(b+)}_{0,H} f := f(b+), \qquad \Gamma^{(b+)}_{1,H} f := f'(b+), \tag{4.15}$$

forms a boundary triplet for the operator H^*_{b+}.

(ii) *The spectrum of the operator* $H_{b+,0} := H^*_{b+,H} \upharpoonright \ker \Gamma^{(b+)}_{0,H} = H^*_{b+,0}$ *is absolutely continuous,*

$$\sigma(H_{b+,0}) = \sigma_{ac}(H_{b+,0}) = [0, +\infty). \tag{4.16}$$

(iii) *The* γ*-field* $\gamma_{b+,H}(\cdot) : \mathbb{C} \to L^2(\mathbb{R}^+_b)$ *corresponding to the triplet* $\Pi^{(b+)}_H$ *is given by*

$$\gamma_{b+}(z)w = we^{-i\sqrt{z}b} f^+_{b,z,H}, \quad w \in \mathbb{C},\ z \in \rho(H_{b+,0}). \tag{4.17}$$

(iv) *The Weyl function* $M_{b+,H}(\cdot)$ *corresponding to the triplet* $\Pi^{(b+)}_H$ *is*

$$M_{b+,H}(z) = i\sqrt{z}, \quad z \in \rho(H_{b+,0}). \tag{4.18}$$

4.2. Boundary triplet for Schrödinger operators with point interactions

Let $X = \{x_n\}_{n=1}^{\infty}$, $a = x_0$, $x_{n-1} < x_n$, be a discrete set as in Section 3.2. We define H_X by

$$H_X := \bigoplus_{n\in\mathbb{N}} H_n. \tag{4.19}$$

Now we are ready to construct a boundary triplet for the operator H^*_X.

Proposition 4.4. *(See [44].) Let* $\widetilde{\Pi}^{(n)}_H = \{\mathbb{C}^2, \widetilde{\Gamma}^{(n)}_{0,H}, \widetilde{\Gamma}^{(n)}_{1,H}\}$ *be the boundary triplet for the operator* H^*_n, $n \in \mathbb{N}$, *defined in* (4.3)*. Then the direct sum*

$$\widetilde{\Pi} = \bigoplus_{n=1}^{\infty} \widetilde{\Pi}^{(n)}_H = \{\mathcal{H}, \widetilde{\Gamma}_{0,H}, \widetilde{\Gamma}_{1,H}\}, \quad \mathcal{H} = l_2(\mathbb{N}) \otimes \mathbb{C}^2, \qquad \widetilde{\Gamma}_{j,H} = \bigoplus_{n=1}^{\infty} \widetilde{\Gamma}^{(n)}_{j,H}, \quad j \in \{0,1\}, \tag{4.20}$$

forms a boundary triplet for the operator $H^*_X = \bigoplus_{n=1}^{\infty} H^*_n$ *if and only if* $0 < d_*(X) < d^*(X) < \infty$.

In the case $d_*(X) = 0$ a boundary triplet for the operator H^*_X was constructed in [44, Theorems 4.1, 4.7] by applying to triplets $\widetilde{\Pi}^{(n)}_H$ the regularization procedure described in Corollary 2.13.

Theorem 4.5. *(See [44, Theorem 4.7].) Assume that* $d^*(X) < +\infty$ *and define the mappings* $\Gamma^{(n)}_{j,H} : W^{2,2}[x_{n-1}, x_n] \to \mathbb{C}^2$, $n \in \mathbb{N}$, $j \in \{0,1\}$, *by setting*

$$\Gamma^{(n)}_{0,H} f := \begin{pmatrix} d_n^{1/2} f(x_{n-1}+) \\ d_n^{3/2} f'(x_n-) \end{pmatrix}, \tag{4.21}$$

$$\Gamma^{(n)}_{1,H} f := \begin{pmatrix} d_n^{-1/2}(f'(x_{n-1}+) - f'(x_n-)) \\ d_n^{-3/2}(f(x_n-) - f(x_{n-1}+)) - d_n^{-1/2} f'(x_n-) \end{pmatrix}. \tag{4.22}$$

Then:

(i) *For any* $n \in \mathbb{N}$, $\Pi^{(n)}_H = \{\mathbb{C}^2, \Gamma^{(n)}_{0,H}, \Gamma^{(n)}_{1,H}\}$ *is a boundary triplet for* H^*_n.
(ii) *The direct sum* $\Pi = \bigoplus_{n=1}^{\infty} \Pi^{(n)}_H = \{\mathcal{H}, \Gamma_{0,H}, \Gamma_{1,H}\}$ *with* $\mathcal{H} = l^2(\mathbb{N}, \mathbb{C}^2)$ *and* $\Gamma_{j,H} = \bigoplus_{n=1}^{\infty} \Gamma^{(n)}_{j,H}$, $j \in \{0,1\}$, *is a boundary triplet for the operator* $H^*_X = \bigoplus_{n=1}^{\infty} H^*_n$.

Corollary 4.6. *Let $\Pi_H = \{\mathcal{H}, \Gamma_{0,H}, \Gamma_{1,H}\} := \bigoplus_{n=1}^{\infty} \Pi_H^{(n)}$ be the boundary triplet for H_X^* defined in Theorem 4.5. Then:*

(i) *The set of closed proper extensions of H_X is parameterized as follows:*

$$\widetilde{H}_X = H_{X,\Theta} := H_X^* \upharpoonright \operatorname{dom}(H_{X,\Theta}),$$
$$\operatorname{dom}(H_{X,\Theta}) = \bigl\{ f \in W^{2,2}(\mathcal{I} \setminus X) \colon \{\Gamma_{0,H} f, \Gamma_{1,H} f\} \in \Theta \bigr\}, \tag{4.23}$$

where $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \setminus \{\{0\} \cup \mathcal{H} \oplus \mathcal{H}\}$.

(ii) *$H_{X,\Theta}$ is symmetric (self-adjoint) if and only if so is Θ. Moreover, $n_{\pm}(H_{X,\Theta}) = n_{\pm}(\Theta)$.*

(iii) *$H_{X,\Theta}$ is m-dissipative (m-accumulative) if and only if so is Θ.*

(iv) *$\widetilde{H}_X = H_{X,\Theta}$ is disjoint with $H_{X,0} := H_X^* \upharpoonright \ker \Gamma_{0,H} (= H_{X,0}^*)$ if and only if Θ is a closed operator. In this case relation (4.23) takes the form*

$$\widetilde{H}_X = H_{X,\Theta} := H_X^* \upharpoonright \ker(\Gamma_{1,H} - \Theta \Gamma_{0,H}). \tag{4.24}$$

Moreover, $\widetilde{H}_X = H_{X,\Theta}$ and $H_{X,0}$ are transversal if and only if (4.24) holds with $\Theta \in [\mathcal{H}]$.

Remark 4.7. Consider Schrödinger operator with point interactions supported on the set $X = \{x_k\}_{k\in I} \subset \mathcal{I} = (a,b)$, $x_{k-1} < x_k$, $k \in I$, where either $I = \mathbb{N}$ or $I = \mathbb{Z}$. Moreover, we assume as usual that $\lim_{k\to\infty} x_k = b \leqslant \infty$ and $\lim_{k\to-\infty} x_k = a \geqslant -\infty$ if $I = \mathbb{Z}$. Now in place of (4.19) the minimal operator is $H_X := \bigoplus_{n\in I} H_n$. In the next section we will also use the operators

$$H_{a-} \oplus H_X \oplus H_{b+}, \tag{4.25}$$

where the first (resp. the third) summand is missing whenever $a = -\infty$ (resp. $b = \infty$). The corresponding maximal operators are given by $H_X^* = \bigoplus_{n\in I} H_n^*$ and $H_{a-}^* \oplus H_X^* \oplus H_{b+}^*$, respectively.

The appropriate boundary triplets for the maximal operators are of the form $\bigoplus_{n\in I} \Pi_H^{(n)}$ and $\Pi_H^{(a-)} \oplus (\bigoplus_{n\in I}^{\infty} \Pi_H^{(n)}) \oplus \Pi_H^{(b+)}$, respectively. Here $\Pi_H^{(a-)}$ and $\Pi_H^{(b+)}$ are the boundary triplets defined in Lemmas 4.2 and 4.3. Using these boundary triplets one parameterizes the set of proper extensions of the operators H_X and (4.25) in just the same way as in Corollary 4.6.

4.3. The non-relativistic limit for general realizations

In this section we investigate the non-relativistic limit of any m-accumulative, m-dissipative (in particular self-adjoint) extension $\widetilde{D}_X$ of D_X. We confine ourself to the case of the half-line $\mathbb{R}_a = (a, +\infty)$ only, i.e. assume that $b = +\infty$. The cases of Dirac operators either in $L^2(-\infty, b) \otimes \mathbb{C}^2$ or in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, with $b < +\infty$, can be treated similarly by using the boundary triplets discussed in Remarks 3.13 and 4.7.

Denote by $X = \{x_n\}_{n=1}^{\infty}$ the discrete subset of $\mathbb{R}_a$, $x_0 = a$ (cf. (3.27)). We will use the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} = \bigoplus_{n=1}^{\infty} \Pi^{(n)}$ defined in Theorem 3.10 where $\Pi^{(n)}$ is given by (3.54) and (3.55). In what follows we equip all objects related to the Dirac operators by index c to exhibit dependence on the velocity of light. For instance, we write $D_X^c, D_n^c, M_{n,c}(\cdot)$ in place of $D_X, D_n, M_n(\cdot)$.

Theorem 4.8. *Assume that $d^*(X) < +\infty$, $\Pi_D = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\Pi_H = \{\mathcal{H}, \Gamma_{0,H}, \Gamma_{1,H}\}$ are the boundary triplets for $(D_X^c)^*$ and H_X^* defined in Corollaries 3.12 and 4.6, respectively. Let also $\widetilde{D}_X^c$ and $\widetilde{H}_X$ be m-accumulative (m-dissipative), in particular, self-adjoint, extensions of D_X^c and H_X, respectively, and let Θ_c and Θ be the corresponding boundary relations in the boundary triplets Π_D and Π_H, i.e. $\widetilde{D}_X^c = D_{X,\Theta_c}^c$ and*

$\widetilde{H}_X = H_{X,\Theta}$ according to formulae (3.62) and (4.23), respectively. If $l_0^2(\mathbb{N},\mathbb{C}^2)$ is a core for Θ, $l_0^2(\mathbb{N},\mathbb{C}^2) \subset \bigcap_{c>1} \operatorname{dom}(\Theta_c) \cap \operatorname{dom}(\Theta)$ and

$$\lim_{c\to+\infty} \Theta_c h = \Theta h, \quad h \in l_0^2\big(\mathbb{N},\mathbb{C}^2\big), \tag{4.26}$$

then

$$s-\lim_{c\to+\infty}\big(D^c_{X,\Theta_c} - \big(z + c^2/2\big)\big)^{-1} = (H_{X,\Theta} - z)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{C}_+ \ (z \in \mathbb{C}_-). \tag{4.27}$$

Proof. (i) First we investigate the limit of the Weyl functions $M_{n,c}(\cdot)$ given by (3.59). It follows from (3.3) and (3.5) that

$$\lim_{c\to+\infty} k\big(z + c^2/2\big) = \lim_{c\to+\infty} ck_1\big(z + c^2/2\big) = \sqrt{z}, \quad z \in \mathbb{C}_\pm. \tag{4.28}$$

Next we find the limits as $c \to \infty$ of the basis defect vectors defined by (3.7), (3.15) and (3.22), respectively. Taking into account (4.28) and relation (4.2) we obtain

$$\lim_{c\to+\infty} f_n^\pm\big(\cdot, z + c^2/2\big) = \begin{pmatrix} f^\pm_{n,H}(\cdot, z) \\ 0 \end{pmatrix},$$

where the convergence is in the Hilbert spaces $L^2(x_{n-1}, x_n) \otimes \mathbb{C}^2$.

According to (3.59) and [44, Eq. (83)]

$$\begin{aligned} M_{n,c}(z) &= R_n(c)^{-1}\big(\widetilde{M}_{n,c}(z) - Q_n\big)R_n(c)^{-1}, \\ M_{n,H}(z) &= R_{n,H}^{-1}\big(\widetilde{M}_{n,H}(z) - Q_n\big)R_{n,H}^{-1}, \end{aligned} \tag{4.29}$$

where $\widetilde{M}_{n,c}(\cdot)$ and $\widetilde{M}_{n,H}(\cdot)$ are defined by (3.12) and (4.6), respectively. Besides, $R_n(c) := R_n$ and $R_{n,H}$ are defined by (3.57) and [44, formula (94)], respectively, i.e.

$$R_n(c) := R_n = \begin{pmatrix} d_n^{1/2} & 0 \\ 0 & d_n^{3/2}\sqrt{1 + \frac{1}{c^2 d_n^2}} \end{pmatrix} \quad \text{and} \quad R_{n,H} := \begin{pmatrix} d_n^{1/2} & 0 \\ 0 & d_n^{3/2} \end{pmatrix}, \quad n \in \mathbb{N}. \tag{4.30}$$

Clearly,

$$\big(1 + c^{-2}d_n^{-2}\big)^{-1} \to 1 \quad \text{as } c \to \infty. \tag{4.31}$$

It follows from (4.30) and (4.31) that $\lim_{c\to\infty} R_n(c) = R_{n,H}$, $n \in \mathbb{N}$, and

$$R_n(c)^{-1} := \begin{pmatrix} d_n^{-1/2} & 0 \\ 0 & d_n^{-3/2}(1 + c^{-2}d_n^{-2})^{-1/2} \end{pmatrix} \to R_{n,H}^{-1} \quad \text{as } c \to \infty,\ n \in \mathbb{N}. \tag{4.32}$$

Next, combining (3.12) and (4.6) with (4.29) and taking into account relations (4.28) and (4.32) we arrive at

$$\lim_{c\to+\infty} M_{n,c}\big(z + c^2/2\big) = M_{n,H}(z), \quad n \in \mathbb{N}. \tag{4.33}$$

We emphasize however that the convergence in (4.31), hence the convergence in (4.32) and (4.33), is uniform in $n \in \mathbb{N}$ if and only if $d_*(X) > 0$.

(ii) In this step we show that

$$s - \lim_{c\to+\infty}\big(\Theta_c - M_c\big(z + c^2/2\big)\big)^{-1} = \big(\Theta - M_H(z)\big)^{-1}, \quad z \in \mathbb{C}_+. \tag{4.34}$$

We consider the case of m-accumulative Θ, the case of m-dissipative Θ is treateded similarly.

Straightforward calculations show that the matrices $M'_{n,c}(c^2/2)$ are uniformly positive in $n \in \mathbb{N}$ and $c \in (0,\infty)$. Indeed, it easily follows from (3.60) that the following inequalities hold

$$M'_{n.c}\big(c^2/2\big) = \begin{pmatrix} 1 & \frac{1}{2}\big(1+\frac{1}{c^2 d_n^2}\big)^{-1/2} \\ \frac{1}{2}\big(1+\frac{1}{c^2 d_n^2}\big)^{-1/2} & \frac{1}{3}\frac{3+c^2d_n^2}{1+c^2d_n^2} \end{pmatrix} > \frac{1}{16} I_2, \quad c \in (0,\infty),\ n \in \mathbb{N}. \tag{4.35}$$

Note that the Weyl function $M_c(\cdot)$ corresponding to the triplet $\Pi(c) = \bigoplus_1^\infty \Pi_n(c)$ is $M_c(\cdot) = \bigoplus_1^\infty M_{n,c}(\cdot) \in [\mathcal{H}]$. Combining this fact with inequalities (4.35) and the integral representation of the $R[\mathcal{H}]$-function $M_c(\cdot)$ (see (2.8)) we obtain

$$\begin{aligned} M'_c\big(c^2/2\big) = \bigoplus_{n=1}^{\infty} M'_{n,c}\big(c^2/2\big) &= \int\limits_{\mathbb{R}\setminus(-\alpha_c,\alpha_c)} \frac{1}{(t-c^2/2)^2}\, d\Sigma_c(t) \\ &> \frac{1}{16} I_{\mathcal{H}}, \quad c \in (0,\infty), \end{aligned} \tag{4.36}$$

where $(-\alpha_c, \alpha_c) := \bigcap_{n=1}^\infty (-\alpha_{n,c}, \alpha_{n,c})$ and $\alpha_c > c^2/2$. Further, it follows from (3.10) that with some $\varepsilon_0 > 0$

$$\begin{aligned} \alpha_c - \frac{c^2}{2} &= \sqrt{\frac{c^2\pi^2}{4d^*(X)^2} + \left(\frac{c^2}{2}\right)^2} - \frac{c^2}{2} = \frac{c^2}{2}\left[\sqrt{\frac{\pi^2}{d^*(X)^2c^2} + 1} - 1\right] \\ &= \frac{2^{-1}\pi^2 d^*(X)^{-2}}{\sqrt{\pi^2 d^*(X)^{-2}c^{-2} + 1} + 1} \geqslant \varepsilon_0 \quad \text{for } c \geqslant 1. \end{aligned} \tag{4.37}$$

Hence $|t - c^2/2| > \alpha_c - c^2/2 \geqslant \varepsilon_0$, $t \in \mathbb{R} \setminus (-\alpha_c, \alpha_c)$ and $c > 1$. In turn, this inequality yields

$$\frac{(t-c^2/2)^2}{(t-c^2/2)^2 + 1} \geqslant \varepsilon_1 := \frac{\varepsilon_0^2}{1+\varepsilon_0^2}, \quad t \in \mathbb{R} \setminus (-\alpha_c, \alpha_c),\ c > 1. \tag{4.38}$$

Combining this inequality with (4.36) and using the integral representation (2.8) of the Weyl function $M_c(\cdot)$ we arrive at the following uniform estimate

$$\operatorname{Im} M_c\big(i + c^2/2\big) = \int\limits_{\mathbb{R}\setminus(-\alpha_c,\alpha_c)} \frac{1}{(t-c^2/2)^2+1}\, d\Sigma_c(t) \geqslant \frac{\varepsilon_1}{16} I_{\mathcal{H}}, \quad c \geqslant 1. \tag{4.39}$$

Since Θ_c is m-accumulative, we easily get from (4.39) that

$$\begin{aligned} &\big\|\big(\Theta_c - M_c\big(i + c^2/2\big)\big)h\big\| \cdot \|h\| \\ &\quad \geqslant \big|\big(\big(\Theta_c - M_c\big(i+c^2/2\big)\big)h, h\big)\big| \geqslant \big|\operatorname{Im}\big(\big(\Theta_c - M_c\big(i+c^2/2\big)\big)h, h\big)\big| \\ &\quad = -\operatorname{Im}(\Theta_c h, h) + \operatorname{Im}\big(M_c\big(i+c^2/2\big)h, h\big) \geqslant \frac{1}{16}\varepsilon_1 \|h\|^2, \quad c \geqslant 1,\ h \in l_0^2\big(\mathbb{N}, \mathbb{C}^2\big). \end{aligned}$$

Since $l_0^2(\mathbb{N},\mathbb{C}^2)$ is dense in $\mathcal{H}=l^2(\mathbb{N},\mathbb{C}^2)$, this inequality yields

$$\left\|\left(\Theta_c - M_c\left(i+c^2/2\right)\right)^{-1}\right\| \leqslant 16\varepsilon_1^{-1}, \quad c \geqslant 1. \tag{4.40}$$

Further, relations (4.33) immediately imply $\lim_{c\to+\infty} M_c(z+c^2/2)h = M_H(z)h$, $h\in l_0^2(\mathbb{N},\mathbb{C}^2)$. Combining this relation with (4.26), yields

$$\lim_{c\to+\infty}\left(\Theta_c - M_c\left(z+c^2/2\right)\right)h = \left(\Theta - M_H(z)\right)h, \quad h\in l_0^2\left(\mathbb{N},\mathbb{C}^2\right),\ z\in\mathbb{C}_+. \tag{4.41}$$

In turn, combining (4.41) with the uniform estimate (4.40) and applying [41, Theorem 8.1.5] we arrive at (4.34).

