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# Multiple periodic solutions for lattice dynamical systems with superquadratic potentials <sup>☆</sup>

Jijiang Sun, Shiwang Ma <sup>\*</sup>

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China

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## ABSTRACT

In this paper, we consider one dimensional lattices consisting of infinitely many particles with nearest neighbor interaction. The autonomous dynamical system is described by the following infinite system of second order differential equations

$$\ddot{q}_i = \Phi'_{i-1}(q_{i-1} - q_i) - \Phi'_i(q_i - q_{i+1}), \quad i \in \mathbb{Z},$$

where  $\Phi_i$  denotes the interaction potential between two neighboring particles and  $q_i(t)$  is the state of the  $i$ -th particle. Supposing  $\Phi_i$  is superquadratic at infinity, for all  $T > 0$ , we obtain a nonzero  $T$ -periodic solution of finite energy which may be nonconstant in some range of period. If in addition  $\Phi_i(x)$  is even in  $x$ , we also obtain infinitely many geometrically distinct solutions for any period  $T > 0$ . In particular, a prescribed number of geometrically distinct nonconstant periodic solutions is obtained for some range of period. Since the functional associated to the above system is invariant under the actions of the non-compact group  $\mathbb{Z}$  and the continuous compact group  $S^1$  under our assumptions, in order to prove our results, we need to extend the abstract critical point theorem about strongly indefinite functional developed by Bartsch and Ding [Math. Nachr. 279 (2006) 1267–1288] to a more general class of symmetry.

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<sup>\*</sup> Corresponding author.

E-mail addresses: [sunjijiang2005@163.com](mailto:sunjijiang2005@163.com) (J. Sun), [shiwangm@163.net](mailto:shiwangm@163.net) (S. Ma).

## 1. Introduction and main results

In this paper, we consider one dimensional lattices consisting of infinitely many particles with nearest neighbor interaction. We represent the state of the autonomous dynamical system at time  $t$  by a sequence of functions  $q(t) = \{q_i(t)\}$ ,  $i \in \mathbb{Z}$ , where  $q_i(t)$  is the state of the  $i$ -th particle. Let  $\Phi_i$  denote the potential of the interaction between the  $i$ -th and the  $(i+1)$ -th particle (whose displacement is  $q_i - q_{i+1}$ ), then the equation governing the state of  $q_i(t)$  reads

$$\ddot{q}_i = \Phi'_{i-1}(q_{i-1} - q_i) - \Phi'_i(q_i - q_{i+1}), \quad i \in \mathbb{Z}. \quad (1.1)$$

We define the potential  $\Phi : \mathbb{R}^\infty \rightarrow \mathbb{R}$  by

$$\Phi(q) = \sum_{i \in \mathbb{Z}} \Phi_i(q_i - q_{i+1}).$$

Then infinitely many equations (1.1) can be written in a vectorial form

$$\ddot{q} = -\Phi'(q). \quad (1.2)$$

After the pioneering numerical experiment of Fermi, Pasta and Ulam [17] on finite lattices, this autonomous Hamiltonian system with different kinds of potentials has been widely studied, see e.g., [7,18,20,21,24,26,27] and the references therein.

Before we state the main results of this paper, let us recall firstly some recent results concerning the existence of nontrivial periodic motions of finite energy for an infinite lattice of particles via variational techniques, see [2–5] and the references therein. In [23], Ruf and Srikanth proved that a finite chain of particles with a Toda type potential (see [27]) admits periodic solutions. Later, Arioli and Gazzola [4] extended in some sense Ruf–Srikanth result to an infinite dimensional system. They obtained the existence of a  $T$ -periodic nonconstant solution of Eq. (1.2) for  $T$  large enough where the following conditions are satisfied for all  $i \in \mathbb{Z}$ :

- (i)  $\Phi_i(x) = -\alpha_i x^2 + V_i(x)$  and  $\alpha_i > 0$ ;
- (ii) there exists  $\delta > 0$  such that  $V'_i(x)x \geq (2 + \delta)V_i(x) \geq 0$ , for all  $x \in \mathbb{R}$ ;
- (iii)  $V_i \in C^{1,1}_{loc}$  and  $V_i(x) \rightarrow +\infty$  as  $x \rightarrow \pm\infty$ ;
- (iv) there exists  $m \in \mathbb{N}$  such that  $\Phi_{i+m} = \Phi_i$ .

We remark that the above hypothesis  $\alpha_i > 0$  implies that the potential  $\Phi_i$  is quadratically repulsive for small displacements and superquadratically attractive for large displacements; in this case the minimum of the spectrum of the quadratic part of the Lagrangian functional corresponding to problem (1.2) is strictly positive. In [3], Arioli and Gazzola also extended the result to the purely attractive potentials ( $\alpha_i = 0$  for all  $i \in \mathbb{Z}$ ) which are strictly superquadratic at both the origin and at infinity by using a finite lattice approximation and a suitable limit procedure. Arioli, Gazzola and Terracini [5] proved the existence of infinitely many periodic nonconstant motions of multibump type for system (1.2) under the same assumptions taken in [4]. In [6], Arioli and Szulkin extended the result of [3] to the case of potentials quadratically attractive and they also considered the case with potentials asymptotically quadratic, in the sense that  $\Phi_i(x)$  tends to a quadratic function as  $i \rightarrow \pm\infty$ . It is worth mentioning that these two cases are more delicate to handle with respect to the case treated in the previous works because the quadratic part of the functional is strongly indefinite. By means of variational methods, they proved that system (1.2) admits a nonzero  $T$ -periodic solution of finite energy for all  $T$  in a given range of values and given a bifurcate result under some additional conditions.

Note that if  $q = \{q_i\}$  is a solution of the system (1.2), then so is  $\hat{q} = \{q_i + \sigma\}$  for any constant  $\sigma$ . We can normalize (1.2) by requiring that  $\int_0^T q_0 dt = 0$ . For the periodic potentials, that is  $\Phi_i = \Phi_{i+m}$  for some  $m \in \mathbb{N}$ , it is easy to see that Eq. (1.1) is invariant with respect to the action of  $\mathbb{Z}$  given by  $(k * q)_i = q_{i+mk} + \sigma$ , where  $\sigma$  is chosen so that  $\int_0^T (k * q)_0 dt = 0$ . Furthermore, Eq. (1.1) is invariant with respect to the action of  $S^1 := [0, T]/\{0, T\}$  defined by  $\Omega_\theta q(t) = q(t + \theta)$  where  $\theta \in S^1$ . For the periodic

potentials, if  $\alpha_i \leq 0$  for all  $i$ , Arioli and Szulkin in [6] obtained infinitely many geometrically distinct nonconstant solutions of any period  $T > 0$  for system (1.2) where two (normalized) solutions  $q^1, q^2$  of Eq. (1.2) are geometrically distinct if  $k * (\Omega_\theta q^1) \neq q^2$  for all  $k \in \mathbb{Z}$  and all  $\theta \in S^1$ . For asymptotically quadratic potentials, Eq. (1.1) is invariant only with respect to the action of  $S^1$ . In this case, Arioli and Szulkin in [6] also proved that system (1.2) admits infinitely many geometrically distinct solutions for all  $T > 0$  where  $q^1, q^2$  are called geometrically distinct if  $\Omega_\theta q^1 \neq q^2$  for all  $\theta \in S^1$ .

Observe that the condition (ii) plays an important role in their argument in order to show that every Palais–Smale sequence for the Lagrangian functional associated with system (1.2) is bounded. This kind of technical condition was firstly introduced by Ambrosetti and Rabinowitz [2], and often appears as necessary to study superlinear problems via variational methods such as elliptic problems, Hamiltonian systems and wave equations. In the present paper we are interested in periodic potentials satisfying superquadratic conditions at infinity which are more general than (ii). More precisely, we assume the potentials  $\Phi_i$  to be defined by

$$\Phi_i(x) = -\frac{\alpha_i}{2}x^2 + V_i(x),$$

where for all  $i \in \mathbb{Z}$ :

- (A<sub>1</sub>)  $V_i(x) \geq 0$  and  $V_i(x)/x^2 \rightarrow \infty$  as  $|x| \rightarrow \infty$ ;
- (A<sub>2</sub>)  $V_i \in C^1(\mathbb{R}, \mathbb{R})$ ,  $V'_i(x) = o(x)$  as  $x \rightarrow 0$ ;
- (A<sub>3</sub>)  $\tilde{V}_i(x) := \frac{1}{2}V'_i(x)x - V_i(x) > 0$  if  $x \neq 0$  and there exist  $\sigma > 1$  and  $r_0, a > 0$  such that  $|V'_i(x)|^\sigma \leq a\tilde{V}_i(x)|x|^\sigma$  for  $|x| \geq r_0$ ;
- (A<sub>4</sub>) there exists  $m \in \mathbb{N}$  such that  $\Phi_{i+m} = \Phi_i$ .

It is not difficult to check that conditions (A<sub>1</sub>) and (A<sub>2</sub>) imply that for all  $i \in \mathbb{Z}$ ,  $\Phi_i(0) = 0$ ,  $V_i$  is superquadratic both at the origin and at infinity and  $\Phi_i$  has either a strict local maximum if  $\alpha_i > 0$  or a strict global minimum at 0 when  $\alpha_i \leq 0$ .

Now we state our main theorem.

**Theorem 1.1.** Suppose (A<sub>1</sub>)–(A<sub>4</sub>) hold, the coefficients  $\alpha_i \neq 0$  and take both signs. (1) System (1.2) admits a nonzero  $T$ -periodic solution of finite energy for any  $T > 0$ ; moreover there exists  $T_{\min} > 0$ , where  $T_{\min}$  only depends on the positive  $\alpha_i$  and  $V_i$ , such that the solution obtained above is nonconstant if  $T_{\min} < \pi/\sqrt{\beta}$  and its period  $T \in (T_{\min}, \pi/\sqrt{\beta})$  where  $\beta = -\inf\{\alpha_i\} > 0$ . (2) If in addition we assume that  $V_i(x)$  is even in  $x$  for all  $i \in \mathbb{Z}$ , then system (1.2) possesses infinitely many geometrically distinct solutions for any period  $T > 0$ ; moreover for any positive integer  $N$ , there exists  $\bar{T} > 0$  which depends on  $N$ , positive  $\alpha_i$  and corresponding  $V_i$ , such that  $N$  different geometrically distinct  $T$ -periodic solutions we have obtained can be nonconstant if  $\bar{T} < \pi/\sqrt{\beta}$  and  $T \in (\bar{T}, \pi/\sqrt{\beta})$ .

**Remark 1.2.** As a  $T/l$  ( $l \in \mathbb{N}$ ) periodic solution is  $T$  periodic as well, we only consider the existence of  $T$ -periodic solution of finite energy and the multiplicity of periodic solutions for  $T < \pi/\sqrt{\beta}$ .

Notice that condition (A<sub>3</sub>) does not imply that  $V_i(x)$  is superquadratic. Indeed, the asymptotically quadratic function  $V_i(x) = |x|^2(1 - \frac{1}{\ln(e+|x|)})$  also verifies (A<sub>3</sub>). Recall that (ii) and (iii) imply that  $V_i(x) \geq c|x|^{2+\delta}$  for  $|x|$  large, so it is a stronger condition than (A<sub>1</sub>). We mention that the functions given in the following examples satisfy our conditions (A<sub>1</sub>)–(A<sub>3</sub>):

Ex1.  $V'_i(x) = x \ln(1 + |x|)$ .

Ex2.  $V_i(x) = |x|^\mu + (\mu - 2)|x|^{\mu-\epsilon} \sin^2(|x|^\epsilon/\epsilon)$  where  $\mu > 2$  and  $\epsilon \in (0, \mu - 2)$ .

Remark that these functions do not satisfy (ii). Thus the superlinear assumptions of Theorem 1.1 are indeed more general than [6]. For getting more examples satisfying the superlinear conditions we show that (A<sub>3</sub>) holds provided that  $V'_i(x)$  satisfies:

- (B<sub>1</sub>) there exist  $r, c_1 > 0$  and  $p > 2$  such that  $|V'_i(x)| \leq c_1|x|^{p-1}$  if  $|x| \geq r$ ;  
 (B<sub>2</sub>)  $2V_i(x) < V'_i(x)x$  if  $x \neq 0$ , and there exist  $r, c_2 > 0$  and  $\omega \in (0, 2)$  such that, for all  $|x| \geq r$ ,

$$V_i(x) \leq \left( \frac{1}{2} - \frac{1}{c_2|x|^\omega} \right) V'_i(x)x.$$

In fact, it is easy to check that (B<sub>1</sub>) and (B<sub>2</sub>) imply that  $|V'_i(x)|^\sigma \leq c_3 \tilde{V}_i(x)|x|^\sigma$  for all  $|x| \geq r$  with  $1 < \sigma \leq (p - \omega)/(p - 2)$ .

Our argument is variational and follows the lines of Arioli and Szulkin [6]. The most interesting result here, in Theorem 1.1, refers to the multiplicity. In [6] Arioli and Szulkin proved that system (1.2) admits infinitely many geometrically distinct solutions with  $\alpha_i \leq 0$  for all  $i$ , but in our result we assume the coefficients  $\alpha_i$  can take both signs. Indeed, if  $\alpha_i \leq 0$  for all  $i$ , then  $J(q) \leq 0$  for any  $q$  which is constant; then if one obtains  $J(u) > 0$  for some  $u$ , it must be nonconstant. Hence it is easy to obtain infinitely many geometrically distinct solutions of period  $T > 0$  since any nonconstant solution has minimum positive period and  $T/l$  ( $l \in \mathbb{N}$ ) periodic solution is also  $T$  periodic (see [6] for more details). But in our case, it is not a trivial fact and becomes quite complicated.

Obviously, under our assumptions, the functional associated to system (1.2) is invariant under the action of the non-compact group  $\mathbb{Z}$  and the continuous compact group  $S^1$ . Due to the translation invariance, the functional does not satisfy the Palais–Smale condition. It is worth mentioning that under our assumptions, we do not know whether the Palais–Smale sequences for our functional are bounded. However, we can check that any Cerami sequence is bounded (see Lemma 4.1 in Section 4). Cerami sequences are often used instead of Palais–Smale sequences in order to weaken the classical Ambrosetti–Rabinowitz condition.

The remainder of this paper is organized as follows. In Section 2 we will study  $\sigma(L)$ , the spectrum of the operator  $L$ . Based on the description on  $\sigma(L)$ , we derive a variational setting for system (1.2) and represent the associated functional in the form  $J(q) = \frac{1}{2}(\|q^-\|^2 - \|q^+\|^2) - V(q)$  defined on the Hilbert space  $H$  with decomposition  $H = H^- \oplus H^+$ , where  $q = q^- + q^+$  and  $\dim H^\pm = \infty$ . In Section 3 we show the linking structure of  $J$ , that is,  $\inf J(S_\rho H^+) > 0$  for some  $\rho > 0$  and that there is a sequence  $(Y_n) \subset H^+$  of finite-dimensional subspaces such that  $J(q) \rightarrow -\infty$  as  $\|q\| \rightarrow \infty$  in  $H_n = H^- \oplus Y_n$ . In Section 4 we verify the boundedness of Cerami sequences for  $J$  and prove that, for any bounded interval  $I \subset \mathbb{R}$ , there is a discrete set  $\mathcal{A}$ , called  $(C)_I$ -attractors, consisting of finite sums of critical points of  $J$  so that any Cerami sequence at level  $c \in I$  converges to  $\mathcal{A}$ . We recall some abstract critical point theorem about existence recently developed in [10] that we present in Section 5 and we also show the existence of Theorem 1.1 in this section. Finally, we extend the abstract critical point theorem about multiplicity of a strongly indefinite functional developed by Bartsch and Ding in [10] to a more general class of symmetry and establish the multiplicity result of Theorem 1.1 in Section 6.

