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Existence and concentration of solutions for the Schrödinger–Poisson equations with steep well potential

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ABSTRACT

In this paper we study a system of Schrödinger–Poisson equations

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter, $2 < p < 6$. Under suitable assumptions on V and K, the existence of nontrivial solution and concentration results are obtained via variational methods. In particular, the potential V is allowed to be sign-changing for the case $p \in (4, 6)$.

1. Introduction

In this paper, we consider the following system of Schrödinger–Poisson equations

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $2 < p < 6$. This system has been first introduced in [6] as a physical model describing a charged wave interacting with its own electrostatic field in quantum mechanic. The unknowns u and ϕ represent the wave functions associated to the particle and electric potential, and functions V and K are respectively an external potential and nonnegative density charge. We refer to [6] and the references therein for more physical background.

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On the potential V, we make the following assumptions:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and V is bounded from below.
(V_2) There exists $b > 0$ such that the set $\{x \in R^3\colon\ V(x) < b\}$ is nonempty and has finite measure.
(V_3) $\Omega = int\, V^{-1}(0)$ is nonempty and has smooth boundary and $\overline{\Omega} = V^{-1}(0)$.

This kind of hypotheses was first introduced by Bartsch and Wang [4] in the study of a nonlinear Schrödinger equation and the potential $\lambda V(x)$ with V satisfying (V_1)–(V_3) is referred as the steep well potential.

In recent years, system (1.1) has been widely studied under variant assumptions on V and K. The greatest part of the literature focuses on the study of the system for V and K being constants or radially symmetric functions, and existence, nonexistence and multiplicity results are obtained in many papers, see e.g. [1,2,9–11,22,25,27]. Recently, the case of a positive and non-radial potential V has been studied in [32] for asymptotically linear nonlinearity, in [3] for $p \in (4, 6)$ and in [36] for $p \in (3, 4)$. In [3,36], it was assumed that

($\widetilde{V}$) $V_\infty = \lim_{|y|\to+\infty} V(y) \geqslant V(x)$ for a.e. $x \in \mathbb{R}^3$, and the strict inequality holds on a positive measure set.

We note that ($\widetilde{V}$) is crucial to use concentration compactness method as in [3,36].

Very recently, under the assumptions (V_1)–(V_3) and $V \geqslant 0$, Jiang and Zhou [21] considered a similar problem

$$\begin{cases} -\Delta u + \big(\lambda V(x) + 1\big)u + \mu\phi u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

with $\lambda > 0$, $\mu > 0$, and obtained the existence and concentration results for $p \in (2, 3) \cup [4, 6)$ by combining domains approximation with priori estimates. In [8], Cerami and Vaira studied system (1.1) with $p \in (4, 6)$, $\lambda V(x) \equiv 1$ and $K \in L^2(\mathbb{R}^3)$ satisfying $K(x) \to 0$ as $|x| \to \infty$. They proved the existence of positive ground state by minimization on Nehari manifold and concentration compactness method.

Moreover, the semiclassical limit of the system (1.1) was also discussed recently. More precisely, replacing $-\triangle$ by $-\varepsilon\triangle$, Ruiz [28] and D'Aprile and Wei [11] showed that system (1.1) with $\lambda V(x) \equiv K(x) \equiv 1$ possesses a family of solutions concentrating around a sphere when $\varepsilon \to 0$ for $p \in (2, 18/7)$. Their results were generalized in [15,16] for the radial V and K. In [30], Ruiz and Vaira proved the existence of multi-bump solutions whose bumps concentrated around local minimums of the potential V. The proofs explored in [15,16,30] are based on a singular perturbation, essentially a Lyapunov–Schmitt reduction method. For other concentration phenomena for this system, see for example [1,14,17,23,29,34,35] and the references therein.

Motivated by the above works, in the present paper we consider system (1.1) with more general potential V, K and the range of p. The existence and concentration of nontrivial solutions of system (1.1) are established via variational method. Let $H^1(\mathbb{R}^3)$ and $D^{1,2}(\mathbb{R}^3)$ be the usual Sobolev spaces and denote by $|\cdot|_s$ the usual norm of $L^s(\mathbb{R}^3)$ for $s \in [2, \infty]$. Our main results are as follows:

Theorem 1.1. *Assume that $4 < p < 6$, (V_1)–(V_3) are satisfied and $K(x) > 0$ for $x \in \mathbb{R}^3$, $K(x) \in L^2(\mathbb{R}^3)$ (or $K(x) \in L^\infty(\mathbb{R}^3)$). Then there exist $\Lambda > 0$ and $k_\lambda^* > 0$, for $\lambda > \Lambda$, such that problem (1.1) has at least a nontrivial solution $(u_\lambda, \phi_\lambda)$ in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ for $\lambda > \Lambda$ and $|K|_2 < k_\lambda^*$ (or $|K|_\infty < k_\lambda^*$).*

Remark 1.1. In Theorem 1.1, V is allowed to be sign-changing for $p \in (4, 6)$. The positivity of the infimum of the potential $V_\lambda(x) := \lambda V(x) + 1$ is important in the arguments of the paper [21], and it is possible that $\liminf_{|x|\to\infty} V(x) = 0$ in our settings, so our results extend the corresponding one in [21]. To the best of our knowledge, it seems that the only earlier work on (1.1) with sign-changing potential is the paper [20], where different assumptions were used and existence result was obtained by using sub-supersolution method. The works on nonlinear Schrodinger equation with sign-changing potential can be found in [12,13] by Ding and Szulkin.

If $V \geqslant 0$, the restriction on the norm of K can be removed and we have the following theorem.

Theorem 1.2. *Assume that* $4 \leqslant p < 6$, (V_1)–(V_3) *are satisfied and* $V(x) \geqslant 0$, $K(x) \geqslant 0$ *for* $x \in \mathbb{R}^3$ *and* $K(x) \in L^2(\mathbb{R}^3)$ *(or* $K(x) \in L^\infty(\mathbb{R}^3)$*), if* $p = 4$, *assume moreover* $K(x) \in L^2(\mathbb{R}^3) \cap L^\infty_{loc}(\mathbb{R}^3)$ *(or* $K(x) \in L^\infty(\mathbb{R}^3)$*). Then there exists* $\Lambda > 0$ *such that problem* (1.1) *has at least a nontrivial solution* $(u_\lambda, \phi_\lambda)$ *in* $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *for all* $\lambda > \Lambda$.

For the case $3 < p < 4$, we need more assumptions on V and K.

Theorem 1.3. *Assume that* $3 < p < 4$, (V_1)–(V_3) *are satisfied and* $V \geqslant 0$, $K(x) \geqslant 0$ *for* $x \in \mathbb{R}^3$ *and* $K(x) \in L^2(\mathbb{R}^3) \cap L^\infty_{loc}(\mathbb{R}^3)$. *Assume moreover*

(V_4) $V(x)$ *is weakly differentiable function such that* $(\nabla V(x), x) \in L^{r_1}(\mathbb{R}^3)$ *for some* $r_1 \in [3/2, \infty]$, *and*

$$2V(x) + \big(\nabla V(x), x\big) \geqslant 0, \quad \textit{for a.e. } x \in \mathbb{R}^3.$$

(K_1) $K(x)$ *is weakly differentiable function such that* $(\nabla K(x), x) \in L^{r_2}(\mathbb{R}^3)$ *for some* $r_2 \in [2, \infty]$, *and*

$$\frac{2(p-3)}{p} K(x) + \big(\nabla K(x), x\big) \geqslant 0, \quad \textit{for a.e. } x \in \mathbb{R}^3,$$

hold, where $(\cdot,\cdot)$ *is the usual inner product in* $\mathbb{R}^3$. *Then there exists* $\Lambda > 0$ *such that problem* (1.1) *has at least a nontrivial solution* $(u_\lambda, \phi_\lambda)$ *in* $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *for all* $\lambda > \Lambda$.

Corollary 1.4. *Under the assumptions of Theorem* 1.3 *with* (V_4) *replaced by*

(V_4') $V(x)$ *is weakly differentiable function such that* $(\nabla V(x), x) \in L^{r_1}(\mathbb{R}^3)$ *for some* $r_1 \in [3/2, \infty]$, *and*

$$\big(\nabla V(x), x\big) \in L^{p/(p-2)}\big(\mathbb{R}^3\big) \quad \textit{or} \quad 2V(x) + \big(\nabla V(x), x\big) \in L^{p/(p-2)}\big(\mathbb{R}^3\big).$$

Then the conclusion of Theorem 1.3 *holds.*

Remark 1.2. The condition (V_4) was introduced in [36] to get a special bounded (PS) sequence by the monotonicity trick of Jeanjean [18], since it is hard to obtain the boundness of every (PS) sequence when $p \in (3, 4)$. Similar method was used in [34]. The condition (V_4') is a different type one from (V_4).

On the concentration of solutions we have the following result.