(iii) In this step we prove the following limit relation

$$\begin{aligned} s-\lim_{c\to+\infty}&\gamma_c\left(z+c^2/2\right)\left(\Theta_c - M_c\left(z+c^2/2\right)\right)^{-1}\gamma_c^*\left(\bar z+c^2/2\right)\\ &= \left(\gamma_H(z)\left(\Theta - M_H(z)\right)^{-1}\gamma_H^*(\bar z)\right)\otimes\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad z\in\mathbb{C}_+. \end{aligned} \tag{4.42}$$

It follows from the first identity in (3.58) and Definition 2.4 (see formula (2.4)) that

$$\gamma_{n,c}(z) = \widetilde{\gamma}_{n,c}(z)R_n^{-1} \quad\text{and}\quad \gamma_{n,H}(z) = \widetilde{\gamma}_{n,H}(z)R_{n,H}^{-1}, \quad n\in\mathbb{N}.$$

Combining these identities with (3.11) and (4.5) and taking (4.28) and (4.32) into account we obtain

$$\begin{aligned} \lim_{c\to+\infty}\gamma_{n,c}\left(z+c^2/2\right) &= \begin{pmatrix}(\gamma_{n,H}(z))_1 & (\gamma_{n,H}(z))_2\\ 0 & 0\end{pmatrix},\\ \lim_{c\to+\infty}\gamma_{n,c}^*\left(z+c^2/2\right) &= \begin{pmatrix}(\gamma_{n,H}^*(z))_1 & 0\\ (\gamma_{n,H}^*(z))_2 & 0\end{pmatrix}. \end{aligned} \tag{4.43}$$

Here $(\gamma_{n,H}(z))_j$ denotes the jth component of the vector function $\gamma_{n,H}(z)$ and the convergence is understood in $L^2(x_{n-1},x_n)\otimes\mathbb{C}^4$ and $\mathcal{H}_n$, respectively.

Next we prove that the family $\gamma_c(z+c^2/2) = \bigoplus_{n=1}^\infty \gamma_{n,c}(i+c^2/2)$ is uniformly bounded in $c>1$. More precisely, assuming for simplicity that $z=i$ we show that

$$\sup_{c>1}\left\|\gamma_c\left(c^2/2\pm i\right)\right\| = \sup_{c>1}\left\|\gamma_c^*\left(c^2/2\pm i\right)\right\| \leqslant 8\sqrt{3}. \tag{4.44}$$

It follows from (2.6) that

$$\begin{aligned} \operatorname{Im} M_c\left(c^2/2\pm i\right) &= \operatorname{Im}\left(c^2/2\pm i\right)\gamma_c^*\left(c^2/2\pm i\right)\gamma_c\left(c^2/2\pm i\right)\\ &= \pm\gamma_c^*\left(c^2/2\pm i\right)\gamma_c\left(c^2/2\pm i\right). \end{aligned} \tag{4.45}$$

So, it suffices to estimate $\operatorname{Im} M_c(c^2/2\pm i)$ from above.

Taking into account formula (3.61) where $\Delta(c^2/2) := 12(1+c^2d_n^2)(12+c^2d_n^2)^{-1} < 12$, we obtain from (3.61) and (4.35) that

$$\begin{aligned}\big(M'_{n,c}\big(c^2/2\big)\big)^{-1} &= \frac{1}{\Delta(c^2/2)}\begin{pmatrix} \frac{1}{3}\frac{3+c^2d_n^2}{1+c^2d_n^2} & -\frac{1}{2}(1+\frac{1}{c^2d_n^2})^{-1/2} \\ -\frac{1}{2}(1+\frac{1}{c^2d_n^2})^{-1/2} & 1\end{pmatrix} \\ &> \frac{1}{12}\cdot\frac{1}{16}, \quad n\in\mathbb{N},\ c>1.\end{aligned} \tag{4.46}$$

Hence

$$16^{-1} < M'_{n,c}\big(c^2/2\big) < 192 \quad \text{and} \quad M'_c\big(c^2/2\big) = \bigoplus_{n=1}^{\infty} M'_{n,c}\big(c^2/2\big) < 192\cdot I_{\mathcal{H}}. \tag{4.47}$$

On the other hand, it follows from (4.39) and (4.36) with account of (4.47) that

$$\begin{aligned}\pm\operatorname{Im} M_c\big(c^2/2\pm i\big) &= \int\limits_{\mathbb{R}\setminus(-\alpha_c,\alpha_c)} \frac{1}{(t-c^2/2)^2+1}\,d\Sigma_c(t) \leqslant \int\limits_{\mathbb{R}\setminus(-\alpha_c,\alpha_c)} \frac{1}{(t-c^2/2)^2}\,d\Sigma_c(t) \\ &= M'_c\big(c^2/2\big) < 192\cdot I_{\mathcal{H}}, \quad c\geqslant 1.\end{aligned} \tag{4.48}$$

Combining this estimate with (4.45) we arrive at (4.44).

Further, note that the convergence in (4.43) implies the convergence of finite direct sums. Finally, combining this fact with the uniform estimate (4.44) we obtain

$$\begin{aligned} s-\lim_{c\to+\infty}\gamma_c\big(c^2/2\pm i\big) &= \begin{pmatrix} (\gamma_H(\pm i))_1 & (\gamma_H(\pm i))_2 \\ 0 & 0\end{pmatrix}, \\ s-\lim_{c\to+\infty}\gamma_c^*\big(c^2/2\pm i\big) &= \begin{pmatrix} (\gamma_H^*(\pm i))_1 & 0 \\ (\gamma_H^*(\pm i))_2 & 0\end{pmatrix},\end{aligned} \tag{4.49}$$

where $(\gamma_H(\cdot))_j$, $j\in\{1,2\}$, denotes the jth component of the vector function $\gamma_H(\cdot)$. The convergence in (4.49) holds in the spaces $L^2(a,b)\otimes\mathbb{C}^4$ and $\mathcal{H}$, respectively. Combining relations (4.49) with (4.34) we arrive at (4.42).

(iv) In this step we prove formula (4.27) for the operators $D_X^c = \bigoplus_{n=1}^{\infty} D_n^c$ and $H_X = \bigoplus_{n=1}^{\infty} H_n$ assuming for the moment that the following limit formula holds

$$u-\lim_{c\to+\infty}\big(D_{n,0}^c - \big(z+c^2/2\big)\big)^{-1} = (H_{n,0}-z)^{-1}\otimes\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad n\in\mathbb{N}. \tag{4.50}$$

Here $D_{n,0}^c := D_n^*\restriction\ker\varGamma_0^{(n)}$, $H_{n,0} := H_n^*\restriction\ker\varGamma_{0,H}^{(n)}$ and

$$\operatorname{dom}\big(D_{n,0}^c\big) = \ker\varGamma_0^{(n)} = \big\{\{f_1,f_2\}^\tau\in W^{1,2}[x_{n-1},x_n]\otimes\mathbb{C}^2\colon\ f_1(x_{n-1}+) = f_2(x_n-) = 0\big\} \quad\text{and}$$
$$\operatorname{dom}(H_{n,0}) = W_0^{2,2}[x_{n-1},x_n].$$

The proof of (4.50) is postponed to the next step. Note that convergence in (4.50) is uniform in $L^2[x_{n-1},x_n]\otimes\mathbb{C}^2$.

According to the Krein type formula for resolvents (see (2.9))

$$\big(D_{X,\Theta_c}^c - z\big)^{-1} = \big(D_{X,0}^c - z\big)^{-1} + \gamma_c(z)\big(\Theta_c - M_c(z)\big)^{-1}\gamma_c^*(\bar z) \tag{4.51}$$

and

$$(H_{X,\Theta}-z)^{-1}=(H_{X,0}-z)^{-1}+\gamma_H(z)\big(\Theta-M_H(z)\big)^{-1}\gamma_H^*(\bar z). \tag{4.52}$$

Here the realizations $D^c_{X,0}$ and $H_{X,0}$ are given by

$$D^c_{X,0}=\bigoplus_{n=1}^{\infty}D^c_{n,0}=\bigoplus_{n=1}^{\infty}\big(D^c_{n,0}\big)^*=\big(D^c_{X,0}\big)^* \quad \text{and}$$
$$H_{X,0}=\bigoplus_{n=1}^{\infty}H_{n,0}=\bigoplus_{n=1}^{\infty}(H_{n,0})^*=H^*_{X,0}. \tag{4.53}$$

Combining relations (4.50) with (4.53) and noting that $\|(D^c_{n,0}-(z+c^2/2))^{-1}\|\leqslant|\mathrm{Im}\,z|^{-1}$ for any n and $c>0$, we obtain

$$s-\lim_{c\to+\infty}\big(D^c_{X,0}-\big(z+c^2/2\big)\big)^{-1}=(H_{X,0}-z)^{-1}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad z\in\mathbb{C}_+. \tag{4.54}$$

Finally, combining this relation with (4.42) and applying the Krein type formulae (4.51) and (4.52) we arrive at (4.27).

(v) In this step we prove formula (4.50) as well as the following formulas

$$u-\lim_{c\to+\infty}\big(D^c_\tau-\big(z+c^2/2\big)\big)^{-1}=(H_\tau-z)^{-1}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \tau=a-,b+,\ z\in\mathbb{C}_+. \tag{4.55}$$

All formulae can be obtained by direct calculations but we prefer to extract them from the classical result for the "free" Dirac operator considered on the whole line. To be precise denote by D^c_{free} and H_{free} the "free" Dirac and Schrödinger operators generated by deferential expressions (3.1) and $-\frac{d^2}{dx^2}$ on $L^2(\mathbb{R})\otimes\mathbb{C}^2$ and $L^2(\mathbb{R})$, respectively. By definition, $\mathrm{dom}(D^c_{\mathrm{free}})=W^{1,2}(\mathbb{R})\otimes\mathbb{C}^2$ and $\mathrm{dom}(H_{\mathrm{free}})=W^{2,2}(\mathbb{R})$. Then according to the classical result (see e.g. [72, Chapter 6])

$$u-\lim_{c\to+\infty}\big(D^c_{\mathrm{free}}-\big(z+c^2/2\big)\big)^{-1}=(H_{\mathrm{free}}-z)^{-1}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix}. \tag{4.56}$$

To this end we introduce a two points set $Y:=\{x_{n-1},x_n\}=:\{a,b\}$ and consider the boundary triplet $\Pi^c_Y=\Pi^{(a-)}\oplus\widetilde{\Pi}^{(n)}\oplus\Pi^{(b+)}$ constructed in Corollary 3.12 for the operator $(D^c_Y)^*=(D^c_{a^-}\oplus D^c_n\oplus D^c_{b^+})^*$. In other words, $\Pi^c_Y=\{\mathbb{C}^4,\Gamma^c_0,\Gamma^c_1\}$ where $\Gamma^c_j:=\Gamma^{(a-)}_j\oplus\widetilde{\Gamma}^{(n)}_j\oplus\Gamma^{(b+)}_j$, $j\in\{0,1\}$, and $\Gamma^{(a-)}_j$, $\widetilde{\Gamma}^{(n)}_j$, and $\Gamma^{(b+)}_j$ are given by (3.16), (3.8) and (3.23), respectively.

It is easily seen that in the triplet Π^c_Y the operator D^c_{free} is given by

$$\Gamma^c_1 f=\begin{pmatrix}f_1(a-)\\ icf_2(a+)\\ f_1(b-)\\ icf_2(b+)\end{pmatrix}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}icf_2(a-)\\ f_1(a+)\\ icf_2(b-)\\ f_1(b+)\end{pmatrix}$$
$$=:\Theta_{\mathrm{free}}\Gamma^c_0 f,\quad f\in\mathrm{dom}\big(\big(D^c_Y\big)^*\big), \tag{4.57}$$

i.e. $D^c_{\mathrm{free}}=(D^c_Y)^*\upharpoonright\ker(\Gamma^c_1-\Theta_{\mathrm{free}}\Gamma^c_0)$. We emphasize that despite of the dependence of the triplets Π^c_Y on c, the boundary operators $\Theta_{\mathrm{free}}=\sigma_1\oplus\sigma_1$ do not depend on c. Here $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ (see definition (3.2)).

Alongside the triplet Π_Y^c we consider the boundary triplet $\Pi_{Y,H}$ for the maximal Schrödinger operator

$$H_Y^* = H_{a_-}^* \oplus H_n^* \oplus H_{b+}^*,$$

given in Remark 4.7 (see also Theorem 4.5). Clearly,

$$\Pi_{Y,H} = \{\mathbb{C}^n, \Gamma_{0,H}, \Gamma_{1,H}\} := \Pi_H^{(a-)} \oplus \Pi_H^{(n)} \oplus \Pi_H^{(b+)} \quad \text{with } \Gamma_{j,H} := \Gamma_{j,H}^{(a-)} \oplus \widetilde{\Gamma}_{j,H}^{(n)} \oplus \Gamma_{j,H}^{(b+)}.$$

Here $\Gamma_{j,H}^{(a-)}$, $\widetilde{\Gamma}_{j,H}^{(n)}$ and $\Gamma_{j,H}^{(b+)}$, $j \in \{0,1\}$, are given by (4.10), (4.3) and (4.15), respectively. It is easily seen that in the boundary triplet $\Pi_{Y,H}$ the free Schrödinger operator H_{free} is given by $H_{\mathrm{free}} = H_Y^* \upharpoonright \ker(\Gamma_{1,H} - \Theta_{\mathrm{free}}\Gamma_{0,H})$ with the same boundary operator Θ_{free} as in (4.57).

Consider formulae (4.51), (4.52) and the limit relation (4.42) with the set $Y = \{a,b\}$ in place of X and $\Theta_c = \Theta = \Theta_{\mathrm{free}}$. In this case (4.42) holds in the uniform sense since Y is finite. Taking this relation into account and passing to the limit as $c \to \infty$ in the Krein type formulae (4.51), (4.52), with account of (4.56) we arrive at the identity

$$u - \lim_{c\to+\infty} \left(D_{Y,0}^c - \left(z + c^2/2\right)\right)^{-1} = (H_{Y,0} - z)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{C}_+. \tag{4.58}$$

In turn, it implies (4.50) as well as relations (4.55). □

Corollary 4.9. *Assume the conditions of Theorem 4.8. Assume, in addition, that $d_*(X) > 0$ (in particular, X is finite) and that in place of (4.26) the uniform resolvent convergence holds, i.e.*

$$\lim_{c\to+\infty} \left\| (\Theta_c - z)^{-1} - (\Theta - z)^{-1} \right\| = 0, \quad z \in \mathbb{C}_+ \ (z \in \mathbb{C}_-). \tag{4.59}$$

Then in place of (4.27) the uniform resolvent convergence holds, i.e.

$$u - \lim_{c\to+\infty} \left(D_{X,\Theta_c}^c - \left(z + c^2/2\right)\right)^{-1} = (H_{X,\Theta} - z)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{C}_+ \ (z \in \mathbb{C}_-). \tag{4.60}$$

Proof. It can be proved that in the case $d_*(X) > 0$ condition (4.59) implies uniform convergence in (4.34). It can be done using uniform counterpart of [22, Lemma 3.1]. Moreover, in this case the convergence in (4.49), hence the convergence in (4.42), is uniform too. Besides, in the case $d_*(X) > 0$ the convergence in (4.54) is also uniform. Finally, combining these relations and applying the Krein type formulae (4.51) and (4.52) we arrive at (4.60). □

Remark 4.10. Theorem 4.8 with its proof remains valid in the case of Dirac operators D_X on the line with interaction set $X = \{x_k\}_{k\in\mathbb{Z}}$, $x_{k-1} < x_k$. Indeed, it can be adapted to this case by using the boundary triplets defined in Remarks 3.13 and 4.7.

In conclusion, note that Theorem 4.8 comprises (see also Theorem 5.43 below) and extends known results on the non-relativistic limits of Dirac operators with point interactions (see [9,30], [3, Appendix J] and references therein).

5. Gesztesy–Šeba realizations

Following [30] (see also [3]) we define two families of symmetric extensions, which turn out to be closely related to their non-relativistic counterparts δ- and δ'-interactions. First we consider the case of a finite or infinite interval $\mathcal{I} = (a, b) \subseteq \mathbb{R}_a$, $-\infty < a$. Let, as in the previous sections,

$$X = \{x_n\}_{n\in\mathbb{N}}, \quad -\infty < a =: x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots, \quad \lim_{n\to+\infty} x_n = b \leqslant \infty,$$

and let

$$\alpha := \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R} \cup \{+\infty\}, \qquad \beta := \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R} \cup \{+\infty\}.$$

Then the two families of Gesztesy–Šeba operators (in short, GS-operators or GS-realizations) on the interval (a, b) are defined to be the closures of the operators

$$\begin{aligned} D^0_{X,\alpha} &= D \restriction \operatorname{dom}\big(D^0_{X,\alpha}\big), \\ \operatorname{dom}\big(D^0_{X,\alpha}\big) &= \Big\{ f \in W^{1,2}_{\mathrm{comp}}(\mathcal{I}\backslash X) \otimes \mathbb{C}^2\colon\ f_1 \in AC_{\mathrm{loc}}(\mathcal{I}),\ f_2 \in AC_{\mathrm{loc}}(\mathcal{I}\backslash X); \\ &\qquad f_2(a+) = 0,\ f_2(x_n+) - f_2(x_n-) = -\frac{i\alpha_n}{c} f_1(x_n),\ n \in \mathbb{N} \Big\}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} D^0_{X,\beta} &= D \restriction \operatorname{dom}\big(D^0_{X,\beta}\big), \\ \operatorname{dom}\big(D^0_{X,\beta}\big) &= \big\{ f \in W^{1,2}_{\mathrm{comp}}(\mathcal{I}\backslash X) \otimes \mathbb{C}^2\colon\ f_1 \in AC_{\mathrm{loc}}(\mathcal{I}\backslash X),\ f_2 \in AC_{\mathrm{loc}}(\mathcal{I}); \\ &\qquad f_2(a+) = 0,\ f_1(x_n+) - f_1(x_n-) = i\beta_n c f_2(x_n),\ n \in \mathbb{N} \big\}, \end{aligned} \tag{5.2}$$

respectively, i.e. $D_{X,\alpha} = \overline{D^0_{X,\alpha}}$ and $D_{X,\beta} = \overline{D^0_{X,\beta}}$.

It is easily seen that both operators $D_{X,\alpha}$ and $D_{X,\beta}$ are symmetric, but not necessarily self-adjoint, in general. However, both $D_{X,\alpha}$ and $D_{X,\beta}$ are either symmetric or self-adjoint only simultaneously. If $D_{X,\alpha}$ and $D_{X,\beta}$ are self-adjoint, then their domains are described explicitly (see Theorem 5.9(i)). Moreover, the character feature of GS-realizations $D_{X,\alpha}$ and $D_{X,\beta}$ is that they are always self-adjoint provided that $\mathcal{I} = \mathbb{R}_\pm, \mathbb{R}$ (see Proposition 5.5 and Theorem 5.9(ii)).

Remark 5.1.

