



# Global strong solutions to radial symmetric compressible Navier–Stokes equations with free boundary

Hai-liang Li<sup>a,1</sup>, Xingwei Zhang<sup>b,\*</sup>

<sup>a</sup> School of Mathematical Sciences, Capital Normal University, Beijing 100048, PR China

<sup>b</sup> College of Teacher Education, Quzhou University, Quzhou 324000, PR China

Received 14 March 2016; revised 20 July 2016

Available online 5 September 2016

---

## Abstract

In this paper, we consider the two-dimensional barotropic compressible Navier–Stokes equations with stress free boundary condition imposed on the free surface. As the viscosity coefficients satisfies  $\mu(\rho) = 2\mu$ ,  $\lambda(\rho) = \rho^\beta$ ,  $\beta > 1$ , we establish the existence of global strong solution for arbitrarily large spherical symmetric initial data even if the density vanishes across the free boundary. In particular, we show that the density is strictly positive and bounded from the above and below in any finite time if the initial density is strictly positive, and the free boundary propagates along the particle path and expand outwards at an algebraic rate.

© 2016 Elsevier Inc. All rights reserved.

MSC: 76N15; 35Q30

Keywords: Compressible Navier–Stokes equations; Free boundary; Global strong solution

---

---

\* Corresponding author.

E-mail addresses: [hailiang.li.math@gmail.com](mailto:hailiang.li.math@gmail.com) (H.-l. Li), [xingweizhang2014@163.com](mailto:xingweizhang2014@163.com) (X. Zhang).

<sup>1</sup> The research is supported by the NNSFC grants No. 11171228, 11231006 and 11225102, NSFC-RGC Grant 11461161007, and by the Key Project of Beijing Municipal Education Commission no. CIT&TCD20140323.

## 1. Introduction

In this paper, we consider the following barotropic compressible Navier–Stokes equations (CNS) with density-dependent viscosity coefficient in  $\mathbb{R}^2$ ,

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \\ (\rho \mathbf{U})_t + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla P(\rho) = \mu \Delta \mathbf{U} + \nabla((\mu + \lambda(\rho)) \operatorname{div} \mathbf{U}), \end{cases} \quad (1.1)$$

where  $t \geq 0$ ,  $\mathbf{x} \in \mathbb{R}^2$ ,  $\rho = \rho(\mathbf{x}, t)$  and  $\mathbf{U} = \mathbf{U}(\mathbf{x}, t) = (U_1(\mathbf{x}, t), U_2(\mathbf{x}, t))$  represent the density and velocity of the fluid respectively, and the pressure function  $P(\rho)$  is given by  $P(\rho) = \rho^\gamma$  with  $\gamma > 1$ . The shear viscosity  $\mu > 0$  is a positive constant, and the bulk one is  $\lambda(\rho) = \rho^\beta$  with  $\beta > 1$ . The compressible Navier–Stokes equations with density-dependent viscosity coefficients have been taken into consideration, among which one of the typical equations consists of the viscous Saint-Venant model in the description of flow motion in shallow region [10,34,28] where both shear and bulk viscosity coefficients are functions of flow density. The system (1.1) under the consideration in the present paper was first proposed by Vaigant and Kazhikhov [38] who obtained the unique global strong solution for general initial data away from vacuum with two-dimensional periodic boundary condition.

The global existence of solution to (multi-dimensional) compressible isentropic Navier–Stokes equations for general initial data is an attractive issue, and there are important progress and huge literature on it and related topics, refer to [7,29,13,15,38,33,16,25,9,19,40,46] and references therein. For instance, in the case when both the shear and bulk viscosities are constants, the global existence and/or the qualitative behaviors of classic/strong solutions to multi-dimensional compressible Navier–Stokes equations were shown if the regular initial data is a small perturbation to a non-vacuum equilibrium in some Sobolev norm with higher regularity [29,13] or in some Besov norm but with lower regularity [7]. Therein, the flow density is strictly positive with the lower and upper bounds, namely solution can not contain any vacuum states, which is one of the essential parts to carry out the analysis. Recently, the global existence of classical solutions to the Cauchy problem for the isentropic compressible Navier–Stokes equations in three-dimensional space is made in [15] for initial data with small total energy but possibly large oscillations, where the solution is allowed to contain vacuum state and the flow density is uniformly bounded from the above. For the CNS (1.1), the global existence and large time behavior of the solutions for large initial data were made in periodic domain [16,33,38] and in whole two-dimensional space [17,20].

The global existence of multi-dimensional weak solutions to multi-dimensional CNS with constant viscosity coefficients for general large initial data with finite total energy and possibly vacuum included were also established in the compactness framework of renormalized solutions under some constraints either on the specific heat ratio  $\gamma$  of pressure law [25,9] or on the symmetry structure of the solution [19]. Based on the Bresch–Desjardins entropy estimate [1,4] and Mellet–Vasseur compactness estimates [30], some important progresses were also achieved recently on the construction of approximate solutions, global existence and dynamic behavior of global weak solutions to the multi-dimensional compressible Navier–Stokes equations with density-dependent viscosity coefficients, refer to [1–4,11,12,23,39] and references therein.

However, the regularity, uniqueness and dynamical behavior of the weak solutions for arbitrary initial data remain largely open for the compressible Navier–Stokes equations with constant viscosity coefficients. As emphasized in many related papers (refer to [14,27,35,41] for instance),

the possible appearance of vacuum is one of the main difficulties, which indeed leads to the singular behaviors of solutions in the presence of vacuum, for instance the finite time blow-up of smooth solutions [41] and the finite time vanishing of finite vacuum [22].

Thus, it is a natural and interesting problem to investigate the influence of vacuum state on the existence and dynamics of global solutions to compressible Navier–Stokes equations. One of the prototypes is the free boundary value problem (FBVP) for the compressible Navier–Stokes equations with the stress boundary condition imposed on the free surface [31,36], which may describe the time-evolution of the compressible viscous flow of finite mass expanding into infinite vacuum. The study of free boundary value problem of compressible Navier–Stokes equations has been made recently by many authors, and abundant interesting results concerning the well-posedness and long time behaviors of global solutions to one-dimensional free boundary problem for CNS are obtained, refer to [18,21,27,26,22,24,32,42,43,47] and references therein. In the case when across the free surface the stress tensor is balanced by additional force such as surface tension and/or exterior pressure [37,44,45], global existence of classical solutions with small amplitude and positive densities in fluid region to the free boundary value problem (FBVP) for multi-dimensional compressible Navier–Stokes equations with constant viscosity coefficients is established, where initial data is assumed to be near to non-vacuum equilibrium state. Global existence of spherically symmetric weak solution to FBVP for multi-dimensional compressible Navier–Stokes equations was shown in [5,6] on the spatial exterior domain excluding the symmetry center. Later on, the global existence and the dynamical behaviors of the spherically symmetric solutions to the free boundary value problem for CNS with the stress free boundary condition were made in [12] where yet the flow density is required to be positive up to the free interface.

In this paper, we consider the existence and large time behavior of global spherically symmetric strong solution to free boundary value problem for the compressible Navier–Stokes equations (1.1) in two dimensions for  $\beta > 1$  with stress free boundary condition even if the flow density vanishes at the free boundary as a continuous study of [12]. In particular, we show that the flow density is strictly positive and bounded from the above and below in any finite time along the particle so long as the initial density is positive, and the free boundary propagates along the particle path and expands outwards at an algebraic rate. To state the main results, we consider the two-dimensional CNS (1.1) in spherically symmetric coordinates, that is

$$\rho(\mathbf{x}, t) = \rho(r, t), \quad \mathbf{U}(\mathbf{x}, t) = u(r, t) \frac{\mathbf{x}}{r},$$

where  $u$  is a scalar function and  $r = |\mathbf{x}|$ , then the CNS (1.1) is changed to

$$\begin{cases} \rho_t + (\rho u)_r + \frac{1}{r} \rho u = 0, \\ (\rho u)_t + (\rho u^2)_r + \frac{1}{r} \rho u^2 + (\rho^\gamma)_r = \left( (2\mu + \lambda(\rho))(u_r + \frac{1}{r} u) \right)_r, \end{cases} \quad (1.2)$$

for  $(r, t) \in \Omega_t \times [0, T]$  with

$$\Omega_t = \{(r, t) | 0 \leq r \leq a(t), 0 \leq t \leq T\}.$$

The initial data is taken as

$$\rho(r, 0) = \rho_0(r), \quad \rho u(r, 0) = m_0(r). \quad (1.3)$$

The free boundary is described by

$$a'(t) = u(a(t), t), \quad t > 0, \quad a(0) = a_0. \quad (1.4)$$

At the center of symmetry we impose the Dirichlet boundary condition  $u(0, t) = 0$ , and stress-free boundary condition

$$\left( \rho^\gamma - (2\mu + \lambda(\rho))(u_r + \frac{1}{r}u) \right) (a(t), t) = 0, \quad t > 0 \quad (1.5)$$

at the free surface.

Denote the effective viscous flux by

$$F = (2\mu + \lambda(\rho))\operatorname{div}\mathbf{U} - P(\rho). \quad (1.6)$$

Thanks to the non-curl condition in the considering case, we thus rewrite the momentum equations (1.1)<sub>2</sub> as

$$\rho \dot{\mathbf{U}} = \nabla F \quad (1.7)$$

where  $\dot{\mathbf{U}} = \mathbf{U}_t + \mathbf{U} \cdot \nabla \mathbf{U} = (u_t + uu_r)_r \frac{\mathbf{x}}{r} = \dot{u} \frac{\mathbf{x}}{r}$ .

In what follows, we use  $C_T$  to denote a generic positive constant depending only on the initial data,  $\beta, \gamma$  and the given time  $T > 0$ . And  $C_{x_0, T}$  denotes a generic positive constant depending only on the initial data,  $\beta, \gamma, x_0$  and the given time  $T > 0$ .

We have the following main results on the global existence and long time behaviors of solution.