Theorem 1.5. *Let* $(u_\lambda, \phi_\lambda)$ *be the solutions obtained in Theorem* 1.2 *or Theorem* 1.3. *Then* $u_\lambda \to \bar{u}$ *in* $H^1(\mathbb{R}^3)$, $\phi_\lambda \to \phi_{\bar{u}}$ *in* $D^{1,2}(\mathbb{R}^3)$, *as* $\lambda \to \infty$, *where* $\bar{u} \in H^1_0(\Omega)$ *is a nontrivial solution of*

$$\begin{cases} -\Delta u + \dfrac{1}{4\pi}\left(\left(K(x)u^2\right) * \dfrac{1}{|x|}\right)K(x)u = |u|^{p-2}u, & \textit{in } \Omega, \\ u = 0, & \textit{on } \partial\Omega. \end{cases} \tag{1.2}$$

To obtain our results, we have to overcome several difficulties in using variational method. The main difficulty consists in the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, $p \in (2,6)$. Since we assume neither that the potential is radially symmetric nor the condition $(\widetilde{\mathrm{V}})$, we cannot use the usual way to recover compactness, for example, restricting in the subspace $H_r^1(\mathbb{R}^3)$ of radially symmetric functions or using concentration compactness methods. To recover the compactness, we borrow some ideas used in [5,13] and establish the parameter dependent compactness conditions. Let us point out that the adaptation of the ideas to the procedure of our problem is not trivial at all, because of the presence of the nonlocal term $K(x)\phi u$.

On the other hand, the situation is more delicate when dealing with the case $p \in (3,4)$. First, it is known that it is difficult to get the boundedness of a (PS) sequence in this case. To overcome the difficulty, motivated by [19,22], we use Jeanjean's monotonicity trick [18] (see also Struwe [31]) to construct a special (PS) sequence. Moreover, in the process of proving the convergence of a bounded (PS) sequence, we use the observation that the condition $K \in L^2(\mathbb{R}^3)$ makes the less strong influence of the nonlocal term, see Lemma 2.1.

Throughout the paper, by C we mean a positive constant that may vary from line to line but remains independent of the relevant quantities. We denote by "$\rightharpoonup$" weak convergence and by "$\to$" strong convergence.

The paper is organized as follows. In Section 2, we give the variational setting for (1.1) and establish the compactness conditions. In Section 3, we consider the case $4 < p < 6$ and prove Theorem 1.1 and Theorem 1.2. Section 4 is devoted to dealing with the case $3 < p < 4$ and the proof of Theorem 1.3 and Corollary 1.4. In the last section, we study the concentration of solutions and prove Theorem 1.5.

2. Variational setting and compactness condition

In this section, we give the variational setting for (1.1) following [13] and establish the compactness conditions.

Let $V(x) = V^+(x) - V^-(x)$, where $V^\pm(x) = \max\{\pm V(x), 0\}$. Let

$$E = \left\{ u \in H^1\left(\mathbb{R}^3\right): \int_{\mathbb{R}^3} V^+(x)u^2\,dx < \infty \right\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^3} \left(\nabla u \nabla v + V^+(x)uv\right)dx, \qquad \|u\| = \langle u, u\rangle^{1/2}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^3} \left(\nabla u \nabla v + \lambda V^+(x)uv\right)dx, \qquad \|u\|_\lambda = \langle u, u\rangle_\lambda^{1/2}.$$

It is clear that $\|u\| \leqslant \|u\|_\lambda$ for $\lambda \geqslant 1$. Set $E_\lambda = (E, \|\cdot\|_\lambda)$. It follows from (V_1)–(V_2) and Poincaré inequality that the embedding $E \hookrightarrow H^1(\mathbb{R}^3)$ is continuous. Thus for each $s \in [2,6]$, there exists $d_s > 0$ (independent of $\lambda \geqslant 1$) such that

$$|u|_s \leqslant d_s\|u\| \leqslant d_s\|u\|_\lambda, \quad \text{for } u \in E. \tag{2.1}$$

Let

$$F = \left\{ u \in E: \mathit{supp}\, u \subset V^{-1}\left([0,\infty)\right)\right\}$$

and denote the orthogonal complement of F in E_λ by $F_\lambda^\perp$. If $V \geqslant 0$, then $E = F$, otherwise $F_\lambda^\perp \neq \{0\}$. Let $A_\lambda = -\Delta + \lambda V$, then A_λ is formally self-adjoint in $L^2(\mathbb{R}^3)$ and the associated bilinear form

$$a_\lambda(u, v) = \int_{\mathbb{R}^3} \big(\nabla u \nabla v + \lambda V(x) u v\big)\, dx$$

is continuous in E_λ. For fixed $\lambda > 0$, consider the eigenvalue problem

$$-\Delta u + \lambda V^+(x) u = \alpha \lambda V^-(x) u, \quad u \in F_\lambda^\perp. \tag{2.2}$$

Since *supp* V^- is of finite measure, we see that the quadratic form $u \mapsto \int_{\mathbb{R}^3} \lambda V^-(x) u^2\, dx$ is weakly continuous. Hence following Theorem 4.45 and Theorem 4.46 in [33], there exists a sequence of positive eigenvalues $\{\alpha_j(\lambda)\}$, which may be characterized by

$$\alpha_j(\lambda) = \inf_{\dim M \geqslant j,\, M \subset F_\lambda^\perp} \sup\bigg\{\|u\|_\lambda^2\colon u \in M, \int_{\mathbb{R}^3} \lambda V^-(x) u^2\, dx = 1\bigg\}, \quad j = 1, 2, 3, \ldots.$$

Moreover, $\alpha_1(\lambda) \leqslant \alpha_2(\lambda) \leqslant \cdots \leqslant \alpha_j(\lambda) \to \infty$ as $j \to \infty$, and the corresponding eigenfunctions e_j, which may be chosen so that $\langle e_i, e_j\rangle = \delta_{i,j}$, are a basis for $F_\lambda^\perp$. Let

$$\widehat{E}_\lambda = span\big\{e_j\colon \alpha_j(\lambda) \leqslant 1\big\} \quad \text{and} \quad E_\lambda^+ = span\big\{e_j\colon \alpha_j(\lambda) > 1\big\}.$$

Then $E_\lambda = \widehat{E}_\lambda \oplus E_\lambda^+ \oplus F$ is an orthogonal decomposition, $\dim \widehat{E}_\lambda < \infty$, the quadratic form a_λ is negative semidefinite on $\widehat{E}_\lambda$, positive definite on $E_\lambda^+ \oplus F$ and it is easy to see that $a_\lambda(u, v) = 0$ if u, v are in different subspaces of the above decomposition of E_λ.

It is well known that problem (1.1) can be reduced to a single equation with a nonlocal term. Actually, for each $u \in E \subset H^1(\mathbb{R}^3)$, the linear functional T_u in $D^{1,2}(\mathbb{R}^3)$ defined by

$$T_u(v) = \int_{\mathbb{R}^3} K(x) u^2 v\, dx$$

is continuous. In fact, if $K \in L^\infty(\mathbb{R}^3)$, Hölder inequality and Sobolev inequality yield that there is a constant $C > 0$ such that

$$\big|T_u(v)\big| \leqslant |K|_\infty \big|u^2\big|_{6/5} |v|_6 \leqslant C|K|_\infty |u|_{12/5}^2 \|v\|_D, \tag{2.3}$$

while for $K \in L^2(\mathbb{R}^3)$, we have

$$\big|T_u(v)\big| \leqslant |K|_2 \big|u^2\big|_3 |v|_6 \leqslant C|K|_2 |u|_6^2 \|v\|_D. \tag{2.4}$$

It follows from the Lax–Milgram theorem that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta \phi_u = K(x) u^2, \tag{2.5}$$

and ϕ_u can be represented by

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|}\, dy.$$

By (2.3), it is easy to see that, if $K \in L^\infty(\mathbb{R}^3)$,

$$\|\phi_u\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2\, dx \leqslant C|K|_\infty^2 |u|_{12/5}^4. \tag{2.6}$$

Similarly, if $K \in L^2(\mathbb{R}^3)$, (2.4) implies that

$$\|\phi_u\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2\, dx \leqslant C|K|_2^2 |u|_6^4. \tag{2.7}$$

It can be proved that $(u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in E$ is a critical point of the functional $I_\lambda : E \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u u^2\, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p\, dx$$

and $\phi = \phi_u$. We refer the readers to [10] for the details.

Set

$$N(u) = \int_{\mathbb{R}^3} K(x)\phi_u u^2\, dx = \frac{1}{4\pi} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{K(x)K(y)u^2(x)u^2(y)}{|x-y|}\, dx\, dy. \tag{2.8}$$

Now we give some properties about the functional N.

Lemma 2.1.

(i) *Let $K \in L^\infty(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then*

$$\phi_{u_n} \rightharpoonup \phi_u \quad \text{in } D^{1,2}\left(\mathbb{R}^3\right),$$

in particular, $N(u) \leqslant \liminf_{n\to\infty} N(u_n)$.

(ii) *Let $K \in L^2(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then up to a subsequence,*

$$\phi_{u_n} \to \phi_u \quad \text{in } D^{1,2}\left(\mathbb{R}^3\right),$$

in particular, $N(u) = \lim_{n\to\infty} N(u_n)$.

