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# Impacts of noise on a class of partial differential equations

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**Abstract**

This paper is concerned with effects of noise on the solutions of partial differential equations. We first provide a sufficient condition to ensure the existence of a unique positive solution for a class of stochastic partial differential equations. Then, we prove that noise could induce singularities (finite time blow up of solutions). Finally, we show that a stochastic Allen–Cahn equation does not have finite time singularities and the unique solution exists globally.
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## 1. Introduction

Stochastic partial differential equations (SPDEs) are playing an increasingly important role in modeling complex phenomena in physics, geophysics and biology. In recent years, existence, uniqueness, stability, blow-up phenomenon, invariant measures and other properties of the solutions to SPDEs have been extensively investigated [2,14,15]. It is known that the existence and uniqueness of global solutions to SPDEs can be established under appropriate conditions [2,5].

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It is also known that certain deterministic parabolic or hyperbolic partial differential equations (even with polynomial nonlinearity) tend to develop singularities in finite time [8,22]. These equations only have solutions which are defined locally in time. For example, consider the following equation

$$\begin{cases} u_t - \Delta u = u^{1+\alpha}, & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{1.1}$$

where $\alpha > 0$, and $D$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial D$. It was shown [8] that for a nonnegative initial condition $u_0 \in L^2(D)$ satisfying

$$\int_D u_0(x)\phi(x)dx > \lambda_1^{\frac{1}{\alpha}}, \tag{1.2}$$

the solution develops finite time blow-up. Here $\lambda_1$ is the first eigenvalue of the Laplacian operator $-\Delta$, with zero Dirichlet boundary condition on $D$, and $\phi$ is the corresponding eigenfunction normalized so that $\|\phi\|_{L^1(D)} = 1$. Kaplan [13] showed that the solution of (1.1) will blow up if the initial datum is large enough. Fujita [8,9] proved that the Cauchy problem (1.1), with $D = \mathbb{R}^n$, has no global positive nontrivial solutions if $0 < \alpha < 2/n$, and every solution with arbitrarily small initial datum blows up. The same is true for $\alpha = 2/n$ as shown by Hayakawa [12]. When $\alpha > 2/n$, solutions with small initial data tend to zero as time goes to infinity. In this paper, we will prove that noise can lead to finite time blow-up.

For stochastic parabolic equations, the existence of solutions has been well studied [14,15,24]. For instance, for the following equation

$$\begin{cases} du = \big(\Delta u + f(u)\big)dt + \sigma(u)dW_t, & t > 0,\ x \in D, \\ u(x,0) = u_0(x) \geq 0, & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{1.3}$$

Da Prato–Zabczyk [21] considered the existence of global solutions with additive noise ($\sigma$ is constant). Manthey–Zausinger [16] considered (1.3), where $\sigma$ satisfied the global Lipschitz condition. Dozzi and López-Mimbela [6] considered Eq. (1.3) with $\sigma(u) = u$. They proved that if $f(u) \geq u^{1+\alpha}$ ($\alpha > 0$) and initial data is large enough, the solution will blow up in finite time, and that if $f(u) \leq u^{1+\beta}$ ($\beta$ is a certain positive constant) and the initial data is small enough, the solution will exist globally, see also [19]. A natural question arises: If $\sigma$ does not satisfy the global Lipschitz condition, what can we say about the solution? Will it blow up in finite time or exist globally? In a somewhat different case, Mueller [17] and, later, Mueller–Sowers [18] investigated the problem of a noise-induced explosion for a special case of Eq. (1.3), where $f(u) \equiv 0$, $\sigma(u) = u^\gamma$ with $\gamma > 0$ and $W(x,t)$ is a space-time white noise. It was shown that the solution will explode in finite time with positive probability for some $\gamma > 3/2$.

In the present paper, we shall provide separate sufficient conditions to ensure that the solutions of (1.3) remain positive, or blow up in finite time. Here blowup means that the solutions will blow up in finite time in mean $L^\infty$-norm or mean square $L^\infty$-norm; see Theorems 4.1–4.3. Moreover, we will consider a special case, i.e., stochastic Allen–Cahn equation, whose solution will not blow up in finite time and thus exists globally.

This paper is arranged as follows. After some preliminaries in the next section, we prove that the solutions of (3.1) remain positive under some assumptions in Section 3. Section 4 is concerned with the blow-up phenomenon of solution to (3.1) and we will obtain a new result, which shows that noise can indeed lead to finite time blow-up. In Section 5, we consider the existence of global solution, with help of a Lyapunov functional, for a stochastic Allen–Cahn equation, where the noise intensity $\sigma(u) = u^{1+\beta}$ ($\beta > 0$) is not globally Lipschitz continuous.

## 2. Preliminaries

To set the stage for our study, we recall Chow's recent works [3,4] on finite time blow-up for the following SPDE

$$\begin{cases} du = \big(Au + f(u,x,t)\big)dt + \sigma(u, \nabla u, x, t)dW_t, & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{2.1}$$

where $A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$ is a symmetric, uniformly elliptic operator with smooth coefficients, $\sigma$ is a given function, and $W(x,t)$ is a Wiener random field defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_t$. The Wiener random field has mean $\mathbb{E}W(x,t) = 0$ and its covariance function $q(x,y)$ is defined by

$$\mathbb{E}W(x,t)W(y,s) = (t \wedge s)q(x,y), \quad x, y \in \mathbb{R}^n,$$

where $(t \wedge s) = \min\{t, s\}$ for $0 \le t, s \le T$. The existence of strong solutions of (2.1) has been studied by many authors [2,20]. To consider positive solutions, they start with the unique solution $u \in C(\bar{D} \times [0,T]) \cap L^2((0,T); H^2)$ for Eq. (2.1). Under the following conditions

(P1) There exists a constant $\delta \ge 0$ such that

$$\frac{1}{2}q(x,x)\sigma^2(r,\xi,x,t) - \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \le \delta r^2$$

for all $r \in \mathbb{R}$, $x \in \bar{D}$, $\xi \in \mathbb{R}^n$ and $t \in [0,T]$;

(P2) The function $f(r,x,t)$ is continuous on $\mathbb{R} \times \bar{D} \times [0,T]$ and such that $f(r,x,t) \ge 0$ for $r \le 0$ and $x \in \bar{D}$, $t \in [0,T]$; and

(P3) The initial datum $u_0(x)$ on $\bar{D}$ is positive and continuous,

Chow obtained the following result [3].

**Proposition 2.1.** *(See [3, Theorem 3.3].) Suppose that the conditions* (P1), (P2) *and* (P3) *hold true. Then the solution of the initial-boundary problem for the parabolic Itô's equation* (2.1) *remains positive, i.e.,* $u(x,t) \ge 0$, *a.s. for almost every* $x \in D$ *and for all* $t \in [0,T]$.

From (P1), it follows that $\sigma = ku$ ($k$ is a constant) if we only consider the case that $\sigma = \sigma(u)$. The similar result can be found in [6,23]. The previous results on existence of global solution

to (2.1) require that $\sigma(u)$ satisfies a global positive Lipschitz condition. A natural question is what the solution becomes if $\sigma(u)$ does not satisfy the global Lipschitz condition. We shall study the positive solutions and global solutions of (2.1) with $\sigma(u) = u^\gamma$ (for $\gamma > 1$) in Sections 3 and 5, respectively.

We consider the eigenvalue problem for the elliptic equation

$$\begin{cases} -\Delta\phi = \lambda\phi, & \text{in } D, \\ \phi = 0, & \text{on } \partial D. \end{cases} \tag{2.2}$$

Then, all the eigenvalues are strictly positive, increasing and the eigenfunction $\phi$ corresponding to the smallest eigenvalue $\lambda_1$ does not change sign in domain $D$, as shown in [10]. Therefore, we normalize it in such a way that

$$\phi(x) \geq 0, \quad \int_D \phi(x)dx = 1.$$

In paper [4], Chow assumed that the following conditions hold, where $\lambda_1$ is the first eigenvalue of (2.2) with $\Delta$ replaced by $A$.