**Theorem 1.1** (Global existence). *Let  $\beta > 1$ ,  $\gamma > 1$ . Assume that the initial data  $(\rho_0, \mathbf{U}_0)$  satisfy*

$$\rho_0(r) > 0 \quad \text{for } r \in [0, a_0), \quad \rho_0(a_0) \geq 0, \quad \text{and} \quad \int_{\Omega_0} \rho_0 d\mathbf{x} = 1, \quad (1.8)$$

$$\rho_0 \in L^1(\Omega_0) \cap W^{1,p}(\Omega_0), \quad \mathbf{U}_0 \in H^2(\Omega_0), \quad (1.9)$$

for some  $p > 2$ . Then for any fixed  $T > 0$ , the free boundary value problem (1.2)–(1.5) has a unique global spherically symmetric strong solution

$$(\rho, \mathbf{U}, a) = (\rho(r, t), u(r, t) \frac{\mathbf{x}}{r}, a(t))$$

which satisfies

$$\begin{aligned} \rho_0(r_0)e^{-C_T} &\leq \rho(r_{x_0}(t), t) \leq \rho_0(r_0)e^{C_T}, \\ \rho &\in L^\infty(0, T; L^1(\Omega_t) \cap W^{1,p}(\Omega_t)), \quad \mathbf{U} \in L^\infty(0, T; H^2(\Omega_t)), \end{aligned}$$

where  $r = r_{x_0}(t)$  is a particle path uniquely defined by

$$\frac{d}{dt}r_{x_0}(t) = u(r_{x_0}(t), t), \quad (1.10)$$

with  $r_{x_0}(0) = r_0 \in [0, a_0]$  and  $x_0 = 1 - \int_{r_0}^{a_0} \rho_0(r)rdr$ . Moreover, it holds

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\Omega_t} \left( \frac{1}{2} \rho |\mathbf{U}|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) d\mathbf{x} + \int_0^T \int_{\Omega_t} (2\mu + \lambda(\rho)) |\operatorname{div} \mathbf{U}|^2 d\mathbf{x} dt \leq C_T, \\ & \sup_{t \in [0, T]} \int_{\Omega_t} F^2 d\mathbf{x} + \int_0^T \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 d\mathbf{x} \leq C_{x_0, T}, \\ & \sup_{t \in [0, T]} \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 d\mathbf{x} + \int_0^T \int_{\Omega_t} (2\mu + \lambda(\rho)) |\operatorname{div} \dot{\mathbf{U}}|^2 d\mathbf{x} dt \leq C_{x_0, T}, \\ & \sup_{t \in [0, T]} (\|\rho\|_{L^1(\Omega_t) \cap W^{1,p}(\Omega_t)} + \|\mathbf{U}\|_{H^2(\Omega_t)}) + \int_0^T (\|\nabla \mathbf{U}\|_{L^\infty(\Omega_t)}^2 + \|\mathbf{U}_t\|_{H^1(\Omega_t)}^2) dt \leq C_{x_0, T}. \end{aligned}$$

**Theorem 1.2** (Long time behavior). Let  $1 < \beta \leq \gamma$ . Assume that the initial data  $(\rho_0, \mathbf{U}_0)$  satisfy (1.8)–(1.9). Then the global solution  $(\rho, \mathbf{U}, a)$  of the problem (1.2)–(1.5) satisfies

$$a_M(t) = \sup_{s \in [0, t]} a(s) \geq \begin{cases} C(1+t)^{\frac{\gamma-1}{\gamma}}, & \gamma \in (1, \frac{3}{2}), \\ C(1+t)^{\frac{1}{2\gamma}} [1 + \ln(1+t)]^{-\frac{1}{2\gamma}}, & \gamma = \frac{3}{2}, \\ C(1+t)^{\frac{1}{2\gamma}}, & \gamma > \frac{3}{2}. \end{cases} \quad (1.11)$$

Moreover, if  $\gamma \geq \frac{3}{2}$ ,

$$a(t) \geq \begin{cases} C(1+t)^{\frac{1}{2\gamma}} [1 + \ln(1+t)]^{-\frac{1}{2\gamma}}, & \gamma = \frac{3}{2}, \\ C(1+t)^{\frac{1}{2\gamma}}, & \gamma > \frac{3}{2}. \end{cases} \quad (1.12)$$

The rest of this paper is organized as follows. The upper and lower bounds of the density are established in Section 2.1. Based on the a priori estimates obtained in Section 2.1, the higher order estimates are established in Section 2.2. Finally, in Section 3, we prove our main results.

## 2. The a priori estimates

Before establishing the main a priori estimates, we introduce the following Beale–Kato–Majda-type inequality,

**Lemma 2.1.** For  $2 < p < \infty$ , there exists a constant  $C(p)$  such that the following estimate holds for all  $\nabla \mathbf{V} \in L^2(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$ ,

$$\begin{aligned} \|\nabla \mathbf{V}\|_{L^\infty(\mathbb{R}^2)} &\leq C(\|\operatorname{div} \mathbf{V}\|_{L^\infty(\mathbb{R}^2)} + \|\operatorname{curl} \mathbf{V}\|_{L^\infty(\mathbb{R}^2)}) \log(e + \|\nabla^2 \mathbf{V}\|_{L^p(\mathbb{R}^2)}) \\ &\quad + C\|\nabla \mathbf{V}\|_{L^2(\mathbb{R}^2)} + C. \end{aligned}$$

The following Poincaré type inequality will be frequently used,

**Lemma 2.2.** ([8]) Let  $\mathbf{V} \in W^{1,2}(\Omega)$ , and let  $\rho$  be a non-negative function such that

$$0 < M_1 \leq \int_{\Omega} \rho d\mathbf{x}, \quad \int_{\Omega} \rho^\gamma d\mathbf{x} \leq M_2,$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain and  $\gamma > 1$ ,  $M_1 > 0$ ,  $M_2 > 0$ . Then there exist a constant  $C$  depending only on  $M_1$ ,  $M_2$  and  $\Omega$  such that

$$\|\mathbf{V}\|_{L^2(\Omega)}^2 \leq C(M_1, M_2)(\|\nabla \mathbf{V}\|_{L^2(\Omega)}^2 + (\int_{\Omega} \rho |\mathbf{V}| dx)^2).$$

### 2.1. Upper and lower bounds of density

Define

$$\xi(\mathbf{x}, t) = \xi(|\mathbf{x}|, t) = \int_{a(t)}^{|\mathbf{x}|} \rho u(r, t) dr, \quad (2.1)$$

$$\eta(\mathbf{x}, t) = \eta(|\mathbf{x}|, t) = \rho u^2(|\mathbf{x}|, t) - \rho u^2(a(t), t) + \int_{a(t)}^{|\mathbf{x}|} \rho u^2(r, t) \frac{1}{r} dr. \quad (2.2)$$

The exact definitions (2.1) and (2.2) are different from the following definitions due to [20] and [17]

$$\begin{cases} \Delta \xi = \operatorname{div}(\rho \mathbf{U}), \\ \Delta \eta = \operatorname{div}(\operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U})). \end{cases} \quad (2.3)$$

**Remark 2.1.** Compared with the Cauchy problem, the free boundary value problem yields new phenomena and difficulties and new arguments are introduced in the present paper to derive the positive bounds of density and obtain the regularities of solution. In our free boundary value problem in spatial bounded domain, we can't make use of the equations (2.3) which is very important to obtain the upper bound of the density in [17,20]. But, we can have (2.1)–(2.2) instead of (2.3) by the spherical symmetry property of velocity. Unlike [17] or [20], we will make use of Lemma 2.2 instead of weighted estimates to obtain the  $L^q$  ( $q > 2$ ) bound of velocity  $\mathbf{U}$ .

Integrating the equation (1.2)<sub>2</sub> from  $a(t)$  to  $|\mathbf{x}|$ , and using (2.1) and (2.2), we have

$$\xi_t + \eta - F = -\rho u^2(a(t), t). \quad (2.4)$$

Define

$$\theta(\rho) = \int_1^\rho \frac{2\mu + \lambda(s)}{s} ds = 2\mu \ln \rho + \frac{1}{\beta}(\rho^\beta - 1),$$

then the continuity equation (1.1)<sub>1</sub> yields that

$$\theta(\rho)_t + \mathbf{U} \cdot \nabla \theta(\rho) + F + P(\rho) = 0. \quad (2.5)$$

The combination of (2.4) and (2.5) yields that

$$(\xi + \theta(\rho))_t + \mathbf{U} \cdot \nabla (\xi + \theta(\rho)) + P(\rho) + \int_{a(t)}^{|\mathbf{x}|} \rho u^2(r, t) \frac{1}{r} dr = 0, \quad (2.6)$$

where we have used the fact

$$\eta - \mathbf{U} \cdot \nabla \xi = -\rho u^2(a(t), t) + \int_{a(t)}^{|\mathbf{x}|} \rho u^2(r, t) \frac{1}{r} dr.$$

We begin with the following basic energy estimates.

