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Low Mach number limit of full Navier–Stokes equations in a 3D bounded domain

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Abstract

This paper studies the low Mach number limit of the full compressible Navier–Stokes equations in a three-dimensional bounded domain where the velocity field and the temperature satisfy the slip boundary conditions and the Neumann boundary condition, respectively. The uniform estimates in the Mach number for the strong solutions are derived in a short time interval, provided that the initial density and temperature are close to the constant states and satisfy the "bounded derivative conditions". Thus the solutions of the full compressible Navier–Stokes equations converge to the one of the isentropic incompressible Navier–Stokes equations, as the Mach number vanishes.

Keywords: Low Mach number limit; Full Navier–Stokes equations; Slip boundary conditions; Bounded domain; Polytropic gas

1. Introduction

This paper is concerned with the low Mach number limit of compressible fluid flows which are described by the following non-dimensional Navier–Stokes equations in $\mathbb{R}^3$:

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$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\bigl(2\mu D(u) + \zeta \operatorname{div} u I\bigr) + \frac{1}{\epsilon^2}\nabla P = 0, \tag{2}$$

$$(\rho e)_t + \operatorname{div}(\rho u e) + P \operatorname{div} u - \operatorname{div}(\kappa \nabla \mathcal{T}) = \epsilon^2\bigl(2\mu\bigl|D(u)\bigr|^2 + \zeta(\operatorname{div} u)^2\bigr), \tag{3}$$

where ρ, $u = (u^1, u^2, u^3)$, P, e, $\mathcal{T}$ stand for the density, velocity, pressure, internal energy and temperature, respectively. The constants μ, ζ are viscosity coefficients satisfying $\mu > 0$ and $\mu + 3\zeta/2 \geq 0$, $\epsilon > 0$ is the Mach number, $\kappa > 0$ is the heat conductivity coefficient, and $D(u) = (\nabla u + \nabla u^t)/2$. Moreover, we assume that the fluid is a polytropic ideal gas, that is,

$$e = c_v \mathcal{T}, \qquad P = R\rho\mathcal{T} \tag{4}$$

with $c_v > 0$ and R being the specific heat at constant volume and the generic gas constant, respectively. The ratio of specific heats is denoted by $\gamma = 1 + R/c_v$.

By a formal derivation, see [18], for instance, as ϵ tends to zero, the solutions of (1)–(4) converge to the solution (ρ, u, π) of the following problem:

$$\rho_t + \operatorname{div}(\rho u) = 0,$$
$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\bigl(2\mu D(u) + \zeta \operatorname{div} u I\bigr) + \nabla\pi = 0,$$
$$\gamma \operatorname{div} u = \operatorname{div}\left[\frac{\kappa}{R}\nabla\left(\frac{1}{\rho}\right)\right].$$

This procedure is a singular limit, namely, the low Mach number limit, which is a physically interesting problem. Due to the large parameter $1/\epsilon^2$ in the momentum equations (2), mathematically, it is difficult to obtain uniform estimates (of higher derivatives) in Mach number, which are necessary for the strong convergence to the background incompressible flows.

One of the major physical factors concerning the low Mach number limit is that the fluid is isentropic or non-isentropic. When the flows are isentropic, the limit velocity u is divergence-free and the limit equations are the incompressible Navier–Stokes equations in the usual sense. In this situation, the singular limit reduces to the incompressible limit, which was well investigated in the past decades. The readers may refer to [3,6–8,14,17,19,20,25,26], for instance, and the references therein for details.

In the non-isentropic case, the large pressure gradient of $O(1/\epsilon^2)$ in the momentum equations is closely related to the behavior of both density and temperature. Thus, the low Mach number analysis is much more complicated. One may refer to [1,21,22] for the literature on the non-isentropic Euler equations ($\mu = \zeta = 0$).

The analysis for the non-isentropic Navier–Stokes equations is more difficult, due to the complexity of the system structure. Bresch, Desjardins, Grenier and Lin [5] analyzed the acoustic waves by a method of characteristic expansions and gave a formal asymptotics as $\epsilon \to 0$ in $\mathbb{T}^n$ under the assumption that the viscous heating and thermal diffusion are negligible. As an improvement of [5], Feireisl and Novotný [10] considered the low Mach number limit for the periodic "variational solutions" to the full Navier–Stokes–Fourier equations of certain radiative gases for "ill-prepared" initial data. Alazard [2] studied this singular limit for local H^s solutions ($s > 2 + n/2$) in $\mathbb{R}^n$ for general initial data by employing the technique of pseudo-differential operators, and Jiang, Ju, Li and Xin in [15] by careful a priori (energy) estimates only. Unfortunately, the arguments in [2,15] do not apply to the cases with boundary. Recent progress for the situations of initial boundary value problems for the full Navier–Stokes–Fourier equations can be found, for example, in [9,12]. More recently, an interesting result on the inviscid and low Mach number limits of the Navier–Stokes–Fourier equations in $\mathbb{R}^3$ was obtained in [11].

To our best knowledge, the low Mach number limit for the full Navier–Stokes equations of perfect gases in bounded domains is not yet proved so far. One of the difficulties is to deal with the boundary effect due to the acoustic waves. In a recent work [16], the authors studied the low Mach number limit of the non-isentropic Navier–Stokes equations with zero thermal conductivity in a three-dimensional bounded domain, provided that the initial data are "well-prepared" and the velocity satisfies the "non-slip" boundary condition. The situation in 2D with Navier's slip boundary condition was considered in [23]. The aim of the current paper is to extend the result in [16,23] to the three-dimensional full Navier–Stokes equations, that is, to verify the low Mach number limit for (1)–(4) with positive heat conductivity in a bounded domain in $\mathbb{R}^3$. It should be noted that our result is not included in the cases considered in [9,12]. Uniform estimates for the full compressible Navier–Stokes equations are shown by careful energy estimates, thus the strong solutions converge to the one of the isentropic incompressible Navier–Stokes equations as the Mach number vanishes. The result in this paper is a nontrivial generalization since the structure of the penalty operator of order ϵ^{-1} is essentially different from the ones in [16,23], which leads to the difficulties in controlling the high-order spatial derivatives of the velocity and the temperature. We shall overcome these difficulties by changing variables for both density and temperature, and showing some new energy estimates. It is worthy to note that all the second-order spatial derivatives of the velocity are uniformly bounded in this paper, while in [16] the normal component of the second order derivatives near the boundary is not uniformly bounded with respect to the Mach number. Moreover, here the boundary estimates are simplified in the sense that only the derivatives of the vorticity are needed to be evaluated near the boundary. Furthermore, compared with [23], the time derivatives of second order of the solutions are no longer required to be bounded initially in the current paper.

To state the main result of this paper, we denote the density and temperature variations by σ and θ:

$$\rho = 1 + \epsilon\sigma, \qquad \mathcal{T} = 1 + \epsilon\theta.$$

Then we can rewrite the non-dimensional system (1)–(3) as follows

$$\sigma_t + \operatorname{div}(\sigma u) + \frac{1}{\epsilon}\operatorname{div} u = 0, \tag{5}$$

$$\rho(u_t + u\cdot\nabla u) + \frac{R}{\epsilon}(\nabla\sigma + \nabla\theta) + R\nabla(\sigma\theta) = \mu\Delta u + \lambda\nabla\operatorname{div} u, \tag{6}$$

$$\begin{aligned} &c_v\rho(\theta_t + u\cdot\nabla\theta) + R(\rho\theta + \sigma)\operatorname{div} u + \frac{R}{\epsilon}\operatorname{div} u \\ &\quad = \kappa\Delta\theta + \epsilon\big(2\mu\big|D(u)\big|^2 + \zeta(\operatorname{div} u)^2\big), \end{aligned} \tag{7}$$

where $\lambda := \mu + \zeta$. We impose the following initial and boundary conditions:

$$(\sigma, u, \theta)|_{t=0} = (\sigma_0, u_0, \theta_0) \quad \text{in } \Omega, \tag{8}$$

$$u \cdot n = 0, \qquad n \times \operatorname{curl} u = 0, \qquad \frac{\partial \theta}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{9}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, n is the unit outer normal to $\partial\Omega$, and the vorticity $\operatorname{curl} u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^t$.