**Notation.** Throughout this paper we shall denote by  $c > 0$  for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of problem.  $B(a, \rho)$  denotes an open ball centered at  $a$  and having radius  $\rho$ . Furthermore,  $B_\rho := B(0, \rho)$ . The closure of a set  $A$  is denoted by  $\bar{A}$ . We denote by  $\|q - V\|$  and  $\|U - V\|$  (in the topology induced by the norm  $\|\cdot\|$ ) the distance from the point  $q$  and the set  $U$  to the set  $V$ , respectively.

## 2. Variational setting and preliminary lemmas

In this paper, we work in the following Hilbert space

$$H = \left\{ q \in H^1(S^1, \mathbb{R})^{\mathbb{Z}} : \int_0^T q_0(t) dt = 0, \sum_i \int_0^T [\dot{q}_i^2(t) + (q_i(t) - q_{i+1}(t))^2] dt < \infty \right\}$$

which is endowed with the scalar product

$$(q, p)_H = \sum_i \int_0^T [\dot{q}_i(t) \dot{p}_i(t) + (q_i(t) - q_{i+1}(t))(p_i(t) - p_{i+1}(t))] dt. \quad (2.1)$$

Here  $H^1(S^1, \mathbb{R})$  is the usual Hilbert space equipped with the norm

$$\|q_i\|_{H^1} = \left( \int_0^T (|\dot{q}_i(t)|^2 + |q_i(t)|^2) dt \right)^{1/2}.$$

Note that  $\int_0^T q_0(t) dt = 0$  in the definition of  $H$  in order to have (2.1) defining a scalar product; this causes no loss of generality as has been observed in the preceding section. We denote by  $\|\cdot\|_H$  the norm induced by (2.1) and  $\|\cdot\|_p$  the norm of  $L^p(S^1, \mathbb{R})$  for  $p \in [1, +\infty]$ . For  $q \in H$  we denote by  $\Omega q = \bigcup_{\theta \in S^1} \{\Omega_\theta q\}$  the orbit of such a point under the  $S^1$ -action. Obviously,  $\Omega q$  is a compact set. It is well known that  $H^1(S^1, \mathbb{R})$  is compactly embedded in  $L^\infty(S^1, \mathbb{R})$ . We will seek solutions of (1.2) as critical points of the functional  $J : H \rightarrow \mathbb{R}$  associated with (1.2) and given by

$$J(q) = \frac{1}{2} \int_0^T |\dot{q}(t)|^2 dt - \int_0^T \Phi(q(t)) dt.$$

We define a self-adjoint linear operator  $L : H \rightarrow H$  by

$$(Lq, p)_H = \sum_i \int_0^T [\dot{q}_i(t) \dot{p}_i(t) + \alpha_i (q_i(t) - q_{i+1}(t))(p_i(t) - p_{i+1}(t))] dt,$$

and a functional  $V : H \rightarrow \mathbb{R}$  by

$$V(q) = \sum_i \int_0^T V_i(q_i(t) - q_{i+1}(t)) dt,$$

then we have  $J(q) = \frac{1}{2}(Lq, q)_H - V(q)$ .

**Proposition 2.1.** Assume that  $(A_2)$  and  $(A_4)$  hold. Then  $J \in C^1(H, \mathbb{R})$ .

**Proof.** The proof is similar to that of Proposition 2.1 in [6] and is omitted.  $\square$

In order to establish a variational setting for system (1.2) we study the spectrum of the linear operator  $L$  (we denote it by  $\sigma(L)$ ). The original idea goes back to Arioli and Szulkin [6].

If  $\alpha_j = 0$  for some integer  $j$ , it is clear that the operator  $L$  is not invertible. Indeed (assume without loss of generality  $j \geq 0$ ) we can define the vector  $q \in H$  by  $q_i = 0$  if  $i \leq j$ ; and  $q_i = 2$  if  $i > j$ . Then it is easy to check that  $Lq = 0$ . We remark that this case was proved in [5] (also in [6]) via a finite lattice approximation and a suitable limit procedure. On the other hand, it is clear that if  $\alpha_i > 0$  for all  $i$  and  $(A_4)$  holds, then  $\sigma(L) \subset (0, +\infty)$  and the quadratic form  $q \mapsto (Lq, q)_H$  is positive definite; this case has already been considered in [4]. In this paper, we always assume the coefficients  $\alpha_i \neq 0$  and take both signs.

We need the following notation. Let  $I \subsetneq \mathbb{Z}$  be the set of indices  $i$  such that  $\alpha_i > 0$ , let  $H^-$  be the subspace of constant functions  $q \in H$  satisfying  $q_i - q_{i+1} = 0$  for all  $i \in I$  and  $H^+$  its orthogonal complement. We have the following lemma.

**Lemma 2.2.** Assume  $(A_4)$  holds. If the coefficients  $\alpha_i \neq 0$  and take both signs and  $T < \pi/\sqrt{\beta}$ , then there exists  $\lambda > 0$  such that  $(Lq, q)_H \leq -\lambda \|q\|_H^2$  for all  $q \in H^-$  and  $(Lq, q)_H \geq \lambda \|q\|_H^2$  for all  $q \in H^+$ . In particular  $L$  is invertible.

**Proof.** The proof is similar to that of Lemma 3.1 in [6] and is omitted.  $\square$

**Remark 2.3.** If the coefficients  $\alpha_i$  do not depend on  $i$ , then the condition  $T < \pi/\sqrt{\beta}$  assumed in Lemma 2.2 is also necessary for  $L$  to be invertible (see Lemma 3.2 in [6]).

In virtue of Lemma 2.2, we may define a new scalar product  $(\cdot, \cdot)$  on  $H$  with corresponding norm  $\|\cdot\|$  such that  $(Lq, q)_H = -\|q\|^2$  for  $q \in H^-$ , and  $(Lq, q)_H = \|q\|^2$  for  $q \in H^+$ . In fact  $(q, p) = (Lq^+, p^+)_H - (Lq^-, p^-)_H$  for  $q = q^- + q^+$ ,  $p = p^- + p^+ \in H^- \oplus H^+$ . Since  $\sigma(L) \subset \mathbb{R} \setminus (-\lambda, \lambda)$ , it is easy to check that the norm  $\|\cdot\|$  is equivalent to the standard norm  $\|\cdot\|_H$  in  $H$ . Hence we have the decomposition  $H = H^- \oplus H^+$  orthogonal with respect to  $(\cdot, \cdot)$ . Moreover, on  $H$  we can rewrite the functional  $J(q)$  as follows

$$J(q) := \frac{1}{2} \|q^+\|^2 - \frac{1}{2} \|q^-\|^2 - V(q), \quad (2.2)$$

where  $q = q^- + q^+ \in H^- \oplus H^+$ . Let  $P^- : H \rightarrow H^-$  and  $P^+ : H \rightarrow H^+$  be the orthogonal projections. Since the spaces  $H^-$  and  $H^+$  are  $L$ -invariant, they are  $\mathbb{Z}$ -invariant. Indeed, spectral theory asserts that the projectors  $P^-, P^+$  commute with any operator which commutes with  $L$ ; in particular, they commute with the  $\mathbb{Z}$ -action.

In order to prove our main results, we need the following important lemma.

**Lemma 2.4.** If  $(q^{(n)})$  is a bounded sequence in  $H$ , then passing to a subsequence, there exists  $q \in H$  such that  $q^{(n)} \rightharpoonup q$  in  $H$ . Moreover, we have  $q_i^{(n)} \rightharpoonup q_i$  in  $H^1$  and  $q_i^{(n)} \rightarrow q_i$  in  $L^\infty$  for all  $i \in \mathbb{Z}$ .

**Proof.** Since  $H$  is a Hilbert space, passing to a subsequence, still denoted by  $(q^{(n)})$ , there exists  $q \in H$  such that  $q^{(n)} \rightharpoonup q$  in  $H$ . In the following, the proof will be divided into three steps.

*Step 1.* For every  $i \in \mathbb{Z}$ ,  $(q_i^{(n)})_{n \in \mathbb{N}}$  is bounded in  $H^1(S^1, \mathbb{R})$ .

Since  $(q^{(n)})$  is bounded, there exists  $c > 0$  such that  $\|q^{(n)}\| \leq c$ . Obviously, for all  $i \in \mathbb{Z}$  and  $n \in \mathbb{N}$ ,  $\int_0^T |\dot{q}_i^{(n)}|^2$ ,  $\int_0^T |q_i^{(n)} - q_{i+1}^{(n)}|^2 \leq c$ . Since  $\int_0^T q_0^{(n)} = 0$ , by Wirtinger's inequality, one has

$$\int_0^T |q_0^{(n)}|^2 \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{q}_0^{(n)}|^2 \leq c.$$

Then for  $i \in \mathbb{Z} \setminus \{0\}$ , we conclude that

$$\int_0^T |q_i^{(n)}|^2 \leq \int_0^T |q_0^{(n)}|^2 + \sum_{j=0}^{i-1} \int_0^T |q_j^{(n)} - q_{j+1}^{(n)}|^2 \leq c, \quad \text{if } i \geq 1,$$

and

$$\int_0^T |q_i^{(n)}|^2 \leq \int_0^T |q_0^{(n)}|^2 + \sum_{j=i}^{-1} \int_0^T |q_j^{(n)} - q_{j+1}^{(n)}|^2 \leq c, \quad \text{if } i \leq -1.$$

Therefore, for every  $i$ ,  $\|q_i^{(n)}\|_{H^1}^2 = \int_0^T |\dot{q}_i^{(n)}|^2 + |q_i^{(n)}|^2 \leq c$  for all  $n$ . Consequently, for all  $i \in \mathbb{Z}$ , there exists  $p_i \in H^1$ , such that  $q_i^{(n)} \rightharpoonup p_i$  in  $H^1$  and  $q_i^{(n)} \rightarrow p_i$  in  $L^\infty$  by the compact embedding of  $H^1$  into  $L^\infty$ . We denote  $p = \{p_i\}_{i \in \mathbb{Z}}$ .

Step 2.  $p \in H$ .

Indeed, using Fatou's lemma and the fact that  $q_i^{(n)} - q_{i+1}^{(n)} \rightarrow p_i - p_{i+1}$  in  $L^2$  for all  $i$ , we have

$$\sum_i \|p_i - p_{i+1}\|_2^2 = \sum_i \lim_{n \rightarrow \infty} \|q_i^{(n)} - q_{i+1}^{(n)}\|_2^2 \leq \liminf_{n \rightarrow \infty} \sum_i \|q_i^{(n)} - q_{i+1}^{(n)}\|_2^2 \leq c. \quad (2.3)$$

On the other hand, since  $q_i^{(n)} \rightharpoonup p_i$  in  $H^1$  and  $q_i^{(n)} \rightarrow p_i$  in  $L^2$ , using the weakly lower semi-continuity of  $\|\cdot\|_{H^1}^2$ , one has  $\|\dot{p}_i\|_2^2 \leq \liminf_{n \rightarrow \infty} \|\dot{q}_i^{(n)}\|_2^2$ . It follows from Fatou's lemma that

$$\sum_i \|\dot{p}_i\|_2^2 \leq \sum_i \liminf_{n \rightarrow \infty} \|\dot{q}_i^{(n)}\|_2^2 \leq \liminf_{n \rightarrow \infty} \sum_i \|\dot{q}_i^{(n)}\|_2^2 \leq c,$$

which, jointly with (2.3) and the obvious fact that  $\int_0^T p_0 = 0$ , implies that  $p \in H$ .

Step 3.  $q = p$ .

Since  $q_i^{(n)} \rightharpoonup p_i$  in  $H^1$ , for every  $h \in H$ , we have

$$\int_0^T \dot{q}_i^{(n)} \dot{h}_i \rightarrow \int_0^T \dot{p}_i \dot{h}_i \quad (2.4)$$

and

$$\int_0^T (q_i^{(n)} - q_{i+1}^{(n)})(h_i - h_{i+1}) \rightarrow \int_0^T (p_i - p_{i+1})(h_i - h_{i+1}).$$

Since  $h \in H$ , for any  $\varepsilon > 0$ , there exists a positive integer  $N$  such that

$$\sum_{|i| \geq N} (\|\dot{h}_i\|_2^2 + \|h_i - h_{i+1}\|_2^2) < \varepsilon,$$

which implies, by Hölder's inequality in  $L^2$  and  $l^2$ ,

$$\begin{aligned} \sum_{|i| \geq N} \int_0^T (\dot{q}_i^{(n)} - \dot{p}_i) \dot{h}_i &\leq \sum_{|i| \geq N} \|\dot{q}_i^{(n)} - \dot{p}_i\|_2 \|\dot{h}_i\|_2 \\ &\leq \left( \sum_{|i| \geq N} \|\dot{q}_i^{(n)} - \dot{p}_i\|_2^2 \right)^{1/2} \left( \sum_{|i| \geq N} \|\dot{h}_i\|_2^2 \right)^{1/2} \\ &< c\varepsilon^{1/2}. \end{aligned} \quad (2.5)$$

Furthermore, by (2.4), for  $n$  large enough, one has

$$\sum_{|i| < N} \int_0^T (\dot{q}_i^{(n)} - \dot{p}_i) \dot{h}_i < \varepsilon. \quad (2.6)$$

Then, by (2.5) and (2.6), we conclude that

$$\sum_i \int_0^T \dot{q}_i^{(n)} \dot{h}_i \longrightarrow \sum_i \int_0^T \dot{p}_i \dot{h}_i. \quad (2.7)$$

Similarly, we can get

$$\sum_i \int_0^T (q_i^{(n)} - q_{i+1}^{(n)})(h_i - h_{i+1}) \longrightarrow \sum_i \int_0^T (p_i - p_{i+1})(h_i - h_{i+1}). \quad (2.8)$$

From (2.7) and (2.8), we have  $(q^{(n)}, h) \rightarrow (p, h)$ . Since  $h$  is arbitrary, it follows from  $q^{(n)} \rightharpoonup q$  that  $q = p$ , implying  $q_i^{(n)} \rightarrow q_i$  in  $H^1$  and  $q_i^{(n)} \rightarrow q_i$  in  $L^\infty$  for all  $i \in \mathbb{Z}$ .  $\square$

**Remark 2.5.** Though  $q^{(n)} \rightharpoonup q$  in  $H$ , it is not easy to see that  $q_i^{(n)} \rightarrow q_i$  in  $H^1$ ; notice that this fact is not trivial as we deal with a series of integrals and the definition of  $\|\cdot\|$  in  $H$ .