Proof. (i) The proof was given in [10] for $K \equiv 1$ and it is easy to show the conclusion for $K \in L^\infty(\mathbb{R}^3)$ by following the same method in [10].

(ii) We modify the proof in [8]. First, we show that $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$. Indeed, for any $v \in C_0^\infty(\mathbb{R}^3)$, since u_n is bounded in $H^1(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$, by Hölder inequality we have

$$(\phi_{u_n}, v)_{D^{1,2}} - (\phi_u, v)_{D^{1,2}} = \int_{\mathbb{R}^3} \big(K(x)u_n^2 v - K(x)u^2 v\big)\,dx$$
$$\leqslant |v|_\infty |K|_2 |u_n + u|_6 |u_n - u|_{L^3(\Omega v)}$$
$$\leqslant C|u_n - u|_{L^3(\Omega v)} \to 0$$

where Ωv is the support set of v. This implies that $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ by the density of $C_0^\infty(\mathbb{R}^3)$ in $D^{1,2}(\mathbb{R}^3)$.

To complete the proof, it suffices to show that, up to a subsequence,

$$\|\phi_{u_n}\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2\,dx \to \|\phi_u\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx \tag{2.9}$$

as $n \to \infty$. First, by Sobolev embedding, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ implies that

$$\phi_{u_n} \rightharpoonup \phi_u \quad \text{in } L^6\big(\mathbb{R}^3\big),$$

and then

$$\int_{\mathbb{R}^3} K(x)u^2(\phi_{u_n} - \phi_u)\,dx \to 0, \tag{2.10}$$

since $K(x)u^2 \in L^{6/5}(\mathbb{R}^3)$ by Hölder inequality and the fact that $K \in L^2(\mathbb{R}^3)$. Furthermore, from $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, we can assume that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } L^s\big(\mathbb{R}^3\big), \qquad u_n \to u \quad \text{in } L^s_{loc}\big(\mathbb{R}^3\big), \quad \text{for } 2 \leqslant s < 6.$$

Thus, by Hölder inequality, we have

$$\int_{\mathbb{R}^3} \big(K(x)\phi_{u_n} u_n^2 - K(x)\phi_{u_n} u^2\big)\,dx = \int_{\mathbb{R}^3} K(x)\phi_{u_n}\big(u_n^2 - u^2\big)\,dx$$
$$\leqslant |\phi_{u_n}|_6 |u_n + u|_6 \bigg(\int_{\mathbb{R}^3} \big|K(x)(u_n - u)\big|^{3/2}\,dx\bigg)^{2/3}$$
$$\leqslant C\bigg(\int_{\mathbb{R}^3} \big|K(x)(u_n - u)\big|^{3/2}\,dx\bigg)^{2/3} \to 0 \tag{2.11}$$

as $n \to \infty$, since the sequence $w_n := (u_n - u)^{3/2} \rightharpoonup 0$ in $L^4(\mathbb{R}^3)$ and $K(x)^{3/2} \in L^{4/3}(\mathbb{R}^3)$. Thus, by (2.10) and (2.11), we obtain (2.9). The proof is complete. □

In [36], it was shown that the functional N and its derivative N' possess *BL-splitting property*, which is similar to Brezis–Lieb Lemma [7].

Lemma 2.2. *Let $K \in L^\infty(\mathbb{R}^3) \cup L^2(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then as $n \to \infty$,*

(i) *$N(u_n - u) = N(u_n) - N(u) + o(1)$;*
(ii) *$N'(u_n - u) = N'(u_n) - N'(u) + o(1)$, in $H^{-1}(\mathbb{R}^3)$.*

Proof. The proof was given in [36] for $K \equiv 1$ and it is easy to show the conclusion for $K \in L^\infty(\mathbb{R}^3)$ by following the same method in [36]. In what follows, we give the proof for $K \in L^2(\mathbb{R}^3)$.

(i) It is a straight consequence of Lemma 2.1(ii).

(ii) Let $v_n = u_n - u$, it suffices to show that

$$\int_{\mathbb{R}^3} K(x)\phi_{v_n} v_n h\,dx \to 0 \tag{2.12}$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n h - K(x)\phi_u u h\,dx \to 0 \tag{2.13}$$

uniformly for $h \in E$ with $\|h\| \leqslant 1$ as $n \to \infty$. In fact, similar to (2.11), we have

$$\int_{\mathbb{R}^3} K(x)\phi_{v_n} v_n h\,dx \leqslant |\phi_{v_n}|_6 |h|_6 \left(\int_{\mathbb{R}^3} \left|K(x)v_n\right|^{3/2} dx\right)^{2/3} \leqslant C\|h\| \left(\int_{\mathbb{R}^3} \left|K(x)v_n\right|^{3/2} dx\right)^{2/3} \to 0,$$

which implies (2.12). Analogously,

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n - u)h\,dx \leqslant C\|h\| \left(\int_{\mathbb{R}^3} \left|K(x)(u_n - u)\right|^{3/2} dx\right)^{2/3} \to 0. \tag{2.14}$$

Similar to (2.10), we have

$$\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)uh\,dx \leqslant C\|h\| \left(\int_{\mathbb{R}^3} \left|K(x)u\right|^{6/5} (\phi_{u_n} - \phi_u)^{6/5}\,dx\right)^{5/6} \to 0, \tag{2.15}$$

since the sequence $(\phi_{u_n} - \phi_u)^{6/5} \rightharpoonup 0$ in $L^5(\mathbb{R}^3)$ and $|K(x)u|^{6/5} \in L^{5/4}(\mathbb{R}^3)$ by Hölder inequality and the fact that $K \in L^2(\mathbb{R}^3)$. Now (2.14) and (2.15) yield (2.13). The proof is complete. □

Next, we investigate the compactness conditions for the functional I_λ. Recall that a C^1 functional J satisfies Cerami condition at level c ($(\mathrm{C})_c$ condition for short) if any sequence $\{u_n\} \subset E$ such that $J(u_n) \to c$ and $(1 + \|u_n\|)J'(u_n) \to 0$ has a convergent subsequence, and such sequence is called a $(\mathrm{C})_c$ sequence.

Lemma 2.3. *Let $4 < p < 6$ and (V_1)–(V_2) be satisfied. Then every $(\mathrm{C})_c$ sequence of I_λ is bounded in E_λ for each $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(\mathrm{C})_c$ sequence of I_λ, that is

$$I_\lambda(u_n) \to c, \qquad \bigl(1 + \|u_n\|_\lambda\bigr) I'_\lambda(u_n) \to 0 \quad \text{in } E_\lambda^{-1}. \tag{2.16}$$

Thus, for n large enough,

$$I_\lambda(u_n) - \frac{1}{p}\langle I'_\lambda(u_n), u_n\rangle = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 - \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^3} \lambda V^-(x)u_n^2\,dx + \left(\frac{1}{4} - \frac{1}{p}\right)N(u_n)$$
$$\leqslant c + 1. \tag{2.17}$$

Since $4 < p < 6$, it follows from (V_1) and (2.17) that there exists $C > 0$ such that

$$\|u_n\|_\lambda^2 \leqslant C \int_{\mathbb{R}^3} \lambda u_n^2\,dx + (c+1)\frac{2p}{p-2}.$$

Thus it suffices to show that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^3)$. Assume by contradiction that $|u_n|_2 \to \infty$ as $n \to \infty$. Let $v_n = u_n/|u_n|_2$, then $|v_n|_2 = 1$. By (2.17), we have

$$\|v_n\|_\lambda^2 - \lambda \int_{\mathbb{R}^3} V^-(x)v_n^2\,dx + N(v_n)|u_n|_2^2 \leqslant \frac{C}{|u_n|_2^2}, \tag{2.18}$$

which implies that $\|v_n\|_\lambda$ and $N(v_n)|u_n|_2^2$ are both bounded. Passing to a subsequence, we can assume that

$$v_n \rightharpoonup v \quad \text{in } E_\lambda; \qquad v_n \to v \quad \text{a.e. in } \mathbb{R}^3; \qquad v_n \to v \quad \text{in } L^s_{loc}\big(\mathbb{R}^3\big),\ \text{for } 2 \leqslant s < 6.$$

By (2.18), we have

$$\lambda \int_{\mathbb{R}^3} V(x)v_n^2\,dx \leqslant \frac{C}{|u_n|_2^2} \to 0 \tag{2.19}$$

as $n \to \infty$. On the other hand, by (2.18) and Fatou's Lemma

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{K(x)K(y)v^2(x)v^2(y)}{|x-y|}\,dx\,dy \leqslant \liminf_{n\to\infty} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{K(x)K(y)v_n^2(x)v_n^2(y)}{|x-y|}\,dx\,dy$$
$$\leqslant \liminf_{n\to\infty} \frac{C}{|u_n|_2^4} = 0.$$