**Lemma 2.3.** *Let  $(\rho, \mathbf{U}, a(t))$  with  $\mathbf{U} = u \frac{\mathbf{x}}{r}$  be a smooth solution of free boundary value problem (1.2)–(1.5), and*

$$E_0 = \int_{\Omega_0} \left( \frac{1}{2} \rho_0 |\mathbf{U}_0|^2 + \frac{1}{\gamma - 1} \rho_0^\gamma \right) d\mathbf{x},$$

then one has

$$\sup_{t \in (0, T)} \int_{\Omega_t} \left( \frac{1}{2} \rho |\mathbf{U}|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) d\mathbf{x} + \int_0^T \int_{\Omega_t} (2\mu + \lambda(\rho)) |\operatorname{div} \mathbf{U}|^2 d\mathbf{x} dt \leq E_0, \quad (2.7)$$

$$\int_0^T u^2(a(s), s) ds \leq E_0, \quad a(t) \leq C(1 + t)^{\frac{1}{2}}. \quad (2.8)$$

**Proof.** It is easy to get (2.7) by taking the inner product of (1.1)<sub>2</sub> with  $\mathbf{U}$ . We omit the details here. By the spherical symmetry of velocity  $\mathbf{U}$ , one can obtain  $|\operatorname{div} \mathbf{U}|^2 = u_r^2 + \frac{u^2}{r^2} + \frac{2}{r} u u_r$ , and we get from (2.7)

$$\int_0^t |a'(s)|^2 ds = \int_0^t u^2(a(s), s) ds = \int_0^t \int_0^{a(s)} 2uu_r dr ds \leq E_0,$$

$$a(t) = a_0 + \int_0^t a'(s) ds \leq a_0 + Ct^{\frac{1}{2}} \left( \int_0^t a'(s)^2 ds \right)^{\frac{1}{2}} \leq C(1+t)^{\frac{1}{2}}. \quad \square$$

**Lemma 2.4.** Assume  $\beta > 1$ ,  $\gamma > 1$ , then there is a positive constant  $C_T$  depending only on  $E_0, \beta, \gamma, T$  such that

$$\sup_{t \in [0, T]} \int_{\Omega_t} \rho^{2\beta\gamma+1} d\mathbf{x} \leq C_T. \quad (2.9)$$

**Proof.** Denoting  $f = (\theta(\rho) + \xi)_+$ , multiplying the equation (2.6) by  $\rho f^{2\gamma-1}$ , and integrating the resulting equality over  $\Omega_t$  leads to

$$\begin{aligned} \frac{1}{2\gamma} \frac{d}{dt} \int_{\Omega_t} \rho f^{2\gamma} d\mathbf{x} &\leq \int_{\Omega_t} \rho f^{2\gamma-1} \int_{|\mathbf{x}|}^{a(t)} \rho u^2 \frac{1}{r} dr d\mathbf{x} \\ &\leq C \|\rho^{\frac{1}{2\gamma}} f\|_{L^{2\gamma}(\Omega_t)}^{2\gamma-1} \|\rho\|_{L^{2\beta\gamma+1}(\Omega_t)}^{\frac{1}{2\gamma}} \left\| \int_{|\mathbf{x}|}^{a(t)} \rho u^2 \frac{1}{r} dr \right\|_{L^{\frac{2\beta\gamma+1}{\beta}}(\Omega_t)} \\ &\leq C \|\rho^{\frac{1}{2\gamma}} f\|_{L^{2\gamma}(\Omega_t)}^{2\gamma-1} \|\rho\|_{L^{2\beta\gamma+1}(\Omega_t)}^{\frac{1}{2\gamma}} \|\rho u^2 \frac{x_i}{r^2}\|_{L^{\frac{2(2\beta\gamma+1)}{2\beta\gamma+1+2\beta}}(\Omega_t)} \\ &\leq C \|\rho^{\frac{1}{2\gamma}} f\|_{L^{2\gamma}(\Omega_t)}^{2\gamma-1} \|\rho\|_{L^{2\beta\gamma+1}(\Omega_t)}^{\frac{1}{2\gamma}} \|\rho \mathbf{U}\|_{L^{\frac{2\beta\gamma+1}{\beta}}(\Omega_t)} \left\| u \frac{1}{r} \right\|_{L^2(\Omega_t)}, \end{aligned} \quad (2.10)$$

where we have used Hölder's inequality and Gagliardo–Nirenberg inequality. Due to Lemma 2.2, we have

$$\|\rho \mathbf{U}\|_{L^{\frac{2\beta\gamma+1}{\beta}}(\Omega_t)} \leq C \|\rho\|_{L^{2\beta\gamma+1}(\Omega_t)} \|\mathbf{U}\|_{L^{\frac{2\beta\gamma+1}{\beta-1}}(\Omega_t)} \leq C_T \|\rho\|_{L^{2\beta\gamma+1}(\Omega_t)} (1 + \|\nabla \mathbf{U}\|_{L^2(\Omega_t)}).$$

Substituting the above inequality into (2.10), and using Lemma 2.3, one obtains

$$\frac{d}{dt} \int_{\Omega_t} \rho f^{2\gamma} d\mathbf{x} \leq C_T \left( 1 + \int_{\Omega_t} \rho f^{2\gamma} d\mathbf{x} + \int_{\Omega_t} \rho^{2\beta\gamma+1} d\mathbf{x} \right) (1 + \|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)}^2). \quad (2.11)$$

The term  $\int_{\Omega_t} \rho^{2\beta\gamma+1} d\mathbf{x}$  in the above inequality can be estimated as follows,

$$\begin{aligned}
\int_{\Omega_t} \rho^{2\beta\gamma+1} d\mathbf{x} &= \int_{\Omega_t \cap \{\rho \leq 2\}} \rho^{2\beta\gamma+1} d\mathbf{x} + \int_{\Omega_t \cap \{\rho > 2\}} \rho^{2\beta\gamma+1} d\mathbf{x} \\
&\leq C \int_{\Omega_t} \rho d\mathbf{x} + C \int_{\Omega_t \cap \{\rho > 2\}} \rho f^{2\gamma} d\mathbf{x} + C \int_{\Omega_t} \rho |\xi|^{2\gamma} d\mathbf{x} \\
&\leq C + \int_{\Omega_t} \rho f^{2\gamma} d\mathbf{x} + C \|\rho\|_{L^{\frac{2\beta\gamma+1}{2\beta\gamma+1-\gamma}}(\Omega_t)} \|\xi\|_{L^{2(2\beta\gamma+1)}(\Omega_t)}^{2\gamma} \\
&\leq C + \int_{\Omega_t} \rho f^{2\gamma} d\mathbf{x} + C \|\rho\|_{L^{\frac{2\beta\gamma+1}{2\beta\gamma+1-\gamma}}(\Omega_t)} \|\nabla \xi\|_{L^{\frac{2\beta\gamma+1}{\beta\gamma+1}}(\Omega_t)}^{2\gamma} \\
&\leq C + \int_{\Omega_t} \rho f^{2\gamma} d\mathbf{x} + C \|\rho\|_{L^{\frac{2\beta\gamma+1}{2\beta\gamma+1-\gamma}}(\Omega_t)} \|\rho^{\frac{1}{2}}\|_{L^{2(2\beta\gamma+1)}(\Omega_t)}^{2\gamma} \|\rho^{\frac{1}{2}} \mathbf{U}\|_{L^2(\Omega_t)}^{2\gamma},
\end{aligned}$$

which leads to

$$\int_{\Omega_t} \rho^{2\beta\gamma+1} d\mathbf{x} \leq C + C \int_{\Omega_t} \rho f^{2\gamma} d\mathbf{x}, \quad (2.12)$$

where we have used Gagliardo–Nirenberg inequality, Hölder’s inequality and Young’s inequality. Substituting (2.12) into (2.11) and using Gronwall’s inequality, we have

$$\sup_{t \in [0, T]} \int_{\Omega_t} \rho f^{2\gamma} d\mathbf{x} \leq C_T.$$

This combined with (2.12) directly gives (2.9). The proof of Lemma 2.4 is completed.  $\square$

**Lemma 2.5.** *There exists a positive constant  $C_T$  depending only on  $\beta, \gamma, T$ , initial data and a small constant  $\nu > 0$ , such that*

$$\sup_{t \in [0, T]} \int_{\Omega_t} \rho |\mathbf{U}|^{2+\nu} d\mathbf{x} \leq C_T. \quad (2.13)$$

**Proof.** Multiplying (1.1)<sub>2</sub> by  $(2 + \nu)\mathbf{U}|\mathbf{U}|^\nu$  and integrating the resulted equality over  $\Omega_t$  with respect to  $\mathbf{x}$ , one obtains

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega_t} \rho |\mathbf{U}|^{2+\nu} d\mathbf{x} + (2 + \nu) \int_{\Omega_t} (2\mu + \lambda(\rho)) |\mathbf{U}|^\nu |\operatorname{div} \mathbf{U}|^2 d\mathbf{x} \\
&\quad + \nu(2 + \nu) \int_{\Omega_t} (2\mu + \lambda(\rho)) (\mathbf{U} \cdot \nabla |\mathbf{U}|) |\mathbf{U}|^{\nu-1} \operatorname{div} \mathbf{U} d\mathbf{x} \\
&\leq C \int_{\Omega_t} \rho^\gamma |\mathbf{U}|^\nu (|\operatorname{div} \mathbf{U}| + |\nabla \mathbf{U}|) d\mathbf{x}.
\end{aligned} \quad (2.14)$$

The third term on the left hand side of (2.14) can be estimated as

$$\begin{aligned}
& v(2+v) \int_{\Omega_t} (2\mu + \lambda(\rho)) \mathbf{U} \cdot \nabla |\mathbf{U}| |\mathbf{U}|^{v-1} \operatorname{div} \mathbf{U} d\mathbf{x} \\
&= v(2+v) \int_{\Omega_t} (2\mu + \lambda(\rho)) \left( u_r^2 + \frac{1}{r} u u_r \right) |\mathbf{U}|^v d\mathbf{x} \\
&\geq v(2+v) \int_{\Omega_t} (2\mu + \lambda(\rho)) u_r^2 |\mathbf{U}|^v d\mathbf{x} - \frac{v}{2} (2+v) \int_{\Omega_t} (2\mu + \lambda(\rho)) |\operatorname{div} \mathbf{U}|^2 |\mathbf{U}|^v d\mathbf{x}, \quad (2.15)
\end{aligned}$$

where we have used the fact  $\frac{1}{r} u u_r \leq \frac{1}{2} (u_r + \frac{1}{r} u)^2 = \frac{1}{2} |\operatorname{div} \mathbf{U}|^2$ . Substituting (2.15) into (2.14) gives rise to