(N1) There exist a continuous function $F(r)$ and a constant $r_1 > 0$ such that $F$ is positive, convex and strictly increasing for $r \geq r_1$ and satisfies

$$f(r, x, t) \geq F(r)$$

for $r \geq r_1$, $x \in \bar{D}$, $t \in [0, \infty)$;

(N2) There exists a constant $M_1 > r_1$ such that $F(r) > \lambda_1 r$ for $r \geq M_1$;

(N3) The positive initial datum satisfies the condition

$$(\phi, u_0) = \int_D u_0(x)\phi(x)dx > M_1;$$

(N4) The following condition holds

$$\int_{M_1}^{\infty} \frac{dr}{F(r) - \lambda_1 r} < \infty.$$

Alternatively, he imposes the following conditions $S$ on the noise term:

(S1) The correlation function $q(x, y)$ is continuous and positive for $x, y \in \bar{D}$ such that

$$\int_D \int_D q(x, y)v(x)v(y)dxdy \geq q_1 \int_D v^2(x)dx$$

for any positive $v \in H$ and for some $q_1 > 0$;

(S2) There exist a positive constant $r_2$, continuous functions $\sigma_0(r)$ and $G(r)$ such that they are both positive, convex and strictly increasing for $r \geq r_2$ and satisfy

$$\sigma(r,x,t) \geq \sigma_0(r) \quad \text{and} \quad \sigma_0^2(r) \geq 2G\big(r^2\big)$$

for $x \in \bar{D}$, $t \in [0,\infty)$;

(S3) There exists a constant $M_2 > r_2$ such that $q_1 G(r) > \lambda_1 r$ for $r \geq M_2$;

(S4) The positive initial datum satisfies the condition

$$(\phi, u_0) = \int_D u_0(x)\phi(x)dx > M_2;$$

(S5) The following integral is convergent so that

$$\int_{M_2}^{\infty} \frac{dr}{q_1 G(r) - \lambda_1 r} < \infty.$$

**Proposition 2.2.** *(See [4, Theorem 3.1].) Suppose the initial-boundary value problem (2.1) has a unique local solution and the conditions* (P1)–(P3) *are satisfied, where $\sigma$ does not depend on $\nabla u$. In addition, we assume that either the conditions* (N1)–(N4) *or the alternative conditions* (S1)–(S5) *given above hold true. Then, for a real number $p > 0$, there exists a constant $T_p > 0$ such that*

$$\lim_{t \to T_p -} \mathbb{E}\|u\|_p = \lim_{t \to T_p -} \mathbb{E}\left(\int_D \big|u(x,t)\big|^p dx\right)^{\frac{1}{p}} = \infty, \tag{2.3}$$

*or the solution explodes in the mean $L^p$-norm as shown by (2.3), where $p \geq 1$ under conditions N, while $p \geq 2$ under conditions S.*

Looking at the conditions in Propositions 2.1 and 2.2, it is clear that the condition (P1) is very stringent. A noise intensity like $\sigma(u) = u^{1+\beta}$, $\beta > 0$, does not satisfy the condition (P1). But the condition (S5) implies that $G(r) \geq r^{1+\varepsilon}$, where $\varepsilon$ is a positive constant. Therefore, in order to prove that noise can lead to blow up, we should delete or change the condition (P1). Moreover, if we assume that $\sigma = \sigma(u)$, we see that the term $-\sum_{i,j=1}^n a_{ij}(x)\partial_{x_i} u \partial_{x_j} u$ will not play any role in proving that the solutions are positive. When the elliptic operator is replaced by the $p$-Laplace operator, the situation is different.

## 3. Positive solutions

In this section, we will consider the positive solution to (3.1), which will be used to examine the finite time blow-up phenomenon.

For simplicity, we first consider the following stochastic parabolic Itô equation

$$\begin{cases} du = \big(\Delta u + f(u,x,t)\big)dt + \sigma(u,\nabla u,x,t)dW_t, & t>0,\ x\in D,\\ u(x,0)=u_0(x), & x\in D,\\ u(x,t)=0, & t>0,\ x\in \partial D. \end{cases} \tag{3.1}$$

We assume that the covariance function $q(x,y)$ is bounded, continuous and there is a constant $q_0>0$ such that

$$\sup_{x,y\in D}\big|q(x,y)\big|\le q_0 \quad \text{and} \quad \int_{\mathbb{R}} q(x,x)dx<\infty.$$

In addition, we assume that

$$\begin{cases} f(u,x,t)\ge a_1u^{\beta}+a_2u,\\ \dfrac{q_0}{2}\sigma^2(u,\nabla u,x,t)-|\nabla u|^2\le b_1u^{2m}+b_2u^2, \end{cases} \tag{3.2}$$

where $a_2\in\mathbb{R}$, $b_i,\beta\ge 0$, $(-1)^{\beta}\in\mathbb{R}$ and

$$a_1\begin{cases} >0, & \text{if } (-1)^{\beta}=1,\\ <0, & \text{if } (-1)^{\beta}=-1, \end{cases} \quad \text{and}$$
$$1\le m<(1+\beta)/2.$$

As in [3,4], let $\eta(r)=r^-$ denote the negative part of $r$ for $r\in\mathbb{R}$. Set

$$k(r)=\eta^2(r),$$

so that $k(r)=0$ for $r\ge 0$ and $k(r)=r^2$ for $r<0$. For $\varepsilon>0$, let $k_\varepsilon(r)$ be a $C^2$-regularization of $k(r)$ defined by

$$k_\varepsilon(r)=\begin{cases} r^2-\frac{\varepsilon^2}{6}, & r<-\varepsilon,\\ -\frac{r^3}{\varepsilon}\big(\frac{r}{2\varepsilon}+\frac{4}{3}\big), & -\varepsilon\le r<0,\\ 0, & r\ge 0. \end{cases}$$

Then one can check that $k_\varepsilon(r)$ has the following properties.

**Lemma 3.1.** *(See [3, Lemma 3.1].) The first two derivatives $k_\varepsilon'$, $k_\varepsilon''$ of $k_\varepsilon$ are continuous and satisfy the conditions: $k_\varepsilon'(r)=0$ for $r\ge 0$; $k_\varepsilon'\le 0$ and $k_\varepsilon''\ge 0$ for any $r\in\mathbb{R}$. Moreover, as $\varepsilon\to 0$, we have*

$$k_\varepsilon(r)\to k(r), \quad k_\varepsilon'(r)\to -2\eta(r) \quad \text{and} \quad k_\varepsilon''(r)\to 2\theta(r),$$

*where $\theta(r)=0$ for $r\ge 0$, $\theta=1$ for $r<0$, and the convergence is uniform for $r\in\mathbb{R}$.*

**Lemma 3.2.** *(See [10, Lemma 7.6].) If $u \in W^1(D)$; then $u^+$, $u^-$, $|u| \in W^1(D)$ and*

$$\nabla u^- = \begin{cases} 0, & u \geq 0, \\ \nabla u, & u < 0. \end{cases}$$

With the aid of the above lemmas, we can obtain the following positivity result.

**Theorem 3.1.** *Suppose that (3.2) holds with $1 \leq m < (\beta + 1)/2$. Then the solution of initial-boundary value problem (3.1) with nonnegative initial datum remains positive, i.e., $u(x,t) \geq 0$, a.s., for almost every $x \in D$ and for all $t \in [0,T]$.*

**Proof.** We remark that when $m = 1$, Theorem 3.1 has been proved in [3]. Let $u_t = u(\cdot, t)$ and

$$\Phi_\varepsilon(u_t) = \big(1, k_\varepsilon(u_t)\big) = \int_D k_\varepsilon\big(u(x,t)\big)dx.$$

By Itô's formula, we have

$$\begin{aligned}
\Phi_\varepsilon(u_t) &= \Phi_\varepsilon(u_0) + \int_0^t \int_D k'_\varepsilon\big(u(x,s)\big)\Delta u(x,s)dxds \\
&\quad + \int_0^t \int_D k'_\varepsilon\big(u(x,s)\big) f\big(u(x,s),x,s\big)dxds \\
&\quad + \int_0^t \int_D k'_\varepsilon\big(u(x,s)\big)\sigma\big(u(x,s),\nabla u(x,s),x,s\big)dW(x,s)dx \\
&\quad + \frac{1}{2}\int_0^t \int_D k''_\varepsilon\big(u(x,s)\big)q(x,x)\sigma^2\big(u(x,s),\nabla u(x,s),x,s\big)dxds \\
&= \Phi_\varepsilon(u_0) + \int_0^t \int_D k''_\varepsilon\big(u(x,s)\big)\left(\frac{1}{2}q(x,x)\sigma^2\big(u(x,s),\nabla u(x,s),x,s\big) - |\nabla u|^2\right)dxds \\
&\quad + \int_0^t \int_D k'_\varepsilon\big(u(x,s)\big) f\big(u(x,s),x,s\big)dxds \\
&\quad + \int_0^t \int_D k'_\varepsilon\big(u(x,s)\big)\sigma\big(u(x,s),\nabla u(x,s),x,s\big)dW(x,s)dx.
\end{aligned}$$