Definition 1.1.

$$|||u|||_{k,j} \equiv \sum_{i=0}^{j} \left\| \partial_t^i u \right\|_{H^{k-i}(\Omega)},$$

$$|||u|||_{k,j}(0) = \sum_{i=0}^{j} \left\| \partial_t^i u(0) \right\|_{H^{k-i}(\Omega)},$$

where $H^0(\Omega) \equiv L^2(\Omega)$.

Remark 1.2. To simplify the statement, we have used "$u_t(0)$" to signify the quantity $u_t|_{t=0} := -u_0 \cdot \nabla u_0 + (-\frac{R}{\epsilon}(\nabla\sigma_0 + \nabla\theta_0) - R\nabla(\sigma\theta_0) + \mu\Delta u_0 + \nu\nabla \operatorname{div} u_0)/\rho_0$ obtained through Eqs. (6), and "$u_{tt}(0)$" is given recursively by ∂_t(6) in the same manner. Similarly, we can define $\rho_t(0)$, $\sigma_t(0)$, $u_t(0)$, $\rho_{tt}(0)$, $\sigma_{tt}(0)$ and $u_{tt}(0)$.

A local existence result for (5)–(9) in the following sense is indeed already shown in [28], where (5)–(7) are written equivalently in the form of (1)–(3). Thus, we omit the details of the proof and only describe the result as follows.

Theorem 1.3 *(Local existence). Let $\epsilon \in (0, 1]$ and $\Omega \subset \mathbb{R}^3$ be a simply connected, bounded domain with smooth boundary $\partial\Omega$. Suppose that the initial datum $(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon)$ satisfies $1 + \epsilon\sigma_0^\epsilon \geq m > 0$ for some constant m, and*

$$\left(\partial_t^k \sigma^\epsilon(0), \partial_t^k u^\epsilon(0), \partial_t^k \theta^\epsilon(0)\right) \in H^{2-k}(\Omega), \quad k = 0, 1, 2.$$

Assume the following compatibility conditions are satisfied:

$$u_0^\epsilon \cdot n = u_t^\epsilon(0) \cdot n = 0, \quad n \times \operatorname{curl} u_0^\epsilon = n \times \operatorname{curl} u_t^\epsilon(0) = 0 \quad \text{on } \partial\Omega \tag{10}$$

and

$$\frac{\partial \theta_0^\epsilon}{\partial n} = \frac{\partial \theta_t^\epsilon(0)}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{11}$$

Then there exists a positive constant $T^\epsilon = T^\epsilon(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, m, \epsilon)$, such that the initial–boundary problem (5)–(9) admits a unique solution $(\sigma^\epsilon, u^\epsilon, \theta^\epsilon)$, satisfying $1 + \epsilon\sigma^\epsilon > 0$ in $\Omega \times (0, T^\epsilon)$ and

$$\left(\partial_t^k \sigma^\epsilon, \partial_t^k \theta^\epsilon\right) \in C\left(\left[0, T^\epsilon\right], H^{2-k}\right),$$
$$\partial_t^k u^\epsilon \in C\left(\left[0, T^\epsilon\right], H^{2-k}\right) \cap L^2\left(0, T^\epsilon; H^{3-k}\right), \quad k = 0, 1, 2.$$

Then, the main result of this paper is the following, which gives a uniform estimate for strong solutions of (5)–(9), and the corresponding low Mach number limit.

Theorem 1.4. *Assume that $(\sigma^\epsilon, u^\epsilon, \theta^\epsilon)$ is the solution obtained in Theorem 1.3, where the initial datum $(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon)$ satisfies*

$$\left\| \left(\sigma^\epsilon, u^\epsilon, \theta^\epsilon\right) \right\|_{2,1}(0) + \epsilon \left\| \left(\sigma_{tt}^\epsilon, u_{tt}^\epsilon, \theta_{tt}^\epsilon\right)(0) \right\|_{L^2} + \left\| \left(1+\epsilon\rho_0^\epsilon\right)^{-1} \right\|_{L^\infty} \le D_0$$

for some given constant $D_0 > 0$ independent of $\epsilon \in (0,1]$, and the compatibility conditions (10) and (11). Then there exist positive constants $T_0 = T_0(D_0, \Omega)$ and $D = D(D_0, \Omega)$, independent of $\epsilon \in (0,1]$, such that $(\sigma^\epsilon, u^\epsilon, \theta^\epsilon)$ satisfies the uniform estimate:

$$\begin{aligned} &\sup_{0\le t\le T_0} \left(\left\| \left(\sigma^\epsilon, u^\epsilon, \theta^\epsilon\right) \right\|_{2,1} + \epsilon \left\| \left(\sigma_{tt}^\epsilon, u_{tt}^\epsilon, \theta_{tt}^\epsilon\right) \right\|_{L^2} + \left\| \left(1+\sigma^\epsilon\right)^{-1} \right\|_{L^\infty} \right)(t) \\ &\quad + \left[\int_0^{T_0} \left(\left\| \left(u^\epsilon, \theta^\epsilon\right) \right\|_{3,1}^2 + \left\| \epsilon \left(\sigma_{tt}^\epsilon, u_{tt}^\epsilon, \theta_{tt}^\epsilon\right) \right\|_{H^1}^2 \right) dt \right]^{1/2} \le D. \end{aligned} \tag{12}$$

Furthermore, $(\sigma^\epsilon, u^\epsilon, \theta^\epsilon)$ converges to (σ, u, θ) in certain Sobolev spaces as $\epsilon \to 0$, and there exists a function $\pi(x,t)$, such that $(u, \pi) \in C([0, T_0], H^2)$ solves the following initial–boundary value problem for the isentropic incompressible Navier–Stokes equations:

$$\begin{aligned} &u_t + u\cdot\nabla u + \nabla\pi = \mu\Delta u, \qquad \operatorname{div} u = 0 \quad \text{in } \Omega \times [0, T_0] \\ &u\cdot n = 0, \qquad n \times \operatorname{curl} u = 0 \quad \text{on } \partial\Omega, \\ &u|_{t=0} = u_0 \quad \text{in } \Omega, \end{aligned} \tag{13}$$

where u_0 is the weak limit of u_0^ϵ in $H^2(\Omega)$ with $\operatorname{div} u_0 = 0$ in Ω.

Definition 1.5.

$$\begin{aligned} M^\epsilon(t) \equiv &\sup_{0\le s\le t} \left(\left\| \left(\sigma^\epsilon, u^\epsilon, \theta^\epsilon\right) \right\|_{2,1} + \epsilon \left\| \left(\sigma_{tt}^\epsilon, u_{tt}^\epsilon, \theta_{tt}^\epsilon\right) \right\|_{L^2} + \left\| \left(1+\sigma^\epsilon\right)^{-1} \right\|_{L^\infty} \right)(s) \\ &+ \left[\int_0^{t} \left(\left\| \left(u^\epsilon, \theta^\epsilon\right) \right\|_{3,1}^2 + \left\| \epsilon \left(\sigma_{tt}^\epsilon, u_{tt}^\epsilon, \theta_{tt}^\epsilon\right) \right\|_{H^1}^2 \right) ds \right]^{1/2}, \\ M_0^\epsilon \equiv &M^\epsilon(t=0). \end{aligned}$$

Similar as in [2,21], it suffices to show the following theorem to get the uniform estimate (12):

Theorem 1.6. *Let T_ϵ be the maximal time of existence of the solution to the initial–boundary problem (5)–(9) established in Theorem 1.3. Then for any $t \in [0, T_\epsilon)$, we have*

$$M^\epsilon(t) \le C_0\left(M_0^\epsilon\right) \exp\left[t^{1/4} C\left(M^\epsilon(t)\right)\right],$$

for some given positive nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

This article is organized as follows. In Section 2, we state some elementary lemmas and calculus inequalities for the convenience of the reader. In Section 3, uniform estimates with respect to the Mach number are shown by employing a careful analysis on both temporal and spatial derivatives. To this end, we first estimate the density $\rho = \rho(\epsilon\sigma)$ to simplify the forthcoming calculations. Then we show the L^2-estimate of (σ, u, θ), next the estimate of low-order derivatives and finally the high-order derivatives. Integration by parts is usually invalid in the boundary case, particularly in estimating high-order spatial derivatives. Thus, during the process of a priori estimates, we take the strategy of operator decomposition, so that one can control the divergence and vorticity of the velocity, respectively. When estimating each derivative, we carefully balance the singular differential operators of order ϵ^{-1}, which leads to the uniform estimates for the full norm. In Section 4 we give the proof of Theorems 1.4 and 1.6.