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega_t} \rho |\mathbf{U}|^{2+v} d\mathbf{x} + \mu(2+v) \int_{\Omega_t} |\mathbf{U}|^v |\nabla \mathbf{U}|^2 d\mathbf{x} \leq C \int_{\Omega_t} \rho^\gamma |\mathbf{U}|^v |\nabla \mathbf{U}| d\mathbf{x} \\
&\leq \frac{\mu}{2} \int_{\Omega_t} |\mathbf{U}|^v |\nabla \mathbf{U}|^2 d\mathbf{x} + C \int_{\Omega_t} \rho |\mathbf{U}|^{2+v} d\mathbf{x} + C \int_{\Omega_t} \rho^{\frac{4\gamma-v}{2-v}} d\mathbf{x},
\end{aligned}$$

where we have used Lemma 2.4 and the fact due to the spherical symmetry of velocity

$$|\nabla \mathbf{U}|^2 = u_r^2 + \frac{1}{r^2} u^2, \quad \int_{\Omega_t} |\mathbf{U}|^v |\nabla \mathbf{U}|^2 d\mathbf{x} \leq \int_{\Omega_t} |\mathbf{U}|^v |\operatorname{div} \mathbf{U}|^2 d\mathbf{x}.$$

Choosing  $v > 0$  sufficiently small such that  $\frac{4\gamma-v}{2-v} \leq 2\beta\gamma + 1$ , using Lemma 2.3, and then by Gronwall's inequality, we obtain (2.13). This complete the proof of Lemma 2.5.  $\square$

**Lemma 2.6.** *Let  $(\rho(r, t), u(r, t), a(t))$  be a smooth solution of free boundary value problem (1.2)–(1.5), then there exist positive constants  $C_T$  depending only on  $T, \beta, \gamma$  and initial data, such that*

$$\rho_0(r_0) e^{-C_T} \leq \rho(r_{x_0}(t), t) \leq \rho_0(r_0) e^{C_T}.$$

Here  $r = r_{x_0}(t)$  is a particle path uniquely defined by

$$\frac{d}{dt} r_{x_0}(t) = u(r_{x_0}(t), t), \quad (2.16)$$

with  $r_{x_0}(0) = r_0 \in [0, a_0]$  and  $x_0 = 1 - \int_{r_0}^{a_0} \rho_0(r) r dr$ .

**Proof.** Due to the equation (2.6), we have

$$(\xi + \theta(\rho))_t + u \partial_r (\xi + \theta(\rho)) + P(\rho) + \int_{a(t)}^r \rho u^2(r, t) \frac{1}{r} dr = 0. \quad (2.17)$$

Along the particle path  $r = r_{x_0}(t)$ , there holds the following ODE

$$\frac{d}{dt}(\xi + \theta(\rho))(r_{x_0}(t), t) + P(\rho)(r_{x_0}(t), t) + \int_{a(t)}^{r_{x_0}(t)} \rho u^2(r, t) \frac{1}{r} dr = 0.$$

Integrating the above equality over  $[0, t]$  yields that

$$\begin{aligned} & 2\mu \ln \frac{\rho(r_{x_0}(t), t)}{\rho_0(r_0)} + \frac{1}{\beta} (\rho^\beta(r_{x_0}(t), t) - \rho_0^\beta(r_0)) + \xi(r_{x_0}(t), t) - \xi(r_0) + \int_0^t P(\rho) d\tau \\ & \leq \int_0^t \int_{|r_{x_0}(\tau)|}^{a(\tau)} \rho u^2(r, \tau) \frac{1}{r} dr d\tau. \end{aligned} \quad (2.18)$$

By [Lemma 2.5](#) and Hölder's inequality, we obtain

$$\begin{aligned} |\xi(r_{x_0}(t), t)| &= \left| \int_{a(t)}^{|r_{x_0}(t)|} \rho u(r, t) dr \right| \leq \left| \int_0^{a(t)} \rho^{\frac{1}{2+v}} u(r, t) r^{\frac{1}{2+v}} \rho^{\frac{1+v}{2+v}} r^{\frac{-1}{2+v}} dr \right| \\ &\leq a(t)^{\frac{v}{1+v}} \sup_{t \in [0, T]} \|\rho(r_{x_0}(t), t)\|_{L^\infty}^{\frac{1+v}{2+v}} \int_0^{a(t)} \rho u^{2+v} r dr \\ &\leq C_T \sup_{t \in [0, T]} \|\rho(r_{x_0}(t), t)\|_{L^\infty}^{\frac{1+v}{2+v}}. \end{aligned} \quad (2.19)$$

In the case  $\ln \frac{\rho(r_{x_0}(t), t)}{\rho_0(r_0)} < 0$ , we have

$$\rho(r_{x_0}(t), t) < \rho_0(r_0). \quad (2.20)$$

If  $\ln \frac{\rho(r_{x_0}(t), t)}{\rho_0(r_0)} \geq 0$ , then by substituting (2.19) into (2.18), and using [Lemma 2.3](#), we have

$$\begin{aligned} \sup_{t \in [0, T]} \|\rho(r_{x_0}(t), t)\|_{L^\infty}^\beta &\leq C + C_T \sup_{t \in [0, T]} \|\rho(r_{x_0}(t), t)\|_{L^\infty}^{\frac{1+v}{2+v}} + C \sup_{t \in [0, T]} \|\rho(r_{x_0}(t), t)\|_{L^\infty}, \\ &\text{if } \rho > 1. \end{aligned}$$

Therefore, if  $\beta > 1$ , then we have

$$\sup_{t \in [0, T]} \|\rho(r_{x_0}(t), t)\|_{L^\infty} \leq C_T. \quad (2.21)$$

By (2.18), (2.19) and (2.21), one obtains

$$2\mu \ln \frac{\rho(r_{x_0}(t), t)}{\rho_0(r_0)} \leq C_T.$$

From the above inequality and (2.20), it is easy to see

$$\rho(r_{x_0}(t), t) \leq \rho_0(r_0)e^{C_T}, \quad (2.22)$$

for some positive constant  $C_T$ .

By (2.18), (2.19) and (2.22), it holds that

$$\sup_{t \in [0, T]} \left\| \ln \frac{\rho(r_{x_0}(t), t)}{\rho_0(r_0)} \right\|_{L^\infty} \leq C_T,$$

which implies that

$$\inf_{t \in [0, T]} \|\rho(r_{x_0}(t), t)\|_{L^\infty} \geq \rho_0(r_0)e^{-C_T}.$$

Thus the proof of Lemma 2.6 is completed.  $\square$

## 2.2. Higher order estimate

First, we rewrite the equations (1.2) in Lagrangian mass coordinate:

$$x(r, t) = \int_0^r \rho dr,$$

which translates the domain  $[0, a(t)] \times [0, T]$  into  $[0, 1] \times [0, T]$  and satisfies

$$\frac{\partial x}{\partial r} = \rho r, \quad \frac{\partial x}{\partial t} = -\rho u r, \quad \frac{\partial \tau}{\partial r} = 0, \quad \frac{\partial \tau}{\partial t} = 1.$$

The free boundary value problem (1.2)–(1.5) is transformed into the following fixed boundary problem:

$$\begin{cases} \rho_\tau + \rho^2 (ru)_x = 0, \\ \frac{1}{r} u_\tau + (\rho^\gamma - (2\mu + \lambda(\rho))\rho(ru)_x)_x = 0, \end{cases} \quad (2.23)$$

for  $(x, \tau) \in [0, 1] \times [0, T]$ , with the initial data and boundary conditions given by

$$\begin{aligned} (\rho, u)(x, 0) &= (\rho_0, u_0), \quad x \in [0, 1], \\ u(0, \tau) &= 0, \quad (\rho^\gamma - (2\mu + \lambda(\rho))\rho(ru)_x)(1, \tau) = 0, \quad \tau \in [0, T], \end{aligned}$$

where  $r = r(x, \tau)$  is defined by

$$\frac{d}{d\tau} r(x, \tau) = u(x, \tau), \quad x \in [0, 1], \quad \tau \in [0, T],$$

and the fixed boundary  $x = 1$  corresponds to the free boundary  $a(\tau) = r(1, \tau)$  in Eulerian form determined by

$$\frac{d}{d\tau}a(\tau) = u(1, \tau), \quad \tau \in [0, T], \quad a(0) = a_0.$$

Now, we rewrite (2.7) in Lagrangian coordinate

$$\sup_{t \in (0, T)} \int_0^1 (|u|^2 + \frac{1}{\gamma-1} \rho^{\gamma-1}) dx + \int_0^T \int_0^1 (2\mu + \lambda(\rho)) \rho |(ru)_x|^2 dx dt \leq E_0.$$

By Hölder's inequality and Lemma 2.3, we have

$$1 = \int_{\Omega_t} \rho d\mathbf{x} \leq \left( \int_{\Omega_t} \rho^\gamma d\mathbf{x} \right)^{\frac{1}{\gamma}} |\Omega_t|^{1-\frac{1}{\gamma}} \leq C(\gamma-1) E_0 a(t)^{2-\frac{2}{\gamma}}.$$

Then there exists a constant  $\alpha$  depending only on  $E_0$  and  $\gamma$  such that  $\forall t \in [0, T]$ ,  $a(t) \geq 3\alpha$ .

Now, we introduce the following cut-off functions  $\chi$  and  $\zeta$ . Let  $\zeta \in C^\infty$  be a smooth function of  $r$  satisfying

$$\begin{aligned} 0 \leq \zeta \leq 1, \quad \zeta &= 1 \quad \text{for } r \in [0, \alpha], \quad \text{Supp}(\zeta) \subset [0, 2\alpha], \\ \zeta &= 0 \quad \text{for } r \in [2\alpha, a(t)], \quad \forall t \in [0, T], \quad |\zeta'| \leq \frac{2}{\alpha} \quad \text{and} \quad |\zeta''| \leq \frac{10}{\alpha^2}. \end{aligned}$$

By Lemma 2.6, there exists a constant  $x_1$  depending only on  $T, \beta, \gamma, E_0$  such that

$$0 < x_1 \leq \int_0^\alpha \rho r dr < 1.$$

Similarly, construct a smooth function  $\chi$  of Lagrange coordinates  $x$  such that

$$\begin{aligned} 0 \leq \chi \leq 1, \quad \chi &= 1 \quad \text{for } x \in [x_1, 1], \quad \text{Supp}(\chi) \subset [x_0, 1], \\ \chi &= 0 \quad \text{for } x \in [0, x_0], \quad \text{and} \quad |\chi'| \leq \frac{2}{x_1 - x_0}, \end{aligned}$$

with  $0 < x_0 < x_1 < 1$ . Also  $\chi$  is a function of both  $r$  and  $t$  in Eulerian coordinate.