Taking expectation over the above equality, we get

$$\mathbb{E}\Phi_\varepsilon(u_t) = \Phi_\varepsilon(u_0) + \mathbb{E}\int_0^t \int_D k''_\varepsilon\big(u(x,s)\big)$$

$$\times\left(\frac{1}{2}q(x,x)\sigma^2\big(u(x,s),\nabla u(x,s),x,s\big)-|\nabla u|^2\right)dxds$$
$$+\mathbb{E}\int_0^t\int_D k_\varepsilon'\big(u(x,s)\big)f\big(u(x,s),x,s\big)dxds.$$

Note that $\lim_{\varepsilon\to 0}\mathbb{E}\Phi_\varepsilon(u_t)=\mathbb{E}\|\eta(u_t)\|^2$, by taking the limits termwise as $\varepsilon\to 0$ and using Lemma 3.1, we have

$$\mathbb{E}\big\|\eta(u_t)\big\|^2=\mathbb{E}\big\|\eta(u_0)\big\|^2+2\mathbb{E}\int_0^t\int_D\left(\frac{1}{2}q(x,x)\sigma^2\big(u^-(x,s),\nabla u^-(x,s),x,s\big)-\big|\nabla u^-\big|^2\right)dxds$$
$$-2\mathbb{E}\int_0^t\int_D\eta\big(u(x,s)\big)f\big(u(x,s),x,s\big)dxds, \tag{3.3}$$

where $\|\cdot\|$ denotes the norm of $L^2(D)$. We remark that $\nabla u^-$ exists by Lemma 3.2. Moreover, it follows from Lemma 3.2 that (3.3) is well defined. The authors of [14] proved that $u\in L^p(D)$ if $u_0\in L^p(D)$, where $p\geq 1$ and $u$ is the solution of (1.3). We also remark that in [14] they assumed $\sigma$ satisfied the linear growth. Taniguchi [24] obtained the well-posedness of (1.3) under the condition that the nonlinear terms $f$ and $\sigma$ satisfy the local Lipschitz condition. Chow [2] obtained the well posedness of (3.1), where the nonlinear terms $f$ and $\sigma$ satisfy the global Lipschitz condition (see pp. 74–84 in [2]). One can use the method of [24] to obtain the well-posedness of (3.1), where $f$ and $\sigma$ satisfy the local Lipschitz condition. By using (3.2) and $\eta(u)\geq 0$, we obtain

$$\mathbb{E}\big\|\eta(u_t)\big\|^2\leq\mathbb{E}\big\|\eta(u_0)\big\|^2-2\mathbb{E}\int_0^t\int_D\eta\big(u(x,s)\big)\big(a_1u^\beta(x,s)+a_2u\big)dxds$$
$$+2\mathbb{E}\int_0^t\int_D\left(\frac{1}{2}q_0\sigma^2\big(u^-(x,s),\nabla u^-(x,s),x,s\big)-\big|\nabla u^-\big|^2\right)dxds$$
$$\leq\mathbb{E}\big\|\eta(u_0)\big\|^2-2\mathbb{E}\int_0^t\int_D\big(|a_1|\big(u^-\big)^{\beta+1}(x,s)-a_2\big(u^-\big)^2(x,s)\big)dxds$$
$$+2\mathbb{E}\int_0^t\int_D\big(b_1\big(u^-\big)^{2m}(x,s)+b_2\big(u^-\big)^2(x,s)\big)dxds, \tag{3.4}$$

where we have used the condition on $a_1$, that is, $(-1)^\beta a_1=|a_1|$.

Now, we use $\epsilon$-Young's inequality and the following interpolation inequality of $L^p$ to deal with the last two terms of (3.4),

$$\|u\|_{L^r}\leq\|u\|_{L^p}^{\theta}\|u\|_{L^q}^{1-\theta}, \tag{3.5}$$

where $0 < \theta < 1$ and

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, \quad 0 < p < q \leq \infty. \tag{3.6}$$

For simplicity, we write $u$ instead of $u^-$. Notice that $1 < m < \frac{\beta+1}{2}$ and $0 \leq q(x,x) \leq q_0$, by using (3.5), we have

$$\begin{aligned} 2b_1 \int_D u^{2m}(x,t)dx &= 2b_1 \|u\|_{L^{2m}}^{2m} \\ &\leq C\|u\|_{L^2}^{2m\theta} \|u\|_{L^{\beta+1}}^{2m(1-\theta)} \\ &\leq \epsilon \|u\|_{L^{\beta+1}}^{2m(1-\theta)\frac{1}{1-m\theta}} + C(\epsilon)\|u\|_{L^2}^2 \\ &= \epsilon \|u\|_{L^{\beta+1}}^{\beta+1} + C(\epsilon)\|u\|_{L^2}^2, \end{aligned} \tag{3.7}$$

where $\theta = \frac{\beta+1-2m}{m(\beta-1)}$ satisfying (3.6). Substituting (3.7) into (3.4) and using the fact $\|\eta(u_0)\| = 0$ because $u_0$ is non-negative, we get

$$\mathbb{E}\|\eta(u_t)\|^2 \leq 2\int_0^t \mathbb{E}\big(\epsilon \|u_s^-\|_{L^{\beta+1}}^{\beta+1} + \big(C(\epsilon) + b_2 + a_2\big)\|u_s^-\|_{L^2}^2\big)ds - 2|a_1| \int_0^t \mathbb{E}\|u_s^-\|_{L^{\beta+1}}^{\beta+1} ds.$$

Let $0 < \epsilon < |a_1|$. We observe that

$$\mathbb{E}\|\eta(u_t)\|^2 \leq C\int_0^t \mathbb{E}\|\eta(u_s)\|_{L^2}^2 ds,$$

which, by means of Gronwall's inequality, implies that

$$\mathbb{E}\|\eta(u_t)\|^2 = 0, \quad \forall t \in [0,T].$$

It follows that $\eta(u_t) = u^-(x,t) = 0$ a.s. for a.e. $x \in D$, $\forall t \in [0,T]$. This completes the proof. □

**Remark 3.1.** 1. Comparing Theorem 3.1 with Proposition 2.1, it is clear that our assumption is weaker. For example, $f(u) = u(1-u^2)$, will not satisfy the condition (P2), but it is covered in our theorem. By using a similar method, one can deal with the nonlinearity term depending on the $x$ and $t$. In this section, we only consider the case that $m > 1$ and it is possible to use the similar method to deal with the case that $0 < m < 1$.

2. Obviously, if $f(u) \equiv 0$, Theorem 3.1 will fail, that is, we cannot obtain the positivity of solutions to (3.1) with $f(u) \equiv 0$, even for one dimension. Because in the proof Theorem 3.1, we use $f(u)$ to control $\sigma(u)$. But we can obtain the positivity of solutions of (3.1) with $f(u) \equiv 0$ under the condition that $\Delta$ is replaced by the $p$-Laplacian $\Delta_p$.

3. From the proof of Theorem 3.1, we know that the term $-|\nabla u|^2$ is a good term, as we can use it to control the stochastic term. But when we use the embedding theorem and interpolation

inequality, we find that the term $\|u\|_{L^m}^m$ would be changed to $\|u\|_{L^2}^\nu$, where $\nu > 2$. Due to the convexity of the function $x^\nu$, we cannot get the desired result. However, if we change $-|\nabla u|^2$ to $-|\nabla u|^p$, $p > n \geq 2$, we can show that the solution is positive; see Theorem 3.2 below.