Hence $v = 0$, since $K(x) > 0$ for $x \in \mathbb{R}^3$. By (V_2), for any given $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $meas(B^c_{R_\varepsilon}(0) \cap \{V(x) < b\}) < \varepsilon$, where $B_{R_\varepsilon}(0) = \{x \in \mathbb{R}^3\colon\ |x| \leqslant R_\varepsilon\}$, $B^c_{R_\varepsilon}(0) = \mathbb{R}^3 \setminus B_{R_\varepsilon}(0)$. Therefore, for n large

$$\int_{\{V(x)<b\}} V(x)v_n^2\,dx \leqslant \int_{B_{R_\varepsilon}(0)\cap\{V(x)<b\}} bv_n^2\,dx + \int_{B^c_{R_\varepsilon}(0)\cap\{V(x)<b\}} bv_n^2\,dx$$
$$\leqslant \varepsilon + b|v_n|_{2s}^2\, meas\big(B^c_{R_\varepsilon}(0) \cap \big\{V(x) < b\big\}\big)^{(s-1)/s}$$
$$\leqslant C\varepsilon, \tag{2.20}$$

for some $s \in (1,3)$. Thus, by (2.20) and the fact that $|v_n|_2^2 = 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} V(x)v_n^2\,dx &= \int_{\{V(x)\geqslant b\}} V(x)v_n^2\,dx + \int_{\{V(x)<b\}} V(x)v_n^2\,dx \\
&= \int_{\{V(x)\geqslant b\}} V(x)v_n^2\,dx + o(1) \\
&\geqslant b \int_{\{V(x)\geqslant b\}} v_n^2\,dx + o(1) \\
&\geqslant b\left(1 - \int_{\{V(x)<b\}} v_n^2\,dx\right) + o(1) \\
&= b + o(1) > 0,
\end{aligned}$$

which contracts (2.19). The proof is complete. □

With the aid of Lemma 2.2, we obtain the following parameter dependent compactness condition for $4 \leqslant p < 6$, $V \geqslant 0$, see Lemma 2.4. However, for the case that V is sign-changing or the case $3 < p < 4$, the situation is more delicate. In these cases, we only have weak version of compactness conditions, applied to a bounded (PS) sequence, see Corollaries 2.5–2.6.

Lemma 2.4. *Let $4 \leqslant p < 6$, (V_1)–(V_2) and $V \geqslant 0$ be satisfied, $K(x) \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$. Then for any $M > 0$, there exists $\Lambda = \Lambda(M) > 0$ such that I_λ satisfies $(\mathrm{C})_c$ condition for all $c < M$, $\lambda > \Lambda$.*

Proof. We give the proof for the case $K \in L^2(\mathbb{R}^3)$, and the proof is similar for the case $K \in L^\infty(\mathbb{R}^3)$. Let $\{u_n\}$ be a $(\mathrm{C})_c$ sequence with $c \leqslant M$. By Lemma 2.3, $\{u_n\}$ is bounded in E_λ and there exists C_λ such that $\|u_n\|_\lambda \leqslant C_\lambda$ (if $p = 4$, this follows from (2.17) since $V \geqslant 0$). Therefore, up to a subsequence, we can assume that

$$u_n \rightharpoonup u \quad \text{in } E_\lambda; \qquad u_n \to u \quad \text{in } L^s_{loc}\big(\mathbb{R}^3\big), \text{ for } 2 \leqslant s < 6.$$

Firstly, it is easy to check that $I'_\lambda(u) = 0$. In fact, by (2.7), we have

$$\phi_{u_n} \rightharpoonup \phi_u \quad \text{in } L^6\big(\mathbb{R}^3\big).$$

For any $\varphi \in C_0^\infty(\mathbb{R}^3)$, by Hölder inequality and $K \in L^2(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \big|K(x)u\varphi\big|^{6/5}\,dx \leqslant |\varphi|_\infty^{6/5}|K|_2^{6/5}|u|_3^{6/5},$$

that is $K(x)u\varphi \in L^{6/5}(\mathbb{R}^3)$. Thus

$$\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)u\varphi\,dx \to 0, \quad \text{as } n \to \infty.$$

Similarly, by Hölder inequality and (2.7), we have

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n-u)\varphi\,dx &\leqslant |\varphi|_\infty |K|_2|\phi_{u_n}|_6|u_n-u|_{L^3(\Omega_\varphi)}\\ &\leqslant C|\varphi|_\infty|K|_2^2|u_n-u|_{L^3(\Omega_\varphi)}\\ &\to 0\end{aligned}$$

as $n\to\infty$, where Ω_φ is the support set of φ. Consequently,

$$\begin{aligned}&\int_{\mathbb{R}^3}\big(K(x)\phi_{u_n}u_n\varphi - K(x)\phi_u u\varphi\big)\,dx\\ &\quad=\int_{\mathbb{R}^3}K(x)\phi_{u_n}(u_n-u)\varphi\,dx+\int_{\mathbb{R}^3}K(x)(\phi_{u_n}-\phi_u)u\varphi\,dx\to 0\end{aligned}$$

as $n\to\infty$, thus we see that $I_\lambda'(u)=0$. Moreover, since $V\geqslant 0$ and $4\leqslant p<6$, we have $a_\lambda(u,u)=\|u\|_\lambda^2$ and

$$I_\lambda(u)=I_\lambda(u)-\frac{1}{4}\big\langle I_\lambda'(u),u\big\rangle=\frac{1}{4}\|u\|_\lambda^2+\left(\frac{1}{4}-\frac{1}{p}\right)|u|_p^p\geqslant 0. \tag{2.21}$$

Now we show that $u_n\to u$ in E_λ. Let $v_n:=u_n-u$. It follows from (V_2) that

$$|v_n|_2^2=\int_{\{V(x)\geqslant b\}}v_n^2\,dx+\int_{\{V(x)<b\}}v_n^2\,dx\leqslant\frac{1}{\lambda b}\|v_n\|_\lambda^2+o(1). \tag{2.22}$$

Then, by Hölder and Sobolev inequalities, we have

$$|v_n|_p=|v_n|_2^\theta|v_n|_6^{1-\theta}\leqslant d|v_n|_2^\theta|\nabla v_n|_2^{1-\theta}\leqslant d(\lambda b)^{-\frac{\theta}{2}}\|v_n\|_\lambda+o(1), \tag{2.23}$$

as $n\to\infty$, where $\theta=\frac{6-p}{2p}$ and the constant d is independent of λ. By Lemma 2.2 and Brezis–Lieb Lemma, we have

$$I_\lambda(v_n)=I_\lambda(u_n)-I_\lambda(u)+o(1),\qquad I_\lambda'(v_n)=I_\lambda'(u_n)+o(1). \tag{2.24}$$

Consequently, this together with (2.21) and $4<p<6$, we obtain

$$\begin{aligned}\frac{1}{4}\|v_n\|_\lambda^2+\left(\frac{1}{4}-\frac{1}{p}\right)|v_n|_p^p&=I_\lambda(v_n)-\frac{1}{4}\big\langle I_\lambda'(v_n),v_n\big\rangle\\ &=c-I_\lambda(u)+o(1)\\ &\leqslant M+o(1).\end{aligned} \tag{2.25}$$

Thus

$$|v_n|_p^p\leqslant\frac{4p}{p-4}M+o(1). \tag{2.26}$$

(If $p = 4$, by (2.1) and (2.25), we have

$$|v_n|_p^p \leqslant d_p^p \|v_n\|_\lambda^p \leqslant (2d_p)^p M^{p/2} + o(1),$$

where the constant $d_p > 0$ is independent of $\lambda \geqslant 1$.)

Since $\langle I'_\lambda(v_n), v_n\rangle = o(1)$, it follows from (2.23) and (2.26) that

$$\begin{aligned} o(1) &= \|v_n\|_\lambda^2 + N(v_n) - |v_n|_p^p \\ &\geqslant \|v_n\|_\lambda^2 - |v_n|_p^{p-2}|v_n|_p^2 \\ &\geqslant \left(1 - \max\left\{(2d_p)^p M^{p/2}, \frac{4p}{p-4}M\right\}^{(p-2)/p} d^2(\lambda b)^{-\theta}\right)\|v_n\|_\lambda^2 + o(1). \end{aligned} \tag{2.27}$$

Therefore, there exists $\Lambda = \Lambda(M) > 0$ such that $v_n \to 0$ in E_λ as $n \to \infty$ for $\lambda > \Lambda$. The proof is complete. $\square$

We remark that the relation (2.21) is important in the proof of Lemma 2.4. However, it seems difficult to obtain the fact that $I_\lambda(u) \geqslant 0$ for any critical point u of I_λ, in the case that V is sign-changing under our assumptions. In fact, we have the following result.

Corollary 2.5. *Let $4 < p < 6$ and (V_1)–(V_2) be satisfied, $K(x) \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$. Let $\{u_n\}$ be a $(C)_c$ sequence of I_λ with level $c > 0$. Then for any $M > 0$, there exists $\Lambda = \Lambda(M) > 0$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E_λ with u being a nontrivial critical point of I_λ and satisfying $I_\lambda(u) \leqslant c$ for all $c < M$, $\lambda > \Lambda$.*

Proof. We modify the proof of Lemma 2.4. By Lemma 2.3, $\{u_n\}$ is bounded by C_λ in E_λ, $u_n \rightharpoonup u$ in E_λ and u is a critical point of I_λ. However, since V is allowed to be sign-changing, from

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle = \frac{1}{4}\|u\|_\lambda^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} V^-(x)u^2\,dx + \left(\frac{1}{4} - \frac{1}{p}\right)|u|_p^p,$$

we cannot conclude that $I_\lambda(u) \geqslant 0$. Now we consider two possibilities:

(i) $I_\lambda(u) < 0$,
(ii) $I_\lambda(u) \geqslant 0$.