Recall that  $V$  is said to be weakly sequentially lower semi-continuous if for any  $q^{(n)} \rightharpoonup q$  in  $H$  one has  $V(q) \leq \liminf_{n \rightarrow \infty} V(q^{(n)})$ , and  $V'$  is said to be weakly sequentially continuous if  $\lim_{n \rightarrow \infty} V'(q^{(n)})h = V'(q)h$  for each  $h \in H$ .

**Lemma 2.6.** Suppose  $(A_1) - (A_4)$  hold, then  $V$  is non-negative and weakly sequentially lower semi-continuous, and  $V'$  is weakly sequentially continuous.

**Proof.** By  $(A_1)$ ,  $V_i(x)$  is non-negative, and so is  $V$ . Let  $q^{(n)} \rightharpoonup q$  in  $H$ ; by Lemma 2.4, one has  $q_i^{(n)} \rightarrow q_i$  in  $L^\infty$  for all  $i$ . Then  $V_i(q_i^{(n)} - q_{i+1}^{(n)}) \rightarrow V_i(q_i - q_{i+1})$  for all  $i$ , since  $V_i \in C^1(\mathbb{R}, \mathbb{R})$ . Thus, by Fatou's lemma, we deduce that

$$V(q) = \sum_i \int_0^T \lim_{n \rightarrow \infty} V_i(q_i^{(n)} - q_{i+1}^{(n)}) \leq \liminf_{n \rightarrow \infty} \sum_i \int_0^T V_i(q_i^{(n)} - q_{i+1}^{(n)}) = \liminf_{n \rightarrow \infty} V(q^{(n)}),$$

proving that  $V$  is weakly sequentially lower semi-continuous.

To show that  $V'$  is weakly sequentially continuous, let  $q^{(n)} \rightharpoonup q$  in  $H$ . We have that  $q_i^{(n)} \rightarrow q_i$  in  $L^\infty$  for all  $i$ . Noting that  $H^1(S^1, \mathbb{R})$  (compactly) embeds into  $L^\infty(S^1, \mathbb{R})$ , for all  $p \in H$  and  $i \in \mathbb{Z}$ , we have

$$\|p_i - p_{i+1}\|_\infty \leq c(\|\dot{p}_i - \dot{p}_{i+1}\|_2 + \|p_i - p_{i+1}\|_2) \leq c(\|\dot{p}_i\|_2 + \|\dot{p}_{i+1}\|_2 + \|p_i - p_{i+1}\|_2),$$

which, since  $(\alpha + \beta + \gamma)^2 \leq 3(\alpha^2 + \beta^2 + \gamma^2)$  for all  $\alpha, \beta, \gamma \in \mathbb{R}$ , implies that

$$\sum_i \|p_i - p_{i+1}\|_\infty^2 \leq c\|p\|^2. \quad (2.9)$$



Obviously, for all  $\zeta > 0$ , we have  $\|p_i - p_{i+1}\|_\infty > \zeta$  only for a finite number of indices. Given  $\varepsilon > 0$ , for any  $h \in H$ , by (2.9), there exists  $N \in \mathbb{N}$  such that

$$\sum_{|i| \geq N} \|q_i - q_{i+1}\|_\infty^2 < \varepsilon, \quad (2.10)$$

and

$$\sum_{|i| \geq N} \int_0^T [|\dot{h}_i|^2 + |h_i - h_{i+1}|^2] < \varepsilon. \quad (2.11)$$

For such  $N$ , by the uniform convergence of  $q_i^{(n)}$ , we have

$$\left| \sum_{|i| < N} \int_0^T [V'_i(q_i^{(n)} - q_{i+1}^{(n)}) - V'_i(q_i - q_{i+1})](h_i - h_{i+1}) \right| < \varepsilon$$

for large  $n$ . On the other hand, note that by  $(A_2)$ – $(A_4)$ , for any  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that

$$|V'_i(x)| \leq \varepsilon |x| + C_\varepsilon |x|^{p-1} \quad (2.12)$$

for all  $x \in \mathbb{R}$  and  $i \in \mathbb{Z}$  where  $p \geq 2\sigma/(\sigma - 1)$ . Remark that  $p > 2$  since  $\sigma > 1$ . Then using (2.10)–(2.12),  $(A_2)$  and Hölder's inequality in  $L^2$  and in  $l^2$ , for  $n$  large enough, one has

$$\begin{aligned} & \left| \sum_{|i| \geq N} \int_0^T [V'_i(q_i^{(n)} - q_{i+1}^{(n)}) - V'_i(q_i - q_{i+1})](h_i - h_{i+1}) \right| \\ & \leq \sum_{|i| \geq N} \int_0^T (\varepsilon |q_i^{(n)} - q_{i+1}^{(n)}| + C_\varepsilon |q_i^{(n)} - q_{i+1}^{(n)}|^{p-1} + c|q_i - q_{i+1}|) |h_i - h_{i+1}| \\ & \leq (\varepsilon + C_\varepsilon \sup_i \|d_i^{(n)}\|_\infty^{p-2}) \left( \sum_{|i| \geq N} \|d_i^{(n)}\|_2^2 \right)^{1/2} \left( \sum_{|i| \geq N} \|h_i - h_{i+1}\|_2^2 \right)^{1/2} \\ & \quad + c \left( \sum_{|i| \geq N} \|q_i - q_{i+1}\|_2^2 \right)^{1/2} \left( \sum_{|i| \geq N} \|h_i - h_{i+1}\|_2^2 \right)^{1/2} \\ & < c\varepsilon^{1/2}, \end{aligned}$$

where  $d_i^{(n)} = q_i^{(n)} - q_{i+1}^{(n)}$  and  $\sup_i \|d_i^{(n)}\|_\infty \leq c_0 \|q^{(n)}\|$ . Hence, we obtain that

$$|V'(q^{(n)})h - V'(q)h| < \varepsilon$$

for large  $n$ , proving the weakly sequentially continuity.  $\square$

**Remark 2.7.** Since  $\nu : H \rightarrow \mathbb{R}$  with  $\nu(q) = \|q\|^2$  is  $C^1$  and  $\nu'$  weakly sequentially continuous, by Lemma 2.6, we obtain  $J'$  is weakly sequentially continuous.

### 3. The linking structure

In this section we discuss the linking structure of the functional  $J$ . Firstly we have the following lemma.

**Lemma 3.1.** Assume that  $(A_2)$  and  $(A_4)$  hold, then  $V(q) = o(\|q\|^2)$  as  $q \rightarrow 0$ .

**Proof.** By  $(A_2)$ ,  $V_i(x) = o(|x|^2)$  as  $x \rightarrow 0$ . This, jointly with  $(A_4)$  and (2.9), implies the lemma.  $\square$

From Lemmas 2.2 and 3.1, we can easily obtain

**Lemma 3.2.** Assume  $\Phi$  satisfies  $(A_2)$  and  $(A_4)$ . If the coefficients  $\alpha_i \neq 0$  and take both signs and  $T < \pi/\sqrt{\beta}$ , then there exists  $\rho > 0$  such that  $\kappa := \inf J(S_\rho H^+) > 0$ , where  $S_\rho H^+ := \{q \in H^+ : \|q\| = \rho\}$ .

In the following, for any finite dimensional subspace  $Y$  of  $H^+$ , we set  $H_Y = H^- \oplus Y$ .

**Lemma 3.3.** Let  $(A_1)$ – $(A_4)$  be satisfied. If the coefficients  $\alpha_i \neq 0$  and take both signs and  $T < \pi/\sqrt{\beta}$ , then for any finite dimensional subspace  $Y$  of  $H^+$ ,  $\sup J(H_Y) < \infty$ , and there exists  $R_Y > 0$  such that  $J(q) < \inf J(B_\rho)$  for all  $q \in H_Y$  with  $\|q\| \geq R_Y$ .

**Proof.** It is sufficient to show that  $J(q) \rightarrow -\infty$  as  $q \in H_Y$ ,  $\|q\| \rightarrow \infty$ . Arguing indirectly, assume that for some sequence  $(q^{(n)}) \subset H_Y$  with  $\|q^{(n)}\| \rightarrow \infty$ , there is  $C > 0$  such that  $J(q^{(n)}) \geq -C$  for all  $n$ . Setting  $h^{(n)} = q^{(n)}/\|q^{(n)}\|$ , we have  $\|h^{(n)}\| = 1$ . Then, passing to a subsequence, we may assume that  $h^{(n)} \rightharpoonup h$ ,  $h^{(n)-} \rightharpoonup h^-$ ,  $h^{(n)+} \rightharpoonup h^+ \in Y$  and

$$-\frac{C}{\|q^{(n)}\|^2} \leq \frac{J(q^{(n)})}{\|q^{(n)}\|^2} = \frac{1}{2}\|h^{(n)+}\|^2 - \frac{1}{2}\|h^{(n)-}\|^2 - \frac{V(q^{(n)})}{\|q^{(n)}\|^2}. \quad (3.1)$$

Remark that  $h^+ \neq 0$ . Indeed, if not then it follows from  $(A_1)$  and (3.1) that

$$0 \leq \frac{1}{2}\|h^{(n)-}\|^2 + \frac{V(q^{(n)})}{\|q^{(n)}\|^2} \leq \frac{1}{2}\|h^{(n)+}\|^2 + \frac{C}{\|q^{(n)}\|^2} \rightarrow 0,$$

in particular,  $\|h^{(n)-}\| \rightarrow 0$ , then  $1 = \|h^{(n)}\| \rightarrow 0$ , a contradiction. Hence by the definition of  $H^+$ , there exists some  $j \in \mathbb{Z}$  such that  $h_j - h_{j+1} \neq 0$ . This implies that  $|d_j| = \|q^{(n)}\|\tilde{d}_j \rightarrow \infty$  where  $d_j := q_j^{(n)} - q_{j+1}^{(n)}$  and  $\tilde{d}_j := h_j^{(n)} - h_{j+1}^{(n)}$  (here we omit the index  $(n)$  to simplify the notation). It follows from  $(A_1)$  and Fatou's lemma that

$$\int_0^T \frac{V_j(d_j)}{d_j^2} \tilde{d}_j^2 \rightarrow +\infty.$$

Then by (3.1), we have

$$0 \leq \limsup_{n \rightarrow \infty} \frac{J(q^{(n)})}{\|q^{(n)}\|^2} \leq \frac{1}{2}\|h^+\|^2 - \frac{1}{2}\|h^-\|^2 - \liminf_{n \rightarrow \infty} \int_0^T \frac{V_j(d_j)}{d_j^2} \tilde{d}_j^2 = -\infty,$$

a contradiction.  $\square$

As a special case we have

**Lemma 3.4.** Let  $(A_1)–(A_4)$  be satisfied and  $\kappa > 0$  be given by Lemma 3.2. If the coefficients  $\alpha_i \neq 0$  and take both signs and  $T < \pi/\sqrt{\beta}$ , then there are  $r_0 > 0$  and  $e \in H^+$  with  $\|e\| = 1$  such that  $\sup J(\partial Q) \leq \kappa$ , where  $Q := \{q = q^- + \gamma e: \gamma \geq 0, q^- \in H^-, \|q\| \leq r_0\}$ .

#### 4. The Cerami sequences

In this section we study the properties of Cerami sequences. Recall that a sequence  $(q^{(n)}) \subset H$  is said to be a Cerami sequence at the level  $c$  ( $(C)_c$ -sequence for short) for  $J$  if  $J(q^{(n)}) \rightarrow c$  and  $(1 + \|q^{(n)}\|)J'(q^{(n)}) \rightarrow 0$  (cf. [11]). Firstly, we consider the boundedness of the  $(C)_c$  sequence. We have the following result.

**Lemma 4.1.** Assume that the coefficients  $\alpha_i \neq 0$  and take both signs,  $T < \pi/\sqrt{\beta}$  and  $(A_1)–(A_4)$  are satisfied. Then any  $(C)_c$  sequence is bounded and  $c \geq 0$ .

**Proof.** Let  $(q^{(n)}) \subset H$  be such that

$$J(q^{(n)}) \rightarrow c \quad \text{and} \quad (1 + \|q^{(n)}\|)J'(q^{(n)}) \rightarrow 0. \quad (4.1)$$

Then by  $(A_3)$ , for  $n$  large, we have

$$1 + c \geq J(q^{(n)}) - \frac{1}{2}J'(q^{(n)})q^{(n)} = \sum_i \int_0^T \tilde{V}_i(d_i) \geq 0, \quad (4.2)$$

where  $d_i = q_i^{(n)} - q_{i+1}^{(n)}$ . Arguing indirectly, assume by contradiction that, up to a subsequence  $\|q^{(n)}\| \rightarrow \infty$ . Set  $p^{(n)} = q^{(n)}/\|q^{(n)}\|$  and  $\hat{d}_i = p_i^{(n)} - p_{i+1}^{(n)}$ . Then  $\|p^{(n)}\| = 1$  and  $\hat{d}_i = d_i/\|q^{(n)}\|$ . Noting that

$$J'(q^{(n)})(q^{(n)+} - q^{(n)-}) = \|q^{(n)}\|^2 \left( 1 - \sum_i \int_0^T \frac{V'_i(d_i)(\hat{d}_i^+ - \hat{d}_i^-)}{\|q^{(n)}\|} \right),$$

by (4.1), we have

$$\sum_i \int_0^T \frac{V'_i(d_i)(\hat{d}_i^+ - \hat{d}_i^-)}{\|q^{(n)}\|} \rightarrow 1. \quad (4.3)$$

We denote for  $r \geq 0$ ,

$$g(r) := \inf_i \{ \tilde{V}_i(x): x \in \mathbb{R} \text{ with } |x| \geq r \}.$$

Observe that, from  $(A_3)$  and  $(A_4)$ ,  $g(r) > 0$  for all  $r > 0$ . Moreover, for  $|x| \geq r_0$ , one has

$$a\tilde{V}_i(x) \geq \left( \frac{|V'_i(x)|}{|x|} \right)^\sigma = \left( \frac{|V'_i(x)||x|}{|x|^2} \right)^\sigma \geq \left( \frac{V'_i(x)x}{|x|^2} \right)^\sigma \geq \left( \frac{2V_i(x)}{|x|^2} \right)^\sigma,$$

which, jointly with  $(A_1)$ , implies  $\tilde{V}_i(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$  for all  $i$ , consequently,  $g(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . Fix  $n$  and for all integers  $i$ , set

$$\Omega_i(a, b) = \{t \in [0, T]: a \leq |d_i(t)| < b\},$$

where  $0 \leq a < b$ . Let

$$c_a^b := \inf_i \left\{ \frac{\tilde{V}_i(x)}{x^2} : x \in \mathbb{R} \text{ with } 0 < a \leq |x| \leq b \right\}.$$

It follows from  $(A_3)$  and  $(A_4)$  that,  $c_a^b > 0$  and for all  $i$ ,

$$\tilde{V}_i(d_i(t)) \geq c_a^b |d_i(t)|^2 \quad \text{for any } t \in \Omega_i(a, b).$$

Then by (4.2), one has

$$\begin{aligned} 1 + c &\geq \sum_i \int_{\Omega_i(0, a)} \tilde{V}_i(d_i) + \sum_i \int_{\Omega_i(a, b)} \tilde{V}_i(d_i) + \sum_i \int_{\Omega_i(b, \infty)} \tilde{V}_i(d_i) \\ &\geq \sum_i \int_{\Omega_i(0, a)} \tilde{V}_i(d_i) + c_a^b \sum_i \int_{\Omega_i(a, b)} |d_i|^2 + g(b) \sum_i |\Omega_i(b, \infty)|, \end{aligned}$$

which implies

$$\sum_i |\Omega_i(b, \infty)| \leq \frac{1 + c}{g(b)} \rightarrow 0 \quad (4.4)$$

as  $b \rightarrow \infty$  uniformly in  $n$ , and for any fixed  $0 < a < b$ ,

$$\sum_i \int_{\Omega_i(a, b)} |\hat{d}_i|^2 = \frac{1}{\|q^{(n)}\|^2} \sum_i \int_{\Omega_i(a, b)} |d_i|^2 \leq \frac{1 + c}{c_a^b \|q^{(n)}\|^2} \rightarrow 0 \quad (4.5)$$

as  $n \rightarrow \infty$ .