It follows from Theorem 3.1 that the value of $m$ depends on the nonlinear term $f$. The following result shows that the value of $m$ may not depend on the nonlinearity term $f$. Now, we consider the following Itô parabolic equation with the $p$-Laplacian operator

$$\begin{cases} du = \big(\Delta_p u + f(u,x,t)\big)dt + g(u,x,t)dW_t, & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{3.8}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$. We assume that there exist positive constants $\alpha$ and $\beta$ such that

$$\begin{aligned} &\big(\eta(v),\, f(v,x,t)\big) \geq -\alpha \big\|\eta(v)\big\|^2, \\ &g^2(u,x,t) \leq 2\beta u^{2m}, \quad m > 1. \end{aligned} \tag{3.9}$$

**Theorem 3.2.** *Assume that $p > \max\{2m, n\}$ and (3.9) holds. Then the solution of initial-boundary value problem (3.8) with nonnegative initial datum remains positive: $u(x,t) \geq 0$, a.s., for almost every $x \in D$ and all $t \in [0,T]$.*

**Proof.** Similar to the proof of Theorem 3.1, we get

$$\begin{aligned} \mathbb{E}\big\|\eta(u_t)\big\|^2 &= E\big\|\eta(u_0)\big\|^2 + 2\mathbb{E}\int_0^t \int_D \left(\frac{1}{2}q(x,x)g\big(u^-,x,t\big) - \big|\nabla u^-\big|^p\right)dxds \\ &\quad - 2\mathbb{E}\big(\eta(v),\, f(u,x,t)\big) \\ &\leq \mathbb{E}\big\|\eta(u_0)\big\|^2 + 2\mathbb{E}\int_0^t \int_D \big(\beta q_0\big(u^-\big)^{2m} - \big|\nabla u^-\big|^p\big)dxds \\ &\quad + 2\alpha E\big\|u^-\big\|^2. \end{aligned} \tag{3.10}$$

It follows from Lemma 3.2 that the above inequality is well defined. For simplicity, we write $u$ instead of $u^-$. By the Sobolev embedding inequality and for $p > n$,

$$\|u\|_{L^\infty} \leq C\|u\|_{W^{1,p}}, \tag{3.11}$$

which implies that

$$\begin{aligned} \|u\|_{L^{2m}}^{2m} &= \int_D u^{2m}(x,t)dx \leq \|u\|_{L^\infty}^{2m-2} \int_D u^2(x,t)dx \\ &= \|u\|_{L^\infty}^{2m-2}\|u\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|u\|_{L^\infty}^{2m-2+\gamma}\|u\|_{L^2}^{2-\gamma}\\
&\leq C\|u\|_{W^{1,p}}^{2m-2+\gamma}\|u\|_{L^2}^{2-\gamma}\\
&\leq C(\varepsilon)\|u\|_{L^2}^2+\varepsilon\|u\|_{W^{1,p}}^{(2m-2+\gamma)\cdot\frac{2}{\gamma}},
\end{aligned}\tag{3.12}$$

where $\varepsilon > 0$ and we have used the $\epsilon$-Young's inequality. Noting that $p > \max\{2m, n\}$, there exists a constant $\gamma \in (0, 2)$ such that

$$(2m-2+\gamma)\cdot\frac{2}{\gamma} = p.$$

Letting $kq_0\varepsilon \leq 1$ and submitting (3.12) into (3.10), we get

$$\mathbb{E}\big\|\eta(u_t)\big\|^2 \leq C\int_0^t \mathbb{E}\big\|\eta(u_s)\big\|_{L^2}^2 ds,$$

which, by means of Gronwall's inequality, implies that

$$\mathbb{E}\big\|\eta(u_t)\big\|^2 = 0, \quad \forall t \in [0, T].$$

It follows that $\eta(u_t) = u^-(x,t) = 0$ a.s. for a.e. $x \in D$, $\forall t \in [0, T]$. This completes the proof. □

We remark that the value of $m$ in Theorems 3.1 and 3.2 either depends on the nonlinear term $f$ or the operator $\Delta_p$. In the followings, we will select a new test function $\beta_\varepsilon(r)$, instead of $k_\varepsilon(r)$.

Define new functions

$$\begin{aligned}
&\beta_\varepsilon(r) = \int_r^\infty \rho_\varepsilon(s)ds, \qquad \rho_\varepsilon(r) = \int_{r+\varepsilon}^\infty J_\varepsilon(s)ds, \quad r \in \mathbb{R},\\
&J_\varepsilon(x) = \varepsilon^{-n} J\left(\frac{|x|}{\varepsilon}\right), \qquad J(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1,\\ 0, & |x| \geq 1.\end{cases}
\end{aligned}\tag{3.13}$$

Then by direct verification, we have the following result.

**Lemma 3.3.** *The above constructed functions $\rho_\varepsilon$, $\beta_\varepsilon$ are in $C^\infty(\mathbb{R})$ and have the following properties: $\rho_\varepsilon$ is a non-increasing function and*

$$\beta_\varepsilon'(r) = -\rho_\varepsilon(r) = \begin{cases} 0, & r \geq 0,\\ -1, & r \leq -2\varepsilon.\end{cases}$$

*Additionally, $\beta_\varepsilon$ is convex and*

$$\beta_\varepsilon(r) = \begin{cases} 0, & r \geq 0,\\ -2\varepsilon - r + \varepsilon\hat{C}, & r \leq -2\varepsilon,\end{cases}$$

*where* $\hat{C} = \int_{-2}^{0}\int_{t+1}^{1} J(s)dsdt < 2$. *Furthermore,*

$$0 \le \beta_\varepsilon''(r) = J_\varepsilon(r+\varepsilon) \le \varepsilon^{-n}C, \quad -2\varepsilon \le r \le 0,$$

*which implies that*

$$\begin{aligned} -2^n C \le r^n \beta_\varepsilon''(r) \le 0 \quad & \text{for } -2\varepsilon \le r \le 0, \text{ and } n \text{ is odd;} \\ 0 \le r^n \beta_\varepsilon''(r) \le 2^n C \quad & \text{for } -2\varepsilon \le r \le 0, \text{ and } n \text{ is even.} \end{aligned}$$

Now, we consider the following stochastic parabolic Itô equation

$$\begin{cases} du = \big(\Delta u + f(u,x,t)\big)dt + g(u,x,t)dW_t, & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D. \end{cases} \tag{3.14}$$

**Theorem 3.3.** *Assume that (i) the function* $f(r,x,t)$ *is continuous on* $\mathbb{R} \times \bar{D} \times [0,T]$*; (ii)* $f(r,x,t) \ge 0$ *for* $r \le 0$, $x \in \bar{D}$ *and* $t \in [0,T]$*; and (iii)* $g^2(u,x,t) \le ku^{2m}$, *where* $k > 0$, $2m > n$ *and* $(-1)^{2m-n} \in \mathbb{R}$. *Then the solution of initial-boundary value problem (3.14) with nonnegative initial datum remains positive:* $u(x,t) \ge 0$, *a.s. for almost every* $x \in D$ *and for all* $t \in [0,T]$.

**Proof.** Define

$$\Phi_\varepsilon(u_t) = \big(1, \beta_\varepsilon(u_t)\big) = \int_D \beta_\varepsilon\big(u(x,t)\big)dx.$$

By Itô's formula, we have

$$\begin{aligned} \Phi_\varepsilon(u_t) &= \Phi_\varepsilon(u_0) + \int_0^t\int_D \beta_\varepsilon'\big(u(x,s)\big)\Delta u(x,s)dxds \\ &\quad + \int_0^t\int_D \beta_\varepsilon'\big(u(x,s)\big) f\big(u(x,s),x,s\big)dxds \\ &\quad + \int_0^t\int_D \beta_\varepsilon'\big(u(x,s)\big) g\big(u(x,s),x,s\big)dW(x,s)dx \\ &\quad + \frac{1}{2}\int_0^t\int_D \beta_\varepsilon''\big(u(x,s)\big)q(x,x)g^2\big(u(x,s),x,t\big)dxds \\ &= \Phi_\varepsilon(u_0) + \int_0^t\int_D \beta_\varepsilon''\big(u(x,s)\big)\left(\frac{1}{2}q(x,x)g^2\big(u(x,s),x,s\big) - |\nabla u|^2\right)dxds \end{aligned}$$

$$+\int_0^t\int_D \beta_\varepsilon'\big(u(x,s)\big) f\big(u(x,s),x,s\big)dxds$$

$$+\int_0^t\int_D \beta_\varepsilon'\big(u(x,s)\big) g\big(u(x,s),x,s\big)dW(x,s)dx.$$