We end this section by introducing the notation used throughout this paper. By $W^{k,p}(\Omega)$ and $H^k(\Omega)$ we denote the usual Sobolev spaces with the norms $\|\cdot\|_{W^{k,p}}$ and $\|\cdot\|_{H^k}$, respectively. By C, and $C_0(\cdot)$, $C(\cdot)$, $C_i(\cdot)$, $i = 1, 2, \cdots$, we denote a generic positive constant, and positive non-decreasing continuous functions of their argument respectively, which are independent of ϵ. And we shall use the following abbreviations:

$$L_t^p\big(H^k\big) \equiv L^p\big(0, t; H^k(\Omega)\big), \qquad C_t\big(H^k\big) \equiv C\big([0, t], H^k(\Omega)\big),$$
$$\|\cdot\|_{L_t^p(H^k)} \equiv \|\cdot\|_{L^p(0,t;H^k)}, \qquad \|\cdot\|_{C_t(H^k)} \equiv \|\cdot\|_{C([0,t],H^k)}.$$

Furthermore, we denote the components of a vector by superscripts, for example, u^j means the j-th component of a vector u.

2. Preliminaries

In this section, we list some lemmas which will be frequently used throughout this paper.

Lemma 2.1. *(See [4].) Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outward normal n. Then there exists a constant $C > 0$ independent of u, such that*

$$\|u\|_{H^s(\Omega)} \le C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\operatorname{curl} u\|_{H^{s-1}(\Omega)} + \|u \cdot n\|_{H^{s-1/2}(\partial\Omega)} + \|u\|_{H^{s-1}(\Omega)}\big),$$

for any $u \in H^s(\Omega)^N$, $s \ge 1$.

Lemma 2.2. *(See [27].) Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outward normal n. Then there exists a constant $C > 0$ independent of u, such that*

$$\|u\|_{H^s(\Omega)} \le C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\operatorname{curl} u\|_{H^{s-1}(\Omega)} + \|u \times n\|_{H^{s-1/2}(\partial\Omega)} + \|u\|_{H^{s-1}(\Omega)}\big),$$

for any $u \in H^s(\Omega)^N$, $s \ge 1$.

Lemma 2.3. *(See [13, Part 1, Theorem 10.1].) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^k-boundary and u a function in $W^{k,r}(\Omega) \cap L^q(\Omega)$ with $1 \le r, q \le \infty$. For any integer j with $0 \le j < k$, and for any number a in the interval $[j/k, 1]$, set*

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{k}{N}\right) + (1-a)\frac{1}{q}.$$

If $k - j - N/r$ is not a nonnegative integer, then

$$\left\| D^j u \right\|_{L^p(\Omega)} \le C \|u\|^a_{W^{k,r}(\Omega)} \|u\|^{1-a}_{L^q(\Omega)}. \tag{14}$$

If $k - j - N/r$ is a nonnegative integer, then (14) only holds for $a = j/k$. The constant C depends only on Ω, r, q, k, j, a.

For the reader's convenience, we give some special cases of (14) for $\Omega \subset \mathbb{R}^3$ which will be frequently applied throughout this paper:

$$\left\|\nabla^2 u\right\|_{L^2} \le C\|u\|^{\frac{2}{3}}_{H^3} \|u\|^{\frac{1}{3}}_{L^2}, \qquad \left\|\nabla^2 u\right\|_{L^2} \le C\|u\|^{\frac{1}{2}}_{H^3} \|\nabla u\|^{\frac{1}{2}}_{L^2},$$
$$\|u\|_{L^3} \le C\|u\|^{\frac{1}{2}}_{H^1} \|u\|^{\frac{1}{2}}_{L^2}, \qquad \|u\|_{L^4} \le C\|u\|^{\frac{3}{4}}_{H^1} \|u\|^{\frac{1}{4}}_{L^2}.$$

Moreover, by the Sobolev embedding theorem, we have

$$\|u\|_{L^\infty} \le C\|u\|_{W^{1,4}} \le C\|u\|^{\frac{3}{4}}_{H^2} \|u\|^{\frac{1}{4}}_{H^1} \le C\|u\|^{\frac{7}{8}}_{H^2} \|u\|^{\frac{1}{8}}_{L^2}.$$

3. Uniform estimates

During deriving the uniform energy estimates, we drop the superscript "ϵ" of $\rho^\epsilon, \sigma^\epsilon, u^\epsilon, \theta^\epsilon$, etc. for the sake of simplicity; moreover, we write $M^\epsilon(t)$, M^ϵ_0 as M, M_0, respectively, for short.

3.1. Estimate of $\rho(\epsilon\sigma)$

Note that $\rho(\epsilon\sigma)$ and its derivatives always appear as a coefficient of (σ, u, θ) and their derivatives. Thus, for the convenience of computation, we estimate $\rho(\epsilon\sigma)$ in terms of the initial data and u, by using the conservation of mass, namely Eq. (1). This can be done by the standard energy method (see, e.g. [23]), thus we only sketch the proof here. For any integer $k \ge 2$, we multiply (1) by $-\rho^{-k}$ to derive

$$\frac{1}{k-1}\int_\Omega \left(\partial_t \rho^{1-k} + u \cdot \nabla \rho^{1-k}\right) dx - \int_\Omega \rho^{1-k} \operatorname{div} u\, dx = 0,$$

which gives, by integrating by parts, that

$$\frac{d}{dt}\left\|\rho^{-1}\right\|_{L^{k-1}} \le C\frac{k}{k-1}\|\operatorname{div} u\|_{L^\infty}\left\|\rho^{-1}\right\|_{L^{k-1}},$$

for some constant $C > 0$ independent of k. Then by Gronwall's inequality and letting $k \to \infty$, we obtain

$$\begin{aligned}\|\rho^{-1}(t)\|_{L^\infty} &\le \|\rho_0^{-1}\|_{L^\infty}\exp\big(C\sqrt{t}\|u\|_{L_t^2(H^3)}\big)\\ &\le C_0(M_0)\exp(C\sqrt{t}M).\end{aligned}$$

Applying D^α, $0\le|\alpha|\le 2$, to (1) yields

$$\big(D^\alpha\rho\big)_t + u\cdot\nabla\big(D^\alpha\rho\big) + D^\alpha(\rho\operatorname{div}u) = \big[u, D^\alpha\big]\cdot\nabla\rho,$$

where D^α is the spatial derivative with multi-index α and the commutator $[a,b]$ is defined by $[a,b] := ab - ba$. Then the standard energy method and the commutator estimates give

$$\begin{aligned}\|\rho(t)\|_{H^2} &\le \|\rho_0\|_{H^2}\exp\left(C\int_0^t \|u\|_{H^3}ds\right)\\ &\le C_0(M_0)\exp(C\sqrt{t}M).\end{aligned}$$

Differentiating (1) in temporal variable and calculating as above, we obtain

$$\begin{aligned}\|\rho_t(t)\|_{L^2} &\le \left(\|\rho_t(0)\|_{L^2} + \int_0^t \|u_t\|_{H^1}\|\rho\|_{H^2}ds\right)\exp\left(C\int_0^t \|u\|_{H^3}ds\right)\\ &\le \big(C_0(M_0) + tM\|\rho\|_{C_t(H^2)}\big)\exp(C\sqrt{t}M).\end{aligned}$$