**Lemma 2.7.** *There exists a constant  $C_{x_0, T}$  depending only on  $\beta, \gamma, T, x_0$  and initial data such that*

$$\sup_{t \in [0, T]} \int_{\Omega_t} F^2 d\mathbf{x} + \int_0^T \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 d\mathbf{x} dt \leq C_{x_0, T}, \quad (2.24)$$

where we recall

$$F = (2\mu + \lambda(\rho)) \text{div} \mathbf{U} - P(\rho).$$

**Proof.** The proof of Lemma 2.7 consists of two steps: boundary estimate and interior estimate.

First of all, we deal with the boundary estimates in Lagrangian coordinates. Multiplying the momentum equation (2.23)<sub>2</sub> by  $ru_\tau \chi^2$  and integrating the resulted equation with respect to  $x$ , one obtains

$$\int_0^1 u_\tau^2 \chi^2 dx + \int_0^1 F(ru_\tau)_x \chi^2 dx = -2 \int_0^1 Fru_\tau \chi' \chi dx, \quad (2.25)$$

where we have used the fact  $F = (2\mu + \lambda(\rho))\rho(ru)_x - \rho^\gamma$  in Lagrangian coordinates. A direct calculation together with  $(ru)_x = \frac{F+\rho^\gamma}{(2\mu+\lambda(\rho))\rho}$  yields that

$$\begin{aligned} (ru_\tau)_x &= (ru)_{x\tau} - 2uu_x \\ &= \frac{F_\tau}{(2\mu + \lambda(\rho))\rho} + \frac{(\rho^\gamma)_\tau}{(2\mu + \lambda(\rho))\rho} + \left(\frac{1}{(2\mu + \lambda(\rho))\rho}\right)_\tau (F + \rho^\gamma) - 2uu_x. \end{aligned}$$

Substituting the above equality into (2.25), we have

$$\begin{aligned} &\int_0^1 u_\tau^2 \chi^2 dx + \frac{d}{d\tau} \int_0^1 \frac{F^2}{2(2\mu + \lambda(\rho))\rho} \chi^2 dx \\ &= -2 \int_0^1 Fru_\tau \chi' \chi dx - \frac{1}{2} \int_0^1 F^2 \left(\frac{1}{(2\mu + \lambda(\rho))\rho}\right)_\tau \chi^2 dx \\ &\quad - \int_0^1 F \left(\left(\frac{1}{(2\mu + \lambda(\rho))\rho}\right)_\tau \rho^\gamma + \frac{(\rho^\gamma)_\tau}{(2\mu + \lambda(\rho))\rho}\right) \chi^2 dx + 2 \int_0^1 uu_x F \chi^2 dx. \end{aligned} \quad (2.26)$$

The right hand side terms of (2.26) can be made as follows.

The first term:

$$\begin{aligned} &\left| \int_0^1 Fru_\tau \chi' \chi dx \right| \leq \|u_\tau \chi\|_{L^2((0,1))} \|Fr \chi'\|_{L^2((0,1))} \\ &\leq \frac{1}{4} \|u_\tau \chi\|_{L^2((0,1))}^2 + C_T \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))}^2 + C_T. \end{aligned} \quad (2.27)$$

The second term: by Gagliardo–Nirenberg–Sobolev inequality

$$\begin{aligned} &\left| \int_0^1 F^2 \left(\frac{1}{(2\mu + \lambda(\rho))\rho}\right)_\tau \chi^2 dx \right| \leq C_T \left| \int_0^1 \frac{F^2}{\sqrt{\rho}} \sqrt{\rho}(ru)_x \chi^2 dx \right| \\ &\leq \|F \chi\|_{L^2((0,1))}^{\frac{3}{4}} (\|F_x \chi\|_{L^2((0,1))}^{\frac{1}{4}} + \|F \chi'\|_{L^2((0,1))}^{\frac{1}{4}}) \left\| \frac{F}{\sqrt{\rho}} \chi \right\|_{L^2((0,1))} \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))} \end{aligned}$$

$$\begin{aligned}
&\leq C_{x_0, T} \|F\chi\|_{L^2((0,1))}^{\frac{3}{4}} \|u_\tau \chi\|_{L^2((0,1))}^{\frac{1}{4}} \left\| \frac{F}{\sqrt{\rho}} \chi \right\|_{L^2((0,1))} \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))} \\
&\quad + C_T \left\| \frac{F}{\sqrt{\rho}} \chi \right\|_{L^2((0,1))}^{\frac{7}{4}} \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))} \|F\chi'\|_{L^2((0,1))}^{\frac{1}{4}} \\
&\leq \frac{1}{4} \|u_\tau \chi\|_{L^2((0,1))}^2 + C_{x_0, T} (1 + \left\| \frac{F}{\sqrt{\rho}} \chi \right\|_{L^2((0,1))}^2) (1 + \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))}^2). \tag{2.28}
\end{aligned}$$

The third term:

$$\begin{aligned}
&\left| \int_0^1 F \left( \left( \frac{1}{(2\mu + \lambda(\rho))\rho} \right)_\tau \rho^\gamma + \frac{(\rho^\gamma)_\tau}{(2\mu + \lambda(\rho))\rho} \right) \chi^2 dx \right| \\
&\leq C_T \int_0^1 |F \sqrt{\rho}(ru)_x| \chi^2 dx \leq C_T (1 + \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))}^2). \tag{2.29}
\end{aligned}$$

The fourth term: using the fact

$$\|u\chi\|_{L^\infty((0,1))} \leq C_{x_0, T} \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))}, \quad \int_0^1 \frac{1}{\rho r^2} u^2 dx \leq C \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))},$$

one obtains

$$\begin{aligned}
&\left| \int_0^1 uu_x F \chi^2 dx \right| = \left| \int_0^1 \left( \frac{u}{r} (ru)_x - \frac{1}{\rho r^2} u^2 \right) F \chi^2 dx \right| \\
&\leq C_{x_0, T} \left\| \frac{F}{\sqrt{\rho}} \chi \right\|_{L^2((0,1))} \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))}^2. \tag{2.30}
\end{aligned}$$

Substituting the estimates (2.27)–(2.30) into (2.26), one can arrive at

$$\begin{aligned}
&\frac{d}{d\tau} \int_0^1 \frac{F^2}{(2\mu + \lambda(\rho))\rho} \chi^2 dx + \int_0^1 u_\tau^2 \chi^2 dx \\
&\leq C_{x_0, T} (1 + \left\| \frac{F\chi}{\sqrt{(2\mu + \lambda(\rho))\rho}} \right\|_{L^2((0,1))}^2) (1 + \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))}^2)
\end{aligned}$$

Using Gronwall's inequality, we have

$$\sup_{t \in [0, T]} \int_0^1 \frac{F^2}{(2\mu + \lambda(\rho))\rho} \chi^2 dx + \int_0^T \int_0^1 u_\tau^2 \chi^2 dx dt \leq C_{x_0, T}. \tag{2.31}$$

Then, we go on with the interior estimate in Eulerian coordinates.

A direct computation yields

$$\begin{aligned} \operatorname{div} \dot{\mathbf{U}} &= (\operatorname{div} \mathbf{U})_t + \mathbf{U} \cdot \nabla \operatorname{div} \mathbf{U} + \partial_{x_1} \mathbf{U} \cdot \nabla U_1 + \partial_{x_2} \mathbf{U} \cdot \nabla U_2 \\ &= D_t \left( \frac{F}{2\mu + \lambda(\rho)} \right) + D_t \left( \frac{\rho^\gamma}{2\mu + \lambda(\rho)} \right) + (\operatorname{div} \mathbf{U})^2 - 2\nabla U_1 \cdot \nabla^\perp U_2, \end{aligned} \quad (2.32)$$

where  $D_t V = \partial_t V + \mathbf{U} \cdot \nabla V$  and  $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$ . Multiplying (1.7) by  $\zeta^2 \dot{\mathbf{U}}$ , integrating the resulted equation by parts, and using the fact (2.32), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} \frac{F^2}{2\mu + \lambda(\rho)} \zeta^2 d\mathbf{x} + \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 \zeta^2 d\mathbf{x} \\ &= \frac{1}{2} \int_{\Omega_t} F^2 \operatorname{div} \mathbf{U} \left( \rho \left( \frac{1}{2\mu + \lambda(\rho)} \right)' - \frac{1}{2\mu + \lambda(\rho)} \right) \zeta^2 d\mathbf{x} \\ & \quad - \int_{\Omega_t} F \operatorname{div} \mathbf{U} \left( \rho \left( \frac{\rho^\gamma}{2\mu + \lambda(\rho)} \right)' - \frac{\rho^\gamma}{2\mu + \lambda(\rho)} \right) \zeta^2 d\mathbf{x} \\ & \quad - 2 \int_{\Omega_t} F \nabla U_1 \cdot \nabla^\perp U_2 \zeta^2 d\mathbf{x} - \int_{\Omega_t} F \dot{\mathbf{U}} \cdot \nabla \zeta \zeta d\mathbf{x} \\ &\leq \|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)} \|F\zeta\|_{L^4(\Omega_t)}^2 + \|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)} \left\| \frac{F\zeta}{\sqrt{2\mu + \lambda(\rho)}} \right\|_{L^2(\Omega_t)} \\ & \quad + \|F\zeta\|_{L^2(\Omega_t)} \|\dot{\mathbf{U}} \nabla \zeta\|_{L^2(\Omega_t)} + 2 \left| \int_{\Omega_t} F \nabla U_1 \cdot \nabla^\perp U_2 \zeta^2 d\mathbf{x} \right|. \end{aligned} \quad (2.33)$$

The first term on the right hand side of (2.33) can be estimated as

$$\begin{aligned} & \|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)} \|F\zeta\|_{L^4(\Omega_t)}^2 \leq \|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)} \|F\zeta\|_{L^2(\Omega_t)} (\|F\nabla \zeta\|_{L^2(\Omega_t)} + \|\nabla F\zeta\|_{L^2(\Omega_t)}) \\ &\leq C_{x_0, T} (1 + \|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)}^2) + \frac{1}{2} \|\sqrt{\rho} \dot{\mathbf{U}} \zeta\|_{L^2(\Omega_t)}^2 + C_T \|F\zeta\|_{L^2(\Omega_t)}^2 \|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)}^2, \end{aligned} \quad (2.34)$$

where we have used Gagliardo–Nirenberg–Sobolev inequality, (2.31), and Lemma 2.6.