Taking expectation over the above equality and using Lemma 3.3, we get

$$\begin{aligned}\mathbb{E}\Phi_\varepsilon(u_t) &= \mathbb{E}\Phi_\varepsilon(u_0) + \mathbb{E}\int_0^t\int_D \beta_\varepsilon''\big(u(x,s)\big)\\ &\quad\times\left(\frac{1}{2}q(x,x)g^2\big(u(x,s),x,s\big) - |\nabla u|^2\right)dxds\\ &\quad+\mathbb{E}\int_0^t\int_D \beta_\varepsilon'\big(u(x,s)\big) f\big(u(x,s),x,s\big)dxds\\ &\le \mathbb{E}\Phi_\varepsilon(u_0) + \frac{k}{2}\mathbb{E}\int_0^t\int_D \beta_\varepsilon''\big(u(x,s)\big)q(x,x)u(x,s)^{2m}dxds\\ &\quad+\mathbb{E}\int_0^t\int_D \beta_\varepsilon'\big(u(x,s)\big) f\big(u(x,s),x,s\big)dxds.\end{aligned}$$

Here and after, we denote $\|\cdot\|_{L^1}$ by $\|\cdot\|_1$. Let $\eta(u) = u^-$ denote the negative part of $u$ for $u \in \mathbb{R}$. Then we have $\lim_{\varepsilon\to 0}\mathbb{E}\Phi_\varepsilon(u_t) = \mathbb{E}\|\eta(u_t)\|_1$. It follows from Lemma 3.3 that

$$0 \ge u^{2m}\beta_\varepsilon''(u) \ge \begin{cases} 0, & u \ge 0 \text{ or } u \le -2\varepsilon,\\ -2Cu^{2m-n}, & -2\varepsilon \le u \le 0, \text{ and } u^{2m-1} \ge 0,\end{cases}$$

or

$$0 \le u^{2m}\beta_\varepsilon''(u) \le \begin{cases} 0, & u \ge 0 \text{ or } u \le -2\varepsilon,\\ -2Cu^{2m-n}, & -2\varepsilon \le u \le 0, \text{ and } u^{2m-1} \le 0\end{cases}$$

which implies that $\lim_{\varepsilon\to 0} u^{2m}\beta_\varepsilon''(u) = 0$ provided that $2m > 1$. By taking the limits termwise as $\varepsilon \to 0$ and using Lemma 3.3, we get

$$\begin{aligned}\mathbb{E}\big\|\eta(u_t)\big\|_1 &\le \mathbb{E}\big\|\eta(u_0)\big\|_1 - \mathbb{E}\int_0^t\int_D \eta\big(u(x,s)\big) f\big(u(x,s),x,s\big)dxds\\ &\le 0,\end{aligned} \tag{3.15}$$

which implies that $u^- = 0$ a.s. for a.e. $x \in D$, $\forall t \in [0,T]$. This completes the proof. □

## 4. Blow-up phenomenon

In this section, we shall consider the solutions of (3.1) which blow up in finite time. We first show that a similar result to [8] holds for (3.1), and then we examine how noise induces blow-up in finite time in the mean $L^\infty$-norm. We divide this section into three subsections.

### *4.1. First result on blow-up*

In this subsection, we shall prove that the solution of stochastic parabolic Itô equation will blow up in finite time if the solution of corresponding deterministic equation blows up in finite time. Specifically, there exists a finite time $T^*$ such that $\lim_{t\to T^*-0}\mathbb{E}\sup_{x\in D}u(x,t)=\infty$, where $u(x,t)$ is a positive solution of the stochastic parabolic Itô equation (3.1). We remark that when $\sigma\equiv 0$, then (3.1) becomes the deterministic parabolic equation. Indeed, Fujita [8] presented an existence and non-existence theorem for global solution of (3.1) with $\sigma\equiv 0$. The following result is similar to that in [8].

**Theorem 4.1.** *Suppose the initial-boundary value problem (3.1) has a unique local solution. Assume that all the assumptions in Theorem 3.1 hold, where $a_1>0$. In addition, if $\lambda_1\geq a_2$, we assume that*

$$\int_D u_0(x)\phi(x)dx > \left[a_1^{-1}(\lambda_1-a_2)\right]^{\frac{1}{\beta}}, \tag{4.1}$$

*and if $\lambda_1<a_2$, we assume that $u_0(x)\geq 0$ and $u_0(x)\not\equiv 0$, where $\lambda_1$ is the smallest eigenvalue of the operator $\Delta$ on $D$ and $\phi$ is the corresponding eigenfunction. Then, there exists a constant $T^*>0$ such that*

$$\lim_{t\to T^*-}\mathbb{E}\|u_t\|_{L^\infty}=\lim_{t\to T^*-}\mathbb{E}\sup_{x\in D}\left|u(x,t)\right|=\infty. \tag{4.2}$$

**Proof.** It follows from Theorem 3.1 that (3.1) has a unique positive solution. We will prove the theorem by contradiction. Suppose (4.2) is false. Then there exists a global positive solution $u$ such that for any $T>0$

$$\sup_{0\leq t\leq T}\mathbb{E}\sup_{x\in D}\left|u(x,t)\right|<\infty, \tag{4.3}$$

which implies that

$$\sup_{0\leq t\leq T}\mathbb{E}\int_D u(x,t)\phi(x)dx\leq\sup_{0\leq t\leq T}\mathbb{E}\sup_{x\in D}|u(x,t)|<\infty, \tag{4.4}$$

where $\phi$ is defined as in (2.2) and satisfies $\int_D\phi(x)dx=1$. Define

$$\hat{u}(t):=\int_D u(x,t)\phi(x)dx.$$

Then we have

$$\begin{aligned}\hat{u}(t) &= (u_0,\phi) + \int_0^t \int_D \Delta u(x,s)\phi(x)dxds + \int_0^t \int_D f(u,x,s)\phi(x)dxds \\ &\quad + \int_0^t \int_D \sigma(u,\nabla u,x,s)\phi(x)dW(x,s)dx \\ &= (u_0,\phi) - \lambda_1 \int_0^t \int_D u(x,s)\phi(x)dxds + \int_0^t \int_D f(u,x,s)\phi(x)dxds \\ &\quad + \int_0^t \int_D \sigma(u,\nabla u,x,s)\phi(x)dW(x,s)dx. \end{aligned} \tag{4.5}$$

Taking the expectation over (4.5) and appealing to Fubini's theorem, we obtain

$$\mathbb{E}\hat{u}(t) = (u_0,\phi) - \lambda_1 \int_0^t \mathbb{E}\hat{u}(s)ds + \int_0^t \mathbb{E}\int_D f(u,x,s)\phi(x)dxds,$$

or, in the differential form, for $\xi(t) = \mathbb{E}\hat{u}(t)$,

$$\begin{cases} \dfrac{d\xi(t)}{dt} = -\lambda_1\xi(t) + \mathbb{E}\displaystyle\int_D f(u,x,t)\phi(x)dx \\ \xi(0) = \xi_0, \end{cases} \tag{4.6}$$

where $\xi_0 = (u_0,\phi)$. By Jensen's inequality, (4.6) yields that

$$\begin{cases} \dfrac{d\xi(t)}{dt} \geq -\lambda_1\xi(t) + a_1\xi^{\beta}(t) + a_2\xi(t) \\ \xi(0) = \xi_0. \end{cases} \tag{4.7}$$

For $\xi_0 > (a_1\lambda_1)^{\frac{1}{\beta-1}}$, this implies that $a_1\xi^{\beta}(t) - (\lambda_1 - a_2)\xi(t) > 0$ and $\xi(t) > \xi_0$ for $t > 0$. An integration of Eq. (4.7) gives that

$$T \leq \int_{\xi_0}^{\xi(T)} \frac{dr}{a_1 r^{\beta} - (\lambda_1 - a_2)r} \leq \int_{\xi_0}^{\infty} \frac{dr}{a_1 r^{\beta} - (\lambda_1 - a_2)r} < \infty,$$

which implies $\xi(t)$ must blow up at a time $T^* \leq \int_{\xi_0}^{\infty} \frac{dr}{a_1 r^{\beta} - (\lambda_1 - a_2)r}$. Hence this is a contradiction to (4.4). This completes the proof. □

It is remarked that Proposition 2.2 covers a part of the above result. The following example shows that Theorem 4.1 generalizes Proposition 2.2.