On account of the lower order estimates, one finds that

$$\begin{aligned}&\|\nabla\rho_t(t)\|_{L^2} + \|\rho_{tt}(t)\|_{L^2}\\ &\quad\le \exp(C\sqrt{t}M)\big[C_0(M_0) + \sqrt{t}M\big(\|\rho_t\|_{C_t(H^1)} + \|\rho\|_{C_t(H^2)}\big)\big].\end{aligned}$$

Collecting all the estimates above and applying Gronwall's inequality, we obtain

Lemma 3.1. *For any* $0\le t\le\min(T_\epsilon, 1)$,

$$\big(\|\!|\rho(\epsilon\sigma)|\!\|_2 + \|\rho(\epsilon\sigma)^{-1}\|_{L^\infty}\big)(t) \le C_0(M_0)\exp\big(\sqrt{t}C(M)\big).$$

3.2. Basic estimates

Multiplying (5), (6) and (7) by $R\sigma$, u and θ, respectively, and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|(\sqrt{R}\sigma, \sqrt{\rho}u, \sqrt{c_v\rho}\theta)\big\|_{L^2}^2 + \big\|(\sqrt{\mu}\operatorname{curl}u, \sqrt{\mu+\lambda}\operatorname{div}u, \sqrt{\kappa}\nabla\theta)\big\|_{L^2}^2\\ &\quad= \int_\Omega\left(\operatorname{div}u\left(-\frac{1}{2}\sigma^2 + \sigma\theta - R\rho\theta^2\right) + \epsilon^2\theta\big(2\mu|D(u)|^2 + \zeta(\operatorname{div}u)^2\big)\right)dx\\ &\quad\le \|\nabla u\|_{L^\infty}\big(\big(1+\|\rho\|_{L^\infty}\big)\big(\|(\sigma,\theta)\|_{L^2}^2\big) + \|\nabla u\|_{L^2}\|\theta\|_{L^2}\big).\end{aligned}$$

Then from Gronwall's inequality and Lemmas 2.1 and 3.1, we get

Lemma 3.2. *For any* $0 \le t \le \min(T_\epsilon, 1)$,

$$\|(\sigma, u, \theta)(t)\|_{L^2} + \|(u, \theta)\|_{L_t^2(H^1)} \le C_0(M_0)\exp\big(\sqrt{t}C(M)\big).$$

3.3. Estimates for lower-order derivatives

Note that

$$\int_\Omega (\mu\Delta u + \lambda\nabla \operatorname{div} u)\cdot\nabla \operatorname{div} u\,dx = (\mu+\lambda)\|\nabla \operatorname{div} u\|_{L^2}^2$$

and

$$\int_\Omega \rho u_t\cdot\nabla \operatorname{div} u\,dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega \rho(\operatorname{div} u)^2dx + \int_\Omega\left(\frac{1}{2}\rho_t(\operatorname{div} u)^2 - \nabla\rho\cdot u_t \operatorname{div} u\right)dx.$$

Thus, we integrate the product of $\nabla \operatorname{div} u$ and (6) to obtain

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{\rho}\operatorname{div} u(t)\|_{L^2}^2 + (\mu+\lambda)\|\nabla \operatorname{div} u\|_{L_t^2L^2}^2 - \frac{R}{\epsilon}\int_0^t\int_\Omega(\nabla\sigma+\nabla\theta)\cdot\nabla \operatorname{div} u\,dx\,dt \\
&= \int_0^t\int_\Omega\left(\frac{1}{2}\rho_t(\operatorname{div} u)^2 - \nabla\rho\cdot u_t\operatorname{div} u + \big(\rho u\cdot\nabla u + \nabla(\sigma\theta)\big)\cdot\nabla \operatorname{div} u\right)dx\,dt \\
&\le C_0(M_0) + \sqrt{t}C(M).
\end{aligned}$$

To balance the singular term on the left-hand side of the above inequality, we apply ∇ to (5) and multiply the resulting equality by $R\nabla\sigma$ over $L^2(\Omega)\times(0,t)$ to get

$$\begin{aligned}
&\frac{R}{2}\|\nabla\sigma(t)\|_{L^2} + \frac{R}{\epsilon}\int_0^t\int_\Omega \nabla \operatorname{div} u\cdot\nabla\sigma\,dx\,dt \\
&= \frac{R}{2}\|\nabla\sigma_0\|_{L^2} - R\int_0^t\int_\Omega \nabla \operatorname{div}(\sigma u)\cdot\nabla\sigma\,dx\,dt \\
&\le C_0(M_0) + C\sqrt{t}\|u\|_{L_t^2(H^3)}\|\sigma\|_{L_t^\infty(H^2)}^2 \\
&\le C_0(M_0) + \sqrt{t}C(M).
\end{aligned}$$

Similarly, by (7)–(9) we can deduce

$$
\begin{aligned}
&\frac{c_v}{2}\left\|\sqrt{\rho}\nabla\theta(t)\right\|_{L^2}^2+\frac{\kappa}{2}\|\Delta\theta\|_{L_t^2(L^2)}^2+\frac{R}{\epsilon}\int_0^t\int_\Omega \nabla\theta\cdot\nabla\operatorname{div} u\,dx\,dt \\
&\quad\le \frac{c_v}{2}\|\sqrt{\rho_0}\nabla\theta_0\|_{L^2}^2+C\int_0^t\Big(\|\rho_t\|_{L^2}\|\nabla\theta\|_{H^1}^2+\|\rho\|_{H^2}\|\theta_t\|_{L^2}\|\nabla\theta\|_{H^1} \\
&\qquad+\big(\|\rho\|_{H^2}\|\theta\|_{H^2}+\|\sigma\|_{H^2}\big)\|u\|_{H^2}\|\nabla\theta\|_{H^1}+\epsilon\|\nabla u\|_{H^1}^2\|\nabla\theta\|_{H^1}\Big)dt \\
&\quad\le C_0(M_0)+tC(M).
\end{aligned}
$$

Putting the estimates in the above three inequalities together and applying Lemma 3.1, we obtain

Lemma 3.3. *For any* $0\le t\le \min(T_\epsilon,1)$,

$$
\left\|(\nabla\sigma,\operatorname{div} u,\nabla\theta)(t)\right\|_{L^2}+\left\|(\nabla\operatorname{div} u,\Delta\theta)\right\|_{L_t^2(L^2)}\le C_0(M_0)\exp\big(\sqrt{t}C(M)\big).
$$

In view of Lemma 2.1, we have to bound $w\equiv\operatorname{curl} u$. From (6), we easily derive the following equation for $w\equiv(w_1,w_2,w_3)^t$.

$$
\begin{aligned}
\rho(w_t+u\cdot\nabla w)-\mu\Delta w&=(\partial_j\rho u_{it}-\partial_i\rho u_{jt})+\big(\partial_j(\rho u_k)\partial_k u_i-\partial_i(\rho u_k)\partial_k u_j\big) \\
&\equiv RHS. \qquad (15)
\end{aligned}
$$

Due to the slip boundary condition (9), we see that

$$
\begin{aligned}
\left\|\sqrt{\rho}w(t)\right\|_{L^2}^2+\mu\|\operatorname{curl} w\|_{L_t^2(L^2)}^2&=\|\sqrt{\rho_0}\operatorname{curl} u_0\|_{L^2}^2+\int_0^t\int_\Omega (RHS)w\,dx\,dt \\
&\le C_0(M_0)+\sqrt{t}C(M). \qquad (16)
\end{aligned}
$$

Hence, we have

Lemma 3.4. *For any* $0\le t\le\min(T_\epsilon,1)$, *it holds that*

$$
\left\|\operatorname{curl} u(t)\right\|_{L^2}+\left\|\operatorname{curl}\operatorname{curl} u\right\|_{L_t^2(L^2)}\le C_0(M_0)\exp\big(\sqrt{t}C(M)\big).
$$