The last term on the right hand side of (2.33) can be estimated as

$$\begin{aligned} & \left| \int_{\Omega_t} F \nabla U_1 \cdot \nabla^\perp U_2 \zeta^2 d\mathbf{x} \right| \\ &\leq \left| \int_{\Omega_t} F \nabla (\zeta U_1) \cdot \nabla^\perp (\zeta U_2) d\mathbf{x} \right| + \left| \int_{\Omega_t} F (\nabla U_1 \cdot \nabla^\perp U_2 \zeta^2 - \nabla (\zeta U_1) \cdot \nabla^\perp (\zeta U_2)) d\mathbf{x} \right| \\ &\leq \left| \int_{\mathbb{R}^2} F \zeta \nabla (\zeta U_1) \cdot \nabla^\perp (\zeta U_2) d\mathbf{x} \right| + C_{x_0, T} \int_{\Omega_t} |F u_r \nabla \zeta| d\mathbf{x} + C_{x_0, T} \int_{\Omega_t} |F (\nabla \zeta)^2| d\mathbf{x} \end{aligned}$$

$$\begin{aligned}
&\leq \|F\zeta\|_{BMO} \|\nabla(\zeta U_1) \cdot \nabla^\perp(\zeta U_2)\|_{\mathcal{H}_1} + C_{x_0, T} \|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)}^2 + C_{x_0, T} \|F\|_{L^2(\mathbb{R}^2)} + C_{x_0, T} \\
&\leq \frac{1}{4} \|\sqrt{\rho} \dot{\mathbf{U}} \zeta\|_{L^2(\Omega_t)}^2 + C \|\zeta \operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)}^2 \left\| \frac{\zeta F}{\sqrt{2\mu + \lambda(\rho)}} \right\|_{L^2(\Omega_t)}^2 \\
&\quad + C_{x_0, T} \|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)}^2 + C_{x_0, T},
\end{aligned} \tag{2.35}$$

where we have used [Lemma 2.6](#), [\(2.31\)](#) and

$$\|u\chi\|_{L^\infty((0,1))} \leq C_{x_0, T} \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))} \leq C_{x_0, T}.$$

Substituting [\(2.34\)](#) and [\(2.35\)](#) into [\(2.33\)](#) shows

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega_t} \frac{F^2}{2\mu + \lambda(\rho)} \zeta^2 d\mathbf{x} + \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 \zeta^2 d\mathbf{x} \\
&\leq C_{x_0, T} \left( \left\| \frac{\zeta F}{\sqrt{2\mu + \lambda(\rho)}} \right\|_{L^2(\Omega_t)}^2 + 1 \right) (\|\operatorname{div} \mathbf{U}\|_{L^2(\Omega_t)}^2 + 1) + \|\dot{\mathbf{U}} \nabla \zeta\|_{L^2(\Omega_t)}^2.
\end{aligned}$$

By Gronwall's inequality, we obtain

$$\sup_{t \in [0, T]} \int_{\Omega_t} \frac{F^2}{2\mu + \lambda(\rho)} \zeta^2 d\mathbf{x} + \int_0^T \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 \zeta^2 d\mathbf{x} dt \leq C_{x_0, T}, \tag{2.36}$$

which together with [\(2.31\)](#) yields [\(2.24\)](#), and then the proof of [Lemma 2.7](#) is completed.  $\square$

**Lemma 2.8.** *There exists a constant  $C_{x_0, T}$  depending only on  $\beta, \gamma, T, x_0$  and initial data such that*

$$\sup_{t \in [0, T]} \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 d\mathbf{x} + \int_0^T \int_{\Omega_t} (2\mu + \lambda(\rho)) |\operatorname{div} \dot{\mathbf{U}}|^2 d\mathbf{x} dt \leq C_{x_0, T}. \tag{2.37}$$

**Proof.** First of all, we deal with the boundary estimates in Lagrangian coordinates. Differentiating the equation [\(2.23\)](#) with respect to  $\tau$  gives

$$\frac{1}{r} u_{\tau\tau} - \frac{1}{r^2} u_\tau u + (\rho^\gamma - (2\mu + \lambda(\rho))\rho(ru)_x)_{x\tau} = 0. \tag{2.38}$$

Taking the inner product of [\(2.38\)](#) with  $ru_\tau \chi^2$  over  $[0, 1]$ , one gets

$$\begin{aligned}
&\frac{1}{2} \frac{d}{d\tau} \int_0^1 u_\tau^2 \chi^2 dx + \int_0^1 ((2\mu + \lambda(\rho))\rho(ru)_x - \rho^\gamma)_\tau \chi^2 u_{x\tau} r dx \\
&= \int_0^1 \frac{1}{r} u_\tau^2 u \chi^2 dx - \int_0^1 ((2\mu + \lambda(\rho))\rho(ru)_x - \rho^\gamma)_\tau (2\chi \chi' u_\tau r + u_\tau \chi^2 u) dx.
\end{aligned} \tag{2.39}$$

Using [Lemma 2.6](#), [Lemma 2.7](#), and the following facts:

$$\begin{aligned}
 \|u\chi\|_{L^\infty((0,1))} &\leq C_{x_0,T} \|\sqrt{\rho}(ru)_x\|_{L^2((0,1))} \leq C_{x_0,T} \|div\mathbf{U}\|_{L^2(\Omega_t)} \\
 &\leq C_{x_0,T} \left\| \frac{F + \rho^\gamma}{2\mu + \lambda(\rho)} \right\|_{L^2(\Omega_t)} \leq C_{x_0,T}, \\
 \|\rho\tau\|_{L^2((0,1))} &= \|\rho^2(ru)_x\|_{L^2((0,1))} \leq C_{x_0,T}, \\
 \|u_\tau\chi\|_{L^\infty((0,1))} &\leq C \|u_\tau\chi\|_{L^1((0,1))} + \|u_{x\tau}\chi + u_\tau\chi'\|_{L^1((0,1))} \\
 &\leq C \|u_\tau\chi\|_{L^2((0,1))} + \|\sqrt{\rho}u_{x\tau}\chi\|_{L^1((0,1))} \\
 &\leq C \|u_\tau\chi\|_{L^2((0,1))} + C_{x_0,T} \|\sqrt{\rho}u_{x\tau}\chi\|_{L^2((0,1))}, \\
 \|F\chi\|_{L^\infty((x_0,1))} &\leq C \|F\|_{L^1((0,1))} + C \|F_x\chi\|_{L^1((0,1))} \\
 &\leq C \|F\|_{L^2((0,1))} + C_{x_0,T} \|u_\tau\chi\|_{L^2((0,1))},
 \end{aligned}$$

we can obtain after a complicated computation that

$$\begin{aligned}
 &\frac{d}{d\tau} \int_0^1 (u_\tau^2 + \gamma\rho^{\gamma+1}|(ru)_x|^2)\chi^2 dx + \int_0^1 (2\mu + \lambda(\rho))\rho u_{x\tau}^2 r \chi^2 dx \\
 &\leq C_{x_0,T} (1 + \|u_\tau\|_{L^2((0,1))}^2 + \|F\|_{L^2((0,1))}^2).
 \end{aligned}$$

By Gronwall's inequality, one obtains

$$\sup_{\tau \in [0,T]} \int_0^1 (u_\tau^2 + \gamma\rho^\gamma u_x^2)\chi^2 dx + \int_0^T \int_0^1 (2\mu + \lambda(\rho))\rho u_{x\tau}^2 \chi^2 dx d\tau \leq C_{x_0,T} \quad (2.40)$$

Then, we go on with the interior estimate in Eulerian coordinates. Operating  $\partial_t + \operatorname{div}(\mathbf{U}\cdot)$  to both sides of (1.7), we obtain

$$\begin{aligned}
 (\rho\dot{\mathbf{U}})_t + \operatorname{div}(\rho\mathbf{U} \otimes \dot{\mathbf{U}}) &= \nabla((2\mu + \lambda(\rho))\operatorname{div}(\dot{\mathbf{U}})) - \operatorname{div}(\nabla\mathbf{U}(2\mu + \lambda(\rho))\operatorname{div}\mathbf{U}) \\
 &\quad - \nabla((2\mu + \lambda(\rho))\partial_i\mathbf{U} \cdot \nabla U_i) + \nabla((2\mu + \lambda(\rho) - \rho\lambda'(\rho))(\operatorname{div}\mathbf{U})^2) \\
 &\quad + (\gamma - 1)\nabla(\rho^\gamma \operatorname{div}\mathbf{U}) + \operatorname{div}(\rho^\gamma \nabla\mathbf{U}).
 \end{aligned}$$

Then multiplying the above inequality by  $\dot{\mathbf{U}}\zeta^2$  and integrating the resulted equation by parts, we have