(i) Originally the GS-realizations $D_{X,\alpha}$ and $D_{X,\beta}$ have been introduced (cf. [30]) in the case of point interactions distributed on the line $\mathbb{R}$. In this case $X = \{x_k\}_{k\in\mathbb{Z}}$, $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$, $\beta = \{\beta_k\}_{k\in\mathbb{Z}}$, and $\lim_{n\to-\infty} x_n = -\infty$ and $\lim_{n\to+\infty} x_n = +\infty$. Moreover, in this case boundary conditions in (5.1) and (5.2) are labelled by $n \in \mathbb{Z}$ and the condition $f_2(a+) = 0$ is dropped.

(ii) Note also that if $\alpha_n = \infty$ ($\beta_n = \infty$) for some $n \in \mathbb{N}$, then the nth boundary condition (5.1) (resp. (5.2)) takes the form

$$f_1(x_n) = 0 \quad \big(\text{resp. } f_2(x_n) = 0\big). \tag{5.3}$$

In what follows we call conditions (5.1), (5.2) by Gesztesy–Šeba boundary conditions (in short GS-conditions).

To investigate Gesztesy–Šeba realizations $D_{X,\alpha}$ in the framework of boundary triplets approach we first find boundary relations (operators) Θ that parameterize operators $D_{X,\alpha}$ according to Corollary 3.12. It turns out that, as in the Schrodinger case (cf. [44]), the boundary operator corresponding to $D_{X,\alpha}$ in the boundary triplet constructed in Theorem 3.10, is Jacobi matrix.

5.1. GS-realizations $D_{X,\alpha}$: parametrization by Jacobi matrices

Consider the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for D_X^* constructed in Theorem 3.10, (cf. formulae (3.54), (3.55)). By Corollary 3.12(i), the realization $D_{X,\alpha}$ admits the representation (cf. (3.62))

$$\begin{aligned} D_{X,\alpha} &= D_{\Theta(\alpha)} := D_X^* \lceil \operatorname{dom}(D_{\Theta(\alpha)}), \\ \operatorname{dom}(D_{\Theta(\alpha)}) &= \big\{ f \in W^{1,2}(\mathcal{I}\backslash X) \otimes \mathbb{C}^2 \colon \{\Gamma_0 f, \Gamma_1 f\} \in \Theta(\alpha) \big\}. \end{aligned} \tag{5.4}$$

Since for any α the realizations $D_{X,\alpha}$ and $D_{X,0} := D_X^* \upharpoonright \ker(\Gamma_0)$ are disjoint, $\Theta(\alpha)$ is a (closed) operator in $\mathcal{H} = l^2(\mathbb{N}) \otimes \mathbb{C}^2$, $\Theta(\alpha) \in \mathcal{C}(\mathcal{H})$. We show that $\Theta(\alpha)$ is a Jacobi matrix. More precisely, consider the Jacobi matrix

$$B_{X,\alpha} = \begin{pmatrix} 0 & -\frac{\nu(d_1)}{d_1^2} & 0 & 0 & 0 & \dots \\ -\frac{\nu(d_1)}{d_1^2} & -\frac{\nu(d_1)}{d_1^2} & \frac{\nu(d_1)}{d_1^{3/2} d_2^{1/2}} & 0 & 0 & \dots \\ 0 & \frac{\nu(d_1)}{d_1^{3/2} d_2^{1/2}} & \frac{\alpha_1}{d_2} & -\frac{\nu(d_2)}{d_2^2} & 0 & \dots \\ 0 & 0 & -\frac{\nu(d_2)}{d_2^2} & -\frac{\nu(d_2)}{d_2^2} & \frac{\nu(d_2)}{d_2^{3/2} d_3^{1/2}} & \dots \\ 0 & 0 & 0 & \frac{\nu(d_2)}{d_2^{3/2} d_3^{1/2}} & \frac{\alpha_2}{d_3} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{5.5}$$

where

$$\nu(x) := \frac{1}{\sqrt{1 + \frac{1}{c^2 x^2}}}. \tag{5.6}$$

Let $\tau_{X,\alpha}$ be the second order difference expression associated with (5.5). One defines the corresponding minimal symmetric operator in $l^2(\mathbb{N}) \otimes \mathbb{C}^2$ by (see [1,10])

$$B^0_{X,\alpha} f := \tau_{X,\alpha} f, \quad f \in \operatorname{dom}\big(B^0_{X,\alpha}\big) := l^2_0(\mathbb{N}) \otimes \mathbb{C}^2, \quad \text{and} \quad B_{X,\alpha} = \overline{B^0_{X,\alpha}}. \tag{5.7}$$

Recall that $B_{X,\alpha}$[2] has equal deficiency indices and $\mathrm{n}_+(B_{X,\alpha}) = \mathrm{n}_-(B_{X,\alpha}) \leqslant 1$.

Note that $B_{X,\alpha}$ admits a representation

$$B_{X,\alpha} = R_X^{-1}(\widetilde{B}_\alpha - Q_X)R_X^{-1}, \quad \text{where } \widetilde{B}_\alpha := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & \alpha_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & \alpha_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{5.8}$$

and $R_X = \bigoplus_{n=1}^\infty R_n$, $Q_X = \bigoplus_{n=1}^\infty Q_n$ and R_n, Q_n, are defined by (3.57).

[2] Usually we will identify the Jacobi matrix with (closed) minimal symmetric operator associated with it. Namely, we denote by $B_{X,\alpha}$ the Jacobi matrix (5.5) as well as the minimal closed symmetric operator (5.7).

Proposition 5.2. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for D_X^* constructed in Theorem 3.10 and let $B_{X,\alpha}$ be the minimal Jacobi operator defined by (5.5). Then $\Theta(\alpha) = B_{X,\alpha}$, i.e.,*

$$D_{X,\alpha} = D_{B_{X,\alpha}} = D_X^* \upharpoonright \operatorname{dom}(D_{B_{X,\alpha}}),$$
$$\operatorname{dom}(D_{B_{X,\alpha}}) = \left\{ f \in W^{1,2}(\mathcal{I} \setminus X) \otimes \mathbb{C}^2 \colon \Gamma_1 f = B_{X,\alpha} \Gamma_0 f \right\}.$$

Proof. Let $f \in W^{1,2}_{\mathrm{comp}}(\mathcal{I} \setminus X) \otimes \mathbb{C}^2 = (W^{1,2}(\mathcal{I} \setminus X) \cap L^2_{\mathrm{comp}}(\mathcal{I})) \otimes \mathbb{C}^2$. Then $f \in \operatorname{dom}(D_{X,\alpha})$ if and only if $\widetilde{\Gamma}_1 f = \widetilde{B}_\alpha \widetilde{\Gamma}_0 f$. Here $\widetilde{\Gamma}_j := \bigoplus_{n\in\mathbb{N}} \widetilde{\Gamma}_j^{(n)}$ where $\widetilde{\Gamma}_j^{(n)}$, $j \in \{0,1\}$, are defined by (3.8) and $\widetilde{B}_\alpha$ is given by (5.8). Combining (3.58), (3.57) with (5.8), we rewrite the equality $\widetilde{\Gamma}_1 f = \widetilde{B}_\alpha \widetilde{\Gamma}_0 f$ as $\Gamma_1 f = B_{X,\alpha} \Gamma_0 f$, $f \in W^{1,2}_{\mathrm{comp}}(\mathcal{I} \setminus X) \otimes \mathbb{C}^2$. Taking the closures and applying Corollary 3.12(i) one completes the proof. $\square$

Remark 5.3. Note that the matrix (5.5) has negative off-diagonal entries, although, in the classical theory of Jacobi operators, off-diagonal entries are assumed to be positive. But it is known (see, for instance, [71]) that the (minimal) operator $B_{X,\alpha}$ is unitarily equivalent to the minimal Jacobi operator associated with the matrix

$$B'_{X,\alpha} := \begin{pmatrix} 0 & \frac{\nu(d_1)}{d_1^2} & 0 & 0 & 0 & \dots \\ \frac{\nu(d_1)}{d_1^2} & -\frac{\nu(d_1)}{d_1^2} & \frac{\nu(d_1)}{d_1^{3/2} d_2^{1/2}} & 0 & 0 & \dots \\ 0 & \frac{\nu(d_1)}{d_1^{3/2} d_2^{1/2}} & \frac{\alpha_1}{d_2} & \frac{\nu(d_2)}{d_2^2} & 0 & \dots \\ 0 & 0 & \frac{\nu(d_2)}{d_2^2} & -\frac{\nu(d_2)}{d_2^2} & \frac{\nu(d_2)}{d_2^{3/2} d_3^{1/2}} & \dots \\ 0 & 0 & 0 & \frac{\nu(d_2)}{d_2^{3/2} d_3^{1/2}} & \frac{\alpha_2}{d_3} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{5.9}$$

In the sequel we will identify the operators $B_{X,\alpha}$ and $B'_{X,\alpha}$ when investigating those spectral properties of the operator $H_{X,\alpha}$, which are invariant under unitary transformations.

5.1.1. Self-adjointness

1. Boundary triplets approach. First we study self-adjointness of GS-realizations $D_{X,\alpha}$ using the parametrization by means of the Jacobi matrices $B_{X,\alpha}$.

Combining Corollary 3.12(ii) with Propositions 5.2 we arrive at the following statement.

Proposition 5.4. *The GS-operator $D_{X,\alpha}$ has equal deficiency indices and $\mathrm{n}_+(D_{X,\alpha}) = \mathrm{n}_-(D_{X,\alpha}) \leqslant 1$. Moreover, $\mathrm{n}_\pm(D_{X,\alpha}) = \mathrm{n}_\pm(B_{X,\alpha})$, where $B_{X,\alpha}$ is the minimal Jacobi operator associated with the Jacobi matrix (5.5). In particular, $D_{X,\alpha}$ is self-adjoint if and only if $B_{X,\alpha}$ is.*

Using Carleman's criterium of self-adjointness of Jacobi matrices (see e.g. [1,10,46,47]), we obtain sufficient conditions for the operator $D_{X,\alpha}$ to be self-adjoint in $L^2(\mathcal{I}, \mathbb{C}^2)$.

Proposition 5.5. *Let $\mathcal{I}$ be an infinite interval, i.e. either $\mathcal{I} = \mathbb{R}_\pm$ or $\mathcal{I} = \mathbb{R}$. Then the GS-realization $D_{X,\alpha}$ is self-adjoint for any sequence $\alpha = \{\alpha_n\}_{n=1}^\infty \subset \mathbb{R} \cup \infty$.*

Proof. Let $B'_{X,\alpha}$ be the minimal Jacobi operator of the form (5.9). By Carleman's test (see [1], [10, Chapter VII.1.2]), $B'_{X,\alpha}$ is self-adjoint provided that

$$\sum_{n=1}^\infty \left(d_n^2 + d_n^{3/2} d_{n+1}^{1/2}\right) \sqrt{1 + \frac{1}{c^2 d_n^2}} = \frac{1}{c} \sum_{n=1}^\infty \left(d_n + d_n^{1/2} d_{n+1}^{1/2}\right) \sqrt{1 + c^2 d_n^2} = \infty. \tag{5.10}$$

Since $d_n < d_n + d_n^{1/2} d_{n+1}^{1/2} \leqslant \frac{3}{2} d_n + \frac{1}{2} d_{n+1}$, the series in the left-hand side of (5.10) diverges only simultaneously with the series

$$\sum_{n=1}^{\infty} d_n^2 \sqrt{1 + \frac{1}{c^2 d_n^2}} = \frac{1}{c} \sum_{n=1}^{\infty} d_n \sqrt{1 + c^2 d_n^2}. \tag{5.11}$$

The later series diverges if and only if $\sum_{n=1}^{\infty} d_n = +\infty$, i.e. if and only if the interval $\mathcal{I}$ is infinite. Thus, the minimal Jacobi operator $B_{X,\alpha}$ is self-adjoint whenever the interval $\mathcal{I}$ is infinite. It remains to apply Proposition 5.4. □

Remark 5.6. Note that the condition $\sum_{n=1}^{\infty} d_n = +\infty$ is equivalent to the following one

$$\sum_{n=1}^{\infty} d_n \sqrt{d_n^2 + \frac{1}{c^2}} = +\infty. \tag{5.12}$$

The formal (non-relativistic) limit in (5.12) as $c \to \infty$ leads to the condition $\sum_{n=1}^{\infty} d_n^2 = +\infty$, coinciding with that of [44, Proposition 5.7]. The latter guaranties the self-adjointness of the Schrödinger operator with point interactions.

Next we present sufficient conditions for GS-operators $D_{X,\alpha}$ on a finite interval to be self-adjoint.

Proposition 5.7. *Assume that $|\mathcal{I}| < \infty$. Then the GS-realization $D_{X,\alpha}$ in $L^2(\mathcal{I}, \mathbb{C}^2)$ is self-adjoint provided that*

$$\sum_{n \in \mathbb{N}} \sqrt{d_n d_{n+1}} |\alpha_n| = +\infty. \tag{5.13}$$

Proof. By Proposition 5.4, it suffices to show that the minimal Jacobi operator $B'_{X,\alpha}$ associated with the Jacobi matrix (5.9) is self-adjoint. By the Dennis–Wall test (see [1, Problem 2, p. 25]), $B_{X,\alpha}$ is self-adjoint whenever

$$\sum_{n=1}^{\infty} \frac{d_{n+1}^{3/2}}{\nu(d_{n+1})} \left(\frac{d_n^{3/2} |\alpha_n|}{\nu(d_n)} + d_{n+2}^{1/2} \right) = +\infty. \tag{5.14}$$

The condition $|\mathcal{I}| < \infty$ is equivalent to $\sum_{n=1}^{\infty} d_n < +\infty$. The latter implies $\nu(d_n) \sim c d_n$. Hence $\frac{d_n^{3/2}}{\nu(d_n)} \sim c^{-1} d_n^{1/2}$. Taking these relations into account and noting that

$$2 \sum_{n \in \mathbb{N}} \sqrt{d_{n+1}} \sqrt{d_{n+2}} \leqslant \sum_{n \in \mathbb{N}} (d_{n+1} + d_{n+2}) < +\infty,$$

one concludes that the series (5.13) and (5.14) diverge only simultaneously. □

Example 5.8. Let $\mathcal{I} := (0, 1)$ and let the sequence $X = \{x_n\}_{n=1}^{\infty} \subset (0, 1)$ be given by $x_n = 1 - 1/2^n$, so that $d_n = 1/2^n$. Let also $\alpha = \{\alpha_n\}_1^{\infty}$ be given by $\alpha_n = (-3)2^n + 1$, $n \in \mathbb{N}$. By Proposition 5.7, the GS-operator $D_{X,\alpha}$ on $L^2(0, 1) \otimes \mathbb{C}^2$ is self-adjoint since the series $\sum_{n=1}^{\infty} \alpha_n / 2^n$ diverges.

On the other hand, it is easily seen that

$$\{d_n\}_1^\infty \in l^2(\mathbb{N}), \qquad d_{n-1}d_{n+1} = \frac{1}{2^{2n}} = d_n^2 \quad \text{and} \quad \sum_{n=1}^\infty d_{n+1}\left|\alpha_n + \frac{1}{d_n} + \frac{1}{d_{n+1}}\right| = \sum_{n=1}^\infty \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Therefore, by [44, Proposition 5.9], the corresponding Schrödinger operator $H_{X,\alpha}$ on $L^2(0,1)$ is not self-adjoint: it is symmetric with the deficiency indices $n_\pm(H_{X,\alpha}) = 1$.

2. The classical approach. Now we show, by a direct proof, that in the case $\mathcal{I} = \mathbb{R}$, $X = \{x_k\}_{k\in\mathbb{Z}}$, $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ and $\beta = \{\beta_k\}_{k\in\mathbb{Z}}$ (see Remark 5.1(i)) the Gesztesy–Šeba operators are always self-adjoint. This proof can readily be extended for other realizations as well as for Dirac operators $D_{X,\alpha}(Q)$ with unbounded potential matrix Q.

Theorem 5.9. *Let $D_{X,\alpha}$ and $D_{X,\beta}$ be GS-realizations of the Dirac operator in $L^2(\mathcal{I},\mathbb{C}^2)$. Then*:

(i) *The operator $D^*_{X,\alpha} := (D_{X,\alpha})^*$ adjoint to the symmetric operator $D_{X,\alpha}$ is given by*

$$D^*_{X,\alpha} = D \upharpoonright \operatorname{dom}\bigl(D^*_{X,\alpha}\bigr),$$

$$\operatorname{dom}\bigl(D^*_{X,\alpha}\bigr) = \biggl\{ f \in W^{1,2}(\mathcal{I}\backslash X)\otimes\mathbb{C}^2\colon\ f_1 \in W^{1,2}(\mathcal{I}),\ f_2 \in AC_{\mathrm{loc}}(\mathcal{I}\backslash X);$$
$$f_2(a+) = 0,\ f_2(x_n+) - f_2(x_n-) = -\frac{i\alpha_n}{c} f_1(x_n),\ n \in \mathbb{N}\biggr\}. \tag{5.15}$$

Similarly, the operator $D^*_{X,\beta}$ adjoint to $D_{X,\beta}$ is given by the expression (5.2) with $W^{1,2}_{\mathrm{comp}}(\mathcal{I}\backslash X)$ replaced by $W^{1,2}(\mathcal{I}\backslash X)$.

(ii) *If $|\mathcal{I}| = \infty$ (i.e. either $\mathcal{I} = \mathbb{R}_\pm$ or $\mathcal{I} = \mathbb{R}$) then both $D_{X,\alpha}$ and $D_{X,\beta}$ are self-adjoint, i.e.*

$$D^*_{X,\alpha} = D_{X,\alpha} \quad \text{and} \quad D^*_{X,\beta} = D_{X,\beta}. \tag{5.16}$$

Proof. (i) Denote the right-hand side of (5.15) by $W^{1,2}_\alpha(\mathcal{I}\backslash X)$. Then for any $f \in W^{1,2}_\alpha(\mathcal{I}\backslash X)$ integrating by parts one arrives at the identity

$$\bigl(D^*_{X,\alpha} f, \varphi\bigr) = (f, D_{X,\alpha}\varphi), \quad \varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \in \operatorname{dom}(D_{X,\alpha}), \tag{5.17}$$

proving the inclusion $W^{1,2}_\alpha(\mathcal{I}\backslash X) \subset \operatorname{dom}(D^*_{X,\alpha})$.

Let us prove the converse inclusion. Since $D_X \subset D_{X,\alpha}$ and D_X is symmetric, one has

$$\operatorname{dom}\bigl(D^*_{X,\alpha}\bigr) \subset \operatorname{dom}\bigl(D^*_X\bigr) = W^{1,2}(\mathcal{I}\setminus X)\otimes\mathbb{C}^2 = \bigoplus_{n\in\mathbb{N}} W^{1,2}[x_{n-1},x_n]\otimes\mathbb{C}^2. \tag{5.18}$$

Let $f = \binom{f_1}{f_2} \in \operatorname{dom}(D^*_{X,\alpha})$. Then, by definition, (5.17) holds. According to the "regularity property" (5.18) we can integrate by parts in (5.17) over any interval $[x_{n-1},x_n]$, $n\in\mathbb{N}$. Substitute in (5.17) vector functions φ supported on a small neighborhood $(x_j-\varepsilon, x_j+\varepsilon)$ of x_j and integrating by parts we get

$$\bigl[f_2(x_j+) - f_2(x_j-) + i\alpha_j c^{-1} f_1(x_j+)\bigr]\overline{\varphi_1(x_j)} + \bigl[f_1(x_j+) - f_1(x_j-)\bigr]\overline{\varphi_2(x_j-)} = 0.$$

Since $\varphi_1(x_j)$ and $\varphi_2(x_j-)$ are arbitrary, the latter equality holds if and only if $f_1(x_j+) = f_1(x_j-)$ and $f_2(x_j+) - f_2(x_j-) = -i\alpha_j c^{-1} f_1(x_j)$. Since $j \in \mathbb{N}$ is arbitrary, f satisfies boundary conditions in (5.15) and $\operatorname{dom}(D^*_{X,\alpha}) \subset W^{1,2}_\alpha(\mathcal{I}\setminus X)$. Noting that the opposite inclusion is already proved, we arrive at (5.15).