Next, we control the first-order time derivatives σ_t, u_t, θ_t, which satisfy

$$
\sigma_{tt}+\frac{1}{\epsilon}\operatorname{div} u_t=-\operatorname{div}(\rho u)_t, \qquad (17)
$$

$$
\begin{aligned}
\rho(u_{tt}+u\cdot\nabla u_t)+\frac{R}{\epsilon}(\nabla\sigma_t+\nabla\theta_t)-\mu\Delta u_t+\lambda\nabla\operatorname{div} u_t& \\
=-\rho_t u_t-(\rho u)_t\cdot\nabla u-R\nabla(\sigma\theta)_t& \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
&c_v\rho(\theta_{tt}+u\cdot\nabla\theta_t)+\frac{R}{\epsilon}\operatorname{div}u_t-\kappa\Delta\theta_t\\
&\quad=\epsilon\big(2\mu\big|D(u)\big|^2+\zeta(\operatorname{div}u)^2\big)_t\\
&\quad\quad-c_v\big(\rho_t\theta_t+(\rho u)_t\cdot\nabla\theta\big)-R\big((\rho\theta+\sigma)\operatorname{div}u\big)_t.
\end{aligned}
\tag{19}
$$

Multiplying (17), (18) and (19) by σ_t, u_t and θ_t, respectively, we reach

$$
\begin{aligned}
&\frac{1}{2}\big\|(\sqrt{R}\sigma_t,\sqrt{\rho}u_t,\sqrt{c_v\rho}\theta_t)(t)\big\|_{L^2}^2+\big\|(\sqrt{\mu}\operatorname{curl}u_t,\sqrt{\mu+\lambda}\operatorname{div}u_t,\sqrt{\kappa}\nabla\theta_t)\big\|_{L_t^2(L^2)}^2\\
&\quad\le C_0(M_0)+\sqrt{t}C(M).
\end{aligned}
$$

With the help of Lemma 3.1, we obtain

Lemma 3.5. *For any* $0\le t\le\min(T_\epsilon,1)$*, we have*

$$
\big\|(\sigma_t,u_t,\theta_t)(t)\big\|_{L^2}+\big\|(\operatorname{curl}u_t,\operatorname{div}u_t,\nabla\theta_t)\big\|_{L_t^2(L^2)}\le C_0(M_0)\exp\big(\sqrt{t}C(M)\big).
$$

3.4. Estimates for higher-order derivatives

Multiplying (18) by $\nabla\operatorname{div}u$ in $L^2(\Omega\times(0,t))$, we find that

$$
\begin{aligned}
&\frac{\mu+\lambda}{2}\big\|\nabla\operatorname{div}u(t)\big\|_{L^2}^2-\frac{R}{\epsilon}\int_0^t\int_\Omega(\nabla\sigma_t+\nabla\theta_t)\cdot\nabla\operatorname{div}u\,dx\,dt\\
&\quad=\frac{\mu+\lambda}{2}\|\nabla\operatorname{div}u_0\|_{L^2}^2-\int_0^t\int_\Omega\rho(u_{tt}+u\cdot\nabla u_t)\cdot\nabla\operatorname{div}u\,dx\,dt\\
&\quad\quad+\int_0^t\int_\Omega\big(\rho_tu_t-(\rho u)_t\cdot\nabla u\big)\cdot\nabla\operatorname{div}u\,dx\,dt,
\end{aligned}
\tag{20}
$$

where the last two terms, which are denoted by I_1 and I_2 respectively, can be bounded as follows.

$$
\begin{aligned}
|I_1|&\le\int_0^t\|\rho\|_{L^\infty}\big(\|u_{tt}\|_{L^2}+\|u\|_{H^2}\|\nabla u_t\|_{L^2}\big)\|\nabla\operatorname{div}u\|_{L^2}dt\\
&\le t\|\rho\|_{L_t^\infty(L^\infty)}\big(\|u_{tt}\|_{L_t^\infty(L^2)}+\|u\|_{L_t^\infty(H^2)}\|\nabla u_t\|_{L_t^\infty(L^2)}\big)\|\nabla\operatorname{div}u\|_{L_t^\infty(L^2)}\\
&\le tC(M),
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
|I_2| &\le \int_0^t \big(\|\rho_t\|_{H^1}\|u_t\|_{H^1} + \|u\|_{H^2}\|\|\rho\|_{H^1}\|u_t\|_{H^1} \\
&\quad + \|u\|_{H^2}\|\|\rho_t\|_{H^1}\|u\|_{H^1}\big)\|\nabla \operatorname{div} u\|_{L^2} dt \\
&\le tC(M).
\end{aligned}
$$

To eliminate the singular term on the left-hand side of (20), we apply ∇ to (5) and (7), multiply then the resulting equations by $R\nabla\sigma_t$ and $\nabla\theta_t$ in $L^2(\Omega)$ respectively, to arrive at

$$
\begin{aligned}
R\|\nabla\sigma_t\|^2_{L^2_t(L^2)} + \frac{R}{\epsilon}\int_0^t\int_\Omega \nabla \operatorname{div} u \cdot \nabla\sigma_t dxdt \\
= -R\int_0^t\int_\Omega \nabla \operatorname{div}(\sigma u)\nabla\sigma_t \\
\le \sqrt{t}C(M),
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
&\frac{\kappa}{2}\big\|\Delta\theta(t)\big\|^2_{L^2} + c_v\|\sqrt{\rho}\nabla\theta_t\|^2_{L^2_t(L^2)} + \frac{R}{\epsilon}\int_0^t\int_\Omega \nabla \operatorname{div} u \cdot \nabla\theta_t dxdt \\
&\le C_0(M_0) + \sqrt{t}C(M).
\end{aligned}
\tag{22}
$$

Therefore, we summarize (20), (21) and (22) to conclude that

Lemma 3.6. *For any $0 \le t \le \min(T_\epsilon, 1)$, we have*

$$
\big\|(\nabla \operatorname{div} u, \Delta\theta)(t)\big\|_{L^2} + \big\|(\nabla\sigma_t, \nabla\theta_t)\big\|_{L^2_t(L^2)} \le C_0(M_0)\exp\big(\sqrt{t}C(M)\big).
$$

Next, we will estimate the spatial derivatives of $(\sigma_t, u_t, \theta_t)$. Testing (18) and (19) by $\nabla \operatorname{div} u_t$ and $\Delta\theta_t$ in $L^2(\Omega)$ respectively, we obtain

$$
\begin{aligned}
&\frac{1}{2}\big\|\sqrt{\rho}\operatorname{div} u_t(t)\big\|^2_{L^2} + (\mu+\lambda)\big\|\nabla \operatorname{div} u_t\big\|^2_{L^2_t(L^2)} \\
&\quad - \frac{R}{\epsilon}\int_0^t\int_\Omega \nabla \operatorname{div} u_t \cdot (\nabla\sigma_t + \nabla\theta_t) dxdt \\
&\le C_0(M_0) + \int_0^t\int_\Omega \Big(\frac{\epsilon}{2}\sigma_t(\operatorname{div} u_t)^2 - \epsilon u_{tt}\cdot\nabla\sigma \operatorname{div} u_t \\
&\quad - \big(\rho_t u_t + (\rho u\cdot\nabla u)_t\big)\cdot\nabla \operatorname{div} u_t\Big) dxdt \\
&\le C_0(M_0) + \sqrt{t}C(M)
\end{aligned}
$$

and

$$\frac{c_v}{2}\left\|\sqrt{\rho}\nabla\theta_t(t)\right\|_{L^2}^2+\kappa\|\Delta\theta_t\|_{L_t^2(L^2)}^2-\frac{R}{\epsilon}\int_0^t\int_\Omega \operatorname{div} u_t\,\Delta\theta_t\,dxdt$$
$$\le C_0(M_0)+c_v\int_0^t\int_\Omega\left(\frac{\epsilon}{2}\sigma_t|\nabla\theta_t|^2+\epsilon\theta_{tt}\nabla\sigma\cdot\nabla\theta_t\right)dxdt$$
$$+\int_0^t\int_\Omega \Delta\theta_t\big(c_v\big(\rho_t\theta_t+(\rho u\cdot\nabla\theta)_t\big)+R\big((\rho\theta+\sigma)\operatorname{div} u\big)_t-\epsilon\big(S:D(u)\big)_t\big)dxdt$$
$$\le C_0(M_0)+\sqrt{t}C(M).$$