If (i) occurs, it is clear that u is nontrivial and the proof is done. If (ii) occurs, following the argument in the proof of Lemma 2.4 step by step, we can get $u_n \to u$ in E_λ. In fact, by (V_2), we have

$$\lambda \int_{\mathbb{R}^3} V^-(x)v_n^2\,dx \to 0. \tag{2.28}$$

Thus, similar to (2.25), there holds that

$$\frac{1}{4}\|v_n\|_\lambda^2 + \left(\frac{1}{4} - \frac{1}{p}\right)|v_n|_p^p + o(1) \leqslant c - I_\lambda(u) + o(1) \leqslant M + o(1), \tag{2.29}$$

then we have (2.26) and (2.27). Thus $u_n \to u$ in E_λ and $I_\lambda(u) = c > 0$. The proof is complete. $\square$

For the case $3 < p < 4$, it is difficult to prove that $I_\lambda(u) \geqslant 0$ for any critical point u of I_λ. On the other hand, we cannot obtain the boundness of $\{|v_n|_p\}$ by a constant independent of λ from (2.29).

To overcome this difficulty, we decrease the effect of nonlocal term $K(x)\phi_u u$ by assuming that $K(x) \in L^2(\mathbb{R}^3)$. Moreover, as pointed out in the introduction, it is difficult to get the boundness of a (PS) (or (C)) sequence. In this case, we have the following result.

Corollary 2.6. *Let $3 < p < 4$ and (V_1)–(V_2) be satisfied, $K(x) \in L^2(\mathbb{R}^3)$. Let $\{u_n\}$ be a bounded $(PS)_c$ sequence of I_λ with level $c > 0$. Then for any $M > 0$, there exists $\Lambda = \Lambda(M) > 0$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E_λ with u being a nontrivial critical point of I_λ and satisfying $I_\lambda(u) \leqslant c$ for all $c < M$, $\lambda > \Lambda$.*

Proof. We modify the proof of Corollary 2.5. By assumptions, we have that $u_n \rightharpoonup u$ in E_λ with u being a critical point of I_λ. It remains to obtain the boundness of $\{|v_n|_p\}$ by a constant independent of λ if (ii) occurs. In fact, by Lemma 2.1(ii), it follows from $v_n \rightharpoonup 0$ in E_λ that

$$N(v_n) = \int_{\mathbb{R}^3} K(x)\phi_{v_n} v_n^2 \, dx \to 0, \tag{2.30}$$

as $n \to \infty$. Similar to (2.25), we have

$$\begin{aligned} -\frac{1}{4}N(v_n) + \left(\frac{1}{2} - \frac{1}{p}\right)|v_n|_p^p &= I_\lambda(v_n) - \frac{1}{2}\langle I'_\lambda(v_n), v_n\rangle \\ &= c - I_\lambda(u) + o(1) \\ &\leqslant M + o(1). \end{aligned} \tag{2.31}$$

It follows from (2.30) that

$$|v_n|_p^p \leqslant \frac{2p}{p-2} M + o(1). \tag{2.32}$$

Together with (2.27) and (2.28), we obtain the conclusion. □

3. Solutions for the case $4 < p < 6$

In this section, we study the existence of solutions of (1.1) for the case $4 < p < 6$ and give the proofs of Theorems 1.1–1.2. If V is sign-changing, we verify that the functional have the linking geometry to apply the following linking theorem [26].

Proposition 3.1. *Let $E = E_1 \oplus E_2$ be a Banach space with $\dim E_2 < \infty$, $\Phi \in C^1(E, \mathbb{R}^3)$. If there exist $R > \rho > 0$, $\kappa > 0$ and $e_0 \in E_1$ such that*

$$\kappa = \inf \Phi(E_1 \cap S_\rho) > \sup \Phi(\partial Q)$$

where $S_\rho = \{u \in E\colon \|u\| = \rho\}$, $Q = \{u = v + te_0\colon v \in E_2,\ t \geqslant 0,\ \|u\| \leqslant R\}$. Then Φ has a $(C)_c$ sequence with $c \in [\kappa, \sup \Phi(Q)]$.

In our context, we use Proposition 3.1 with $E_1 = E_\lambda^+ \oplus F$ and $E_2 = \widehat{E}_\lambda$. Following Lemma 2.1 in [13], $\alpha_j(\lambda) \to 0$ as $\lambda \to \infty$, for every j fixed. Hence there is $\Lambda_0 > 0$ such that $\widehat{E}_\lambda \neq \varnothing$ and is finite dimensional for $\lambda > \Lambda_0$. As a result, there exists $\widehat{C}_\lambda > 0$ such that

$$|u|_p \geqslant \widehat{C}_\lambda \|u\|, \quad \text{for } u \in \widehat{E}_\lambda, \tag{3.1}$$

where the constant $\widehat{C}_\lambda$ is dependent of λ, since the dimension of $\widehat{E}_\lambda$ is so. Now we investigate the linking structure of the functional.

Lemma 3.2. *For each $\lambda > \Lambda_0$, there exist $\rho_\lambda > 0$ and $\kappa_\lambda > 0$ such that*

$$I_\lambda(u) \geqslant \kappa_\lambda,$$

for all $u \in E_\lambda^+ \oplus F$ with $\|u\|_\lambda = \rho_\lambda$. Moreover, if $V \geqslant 0$, we can choose ρ and κ independent of $\lambda \geqslant 1$.

Proof. Observe that, from the definition of E_λ^+, there exists $\delta_\lambda > 0$ such that

$$a_\lambda(u,u) \geqslant \delta_\lambda \|u\|_\lambda^2, \quad \text{for } u \in E_\lambda^+,$$

and

$$a_\lambda(u,u) = \|u\|_\lambda^2, \quad \text{for } u \in F.$$

Thus for $u = v + w \in E_\lambda^+ \oplus F$, since $\langle v, w\rangle_\lambda = 0$, it holds that

$$I_\lambda(u) = \frac{1}{2}a_\lambda(v,v) + \frac{1}{2}a_\lambda(w,w) + \frac{1}{4}N(u) - \frac{1}{p}|u|_p^p \geqslant \frac{1}{2}\min\{\delta_\lambda, 1\}\|u\|_\lambda^2 - C\|u\|_\lambda^p,$$

where C is a constant independent of $\lambda \geqslant 1$ by (2.1). Choosing $\rho_\lambda > 0$ and $\kappa_\lambda > 0$ small enough, we have the desired conclusion. If $V \geqslant 0$, since $a_\lambda(u,u) = \|u\|_\lambda^2$, we can choose $\rho > 0$ and $\kappa > 0$ independent of $\lambda \geqslant 1$. □

Next, by (V_3), we can choose $e_0 \in C_0^\infty(\Omega)$ fixed, then $e_0 \in F$.

Lemma 3.3. *Under the assumptions of Theorem 1.1, then for each $\lambda > \Lambda_0$, there exist $k_1^*(\lambda) > 0$ and $R_\lambda > 0$ such that for $|K|_2 < k_1^*(\lambda)$ (or $|K|_\infty < k_1^*(\lambda)$)*

$$\sup_{u \in \partial Q} I_\lambda(u) < \kappa_\lambda,$$

where $Q = \{u = v + te_0\colon\ v \in \widehat{E}_\lambda,\ t \geqslant 0,\ \|u\| \leqslant R_\lambda\}$.

Proof. We give the proof for $K \in L^2(\mathbb{R}^3)$, the proof for $K \in L^\infty(\mathbb{R}^3)$ is similar.

(i) For $u = v + w \in \widehat{E}_\lambda \oplus \mathbb{R}e_0$,

$$a_\lambda(u,u) = a_\lambda(v,v) + a_\lambda(w,w) \leqslant |\nabla w|_2^2 \leqslant \|u\|^2.$$

Since all the norms on finite dimensional space are equivalent, by (2.7), we have

$$I_\lambda(u) \leqslant \frac{1}{2}\|u\|^2 + C|K|_2^2|u|_6^4 - \frac{1}{p}|u|_p^p \to -\infty,$$

for $u \in \widehat{E}_\lambda \oplus \mathbb{R}e_0$ with $\|u\|_\lambda \to \infty$. Consequently, there exists $R_\lambda > 0$, independent of $|K|_2$ for $|K|_2 \leqslant 1$, such that $I_\lambda(u) \leqslant 0$ for $u \in \widehat{E}_\lambda \oplus \mathbb{R}e_0$ satisfying $\|u\|_\lambda = R_\lambda$.