(ii) For definiteness we assume that $\mathcal{I} = \mathbb{R}$. It suffices to show that $D^*_{X,\alpha}$ is symmetric. Let $f = \binom{f_1}{f_2}$, $g = \binom{g_1}{g_2} \in \operatorname{dom}(D^*_{X,\alpha})$.

Choosing $a, b \in \mathbb{R}\setminus X$, $a < b$, such that $x_{p-1} < a < x_p < x_{p+1} < \cdots < x_q < b < x_{q+1}$ and integrating by parts we get

$$\begin{aligned}
&\int_a^b D^*_{X,\alpha} f(x)\overline{g(x)}\,dx - \int_a^b f(x)\overline{D^*_{X,\alpha} g(x)}\,dx \\
&\quad= -ic\int_a^b \big[f_2'(x)\overline{g_1(x)} + f_1'(x)\overline{g_2(x)}\big]\,dx - ic\int_a^b \big[f_1(x)\overline{g_2'(x)} + f_2(x)\overline{g_2'(x)}\big]\,dx \\
&\quad= -ic\big[f_2(x)\overline{g_1(x)} + f_1(x)\overline{g_2(x)}\big]\big|_a^{x_p-} - ic\big[f_2(x)\overline{g_1(x)} + f_1(x)\overline{g_2(x)}\big]\big|_{x_q+}^b \\
&\qquad - ic\sum_{k=p+1}^{q} \big[f_2(x)\overline{g_1(x)} + f_1(x)\overline{g_2(x)}\big]\big|_{x_{k-1}+}^{x_k-} \\
&\quad= ic\big[f_2(a)\overline{g_1(a)} + f_1(a)\overline{g_2(a)}\big] - ic\big[f_2(b)\overline{g_1(b)} + f_1(b)\overline{g_2(b)}\big] \\
&\qquad + ic\sum_{k=p}^{q}\big[f_2(x_k+) - f_2(x_k-)\big]\overline{g_1(x_k)} + ic\sum_{k=p}^{q} f_1(x_k)\big[\overline{g_2(x_k+)} - \overline{g_2(x_k-)}\big] \\
&\quad= ic\big[f_2(a)\overline{g_1(a)} + f_1(a)\overline{g_2(a)}\big] - ic\big[f_2(b)\overline{g_1(b)} + f_1(b)\overline{g_2(b)}\big].
\end{aligned} \tag{5.19}$$

Since $f_j, g_j \in L^2(\mathbb{R})$, $j \in \{1,2\}$, there exist (non unique) sequences $\{a_n\}, \{b_n\} \subset \mathbb{R}$ such that $a_n \to -\infty$, $b_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty}\big\{\big|f_1(a_n)\big| + \big|f_2(a_n)\big| + \big|g_1(a_n)\big| + \big|g_2(a_n)\big|\big\} = 0 \tag{5.20}$$

and

$$\lim_{n\to\infty}\big\{\big|f_1(b_n)\big| + \big|f_2(b_n)\big| + \big|g_1(b_n)\big| + \big|g_2(b_n)\big|\big\} = 0. \tag{5.21}$$

Without loss of generality, we can assume that $\{a_n\}, \{b_n\} \subset \mathbb{R}\setminus X$ since (5.20) and (5.21) remain valid with $\{a_n \pm \varepsilon_n\}$ and $\{b_n \pm \varepsilon_n\}$ in place of $\{a_n\}$ and $\{b_n\}$, respectively, provided that ε_n, $n \in \mathbb{N}$, are small enough.

According to (5.19)

$$\begin{aligned}
&\int_{a_n}^{b_n} D^*_{X,\alpha} f(x)\overline{g(x)}\,dx - \int_{a_n}^{b_n} f(x)\overline{D^*_{X,\alpha} g(x)}\,dx \\
&\quad= ic\big[f_2(a_n)\overline{g_1(a_n)} + f_1(a_n)\overline{g_2(a_n)}\big] - ic\big[f_2(b_n)\overline{g_1(b_n)} + f_1(b_n)\overline{g_2(b_n)}\big].
\end{aligned} \tag{5.22}$$

Passing here to the limit as $n \to \infty$ with account of relations (5.20), (5.21) we arrive at the identity

$$\big(D^*_{X,\alpha} f, g\big) = \big(f, D^*_{X,\alpha} g\big), \quad f, g \in \operatorname{dom}\big(D^*_{X,\alpha}\big), \tag{5.23}$$

showing that $D^*_{X,\alpha}$ is symmetric, $D^*_{X,\alpha} \subseteq D^{**}_{X,\alpha} = D_{X,\alpha}$. Since $D_{X,\alpha}$ is also symmetric, one has $D_{X,\alpha} = D^*_{X,\alpha}$.

The case of GS-realizations $D_{X,\beta}$ is considered in much the same way. □

Remark 5.10.

(i) In the case $d_*(X) > 0$ this result is stated in [30] (see also [3, Appendix J]).
(ii) The proof of Theorem 5.9 remains valid for general Dirac operators with arbitrary potential matrix $Q \in L^2_{\mathrm{loc}}(\mathbb{R}) \otimes \mathbb{C}^{2\times 2}$,

$$D_{X,\alpha}(Q) := -ic\frac{d}{dx} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3 + Q(x), \quad Q(x) = Q(x)^*, \tag{5.24}$$

subject to GS-boundary conditions (5.1), (5.2).

Moreover, the GS-boundary conditions (5.1) and (5.2) can be replaced by certain other boundary conditions. For instance, Theorem 5.9 as well as its proof remains valid for operators $D_{X,\gamma}(Q)$ generated by differential expression (5.24) subject to the boundary conditions

$$\begin{aligned} f_1(x_j+) &= \cos(\gamma_j) f_1(x_j-) - i\sin(\gamma_j) f_2(x_j-), \\ f_2(x_j+) &= \cos(\gamma_j) f_2(x_j-) - i\sin(\gamma_j) f_1(x_j-), \end{aligned} \tag{5.25}$$

with $\gamma_j \in \mathbb{R}$, $j \in \mathbb{Z}$. Note that realizations $D_{X,\gamma}$ have been studied in numerous papers under the assumption $d_*(X) > 0$ (see for instance [36,51,7,17], as well [3, Appendix J] and the references therein).

Our proof of Theorem 5.9 generalizes the known proof of self-adjointness in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ of the Dirac operator $D(Q)$ with a continuous potential matrix Q (see [52, Chapter 8]).

Next we complete Proposition 5.7 providing sufficient conditions for GS-realization $D_{X,\alpha}$ on a finite interval $\mathcal{I}$ to have non-trivial deficiency indices $n_\pm(D_{X,\alpha}) = 1$.

Theorem 5.11. *Let $|\mathcal{I}| < +\infty$ and let $D_{X,\alpha}$ be the GS-realization of the Dirac expression on $\mathcal{I}$. Then $D_{X,\alpha}$ is symmetric with $n_\pm(D_{X,\alpha}) = 1$ provided that*

$$\sum_{n=2}^{\infty} d_n \prod_{k=1}^{n-1} \left(1 + \frac{1}{c}|\alpha_k|\right)^2 < +\infty. \tag{5.26}$$

Proof. We examine the operator

$$T_{X,\alpha} := D_{X,\alpha} - \frac{c^2}{2} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -ic\frac{d}{dx} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.27}$$

since obviously $n_\pm(D_{X,\alpha}) = n_\pm(T_{X,\alpha})$. It suffices to show that under the assumption (5.26) the equation $(T^*_{X,\alpha} + i)f = 0$ has a non-trivial $L^2(\mathcal{I}, \mathbb{C}^2)$-solution. The equation is equivalent to the system $cf_1' = f_2$ and $cf_2' = f_1$ which has the following piecewise smooth solutions

$$f_1 = \bigoplus_{n=1}^{\infty} f_{1,n}, \qquad f_{1,n}(x) = a_n e^{-(x_n - x)/c} + b_n e^{(x_n - x)/c}, \quad x \in [x_{n-1}, x_n],$$
$$f_2 = \bigoplus_{n=1}^{\infty} f_{2,n}, \qquad f_{2,n}(x) = a_n e^{-(x_n - x)/c} - b_n e^{(x_n - x)/c}, \quad x \in [x_{n-1}, x_n]. \tag{5.28}$$

Let us find sequences $\{a_n\}_1^\infty \subset \mathbb{C}$, $\{b_n\}_1^\infty \subset \mathbb{C}$ such that $f \in \operatorname{dom}(T^*_{X,\alpha}) = \operatorname{dom}(D^*_{X,\alpha})$. According to the description of $\operatorname{dom}(D^*_{X,\alpha})$ (see Theorem 5.9(i)) $\binom{f_1}{f_2}$ should satisfy boundary conditions (5.15). The condition $f_2(x_0+) = 0$ yields $a_1 e^{-d_1/c} - b_1 e^{d_1/c} = 0$. Further, the condition

$$f_{1,n}(x_n+) = f_{1,n}(x_n-), \quad n \in \mathbb{N},$$

is transformed into

$$a_{n+1} e^{-d_{n+1}/c} + b_{n+1} e^{d_{n+1}/c} = a_n + b_n, \quad n \in \mathbb{N}. \tag{5.29}$$

Moreover, the jump condition

$$f_{2,n}(x_n+) - f_{2,n}(x_n-) = -i \frac{\alpha_n}{c} f_{1,n}(x_n), \quad n \in \mathbb{N},$$

is equivalent to

$$a_{n+1} e^{-d_{n+1}/c} - b_{n+1} e^{d_{n+1}/c} - (a_n - b_n) = -i \frac{\alpha_n}{c} (a_n + b_n), \quad n \in \mathbb{N}. \tag{5.30}$$

Clearly, relations (5.29) and (5.30) are equivalent to the following recursive equations

$$a_{n+1} = \left(a_n - i \frac{\alpha_n}{2c}(a_n + b_n)\right) e^{d_{n+1}/c}, \quad n \in \mathbb{N},$$
$$b_{n+1} = \left(b_n + i \frac{\alpha_n}{2c}(a_n + b_n)\right) e^{-d_{n+1}/c}, \quad n \in \mathbb{N}, \tag{5.31}$$

for sequences $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ with the following initial data

$$a_1 = e^{d_1/c} \quad \text{and} \quad b_1 = e^{-d_1/c}.$$

It remains to check that under condition (5.26) the inclusion $f_1, f_2 \in L^2(\mathcal{I})$ holds. It follows from (5.28) that

$$\|f_k\|_2^2 = \sum_{n=1}^{\infty} \|f_{k,n}\|_2^2 \leqslant 2 \sum_{n=1}^{\infty} \int_0^{d_n} \left(|a_n|^2 e^{-2x/c} + |b_n|^2 e^{2x/c}\right) dx$$
$$= c \sum_{n=1}^{\infty} \left(|a_n|^2 \left(1 - e^{-2d_n/c}\right) + |b_n|^2 \left(e^{2d_n/c} - 1\right)\right), \quad k \in \{1, 2\}.$$

Since $\sum_{n=1}^{\infty} d_n = |\mathcal{I}| < +\infty$, $d_n \to 0$ and therefore $(1 - e^{-2d_n/c}) \sim (e^{2d_n/c} - 1) \sim 2d_n/c$ as $n \to \infty$. This implies inequality $\|f_k\|_2 < +\infty$ whenever

$$\sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right) d_n < +\infty. \tag{5.32}$$

Let us prove by induction the following estimates

$$|a_{n+1}|, |b_{n+1}| \leqslant \exp\left(\frac{d_1 + \cdots + d_{n+1}}{c}\right) \cdot \prod_{k=1}^{n}\left(1 + \frac{|\alpha_k|}{c}\right), \quad n \in \mathbb{N}. \tag{5.33}$$

For $n = 1$ these estimates are obvious. Assume that inequalities (5.33) are proved for $n \leqslant m - 1$. Then for $n = m$ we obtain from (5.31) and (5.33) that

$$\begin{aligned} |a_{m+1}| &\leqslant \left(|a_m|\left(1 + \frac{|\alpha_m|}{2c}\right) + \frac{|\alpha_m|}{2c}|b_m|\right) e^{d_{m+1}/c} \\ &\leqslant \prod_{k=1}^{m-1}\left(1 + \frac{|\alpha_k|}{c}\right)\left[\left(1 + \frac{|\alpha_m|}{2c}\right) + \frac{|\alpha_m|}{2c}\right] \exp\left(\frac{d_1 + \cdots + d_m + d_{m+1}}{c}\right) \\ &= \exp\left(\frac{d_1 + \cdots + d_{m+1}}{c}\right) \cdot \prod_{k=1}^{m}\left(1 + \frac{|\alpha_k|}{c}\right). \end{aligned}$$

This inequality proves the inductive hypothesis (5.33) for a_n. The estimate for b_{m+1} is proved similarly. Thus, both inequalities (5.33) are established. Combining (5.32) with (5.33) and the assumption (5.26) we conclude that $f_1, f_2 \in L^2(\mathcal{I})$. This completes the proof. □

Next we extract from Theorem 5.11 certain simple sufficient conditions for the equality $n_{\pm}(D_{X,\alpha}) = 1$ to hold. First we present such conditions involving α and not depending on $X = \{x_n\}_1^{\infty}$.

Corollary 5.12. *The GS-realization $D_{X,\alpha}$ on a finite interval $\mathcal{I}$ is symmetric with $n_{\pm}(D_{X,\alpha}) = 1$ whenever $\alpha = \{\alpha_n\}_1^{\infty} \in l^1(\mathbb{N})$.*

Proof. Clearly, for any positive sequence $\{p_k\}_1^{\infty}$

$$\prod_{k=1}^{\infty}(1 + p_k) \leqslant \exp\left(\sum_{k=1}^{\infty} p_k\right).$$

It follows with account of the inclusion $\alpha \in l^1(\mathbb{N})$ that

$$\sum_{n=2}^{\infty} d_n \prod_{k=1}^{n-1}\left(1 + \frac{1}{c}|\alpha_k|\right)^2 \leqslant \exp\left(\frac{2}{c}\sum_{k=1}^{\infty}|\alpha_k|\right)\sum_{n=2}^{\infty} d_n \leqslant |\mathcal{I}| \exp\left(\frac{2}{c}\sum_{k=1}^{\infty}|\alpha_k|\right).$$

It remains to apply Theorem 5.11. □

Our next test involves both X and α.

Corollary 5.13. *Let $|\mathcal{I}| < +\infty$. Then the GS-realization $D_{X,\alpha}$ is symmetric with $n_\pm(D_{X,\alpha}) = 1$ whenever*

$$\limsup_{n\to\infty} \frac{d_{n+1}}{d_n}\left(1 + \frac{|\alpha_n|}{c}\right)^2 < 1. \tag{5.34}$$

In particular, $n_\pm(D_{X,\alpha}) = 1$ provided that one of the following conditions is satisfied

(i) $\limsup_{n\to\infty}(d_{n+1}/d_n) = 0$ *and the sequence* $\alpha = \{\alpha_n\}_1^\infty$ *is bounded*;
(ii) $\limsup_{n\to\infty}(d_{n+1}/d_n) =: (1/d)$ *with* $d > 1$ *and* $\sup_{n\in\mathbb{N}} \alpha_n < c(\sqrt{d} - 1)$.

Proof. By the ratio test condition (5.34) yields the convergence of the series (5.26). It remains to apply Theorem 5.11. □

Remark 5.14. Note that the condition $\limsup_{n\to\infty}(d_{n+1}/d_n) \leqslant 1$ is always satisfied whenever the interval $\mathcal{I}$ is finite. Indeed, it is implied by the convergence of the series $\sum_{n=1}^\infty d_n = |\mathcal{I}| < +\infty$.

The following cases are more complicated and require more detailed analysis:

(i) $\limsup_{n\to\infty}(d_{n+1}/d_n) = 0$ and the sequence $\{\alpha_n\}_1^\infty$ is unbounded;
(ii) $\limsup_{n\to\infty}(d_{n+1}/d_n) = 1$ although $\lim_{n\to\infty} \alpha_n = 0$.

We discuss the case (i) in the following example.

Example 5.15.

(i) Let $|\alpha_n| \sim \alpha_0/n^s$, $s > 0$, and $\alpha_0 > 0$. Then, by Corollary 5.12, $n_\pm(D_{X,\alpha}) = 1$ for any $X = \{x_n\}_1^\infty$ whenever $s > 1$.
Next let $X = \{x_n\}_1^\infty$ with $x_n = 1 - 1/d^n$, $d > 1$, $n \in \mathbb{N}$. Then, by Corollary 5.13, $n_\pm(D_{X,\alpha}) = 1$ for $s \in (0, 1]$ and any $\alpha_0 \in \mathbb{R}_+$ as well as for $s = 0$ whenever $\alpha_0 < c(\sqrt{d} - 1)$.
(ii) Let $X = \{x_n\}_1^\infty$ with $x_n = 1 - 1/n!$, $|\alpha_n| \sim \alpha_0 n^s$, $n \in \mathbb{N}$, $s \in \mathbb{R}$. Then, by Corollary 5.13, $n_\pm(D_{X,\alpha}) = 1$ for any α_0 whenever $s < 1/2$, and for $\alpha_0 < c$ whenever $s = 1/2$.

On the other hand, if $\alpha_n \geqslant (n-1)!$, then, by Proposition 5.7, the operator $D_{X,\alpha}$ is self-adjoint.

Remark 5.16. Comparing Proposition 5.7 with Theorem 5.11 one might say that very roughly speaking $D_{X,\alpha}$ is self-adjoint on a finite interval whenever the sequence $\{\alpha_n\}_1^\infty$ grows faster than the sequence $\{d_n\}_1^\infty$ decays.