Similarly, we can establish the estimate for $\nabla\sigma_t$ as follows.

$$\frac{R}{2}\left\|\nabla\sigma_t(t)\right\|_{L^2}^2+\frac{R}{\epsilon}\int_0^t\int_\Omega\nabla\operatorname{div} u_t\cdot\nabla\theta_t\,dxdt\le C_0(M_0)+I,$$

where

$$I=\int_0^t\int_\Omega\nabla\operatorname{div}(\rho u)_t\cdot\nabla\sigma_t\,dxdt$$
$$=\int_0^t\int_\Omega\big(\nabla\operatorname{div}(\rho u_t)+\nabla(\rho_t\operatorname{div} u)\big)\cdot\nabla\sigma_t\,dxdt$$
$$+\epsilon\int_0^t\int_\Omega\big((\nabla\sigma_t)^t\nabla u\nabla\sigma_t-\operatorname{div} u|\nabla\sigma_t|^2\big)dxdt$$
$$\le\sqrt{t}C(M).$$

As a consequence, we obtain

Lemma 3.7. *For any* $0\le t\le\min(T_\epsilon,1)$*, we have*

$$\left\|(\nabla\sigma_t,\operatorname{div} u_t,\nabla\theta_t)(t)\right\|_{L^2}+\left\|(\nabla\operatorname{div} u_t,\Delta\theta_t)\right\|_{L_t^2(L^2)}\le C_0(M_0)\exp\big(\sqrt{t}C(M)\big).$$

Next, we derive the crucial estimates for $\|\nabla^2\operatorname{div} u(t)\|_{L_t^2L^2}$ and $\|\nabla^2\sigma(t)\|_{L^2}$. Integrating the inner product of ∂_i(6) and $\partial_i\nabla\operatorname{div} u$ in $L^2(\Omega\times(0,t))$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|\partial_i\nabla\operatorname{div}u\|^2_{L^2_t(L^2)}-\frac{R}{\epsilon}\int_0^t\int_\Omega\partial_i\nabla\operatorname{div}u\cdot(\partial_i\nabla\sigma+\partial_i\nabla\theta)dxdt\\
&\quad\le C\int_0^t\int_\Omega\left(\left|\nabla\big(\rho(u_t+u\cdot\nabla u)\big)\right|^2+\left|\nabla^2(\sigma\theta)\right|^2\right)dxdt\\
&\quad\le Ct\left(\|\rho\|^2_{L^\infty_t(H^2)}\left(\|u_t\|^2_{L^\infty_t(H^1)}+\|u\|^2_{L^\infty_t(H^2)}\right)+\|\sigma\|^2_{L^\infty_t(H^2)}\|\theta\|^2_{L^\infty_t(H^2)}\right)\\
&\quad\le tC(M).
\end{aligned}
$$

To cancel the singular term in the above inequality, we multiply $\partial_i\nabla$(5) by $R(\partial_i\nabla\sigma+\partial_i\nabla\theta)$ and integrate to infer that

$$
\begin{aligned}
&\frac{R}{2}\left(\left\|\partial_i\nabla\sigma(t)\right\|^2_{L^2}+2\int_\Omega\partial_i\nabla\sigma\cdot\partial_i\nabla\theta dx\right)+\frac{R}{\epsilon}\int_0^t\int_\Omega\partial_i\nabla\operatorname{div}u\cdot(\partial_i\nabla\sigma+\partial_i\nabla\theta)dxdt\\
&\quad=\frac{R}{2}\left(\|\partial_i\nabla\sigma_0\|^2_{L^2}+2\int_\Omega\partial_i\nabla\sigma_0\cdot\partial_i\nabla\theta_0dx\right)+I_1+I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&=\int_0^t\int_\Omega\partial_i\nabla\sigma\cdot\partial_i\nabla(\sigma\operatorname{div}u+\nabla\sigma\cdot u)dxdt\\
&=\int_0^t\int_\Omega\partial_i\nabla\sigma\cdot\big(\partial_i\nabla(\sigma\operatorname{div}u)+\partial_i\nabla\sigma\nabla u+\nabla^2\sigma\,\partial_iu+\nabla\sigma\,\partial_i\nabla u\big)dxdt\\
&\quad-\int_0^t\int_\Omega\operatorname{div}u|\partial_i\nabla\sigma|^2dxdt\\
&\le\sqrt{t}\|\sigma\|^2_{L^\infty_t(H^2)}\|\nabla u\|_{L^2_t(H^2)}\\
&\le\sqrt{t}C(M)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2&=\int_0^t\int_\Omega\partial_i\nabla\theta\cdot\partial_i\nabla(\sigma\operatorname{div}u+\nabla\sigma\cdot u)dxdt\\
&=\int_0^t\int_\Omega\partial_i\nabla\theta\cdot\big(\partial_i\nabla(\sigma\operatorname{div}u)+\partial_i\nabla\sigma\nabla u\\
&\quad+\nabla^2\sigma\,\partial_iu+\nabla\sigma\,\partial_i\nabla u-\partial_i\nabla^2\sigma u\big)dxdt
\end{aligned}
$$

$$
\begin{aligned}
&\le \sqrt{t}\|\nabla^2\theta\|_{L_t^\infty(L^2)}\|\sigma\|_{L_t^\infty(H^2)}\|\nabla u\|_{L_t^2(H^2)} \\
&\quad + \left|\int_0^t\int_\Omega \partial_i\nabla\theta\cdot\partial_i\nabla\sigma\,\mathrm{div}\,u\,dxdt\right| \\
&\le \sqrt{t}C(M).
\end{aligned}
$$

Therefore, we obtain

Lemma 3.8. *For any* $0 \le t \le \min(T_\epsilon, 1)$*, we have*

$$
\left\|\nabla^2\sigma(t)\right\|_{L^2} + \left\|\nabla^2\,\mathrm{div}\,u\right\|_{L_t^2(L^2)} \le C_0(M_0)\exp\bigl(\sqrt{t}C(M)\bigr).
$$

Thanks to Lemma 2.1, it suffices to bound derivatives of $\mathrm{curl}\,u$ to close the uniform estimates. Observing that $n \times (\rho w_t)|_{\partial\Omega} = 0$, we multiply (16) by Δw in $L^2(\Omega \times (0,t))$ to get

$$
\begin{aligned}
&\left\|\sqrt{\rho},\,\mathrm{curl}\,w(t)\right\|_{L^2}^2 + \mu\|\Delta w\|_{L_t^2(L^2)}^2 \\
&= \|\sqrt{\rho_0}\,\mathrm{curl}\,\mathrm{curl}\,u_0\|_{L^2}^2 - \int_0^t\int_\Omega (\partial_j\rho w_{it} - \partial_i\rho w_{jt})\,\mathrm{curl}\,w\,dxdt \\
&\quad + \int_0^t\int_\Omega \Delta w\bigl(\partial_j\rho u_{it} - \partial_i\rho u_{jt} + \partial_j(\rho u_k)\partial_k u_i - \partial_i(\rho u_k)\partial_k u_j\bigr)dxdt \\
&\le C_0(M_0) + \sqrt{t}\|\rho\|_{L_t^\infty(H^2)}\bigl(\|u_t\|_{L_t^\infty(H^1)}\left\|\nabla^2 u\right\|_{L_t^2(H^1)}^2 + \|u\|_{L_t^\infty(H^2)}^2\|u\|_{L_t^2(H^3)}\bigr) \\
&\le C_0(M_0) + \sqrt{t}C(M).
\end{aligned}
$$