(ii) For $u \in \widehat{E}_\lambda$ with $\|u\|_\lambda \leqslant R_\lambda$, by (2.7), we have

$$I_\lambda(u) \leqslant \frac{1}{4}N(u) \leqslant C|K|_2^2|u|_6^4 \leqslant C|K|_2^2\|u\|_\lambda^4 \leqslant C|K|_2^2 R_\lambda^4. \tag{3.2}$$

Hence, taking $k_1^*(\lambda) = \min\{1, \kappa_\lambda^{1/2}/(C^{1/2}R_\lambda^2)\}$, we obtain the conclusion. The proof is complete. □

Lemma 3.4. *Under the assumptions of Theorem* 1.1*, then for each $\lambda > \Lambda_0$, there exists $k_2^*(\lambda) > 0$ such that for $|K|_2 < k_2^*(\lambda)$ (or $|K|_\infty < k_2^*(\lambda)$) the quantity $\sup_{u\in Q} I_\lambda(u)$ is bounded from above by a constant independent of λ.*

Proof. Let

$$J_\lambda(u) := \frac{1}{2}a_\lambda(u,u) - \frac{1}{p}\int_\Omega |u|^p\,dx, \quad \text{for } u \in E_\lambda.$$

Following the arguments in [13], we can show that $\sup_{u\in Q} J_\lambda(u)$ is bounded above by a constant independent of λ, then by the continuity of I_λ on K, we have the conclusion. We give the details here for completeness. Observe that for each $\eta > 0$ there is $r_\eta > 0$ such that $\frac{1}{p}|u|^p \geqslant \frac{1}{2}\eta u^2$ if $|u| \geqslant r_\eta$. Let $u = v + w \in \widehat{E}_\lambda \oplus \mathbb{R}e_0$, then

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{2}a_\lambda(u,u) - \frac{1}{p}\int_\Omega |u|^p\,dx \\
&\leqslant \frac{1}{2}a_\lambda(w,w) - \frac{1}{2}\eta|u|^2_{L^2(\Omega)} + \int_\Omega \left(\frac{1}{2}\eta u^2 - \frac{1}{p}|u|^p\right)dx \\
&\leqslant \frac{1}{2}|\nabla w|_2^2 - \frac{1}{2}\eta|u|^2_{L^2(\Omega)} + \int_{\{x\in\Omega:\ |u(x)|\leqslant r_\eta\}} \left(\frac{1}{2}\eta u^2 - \frac{1}{p}|u|^p\right)dx \\
&\leqslant \frac{1}{2}|\nabla w|_2^2 - \frac{1}{2}\eta|u|^2_{L^2(\Omega)} + C_\eta,
\end{aligned} \tag{3.3}$$

where C_η is independent of λ. Since $e_0 \in C_0^\infty(\Omega)$ and $a_\lambda(v,w) = 0$,

$$\begin{aligned}
|\nabla w|_2^2 = a_\lambda(u,w) &= \int_\Omega (-\Delta w)u\,dx \leqslant |\Delta w|_2|u|_{L^2(\Omega)} \\
&\leqslant C_0|\nabla w|_2|u|_{L^2(\Omega)} \leqslant \frac{C_0^2}{2\eta}|\nabla w|_2^2 + \frac{\eta}{2}|u|^2_{L^2(\Omega)}
\end{aligned}$$

where C_0 is a constant dependent of e_0. Choosing $\eta \geqslant C_0^2$, we obtain $|\nabla w|_2^2 \leqslant \eta|u|^2_{L^2(\Omega)}$ and it follows from (3.3) that $J_\lambda(u) \leqslant C_\eta$. Moreover, similar to (3.2), we have for $u \in Q$,

$$I_\lambda(u) \leqslant J_\lambda(u) + \frac{1}{4}N(u) \leqslant C_\eta + C|K|_2^2 R_\lambda^4 \leqslant C_\eta + 1$$

provided $|K|_2 < k_2^*(\lambda) := 1/(C^{1/2}R_\lambda^2)$. The proof is complete. □

Proof of Theorem 1.1. Set $k_\lambda^* := \min(k_1^*(\lambda), k_2^*(\lambda))$, then it follows from Proposition 3.1, Lemma 3.2 and Lemma 3.3 that for any $\lambda > \Lambda_0$ and $0 < |K|_2 < k_\lambda^*$ (or $0 < |K|_\infty < k_\lambda^*$), I_λ possesses a $(\mathrm{C})_c$ sequence $\{u_n\}$ with $c \in [\kappa_\lambda, \sup_{u\in Q} I_\lambda(u)]$. By Lemma 3.4, we set $M = \sup_{u\in Q} I_\lambda(u)$, then the conclusion follows from Lemma 2.3 and Corollary 2.5. The proof is complete. □

Proof of Theorem 1.2. Since we suppose $V \geqslant 0$, the functional I_λ has mountain pass geometry and the existence of nontrivial solutions can be obtained by mountain pass theorem [26]. In fact, by

Lemma 3.2 with $\widehat{E}_\lambda = \{0\}$, $0 \in E$ is the local minimum for I_λ and κ is independent of λ. Let $e_0 \in C_0^\infty(\Omega)$, then

$$I_\lambda(te_0) = \frac{t^2}{2}\int_\Omega |\nabla e_0|^2\,dx + \frac{t^4}{4}N(e_0) - \frac{t^p}{p}\int_\Omega |e_0|^p\,dx \to -\infty$$

as $t \to \infty$, if $4 < p < 6$. It is clear that there exists a constant $C_0 > 0$, independent of λ, such that

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda\bigl(\gamma(t)\bigr) \leqslant \sup_{t\geqslant 0} I_\lambda(te_0) \leqslant C_0, \tag{3.4}$$

where $\Gamma = \{\gamma \in C([0,1], E_\lambda)\colon\ \gamma(0) = 0,\ \ \|\gamma(1)\|_\lambda > \rho,\ \ I_\lambda(\gamma(1)) < 0\}$. (If $p = 4$, (3.4) follows from (4.2) in the next section.) By mountain pass theorem and Lemma 2.4, we obtain a nontrivial critical point u_λ for I_λ with $I_\lambda(u_\lambda) \in [\kappa, C_0]$ for λ large. The proof is complete. □

4. Solutions for the case 3 < *p* < 4

In this section, we consider (1.1) for the case $3 < p < 4$ and give the proof of Theorem 1.3. We need the following abstract result which is due to Jeanjean [18].

Proposition 4.1. *Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $\{\Phi_\mu\}_{\mu\in J}$ of C^1-functionals on X of the form*

$$\Phi_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in J,$$

where $B(u) \geqslant 0$ for all $u \in X$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\|_X \to \infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\mu = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \Phi_\mu\bigl(\gamma(t)\bigr) > \max\bigl\{\Phi_\mu(v_1), \Phi_\mu(v_2)\bigr\}, \quad \forall \mu \in J,$$

where

$$\Gamma = \bigl\{\gamma \in C\bigl([0,1], X\bigr)\colon\ \gamma(0) = v_1,\ \gamma(1) = v_2\bigr\}.$$

Then, for almost every $\mu \in J$, there is a bounded $(\mathrm{PS})_{c_\mu}$ sequence for Φ_μ, that is, there exists a sequence $\{u_n(\mu)\} \subset X$ such that

(i) *$\{u_n(\mu)\}$ is bounded in X,*
(ii) *$\Phi_\mu(u_n(\mu)) \to c_\mu$,*
(iii) *$\Phi'_\mu(u_n(\mu)) \to 0$ in X^*, where X^* is the dual of X.*

To apply Proposition 3.1, we introduce a family of functionals defined by

$$I_{\lambda,\mu}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \bigl(|\nabla u|^2 + \lambda V(x)u^2\bigr)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx - \frac{\mu}{p}\int_{\mathbb{R}^3} |u|^p\,dx, \tag{4.1}$$

for $\mu \in [1/2, 1]$.

The following lemma ensures that $I_{\lambda,\mu}$ has the mountain pass geometry [26]. The corresponding mountain pass level is denoted by $c_{\lambda,\mu}$.