Let  $0 < \varepsilon < 1/3$ . By  $(A_2)$  and  $(A_4)$ , there is  $a_\varepsilon > 0$  such that  $|V'_i(x)| \leq \varepsilon|x|$  for all  $i$  and  $|x| \leq a_\varepsilon$ , consequently,

$$\begin{aligned} \sum_i \int_{\Omega_i(0, a_\varepsilon)} \frac{|V'_i(d_i)|}{|d_i|} |\hat{d}_i^+ - \hat{d}_i^-| |\hat{d}_i| &\leq \sum_i \int_{\Omega_i(0, a_\varepsilon)} \varepsilon |\hat{d}_i^+ - \hat{d}_i^-| |\hat{d}_i| \\ &\leq \varepsilon \sum_i \|\hat{d}_i\|_2^2 \\ &\leq \varepsilon \end{aligned} \quad (4.6)$$

for all  $n$ . Invoking  $(A_3)$ , set  $\tau := 2\sigma/(\sigma - 1)$  and  $\sigma' = \tau/2$ , it follows that  $\tau > 2$  and  $1/\sigma + 1/\sigma' = 1$ . By  $(A_3)$ , (4.2), (4.4) and Hölder's inequality, we can take  $b_\varepsilon \geq r_0$  large so that

$$\begin{aligned} \sum_i \int_{\Omega_i(b_\varepsilon, \infty)} \frac{|V'_i(d_i)|}{|d_i|} |\hat{d}_i^+ - \hat{d}_i^-| |\hat{d}_i| \\ \leq \left( \sum_i \int_{\Omega_i(b_\varepsilon, \infty)} \frac{|V'_i(d_i)|^\sigma}{|d_i|^\sigma} \right)^{1/\sigma} \left( \sum_i \int_{\Omega_i(b_\varepsilon, \infty)} (|\hat{d}_i^+ - \hat{d}_i^-| |\hat{d}_i|)^{\sigma'} \right)^{1/\sigma'} \end{aligned}$$

$$\begin{aligned}
 &\leq \left( \sum_i \int_0^T c_0 \tilde{V}_i(d_i) \right)^{1/\sigma} \left( \sum_i \int_0^T |\hat{d}_i^+ - \hat{d}_i^-|^\tau \right)^{1/\tau} \left( \sum_i \int_{\Omega_i(b_\varepsilon, \infty)} |\hat{d}_i|^\tau \right)^{1/\tau} \\
 &\leq c_1 \sup_i \|\hat{d}_i\|_\infty \left( \sum_i |\Omega_i(b_\varepsilon, \infty)| \right)^{1/\tau} \\
 &< \varepsilon
 \end{aligned} \tag{4.7}$$

for all  $n$ . Note that by (A<sub>4</sub>), there is  $\gamma = \gamma(\varepsilon) > 0$  independent of  $n$  such that  $|V'_i(d_i(t))| \leq \gamma |d_i(t)|$  for all  $i$  and  $x \in \Omega_i(a_\varepsilon, b_\varepsilon)$ . By (4.5) and Hölder's inequality, there is  $n_0$  such that

$$\begin{aligned}
 \sum_i \int_{\Omega_i(a_\varepsilon, b_\varepsilon)} \frac{|V'_i(d_i)|}{|d_i|} |\hat{d}_i^+ - \hat{d}_i^-| |\hat{d}_i| &\leq \gamma \sum_i \int_{\Omega_i(a_\varepsilon, b_\varepsilon)} |\hat{d}_i^+ - \hat{d}_i^-| |\hat{d}_i| \\
 &\leq \gamma \sum_i \|\hat{d}_i\|_2^2 \left( \sum_i \int_{\Omega_i(a_\varepsilon, b_\varepsilon)} |\hat{d}_i|^2 \right)^{1/2} \\
 &< \varepsilon
 \end{aligned} \tag{4.8}$$

for all  $n \geq n_0$ . Now the combination of (4.6)–(4.8) implies that for  $n \geq n_0$

$$\sum_i \int_0^T \frac{V'_i(d_i)(\hat{d}_i^+ - \hat{d}_i^-)}{\|q^{(n)}\|} < 3\varepsilon < 1,$$

which contradicts (4.3). Hence  $(q^{(n)})$  is bounded.  $c \geq 0$  follows from (4.1) and (4.2).  $\square$

**Lemma 4.2.** Under the assumptions of Lemma 4.1, let  $(q^{(n)})$  be a  $(C)_c$  sequence for  $J$ . Then either

- (i)  $q^{(n)} \rightarrow 0$  (and hence  $c = 0$ ), or
- (ii) there exist  $\eta > 0$  and a sequence  $(i_n) \subset \mathbb{Z}$  such that  $\|q_{i_n}^{(n)} - q_{i_n+1}^{(n)}\|_\infty \geq \eta$ .

**Proof.** By Lemma 4.1,  $(q^{(n)})$  is bounded in  $H$  and  $c \geq 0$ . Suppose that (ii) does not hold. Then for all  $\varepsilon > 0$ ,  $i \in \mathbb{Z}$ , we have

$$\|q_i^{(n)} - q_{i+1}^{(n)}\|_\infty < \varepsilon.$$

Hence, setting  $d_i = q_i^{(n)} - q_{i+1}^{(n)}$ , by (A<sub>2</sub>), (A<sub>4</sub>) and Hölder's inequality, we have

$$\begin{aligned}
 \|q^{(n)+}\|^2 &= J'(q^{(n)})q^{(n)+} + \sum_i \int_0^T V'_i(d_i)d_i^+ \\
 &\leq \|J'(q^{(n)})\| \|q^{(n)+}\| + \varepsilon \sum_i \int_0^T |d_i d_i^+|
 \end{aligned}$$

$$\begin{aligned}
&\leq \|J'(q^{(n)})\| \|q^{(n)+}\| + \varepsilon \left( \sum_i \int_0^T |d_i|^2 \right)^{1/2} \left( \sum_i \int_0^T |d_i^+|^2 \right)^{1/2} \\
&\leq \|J'(q^{(n)})\| \|q^{(n)+}\| + \varepsilon \|q^{(n)}\|^2 \\
&\rightarrow 0
\end{aligned}$$

as  $\varepsilon \rightarrow 0$  and  $n$  large enough, which means  $q^{(n)+} \rightarrow 0$ . Similarly, we have  $q^{(n)-} \rightarrow 0$ . Therefore  $q^{(n)} \rightarrow 0$ . It is easy to see that  $c = 0$ .  $\square$

**Lemma 4.3.** Under the assumptions of Lemma 4.1, suppose  $J$  admits a  $(C)_c$  sequence  $(q^{(n)})$  at level  $c > 0$ , then up to a translation of indices,  $q^{(n)} \rightharpoonup q \neq 0$  and  $J'(q) = 0$ .

**Proof.** By Lemma 4.1,  $(q^{(n)})$  is bounded. Since  $c > 0$ , let  $(i_n)$  be the sequence defined in Lemma 4.2(ii). There exist  $j \in \{1, \dots, m\}$  and a sequence  $(n_k) \subset \mathbb{N}$ ,  $n_k \rightarrow \infty$ , such that  $i_{n_k} = j \bmod m$  for all  $k$ . Let  $(\hat{q}^{(k)})$  be defined by  $\hat{q}_i^{(k)} = q_{i+i_{n_k}-j}^{(n_k)} - \sigma^{(k)}$ , where  $\sigma^{(k)}$  is chosen in order to have  $\int_0^T \hat{q}_0^{(k)} = 0$  which means  $(\hat{q}^{(k)}) \subset H$ . Then  $(\hat{q}^{(k)})$  is also a bounded  $(C)_c$  sequence since  $J$  and  $J'$  are both invariant under translation. Then, up to a subsequence, it weakly converges to some  $q$  in  $H$  and  $\hat{q}_i^{(k)} \rightarrow q_i$  in  $L^\infty$  for all  $i$  by Lemma 2.4. It is easy to see  $\|\hat{q}_j^{(k)} - \hat{q}_{j+1}^{(k)}\|_\infty \geq \eta > 0$  which implies  $\|q_j - q_{j+1}\|_\infty \geq \eta$ . Hence  $q \neq 0$ . Moreover, since  $J'$  is weakly sequentially continuous, that is,

$$J'(q)h = \lim_{k \rightarrow \infty} J'(\hat{q}^{(k)})h = 0, \quad \forall h \in H,$$

we have  $J'(q) = 0$ .  $\square$

In order to obtain the multiplicity result of Theorem 1.1, we shall discuss the  $(C)_c$  sequence  $(q^{(n)}) \subset H$  more carefully. By Lemma 4.1, it is bounded, hence, without loss of generality, we may assume that  $q^{(n)} \rightharpoonup q$ . Plainly,  $q$  is a critical point of  $J$ . Set  $q^{(n),1} = q^{(n)} - q$ .

**Lemma 4.4.** Under the assumptions of Lemma 4.1, one has, along a subsequence, as  $n \rightarrow \infty$ ,

- (1)  $J(q^{(n),1}) \rightarrow c - J(q)$ ;
- (2)  $J'(q^{(n),1}) \rightarrow 0$ .

**Proof.** (1) For convenience, we denote  $d_i := q_i - q_{i+1}$ ,  $d_i^{(n)} := q_i^{(n)} - q_{i+1}^{(n)}$  and  $d_i^{(n),1} := q_i^{(n),1} - q_{i+1}^{(n),1}$  for all  $i \in \mathbb{Z}$ . Observe that

$$\begin{aligned}
J(q^{(n)}) &= J(q^{(n),1}) + J(q) \\
&\quad + \sum_i \int_0^T \dot{q}_i^{(n),1} \dot{q}_i + \alpha_i d_i^{(n),1} d_i - (V_i(d_i^{(n),1} + d_i) - V_i(d_i^{(n),1}) - V_i(d_i)).
\end{aligned}$$

As  $J(q^{(n)}) \rightarrow c$ , to prove  $J(q^{(n),1}) \rightarrow c - J(q)$ , it is sufficient to show that

$$\sum_i \int_0^T \dot{q}_i^{(n),1} \dot{q}_i + \alpha_i d_i^{(n),1} d_i - (V_i(d_i^{(n),1} + d_i) - V_i(d_i^{(n),1}) - V_i(d_i)) \rightarrow 0. \quad (4.9)$$

Since  $q \in H$ , for any  $\varepsilon > 0$ , by (2.9), there exists  $N \in \mathbb{N}$  such that

$$\sum_{|i| \geq N} \int_0^T [|\dot{q}_i|^2 + |d_i|^2] < \varepsilon, \quad \text{and} \quad \sum_{|i| \geq N} \|d_i\|_\infty^2 < \varepsilon. \quad (4.10)$$

For such  $N$ , as  $q_i^{(n),1} \rightarrow 0$  in  $H^1$ ,  $d_i^{(n),1} \rightarrow 0$  in  $L^2$  and  $L^\infty$  for all  $i$  (because  $q^{(n),1} \rightarrow 0$  in  $H$  and by Lemma 2.4) and  $(A_2)$ , we first obtain that

$$\left| \sum_{|i| < N} \int_0^T \dot{q}_i^{(n),1} \dot{q}_i + \alpha_i d_i^{(n),1} d_i - (V_i(d_i^{(n),1} + d_i) - V_i(d_i^{(n),1}) - V_i(d_i)) \right| < \varepsilon \quad (4.11)$$

for sufficiently large  $n$ . Next, we also show that

$$\left| \sum_{|i| \geq N} \int_0^T \dot{q}_i^{(n),1} \dot{q}_i + \alpha_i d_i^{(n),1} d_i - (V_i(d_i^{(n),1} + d_i) - V_i(d_i^{(n),1}) - V_i(d_i)) \right| < \varepsilon, \quad (4.12)$$

which, jointly with (4.11), implies (4.9). Indeed, by (4.10) and Hölder's inequality, we have

$$\left| \sum_{|i| \geq N} \int_0^T \dot{q}_i^{(n),1} \dot{q}_i \right| \leq \left( \sum_{|i| \geq N} \int_0^T |\dot{q}_i^{(n),1}|^2 \right)^{1/2} \left( \sum_{|i| \geq N} \int_0^T |\dot{q}_i|^2 \right)^{1/2} < c\varepsilon^{1/2}$$

and

$$\begin{aligned} \left| \sum_{|i| \geq N} \int_0^T \alpha_i d_i^{(n),1} d_i \right| &\leq \max_{1 \leq i \leq m} \{|\alpha_i|\} \left( \sum_{|i| \geq N} \int_0^T |d_i^{(n),1}|^2 \right)^{1/2} \left( \sum_{|i| \geq N} \int_0^T |d_i|^2 \right)^{1/2} \\ &< c\varepsilon^{1/2}. \end{aligned}$$

On the other hand, using the mean value theorem,  $(A_2)$ , (2.12), (4.10) and Hölder's inequality, for some  $\theta_i = \theta_i(t) \in (0, 1)$  one can conclude that

$$\begin{aligned} &\left| \sum_{|i| \geq N} \int_0^T V_i(d_i^{(n),1} + d_i) - V_i(d_i^{(n),1}) - V_i(d_i) \right| \\ &\leq \sum_{|i| \geq N} \int_0^T \varepsilon |d_i^{(n),1} + \theta_i d_i| |d_i| + C_\varepsilon |d_i^{(n),1} + \theta_i d_i|^{p-1} |d_i| + c |d_i|^2 \\ &\leq \left( \varepsilon + cC_\varepsilon \sup_i \|d_i^{(n),1}\|_\infty^{p-2} \right) \left( \sum_{|i| \geq N} \|d_i^{(n),1}\|_2^2 \right)^{1/2} \left( \sum_{|i| \geq N} \|d_i\|_2^2 \right)^{1/2} \\ &\quad + \left( c + \varepsilon + C_\varepsilon \sup_i \|d_i\|_\infty^{p-2} \right) \sum_{|i| \geq N} \|d_i\|_2^2 \\ &< c\varepsilon, \end{aligned}$$

where  $\sup_i \|d_i^{(n),1}\|_\infty \leq c_0 \|q^{(n),1}\|$ ,  $\sup_i \|d_i\|_\infty \leq c_1 \|q\|$ . Hence, we obtain (4.12).