5.1.2. Continuous spectrum and resolvent comparability

Proposition 5.17. *Let $D_{X,\alpha^{(k)}}$ be the Gesztesy–Šeba realization of Dirac operator on the half-line $\mathbb{R}_+$ given by (5.1) with $\alpha^{(k)} := \{\alpha_n^{(k)}\}_{n\in\mathbb{N}} (\subset \mathbb{R})$, $k \in \{1, 2\}$. Let also $B_{X,\alpha^{(k)}}$ be the Jacobi operator defined on $\mathcal{H} = l^2(\mathbb{N}) \otimes \mathbb{C}^2$ by the matrix (5.5) with $\alpha^{(k)}$ in place of α. Then $D_{X,\alpha^{(k)}} = D^*_{X,\alpha^{(k)}}$, $k \in \{1, 2\}$, and for any $p \in (0, \infty]$ the inclusion*

$$(D_{X,\alpha^{(1)}} - z)^{-1} - (D_{X,\alpha^{(2)}} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}), \quad z \in \rho(D_{X,\alpha^{(1)}}) \cap \rho(D_{X,\alpha^{(2)}}) \tag{5.35}$$

is equivalent to the inclusion

$$(B_{X,\alpha^{(1)}} - \zeta)^{-1} - (B_{X,\alpha^{(2)}} - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H}), \quad \zeta \in \rho(B_{X,\alpha^{(1)}}) \cap \rho(B_{X,\alpha^{(2)}}). \tag{5.36}$$

Proof. Consider the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ defined in Theorem 3.10. Then, by Proposition 5.2, $H_{X,\alpha^{(k)}} = H_{B_{X,\alpha^{(k)}}}$ where $B_{X,\alpha^{(k)}}$, $k \in \{1,2\}$, is the corresponding Jacobi operator. Since $\mathcal{I} = \mathbb{R}_+$, both operators $D_{X,\alpha^{(k)}}$ and $B_{X,\alpha^{(k)}}$, $k \in \{1,2\}$, are self-adjoint, by Proposition 5.5. Therefore the resolvents $(D_{X,\alpha^{(k)}} - z)^{-1}$ and $(B_{X,\alpha^{(k)}} - \zeta)^{-1}$, $k \in \{1,2\}$, are well defined for any $z, \zeta \in \mathbb{C}_+$ and relations (5.35) and (5.36) have sense. One completes the proof by applying Proposition 2.6(i). $\square$

Corollary 5.18. *Assume the conditions of Proposition 5.17. Assume, in addition, that either*

$$\left\{\frac{\alpha_n^{(1)} - \alpha_n^{(2)}}{d_{n+1}}\right\}_{n=1}^{\infty} \in l^p(\mathbb{N}), \quad p \in (0,\infty) \quad \text{or} \quad \left\{\frac{\alpha_n^{(1)} - \alpha_n^{(2)}}{d_{n+1}}\right\}_{n=1}^{\infty} \in c_0(\mathbb{N}). \tag{5.37}$$

Then the inclusion (5.35) *holds with $p \in (0,\infty)$ and $p = \infty$, respectively.*

Proof. Let $B_{X,\alpha^{(k)}}$, $k \in \{1,2\}$, be the Jacobi operator given by (5.5). Clearly, $l_0^2(\mathbb{N}) \otimes C^2 \subset \operatorname{dom}(B_{X,\alpha^{(1)}}) \cap \operatorname{dom}(B_{X,\alpha^{(2)}})$. It follows from representation (5.8) for $B_{X,\alpha}$ and formula (3.57) for R_n that

$$B_{X,\alpha^{(1)}} f - B_{X,\alpha^{(2)}} f = R_X^{-1}(\widetilde{B}_{\alpha^{(1)}} - \widetilde{B}_{\alpha^{(2)}}) R_X^{-1} f = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & \frac{\alpha_n^{(1)} - \alpha_n^{(2)}}{d_{n+1}} \end{pmatrix} f, \quad f \in l_0^2(\mathbb{N}).$$

Due to the assumption (5.37) the operator $B_{X,\alpha^{(1)}} - B_{X,\alpha^{(2)}}$ admits the closure and $\overline{B_{X,\alpha^{(1)}} - B_{X,\alpha^{(2)}}} \in \mathfrak{S}_p(\mathcal{H}) \subset [\mathcal{H}]$. Hence $\operatorname{dom}(B_{X,\alpha^{(1)}}) = \operatorname{dom}(B_{X,\alpha^{(2)}})$ and, by Proposition 2.6(ii), the inclusion (5.36) holds. It remains to apply Proposition 5.17. $\square$

Next we slightly generalize Corollary 5.18 allowing one of the sequences $\alpha^{(k)} = \{\alpha_n^{(k)}\}_{n\in\mathbb{N}}$ to take infinite values. Moreover, in the case $d_*(X) > 0$ we can drop dependence on d_n in (5.37).

To state the result we set $(i + \infty)^{-1} := 0$.

Corollary 5.19. *Let $\alpha^{(1)} = \{\alpha_n^{(1)}\}_1^\infty \subset \mathbb{R}$ and $\alpha^{(2)} = \{\alpha_n^{(2)}\}_1^\infty \subset \mathbb{R} \cup \{\infty\}$. Then:*

(i) *The inclusion*

$$\begin{aligned} &\left\{\left(\alpha_n^{(1)}/d_{n+1} - \mathrm{i}\right)^{-1} - \left(\alpha_n^{(2)}/d_{n+1} - \mathrm{i}\right)^{-1}\right\}_{n=1}^{\infty} \in l^p(\mathbb{N}), \\ &p \in (0,\infty) \ \left(\in c_0(\mathbb{N}),\ \text{for } p = \infty\right), \end{aligned} \tag{5.38}$$

yields the inclusion (5.35).

(ii) *If in addition $0 < d_*(X) \leqslant d^*(X) < \infty$, then* (5.35) *is equivalent to the inclusion*

$$\left\{\left(\alpha_n^{(1)} - \mathrm{i}\right)^{-1} - \left(\alpha_n^{(2)} - \mathrm{i}\right)^{-1}\right\}_{n=1}^{\infty} \in l^p(\mathbb{N}), \quad p \in (0,\infty) \ \left(\in c_0(\mathbb{N}),\ \text{if } p = \infty\right). \tag{5.39}$$

Moreover, if $\{\alpha_n^{(j)}\}_{n=1}^\infty \in l^\infty(\mathbb{N})$, $j \in \{1,2\}$, then (5.39) *is equivalent to the inclusion $\{\alpha_n^{(1)} - \alpha_n^{(2)}\}_{n=1}^\infty \in l^p(\mathbb{N})$ (resp. $c_0(\mathbb{N})$).*

The proof is similar to that of Corollary 5.18 and is omitted.

To state the next result we recall the definition of the essential spectrum.

Definition 5.20. It is said that $\lambda_0 = \bar{\lambda}_0$ belongs to the essential spectrum of the operator $T = T^* \in \mathcal{C}(\mathfrak{H})$ (in short $\lambda_0 \in \sigma_{\text{ess}}(T)$) if there exists a bounded non-compact sequence $f_n \in \mathfrak{H}$, $n \in \mathbb{N}$, such that

$$(T - \lambda_0) f_n \to 0 \quad \text{as } n \to \infty. \tag{5.40}$$

In the following theorem we investigate continuous and absolutely continuous spectra of GS-realizations comparing them with the Neumann realization D_N of the Dirac expression given by

$$D_N = D \restriction \operatorname{dom}(D_N), \quad \operatorname{dom}(D_N) = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in W^{1,2}(\mathbb{R}_+) \otimes \mathbb{C}^2 \colon\ f_2(x_0+) = 0 \right\}. \tag{5.41}$$

We also investigate singular spectrum of GS-realizations.

Theorem 5.21. *Let $X = \{x_n\}_{n=1}^{\infty} (\subset \mathbb{R}_+)$, $\alpha = \{\alpha_n\}_1^{\infty} \subset \mathbb{R}$ and let $D_{X,\alpha}$ be the corresponding Gesztesy–Šeba operator on the half-line $\mathbb{R}_+$. Then the following holds:*

(i) *If $\{\alpha_n/d_{n+1}\}_1^{\infty} \in c_0(\mathbb{N})$, i.e. $\lim_{n\to\infty} \alpha_n/d_{n+1} = 0$, then*

$$\sigma_{\mathrm{ess}}(D_{X,\alpha}) = \sigma_{\mathrm{ess}}(D_N) = \mathbb{R} \setminus \left(-c^2/2, c^2/2\right). \tag{5.42}$$

(ii) *Assume that $\{\alpha_n/d_{n+1}\}_1^{\infty} \in l^1(\mathbb{N})$, i.e. $\sum_{n\in\mathbb{N}} |\alpha_n/d_{n+1}| < \infty$. Then the ac-part $D_{X,\alpha}^{ac}$ of $D_{X,\alpha}$ is unitarily equivalent to the Neumann realization D_N. In particular,*

$$\sigma_{ac}(D_{X,\alpha}) = \sigma_{ac}(D_N) = \mathbb{R} \setminus \left(-c^2/2, c^2/2\right). \tag{5.43}$$

(iii) *Assume that*

$$\limsup_{n\to\infty} \frac{|\alpha_n|}{d_{n+1}} = \infty. \tag{5.44}$$

Then the GS-operator $D_{X,\alpha}$ is purely singular, i.e.

$$\sigma_{ac}(D_{X,\alpha}) = \emptyset. \tag{5.45}$$

(iv) *Assume in addition that $d_*(X) > 0$. Then the above assumptions on the sequence $\{\alpha_n/d_{n+1}\}_1^{\infty}$ in* (i), (ii), *and* (5.44) *can be replaced by*

$$\{\alpha_n\}_1^{\infty} \in c_0(\mathbb{N}), \qquad \{\alpha_n\}_1^{\infty} \in l^1(\mathbb{N}) \quad \text{and} \quad \limsup_{n\to\infty} |\alpha_n| = \infty,$$

respectively.

Proof. (i) We choose $\alpha^{(2)} := \mathbf{0} = \{0\}_1^{\infty}$ to be a zero sequence and set $\alpha^{(1)} := \alpha = \{\alpha_n\}_1^{\infty}$. It is easily seen that $D_{X,\alpha^{(2)}} = D_{X,\mathbf{0}}$ coincides with the Neumann realization D_N given by (5.41). Moreover, noting that $D_N = D_{b+,0}$ with $b = 0$ (see formula (3.20)), Lemma 3.3(ii) yields $\sigma_{\mathrm{ess}}(D_N) = \mathbb{R} \setminus (-c^2/2, c^2/2)$. On the other hand, due to the assumption on $\{\alpha_n/d_{n+1}\}_1^{\infty}$, the above sequences $\alpha^{(1)}$ and $\alpha^{(2)}$ satisfy the second condition in (5.37), hence, by Corollary 5.18, $(D_{X,\alpha} - i)^{-1} - (D_N - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$. By the Weyl theorem (see [65, Corollary XIII.4.1]), the later inclusion implies $\sigma_{\mathrm{ess}}(D_{X,\alpha}) = \sigma_{\mathrm{ess}}(D_N) = \mathbb{R} \setminus (-c^2/2, c^2/2)$.

(ii) Now, by Corollary 5.18, the condition $\{\alpha_n/d_{n+1}\}_1^{\infty} \in l^1(\mathbb{N})$ implies $(D_{X,\alpha} - i)^{-1} - (D_N - i)^{-1} = (D_{X,\alpha^{(1)}} - i)^{-1} - (D_{X,\alpha^{(2)}} - i)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$. By the Kato–Rosenblum theorem (see [41, Chapter 10.4], [66, Theorem XI.9]), the *ac*-part $D_{X,\alpha}^{ac}$ of $D_{X,\alpha}$ is unitarily equivalent to $D_N^{ac} = D_N$. It remains to apply Lemma 3.3(ii) and note that $D_N = D_{b+,0}$ with $b = 0$.

(iii) According to (5.44) there exists a subsequence $\{\alpha_{n_k}\}$ such that

$$\lim_{n\to\infty} \frac{|\alpha_{n_k}|}{d_{n_k}} = \infty. \tag{5.46}$$

Set

$$\widetilde{\alpha}_n := \begin{cases} \alpha_n, & n \notin \{n_k\}, \\ \infty, & n \in \{n_k\}, \end{cases} \tag{5.47}$$

and $\widetilde{\alpha} := \{\widetilde{\alpha}_n\}_1^\infty$. Without loss of generality we assume that the subsequence $\{\alpha_{n_k}\}_{k=1}^\infty$ satisfies

$$\sum_{k=1}^{\infty} d_{n_k+1} |\alpha_{n_k}|^{-1} < \infty, \tag{5.48}$$

i.e. $\{d_{n_k+1}\alpha_{n_k}^{-1}\}_{k\in\mathbb{N}} \in l^1(\mathbb{N})$. Otherwise we replace $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ by its appropriate subsequence. It follows that

$$\sum_{n=1}^{\infty} \bigl| \bigl(\alpha_n d_{n+1}^{-1} - i\bigr)^{-1} - \bigl(\widetilde{\alpha}_n d_{n+1}^{-1} - i\bigr)^{-1} \bigr| = \sum_{n=1}^{\infty} \bigl| \bigl(\alpha_{n_k} d_{n_k+1}^{-1} - i\bigr)^{-1} \bigr| < \infty. \tag{5.49}$$

By Corollary 5.19(i), this relation yields

$$(D_{X,\alpha} - i)^{-1} - (D_{X,\widetilde{\alpha}} - i)^{-1} \in \mathfrak{S}_1(\mathfrak{H}). \tag{5.50}$$

According to Remark 5.1 (see formula (5.3)) it follows from (5.47) that the operator $D_{X,\widetilde{\alpha}}$ admits the following orthogonal decomposition

$$D_{X,\widetilde{\alpha}} = \bigoplus_{k=1}^{\infty} D_{n_k}, \qquad L^2\bigl(\mathbb{R}_+, \mathbb{C}^2\bigr) = \bigoplus_{k=1}^{\infty} L^2\bigl([x_{n_{k-1}}, x_{n_k}], \mathbb{C}^2\bigr), \tag{5.51}$$

where

$$\begin{aligned} D_{n_k} &= D \upharpoonright \operatorname{dom}(D_{n_k}), \\ \operatorname{dom}(D_{n_k}) &= \Biggl\{ f \in \bigoplus_{j=n_{k-1}}^{n_k - 1} W^{1,2}\bigl([x_j, x_{j+1}], \mathbb{C}^2\bigr) \colon\ f_1(x_j+) = f_1(x_j-), \\ &\qquad f_2(x_j+) - f_2(x_j-) = -ic^{-1}\alpha_j f_1(x_j), \\ &\qquad x_j \in X \cap (x_{n_{k-1}}, x_{n_k}),\ f_1(x_{n_{k-1}}) = f_1(x_{n_k}) = 0 \Biggr\}. \end{aligned}$$

Clearly, D_{n_k} is a self-adjoint extension of the minimal operator $D'_{n_k} := D_{n_k,\min}$ given by

$$\begin{aligned} D'_{n_k} &= D \upharpoonright \operatorname{dom}\bigl(D'_{n_k}\bigr), \\ \operatorname{dom}\bigl(D'_{n_k}\bigr) &= \bigl\{ f \in W_0^{1,2}\bigl([x_{n_{k-1}}, x_{n_k}], \mathbb{C}^2\bigr) \colon\ f(x_j) = 0,\ x_j \in X \cap [x_{n_{k-1}}, x_{n_k}] \bigr\}. \end{aligned}$$

Clearly, the operator D'_{n_k} admits a self-adjoint extension $\bigoplus_{j=n_{k-1}+1}^{n_k} D_{n_j,0}$ with discrete spectrum (see Lemma 3.1(ii)) wher e $D_{n_j,0}$ is given by (3.9). Since D'_{n_k} is a symmetric operator with finite deficiency indices, each its self-adjoint extension has also discrete spectrum. In particular, D_{n_k}, $k \in \mathbb{N}$, has discrete spectrum.

Therefore due to the representation (5.51) the spectrum of the operator $D_{X,\widetilde{\alpha}}$ is purely point, in particular $\sigma_{ac}(D_{X,\widetilde{\alpha}}) = \emptyset$. On the other hand, it follows from (5.50) and the Kato–Rosenblum theorem that the ac-parts $D^{ac}_{X,\alpha}$ and $D^{ac}_{X,\widetilde{\alpha}}$ of the operators $D_{X,\alpha}$ and $D_{X,\widetilde{\alpha}}$ are unitarily equivalent. In particular, $\sigma_{ac}(D_{X,\alpha}) = \sigma_{ac}(D_{X,\widetilde{\alpha}}) = \emptyset$.

(iv) This statement is immediate from the previous ones. $\square$

Next we extend Theorem 5.21 to the case of GS-realizations of the Dirac expression $D(Q)$ with a bounded potential matrix $Q \in L^\infty(\mathcal{I}) \otimes \mathbb{C}^{2\times 2}$. Namely, consider differential expression

$$D(Q) := D^c(Q) := -ic\frac{d}{dx} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3 + Q(x), \quad Q(x) = Q(x)^*, \tag{5.52}$$

and denote by $D_X(Q) := D^c_X(Q)$ the minimal operator associated on $\mathcal{I} \setminus X$ with the expression $D^c(Q)$. As in (3.29) one has

$$\begin{aligned} D_X(Q) &= D(Q) \upharpoonright \operatorname{dom}\bigl(D_X(Q)\bigr), \\ \operatorname{dom}\bigl(D_X(Q)\bigr) &= W^{1,2}_0\bigl(\mathcal{I} \setminus X, \mathbb{C}^2\bigr) = \bigoplus_{n=1}^{\infty} W^{1,2}_0\bigl([x_{n-1}, x_n], \mathbb{C}^2\bigr). \end{aligned} \tag{5.53}$$

Further, let $D_{X,\alpha}(Q) := D_{X,\alpha} + Q$ be the GS-realization of $D(Q)$. If $\alpha := \mathbf{0} = \{0\}_1^\infty$ is a zero sequence we set $D_N(Q) := D_{X,\mathbf{0}}(Q)$ and note that $D_N(Q)$, the Neumann realization of $D(Q)$, is given by the expression (5.52) on the domain (5.41), i.e.

$$\begin{aligned} D_N(Q) &= D(Q) \upharpoonright \operatorname{dom}\bigl(D_N(Q)\bigr), \\ \operatorname{dom}\bigl(D_N(Q)\bigr) &= \operatorname{dom}(D_N) = \bigl\{ f \in W^{1,2}(\mathbb{R}_+) \otimes \mathbb{C}^2\colon\ f_2(x_0+) = 0 \bigr\}. \end{aligned}$$

Proposition 5.22. *Assume that $Q \in L^\infty(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$, $Q(x) = Q^*(x)$ for a.e. $x \in \mathbb{R}_+$, and $\alpha = \{\alpha_n\}_1^\infty \subset \mathbb{R}$. Then the following hold*:

(i) *If $\{\alpha_n/d_{n+1}\}_1^\infty \in c_0(\mathbb{N})$, then*

$$\sigma_{\mathrm{ess}}\bigl(D_{X,\alpha}(Q)\bigr) = \sigma_{\mathrm{ess}}\bigl(D_N(Q)\bigr). \tag{5.54}$$

Moreover, if in addition, $Q(x) \to 0$ as $x \to \infty$, then

$$\sigma_{\mathrm{ess}}\bigl(D_{X,\alpha}(Q)\bigr) = \sigma_{\mathrm{ess}}\bigl(D_N(Q)\bigr) = \mathbb{R} \setminus \bigl(-c^2/2, c^2/2\bigr). \tag{5.55}$$

(ii) *If $\{\alpha_n/d_{n+1}\}_1^\infty \in l^1(\mathbb{N})$, then*

$$\sigma_{ac}\bigl(D_{X,\alpha}(Q)\bigr) = \sigma_{ac}\bigl(D_N(Q)\bigr). \tag{5.56}$$

Moreover, if additionally, $Q \in L^1(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$, then

$$\sigma_{ac}\bigl(D_{X,\alpha}(Q)\bigr) = \sigma_{ac}\bigl(D_N(Q)\bigr) = \mathbb{R} \setminus \bigl(-c^2/2, c^2/2\bigr). \tag{5.57}$$

(iii) *If condition (5.44) is satisfied, then the spectrum of $D_{X,\alpha}(Q)$ is purely singular, i.e.*

$$\sigma_{ac}\bigl(D_{X,\alpha}(Q)\bigr) = \emptyset. \tag{5.58}$$

(iv) *Assume in addition that $d_*(X) > 0$. Then the above assumptions can be replaced by*

$$\{\alpha_n\}_1^\infty \in c_0(\mathbb{N}), \qquad \{\alpha_n\}_1^\infty \in l^1(\mathbb{N}) \quad \textit{and} \quad \limsup_{n\to\infty} |\alpha_n| = \infty,$$

respectively.