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 \zeta^2 d\mathbf{x} + \int_{\Omega_t} (2\mu + \lambda(\rho)) |\operatorname{div}\dot{\mathbf{U}}|^2 \zeta^2 d\mathbf{x} \\
 &\leq C_{x_0,T} + C \|\nabla\mathbf{U}\|_{L^4(\Omega_t)}^4 + C \|\nabla\mathbf{U}\|_{L^2(\Omega_t)}^2 + \frac{\mu}{2} \|\zeta \operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)}^2 \\
 &\leq C_{x_0,T} + C \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 \zeta^2 d\mathbf{x} + C \|\operatorname{div}\mathbf{U}\|_{L^2(\Omega_t)}^2 + \frac{\mu}{2} \|\zeta \operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)}^2, \quad (2.41)
 \end{aligned}$$

where we have used the following fact

$$\begin{aligned}
 \|\nabla \mathbf{U}\|_{L^4(\Omega_t)}^4 &\leq \|\nabla(\zeta \mathbf{U})\|_{L^4(\Omega_t)}^4 + \|(1-\zeta)\nabla \mathbf{U} - \nabla \zeta \mathbf{U}\|_{L^4(\Omega_t)}^4 \\
 &\leq C_{x_0, T} + \|\operatorname{div}(\zeta \mathbf{U})\|_{L^4(\Omega_t)}^4 + \|(1-\zeta)F\|_{L^4(\Omega_t)}^4 \\
 &\leq C_{x_0, T} + \|\zeta \operatorname{div} \mathbf{U}\|_{L^4(\Omega_t)}^4 = C_{x_0, T} + \|\zeta \frac{F + P(\rho)}{2\mu + \lambda(\rho)}\|_{L^4(\Omega_t)}^4 \\
 &\leq C_{x_0, T} + \|\zeta F\|_{L^4(\Omega_t)}^4 \leq C_{x_0, T} + \|\zeta F\|_{L^2(\Omega_t)}^2 \|\nabla(\zeta F)\|_{L^2(\Omega_t)}^2 \\
 &\leq C_{x_0, T} + C_{x_0, T} \|\zeta \nabla F\|_{L^2(\Omega_t)}^2 \leq C_{x_0, T} + C_{x_0, T} \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 \zeta^2 d\mathbf{x}.
 \end{aligned}$$

Then, by Gronwall's inequality, we have

$$\sup_{t \in [0, T]} \int_{\Omega_t} \rho |\dot{\mathbf{U}}|^2 \zeta^2 d\mathbf{x} + \int_0^T \int_{\Omega_t} (2\mu + \lambda(\rho)) |\operatorname{div} \dot{\mathbf{U}}|^2 \zeta^2 d\mathbf{x} dt \leq C_{x_0, T}, \quad (2.42)$$

which together with (2.40) yields (2.37). The proof of Lemma 2.8 is completed.  $\square$

**Lemma 2.9.** For any  $p > 2$ , there exists a constant  $C_{x_0, T}$  depending only on  $\beta, \gamma, T, x_0$  and initial data, such that

$$\sup_{t \in [0, T]} (\|\rho\|_{L^1(\Omega_t) \cap W^{1, p}(\Omega_t)} + \|\mathbf{U}\|_{H^2(\Omega_t)}) + \int_0^T (\|\nabla \mathbf{U}\|_{L^\infty(\Omega_t)}^2 + \|\mathbf{U}_t\|_{H^1(\Omega_t)}^2) dt \leq C_{x_0, T}. \quad (2.43)$$

**Proof.** Applying operator  $\nabla$  to the continuity equation (1.1)<sub>1</sub>, multiplying the resulting equation by  $p|\nabla \rho|^{p-2} \nabla \rho$  with  $p > 2$ , one obtains

$$\begin{aligned}
 &(|\nabla \rho|^p)_t + \operatorname{div}(\mathbf{U} |\nabla \rho|^p) + (p-1) |\nabla \rho|^p \operatorname{div} \mathbf{U} + p |\nabla \rho|^{p-2} \nabla \rho \cdot (\nabla \mathbf{U} \cdot \nabla \rho) \\
 &+ p \rho |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \operatorname{div} \mathbf{U} = 0.
 \end{aligned}$$

Integrating the above equation with respect to  $\mathbf{x}$  over  $\Omega_t$  gives that

$$\begin{aligned}
 \frac{d}{dt} \|\nabla \rho\|_{L^p} &\leq C (\|\nabla \mathbf{U}\|_{L^\infty(\Omega_t)} \|\nabla \rho\|_{L^p(\Omega_t)} + \|\nabla \operatorname{div} \mathbf{U}\|_{L^p(\Omega_t)}) \\
 &\leq C (\|\nabla \mathbf{U}\|_{L^\infty(\Omega_t)} \|\nabla \rho\|_{L^p(\Omega_t)} + \|\nabla (\frac{F + P(\rho)}{2\mu + \lambda(\rho)})\|_{L^p(\Omega_t)}) \\
 &\leq C (1 + \|F\|_{L^\infty(\Omega_t)} + \|\nabla \mathbf{U}\|_{L^\infty(\Omega_t)}) \|\nabla \rho\|_{L^p(\Omega_t)} + C_{x_0, T} \|\dot{\mathbf{U}}\|_{L^p(\Omega_t)} \\
 &\leq C (1 + \|F\|_{L^{\frac{p-2}{2}}(\Omega_t)}^{\frac{p-2}{2}} \|\nabla F\|_{L^{\frac{p}{2p-2}}(\Omega_t)}^{\frac{p}{2p-2}} + \|\nabla \mathbf{U}\|_{L^\infty(\Omega_t)}) \|\nabla \rho\|_{L^p(\Omega_t)} + C_{x_0, T} \|\dot{\mathbf{U}}\|_{L^p(\Omega_t)} \\
 &\leq C_{x_0, T} (1 + \|\dot{\mathbf{U}}\|_{L^p(\Omega_t)}^{\frac{p}{2p-2}} + \|\nabla \mathbf{U}\|_{L^\infty(\Omega_t)}) \|\nabla \rho\|_{L^p(\Omega_t)} + C_{x_0, T} \|\dot{\mathbf{U}}\|_{L^p(\Omega_t)}. \quad (2.44)
 \end{aligned}$$

Next, we will estimate the terms on the right hand side of (2.44) one by one. First, we have

$$\begin{aligned}\|\dot{\mathbf{U}}\|_{L^p(\Omega_t)} &\leq C\|\dot{\mathbf{U}}\|_{L^2(\Omega_t)} + C\|\dot{\mathbf{U}}\|_{L^2(\Omega_t)}^{\frac{2}{p}}\|\nabla\dot{\mathbf{U}}\|_{L^2(\Omega_t)}^{1-\frac{2}{p}} \\ &\leq C_T(\|\sqrt{\rho}\dot{\mathbf{U}}\|_{L^2(\Omega_t)} + \|\nabla\dot{\mathbf{U}}\|_{L^2(\Omega_t)}) \\ &\leq C_T(1 + \|\operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)}),\end{aligned}\quad (2.45)$$

here we have used Gagliardo–Nirenberg inequality, Lemma 2.2, and the fact

$$\int_0^{a(t)} (|\dot{u}_r|^2 r + \frac{1}{r}\dot{u}^2) dr = \|\nabla\dot{\mathbf{U}}\|_{L^2(\Omega_t)} \leq \|\operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)} = \int_0^{a(t)} (|\dot{u}_r|^2 r + \frac{1}{r}\dot{u}^2) dr + \dot{u}^2(a(t), t).$$

By Lemma 2.1, (2.45) and standard  $L^p$ -estimate for elliptic system, one has

$$\begin{aligned}\|\nabla\mathbf{U}\|_{L^\infty(\Omega_t)} &\leq \|\nabla(\zeta\mathbf{U})\|_{L^\infty(\Omega_t)} + \|(1-\zeta)\nabla\mathbf{U} + \nabla\zeta\mathbf{U}\|_{L^\infty(\Omega_t)} \\ &\leq C_{x_0,T} + \|\operatorname{div}(\zeta\mathbf{U})\|_{L^\infty(\Omega_t)} \log(e + \|\nabla^2(\zeta\mathbf{U})\|_{L^p(\Omega_t)}) + C\|\nabla(\zeta\mathbf{U})\|_{L^2(\Omega_t)} \\ &\leq C_{x_0,T} + C_{x_0,T}(1 + \|F\|_{L^\infty(\Omega_t)}) \log(e + \|\nabla\operatorname{div}(\zeta\mathbf{U})\|_{L^p(\Omega_t)}) \\ &\quad + C_{x_0,T}\|\operatorname{div}\mathbf{U}\|_{L^2(\Omega_t)} \\ &\leq C_{x_0,T} + C_{x_0,T}(1 + \|F\|_{L^2(\Omega_t)}^{\frac{p-2}{2p-2}}\|\nabla F\|_{L^p(\Omega_t)}^{\frac{p}{2p-2}}) \log(e + \|\nabla\operatorname{div}(\zeta\mathbf{U})\|_{L^p(\Omega_t)}) \\ &\quad + C_{x_0,T}\|\operatorname{div}\mathbf{U}\|_{L^2(\Omega_t)} \\ &\leq C_{x_0,T} + C_{x_0,T}(1 + \|\dot{\mathbf{U}}\|_{L^p(\Omega_t)}^{\frac{p}{2p-2}}) \log(e + \|\nabla F\|_{L^p(\Omega_t)} + \|\nabla\rho\|_{L^p(\Omega_t)}) \\ &\leq C_{x_0,T}(1 + \|\operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)}^{\frac{p}{2p-2}}) \\ &\quad + C_{x_0,T}\|\operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)}^{\frac{p}{2p-2}} \log(e + \|\dot{\mathbf{U}}\|_{L^p(\Omega_t)} + \|\nabla\rho\|_{L^p(\Omega_t)}) \\ &\leq C_{x_0,T}(1 + \|\operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)}^{\frac{p}{2p-2}}) \\ &\quad + C_{x_0,T}\|\operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)}^{\frac{p}{2p-2}} \log(e + \|\operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)} + \|\nabla\rho\|_{L^p(\Omega_t)}) \\ &\leq C_{x_0,T}(1 + \|\operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)}^{\frac{p}{2p-2}}) \\ &\quad + C_{x_0,T}\|\operatorname{div}\dot{\mathbf{U}}\|_{L^2(\Omega_t)} \log(e + \|\nabla\rho\|_{L^p(\Omega_t)}),\end{aligned}\quad (2.46)$$

here we have used the fact:

$$\begin{aligned}\|\nabla\zeta\mathbf{U}\|_{L^\infty(\Omega_t)} &\leq C_{x_0,T}\|\operatorname{div}\mathbf{U}\|_{L^2(\Omega_t)} \leq C_{x_0,T}\left\|\frac{F + \rho^\gamma}{2\mu + \lambda(\rho)}\right\|_{L^2(\Omega_t)} \\ &\leq C_{x_0,T}(1 + \|F\|_{L^2(\Omega_t)}) \leq C_{x_0,T},\end{aligned}$$

and

$$\begin{aligned}\|(1-\zeta)\nabla\mathbf{U}\|_{L^\infty(\Omega_t)} &\leq C(1+\|(1-\zeta)F\|_{L^\infty(\Omega_t)}) \leq C_{x_0,T}(1+\|F\|_{H^1(\Omega_t)}) \\ &\leq C_{x_0,T}(1+\|F\|_{L^2(\Omega_t)}+\|\sqrt{\rho}\dot{\mathbf{U}}\|_{L^2(\Omega_t)}) \leq C_{x_0,T}.\end{aligned}$$