(2) Note that  $J'(q) = 0$ , for any  $q \in H$ , we have

$$J'(q^{(n),1})\varphi = J'(q^{(n)})\varphi + \sum_i \int_0^T (V'_i(d_i^{(n)}) - V'_i(d_i^{(n),1}) - V'_i(d_i))(\varphi_i - \varphi_{i+1}).$$

Since  $J'(q^{(n)}) \rightarrow 0$ , to prove  $J'(q^{(n),1}) \rightarrow 0$ , it suffices to show that

$$\sup_{\|\varphi\| \leq 1} \left| \sum_i \int_0^T (V'_i(d_i^{(n)}) - V'_i(d_i^{(n),1}) - V'_i(d_i))(\varphi_i - \varphi_{i+1}) \right| \rightarrow 0. \quad (4.13)$$

We adapt a technique in the proof of Lemma 5.2 in [13] (see also [14]). By (2.12), we choose  $p > 2$  such that  $|V'_i(x)| \leq |x| + c|x|^{p-1}$  for all  $x$  and  $i \in \mathbb{Z}$ . Let  $\mu$  stand for either 2 or  $p$ . First, we claim that there is a subsequence  $(q^{(n_j)})$  such that for any  $\varepsilon > 0$  there exists  $r_\varepsilon > 0$  satisfying

$$\limsup_{j \rightarrow \infty} \sum_{r < |i| \leq j} \int_0^T |d_i^{(n_j)}|^\mu \leq \varepsilon \quad (4.14)$$

for all  $r \geq r_\varepsilon$ . As

$$\sum_{r < |i| \leq j} \int_0^T |d_i^{(n_j)}|^\mu \leq \sup_i \|d_i^{(n_j)}\|_\infty^{\mu-2} \sum_{r < |i| \leq j} \int_0^T |d_i^{(n_j)}|^2 \leq c \sum_{r < |i| \leq j} \int_0^T |d_i^{(n_j)}|^2,$$

it suffices to show that (4.14) holds for  $\mu = 2$ . To verify this, note that, for any  $j \in \mathbb{N}$ ,  $\sum_{|i| \leq j} \int_0^T |d_i^{(n)}|^2 \rightarrow \sum_{|i| \leq j} \int_0^T |d_i|^2$  as  $n \rightarrow \infty$ . There exists  $k_j$  such that

$$\sum_{|i| \leq j} \int_0^T (|d_i^{(n)}|^2 - |d_i|^2) < \frac{1}{j} \quad \text{for all } n = k_j + m, \quad m = 1, 2, \dots$$

Without loss of generality, we can assume that  $k_{j+1} \geq k_j$ . In particular, for  $n_j = k_j + j$ , we have

$$\sum_{|i| \leq j} \int_0^T (|d_i^{(n_j)}|^2 - |d_i|^2) < \frac{1}{j}.$$

Observe that there is  $r_\varepsilon$  satisfying

$$\sum_{|i| > r} \int_0^T |d_i|^2 < \varepsilon \quad (4.15)$$

for all  $r \geq r_\varepsilon$ . Since



$$\begin{aligned} \sum_{r < |i| \leq j} \int_0^T |d_i^{(n_j)}|^2 &= \sum_{|i| \leq j} \int_0^T (|d_i^{(n_j)}|^2 - |d_i|^2) + \sum_{r < |i| \leq j} \int_0^T |d_i|^2 + \sum_{|i| \leq r} \int_0^T (|d_i|^2 - |d_i^{(n_j)}|^2) \\ &\leq \frac{1}{j} + \sum_{r < |i| \leq j} \int_0^T |d_i|^2 + \sum_{|i| \leq r} \int_0^T (|d_i|^2 - |d_i^{(n_j)}|^2), \end{aligned}$$

(4.14) now follows.

In the following, we make use of a cut-off technique first developed in [1]. Let  $\eta : [0, \infty) \rightarrow [0, 1]$  be a smooth function satisfying  $\eta(t) = 1$  if  $t \leq 1$  and  $\eta(t) = 0$  if  $t \geq 2$ . Set  $\tilde{q}_i^{(j)} = \eta(2|i|/j)q_i$  and let  $h_i^{(j)} := q_i - \tilde{q}_i^{(j)}$ ,  $l_i^{(j)} := h_i^{(j)} - h_{i+1}^{(j)}$ . Since  $q \in H$ , we have by definition that  $h^{(j)} = \{h_i^{(j)}\}_{i \in \mathbb{Z}} \in H$  and by (2.9),

$$\|h^{(j)}\| \rightarrow 0 \quad \text{and} \quad \sum_i \|l_i^{(j)}\|_\infty^2 \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (4.16)$$

Observe that for any  $\varphi \in H$

$$\begin{aligned} &\sum_i \int_0^T (V'_i(d_i^{(n_j)}) - V'_i(d_i^{(n_j),1}) - V'_i(d_i))\tilde{\varphi}_i \\ &= \sum_i \int_0^T (V'_i(d_i^{(n_j)}) - V'_i(d_i^{(n_j)} - \tilde{d}_i^{(j)}) - V'_i(\tilde{d}_i^{(j)}))\tilde{\varphi}_i \\ &\quad + \sum_i \int_0^T (V'_i(d_i^{(n_j),1} + l_i^{(j)}) - V'_i(d_i^{(n_j),1}))\tilde{\varphi}_i + \sum_i \int_0^T (V'_i(\tilde{d}_i^{(j)}) - V'_i(d_i))\tilde{\varphi}_i \end{aligned}$$

where  $\tilde{\varphi}_i = \varphi_i - \varphi_{i+1}$ . Obviously, by (4.16)

$$\lim_{j \rightarrow \infty} \left| \sum_i \int_0^T (V'_i(\tilde{d}_i^{(j)}) - V'_i(d_i))\tilde{\varphi}_i \right| = 0$$

uniformly in  $\|\varphi\| \leq 1$ . It remains for checking (4.13) to show that

$$\lim_{j \rightarrow \infty} \left| \sum_i \int_0^T (V'_i(d_i^{(n_j)}) - V'_i(d_i^{(n_j)} - \tilde{d}_i^{(j)}) - V'_i(\tilde{d}_i^{(j)}))\tilde{\varphi}_i \right| = 0 \quad (4.17)$$

and

$$\lim_{j \rightarrow \infty} \left| \sum_i \int_0^T (V'_i(d_i^{(n_j),1} + l_i^{(j)}) - V'_i(d_i^{(n_j),1}))\tilde{\varphi}_i \right| = 0 \quad (4.18)$$

uniformly in  $\|\varphi\| \leq 1$ .

To check (4.17), note that (4.16) and the local compactness of Sobolev embedding imply that, for any  $r > 0$ ,

$$\lim_{j \rightarrow \infty} \left| \sum_{|i| \leq r} \int_0^T (V'_i(d_i^{(n_j)}) - V'_i(d_i^{(n_j)} - \tilde{d}_i^{(j)}) - V'_i(\tilde{d}_i^{(j)})) \tilde{\varphi}_i \right| = 0$$

uniformly in  $\|\varphi\| \leq 1$ . For any  $\varepsilon > 0$ , choose  $r_\varepsilon > 0$  so large that (4.14) and (4.15) hold. Consequently,

$$\limsup_{j \rightarrow \infty} \sum_{r < |i| \leq j} \int_0^T |\tilde{d}_i^{(j)}|^\mu \leq \sum_{|i| > r} \int_0^T |d_i|^\mu \leq \sup_i \|d_i\|^{\mu-2} \sum_{|i| > r} \int_0^T |d_i|^2 \leq c\varepsilon$$

for all  $r \geq r_\varepsilon$ . Then using (4.14) for  $\mu = 2, p$  and Hölder's inequality, we have

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \left| \sum_i \int_0^T (V'_i(d_i^{(n_j)}) - V'_i(d_i^{(n_j)} - \tilde{d}_i^{(j)}) - V'_i(\tilde{d}_i^{(j)})) \tilde{\varphi}_i \right| \\ &= \limsup_{j \rightarrow \infty} \left| \sum_{r < |i| \leq j} \int_0^T (V'_i(d_i^{(n_j)}) - V'_i(d_i^{(n_j)} - \tilde{d}_i^{(j)}) - V'_i(\tilde{d}_i^{(j)})) \tilde{\varphi}_i \right| \\ &\leq c \limsup_{j \rightarrow \infty} \sum_{r < |i| \leq j} \int_0^T (|d_i^{(n_j)}| + |\tilde{d}_i^{(j)}| + |d_i^{(n_j)}|^{p-1} + |\tilde{d}_i^{(j)}|^{p-1}) |\tilde{\varphi}_i| \\ &\leq c \limsup_{j \rightarrow \infty} \left( \left( \sum_{r < |i| \leq j} \|d_i^{(n_j)}\|_2^2 \right)^{1/2} + \left( \sum_{r < |i| \leq j} \|\tilde{d}_i^{(j)}\|_2^2 \right)^{1/2} \right) \left( \sum_i \|\tilde{\varphi}_i\|_2^2 \right)^{1/2} \\ &\quad + c \limsup_{j \rightarrow \infty} \left( \left( \sum_{r < |i| \leq j} \|d_i^{(n_j)}\|_p^p \right)^{\frac{p-1}{p}} + \left( \sum_{r < |i| \leq j} \|\tilde{d}_i^{(j)}\|_p^p \right)^{\frac{p-1}{p}} \right) \left( \sum_i \|\tilde{\varphi}_i\|_p^p \right)^{\frac{1}{p}} \\ &\leq c\varepsilon^{1/2} + c\varepsilon^{(p-1)/p}, \end{aligned}$$

which implies (4.17).

In order to verify (4.18), for any  $i \in \mathbb{Z}$ , we define  $f_i(x) = 0$  and  $f_i(x) = V'_i(x)/|x|$  if  $x \neq 0$ . By  $(A_2)$ ,  $f_i$  is continuous at  $x = 0$ , hence in  $\mathbb{R}$ . It follows from  $(A_4)$  that  $f_{i+m} = f_i$ . Plainly,  $f_i$  is uniformly continuous in  $I_a := \{x \in \mathbb{R} : |x| \leq a\}$ . Furthermore, by (2.12),  $|f_i(x)| \leq c(1 + |x|^{p-2})$  for all  $x$  and  $i$ . Set

$$C_i^{a,j} := \{t \in [0, T] : |d_i^{(n_j),1}(t)| \leq a\} \quad \text{and} \quad D_i^{a,j} := [0, T] \setminus C_i^{a,j}.$$

Since  $(q^{(n_j),1})$  is bounded,  $\sum_i \int_0^T |d_i^{(n_j),1}|^p \leq c$ . Then the Lebesgue measure

$$\sum_i |D_i^{a,j}| \leq \frac{1}{a^p} \sum_i \int_{D_i^{a,j}} |d_i^{(n_j),1}|^p \leq \frac{c}{a^p} \rightarrow 0 \quad \text{as } a \rightarrow \infty,$$

one has, for any  $\varepsilon > 0$ , there is  $\hat{a} > 0$  such that

$$\left| \sum_i \int_{D_i^{a,j}} (v'_i(d_i^{(n_j),1} + l_i^{(j)}) - v'_i(d_i^{(n_j),1})) \tilde{\varphi}_i \right| \leq \varepsilon \quad (4.19)$$

uniformly in  $\|\varphi\| \leq 1$  for all  $a \geq \hat{a}$  and all  $j$ . Since  $f_{i+m} = f_i$  and the uniform continuity of  $f_i$  on  $I_{\hat{a}}$ , there is  $\delta > 0$  satisfying

$$|f_i(x+l) - f_i(x)| < \varepsilon \quad \text{for all } x \in I_{\hat{a}}, i, \text{ and } |l| \leq \delta,$$

and by (4.16), there exists  $j_0$  such that  $\|l_i^{(j)}\|_{\infty} \leq \delta$  for all  $j \geq j_0$ , hence

$$|f_i(d_i^{(n_j)} + l_i^{(j)}) - f_i(d_i^{(n_j)})| < \varepsilon \quad \text{for all } j \geq j_0, i, \text{ and } t \in C_i^{\hat{a},j}.$$

Note that

$$\begin{aligned} (v'_i(d_i^{(n_j),1} + l_i^{(j)}) - v'_i(d_i^{(n_j),1})) \tilde{\varphi}_i &= f_i(d_i^{(n_j),1} + l_i^{(j)}) (|d_i^{(n_j),1} + l_i^{(j)}| - |d_i^{(n_j),1}|) \tilde{\varphi}_i \\ &\quad + (f_i(d_i^{(n_j),1} + l_i^{(j)}) - f_i(d_i^{(n_j),1})) |d_i^{(n_j),1}| \tilde{\varphi}_i \end{aligned}$$

and by (4.16),  $\sum_i \int_0^T |l_i^{(j)}|^2 < \varepsilon$ ,  $\sum_i \int_0^T |l_i^{(j)}|^p < \varepsilon$  for all  $n \geq n_1$  with some  $n_1 \geq n_0$ . Thus, for all  $\|\varphi\| \leq 1$  and  $n \geq n_1$ ,

$$\begin{aligned} &\left| \sum_i \int_{C_i^{\hat{a},j}} (v'_i(d_i^{(n_j),1} + l_i^{(j)}) - v'_i(d_i^{(n_j),1})) \tilde{\varphi}_i \right| \\ &= \sum_i \int_{C_i^{\hat{a},j}} c(1 + |d_i^{(n_j),1} + l_i^{(j)}|^{p-2}) |l_i^{(j)}| |\tilde{\varphi}_i| + \varepsilon \sum_i \int_{C_i^{\hat{a},j}} |d_i^{(n_j),1}| |\tilde{\varphi}_i| \\ &\leq c \left( \sum_i \|l_i^{(j)}\|_2^2 \right)^{1/2} \left( \sum_i \|\tilde{\varphi}_i\|_2^2 \right)^{1/2} \\ &\quad + c \left( \sum_i \|d_i^{(n_j),1} + l_i^{(j)}\|_p^p \right)^{\frac{p-2}{p}} \left( \sum_i \|l_i^{(j)}\|_p^p \right)^{1/p} \left( \sum_i \|\tilde{\varphi}_i\|_p^p \right)^{1/p} \\ &\quad + \varepsilon \left( \sum_i \|d_i^{(n_j),1}\|_2^2 \right)^{1/2} \left( \sum_i \|\tilde{\varphi}_i\|_2^2 \right)^{1/2} \\ &\leq c\varepsilon, \end{aligned}$$

which, jointly with (4.19), implies (4.18). The proof is complete.  $\square$

Let  $\mathcal{K} := \{q \in H : J'(q) = 0\}$  denote the set of all the critical points of  $J$ .

**Lemma 4.5.** *Under the assumptions of Lemma 4.1, we obtain*

- (a)  $\nu := \inf\{\|q\| : q \in \mathcal{K} \setminus \{0\}\} > 0$ ;
- (b)  $\theta := \inf\{J(q) : q \in \mathcal{K} \setminus \{0\}\} > 0$ .