In the same manner, we apply ∂_t to (16) and multiply the resulting equations by w_t in $L^2(\Omega \times (0,t))$ to deduce that

$$
\begin{aligned}
&\left\|\sqrt{\rho}w_t(t)\right\|_{L^2}^2 + \mu\|\,\mathrm{curl}\,w_t\|_{L_t^2(L^2)}^2 \\
&\le \left\|\sqrt{\rho_0}\,\mathrm{curl}\,u_t(0)\right\|_{L^2}^2 + C\int_0^t\int_\Omega |w_t|\bigl(|\rho_t| + \left|(\rho u)_t\cdot\nabla w\right|\bigr)dxdt \\
&\quad + C\int_0^t\int_\Omega |w_t|\bigl(|\epsilon u_{itt}\partial_j\sigma| + |\epsilon u_{it}\partial_j\sigma_t| + \left|\bigl(\partial_j(\rho u_k)\partial_k u_i\bigr)_t\right|\bigr)dxdt \\
&\le C_0(M_0) + \sqrt{t}C(M).
\end{aligned}
$$

Thus, we conclude

Lemma 3.9. *For any* $0 \le t \le \min(T_\epsilon, 1)$*, we have*

$$\begin{aligned}&\left\|(\operatorname{curl}\operatorname{curl} u, \operatorname{curl} u_t)(t)\right\|_{L^2} + \|\Delta \operatorname{curl} u\|_{L_t^2(L^2)} + \|\operatorname{curl}\operatorname{curl} u_t\|_{L_t^2(H^1)}\\&\quad \le C_0(M_0)\exp\big(\sqrt{t}C(M)\big).\end{aligned}$$

Due to (5) and (7), we have

$$\begin{aligned}\kappa\Delta\theta &= c_v\rho(\theta_t + u\cdot\nabla\theta) + R(\rho\theta+\sigma)\operatorname{div} u\\&\quad - \epsilon\big(2\mu\big|D(u)\big|^2 + \zeta(\operatorname{div} u)^2\big) + R\big(\sigma_t + \operatorname{div}(\sigma u)\big),\end{aligned}$$

which gives

$$\begin{aligned}\|\Delta\theta\|_{L_t^2(H^1)} &\le C\sqrt{t}\big(\|\rho\|_{L_t^\infty(H^2)}\big(\|\theta_t\|_{L_t^\infty(H^1)} + \|u\|_{L_t^\infty(H^2)}\|\theta\|_{L_t^\infty(H^2)}\big)\big)\\&\quad + C\sqrt{t}\big(\|\sigma_t\|_{L_t^\infty(H^1)} + \|\sigma\|_{L_t^\infty(H^2)}\|u\|_{L_t^\infty(H^2)}\big)\\&\quad + t^{1/4}\|u\|_{L_t^\infty(H^2)}^{3/2}\|u\|_{L_t^2(H^2)}^{1/2}.\end{aligned}$$

Hence, we obtain the estimate for the first-order derivatives of $\Delta\theta$ as follows.

Lemma 3.10. *For any* $0 \le t \le \min(T_\epsilon, 1)$ *it holds that*

$$\|\Delta\theta\|_{L_t^2(H^1)} \le C_0(M_0)\exp\big(t^{1/4}C(M)\big).$$

Finally, we estimate $\epsilon\sigma_{tt}, \epsilon u_{tt}$ and $\epsilon\theta_{tt}$ in order to close the energy estimates. Multiplying ∂_{tt}(5), ∂_{tt}(6) and ∂_{tt}(7) by $R\epsilon^2\sigma_{tt}$, $\epsilon^2 u_{tt}$ and $\epsilon^2\theta_{tt}$ respectively, and integrating over $\Omega\times(0,t)$, we use Lemma 2.1 to get

Lemma 3.11. *For any* $0 \le t \le \min(T_\epsilon, 1)$*, we have*

$$\epsilon\left\|(\sigma_{tt}, u_{tt}, \theta_{tt})(t)\right\|_{L^2} + \epsilon\left\|(u_{tt}, \theta_{tt})\right\|_{L_t^2(H^1)} \le C_0(M_0)\exp\big(t^{1/4}C(M)\big).$$

The proof is similar to that of Lemma 3.2, since σ_{tt}, u_{tt} and θ_{tt} are indeed the tangential derivatives of σ, u and θ, respectively. Hence, we omit the details here. The algebraic growth rate $t^{1/4}$ on the left-hand side of the above inequality arises from the estimate of $\int_0^t\int_\Omega \epsilon^2\theta_{tt}\partial_{tt}(6)dxdt$, e.g.,

$$\begin{aligned}&\int_0^t\int_\Omega \epsilon^3\big(2\mu\big|D(u)\big|^2 + \zeta(\operatorname{div} u)^2\big)_{tt}\theta_{tt}dxdt\\&\quad \le \int_0^t\int_\Omega \big(|\epsilon\nabla u_{tt}||\nabla u| + |\nabla u_t|^2\big)|\epsilon\theta_{tt}|dxdt\end{aligned}$$

$$\leq \int_{\Omega} \big(\|\epsilon \nabla u_{tt}\|_{L^2}\|\nabla u\|_{H^1} + \|\nabla u_t\|_{L^2}\|\nabla u_t\|_{H^1}\big)\|\epsilon\theta_{tt}\|_{L^3} dt$$

$$\leq t^{1/4}\big(\|\epsilon u_{tt}\|_{L^2_t(H^1)}\|u\|_{L^\infty_t(H^2)} + \|u_t\|_{L^\infty_t(H^1)}\|u_t\|_{L^2_t(H^2)}\big)\|\epsilon\theta_{tt}\|^{1/2}_{L^\infty_t(L^2)}\|\epsilon\theta_{tt}\|^{1/2}_{L^2_t(H^1)}.$$

4. Proof of the main theorems

Proof of Theorem 1.6. Collecting the estimates obtained in Lemmas 3.2–3.11, we easily obtain

$$\begin{aligned}
&\big\|(\sigma, u, \theta)(t)\big\|_{L^2} + \big\|(\nabla\sigma, \operatorname{div} u)(t)\big\|_{H^1} + \big\|(\operatorname{curl} u, \nabla\theta, \operatorname{curl}\operatorname{curl} u, \Delta\theta)(t)\big\|_{L^2} \\
&\quad + \big\|(\sigma_t, u_t, \theta_t)(t)\big\|_{H^1} + \epsilon\big\|(\sigma_{tt}, u_{tt}, \theta_{tt})(t)\big\|_{L^2} \\
&\quad + \big\|(\sigma, u, \theta)\big\|_{L^2_t(H^1)} + \big\|(\nabla \operatorname{div} u, \Delta\theta)\big\|_{L^2_t(H^1)} \\
&\quad + \big\|(\operatorname{curl}\operatorname{curl} u, \Delta \operatorname{curl} u)\big\|_{L^2_t(L^2)} + \big\|(\nabla\sigma_t, \operatorname{div} u_t, \operatorname{curl} u_t, \nabla\theta_t)\big\|_{L^2_t(L^2)} \\
&\quad + \big\|(\nabla \operatorname{div} u_t, \operatorname{curl}\operatorname{curl} u_t, \Delta\theta_t)\big\|_{L^2_t(L^2)} + \epsilon\big\|(u_{tt}, \theta_{tt})\big\|_{L^2_t(H^1)} \\
&\leq C_0(M_0)\exp\big(t^{1/4}C(M)\big), \quad \forall 0 \leq t \leq \min(T_\epsilon, 1),\ 0 < \epsilon \leq 1.
\end{aligned} \tag{23}$$

In view of the boundary conditions (9), we make use of Lemmas 2.1–2.2 to get

$$\|u\|_{H^s} \leq C\big(\|u\|_{H^{s-1}} + \|\operatorname{div} u\|_{H^{s-1}} + \|\operatorname{curl} u\|_{H^{s-1}}\big), \quad s = 1, 2, 3, \tag{24}$$

$$\|\operatorname{curl} u\|_{H^1} \leq C\big(\|\operatorname{curl}\operatorname{curl} u\|_{L^2} + \|\operatorname{curl} u\|_{L^2}\big), \tag{25}$$

$$\begin{aligned}
\|\operatorname{curl} u\|_{H^2} &\leq C\big(\|\operatorname{curl}\operatorname{curl} u\|_{H^1} + \|u\|_{H^2} + \|\operatorname{curl}\operatorname{curl} u\cdot n\|_{H^{1/2}(\partial\Omega)}\big) \\
&\leq C(\|\Delta \operatorname{curl} u\|_{L^2} + \|u\|_{H^2} + \|\operatorname{curl}\operatorname{curl} u\cdot n\|_{H^{1/2}(\partial\Omega)}).
\end{aligned} \tag{26}$$

Therefore, to obtain the boundedness of $\|u\|_{L^2_t(H^3)}$, it remains to estimate $\|\operatorname{curl}\operatorname{curl} u\cdot n\|_{H^{1/2}(\partial\Omega)}$ (see also [24]).