**Example.** Consider the following stochastic parabolic Itô equation

$$\begin{cases} du = \left(\Delta u + u^2\right)dt + ku^{1+\frac{1}{3}}dW_t, & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{4.8}$$

where $k \in \mathbb{R}$ and $D$ is defined as in (1.1). Fujita [8] obtained that the solution of (4.8) with $k = 0$ and $u_0 \geq 0$ will blow up in finite time. By Theorem 3.1, we know that the solution of (4.8) remains positive if $u_0 \geq 0$. It follows from Theorem 4.1 that the solution of (4.8) will blow up in finite time under the same assumptions as in [8]. We also remark that Proposition 2.2 is not suitable for (4.8).

We also have the following remarks.

**Remark 4.1.** 1. From the proof of Theorem 4.1, we conclude that the stochastic term does not play a role because the first moment of white noise is zero. White noise cannot prevent the blow-up of the solution. If we want to study whether the noise can prevent singularities (see [7]), perhaps we should consider the colored noise or complex noise.

2. In [6], the authors obtained a similar result to Theorem 4.1. They assumed that the nonlinearity $f(u) \geq u^{1+\beta}$ $(\beta > 0)$ and $\sigma(u, \nabla u, x, t) = u$.

3. If $a_1 = 1$ and $a_2 = 0$, then the condition (4.1) becomes (1.2). That is, under the same conditions on initial data, the solutions of (1.1) and (3.1) will blow up in finite time. Thus we can say we obtain a similar result to [8].

### *4.2. Second result on blow-up*

In this subsection, we consider the issue about how noise may induce finite time blow-up of the solution of stochastic partial differential equations.

Consider the following stochastic parabolic Itô equation

$$\begin{cases} du = \left(\Delta u + |u|^{1+\alpha}\right)dt + bu^m dW_t, & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{4.9}$$

where $b \in \mathbb{R}, \alpha > 0$ and $1 \leq m < 1 + \frac{\alpha}{2}$. When $m = 1$, Dozzi and López-Mimbela [6] obtained the global solution of (4.9) if the initial data and the noise are small enough (see Theorem 5 in [6]), which is similar to the deterministic case [8]. It is known that when $b = 0$ and the nonnegative initial datum is small enough, (4.9) has a unique global solution [8]. In this subsection, we will show that noise can induce blow-up.

**Theorem 4.2.** *Assume that $u_0$ is a nonnegative continuous function and*

$$\inf_{x,y \in D} q(x,y) \geq q_1, \quad r^{1+\frac{\alpha}{2}} + \frac{b^2 q_1}{2} r^m - \lambda_1 r > 0, \quad r = \left( \int_D u_0(x)\phi(x)dx \right)^2, \tag{4.10}$$

*where $\lambda_1$ is defined as in (2.2) and $q(x,y)$ is the correlation function. Then the solution of (4.9) will blow up in finite time in $L^2(\Omega)\times L^\infty(D)$, that is, there exists a constant $T^*>0$ such that*

$$\lim_{t\to T^*-0}\left(\mathbb{E}\|u\|^2_{L^\infty(D)}\right)^{\frac{1}{2}}=\lim_{t\to T^*-0}\left(\mathbb{E}\sup_{x\in D}u(x,t)^2\right)^{\frac{1}{2}}=\infty. \tag{4.11}$$

**Proof.** By [2,14,20], we know that (4.9) has a unique local solution. It follows from Theorem 3.1 that the solution of (4.9) remains positive. Since

$$\mathbb{E}\hat{u}^2(t)\le\mathbb{E}\sup_{x\in D}u(x,t)^2,$$

it suffices to show that $\mathbb{E}\hat{u}^2(t)$ blows up in finite time, where $\hat{u}(t)=(u,\phi)$.

By applying Itô's formula to $\hat{u}^2(t)$ and making use of (2.2), we get

$$\begin{aligned}\hat{u}^2(t)=(u_0,\phi)^2&-2\lambda_1\int_0^t\hat{u}^2(s)ds+2\int_0^t\int_D\hat{u}(s)u^{1+\alpha}(x,s)\phi(x)dxds\\&+2b\int_0^t\int_D\hat{u}(s)u^m(x,s)\phi(x)dW(x,s)ds\\&+b^2\int_0^t\int_D\int_D q(x,y)u^m(x,s)\phi(x)u^m(y,s)\phi(y)dxdyds\end{aligned} \tag{4.12}$$

Let $\eta(t)=\mathbb{E}\hat{u}^2(t)$. By taking an expectation over (4.12), we obtain

$$\begin{aligned}\eta(t)=(u_0,\phi)^2&-2\lambda_1\int_0^t\eta(s)ds+2\mathbb{E}\int_0^t\int_D\hat{u}(s)u^{1+\alpha}(x,s)\phi(x)dxds\\&+b^2\int_0^t\mathbb{E}\int_D\int_D q(x,y)u^m(x,s)\phi(x)u^m(y,s)\phi(y)dxdyds,\end{aligned} \tag{4.13}$$

or, in the differential form

$$\begin{cases}\dfrac{d\eta(t)}{dt}=-2\lambda_1\eta(t)+2\mathbb{E}\hat{u}(t)\displaystyle\int_D u^{1+\alpha}(x,t)\phi(x)dx\\\qquad\qquad+b^2\mathbb{E}\displaystyle\int_D\int_D q(x,y)u^m(x,t)\phi(x)u^m(y,t)\phi(y)dxdy\\\eta(0)=\eta_0=(u_0,\phi)^2.\end{cases} \tag{4.14}$$

By Jensen's inequality, (4.14) yields

$$\begin{cases} \dfrac{d\eta(t)}{dt} \geq -2\lambda_1\eta(t) + 2\eta^{1+\frac{\alpha}{2}}(t) + q_1 b^2 \eta^m(t) \\ \eta(0) = \eta_0. \end{cases} \tag{4.15}$$

This implies that, for $\eta_0^{1+\frac{\alpha}{2}} + \frac{b^2 q_1}{2}\eta_0^m - \lambda_1\eta_0 > 0$, we have $\eta^{1+\frac{\alpha}{2}}(t) + \frac{b^2 q_1}{2}\eta(t)^m - \lambda_1\eta(t) > 0$ and $\eta(t) > \eta_0$, for $t > 0$. An integration of Eq. (4.15) gives that

$$T \leq \int_{\eta_0}^{\eta(T)} \frac{dr}{2r^{1+\frac{\alpha}{2}} + b^2 q_1 r^m - 2\lambda_1 r} \leq \int_{\eta_0}^{\infty} \frac{dr}{2r^{1+\frac{\alpha}{2}} + b^2 q_1 r^m - 2\lambda_1 r} < \infty,$$

which implies that $\eta(t)$ must blow up at a time $T^* \leq \int_{\eta_0}^{\infty} \frac{dr}{2r^{1+\frac{\alpha}{2}} + b^2 q_1 r^m - 2\lambda_1 r}$. Hence this is a contradiction. This completes the proof. □

Before ending this section, we make the following remarks.

**Remark 4.2.** 1. Theorem 4.2 contains a new result. First, we suppose there exists a positive constant $q_1$ such that $\inf_{x,y\in D} q(x,y) \geq q_1$. When $b = 0$, Fujita [8] showed that the solution of (4.9) will exist globally if the initial data is sufficiently small. Then we fixed the initial data sufficiently small such that (4.9) with $b = 0$ has a unique global solution. Finally, we take the suitable value of $b$ such that (4.10) holds and it follows from Theorem 4.2 that the unique positive solution of (4.9) will blow up in finite time. Hence we can say that the noise induces the finite time blow-up.

2. From the proof of Theorem 4.2, we know that if one can prove that the solution of (2.1) is positive without using the property of $f(u)$, then the solution of (2.1) with $\sigma = u^m$ $(m > 1)$ will blow up in finite time under the condition that $f(u) \geq 0$ for $u \geq 0$. Similar to [4], one can prove that Theorems 4.1 and 4.2 also hold for $D = \mathbb{R}^n$ (Theorem 3.2 in [4]).

3. From (4.10), we see that for $m = 1$ and $b^2 q_1/2 \geq \lambda_1$, the solution of (4.9) will blow up in finite time for any nonnegative initial data. On the other hand, it follows from the proof of Theorem 4.2 that noise can make the existence time shorter.