**Proof.** (a) Arguing indirectly, assume that there is a sequence  $(q^{(n)}) \subset \mathcal{K} \setminus \{0\}$  such that  $\|q^{(n)}\| \rightarrow 0$ . Choosing  $\varepsilon > 0$ , by (2.9), for  $n$  large enough, one has

$$\sum_i \|q_i^{(n)} - q_{i+1}^{(n)}\|_\infty^2 < \varepsilon. \quad (4.20)$$

Note that  $J'(q^{(n)})q^{(n)\pm} = 0$ , we have

$$\|q^{(n)}\|^2 = \sum_i \int_0^T v'_i(d_i)(d_i^+ - d_i^-),$$

where  $d_i = q_i^{(n)} - q_{i+1}^{(n)}$ . Since  $\sum_i \|d_i\|_2^2$  and  $\sum_i \|d_i^\pm\|_2^2 \leq \|q^{(n)}\|^2$ , by using (4.20),  $(A_2)$ ,  $(A_4)$  and Hölder's inequality, for  $n$  large and  $\varepsilon$  sufficiently small, we can conclude that

$$\begin{aligned} \|q^{(n)}\|^2 &\leq \varepsilon \sum_i \int_0^T |d_i| |d_i^+ - d_i^-| \leq \varepsilon \left( \sum_i \int_0^T |d_i|^2 \right)^{\frac{1}{2}} \left( \sum_i \int_0^T |d_i^+|^2 \right)^{\frac{1}{2}} \\ &\quad + \varepsilon \left( \sum_i \int_0^T |d_i|^2 \right)^{\frac{1}{2}} \left( \sum_i \int_0^T |d_i^-|^2 \right)^{\frac{1}{2}} \\ &\leq 2\varepsilon \|q^{(n)}\|^2, \end{aligned}$$

a contradiction.

(b) Assume that there is a sequence  $(q^{(n)}) \subset \mathcal{K} \setminus \{0\}$  such that  $J(q^{(n)}) \rightarrow 0$ . Then one has

$$\|q^{(n)}\|^2 = \sum_i \int_0^T v'_i(d_i)(d_i^+ - d_i^-) \quad (4.21)$$

and

$$o(1) = J(q^{(n)}) = J(q^{(n)}) - \frac{1}{2} J'(q^{(n)})q^{(n)} = \sum_i \int_0^T \tilde{v}_i(q_i^{(n)} - q_{i+1}^{(n)}). \quad (4.22)$$

Obviously,  $(q^{(n)})$  is a  $(C)_{c=0}$  sequence, hence is bounded by Lemma 4.1. By (a),

$$\|q^{(n)}\| \geq v. \quad (4.23)$$

Using (4.22) and the notations introduced in the proof of Lemma 4.1, we see that, for any  $0 < a < b$  and  $s > 2$ ,

$$\sum_i \int_{\Omega_i(a,b)} |q_i^{(n)} - q_{i+1}^{(n)}|^2 \rightarrow 0 \quad \text{and} \quad \sum_i \int_{\Omega_i(b,\infty)} |q_i^{(n)} - q_{i+1}^{(n)}|^s \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore, by (2.12) and (4.21), for any  $\varepsilon > 0$ , we obtain as in the proof of Lemma 4.1

$$\limsup_{n \rightarrow \infty} \|q^{(n)}\|^2 \leq \varepsilon,$$

contradicting (4.23). The proof is complete.  $\square$

Let  $[r]$  denote the integer part of  $r \in \mathbb{R}$ . We have the following result.

**Lemma 4.6.** *Under the assumptions of Lemma 4.1 and letting  $(q^{(n)}) \subset H$  be a  $(C)_c$  sequence, then either*

- (i)  $q^{(n)} \rightarrow 0$  (and hence  $c = 0$ ), or
- (ii)  $c \geq \theta$  and there exist a positive integer  $l \leq [\frac{c}{\theta}]$ , nonzero critical points  $q^1, \dots, q^l \in \mathcal{K} \setminus \{0\}$ , a subsequence denoted again by  $(q^{(n)})$  and sequences  $(k_n^i) \subset \mathbb{Z}$ ,  $i = 1, \dots, l$ , such that

$$\left\| q^{(n)} - \sum_{i=1}^l k_n^i * q^i \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

$$|k_n^i - k_n^j| \rightarrow \infty \quad \text{for } i \neq j \text{ as } n \rightarrow \infty,$$

and

$$\sum_{i=1}^l J(q^i) = c.$$

**Proof.** It follows from Lemma 4.1 that  $(q^{(n)})$  is bounded and  $c \geq 0$ . Assume that (i) does not hold, then, by Lemma 4.2(ii), there exists a sequence  $(i_n) \subset \mathbb{Z}$  such that  $\|q_{i_n}^{(n)} - q_{i_n+1}^{(n)}\|_\infty \geq \eta$ . We may choose  $k_n \in \mathbb{Z}$  such that setting  $h^{(n)} := k_n * q^{(n)}$  and passing to a subsequence (still denoted by  $(h^{(n)})$ ),

$$\|h_0^{(n)} - h_1^{(n)}\|_\infty \geq \eta. \quad (4.24)$$

Moreover, by  $\mathbb{Z}$  and  $S^1$ -invariance,  $J(h^{(n)}) = J(q^{(n)})$ ,  $\|J'(h^{(n)})\| = \|J'(q^{(n)})\|$  and  $\|h^{(n)}\| = \|q^{(n)}\|$ . Hence  $(h^{(n)})$  is bounded, a subsequence of  $(h^{(n)})$  (still denoted by the same symbol) weakly converges to some  $h \in H$ . It is easy to see  $h \in \mathcal{K}$ . Since  $h_i^{(n)} \rightarrow h_i$  in  $L^\infty$  for all  $i$ , by (4.24) we have  $\|h_0 - h_1\|_\infty \geq \eta$ , which means  $h \neq 0$ . Moreover, Lemma 4.5(b) implies  $J(h) \geq \theta$ .

Set  $h^{(n),1} := h^{(n)} - h$ . Clearly,  $h^{(n),1} \rightarrow 0$ , and by Lemma 4.4, we have  $J(h^{(n),1}) \rightarrow c - J(h)$  and  $J'(h^{(n),1}) \rightarrow 0$ . Therefore  $(h^{(n),1})$  is a  $(C)_{c-J(h)}$ -sequence and, by Lemma 4.1,  $J(h) \leq c$ . There are now two possibilities to consider:

If  $J(h) = c$ , then  $h^{(n),1} \rightarrow 0$ . Indeed, if  $h^{(n),1} \not\rightarrow 0$ , replacing  $(q^{(n)})$  and  $c$  by  $(h^{(n),1})$  and  $c' = c - J(h) = 0$ , in a similar way we can obtain  $\tilde{h} \in \mathcal{K} \setminus \{0\}$  such that  $J(\tilde{h}) \leq 0$ , a contradiction to Lemma 4.5(b). Hence our lemma holds with  $l = 1$ ,  $q^1 = h$  and  $k_n^1 = -k_n$ .

If  $J(h) < c$ , then we are back to our original situation with  $(q^{(n)})$  and  $c$  replaced by  $(h^{(n),1})$  and  $c' = c - J(h)$  respectively, and we obtain  $q^2 \in \mathcal{K} \setminus \{0\}$  with  $\theta \leq J(q^2) \leq c - J(q^1)$ . After at most  $[\frac{c}{\theta}]$  steps, we obtain the desired conclusion.  $\square$

**Remark 4.7.** (a) From Lemma 4.1 we see that any  $(C)_c$  sequence is also a bounded Palais–Smale sequence. Our argument is modeled on Theorem 3 in [4] (see also Theorem 1 in [5]).

(b) In fact, we can guarantee that these critical points  $q^1, \dots, q^l$  are nonconstant for some suitable periodic  $T$  (see more details in Section 5). Note that even if  $l > 1$ , we cannot state that there exist  $l$  different critical points, as it could be  $q^i = q^j$  for  $i \neq j$ .

## 5. Proof of the existence of Theorem 1.1

In this section we will give the proof of the existence of Theorem 1.1. In order to obtain the existence of critical points of  $J$ , we recall the abstract critical point theory which was developed recently in [10]; see also [8,19,28] for earlier results on that direction. This abstract critical point theory has been used widely to handle strongly indefinite problems in different equations, such as Dirac equations [9], Hamiltonian systems [13], Schrödinger equations [15], and Diffusion equations [16]. Also see the monograph [14] for an overview and more references.

### 5.1. An abstract critical point theorem about existence

Let  $H$  be a Banach space with direct sum decomposition  $H = X \oplus Y$  and corresponding projections  $P^-, P^+$  onto  $X, Y$ , respectively. For a functional  $J \in C^1(H, \mathbb{R})$  we write  $J_a = \{q \in H: J(q) \geq a\}$ ,  $J^b = \{q \in H: J(q) \leq b\}$  and  $J_a^b = J_a \cap J^b$ . We write  $J'$  for the derivative of  $J$ .

From now on we assume that  $H$  is separable and reflexive, and we fix a countable dense subset  $S \subset X^*$  (the dual space of  $X$ ). For any  $s \in S$ , there is a semi-norm on  $H$  defined by

$$p_s: H \rightarrow \mathbb{R}, \quad p_s(q) = |s(x)| + \|y\| \quad \text{for } q = x + y \in X \oplus Y.$$

We denote by  $\mathcal{T}_S$  the induced topology. Let  $(H^*, w^*)$  be the dual space  $H^*$  equipped with the usual weak\*-topology  $w^*$  on  $H^*$ .

Suppose that:

- ( $J_0$ )  $J \in C^1(H, \mathbb{R})$ . For any  $c \in \mathbb{R}$ ,  $J_c$  is  $\mathcal{T}_S$ -closed, and  $J': (J_c, \mathcal{T}_S) \rightarrow (H^*, w^*)$  is continuous.
- ( $J_1$ ) For any  $c > 0$ , there exists  $\zeta > 0$  such that  $\|q\| < \zeta \|P^+q\|$  for all  $q \in J_c$ .
- ( $J_2$ ) There exists  $\rho > 0$  such that  $\kappa := \inf J(S_\rho Y) > 0$ , where  $S_\rho Y := \{q \in Y: \|q\| = \rho\}$ .

The following theorem is a special case of Theorem 4.4 of [10] (see also [9]).

**Theorem 5.1.** *Let ( $J_0$ )–( $J_2$ ) be satisfied and suppose that there are  $R > \rho > 0$  and  $e \in Y$  with  $\|e\| = 1$  such that  $\sup J(\partial Q) \leq \kappa$ , where  $Q := \{q = x + \gamma e: \gamma \geq 0, x \in X, \|q\| < R\}$ . Then  $J$  has a  $(C)_c$  sequence with  $\kappa \leq c \leq \sup J(Q)$ .*

### 5.2. Proof of the existence of Theorem 1.1

We choose  $X = H^-$  and  $Y = H^+$  with  $H^\pm$  given in Section 2. Then  $H = X \oplus Y$ . It is easy to see that  $J$  defined by (2.2) fits the general framework of Section 5.1, which suggests us applying Theorem 5.1.

In our applications we take  $S = X^*$  so that  $\mathcal{T}_S$  is the product topology on  $H = X \oplus Y$  given by the weak topology on  $X$  and the strong topology on  $Y$ . In our paper, we will use a weaker topology in the space  $H$ , that is, the so-called  $\tau$ -topology from [19] (see also [8,10] for more general topological settings). Let  $(e_n)_{n \in \mathbb{N}} \subset X$  be an orthonormal base of  $X$  and define a norm on  $H$  by setting

$$\|q\|_\tau^2 := \max \left\{ \|q^+\|^2, \sum_{n=1}^{\infty} \frac{|(q^-, e_n)|^2}{2^n} \right\}$$

for all  $q = q^- + q^+ \in H$ . The  $\tau$ -topology is the topology generated by  $\|\cdot\|_\tau$ . Therefore we can denote by  $\mathcal{T}_S$  the topology on  $H$  generated by  $\|\cdot\|_\tau$ . For convenience we denote by  $s$ -topology the topology induced by  $\|\cdot\|$ . In what follows,  $H_\tau$  denotes the linear space  $H$  equipped with the  $\tau$ -topology. Observe that  $\|q^+\| \leq \|q\|_\tau \leq \|q\|$  for all  $q \in H$ . Moreover, given a finite dimensional subspace  $A \subset Y$ , if  $F \subset X \oplus A$  is a bounded set then on  $F$  the  $\tau$ -topology and the usual weak topology are equivalent, and if  $F$  is moreover convex and  $s$ -closed then it is  $\tau$ -compact.

**Proof of the existence of Theorem 1.1.** In virtue of Lemma 2.6 and the form of  $J$ , we see that  $J$  satisfies  $(J_0)$  (cf. Proposition 4.1 in [10]). Since  $V(q) \geq 0$ , the representation (2.2) of  $J$  implies that  $2\|q^+\|^2 > \|q\|^2$  for all  $q \in J_c$  with  $c > 0$  which means the condition  $(J_1)$  holds. Obviously, Lemma 3.2 is nothing else than  $(J_2)$ , which jointly with Lemma 3.4 gives the linking condition of Theorem 5.1. Therefore  $J$  has a  $(C)_c$  sequence  $(q^{(n)})$  with  $0 < \kappa \leq c \leq \sup J(Q)$ . It follows from Lemma 4.1 that  $(q^{(n)})$  is bounded. Then by Lemma 4.3 we have, up to a translation of indices,  $q^{(n)} \rightharpoonup q$  in  $H$  with  $q \neq 0$  and  $J'(q) = 0$ , that is,  $q$  is a nontrivial solution of (1.2), and the existence is proved. Now we show the solution  $q$  obtained above is nonconstant for some suitable  $T$ . Indeed, we have the following

**Claim.** *There exists  $T_{\min} > 0$  which depends on  $\{\alpha_i: \alpha_i > 0\}$  and corresponding  $V_i$  such that if  $T_{\min} < T$ , then the solution obtained above is nonconstant.*

Here we adapt a technique in the proof of Proposition 2 in [4] (see also Lemma 5.5 in [6]). We divide our proof into four steps.

**Step 1.** We first show that if  $q$  is a constant solution, then  $\Phi'_i(q_i - q_{i+1}) = 0$  for all  $i \in \mathbb{Z}$ , that is no pair of particles undergoes any force. Arguing indirectly, we assume that there exists some  $j \in \mathbb{Z}$  such that  $\Phi'_j(q_j - q_{j+1}) = m \neq 0$ . It follows from  $\dot{q}_i(t) = 0$  for all  $i \in \mathbb{Z}$  and  $q_i$  satisfies (1.1) that  $\Phi'_i(q_i - q_{i+1}) = m$  for all  $i \in \mathbb{Z}$ , and this is impossible because  $q_i - q_{i+1} \rightarrow 0$  as  $i \rightarrow \pm\infty$  and  $\Phi'_i(x) \rightarrow 0$  for  $x \rightarrow 0$ .