We construct the local coordinates by using the isothermal coordinates $\lambda(\psi, \varphi)$ to derive an estimate near the boundary (see [16] for instance), where $\lambda(\psi, \varphi)$ satisfies

$$\lambda_\psi \cdot \lambda_\psi > 0, \qquad \lambda_\varphi \cdot \lambda_\varphi > 0 \quad \text{and} \quad \lambda_\psi \cdot \lambda_\varphi = 0.$$

We cover the boundary $\partial\Omega$ by a finite number of bounded open sets $W^k \subset \mathbb{R}^3$, $k = 1, 2, ..., L$, such that for any $x \in W^k \cap \Omega$,

$$x = \lambda^k(\psi, \varphi) + rn\big(\lambda^k(\psi, \varphi)\big) = \Lambda^k(\psi, \varphi, r),$$

where $\lambda^k(\psi, \varphi)$ is the isothermal coordinate and n is the unit outer normal to $\partial\Omega$. For simplicity, in what follows we will omit the superscript k in each W^k. Then we construct the orthonormal system corresponding to the local coordinates by

$$e_1 = \frac{\lambda_\psi}{|\lambda_\psi|}, \qquad e_2 = \frac{\lambda_\varphi}{|\lambda_\varphi|}, \qquad e_3 = n(\lambda) = e_1 \times e_2.$$

By a straightforward calculation, we see that $J \equiv \det Jac\Lambda \in C^2$ and

$$\begin{aligned} J = \det Jac\Lambda &= (\Lambda_\psi \times \Lambda_\varphi) \cdot e_3 \\ &= |\lambda_\psi||\lambda_\varphi| + r\big(|\lambda_\psi| n_\varphi \cdot e_2 + |\lambda_\varphi| n_\psi \cdot e_1\big) \\ &\quad + r^2\big[(n_\psi \cdot e_1)(n_\varphi \cdot e_2) - (n_\psi \cdot e_2)(n_\varphi \cdot e_1)\big] > 0, \end{aligned}$$

for sufficiently small $r > 0$. Obviously, $Jac(\Lambda^{-1}) = (Jac\Lambda)^{-1}$. Set $y := (y_1, y_2, y_3) := (\psi, \varphi, r)$, $a_{ij} = ((Jac\Lambda)^{-1})_{ij}$. Then

$$n = (a_{31}, a_{32}, a_{33}), \tag{27}$$

and the tangential directions $\tau_i = (a_{i1}, a_{i2}, a_{i3})$ $(i = 1, 2)$ with

$$\sum_{j=1}^{3} a_{ij} a_{3j} = 0 \quad \text{for } i = 1, 2.$$

Then we denote by D_i the partial derivative with respect to y_i in the local coordinates. To be precise, D_3 is the normal derivative and D_i for $i = 1, 2$ are the tangential derivatives in the original coordinates. Moreover, we have

$$\partial_{x_j} = \sum_{k=1}^{3} a_{kj} D_k, \qquad j = 1, 2, 3.$$

Next, we denote the vorticity $w = \operatorname{curl} u$ near the boundary by $\widetilde{w} := (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)^t := w(t, \Lambda(y))$, thus

$$\operatorname{curl} w\big(t, \Lambda(y)\big) = (a_{k2} D_k \widetilde{w}_3 - a_{k3} D_k \widetilde{w}_2, a_{k3} D_k \widetilde{w}_1 - a_{k1} D_k \widetilde{w}_3, a_{k1} D_k \widetilde{w}_2 - a_{k2} D_k \widetilde{w}_1).$$

By (27) and the above identity we get

$$\begin{aligned} (\operatorname{curl} w \cdot n)\big(t, \Lambda(y)\big) &= \big[(a_{32}a_{13} - a_{33}a_{12}) D_1 + (a_{32}a_{23} - a_{33}a_{22}) D_2\big]\widetilde{w}_1 \\ &\quad + \big[(a_{33}a_{11} - a_{31}a_{13}) D_1 + (a_{33}a_{21} - a_{31}a_{23}) D_2\big]\widetilde{w}_2 \\ &\quad + \big[(a_{31}a_{12} - a_{32}a_{11}) D_1 + (a_{31}a_{22} - a_{32}a_{21}) D_2\big]\widetilde{w}_3 \\ &= \sum_{i=1}^{2} (n \times \tau_i) \cdot D_i \widetilde{w} \\ &= \sum_{i=1}^{2} \big(D_i\big((n \times \tau_i) \cdot \widetilde{w}\big) - D_i(n \times \tau_i) \cdot \widetilde{w}\big) \\ &= \sum_{i=1}^{2} \big(-D_i\big((n \times \widetilde{w}) \cdot \tau_i\big) - D_i(n \times \tau_i) \cdot \widetilde{w}\big). \end{aligned}$$

Recalling $n \times w|_{\partial\Omega} = 0$ and D_i $(i = 1, 2)$ are the tangential derivatives, we find that

$$\|\operatorname{curl} w \cdot n\|_{H^{1/2}(\partial\Omega)} \le C\|w\|_{H^{1/2}(\partial\Omega)} \le \|u\|_{H^2}. \tag{28}$$

Therefore, with (23), (24), (25), (26) and (28), Theorem 1.6 is shown. □

Proof of Theorem 1.4. The proof is in the spirit of [2,21]. Assume that Theorem 1.6 holds and $T^\epsilon < +\infty$ is the maximal life time of existence for the solution obtained in Theorem 1.3. Then for any $0 \le t \le \min\{T^\epsilon, 1\}$, we have

$$M^\epsilon(t) \le C_0\big(M_0^\epsilon\big)\exp\big(t^{1/4}C\big(M^\epsilon(t)\big)\big), \tag{29}$$

where $M_0^\epsilon \le D_0$ for $0 < \epsilon \le 1$. In the sequel, we choose $D > C(D_0)$ and next $T_1 \le 1$ such that

$$C(D_0)\exp\big({T_1}^{1/4}C(D)\big) < D. \tag{30}$$

Let $t < \min\{T^\epsilon, T_1\}$. By combining the inequalities (29) and (30) with the hypothesis $M^\epsilon(0) = M_0^\epsilon$, we have that $M^\epsilon(t) \ne D$. Besides, we can assume without restriction that $D_0 < D$, so that $M^\epsilon(0) < D$. Since the function $M(t)$ is continuous, we obtain

$$M(t) < D \quad \text{for } t < \min\big\{T^\epsilon, T_1\big\} \text{ and } 0 < \epsilon \le 1. \tag{31}$$

Consequently, $T^\epsilon > T_1$ for $0 < \epsilon \le 1$, otherwise, by using the uniform estimates in (31) and applying Theorem 1.3 repeatedly, one can extend the time interval of existence to $[0, T_1]$, which contradicts to the maximality of T^ϵ. Therefore, $M(t) < D$ for any $t \in [0, T_1]$ where T_1 is independent of $\epsilon \in (0, 1]$. Clearly, the conclusion is also true for $T^\epsilon = +\infty$ by applying the same argument. This completes the proof of Theorem 1.4. □

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