Proof. (i)–(ii) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator D_X^* defined in Theorem 3.10. Since $Q = Q^* \in L^\infty(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$, Π is also the boundary triplet for the operator $D_X(Q)^*$. Moreover, due to the inclusion $Q \in L^\infty(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$, one has $\operatorname{dom}(D_{X,\alpha}(Q)) = \operatorname{dom}(D_{X,\alpha})$. Therefore the boundary operator for the extension $D_{X,\alpha}(Q)$ of $D_X(Q)$ coincides with the boundary operator for the extension $D_{X,\alpha}$ of D_X. Thus, by Proposition 5.2,

$$\operatorname{dom}\bigl(D_{X,\alpha}(Q)\bigr) = \operatorname{dom}(D_{X,\alpha}) = \bigl\{f \in W^{1,2}\bigl(\mathbb{R}_+ \setminus X, \mathbb{C}^2\bigr): \Gamma_1 f = B_{X,\alpha}\Gamma_0 f\bigr\}, \tag{5.59}$$

where $B_{X,\alpha}$ is the Jacobi operator given by (5.5). Therefore due to Proposition 2.6(i) inclusion (5.35) is equivalent to the inclusion

$$\bigl(D_{X,\alpha^{(1)}}(Q) - i\bigr)^{-1} - \bigl(D_{X,\alpha^{(2)}}(Q) - i\bigr)^{-1} \in \mathfrak{S}_p(\mathfrak{H}), \quad p \in (0, \infty]. \tag{5.60}$$

We compare the realizations $D_{X,\alpha}(Q)$ and $D_{X,\mathbf{0}}(Q) = D_N(Q)$ using the inclusion (5.60) with $\alpha^{(1)} = \alpha$ and $\alpha^{(2)} = \mathbf{0} = \{0\}_1^\infty$. Namely, the inclusion (5.60) with $p = \infty$ yields (5.54) by applying the Weyl theorem [65, Corollary XIII.4.1]. Similarly, the inclusion (5.60) with $p = 1$ yields equality (5.56) by applying the Kato–Rosenblum theorem ([41], [66, Theorem XI.9]).

It is well known that $\sigma_{\mathrm{ess}}(D_N(Q)) = \sigma_{\mathrm{ess}}(D_N) = \mathbb{R} \setminus (-c^2/2, c^2/2)$ provided that $Q(x) \to 0$ as $x \to \infty$. Combining this relation with (5.54) we arrive at (5.55).

Further, according to [52, Theorem 9.1.1], $\sigma_{ac}(D_N(Q)) = \sigma_{\mathrm{ess}}(D_N(Q)) = \mathbb{R}\setminus(-c^2/2, c^2/2)$ whenever $Q \in L^1(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$. Combining this fact with (5.56) we arrive at (5.57).

(iii) As in the proof of Theorem 5.21(iii) we define a sequence $\widetilde{\alpha} := \{\widetilde{\alpha}_n\}_1^\infty$ by formula (5.47) and find a subsequence $\{\widetilde{\alpha}_{n_k}\}_{k=1}^\infty$ satisfying (5.48). Alongside (5.59) we have the following representation of the domain $\operatorname{dom}(D_{X,\widetilde{\alpha}}(Q))$,

$$\operatorname{dom}\bigl(D_{X,\widetilde{\alpha}}(Q)\bigr) = \operatorname{dom}(D_{X,\widetilde{\alpha}}) = \bigl\{f \in W^{1,2}\bigl(\mathbb{R}_+ \setminus X, \mathbb{C}^2\bigr): \{\Gamma_1 f, \Gamma_0 f\} \in \Theta_{X,\widetilde{\alpha}}\bigr\}, \tag{5.61}$$

where the boundary relation $\Theta_{X,\widetilde{\alpha}}$ corresponding to $D_{X,\widetilde{\alpha}}(Q)$ does not depend on Q. As it is shown in the proof of Theorem 5.21(iii) condition (5.44) yields the inclusion (5.50). In turn, combining relations (5.59) and (5.61) with Proposition 2.6(i) we get that the inclusion (5.50) yields (in fact, is equivalent to) the inclusion

$$\bigl(D_{X,\alpha}(Q) - i\bigr)^{-1} - \bigl(D_{X,\widetilde{\alpha}}(Q) - i\bigr)^{-1} \in \mathfrak{S}_1(\mathfrak{H}). \tag{5.62}$$

The rest of the proof coincides with the proof of Theorem 5.21(iii). □

Remark 5.23. In the case of $Q \not\equiv 0$ an explicit description of $\sigma_{\mathrm{ess}}(D_N(Q))$ and $\sigma_{ac}(D_N(Q))$ is known also for some non-decaying potentials. For instance, if Q is periodic, $Q(x + \tau) = Q(x)$, $x \in \mathbb{R}$, then the essential spectrum of the operator $D(Q)$ in $L^2(\mathbb{R}, \mathbb{C}^2)$ is absolutely continuous and has zone-band structure. This fact allows one to complete the statement (ii) for periodic Q.

Remark 5.24. Note that analogs of the main results of this section are known for Schrödinger operators $H_{X,\alpha}$ with δ-interactions. For instance, in the case $d_*(X) > 0$, the resolvent comparability criterion for Schrödinger operators $H_{X,\alpha}$ (i.e. analogs of Corollaries 5.19 and 5.18) was obtained in [43] (see also [59]). Moreover, the statements similar to Theorem 5.21(iv) and Proposition 5.22(iv) have also been obtained for operators $H_{X,\alpha}$ in [59,60,43]. These authors have also applied boundary triplets technique to the operator H_X^* with $d_*(X) > 0$. Other results on absolutely continuous and singular spectrum of $H_{X,\alpha}$ in the case $d_*(X) > 0$ can be found in [68].

In the case $d_*(X) = 0$ Schrödinger operators $H_{X,\alpha}$ were treated in detail in [44] where one can find analogs of Proposition 5.17, Corollary 5.18 and Theorem 5.21(i)–(ii).

Note also that the proof of Theorem 5.21(iii) is similar to that presented in [68,60]. However the idea of the proof goes back to the paper [70] where it is applied to 1-D Schrödinger operators with $L^1_{\mathrm{loc}}(\mathbb{R}_+)$-potentials. In connection with Theorem 5.21(iii) we mention also a recent interesting paper [53]. In particular, it is shown in [53] that the Schrödinger operator with point interactions on a sparse set has purely point continuous spectrum.

Remark 5.25. Another proof of Theorem 5.21(iii) can also be extracted from [56, Theorem 1.1]. It is based on an explicit block-diagonal form $M(\cdot) = \bigoplus_{n=1}^{\infty} M_n(\cdot)$ of the Weyl function $M(\cdot)$.

5.1.3. Discrete spectrum

Here we investigate the discreteness property of proper extensions of the minimal Dirac operator $D_X(Q)$ defined by (5.53) and associated in $\mathfrak{H} = L^2(\mathbb{R}_+) \otimes \mathbb{C}^2$ with the differential expression (5.52) on $\mathbb{R}_+ \setminus X$. In particular, we show that in the case $d_*(X) > 0$ there are no proper extensions with discrete spectrum.

First we investigate the discreteness property for the minimal Dirac operator $D_X := D_X(0)$ with zero potential $Q = 0$.

Theorem 5.26. *Let $X = \{x_n\}_1^{\infty} (\subset \mathbb{R}_+)$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for D_X^* defined in Theorem 3.10 and let $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ with $\rho(\Theta) \neq \emptyset$. Then:*

(i) *D_X has compact inverse $(D_X)^{-1} (\in [\mathrm{ran}(D_X), \mathfrak{H}])$ if and only if $\lim_{n\to+\infty} d_n = 0$.*
(ii) *A proper extension $\widetilde{D}_X = D_{X,\Theta}$ of D_X has discrete spectrum if and only if $\lim_{n\to+\infty} d_n = 0$ and $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ has discrete spectrum.*

Proof. (i) *Sufficiency*. Let $\lim_{n\to+\infty} d_n = 0$. According to the construction, the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for D_X^* is the direct sum, $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ (see Theorem 3.10, formulae (3.54), (3.55)) and

$$A_0 := D_X^* \restriction \ker \Gamma_0 = \bigoplus_{n=1}^{\infty} A_{n,0}, \quad A_{n,0} = D_{n,0} := D_n^* \restriction \ker\bigl(\Gamma_0^{(n)}\bigr). \tag{5.63}$$

Combining (5.63) with Lemma 3.1(ii) we get

$$\sigma(A_0) = \bigcup_{n=1}^{\infty} \sigma(D_{n,0}) = \bigcup_{n,j=1}^{\infty} \{\lambda_{n,j}^{\pm}\} \tag{5.64}$$

where for any fixed n

$$\lambda_{n,j}^{\pm} = \pm\sqrt{\frac{c^2\pi^2}{d_n^2}\left(j+\frac{1}{2}\right)^2 + \left(\frac{c^2}{2}\right)^2} \sim \pm\frac{\pi c}{d_n}\left(j+\frac{1}{2}\right), \quad \text{as } j \to \infty. \tag{5.65}$$

It follows that any non-zero (finite or infinite) accumulation point of the sequence $\{d_n\}_{n=1}^{\infty}$ generates countably many accumulation points for the sequence $\{\lambda_{n,j}^{\pm}\}_{n,j\in\mathbb{N}}$. Thus, the spectrum $\sigma(A_0)$ is discrete, i.e. $A_0^{-1} \in \mathfrak{S}_{\infty}(\mathfrak{H})$ if and only if $\lim_{n\to+\infty} d_n = 0$. In particular, the later condition yields compactness of $(D_X)^{-1} = A_0^{-1} \restriction \mathrm{ran}(D_X)$.

Necessity. Assume that d_n does not converge to zero, so that we can find a subsequence $\{d_{n_k}\}_{k=1}^{\infty}$ and $\varepsilon > 0$ such that $d_{n_k} \geqslant \varepsilon > 0$, $k \in \mathbb{N}$. Choose a function $\varphi = \binom{\phi}{\phi} \in W_0^{1,2}(\mathbb{R}_+, \mathbb{C}^2)$ such that

$$\phi(x) = \begin{cases} 1, & \varepsilon/4 \leqslant x \leqslant 3\varepsilon/4, \\ 0, & x \notin [0, \varepsilon], \end{cases}$$

and put

$$\varphi_k(x) := \varphi(x - x_{n_k}), \quad k \in \mathbb{N}.$$

Clearly, $\varphi_k \in \operatorname{dom}(D_X)$, $k \in \mathbb{N}$. Moreover, there exist constants C_1 and C_2 such that

$$\|\varphi_k\|_{L^2(\mathbb{R}_+)} = C_1 \quad \text{and} \quad \|D_X \varphi_k\|_{L^2(\mathbb{R}_+, \mathbb{C}^2)} \leqslant C_2, \quad k \in \mathbb{N}. \tag{5.66}$$

Since the functions φ_k have disjoint supports, the sequence $\{\varphi_k\}_1^{\infty}$ is not compact in $L^2(\mathbb{R}_+) \otimes \mathbb{C}^2$. Therefore it follows from (5.66) that the operator $(D_X)^{-1}$ is not compact.

(ii) Let the spectrum $\sigma(D_{X,\Theta})$ of $D_{X,\Theta}$ be discrete, i.e. $\rho(D_{X,\Theta}) \neq \emptyset$ and $(D_{X,\Theta} - z)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$ for $z \in \rho(D_{X,\Theta})$. Then $z \in \widehat{\rho}(D_X)$ and the operator

$$(D_X - z)^{-1} = (D_{X,\Theta} - z)^{-1} \upharpoonright \operatorname{ran}(D_X - z)$$

is also compact. By (i) $\lim_{n\to+\infty} d_n = 0$. Therefore it follows from (5.65) and (5.64) (and was already mentioned) that the spectrum $\sigma(A_0)$ is discrete. Since both operators $D_{X,\Theta}$ and A_0 have compact resolvents, it follows from Proposition 2.6(i), that $(\Theta - \zeta)^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$ for $\zeta \in \rho(\Theta)$, i.e. the spectrum $\sigma(\Theta)$ of Θ is discrete too.

Conversely, let $\lim_{n\to+\infty} d_n = 0$ and let the spectrum $\sigma(\Theta)$ be discrete. Then, by (i), the condition $\lim_{n\to+\infty} d_n = 0$ yields discreteness of the spectrum of A_0. Finally, by Proposition 2.6(i), $(D_{X,\Theta} - z)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$ since both resolvents $(A_0 - z)^{-1}$ and $(\Theta - \zeta)^{-1}$ are compact. □

Corollary 5.27. *Assume the conditions of Theorem 5.26. Let also $Q(\cdot) = Q^*(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$, and let $D_X(Q)$ be a minimal Dirac operator on $\mathbb{R}_+ \setminus X$ given by (5.53). Assume in addition that the multiplication operator $f \to Qf$ in $L^2(\mathbb{R}_+, \mathbb{C}^2)$ is strongly subordinated to the Dirac operator D_X^*, i.e.* $\operatorname{dom}(D_X^*) \subset \operatorname{dom}(Q)$ *and there exist constants $a \in (0, 1)$, $b > 0$, such that*

$$\|Qf\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)} \leqslant a\big\|D_X^* f\big\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)} + b\|f\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)}, \quad 0 < a < 1, \ f \in \operatorname{dom}\big(D_X^*\big). \tag{5.67}$$

Then:

(i) *$D_X(Q)$ has compact inverse $(D_X(Q))^{-1} (\in [\operatorname{ran}(D_X(Q)), \mathfrak{H}])$ if and only if $\lim_{n\to+\infty} d_n = 0$.*

(ii) *A proper extension $\widetilde{D}_X(Q) = D_{X,\Theta}(Q) := D_{X,\Theta} + Q$ of $D_X(Q)$ has discrete spectrum if and only if $\lim_{n\to+\infty} d_n = 0$ and $\Theta (\in \widetilde{\mathcal{C}}(\mathcal{H}))$ has discrete spectrum. In particular, both statements are satisfied whenever $Q(\cdot) = Q^*(\cdot) \in L^\infty(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$.*

Proof. (i) Since Q is strongly subordinated to D_X^* it is also strongly subordinated to its restriction $A_0 := D_X^* \upharpoonright \ker \Gamma_0 = A_0^*$ (see (5.63)). The latter yields boundedness of the operator $Q(A_0 - i)^{-1}$. Moreover, by the Kato–Rellich theorem [41, Theorem 5.4.3], $A_0 + Q$ is self-adjoint.

Further, since Q is strongly subordinated to D_X^* it is also subordinated to $D_X^*(Q) = D_X^* + Q$ (see [41, Chapter 4.1]), hence Q is also subordinated to $A_0(Q) = A_0 + Q$, the restriction of $D_X^*(Q)$, with the $(A_0 + Q)$-bound not exceeding $a(1-a)^{-1}$, i.e.

$$\|Qf\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)} \leqslant (1-a)^{-1}\big(a\big\|(A_0 + Q)f\big\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)} + b\|f\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)}\big), \quad f \in \operatorname{dom}(A_0 + Q).$$

Since $A_0 + Q$ is self-adjoint, the latter is amount to saying that the operator $Q(A_0 + Q - i)^{-1}$ is bounded. Therefore it follows from the identity

$$(A_0 - i)^{-1} - \big(A_0(Q) - i\big)^{-1} = \big(A_0(Q) - i\big)^{-1} Q(A_0 - i)^{-1} = (A_0 - i)^{-1} Q\big(A_0(Q) - i\big)^{-1} \tag{5.68}$$

that the operators $(A_0 - i)^{-1}$ and $(A_0(Q) - i)^{-1}$ are compact only simultaneously. It remains to apply Theorem 5.26(i).

(ii) This statement is immediate from Theorem 5.26(ii) and formula (5.68) with A_0 and $A_0(Q)$ replaced by $D_{X,\Theta}$ and $D_{X,\Theta}(Q)$, respectively. $\square$

Next we stand the "individual" version of Corollary 5.27.

Corollary 5.28. *Assume the conditions of Theorem 5.26 and let $Q(\cdot) = Q^*(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$. Assume in addition that the multiplication operator $f \to Qf$ in $L^2(\mathbb{R}_+, \mathbb{C}^2)$ is strongly subordinated to a realization $D_{X,\Theta} = D^*_{X,\Theta}$, i.e. $\mathrm{dom}(D_{X,\Theta}) \subset \mathrm{dom}(Q)$ and the following estimate holds*

$$\|Qf\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)} \leqslant a\|D_{X,\Theta} f\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)} + b\|f\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)}, \quad 0 < a < 1,\ f \in \mathrm{dom}(D_{X,\Theta}). \tag{5.69}$$

Then a proper extension $D_{X,\Theta}(Q) := D_{X,\Theta} + Q$ of the minimal Dirac operator $D_X(Q)$ on $\mathbb{R}_+ \setminus X$ (see (5.53)) has discrete spectrum if and only if $\lim_{n\to+\infty} d_n = 0$ and $\Theta (\in \widetilde{\mathcal{C}}(\mathcal{H}))$ has discrete spectrum.

The proof is similar to that of Corollary 5.27 and is omitted.

Remark 5.29.

(i) Sufficiency in Theorem 5.26(i) can easily be proved directly. Indeed, since $\mathrm{dom}(D_X) = W^{1,2}_0(\mathbb{R}\backslash X) \otimes \mathbb{C}^2$, it suffices to show that the identical embedding $W^{1,2}_0(\mathbb{R}\backslash X, \mathbb{C}^2) \hookrightarrow L^2(\mathbb{R}, \mathbb{C}^2)$ is compact provided that $\lim_{n\to\infty} d_n = 0$. Let f belong to the unit ball of $W^{1,2}_0(\mathbb{R}\backslash X)$. One has

$$\big|f(x)\big|^2 = \big|f(x) - f(x_n)\big|^2 = \bigg|\int_{\Omega_n} f'(t)\,dt\bigg|^2 \leqslant d_n \int_{\Omega_n} \big|f'(t)\big|^2\,dt, \quad x \in \Omega_n = [x_{n-1}, x_n]. \tag{5.70}$$

Choosing any $\varepsilon > 0$ we find $N \in \mathbb{N}$ such that $d_n \leqslant \varepsilon$. Therefore it follows from (5.70) that

$$\sum_{|n|\geqslant N} \int_{\Omega_n} \big|f(t)\big|^2\,dt \leqslant \sum_{|n|\geqslant N} d_n^2 \|f'\|^2_{L^2(\Omega_n)} \leqslant \varepsilon^2 \|f'\|^2_{L^2(\mathbb{R}_+)} \leqslant \varepsilon^2.$$

Thus the "tails" of functions f running through the unit ball of $W^{1,2}_0(\mathbb{R}\backslash X)$ are uniformly small in $L^2(\mathbb{R})$. It remains to note that the embedding $W^{1,2}[a,b] \otimes \mathbb{C}^2 \hookrightarrow L^2[a,b] \otimes \mathbb{C}^2$ is compact for any finite interval $[a,b]$.

(ii) As it is clear from the proof, Corollary 5.27 remains valid under weaker assumptions. Namely, condition (5.67) can be replaced by the following one: Q is subordinated (in the sense of [41, Chapter 4.1]) to both operators D_X^* and D_X^*+Q.
Note also that an alternative proof of Corollary 5.27 can be obtained as follows. Equipping $\operatorname{dom}(D_{X,\Theta}+Q)$ with the graph norm one obtains the Hilbert space $\mathfrak{H}_+(\Theta,Q)$. It follows from estimate (5.67) that the Hilbert spaces $\mathfrak{H}_+(\Theta,Q)$ and $\mathfrak{H}_+(\Theta):=\mathfrak{H}_+(\Theta,0)$ coincide algebraically and topologically (see [41, Chapter 4.1]). Thus, both embeddings $\mathfrak{H}_+(\Theta,Q)\hookrightarrow\mathfrak{H}$ and $\mathfrak{H}_+(\Theta)\hookrightarrow\mathfrak{H}$ are compact only simultaneously. But the compactness of the embedding $\mathfrak{H}_+(\Theta,Q)\hookrightarrow\mathfrak{H}$ is equivalent to the discreteness of the spectrum of $D_{X,\Theta}$.