**Step 2.** We show that if  $q$  is a nonzero constant solution, then  $q_i - q_{i+1} = 0$  if  $\alpha_i < 0$ . In fact, as  $q \neq 0$ , there exists  $j \in \mathbb{Z}$  such that  $q_j - q_{j+1} \neq 0$ ; if  $\alpha_j < 0$ , we can find  $p \in H$  with  $p_j - p_{j+1} = q_j - q_{j+1}$  and  $p_i - p_{i+1} = 0$  for  $i \neq j$ . Then we can obtain  $J'(q)p = -\int_0^T \Phi'_j(q_j - q_{j+1})(q_j - q_{j+1}) = \alpha_j \int_0^T |q_j - q_{j+1}|^2 - \int_0^T V'_j(q_j - q_{j+1})(q_j - q_{j+1}) < 0$  which implies  $\Phi'_j(q_j - q_{j+1}) \neq 0$ , contradicting Step 1. Therefore  $J(q) = -\sum_{i \in I} \int_0^T \Phi_i(q_i - q_{i+1})$ , where  $I = \{i \in \mathbb{Z}: \alpha_i > 0\}$  is defined in Section 2.

**Step 3.** We now prove that for nonzero constant solution  $q$ , there exists a constant  $\tilde{d} > 0$  such that  $J(q) \geq T\tilde{d}$ . Indeed, let  $\tilde{d} = -\max_i \{\max_j [\Phi_i(\vartheta_{ij})]\} > 0$ , where  $\{\vartheta_{ij}\}$  are the nonzero stationary points of  $\Phi_i$  (remark that if  $\Phi_i(x) \geq 0$  with  $x \neq 0$ , by  $(A_3)$ , one has  $\Phi'_i(x)x = -\alpha_i x^2 + V'_i(x)x > -\alpha_i x^2 + 2V_i(x) = 2\Phi_i(x) \geq 0$  which implies  $\Phi'_i(x) \neq 0$ ); note that by Step 2,  $\tilde{d}$  only depends on the positive  $\alpha_i$  and corresponding  $V_i$ . It is easy to check that  $J(q) = -\sum_{i \in I} \int_0^T \Phi_i(q_i - q_{i+1}) \geq T\tilde{d}$ .

Finally, we show that  $\sup J(Q) < T\tilde{d}$  ( $Q$  is defined in Lemma 3.4) holds to finish the proof of our claim. In order to verify this, we need to revisit Lemma 3.4 and give a more accurate estimate of  $\sup J(Q)$ . With no restriction we choose  $\alpha_0 < 0$ . By  $(A_1)$  and  $(A_4)$ , given  $d > 0$  there exists  $c > 0$  such that

$$V_i(x) \geq c|x|^2 - d \quad \text{for all } x \in \mathbb{R}, i \in \mathbb{Z}. \quad (5.1)$$

Define

$$e_i(t) = \begin{cases} \theta \sin(\frac{2\pi}{T}t), & \text{if } 0 < i \leq m, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta := (2T/(4m\pi^2 + (\alpha_0 + \alpha_m)T^2))^{1/2}$ . Since  $\alpha_0 = \alpha_m < 0$  and  $T < \pi/\sqrt{\beta}$ ,  $\theta$  is admissible and  $\theta \geq (2T/(4m\pi^2))^{1/2}$ . By the definitions of  $H^-$  and  $H^+$ , it is easy to see  $e := \{e_i\}_{i \in \mathbb{Z}} \in H^+$  and  $\|e\| = 1$ . Noting that  $V_i(x) \geq 0$  for all  $x, i$  and  $\alpha_0 = \alpha_m < 0$ , by (5.1), for  $q^- \in H^-$ , we have

$$\begin{aligned} J(q^- + \gamma e) &= \frac{\gamma^2}{2} - \frac{\|q^-\|^2}{2} - V(q^- + \gamma e) \\ &\leq \frac{\gamma^2}{2} - \frac{\|q^-\|^2}{2} + Td - c \int_0^T \left| d_0 - \gamma \theta \sin\left(\frac{2\pi}{T}t\right) \right|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\gamma^2}{2} - \frac{\|q^-\|^2}{2} + Td - c \int_0^T \left| \gamma \theta \sin\left(\frac{2\pi}{T}t\right) \right|^2 \\
&= \frac{\gamma^2}{2} - \frac{\|q^-\|^2}{2} + Td - \frac{c\theta^2 T}{2} \gamma^2 \\
&\leq \frac{\gamma^2}{2} - \frac{\|q^-\|^2}{2} + Td - \frac{cT^2}{4m\pi^2} \gamma^2,
\end{aligned}$$

where  $d_0 = q_0^- - q_1^-$ . Here we use the fact that  $\int_0^T |\sin(\frac{2\pi}{T}t) + \hat{d}|^2 \geq \int_0^T |\sin(\frac{2\pi}{T}t)|^2$  for each  $\hat{d} \in \mathbb{R}$  (observe that  $\int_0^T |\sin(\frac{2\pi}{T}t) + \hat{d}|^2 = \int_0^T |\sin(\frac{2\pi}{T}t) - \hat{d}|^2$ ). For sufficiently large  $T$ , it is easy to check that Lemma 3.4 holds and  $\sup J(Q) \leq Td < T\bar{d}$  if one chooses  $d < \bar{d}$ .

Since the conditions on  $T_{\min}$  and  $\pi/\sqrt{\beta}$  are independent of each other, there are potentials for which this inequality  $T_{\min} < \pi/\sqrt{\beta}$  is satisfied. Therefore when the coefficients  $\alpha_i$  take both signs, our method guarantees the existence of a nonconstant solution for system (1.2) only if  $T_{\min} < \pi/\sqrt{\beta}$ .  $\square$

## 6. Proof of the multiplicity of Theorem 1.1

We are now in a position to establish the multiplicity of Theorem 1.1. We firstly prove our result in a more general setting: extending the abstract critical point theory about the multiplicity of a strongly indefinite functional developed by Bartsch and Ding (see Theorem 4.2 in [8] or Theorem 4.7 in [10]) to a more general class of symmetry. The earlier version of this theory can be found in [19].

### 6.1. An abstract critical point theorem about multiplicity

In what follows, a set  $\mathcal{A} \subset H$  is said to be a  $(C)_c$ -attractor if for any  $\varepsilon, \delta > 0$  and any  $(C)_c$ -sequence  $(q^{(n)})$  one has, along a subsequence,  $q^{(n)} \in U_\varepsilon(\mathcal{A} \cap J_{c-\delta}^{c+\delta})$ . Given an interval  $I \subset \mathbb{R}$ , a set  $\mathcal{A}$  is called a  $(C)_I$ -attractor if it is a  $(C)_c$ -attractor for any  $c \in I$ . Now we use the same notations in Section 5.1 and assume that  $J$  satisfies  $(J_0)-(J_2)$ .

Now we recall the abstract critical point theory about multiplicity of a strongly indefinite functional developed by Bartsch and Ding:

**Theorem 6.1.** (See Theorem 4.7 in [10].) Assume  $J$  is even with  $J(0) = 0$  and satisfies  $(J_0)-(J_2)$  and the following conditions:

- $(J_3)$  There exist an increasing sequence  $Y_n \subset Y$  of finite-dimensional subspaces and a sequence  $R_n > 0$ , such that  $\sup J(H_n) < \infty$  and  $\sup J(H_n \setminus B_n) \leq \inf J(B_\rho)$ , where  $H_n := X \oplus Y_n$  and  $B_n := \{q \in H_n : \|q\| \leq R_n\}$ .
- $(J_4)$  For any compact interval  $I \subset (0, \infty)$ , there is a  $(C)_I$ -attractor  $\mathcal{A}$  with  $P^+\mathcal{A}$  bounded and  $\inf\{\|P^+q - P^+p\| : q, p \in \mathcal{A}, P^+q \neq P^+p\} > 0$ .

Then  $J$  has an unbounded sequence of positive critical values.

Since our functional is invariant under the action of the continuous compact group  $S^1$ , the  $(C)_I$ -attractor  $\mathcal{A}$  will be constructed by the sums of translates of finite compact subsets ( $S^1$ -critical orbits in our situation). In other words, we would weaken the rather strong non-degenerate assumption (in sense that  $\mathcal{K}/\mathbb{Z}$  is finite or countable). In the following, for a set  $A$ , the class consisting of all the maximal connected components of  $A$  will be denoted by  $\mathcal{C}(A)$ . Here we replace  $(J_4)$  by the more general (degenerate) assumption:

- $(J'_4)$  For any compact interval  $I \subset (0, \infty)$  there is a  $(C)_I$ -attractor  $\mathcal{A}$  with  $P^+\mathcal{A}$  bounded and  $\inf\{\|P^+U - P^+V\| : U, V \in \mathcal{C}(\mathcal{A}), P^+U \cap P^+V = \emptyset\} > 0$ .



Then we have the following result.

**Theorem 6.2.** Assume  $J$  is even with  $J(0) = 0$  and satisfies  $(J_0)$ – $(J_3)$  and  $(J'_4)$ . Then  $J$  has an unbounded sequence of positive critical values.

**Remark 6.3.** Since  $J$  is even and we consider any compact interval  $I \subset (0, \infty)$ , in what follows, without loss of generality, we may assume  $\mathcal{A}$  is symmetric and  $0 \notin \mathcal{A}$ . Obviously, if  $\mathcal{A}$  is constructed by the sums of translates of finite points, Theorem 6.2 also holds true. Invoking  $(J'_4)$ , we can construct a new deformation with a locally finite dimensional property to obtain Theorem 6.2.

The proof of Theorem 6.2 will occupy the rest of this subsection. For convenience, we consider the topology  $\mathcal{T}_S$  on  $H$  generated by  $\|\cdot\|_\tau$  (see Section 5.2); the proof can be extended without major changes to more general topological settings (see e.g. [10]). For a closed symmetric subset  $M = -M \subset H$ , we introduce as in [8] the class  $\mathcal{M}(M)$  of mappings  $g: M \rightarrow H$  with the properties:

- (g<sub>1</sub>)  $g$  is a homeomorphism of  $M$  onto  $g(M)$  (in the  $s$ -topology of  $H$ );
- (g<sub>2</sub>)  $g$  is odd and  $J(g(q)) \leq J(q)$  for all  $q \in M$ ;
- (g<sub>3</sub>) each  $q \in M$  has a  $\tau$ -open neighborhood  $W_q$  such that  $\{v - g(v): v \in W_q \cap M\}$  is contained in a finite dimensional subspace of  $H$ .

Clearly,  $\mathcal{M}(M)$  is nonempty, since it contains the identity  $id: M \rightarrow M \subset H$ . Moreover, if  $M \subset N$  (here  $N$  is also closed and symmetric), one can easily obtain that for any  $g \in \mathcal{M}(N)$ ,  $g|_M \in \mathcal{M}(M)$ . Define the pseudo-index for the sublevel sets  $J^c$  by setting

$$\psi(c) := \min\{\text{gen}(g(J^c) \cap S_\rho Y): g \in \mathcal{M}(J^c)\} \in \mathbb{N} \cup \{0, \infty\},$$

where  $\text{gen}(A)$  denotes the Krasnoselskii genus of a closed symmetric subset  $A$  of  $H$ , that is,  $\text{gen}(A)$  is the least integer  $k$  such that there exists an odd continuous map  $A \rightarrow S^{k-1}$ ; if no such map exists then  $\text{gen}(M) := \infty$  (cf. [22,25]). Observe that since  $g(J^c)$  is closed,  $\psi(c)$  is well defined. From  $(J_2)$  it is easy to see that if  $c < \kappa$  then  $\psi(c) = 0$  since  $J^c \cap S_\rho Y = \emptyset$  and  $\text{gen}(\emptyset) = 0$ . Therefore as in the proof of Theorem 4.2 in [8] or Theorem 4.7 in [10], Theorem 6.2 is a consequence of the next three lemmas.

**Lemma 6.4.** If  $c \geq \sup J(E_n)$  then  $\psi(c) \geq n$ .

**Lemma 6.5.** If the set of critical values of  $J$  is bounded above by some  $m > 0$ , then  $\psi$  is constant on  $[m', \infty)$  for  $m' \gg m$ .

**Lemma 6.6.**  $\psi$  achieves only finite values provided the set of critical values of  $J$  is bounded.

The proof of Lemma 6.4 is similar to that of Lemma 4.3 in [8] or Lemma 4.1 in [10] and is omitted. The verifications of Lemmas 6.5 and 6.6 will be carried out via deformations arguments. In order to obtain the deformations, we need to invoke  $(J'_4)$ . Here we define a vector field  $V$  by slightly modifying the construction in the proof of Theorem 4.2 in [8] or Theorem 4.7 in [10].

Below, for any  $b \geq \kappa$ , we set  $a = \kappa/2$ ,  $I = [a, b]$ , and take by  $(J'_4)$ , a  $(C)_I$ -attractor  $\mathcal{A}$  with  $B := P^+ \mathcal{A}$  bounded and  $0 < \sigma < 1$  such that

$$\|P^+U - P^+V\| > 2\sigma \quad \text{for } U, V \in \mathcal{C}(\mathcal{A}), \quad P^+U \cap P^+V = \emptyset. \quad (6.1)$$

For any  $0 < r \leq \sigma$ , we set  $W_r := X \times W_r(B)$ ; here  $W_r(B) = \{y \in Y: \|y - B\| < r\}$ . Clearly  $W_r$  is  $\tau$ -open. Since  $\mathcal{A}$  is a  $(C)_I$ -attractor and  $W_{\sigma/3}(\mathcal{A}) \subset W_{\sigma/3}$ , there is  $\alpha > 0$  such that

$$(1 + \|q\|)\|J'(q)\| \geq 2\alpha \quad \text{for all } q \in J_a^b \setminus W_{\sigma/3}. \quad (6.2)$$

Since  $B$  is bounded, let  $M > 0$  be such that

$$\|P^+q\| \leq M \quad \text{for all } q \in W_\sigma. \quad (6.3)$$

For each  $q \in J_a^b \setminus W_{\sigma/3}$ , there exists a pseudo-gradient vector  $w(q) \in H$  such that  $\|w(q)\| \leq 2$  and  $J'(q)w(q) \geq \|J'(q)\|$ . Therefore it follows from  $(J_0)$  and (6.2) that there is a  $\tau$ -open neighborhood  $N(q) \subset X \times W_{\sigma/3}(P^+q)$  in  $H$  such that

$$(1 + \|v\|)J'(v)w(q) > \alpha \quad \text{for } v \in N(q) \cap J_a^b. \quad (6.4)$$

By  $(J_1)$ , there is  $\zeta > 0$  such that the set  $H_q := \{v \in H: \zeta\|P^+v\| > \|q\|\}$  is a  $\tau$ -open neighborhood of  $q$  for  $q \in J_a$ . Then we may also assume that  $\|q\| < \zeta\|P^+v\|$  holds for  $v \in N(q) \cap J_a^b$ . If  $q \in J_a^b \cap W_{\sigma/3}$  we define  $w(q) := 0$  and  $N(q) := W_{\sigma/3}$ . If  $J(q) < a$  we set  $w(q) := 0$  and  $N(q) := H \setminus J_a$ . Then  $\mathcal{N} := \{N(q): q \in J^b\}$  is a  $\tau$ -open cover of  $J^b$ . Let  $\mathcal{U} = \{U_k: k \in K\}$  be a  $\tau$ -locally finite  $\tau$ -open refinement of  $\mathcal{N}$  and  $\{\pi_k: k \in K\}$  is a  $\tau$ -Lipschitz continuous partition of unity subordinated to  $\mathcal{U}$ . For  $k \in K$  we choose  $q_k \in J^b$  with  $U_k \subset N(q_k)$ , and we define  $w_k := (1 + \|q_k\|)w(q_k)$ . It is easy to see that if  $q \in U_k \subset N(q_k)$  and  $w_k \neq 0$  then  $\|q_k\| < \zeta\|P^+q\|$ , hence  $\|w_k\| \leq 2(1 + \|q_k\|) \leq 2(1 + \zeta\|P^+q\|)$ . Set  $D := \bigcup_{k \in K} U_k$  and define  $V_0(q) := \sum_{k \in K} \pi_k(q)w_k$  and

$$V(q) := \frac{1}{2}(V_0(q) - V_0(-q)) \quad \text{for } q \in D.$$

By construction,  $V$  is locally Lipschitz continuous with respect to both  $\tau$  and  $s$ -topologies, and for  $q \in D$ ,

$$\|V(q)\|_\tau \leq \|V(q)\| \leq 2(1 + \zeta\|P^+q\|) \leq 2(1 + \zeta\|q\|). \quad (6.5)$$