Lemma 4.2. *Let $3 < p < 4$, (V_1)–(V_3) and $V \geqslant 0$ be satisfied, $K(x) \geqslant 0$ for $x \in \mathbb{R}^3$ and $K(x) \in L^2(\mathbb{R}^3) \cap L^\infty_{loc}(\mathbb{R}^3)$. Then:*

(i) *There exists a $v_1 \in E \setminus \{0\}$ independent of μ such that $I_{\lambda,\mu}(v_1) \leqslant 0$ for all $\mu \in [1/2, 1]$.*
(ii) *$c_{\lambda,\mu} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)) > \max\{I_{\lambda,\mu}(0), I_{\lambda,\mu}(v_1)\}$ for all $\mu \in [1/2, 1]$, where*

$$\Gamma = \big\{\gamma \in C\big([0,1], E\big)\colon\ \gamma(0) = 0,\ \gamma(1) = v_1\big\}.$$

(iii) *There exists $M > 0$ independent of μ and λ such that $c_{\lambda,\mu} \leqslant M$.*

Proof. (i) Let $x_0 \in \Omega$ and $B_{\varepsilon_0}(x_0) \subset \Omega$ for some $\varepsilon_0 > 0$. Let $v \in C_0^\infty(\mathbb{R}^3)$ be such that *supp* $v \subset B_{\varepsilon_0}(0)$. Set $v_t := t^2 v(t(x - x_0))$, then *supp* $v_t \subset B_{\varepsilon_0}(x_0)$ for $t > 1$. Thus, by direct computation, we have

$$\int_{\mathbb{R}^3} |\nabla v_t|^2\,dx = t^3 \int_{B_{\varepsilon_0}(0)} |\nabla v|^2\,dx, \qquad \int_{\mathbb{R}^3} |v_t|^p\,dx = t^{2p-3} \int_{B_{\varepsilon_0}(0)} |v|^p\,dx,$$

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{v_t} v_t^2\,dx &= t^3 \int_{B_{\varepsilon_0}(0)} \int_{B_{\varepsilon_0}(0)} K\left(\frac{x}{t} + x_0\right) K\left(\frac{y}{t} + x_0\right) \frac{v^2(x)v^2(y)}{|x-y|}\,dx\,dy \\ &\leqslant t^3 |K|^2_{L^\infty_{loc}} \int_{B_{\varepsilon_0}(0)} \phi_v v^2\,dx.\end{aligned}$$

Therefore

$$I_{\lambda,1/2}(v_t) \leqslant \frac{t^3}{2} \int_{B_{\varepsilon_0}(0)} |\nabla v|^2\,dx + \frac{t^3}{4} |K|^2_{L^\infty_{loc}} \int_{B_{\varepsilon_0}(0)} \phi_v v^2\,dx - \frac{t^{2p-3}}{2p} \int_{B_{\varepsilon_0}(0)} |v|^p\,dx. \tag{4.2}$$

Since $2p - 3 > 3$ for $p > 3$, we see that $I_{\lambda,1/2}(v_t) \to -\infty$ as $t \to +\infty$. Taking $v_1 := v_t$ for t large enough, we have $I_{\lambda,\mu}(v_1) \leqslant I_{\lambda,1/2}(v_1) < 0$ for all $\mu \in [1/2, 1]$.

(ii) Since

$$I_{\lambda,\mu}(u) \geqslant \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,dx \geqslant \frac{1}{2}\|u\|_\lambda^2 - C\|u\|_\lambda^p$$

and $p > 3$, we deduce that I_λ has a strict local minimum at 0 and $c_{\lambda,\mu} > 0$.

(iii) Since $c_{\lambda,\mu} \leqslant \max_{t>0} I_{\lambda,\mu}(v_t) \leqslant \max_{t>0} I_{\lambda,1/2}(v_t)$ for all $\mu \in [1/2, 1]$, the conclusion follows from (4.2). $\square$

It follows from Proposition 4.1, Lemma 4.2 and Corollary 2.6 that for almost every $\mu \in [1/2, 1]$, $I_{\lambda,\mu}$ has a nontrivial critical point u_μ. In general, it is not known whether it is true for $\mu = 1$. However we have

Lemma 4.3. *Under the assumptions of Lemma 4.2, there exists a sequence $\mu_n \in [1/2, 1]$ and $u_n \in E_\lambda \setminus \{0\}$ such that*

$$\mu_n \to 1, \qquad I'_{\lambda,\mu_n}(u_n) = 0 \quad \textit{and} \quad I_{\lambda,\mu_n}(u_n) \leqslant c_{\lambda,\mu_n}.$$

Next we are to show that the sequence $\{u_n\}$ obtained in Lemma 4.3 is a bounded (PS) sequence for $I_\lambda = I_{\lambda,1}$. For this purpose, we need the following Pohozaev type identity. The proof is standard and we omit it here, see e.g. [19,25]. Note that we use here the integrality of $(\nabla V(x), x)$ and $(\nabla K(x), x)$.

Lemma 4.4. *Let u be a critical point of $I_{\lambda,\mu}$ in E_λ, then*

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{3}{2}\int_{\mathbb{R}^3} \lambda V(x)u^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \lambda\big(\nabla V(x), x\big)u^2\,dx$$
$$+ \frac{5}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \big(\nabla K(x), x\big)\phi_u u^2\,dx - \frac{3\mu}{p}\int_{\mathbb{R}^3} |u|^p\,dx = 0. \tag{4.3}$$

Lemma 4.5. *Under the assumptions of Theorem 1.3, the sequence $\{u_n\}$ obtained in Lemma 4.3 is bounded in E_λ.*

Proof. Denote

$$a_n := \int_{\mathbb{R}^3} |\nabla u_n|^2\,dx, \qquad b_n := \int_{\mathbb{R}^3} \lambda V(x)u_n^2\,dx, \qquad \bar{b}_n := \int_{\mathbb{R}^3} \lambda\big(\nabla V(x), x\big)u_n^2\,dx,$$
$$c_n := \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2\,dx, \qquad \bar{c}_n := \int_{\mathbb{R}^3} \big(\nabla K(x), x\big)\phi_{u_n} u_n^2\,dx, \qquad d_n := \int_{\mathbb{R}^3} |u_n|^{p+1}\,dx.$$

Then by Lemma 4.2 and Lemma 4.3, we have

$$\begin{cases} \frac{1}{2}a_n + \frac{3}{2}b_n + \frac{1}{2}\bar{b}_n + \frac{5}{4}c_n + \frac{1}{2}\bar{c}_n - \frac{3\mu_n}{p}d_n = 0, \\ a_n + b_n + c_n - \mu_n d_n = 0, \\ \frac{1}{2}a_n + \frac{1}{2}b_n + \frac{1}{4}c_n - \frac{\mu_n}{p}d_n \leqslant c_{\lambda,\mu_n}. \end{cases} \tag{4.4}$$

The first equation comes from the Pohozaev equality (4.3), the second one is $\langle I'_{\lambda,\mu}(u_n), u_n\rangle = 0$, and the last one is from the definition of $I_{\lambda,\mu}$. From these relations, we have

$$\frac{1}{4}a_n + \frac{1}{4}b_n + \left(\frac{1}{4} - \frac{1}{p}\right)\mu_n d_n \leqslant c_{\lambda,\mu_n} \tag{4.5}$$

and

$$b_n + \frac{1}{2}\bar{b}_n + \frac{1}{2}\bar{c}_n + \frac{2(p-3)}{p}\mu_n d_n \leqslant 3c_{\lambda,\mu_n}. \tag{4.6}$$

On the other hand, by eliminating the term d_n from (4.4), we obtain

$$\frac{2(p-3)}{p}a_n + \frac{2(p-3)}{p}b_n + b_n + \frac{1}{2}\bar{b}_n + \frac{2(p-3)}{p}c_n + \frac{1}{2}\bar{c}_n \leqslant 3c_{\lambda,\mu_n}. \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$\frac{2(p-3)}{p}a_n + \frac{2(p-3)}{p}b_n + 2b_n + \bar{b}_n + \frac{2(p-3)}{p}c_n + \bar{c}_n + \frac{2(p-3)}{p}\mu_n d_n \leqslant 6c_{\lambda,\mu_n}. \tag{4.8}$$

With this inequality, we show that $\{u_n\}$ is bounded in E_λ under our assumptions on V and K. In fact, if (V_4) and (K_1) hold, the conclusion is obvious from (4.8) since $p > 3$ and $c_{\lambda,\mu_n} \leqslant c_{\lambda,1/2}$. If (V_4') and (K_1) hold, by Hölder inequality and Young inequality, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} |\bar{b}_n| &\leqslant \lambda \big|\big(\nabla V(x), x\big)\big|_{p/(p-2)} |u_n|_p^2 \\ &\leqslant \lambda^{p/(p-2)} C_\varepsilon \big|\big(\nabla V(x), x\big)\big|_{p/(p-2)}^{p/(p-2)} + \varepsilon |u_n|_p^p. \end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8), together with (K_1), we have

$$\begin{aligned} &\frac{2(p-3)}{p}a_n + \frac{2(p-3)}{p}b_n + \left(\frac{2(p-3)}{p} - \varepsilon\right)\mu_n d_n \\ &\quad\leqslant 6c_{\lambda,1/2} + \lambda^{p/(p-2)} C_\varepsilon \big|\big(\nabla V(x), x\big)\big|_{p/(p-2)}^{p/(p-2)}. \end{aligned} \tag{4.10}$$

Taking ε small enough, we see that $\|u_n\|_\lambda^2 = a_n + b_n$ is bounded. The proof is complete. $\square$

Proof of Theorem 1.3 and Corollary 1.4. By Lemma 4.3 and Lemma 4.5, we obtain a bounded sequence of $\{u_n\} \in E_\lambda \setminus \{0\}$ satisfying

$$\mu_n \to 1, \qquad I'_{\lambda,\mu_n}(u_n) = 0 \quad \text{and} \quad I_{\lambda,\mu_n}(u_n) \leqslant c_{\lambda,\mu_n}.$$

Moreover, recalling the proof of Corollary 2.6, there are two possibilities for the energy of u_n,

(i) $I_{\lambda,\mu_n}(u_n) < 0$,
(ii) $I_{\lambda,\mu_n}(u_n) = c_{\lambda,\mu_n}$.