Theorem 5.26 establishes a connection between the discreteness property of extensions $D_{X,\Theta}(Q)$ of $D_X(Q)$ and the same property of the corresponding boundary relations Θ with respect to the boundary triplet for D_X^* defined in Theorem 3.10. Now we are in position to investigate discreteness property for GS-realizations $D_{X,\alpha}^c(Q)$ in terms of the corresponding distances d_n and intensities α_n. To this end we exploit a connection between the GS-realizations and Jacobi matrices on the one hand and the known results on discreteness property of Jacobi matrices on the other hand.

Proposition 5.30. *Let $X=\{x_n\}_1^\infty(\subset\mathbb{R}_+)$, $\alpha=\{\alpha_j\}_1^\infty\subset\mathbb{R}$ and let $Q(\cdot)=Q^*(\cdot)\in L^2_{\mathrm{loc}}(\mathbb{R}_+)\otimes\mathbb{C}^{2\times2}$ be strongly subordinated to the GS-realization $D_{X,\alpha}^c=D_{X,\alpha}^c(0)$ on $\mathbb{R}_+$. Assume also that $\lim_{n\to\infty}d_n=0$ and*

$$\lim_{n\to\infty}\frac{|\alpha_n|}{d_n}=\infty\quad\textit{and}\quad\lim_{n\to\infty}\frac{c}{\alpha_n}>-\frac{1}{4}.\tag{5.71}$$

Then the GS-operator $D_{X,\alpha}^c(Q)$ on the half-line $\mathbb{R}_+$ has discrete spectrum.

Proof. First we consider the case of the Dirac operator $D_{X,\alpha}^c$ with zero potential $Q=0$. Since $\lim_{n\to\infty}d_n=0$, one has

$$\lim_{n\to\infty}\frac{1}{\alpha_n\sqrt{d_n^2+1/c^2}}=\lim_{n\to\infty}\frac{c}{\alpha_n}.\tag{5.72}$$

By the Carleman test (see Proposition 5.5), the Jacobi matrix $B'_{X,\alpha}$ given by (5.9) is self-adjoint. Therefore, by [16, Theorem 8], the operator $B'_{X,\alpha}$ has discrete spectrum provided that $\lim_{n\to\infty}d_n=0$ and conditions (5.71) are satisfied.

Next we consider the boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for the operator D_X^* constructed in Theorem 3.10. By Proposition 5.2, the boundary operator corresponding to the GS-realization $D_{X,\alpha}=D_{X,\alpha}^c$ is given by the Jacobi operator $B_{X,\alpha}$ of the form (5.5), (5.8). Since the operators $B_{X,\alpha}$ and $B'_{X,\alpha}$ are unitarily equivalent (see Remark 5.3), the spectrum of $B_{X,\alpha}$ is discrete too. To prove the discreteness property of the operator $D_{X,\alpha}$ it remains to apply Theorem 5.26.

Let now $Q\neq0$. Since $Q(\cdot)$ is strongly subordinated to $D_{X,\alpha}^c$, general case is reduced to the previous one by applying Corollary 5.28. □

To apply Proposition 5.30 to GS-operators $D_{X,\alpha}^c(Q)$ with certain unbounded potentials we establish the following analog of the classical Hardy inequality.

Lemma 5.31. *Assume that $d^*(X)<\infty$. Then for any $f\in W^{1,2}(\mathbb{R}_+\setminus X)$ and satisfying $f(0)=0$ the following inequality holds*

$$\int_0^\infty\frac{|f(x)|^2}{x^2}\,dx\leqslant\frac14\int_0^{x_1}\left|f'(x)\right|^2dx+\frac{2}{x_1^2}\left(3d^*(X)^2\int_{x_1}^\infty\left|f'(x)\right|^2dx+2\int_{x_1}^\infty\left|f(x)\right|^2dx\right).\tag{5.73}$$

Proof. Indeed, by the classical Hardy inequality,

$$\int_0^{x_1} \frac{|f(x)|^2}{x^2}\,dx \leqslant \frac{1}{4}\int_0^{x_1} \left|f'(x)\right|^2 dx, \quad f \in W^{1,2}[0,x_1],\ f(0)=0. \tag{5.74}$$

Further, since $f \in W^{1,2}[x_k, x_{k+1}]$ for any $k \in \mathbb{N}$, one easily gets

$$\int_{x_k}^{x_{k+1}} \frac{|f(x)|^2}{x^2}\,dx \leqslant 2d_k\left(\frac{|f(x_k-)|^2}{x_k x_{k+1}} + \int_{x_k}^{x_{k+1}} \frac{dx}{x^2}\int_{x_k}^{x} \left|f'(t)\right|^2 dt\right) \leqslant \frac{2d_k}{x_k x_{k+1}}\left(\left|f(x_k-)\right|^2 + d_k\left\|f'\right\|^2_{L^2(\Delta_k)}\right)$$

$$\leqslant \frac{2}{x_k x_{k+1}}\left(d_k\left|f(x_k-)\right|^2 + d^*(X)^2\left\|f'\right\|^2_{L^2(\Delta_k)}\right), \quad k \geqslant 1.$$

Taking a sum of these inequalities and applying Proposition 3.5(ii) (see formula (3.45)) we obtain

$$\int_{x_1}^{\infty} \frac{|f(x)|^2}{x^2}\,dx \leqslant \frac{2}{x_1^2}\left(2d^*(X)^2\int_{x_1}^{\infty}\left|f'(x)\right|^2 dx + 2\int_{x_1}^{\infty}\left|f(x)\right|^2 dx + d^*(X)^2\int_{x_1}^{\infty}\left|f'(x)\right|^2 dx\right). \tag{5.75}$$

Combining (5.74) with (5.75) we arrive at (5.73). □

Example 5.32. Let us present an example of GS-operator $D^c_{X,\alpha}(Q) = D^c_{X,\alpha} + Q$ with an unbounded potential matrix $Q(\cdot) = Q^*(\cdot) \notin L^\infty(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$ satisfying conditions of Proposition 5.30. Let $Q(\cdot) = \operatorname{diag}(q_1(\cdot), q_2(\cdot))$ with $q_1(\cdot) \in L^\infty(\mathbb{R}_+)$, and an unbounded measurable function $q_2(\cdot)$ satisfying

$$\left|q_2(x)\right| \leqslant \frac{C_0}{x^\gamma}, \quad x \in \mathbb{R}_+,\ \gamma \in (0,1],\ C_0 > 0. \tag{5.76}$$

Let us show that for sufficiently small C_0 the multiplication operator with the matrix $Q(\cdot)$ is strongly subordinated to the operator $D^c_{X,\alpha}$, i.e. that $\operatorname{dom}(D^c_{X,\alpha}(Q)) \subset \operatorname{dom}(Q)$ and inequality (5.69) holds for $f = \binom{f_1}{f_2} \in \operatorname{dom}(D_{X,\alpha})$. Since $q_1(\cdot) \in L^\infty(\mathbb{R}_+)$, it suffices to estimate $\|q_2 f_2\|_{L^2(\mathbb{R}_+)}$. Noting that $f_2(0) = 0$ for $f = \binom{f_1}{f_2} \in \operatorname{dom}(D_{X,\alpha})$, and combining inequality (5.76) with Lemma 5.31, we get

$$\|q_2 f_2\|_{L^2(\mathbb{R}_+)} \leqslant \widetilde{C}_0\left\|f_2'\right\|^2_{L^2(\mathbb{R}_+)} + 4C_0 x_1^{-2}\|f_2\|^2_{L^2(\mathbb{R}_+)},$$

where $\widetilde{C}_0 = C_0\max\{4^{-1}, 6x_1^{-2}d^*(X)^2\}$. Since $q_1(\cdot) \in L^\infty(\mathbb{R}_+)$, this estimate implies (5.67) whenever C_0 is sufficiently small.

Thus, the operator $D^c_{X,\alpha}(Q) = D^c_{X,\alpha} + Q$ is self-adjoint and has discrete spectrum provided that C_0 is small enough and conditions (5.71) are satisfied. Note that strong subordination of the operator Q holds although $q_2 \notin L^2(0,\varepsilon)$ for $\gamma \geqslant 1/2$.

Remark 5.33. For any fixed $c > 0$ conditions (5.71) are weaker than the corresponding conditions for the discreteness of Schrödinger operator $H_{X,\alpha}$ from [44, Propositions 5.18] that read as follows

$$\lim_{n\to\infty}\frac{|\alpha_n|}{d_n} = \infty \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{d_n\alpha_n} > -\frac{1}{4}. \tag{5.77}$$

They can be obtained by taking the formal limit as $c \to +\infty$ in the left-hand side of (5.72) with account of (5.71).

Note also that if α is negative, conditions (5.71) do not guaranty discreteness for the whole family $D^c_{X,\alpha}$, $c > 0$, of GS-realizations.

Example 5.34. Let $\mathcal{I} = \mathbb{R}_+$ and let the sequence $X = \{x_n\}_{n=1}^\infty$ be given by $x_n = \log(n+1)$, so that $d_n = \log(1 + \frac{1}{n}) \sim \frac{1}{n}$. By Proposition 5.5, the GS-operator $D^c_{X,\alpha}$ is self-adjoint for any sequence $\alpha = \{\alpha_n\}_1^\infty \subset \mathbb{R} \cup \infty$. By Proposition 5.30, the GS-operator $D^c_{X,\alpha}$ has discrete spectrum whenever

$$\lim_{n\to\infty} n|\alpha_n| = \infty \quad \text{and} \quad 4c \lim_{n\to\infty} \alpha_n^{-1} > -1. \tag{5.78}$$

It is interesting to compare the GS-operator $D^c_{X,\alpha}$ with the corresponding Schrödinger operator $H_{X,\alpha}$ with δ-interactions (see formula (5.87) below). Since $\{d_n\}_1^\infty \in l^2(\mathbb{N})$, the self-adjointness of $H_{X,\alpha}$ depends on $\alpha = \{\alpha_n\}_1^\infty$ (see [44, Example 5.12]). Moreover, the pair of discreteness conditions (5.77) for $H_{X,\alpha}$ turn into $\lim_{n\to\infty} n|\alpha_n| = \infty$ and $4 \lim_{n\to\infty} n\alpha_n^{-1} > -1$.

Note that if α is positive, i.e. $\alpha = \{\alpha_n\}_1^\infty \subset \mathbb{R}_+$, both pairs of conditions in (5.71) and (5.77) are reduced to the first common condition $\lim_{n\to\infty} \frac{\alpha_n}{d_n} = \infty$. At the same time, if α is negative conditions (5.71) and (5.77) are quite different. For instance, $H_{X,\alpha}$ is discrete (and self-adjoint) whenever $\alpha_n = -(4+\varepsilon)(n+\frac{1}{2}) + O(\frac{1}{n})$, $\varepsilon > 0$ (see [44, Example 5.12(ii) and Proposition 5.18] $H_{X,\alpha}$). On the other hand, $D^c_{X,\alpha}$ is discrete provided that $\alpha_n = -(4c+\varepsilon) + O(\frac{1}{n})$.

5.2. GS-realizations $D_{X,\beta}$: parametrization by Jacobi matrices and spectral properties

Let $X = \{x_n\}_1^\infty (\subset \mathbb{R}_+)$ be as above and let D_X be the minimal Dirac operator given by (3.28), (3.29). In this section we discuss some spectral properties of GS-realizations $D_{X,\beta}$. We compute the corresponding boundary operator $B_{X,\beta}$ parameterizing $D_{X,\beta}$ in the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for D_X^* constructed in Theorem 3.10, and show that the spectral properties of $D_{X,\beta}$ are similar to that of the GS-operators $D_{X,\alpha}$. In what follows we confine ourselves to the case of operators $D_{X,\beta}$ only, although the most part of the results remains valid for operators $D_{X,\beta}(Q)$ depending on a potential matrix $Q \neq 0$.

Consider the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $D_X^* = \bigoplus_{n=1}^\infty D_n^*$ constructed in Theorem 3.10 (see (3.54) for the definitions of Γ_0 and Γ_1). Since $\beta_n \neq 0$, $n \in \mathbb{N}$, the operator $D_{X,\beta}$ is disjoint with the self-adjoint extension $D_X^* \restriction \ker(\Gamma_0)$ of the minimal Dirac operator D_X. Here Γ_0 and Γ_1 are determined by (3.54). Therefore, the boundary relation Θ parameterizing $D_{X,\beta}$ in the triplet Π is a closed operator, $\Theta \in \mathcal{C}(\mathcal{H})$.

Consider the following Jacobi matrix

$$B_{X,\beta} := \begin{pmatrix} 0 & -\frac{\nu(d_1)}{d_1^2} & 0 & 0 & 0 & 0 & \dots \\ -\frac{\nu(d_1)}{d_1^2} & -\frac{\nu^2(d_1)}{d_1^3}(\beta_1 + d_1) & \frac{\nu(d_1)}{d_1^{3/2} {d_2}^{1/2}} & 0 & 0 & 0 & \dots \\ 0 & \frac{\nu(d_1)}{d_1^{3/2} {d_2}^{1/2}} & 0 & -\frac{\nu(d_2)}{d_2^2} & 0 & 0 & \dots \\ 0 & 0 & -\frac{\nu(d_2)}{d_2^2} & -\frac{\nu^2(d_2)}{d_2^3}(\beta_2 + d_2) & \frac{\nu(d_2)}{d_2^{3/2} d_3^{1/2}} & 0 & \dots \\ 0 & 0 & 0 & \frac{\nu(d_2)}{d_2^{3/2} d_3^{1/2}} & 0 & -\frac{\nu(d_3)}{d_3^2} & \dots \\ 0 & 0 & 0 & 0 & -\frac{\nu(d_3)}{d_3^2} & -\frac{\nu^2(d_3)}{d_3^3}(\beta_3 + d_3) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \end{pmatrix}, \tag{5.79}$$

where $\nu(x) := 1/\sqrt{1 + \frac{1}{c^2 x^2}}$. Note that

$$B_{X,\beta} = R_X^{-1}(\widetilde{B}_\beta - Q_X)R_X^{-1}, \quad \widetilde{B}_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\beta_1 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & -\beta_2 & 1 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{pmatrix}, \tag{5.80}$$

where $R_X = \bigoplus_{n_1}^{\infty} R_n$ and $Q_X = \bigoplus_{n=1}^{\infty} Q_n$ are determined by (3.57).

We also denote by $B_{X,\beta}$ the minimal (closed) Jacobi operator associated in $l^2(\mathbb{N}, \mathbb{C}^2)$ with the Jacobi matrix (5.79). Clearly, $B_{X,\beta}$ is symmetric and according to general properties of Jacobi operators $\mathrm{n}_+(B_{X,\beta}) = \mathrm{n}_-(B_{X,\beta}) \leqslant 1$.

Proposition 5.35. *Let $D_X = \bigoplus_{n=1}^{\infty} D_n$ be the minimal Dirac operator in $L^2(\mathcal{I}, \mathbb{C}^2)$. Let also $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $D_X^* = \bigoplus_{n=1}^{\infty} D_n^*$ constructed in Theorem 3.10 and let $B_{X,\beta}$ be the minimal Jacobi operator associated in $l^2(\mathbb{N}, \mathbb{C}^2)$ with the matrix (5.79). Then the boundary operator corresponding to the GS-realization $D_{X,\beta}$ in the triplet Π, is the Jacobi operator $B_{X,\beta}$, i.e.*

$$D_{X,\beta} = D_{B_{X,\beta}} := D_X^* \upharpoonright \mathrm{dom}(D_{B_{X,\beta}}),$$
$$\mathrm{dom}(D_{B_{X,\beta}}) := \big\{ f \in \mathrm{dom}\big(D_X^*\big) \colon \Gamma_1 f = B_{X,\beta} \Gamma_0 f \big\}. \tag{5.81}$$

Corollary 5.36. *The GS-operator $D_{X,\beta}$ has equal deficiency indices and $\mathrm{n}_+(D_{X,\beta}) = \mathrm{n}_-(D_{X,\beta}) \leqslant 1$. Moreover, $\mathrm{n}_\pm(D_{X,\beta}) = \mathrm{n}_\pm(B_{X,\beta})$. In particular, $D_{X,\beta}$ is self-adjoint if and only if $B_{X,\beta}$ is.*

Proof. The proof is implied by combining Proposition 5.35 with Corollary 3.12(ii) and the known properties of Jacobi matrices [1,10]. □

Proposition 5.37. *Let D_X be the minimal Dirac operator in $L^2(\mathcal{I}, \mathbb{C}^2)$ and let $D_{X,\beta}$ be the GS-realization of D_X.*

(i) *If $|\mathcal{I}| = +\infty$, then $D_{X,\beta}$ is self-adjoint.*
(ii) *If $|\mathcal{I}| < \infty$ and*

$$\sum_{n=1}^{\infty} |\beta_n| \sqrt{d_n d_{n+1}} = +\infty, \tag{5.82}$$

then $D_{X,\beta}$ is self-adjoint.

Proof. (i) By Corollary 5.36, $\mathrm{n}_\pm(D_{X,\beta}) = \mathrm{n}_\pm(B_{X,\beta})$. Alongside the Jacobi matrix $B_{X,\beta}$ of the form (5.79) we consider Jacobi matrix $B'_{X,\beta}$ obtained from $B_{X,\beta}$ by replacing its off-diagonal entries by their modulus (cf. with construction of the matrix $B'_{X,\alpha}$ of the form (5.9)). The matrices $B_{X,\beta}$ and $B'_{X,\beta}$ are unitarily equivalent. Self-adjointness of the operator $B'_{X,\beta}$ follows from the Carleman test. In fact, the proof coincides with that of Proposition 5.5 since the off-diagonal entries of the Jacobi matrices $B'_{X,\beta}$ and $B'_{X,\alpha}$ coincide.

(ii) The proof is similar to that of Proposition 5.7. Applying the Dennis–Wall test (see [1, Chapter 1, Problem 2]) to the Jacobi matrix $B'_{X,\beta}$ yields self-adjointness of $B'_{X,\alpha}$ provided that

$$\sum_{n=1}^{\infty} |d_n + \beta_n| \sqrt{d_n d_{n+1}} = +\infty. \tag{5.83}$$

Since $|\mathcal{I}| < +\infty$, one has $\sum_{n=1}^{\infty} d_n\sqrt{d_n d_{n+1}} < 2|\mathcal{I}|^2 < +\infty$. Thus, the series in (5.83) and the series in (5.82) diverge only simultaneously. $\square$

All previous results on spectral properties of the GS-operator $D_{X,\alpha}$ have their counterparts for the realization $D_{X,\beta}$. They can be proved directly in a much the same way but we prefer another way described as follows.

Alongside the GS-realization $D_{X,\alpha}$ we introduce another GS-realization $\widehat{D}_{X,\alpha}$ being the closure of the operator

$$\widehat{D}^0_{X,\alpha} = D \restriction \operatorname{dom}\big(\widehat{D}^0_{X,\alpha}\big),$$
$$\operatorname{dom}\big(\widehat{D}^0_{X,\alpha}\big) = \bigg\{ f \in W^{1,2}_{\mathrm{comp}}(\mathcal{I}\backslash X) \otimes \mathbb{C}^2\colon\ f_1 \in AC_{\mathrm{loc}}(\mathcal{I}),\ f_2 \in AC_{\mathrm{loc}}(\mathcal{I}\backslash X);$$
$$f_1(a+) = 0,\ f_2(x_n+) - f_2(x_n-) = -\frac{i\alpha_n}{c} f_1(x_n),\ n \in \mathbb{N}\bigg\},$$

i.e. $\widehat{D}_{X,\alpha} = \overline{\widehat{D}^0_{X,\alpha}}$. The following statement is immediate from the previous definitions.