### *4.3. Third result on blow-up*

In this subsection, we consider the following equation

$$\begin{cases} du = \big(\Delta u + f(u,x,t)\big)dt + g(u,x,t)dW_t, & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D. \end{cases} \tag{4.16}$$

**Theorem 4.3.** *Assume that all the assumptions in Theorem 3.3 hold. Assume further that $u_0$ is a nonnegative continuous function, $f(u,x,t) \geq 0$ for $u \geq 0$, $x \in D$, $t > 0$ and*

$$g(u,x,t) \geq b^2 u^m, \quad m > 1,\ b \in \mathbb{R},$$
$$\inf_{x,y\in D} q(x,y) \geq q_1, \quad \left(\int_D u_0(x)\phi(x)dx\right)^{2(m-1)} \geq \frac{\lambda_1}{q_1 b^2}, \tag{4.17}$$

*where $\lambda_1$ is defined as in (2.2) and $q(x,y)$ is the correlation function. Then the solution of (4.16) will blow up in finite time in $L^2(\Omega)\times L^\infty(D)$, that is, there exists a constant $T^*>0$ such that*

$$\lim_{t\to T^*-0}\left(\mathbb{E}\|u\|^2_{L^\infty(D)}\right)^{\frac{1}{2}} = \lim_{t\to T^*-0}\left(\mathbb{E}\sup_{x\in D} u(x,t)^2\right)^{\frac{1}{2}} = \infty. \tag{4.18}$$

**Proof.** By [2,14,20], we know that (4.16) has a unique local solution. It follows from Theorem 3.3 that the solution of (4.16) remains positive. Similar to the proof of Theorem 4.2, it suffices to show that $\mathbb{E}\hat{u}^2(t)$ blows up in finite time, where $\hat{u}(t) = (u,\phi)$.

By applying Itô's formula to $\hat{u}^2(t)$ and making use of (2.2), we get

$$\begin{aligned}\hat{u}^2(t) = (u_0,\phi)^2 &- 2\lambda_1\int_0^t \hat{u}^2(s)ds + 2\int_0^t\int_D \hat{u}(s)f(u,x,s)\phi(x)dxds\\ &+ 2\int_0^t\int_D \hat{u}(s)g(u,x,s)\phi(x)dW(x,s)ds\\ &+ \int_0^t\int_D\int_D q(x,y)g(u,x,s)\phi(x)g(u,y,s)\phi(y)dxdyds.\end{aligned} \tag{4.19}$$

Let $\eta(t) = \mathbb{E}\hat{u}^2(t)$. By taking an expectation over (4.19), we conclude that

$$\begin{aligned}\eta(t) = (u_0,\phi)^2 &- 2\lambda_1\int_0^t \eta(s)ds + 2\mathbb{E}\int_0^t\int_D \hat{u}(s)f(u,x,s)\phi(x)dxds\\ &+ \int_0^t \mathbb{E}\int_D\int_D q(x,y)g(u,x,s)\phi(x)g(u,y,s)\phi(y)dxdyds,\end{aligned} \tag{4.20}$$

or, in the differential form

$$\begin{cases}\dfrac{d\eta(t)}{dt} = -2\lambda_1\eta(t) + 2\mathbb{E}\hat{u}(t)\displaystyle\int_D f(u,x,t)\phi(x)dx\\ \qquad\qquad + \mathbb{E}\displaystyle\int_D\int_D q(x,y)g(u,x,t)\phi(x)g(u,y,t)\phi(y)dxdy\\ \eta(0) = \eta_0 = (u_0,\phi)^2.\end{cases} \tag{4.21}$$

Again by Jensen's inequality, (4.21) yields

$$\begin{cases} \dfrac{d\eta(t)}{dt} \geq -2\lambda_1\eta(t) + q_1 b^2 \eta^m(t), \\ \eta(0) = \eta_0. \end{cases} \tag{4.22}$$

This implies that, for $\frac{b^2q_1}{2}\eta_0^m - \lambda_1\eta_0 > 0$, we have $\frac{b^2q_1}{2}\eta(t)^m - \lambda_1\eta(t) > 0$ and $\eta(t) > \eta_0$ for $t > 0$. An integration of Eq. (4.22) gives that

$$T \leq \int_{\eta_0}^{\eta(T)} \frac{dr}{b^2q_1r^m - 2\lambda_1 r} \leq \int_{\eta_0}^{\infty} \frac{dr}{b^2q_1r^m - 2\lambda_1 r} < \infty,$$

which implies $\eta(t)$ must blow up at a time $T^* \leq \int_{\mu_0}^{\infty} \frac{dr}{b^2q_1r^m - 2\lambda_1 r}$. Hence this is a contradiction. This completes the proof. □

**Remark 4.3.** Theorem 4.3 holds for $g(u, x, t) = bu^m$, $m = 2, 3, \cdots$. It shows that the noise can induce a singularity.

## 5. Global solution for a stochastic Allen–Cahn equation

In this section, we show that the solution of a stochastic Allen–Cahn equation does not have finite time singularities and it exists globally. This is an example of SPDEs whose coefficients are not globally Lipschitz continuous.

We consider the following stochastic Allen–Cahn equation.

$$\begin{cases} du = \big(\Delta u + u\big(1 - u^2\big)\big)dt + bu^m dW_t, & t > 0,\ x \in D, \\ u(x, 0) = u_0(x), & x \in D, \\ u(x, t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{5.1}$$

where $1 < m < 2, b \in \mathbb{R}$. If $b = 0$, (5.1) becomes the well-known deterministic Allen–Cahn equation [1], which describes the process of phase separation in iron alloys, including order-disorder transitions. Hairer et al. [11] considered (5.1) with $m = 0$. Eq. (5.1) with $b = 0$ has a global solution. We want to know when the solution of (5.1) exists globally and when the solution blows up. In this section, we shall use the Lyapunov functional method to prove that the solution of (5.1) exists globally, i.e., no finite time blow up.

Throughout this section, we assume that $H = L^2(D)$, $\|\cdot\| = \|\cdot\|_H$. Moreover, $W(x, t)$ is a Wiener random field, and $q(x, y)$ is its covariance function as defined in Section 2.

Let $u(x, t; u_0)$ be a solution of (5.1) with the initial data $u_0$. We first give the definition of global solution.

**Definition 5.1.** A function $u(x, t) \in H(D) \cap H_0^1(D)$ is said to be non-explosive solution of (5.1) if

$$\lim_{r\to\infty} P\Big\{ \sup_{0\leq t\leq T} \|u_t\| > r \Big\} = 0,$$

for any $T > 0$. If the above holds for $T = \infty$, the solution $u(x,t)$ is said to be ultimately bounded, i.e., global solution.

We shall use Lyapunov functional method to obtain the existence of global solution to (5.1). In the following, we recall the definition of Lyapunov functional [2]. We do this for a more general stochastic partial differential equation

$$\begin{cases} du = \big(Au + F(u)\big)dt + \sum(u)dW_t, \quad t \geq 0, \\ u(x,0) = h(x), \end{cases} \tag{5.2}$$

where $A$, $F$ and $\sum$ are assumed to be non-random or deterministic. Let $V$ be a separate Hilbert space. Here we say that an $\mathcal{F}_t$-adapted $V$-valued process $u$ is a *strong solution* of Eq. (5.2) if $u \in L^2(\Omega \times [0,T]; V)$, and for any $\phi \in V$, the following equation

$$(u,\phi) = (h,\phi) + \int_0^t \big\langle Au + F(u), \phi\big\rangle ds + \int_0^t \Big(\phi, \sum(u)dW_s\Big)$$

holds for each $t \in [0,T]$ a.s. Recall that the generator for this stochastic partial differential equation is (see Chapter 7 in [2])

$$\begin{aligned} \mathcal{L}_t\Phi(v,t) = \frac{\partial}{\partial_s}\Phi(v,s) + \frac{1}{2}Tr\Big[\Phi''(v,t)\sum_t(v)Q\overset{*}{\sum_t}(v)\Big] \\ + \big\langle A_t v, \Phi'(v,t)\big\rangle + \big(F_t(v), \Phi'(v,t)\big), \end{aligned} \tag{5.3}$$

where $Q$ is covariance operator. Let $U \subset H$ be a neighborhood of the origin. A function $\Phi$: $U \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a *Lyapunov functional* for Eq. (5.2), if

(1) $\Phi$ is locally bounded and continuous such that its first two partial derivatives $\partial_t\Phi(v,t)$, $\partial_x\Phi(v,t)$ and $\partial_{xx}\Phi(v,t)$ exist, and $\partial_t\Phi(v,t)$, $\partial_x\Phi(v,t)$ are locally bounded.
(2) $\Phi(0,t) = 0$ for all $t \geq 0$, and, for any $r > 0$, there is $\delta > 0$ such that

$$\inf_{t\geq 0, \|v\|\geq r} \Phi(v,t) \geq \delta.$$

(3) For every $t \geq 0$ and $v \in U \cap H_0^1$,

$$\mathcal{L}_t\Phi(v,t) \leq 0,$$

where $\partial_x$ and $\partial_{xx}$ are Fréchet derivative, see [2, pp. 196–201].