It is apparent that

$$J'(q)V(q) \geq 0 \quad \text{for all } q \in J^b, \quad (6.6)$$

and by (6.4)

$$J'(q)V(q) \geq \alpha \quad \text{for all } q \in J_a^b \setminus W_{\sigma/3}. \quad (6.7)$$

In addition, by (6.3) and (6.5), we have

$$\|V(q)\| \leq R := 2(1 + \zeta M) \quad \text{for } q \in W_\sigma. \quad (6.8)$$

Consider the Cauchy problem

$$\frac{d\xi}{dt} = -V(\xi), \quad \xi(0, q) = q \in D. \quad (6.9)$$

Invoking the locally Lipschitz continuity and (6.5), the existence and uniqueness theory of ODE implies that  $\xi$  exists on an interval containing  $[0, \infty)$ , is continuous in  $\tau$  and  $s$ -topologies, and  $\xi(t, \cdot): J^b \rightarrow \xi(t, J^b)$  is a homeomorphism for each  $t \geq 0$ . By (6.6),  $J(\xi(t, q)) \leq J(q)$  for  $q \in J^b$ . Moreover, by construction, each  $u \in D$  has a  $\tau$ -neighborhood which is mapped by  $V$  into a finite dimensional subspace, hence a standard argument (cf. Lemma 6.8 in [28]) shows that each  $(t, q) \in [0, \infty) \times J^b$  has an open neighborhood  $W_{(t, q)}$  (in the product topology on  $\mathbb{R} \times H_\tau$ ) such that  $\{v - \xi(t, v): (s, v) \in W_{(t, q)} \cap [0, \infty) \times J^b\}$  is contained in a finite dimensional subspace.

As mentioned above, in order to prove Theorem 6.2, we only need to prove Lemmas 6.5 and 6.6.

**Proof of Lemma 6.5.** We can assume that the set of critical values of  $J$  is bounded above by some  $m > 0$ ; otherwise, the proof of Theorem 6.2 is complete. For any  $c < d \in [m', \infty)$ , where  $m' \gg m$ , we have  $\psi(c) \leq \psi(d)$  by the monotonicity of the genus. In order to show  $\psi(c) \geq \psi(d)$ , we will construct a map  $h \in \mathcal{M}(J^d)$  with  $h(J^d) \subset J^c$ . Then  $g \circ h \in \mathcal{M}(J^d)$  for all  $g \in \mathcal{M}(J^c)$ , this implies, jointly with the monotonicity of the genus,

$$\begin{aligned} \psi(c) &= \inf\{\text{gen}(g(J^c) \cap S_\rho Y : g \in \mathcal{M}(J^c))\} \\ &\geq \inf\{\text{gen}(g \circ h(J^d) \cap S_\rho Y : g \in \mathcal{M}(J^c))\} \\ &\geq \inf\{\text{gen}(g(J^d) \cap S_\rho Y : g \in \mathcal{M}(J^d))\} \\ &= \psi(d). \end{aligned}$$

Now it suffices to construct a map  $h \in \mathcal{M}(J^d)$  with  $h(J^d) \subset J^c$ . We only modify slightly the construction of the deformation defined above. We replace  $W_{\sigma/3}$  and  $W_\sigma$  by  $N_{\sigma/3}(\mathcal{K})$  and  $N_\sigma(\mathcal{K})$ , respectively, where  $N_r(\mathcal{K}) = \{q \in H : \|q - \mathcal{K}\| < r\}$ . Since  $\mathcal{K} \subset \mathcal{A}$ , (6.5) holds also true. Here it is worth mentioning that for  $q \in J_a^b \cap N_{\sigma/3}(\mathcal{K})$ , in order to ensure that  $N(q)$  is  $\tau$ -open, we may choose  $N(q) := X^- \times N_{\sigma/3}(P^+ \mathcal{K})$ . Hence we can similarly consider a flow  $\xi$  associated with  $V$  (here we use the same notations) which is locally Lipschitz continuous with respect to both  $\tau$  and  $s$ -topologies. Then the existence and uniqueness theory of ODE implies that  $\xi$  exists on an interval containing  $[0, \infty)$ , is continuous in  $\tau$  and  $s$ -topologies, and  $\xi(t, \cdot) : J^b \rightarrow \xi(t, J^b)$  is a homeomorphism for each  $t \geq 0$ . Moreover  $J(\xi(t, q)) \leq J(q)$  for  $q \in J^b$  and by construction, each  $(t, q) \in [0, \infty) \times J^b$  has an open neighborhood  $W_{(t,q)}$  (in the product topology on  $\mathbb{R} \times H_\tau$ ) such that  $\{v - \xi(t, v) : (s, v) \in W_{(t,q)} \cap [0, \infty) \times J^b\}$  is contained in a finite dimensional subspace. Since  $J(q) \leq m$  for all  $q \in \mathcal{K}$ . Then for  $m' \gg m$ , one can easily obtain that  $q \notin N_\sigma(\mathcal{K})$  for  $q \in J_c$ . Therefore, considering the flow  $\xi$ , by (6.7) we have  $J(\xi(\frac{d-c}{\alpha}, q)) \leq c$  for all  $q \in J_c^d$ . Let  $h(\cdot) = \xi(\frac{d-c}{\alpha}, \cdot)$ . Then  $h(J^d) \subset J^c$  and  $h \in \mathcal{M}(J^d)$ , that is,  $h$  is required.  $\square$

**Proof of Lemma 6.6.** The proof is similar to that of Lemma 4.5 in [8]. Since  $\psi(c) = 0$  if  $c < \kappa$ , by Lemma 6.5, we only need to consider any  $c \in [\kappa, m']$ . For any  $b > \kappa$  large, we define

$$\psi_b(c) := \min\{\text{gen}(f(J^c) \cap S_\rho Y) : f \in \mathcal{M}(J^b)\}$$

for  $a = \kappa/2 \leq c < b$ . Since  $\mathcal{M}(J^b) \subset \mathcal{M}(J^c)$  via the restriction  $f \rightarrow f|_{J^c}$ , we have  $\psi(c) \leq \psi_b(c)$ . It is sufficient to show that  $\psi_b$  assumes only finite values. Clearly  $\psi_b(c) = 0$  if  $c < \kappa$  since  $\text{id} \in \mathcal{M}(J^b)$ .

Consider the flow of (6.9). In order to prove our result, we need firstly to prove the following

**Claim.** For any  $c \in (a, b)$ , there is  $\delta > 0$  such that  $\xi(1, J^{c+\delta}) \subset J^{c-\delta} \cup W_\sigma$ .

Assume by contradiction that there exists a sequence  $q_n \in J^{c+1/n}$  with  $\xi(1, q_n) \notin J^{c-1/n}$ . Then for  $n$  large there is  $t_n \in (0, 1)$  such that  $\xi(t_n, q_n) \in W_{\sigma/3}$ . Indeed, if this is not true, then by (6.7), we have  $\xi(1, q_n) \in J^{c+1/n-\alpha} \subset J^{c-1/n}$  for  $n$  large enough, a contradiction. Thus there are  $0 \leq r_n < s_n \leq 1$  such that  $\xi(r_n, q_n) \in \partial W_{\sigma/3}$ ,  $\xi(s_n, q_n) \in \partial W_\sigma$  and  $\xi(t, q_n) \in W_\sigma \setminus W_{\sigma/3}$  for all  $t \in (t_n, r_n)$ . This implies that  $\|\xi(r_n, q_n) - \xi(s_n, q_n)\| \geq 2\sigma/3$ . On the other hand, by (6.8),

$$\|\xi(r_n, q_n) - \xi(s_n, q_n)\| \leq \int_{r_n}^{s_n} \|V(\xi(t, q_n))\| dt \leq R(s_n - r_n).$$

Hence  $s_n - r_n \geq 2\sigma/3R$ ; this jointly with (6.7) implies that

$$\begin{aligned} c - \frac{1}{n} &< J(\xi(s_n, q_n)) \leq J(\xi(r_n, q_n)) - \frac{2\sigma\alpha}{3R} \\ &\leq c + \frac{1}{n} - \frac{2\sigma\alpha}{3R}. \end{aligned}$$

Obviously, this is a contradiction for  $n$  large. The claim is proved.

Now set  $h(\cdot) := \xi(1, \cdot)$ . Then  $h \in \mathcal{M}(J^b)$  (here, by a standard argument, one can obtain  $h(J^b)$  is closed (cf. Lemma 4.6 in [19])) and  $h(J^{c+\delta}) \subset J^{c-\delta} \cup W_\sigma$  for  $\delta > 0$  small. Choose  $f \in \mathcal{M}(J^b)$  such that  $\psi_b(c - \delta) = \text{gen}(f(J^{c-\delta}) \cap S_\rho Y)$ . Then  $f \circ h \in \mathcal{M}(J^b)$ . Consequently, by the monotonicity and subadditivity of the genus, we obtain

$$\begin{aligned} \psi_b(c + \delta) &\leq \text{gen}(f \circ h(J^{c+\delta}) \cap S_\rho Y) \\ &\leq \text{gen}(f(J^{c-\delta} \cup W_\sigma) \cap S_\rho Y) \\ &\leq \text{gen}(f(J^{c-\delta}) \cap S_\rho Y) + \text{gen}(f(W_\sigma)) \\ &\leq \psi_b(c - \delta) + 1. \end{aligned}$$

Here we have used the fact that  $f(W_\sigma)$  is homeomorphic to  $W_\sigma$  which in turn is homotopy equivalent to  $B$ . Note that the homotopy equivalence  $f(W_\sigma) \rightarrow B$  is odd, hence  $\text{gen}(f(W_\sigma)) \leq \text{gen}(B) \leq 1$  follows from the discreteness of  $B$  by (6.1). This implies that  $\psi_b$  achieves only finite values.  $\square$

## 6.2. Proof of the multiplicity of Theorem 1.1

Before we prove our main result, we need firstly to concern with a discreteness property of the set of sums of translates of  $\mathcal{K}/\mathbb{Z}$ . Here we state it in a more general form.

**Lemma 6.7.** *Given  $l \in \mathbb{N}$  and let  $\mathcal{B} := \{B^i\}$ , here  $B^i \subset H$  is compact and  $\mathcal{B}$  consists of finitely many  $B^i$ 's. Set*

$$[\mathcal{B}, l] := \left\{ \sum_{i=1}^j k^i * B^i : 1 \leq j \leq l, k^i \in \mathbb{Z}, B^i \in \mathcal{B} \right\}.$$

Denote  $\mu = \mu([\mathcal{B}, l]) = \inf\{\|U - V\| : U, V \in \mathcal{C}([\mathcal{B}, l]), U \cap V = \emptyset\}$ . Then  $\mu > 0$ .

**Proof.** The proof is similar to that of Proposition 1.55 in [12] and is omitted, we refer the reader to [12] for more details.  $\square$

**Proof of the multiplicity of Theorem 1.1.** The proof will be completed in an indirect way, namely, we show that if

$$\mathcal{K}/\mathbb{Z} \text{ is a finite set consisting of } S^1\text{-critical orbits,} \quad (6.10)$$

then  $J$  has an unbounded sequence of critical values, a contradiction. We do this by checking that if (6.10) is true then  $J$  satisfies all the assumptions of Theorem 6.2.

Since  $V_i(x)$  is even in  $x$  for all  $i \in \mathbb{Z}$ ,  $V(q)$  is even in  $q$ , hence so is  $J$ .  $J(0) = 0$  is deduced from  $(A_2)$ . The assumptions  $(J_0)$ – $(J_2)$  have already been verified as above. Recall that  $\dim H^+ = \infty$ . Let  $(f_k)$  be a base of  $H^+$  and set  $Y_n := \text{span}\{f_1, \dots, f_n\}$ ,  $H_n := H^- \oplus Y_n$ . With such a choice of a sequence of subspaces it follows from Lemma 3.3 that  $(J_3)$  is satisfied. In order to check  $(J'_4)$  assume (6.10) holds.

Denote  $\mathcal{F} := (\mathcal{K} \setminus \{0\})/\mathbb{Z}$ , a set consisting of arbitrarily chosen representatives of  $\mathbb{Z}$ -orbits. Obviously, (6.10) implies that  $\mathcal{F}$  consists of finitely many compact sets. Since  $J'$  is odd, we may assume

that  $\mathcal{F}$  is symmetric. For any compact interval  $I \subset (0, \infty)$  with  $b := \max I$ , set  $l = [b/\theta]$  and take  $\mathcal{A} = [\mathcal{F}, l]$ . Then  $\mathcal{K} \setminus \{0\} \subset \mathcal{A}$  and both sets are symmetric with respect to the origin. Note that  $P^+\mathcal{A} = [P^+\mathcal{F}, l]$ . By (6.10),  $P^+\mathcal{F}$  is a set consisting of finitely many compact sets and

$$\|q\| \leq l \max\{\|\bar{q}\| : \Omega \bar{q} \in \mathcal{F}\} \quad \text{for } q \in \mathcal{A},$$

which implies that  $\mathcal{A}$  is bounded. In addition, by Lemma 4.6,  $\mathcal{A}$  is a  $(C)_I$ -attractor, and by Lemma 6.7,

$$\begin{aligned} & \inf\{\|P^+U_1 - P^+U_2\| : U_1, U_2 \in \mathcal{C}(\mathcal{A}), P^+U_1 \cap P^+U_2 = \emptyset\} \\ &= \inf\{\|U - V\| : U, V \in \mathcal{C}(P^+\mathcal{A}), U \cap V = \emptyset\} > 0. \end{aligned}$$

This argument shows that  $J$  satisfies  $(J'_4)$ . Therefore the multiplicity is obtained.

Moreover, as in the proof of the existence of Theorem 1.1 (see Section 5.2), for any given positive integer  $N$ , we can easily obtain that there exists  $\bar{T} > 0$  depending on  $N$ , positive  $\alpha_i$  and corresponding  $V_i$ , such that  $N$  different geometrically distinct  $T$ -periodic solutions we have just obtained can be nonconstant if  $\bar{T} < \pi/\sqrt{\beta}$  and its period  $T \in (\bar{T}, \pi/\sqrt{\beta})$ . It is worth mentioning that the conditions on  $\bar{T}$  and  $\pi/\sqrt{\beta}$  are independent of each other, there are potentials for which this inequality is satisfied.  $\square$

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