Proposition 5.38. *Let $\alpha = c^2\beta$. Then the realizations $D_{X,\beta}$ and $-\widehat{D}_{X,\alpha}$ are unitarily equivalent. More precisely, the following identity holds*

$$U^{-1} D_{X,\beta} U = -\widehat{D}_{X,\alpha}, \qquad \operatorname{dom}(\widehat{D}_{X,\alpha}) = U^{-1}\operatorname{dom}(D_{X,\beta}), \quad \alpha = c^2\beta,$$

where U is the unitary operator,

$$U : L^2(\mathcal{I}) \otimes \mathbb{C}^2 \to L^2(\mathcal{I}) \otimes \mathbb{C}^2, \quad U := 1 \otimes \sigma_2,$$

and σ_2 is one of the Pauli matrices given by (3.2).

Proposition 5.39. *Let $\alpha = \beta c^2$. Then the GS-realizations $D_{X,\beta}$ and $D_{X,\alpha}$ are self-adjoint only simultaneously. Moreover, the spectrum $\sigma(D_{X,\beta})$ of $D_{X,\beta}$ is either discrete or purely singular if and only if so is the spectrum $\sigma(D_{X,\alpha})$ of $D_{X,\alpha}$.*

Besides, the ac-parts of the operators $D_{X,\alpha}$ and $D_{X,\beta}$ are unitarily equivalent.

Proof. Assume that $D_{X,\alpha}$ is self-adjoint. Then its restriction

$$S := D_{X,\alpha} \restriction \operatorname{dom}(S),$$
$$\operatorname{dom}(S) := \operatorname{dom}(\widehat{D}_{X,\alpha}) \cap \operatorname{dom}(D_{X,\alpha}) = \big\{ f \in \operatorname{dom}(D_{X,\alpha})\colon\ f_1(a+) = 0\big\},$$

is a closed symmetric operator with the deficiency indices $n_\pm(S) = 1$. Therefore, by the second von Neumann formula, $\dim(\operatorname{dom}(\widehat{D}_{X,\alpha})/\operatorname{dom}(S)) = 1$ and $\widehat{D}_{X,\alpha}$ being a symmetric operator is self-adjoint too. Moreover, since $n_\pm(S) = 1$, the resolvent difference

$$(\widehat{D}_{X,\alpha} - z)^{-1} - (D_{X,\alpha} - z)^{-1} \tag{5.84}$$

is rank-one operator. Therefore the operators $\widehat{D}_{X,\alpha}$ and $D_{X,\alpha}$ have either discrete spectrum or purely singular spectrum only simultaneously. Moreover, by the Kato–Rosenblum theorem their *ac*-parts are unitarily equivalent. To complete the proof it remains to apply Proposition 5.38. $\square$

Remark 5.40. Let $\tau : \mathrm{dom}(D_{X,\alpha}) \to \mathbb{C}$ be the trace mapping given by $\tau(f) := f_1(a+)$. Clearly, it is continuous and surjective. Since $\mathrm{dom}(\widehat{D}_{X,\alpha}) \cap \mathrm{dom}(D_{X,\alpha}) = \mathrm{dom}(D_{X,\alpha}) \cap \ker(\tau)$ is dense in $L^2(\mathcal{I})$, the operator $\widehat{D}_{X,\alpha}$ can be treated as a singular perturbation of $D_{X,\alpha}$ in the sense of [64]. This leads to an alternative proof of Proposition 5.39.

Combining Propositions 5.5 and 5.7 with Proposition 5.38 yields an alternative proof of Proposition 5.37. Moreover, using Proposition 5.38 one can obtain the counterparts of the results of Sections 5.4 and 5.5 on spectral properties of $D_{X,\alpha}$.

We demonstrate this possibility by stating the following result on discreteness of $D_{X,\beta}$.

Proposition 5.41. *Let $X = \{x_n\}_1^\infty (\subset \mathbb{R}_+)$, $\beta = \{\beta_j\}_1^\infty \subset \mathbb{R}$. Assume that $\lim_{n\to\infty} d_n = 0$ and*

$$\lim_{n\to\infty} \frac{|\beta_n|}{d_n} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{c\beta_n} > -\frac{1}{4}. \tag{5.85}$$

Then the (self-adjoint) GS-operator $D_{X,\beta}$ on the half-line $\mathbb{R}_+$ has discrete spectrum.

Proof. The statement is immediate by combining Proposition 5.30 with Proposition 5.38. □

5.3. Non-relativistic limit of Gesztesy–Šeba operators

Let, as in Section 3.2, $X = \{x_n\}_{n=1}^\infty (\subset \mathbb{R}_+)$ be a discrete set and $\alpha = \{\alpha_n\}_1^\infty$, $\beta = \{\beta_n\}_1^\infty \subset \mathbb{R}$.

In this section we consider the non-relativistic limits of Gesztesy-Šeba operators $D^c_{X,\alpha} := D_{X,\alpha}$ and $D^c_{X,\beta} := D_{X,\beta}$ using their parameterizations with respect to the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ constructed in Theorem 3.10. By Propositions 5.2 and 5.35,

$$\begin{aligned} D^c_{X,\alpha} &= D_{B^c_{X,\alpha}} := D^*_X \restriction \ker\bigl(\Gamma_1 - B^c_{X,\alpha}\Gamma_0\bigr) \quad \text{and} \\ D^c_{X,\beta} &= D_{B^c_{X,\beta}} := D^*_X \restriction \ker\bigl(\Gamma_1 - B^c_{X,\beta}\Gamma_0\bigr), \end{aligned} \tag{5.86}$$

where $B^c_{X,\alpha}$ and $B^c_{X,\beta}$ are the Jacobi matrices given by (5.5) and (5.79), respectively. Here we indicate explicitly the dependence of all operators on the parameter c by writing $D^c_{X,\alpha}$, $D^c_{X,\beta}$, $B^c_{X,\alpha}$ and $B^c_{X,\beta}$ in place of $D_{X,\alpha}$, $D_{X,\beta}$, $B_{X,\alpha}$ and $B_{X,\beta}$, respectively.

Following [3] we recall the definitions of the operators which describe Schrödinger operators with δ- and δ'-interactions, respectively (cf. also [44, Sections 5, 6]). Let

$$\begin{aligned} H^0_{X,\alpha} &= -\frac{d^2}{dx^2} \restriction \mathrm{dom}\bigl(H^0_{X,\alpha}\bigr), \\ \mathrm{dom}\bigl(H^0_{X,\alpha}\bigr) &= \bigl\{ f \in W^{2,2}_{\mathrm{comp}}(\mathcal{I}\backslash X)\colon\ f \in AC_{\mathrm{loc}}(\mathcal{I}),\ f' \in AC_{\mathrm{loc}}(\mathcal{I}\backslash X); \\ &\qquad f'(a+) = 0,\ f'(x_n+) - f'(x_n-) = \alpha_n f(x_n),\ n \in \mathbb{N} \bigr\}, \end{aligned} \tag{5.87}$$

$$\begin{aligned} H^0_{X,\beta} &= -\frac{d^2}{dx^2} \restriction \mathrm{dom}\bigl(H^0_{X,\beta}\bigr), \\ \mathrm{dom}\bigl(H^0_{X,\beta}\bigr) &= \bigl\{ f \in W^{2,2}_{\mathrm{comp}}(\mathcal{I}\backslash X)\colon\ f \in AC_{\mathrm{loc}}(\mathcal{I}\backslash X),\ f' \in AC_{\mathrm{loc}}(\mathcal{I}); \\ &\qquad f'(a+) = 0,\ f(x_n+) - f(x_n-) = \beta_n f'(x_n),\ n \in \mathbb{N} \bigr\}. \end{aligned} \tag{5.88}$$

Then the operators $H_{X,\alpha}$ and $H_{X,\beta}$ are defined to be the closures of $H^0_{X,\alpha}$ and $H^0_{X,\beta}$, respectively.

If $\mathcal{I} = \mathbb{R}_+$, the operator $H_{X,\beta}$ is self-adjoint in $L^2(\mathbb{R}_+)$ for any β ([15, Theorem 4.7], [44, Theorem 6.3]), although $H_{X,\alpha}$ is only symmetric with equal deficiency indices $n_+(H_{X,\alpha}) = n_-(H_{X,\alpha}) \leqslant 1$

(see [15]). Moreover, $H_{X,\alpha}$ may have non-trivial deficiency indices $n_{\pm}(H_{X,\alpha}) = 1$ (see [68], [44, Section 5.2]). However, the operator $H_{X,\alpha}$ is self-adjoint, $H_{X,\alpha} = H^*_{X,\alpha}$, provided that it is semibounded below ([4, Theorem 1], see also the recent publication [38] for another proof).

It is shown in [44] that certain spectral properties of $H_{X,\alpha}$ closely correlate with the corresponding properties of the following Jacobi matrix

$$B_{X,\alpha}(H) := \begin{pmatrix} 0 & -d_1^{-2} & 0 & 0 & 0 & \dots \\ -d_1^{-2} & -d_1^{-2} & d_1^{-3/2}d_2^{-1/2} & 0 & 0 & \dots \\ 0 & d_1^{-3/2}d_2^{-1/2} & \alpha_1 d_2^{-1} & -d_2^{-2} & 0 & \dots \\ 0 & 0 & -d_2^{-2} & -d_2^{-2} & d_2^{-3/2}d_3^{-1/2} & \dots \\ 0 & 0 & 0 & d_2^{-3/2}d_3^{-1/2} & \alpha_2 d_3^{-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{5.89}$$

As usual we identify the Jacobi matrix $B_{X,\alpha}(H)$ with (closed) minimal symmetric operator associated with it and denote it by the same letter (cf. (5.7)). Recall that $B_{X,\alpha}(H)$ has equal deficiency indices and $\mathrm{n}_{\pm}(B_{X,\alpha}(H)) \leqslant 1$.

The Jacobi matrix $B_{X,\alpha}(H)$ coincides with the matrix $B^{\infty}_{X,\alpha}$ given by (5.5) with $\nu(x) \equiv 1$, i.e. with $c = \infty$. Note however that in the case $\sum_{k\in\mathbb{N}} d_k = \infty$ the matrix $B^c_{X,\alpha}$, $c < \infty$, is always self-adjoint though the matrix $B_{X\alpha}(H)$ might be only symmetric (see [44, Section 5.2]).

Similarly, according to [44] certain spectral properties of $H_{X,\beta}$ closely correlate with the corresponding properties of the Jacobi matrix $B_{X,\beta}(H)$ given by

$$B_{X,\beta}(H) := \begin{pmatrix} 0 & -d_1^{-2} & 0 & 0 & 0 & \dots \\ -d_1^{-2} & -(\beta_1 + d_1)d_1^{-3} & d_1^{-3/2}d_2^{-1/2} & 0 & 0 & \dots \\ 0 & d_1^{-3/2}d_2^{-1/2} & 0 & -d_2^{-2} & 0 & \dots \\ 0 & 0 & -d_2^{-2} & -(\beta_2 + d_2)d_2^{-3} & d_2^{-3/2}d_3^{-1/2} & \dots \\ 0 & 0 & 0 & d_2^{-3/2}d_3^{-1/2} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{5.90}$$

Denote also by $B_{X,\beta}(H)$ the Jacobi matrix defined by (5.79) with $\nu(x) \equiv 1$, i.e. with $c = \infty$.

The Jacobi matrices $B_{X,\alpha}(H)$ and $B_{X,\beta}(H)$ first appeared for the parametrization of Schrödinger operators $H_{X,\alpha}$ and $H_{X,\beta}$ with δ- and δ'-interactions, respectively (cf. [44, Proposition 5.1] and [44, Proposition 6.1]). Let us recall these statements:

Proposition 5.42. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $H^*_{\min}$ constructed in Theorem 4.5. Then the boundary operators corresponding to the realizations $H_{X,\alpha}$ and $H_{X,\beta}$ coincide with the minimal Jacobi operators $B_{X,\alpha} := B_{X,\alpha}(H)$ and $B_{X,\beta} := B_{X,\beta}(H)$, respectively, i.e.*

$$H_{X,\alpha} = H_{B_{X,\alpha}} = H^*_{\min} \upharpoonright \mathrm{dom}(H_{B_{X,\alpha}}), \quad \mathrm{dom}(H_{B_{X,\alpha}}) = \{ f \in W^{2,2}(\mathcal{I} \setminus X)\colon\ \Gamma_1 f = B_{X,\alpha}\Gamma_0 f \},$$
$$H_{X,\beta} = H_{B_{X,\beta}} = H^*_{\min} \upharpoonright \mathrm{dom}(H_{B_{X,\beta}}), \quad \mathrm{dom}(H_{B_{X,\beta}}) = \{ f \in W^{2,2}(\mathcal{I} \setminus X)\colon\ \Gamma_1 f = B_{X,\beta}\Gamma_0 f \}.$$

Moreover, $n_{\pm}(H_{X,\alpha}) = n_{\pm}(B_{X,\alpha}(H)) \leqslant 1$ and $n_{\pm}(H_{X,\beta}) = n_{\pm}(B_{X,\beta}(H)) \leqslant 1$.

Then the following results on the non-relativistic limit hold:

Theorem 5.43. *Let $X = \{x_n\}_1^\infty (\subset \mathbb{R}_+)$ be a discrete set and $\alpha = \{\alpha_n\}_1^\infty$, $\beta = \{\beta_n\}_1^\infty \subset \mathbb{R}$.*

(i) *Assume that $\mathcal{I} = \mathbb{R}_+$ and $H_{X,\alpha}$ is self-adjoint. Then*

$$s - \lim_{c\to+\infty} \left(D^c_{X,\alpha} - \left(z + c^2/2\right)\right)^{-1} = (H_{X,\alpha} - z)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.91}$$

In particular, (5.91) holds provided that $H_{X,\alpha}$ is semibounded below.

(ii) *Assume that $\mathcal{I} = \mathbb{R}_+$. Then the operators $D^c_{X,\beta}$, $c < \infty$, and $H_{X,\beta}$ are self-adjoint and the following relation holds*

$$s - \lim_{c\to+\infty} \left(D^c_{X,\beta} - \left(z + c^2/2\right)\right)^{-1} = (H_{X,\beta} - z)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.92}$$

(iii) *Assume that $\mathcal{I} = (0, b)$ with $b < \infty$. Assume also that*

$$\sum_{n=1}^\infty |\beta_n| \sqrt{d_n d_{n+1}} = +\infty \quad \text{and} \quad \sum_{n=1}^\infty \left(d_{n+1} \left| \sum_{i=1}^n (\beta_i + d_i) \right|^2 \right) = \infty. \tag{5.93}$$

Then relation (5.92) holds.

Proof. (i) Firstly, by Theorem 5.9, the operator $D^c_{X,\alpha}$, $c < \infty$, is self-adjoint for any $\alpha \subset \mathbb{R}$ and any $c > 0$ since $\mathcal{I} = \mathbb{R}_+$. Therefore, by Proposition 5.2, $B^c_{X,\alpha}$, $c < \infty$, is self-adjoint too. Further, let $B_{X,\alpha}(H)$ be the minimal Jacobi operator associated with the matrix (5.89). By Proposition 5.42, $H_{X,\alpha} = H_{X,B_{X,\alpha}}$. Moreover, by Proposition 5.42, $B_{X,\alpha}(H) = B_{X,\alpha}(H)^*$ since $H_{X,\alpha} = H^*_{X,\alpha}$. Combining (5.5), (5.6) and (5.89) with the obvious relation $\lim_{c\to\infty} \nu(cx) = 1$ we get

$$\lim_{c\to+\infty} B^c_{X,\alpha} h = B_{X,\alpha}(H) h \quad \text{for all } h \in l^2_0\left(\mathbb{N}, \mathbb{C}^2\right). \tag{5.94}$$

Note also that, by the definition of a minimal Jacobi operator, $l^2_0(\mathbb{N}, \mathbb{C}^2)$ is a core for both (self-adjoint) Jacobi operators $B_{X,\alpha}(H)$ and $B^c_{X,\alpha}$, $c \in (0, \infty)$. Applying Theorem 4.8 we arrive at the relation (5.91). To complete the proof it remains to note that $H_{X,\alpha}$ is self-adjoint, $H_{X,\alpha} = H^*_{X,\alpha}$, provided that it is semibounded below (see [4, Theorem 1]).

(ii) Let $\mathcal{I} = \mathbb{R}_+$. By [15] (see also [44, Theorem 6.3(i)]), $H_{X,\beta} = H^*_{X,\beta}$. Combining this relation with Proposition 5.2, yields $B_{X,\beta}(H) = B_{X,\beta}(H)^*$.

On the other hand, by Proposition 5.37(i), $D^c_{X,\beta}$ is self-adjoint too, $D^c_{X,\beta} = (D^c_{X,\beta})^*$, $c < \infty$. Further, by Proposition 5.35, $D^c_{X,\beta} = D^c_{B_{X,\beta}} = D^*_X \restriction \ker(\Gamma_1 - B^c_{X,\beta}\Gamma_0)$. Therefore, by Proposition 3.12(ii), $B^c_{X,\beta}$ is self-adjoint too, $B^c_{X,\beta} = (B^c_{X,\beta})^*$. Note that alongside (5.94) we obtain from (5.79) and (5.90) a similar relation $\lim_{c\to+\infty} B^c_{X,\beta} h = B_{X,\beta}(H)h$, $h \in l^2_0(\mathbb{N}, \mathbb{C}^2)$.

Again, by the definition of the minimal Jacobi operators $B^c_{X,\beta}$ and $B_{X,\beta}(H)$, $l^2_0(\mathbb{N}, \mathbb{C}^2)$ is a core for both of them. To arrive at (5.92) it remains to apply Theorem 4.8.

(iii) Let $|\mathcal{I}| < \infty$. By Proposition 5.37(ii), self-adjointness of the GS-operators $D^c_{X,\beta}$, $c \in (0, \infty)$, is implied by the first of conditions (5.93). Combining this fact with Propositions 5.35 and 3.12(ii), we get $B^c_{X,\beta} = (B^c_{X,\beta})^*$, $c < \infty$. Further, by [44, Theorem 6.3(ii)], the second of conditions (5.93) yields $H_{X,\beta} = H^*_{X,\beta}$. In turn, by Proposition 5.42, $B_{X,\beta}(H) = B_{X,\beta}(H)^*$. The proof is completed in much the same way as the proof of statement (i). □

Remark 5.44.

(i) Note that condition $H_{X,\alpha} = H^*_{X,\alpha}$ hence the conclusion (5.91) of Theorem 5.43 is satisfied for any sequence $\alpha = \{\alpha_n\}_1^\infty \subset \mathbb{R}$ provided that $\sum_j d_j^2 = \infty$ [44, Proposition 5.7] (see recent publications

[40,62] for other proofs and generalizations). In particular, the non-relativistic limit (5.91) is valid whenever $d_*(X) > 0$. Under the latter assumption both statements (5.91) and (5.92) were originally obtained by Gesztesy and Šeba [30] (see also [3, Appendix J]). In this case the uniform convergence in (5.91), (5.92) holds.

(ii) Clearly, conditions (5.93) can be replaced by the assumptions of self-adjointness of operators $D^c_{X,\beta}$, $c > 0$, and $H_{X,\beta}$.

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