Now we discuss two cases.

Case 1. There exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that (ii) occurs. In this case, we have

$$\lim_{n\to\infty} I_\lambda(u_n) = \lim_{n\to\infty}\left(I_{\lambda,\mu_n}(u_n) + \frac{\mu_n - 1}{p}\int_{\mathbb{R}^3} |u_n|^p\,dx\right) = \lim_{n\to\infty} c_{\lambda,\mu_n} = c_{\lambda,1}.$$

Here we use the fact that the map $\mu \longmapsto c_{\lambda,\mu}$ is left-continuous, see [18]. Similarly, $I'_\lambda(u_n) \to 0$ in E_λ^{-1}. That is, $\{u_n\}$ is a bounded $(PS)_{c_{\lambda,1}}$ sequence for I_λ. Since $c_{\lambda,1} > 0$, the conclusion follows from Corollary 2.6.

Case 2. There exists $N \in \mathbb{N}$ such that (i) occurs for u_n with $n > N$. Since $\langle I'_{\lambda,\mu_n}(u_n), u_n\rangle = 0$ and $u_n \neq 0$, there exists $C > 0$ such that

$$\|u_n\|_\lambda^2 \leqslant \|u_n\|_\lambda^2 + N(u_n) = \mu_n |u_n|_p^p \leqslant C\|u_n\|_\lambda^p,$$

which implies that

$$\|u_n\|_\lambda \geqslant C^{-1/(p-2)} > 0. \tag{4.11}$$

Let $u_n \rightharpoonup u_0$ in E_λ, as $n \to \infty$. We claim that $u_0 \neq 0$. Otherwise, by Lemma 2.1(ii), we have

$$N(u_n) \to 0, \quad \text{as } n \to \infty.$$

Together with

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 + \left(\frac{1}{4} - \frac{1}{p}\right)N(u_n) = I_{\lambda,\mu_n}(u_n) - \frac{1}{p}\langle I'_{\lambda,\mu_n}(u_n), u_n\rangle \leqslant 0,$$

we have $u_n \to 0$ in E_λ, which contracts (4.11). Thus $u_0 \neq 0$. For any $\varphi \in C_0^\infty(\mathbb{R}^3)$, using the argument in the proof of Lemma 2.4, it is easy to check that

$$\langle I'_\lambda(u_0), \varphi\rangle = \lim_{n\to\infty}\left(\langle I'_{\lambda,\mu_n}(u_n), \varphi\rangle + (\mu_n - 1)\int_{\mathbb{R}^3} |u_n|^{p-2}u_n\varphi\, dx\right) = 0.$$

Therefore u_0 is a nontrivial critical point of I_λ and the proof is complete. $\square$

5. Concentration for solutions

In this section, we investigate the concentration for solutions and give the proof of Theorem 1.5.

Proof of Theorem 1.5. We follows the argument in [5]. For any sequence $\lambda_n \to \infty$, let $u_n := u_{\lambda_n}$ be the critical points of I_{λ_n} obtained in Theorem 1.2 or Theorem 1.3. If $4 \leqslant p < 6$, it follows from (3.4) and

$$c_{\lambda_n} := I_{\lambda_n}(u_n) - \frac{1}{4}\langle I'_{\lambda_n}(u_n)u_n\rangle = \frac{1}{4}\|u_n\|_{\lambda_n}^2 + \left(\frac{1}{4} - \frac{1}{p}\right)|u_n|_p^p$$

that

$$\sup_{n\geqslant 1}\|u_n\|_{\lambda_n}^2 \leqslant 4C_0, \tag{5.1}$$

where the constant C_0 is independent of λ_n. If $3 < p < 4$, then (V_4), (K_1), (4.8) and Lemma 4.2(iii) imply that

$$\frac{2(p-3)}{p}\|u_n\|_{\lambda_n}^2 \leqslant 6c_{\lambda_n} \leqslant 6c_{\lambda_n,1/2} \leqslant 6M \tag{5.2}$$

where the constant M is independent of λ_n. Therefore, we may assume that $u_n \rightharpoonup \bar{u}$ in E and $u_n \to \bar{u}$ in $L^s_{loc}(\mathbb{R}^3)$ for $2 \leqslant s < 6$. By Fatou's Lemma, we have

$$\int_{\mathbb{R}^3} V(x)\bar{u}^2\, dx \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^3} V(x)u_n^2\, dx \leqslant \liminf_{n\to\infty}\frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0,$$

thus $\bar{u} = 0$ a.e. in $\mathbb{R}^3 \setminus V^{-1}(0)$, $\bar{u} \in H_0^1(\Omega)$ by (V_3). Now for any $\varphi \in C_0^\infty(\Omega)$, since $\langle I'_{\lambda_n}(u_n), \varphi\rangle = 0$, it is easy to check that

$$\int_{\mathbb{R}^3} \nabla\bar{u}\nabla\varphi\,dx + \int_{\mathbb{R}^3} K(x)\phi_{\bar{u}}\bar{u}\varphi\,dx = \int_{\mathbb{R}^3} |\bar{u}|^{p-2}\bar{u}\varphi\,dx,$$

that is, $\bar{u}$ is a weak solution of (1.2) by the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$.

We show that $u_n \to \bar{u}$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. Otherwise, by Lions vanishing lemma [24] there exist $\delta > 0$, $\rho > 0$ and $x_n \in \mathbb{R}^3$ such that

$$\int_{B_\rho(x_n)} (u_n - \bar{u})^2\,dx \geqslant \delta.$$

Moreover, $x_n \to \infty$, hence $meas(B_\rho(x_n) \cap \{x \in \mathbb{R}^3 : V(x) < b\}) \to 0$. Similar to (2.20), by Hölder inequality, we have

$$\int_{B_\rho(x_n)\cap\{V(x)<b\}} (u_n - \bar{u})^2\,dx \to 0.$$

Consequently

$$\begin{aligned}
\|u_n\|_{\lambda_n}^2 &\geqslant \lambda_n b \int_{B_\rho(x_n)\cap\{V\geqslant b\}} u_n^2\,dx = \lambda_n b \int_{B_\rho(x_n)\cap\{V\geqslant b\}} (u_n - \bar{u})^2\,dx \\
&= \lambda_n b\left(\int_{B_\rho(x_n)} (u_n - \bar{u})^2\,dx - \int_{B_\rho(x_n)\cap\{V<b\}} (u_n - \bar{u})^2\,dx + o(1)\right) \\
&\to \infty,
\end{aligned}$$

which contradicts (5.1) and (5.2).

To complete the proof, it suffices to show that $u_n \to \bar{u}$ in E. Since $\langle I'_{\lambda_n}(u_n), u_n\rangle = \langle I'_{\lambda_n}(u_n), \bar{u}\rangle = 0$, we have

$$\|u_n\|_{\lambda_n}^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2\,dx = \int_{\mathbb{R}^3} |u_n|^p\,dx, \tag{5.3}$$

$$\langle u_n, \bar{u}\rangle_{\lambda_n} + \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n\bar{u}\,dx = \int_{\mathbb{R}^3} |u_n|^{p-2}u_n\bar{u}\,dx. \tag{5.4}$$

First, we have

$$\int_{\mathbb{R}^3} \big(K(x)\phi_{u_n}u_n^2 - K(x)\phi_{u_n}u_n\bar{u}\big)\,dx \to 0. \tag{5.5}$$

Indeed, if $K(x) \in L^\infty(\mathbb{R}^3)$, then Hölder inequality and $u_n \to \bar{u}$ in $L^3(\mathbb{R}^3)$ imply that

$$\int_{\mathbb{R}^3} \big(K(x)\phi_{u_n}u_n^2 - K(x)\phi_{u_n}u_n\bar{u}\big)\,dx \leqslant |K|_\infty|\phi_{u_n}|_6|u_n|_2|u_n - \bar{u}|_3 \to 0,$$

while for $K(x) \in L^2(\mathbb{R}^3)$, we obtain (5.5) in a similar way to (2.11).

By (5.3), (5.4) and (5.5), we have

$$\lim_{n\to\infty} \|u_n\|_{\lambda_n}^2 = \lim_{n\to\infty} \langle u_n, \bar{u}\rangle_{\lambda_n} = \lim_{n\to\infty} \langle u_n, \bar{u}\rangle = \|\bar{u}\|^2.$$

On the other hand, weakly lower semi-continuity of norm yields that

$$\|\bar{u}\|^2 \leqslant \liminf_{n\to\infty} \|u_n\|^2 \leqslant \lim_{n\to\infty} \|u_n\|_{\lambda_n}^2,$$

thus $u_n \to \bar{u}$ in E. By (5.3) and the fact that $u_n \neq 0$, we have for n large,

$$\|u_n\|^2 \leqslant \|u_n\|_{\lambda_n}^2 \leqslant |u_n|_p^p \leqslant C\|u_n\|^p,$$

which implies that $\bar{u} \neq 0$. The proof is complete. $\square$

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