Let $U \times \mathbb{R}^+ \to \mathbb{R}^+$ be a Lyapunov functional and let $u_t$ denote the strong solution of (5.2) with initial data $u_0$. For $r > 0$, let $B_r = \{h \in H : \|h\| \leq r\}$ such that $B_r \subset U$. Define

$$\tau = \inf\big\{t > 0 : \ u_t \in B_r^c, \ u_0 \in B_r\big\},$$

with $B_r^c = HB_r$. We put $\tau = T$ if the set is empty. Then the process $\phi_t = \Phi(u_{t\wedge\tau}, t\wedge\tau)$ is a local $\mathcal{F}_t$-supermartingale and the following Chebyshev inequality holds

$$P\Big\{\sup_{0\le t\le T}\|u_t\| > r\Big\} \le \frac{\Phi(u_0,0)}{\Phi_r},$$

where

$$\Phi_r = \inf_{0\le t\le T, h\in U\cap B_r^c}\Phi(h,t).$$

In order to obtain the global solution of (5.1), we need the following lemma.

**Lemma 5.1.** *(See [2, p. 200, Theorem 3.2].) Suppose that there exists a Lyapunov functional $\Phi: H\times\mathbb{R}_+\to\mathbb{R}_+$ such that*

$$\Phi_r = \inf_{t\ge 0,\|h\|\ge r}\Phi(h,t)\to\infty, \quad \text{as } r\to\infty.$$

*Then the solution $u_t$ is ultimately bounded.*

Let $\Phi(u,t) = e^{-\alpha t}\Psi(u,t)$. Then it follows from Lemma 5.1 that if there exists a constant $\alpha > 0$ such that

$$\mathcal{L}_t\Psi \le \alpha\Psi(v,t) \quad \text{for any } v\in H_0^1,$$

and the infimum $\inf_{t\ge 0,\|v\|\ge r}\Psi(v,t) = \Psi_r$ exists such that $\lim_{r\to\infty}\Psi_r = \infty$, then the solution $u$ does not explode in finite time.

Now, we use Lemma 5.1 to examine the global solution of (5.1).

**Theorem 5.1.** *Assume that $1 < m < 2$ and $u_0(x)\ge 0$ for $x\in\bar{D}$. Assume further that there exists a positive constant $q_0$ such that the covariance function $q(x,y)$ satisfies the condition $\sup_{x,y\in\bar{D}} q(x,y)\le q_0$. Then (5.1) has a strong global solution.*

**Proof.** It follows from [2,14,20] that (5.1) has a local solution on $[0,T]\times D$. By Theorem 3.1, this local solution is positive. Now, we use Lemma 5.1 to prove the solution does not blow up in finite time. Define $\Psi(v,t) = \|v\|^2$. Then $\lim_{r\to\infty}\inf_{t\ge 0,\|v\|\ge r}\Psi(v,t) = \infty$. Direct calculations show that

$$\begin{aligned}\mathcal{L}_t\Psi(v,t) &= \frac{\partial}{\partial_s}\Psi(v,s) + \frac{1}{2}Tr\big[\Psi''(v,t)v_t^m Q v_t^m\big]\\ &\quad + \big\langle\Delta v, \Psi'(v,t)\big\rangle + \big(v - v^3, \Psi'(v,t)\big)\\ &= -2\int_D |\nabla v|^2 dx + 2\int_D \big(v^2 - v^4\big)dx + \int_D q(x,x)v^{2m}(x)dx\\ &\le 2\|v\|^2 - 2\|v\|_{L^4}^4 + q_0\|v\|_{L^{2m}}^{2m}. \end{aligned} \tag{5.4}$$

By using interpolation inequality (3.5) with $r = 2m$, $p = 2$ and $q = 4$, we have

$$\begin{aligned} q_0\|v\|_{L^{2m}}^{2m} &\le C\|v\|_{L^2}^{2m\theta}\|v\|_{L^4}^{2m(1-\theta)} \\ &\le \epsilon\|v\|_{L^4}^{2m(1-\theta)\frac{1}{1-m\theta}} + C(\epsilon)\|v\|_{L^2}^2 \\ &= \epsilon\|v\|_{L^4}^4 + C(\epsilon)\|v\|_{L^2}^2, \end{aligned} \tag{5.5}$$

where $\theta = \frac{2-m}{m}$ satisfying (3.6). Substituting (5.5) into (5.4), we have

$$\mathcal{L}_t\Psi(v,t) \le C\|v\|^2 - \|v\|_{L^4}^4 \le C\|v\|^2 = C\Psi(v,t),$$

which implies that all the assumptions in Lemma 5.1 hold. Thus by Lemma 5.1 we know that the solution of (5.1) exists globally. This completes the proof. □

It can be shown that Theorem 5.1 also holds if (5.1) is replaced by

$$\begin{cases} du = \big(\Delta u - u^\gamma\big)dt + bu^m dW_t, & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{5.6}$$

for $b \in \mathbb{R}$, $1 < m < (\gamma+1)/2$ and $\gamma > 1$ satisfying $(-1)^\gamma = -1$.

**Corollary 5.1.** *Assume that $\gamma > 1$, $1 < m < (\gamma+1)/2$, $u_0 \ge 0$ and assume also that there exists a positive constant $q_0$ such that the covariance function $q(x,y)$ satisfies the condition $\sup_{x,y\in\bar{D}} q(x,y) \le q_0$. Then (5.6) has a unique strong global solution.*

Theorem 5.1 and Corollary 5.1 imply that if the nonlinearity $f(u) = ku - u^\gamma$ can control the stochastic term $u^m$, i.e., $m < (\gamma+1)/2$, the stochastic partial differential equation also has global solutions, which is different from the earlier results.

Similarly, we can use Lemma 5.1 to study the following stochastic partial differential equation

$$\begin{cases} du = \big(\nu\Delta u + au\big(1-u^2\big)\big)dt + k\displaystyle\sum_{i=1}^{3}\frac{\partial u}{\partial_{x_i}}dW_i(x,t), & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{5.7}$$

where $D \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial D$ and $W_i(x,t)$ are Wiener random fields with bounded covariance functions $q_{jk}(x,y)$ such that

$$\sum_{j,k=1}^{3} q_{jk}(x,x)\xi_j\xi_k \le q_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^3$$

for some $q_0 > 0$. From [2,14,20], we know that Eq. (5.7) has a strong solution $u \in H_0^1$ (see Theorem 6-7.5 in [2]). Similar to the proof of Theorem 5.1, we define $\Psi(v,t) = \|v\|^2$. Then

$\lim_{r\to\infty} \inf_{t\geq 0, \|v\|\geq r} \Psi(v,t) = \infty$. Again, we have

$$\begin{aligned}\mathcal{L}_t\Psi(v,t) &= -2\nu \int_D |\nabla v|^2 dx + 2a \int_D \left(v^2 - v^4\right) dx \\ &\quad + \int_D \sum_{j,k=1}^{3} q_{jk}(x,x) \frac{\partial v(x)}{\partial_{x_j}} \frac{\partial v(x)}{\partial_{x_k}} dx \\ &\leq 2a\|v\|^2 - (2\nu - q_0) \int_D |\nabla v|^2 dx \\ &\leq 2a\|v\|^2\end{aligned}$$

provided that $2\nu - q_0 > 0$. Using Lemma 5.1, we obtain the following result.

**Theorem 5.2.** *Assume that* $2\nu - q_0 > 0$ *and* $u_0(x) \geq 0$ *for* $x \in \bar{D}$. *Then* (5.7) *has a unique strong global solution.*

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