Substituting (2.45) and (2.46) into (2.44), we deduce from Gronwall's inequality that

$$\sup_{t\in[0,T]}\|\nabla\rho\|_{L^p(\Omega_t)}\leq C_{x_0,T}. \quad (2.47)$$

We obtain from (2.24), (2.37) and (2.47)

$$\begin{aligned}\|\nabla^2\mathbf{U}\|_{L^2(\Omega_t)} &\leq \|\nabla^2(\zeta\mathbf{U})\|_{L^2(\Omega_t)}+\|(1-\zeta)\nabla^2\mathbf{U}\|_{L^2(\Omega_t)}+\|\nabla\zeta\nabla\mathbf{U}\|_{L^2(\Omega_t)} \\ &\leq \|\nabla\operatorname{div}(\zeta\mathbf{U})\|_{L^2(\Omega_t)}+C_{x_0,T}(1+\|(1-\zeta)\nabla F\|_{L^2(\Omega_t)} \\ &\quad +\|(1-\zeta)\nabla\rho\|_{L^2(\Omega_t)}+\|F\|_{L^2(\Omega_t)}) \\ &\leq \|\zeta\nabla\operatorname{div}\mathbf{U}\|_{L^2(\Omega_t)}+\|\nabla(\mathbf{U}\cdot\nabla\zeta)\|_{L^2(\Omega_t)}+C_{x_0,T} \\ &\leq \frac{1}{2\mu+\lambda(\rho)}(\rho\dot{\mathbf{U}}+\nabla P(\rho)-\nabla\lambda(\rho)\operatorname{div}\mathbf{U})\|_{L^2(\Omega_t)}+C_{x_0,T}\leq C_{x_0,T},\end{aligned} \quad (2.48)$$

where we have used the fact

$$|(1-\zeta)\nabla^2\mathbf{U}|\leq C_{x_0,T}(1+|(1-\zeta)\nabla F|+|(1-\zeta)\nabla\rho|+|(1-\zeta)F|+|(1-\zeta)\nabla\rho\operatorname{div}\mathbf{U}|).$$

From (2.7), (2.46)–(2.48), Lemma 2.7 and Lemma 2.8, we can immediately obtain (2.43).  $\square$

### 3. The proof of main results

**Proof of Theorem 1.1.** The readers may refer to [11,12] for the construction of approximate solutions to the free boundary value problem (1.2)–(1.5). We here only give some basic steps.

For any  $\varepsilon > 0$ , we consider the approximate system (2.23) in Lagrangian coordinate with initial data  $(\rho_0, u_0)$  replaced by  $(\rho_0^\varepsilon, u_0^\varepsilon)$ , where

$$\rho_0^\varepsilon = \frac{\rho_0 + \varepsilon}{\int_0^{a_0} (\rho_0 + \varepsilon) r dr}, \quad u_0^\varepsilon = \frac{m_0}{\rho_0^\varepsilon}.$$

We construct the approximate system as follows,

$$\begin{cases} \rho_\tau^{(n+1)} + (\rho^{(n+1)})^2 (r^{(n)} u^{(n)})_x = 0, \\ \frac{1}{r^{(n)}} u_\tau^{(n+1)} + ((\rho^{(n)})^\gamma - (2\mu + \lambda(\rho^{(n)}))\rho^{(n)} (r^{(n)} u^{(n+1)})_x)_x = 0, \end{cases}$$

for  $(x, \tau) \in [0, 1] \times [0, T]$ , with initial data and boundary conditions given by

$$(\rho^{(n+1)}, u^{(n+1)})(x, 0) = (\rho_0^\varepsilon, u_0^\varepsilon), \quad x \in [0, 1],$$

$$u^{(n+1)}(0, \tau) = 0, \quad \left( (\rho^{(n)})^\gamma - (2\mu + \lambda(\rho^{(n)}))\rho^{(n)}(r^{(n)}u^{(n+1)})_x \right)(1, \tau) = 0, \quad \tau \in [0, T],$$

where  $r^{(n)} = r^{(n)}(x, \tau)$  is defined by

$$\frac{d}{d\tau} r^{(n)}(x, \tau) = u^{(n)}(x, \tau), \quad x \in [0, 1], \quad \tau \in [0, T].$$

For this problem we can apply standard argument (the energy estimates and the contraction mapping theorem) to obtain the existence of a unique global strong solution  $(\rho^\varepsilon, u^\varepsilon)$  of the free boundary value problem (1.2)–(1.5). Then by standard energy estimate [Lemma 2.3](#), higher order estimate [Lemmas 2.7–2.9](#), we can obtain the global existence of strong solution.

**Proof of Theorem 1.2.** To prove (1.11)–(1.12), we define the following energy functional

$$H(t) = \int_0^{a(t)} (r - (1+t)u)^2 \rho r dr + \frac{2}{\gamma-1} (1+t)^2 \int_0^{a(t)} \rho^\gamma r dr. \quad (3.1)$$

A direct computation implies

$$\begin{aligned} H'(t) &= \frac{8-4\gamma}{\gamma-1} (1+t) \int_0^{a(t)} \rho^\gamma r dr + 4(1+t) \int_0^{a(t)} (2\mu + \lambda(\rho))(ru)_r dr \\ &\quad - 2(1+t)^2 \int_0^{a(t)} (2\mu + \lambda(\rho)) |(ru)_r|^2 \frac{1}{r} dr \\ &\leq \frac{8-4\gamma}{\gamma-1} (1+t) \int_0^{a(t)} \rho^\gamma r dr + 2 \int_0^{a(t)} (2\mu + \lambda(\rho)) r dr \\ &\leq \frac{8-4\gamma}{\gamma-1} (1+t) \int_0^{a(t)} \rho^\gamma r dr + 2\mu a(t)^2 + C. \end{aligned} \quad (3.2)$$

Therefore, we deduce from (3.2) that for  $\gamma \geq 2$

$$H'(t) \leq 2\mu a(t)^2 + C.$$

Integrating the above inequality from 0 to  $t$  and using (3.1), we have

$$\int_0^{a(t)} \rho^\gamma r dr \leq C(1+t)^{-2} \left( (1+t) + \int_0^t a^2(s) ds \right), \quad \gamma \geq 2. \quad (3.3)$$

In the case  $1 < \gamma < 2$ ,  $\gamma \neq \frac{3}{2}$ ,

$$H'(t) \leq \frac{4-2\gamma}{1+t} H(t) + 2\mu a(t)^2 + C.$$

Multiplying the above inequality by  $(1+t)^{2\gamma-4}$  and integrating the resulting inequality with respect to time  $t$ , a direct computation gives

$$\int_0^{a(t)} \rho^\gamma r dr \leq C(1+t)^{2-2\gamma} \left( 1 + \int_0^t (1+s)^{2\gamma-4} a^2(s) ds \right), \quad 1 < \gamma < 2, \gamma \neq \frac{3}{2}. \quad (3.4)$$

For  $\gamma = \frac{3}{2}$ ,

$$\int_0^{a(t)} \rho^\gamma r dr \leq C(1+t)^{-1} \left( \log(1+t) + \int_0^t (1+s)^{-1} a^2(s) ds \right), \quad \gamma = \frac{3}{2}. \quad (3.5)$$

Note that

$$1 = \int_0^{a_0} \rho_0 r dr = \int_0^{a(t)} \rho r dr \leq C a(t)^{\frac{2(\gamma-1)}{\gamma}} \left( \int_0^{a(t)} \rho^\gamma r dr \right)^{\frac{1}{\gamma}}. \quad (3.6)$$

Combining (3.2) with (3.6) implies that

$$a(t)^{\frac{2(\gamma-1)}{\gamma}} \left( (1+t)^{\frac{1}{\gamma}} + \left( \int_0^t a^2(s) ds \right)^{\frac{1}{\gamma}} \right) \geq C(1+t)^{\frac{2}{\gamma}},$$

which leads to

$$a_M(t) = \sup_{s \in [0, t]} a(s) \geq C(1+t)^{\frac{1}{2\gamma}}, \quad \gamma \geq 2. \quad (3.7)$$

Similarly, combining (3.4) with (3.6), (3.5) with (3.6), we can obtain

$$a_M(t) \geq \begin{cases} C(1+t)^{\frac{\gamma-1}{\gamma}}, & \gamma \in (1, \frac{3}{2}), \\ C(1+t)^{\frac{1}{2\gamma}} [1 + \ln(1+t)]^{-\frac{1}{2\gamma}}, & \gamma = \frac{3}{2}, \\ C(1+t)^{\frac{1}{2\gamma}}, & \gamma > \frac{3}{2}. \end{cases} \quad (3.8)$$

The proof of (1.11) is completed.

Now, we are in a position to prove (1.12). Applying a similar argument as to (3.2), we have

$$\begin{aligned}
H'(t) &= \frac{8-4\gamma}{\gamma-1}(1+t) \int_0^{a(t)} \rho^\gamma r dr + 4(1+t) \int_0^{a(t)} (2\mu + \lambda(\rho))(ru)_r dr \\
&\quad - 2(1+t)^2 \int_0^{a(t)} (2\mu + \lambda(\rho))|(ru)_r|^2 \frac{1}{r} dr \\
&\leq \frac{8-4\gamma}{\gamma-1}(1+t) \int_0^{a(t)} \rho^\gamma r dr + 2 \int_0^{a(t)} \lambda(\rho) r dr + 8\mu(1+t) \int_0^{a(t)} (ru)_r dr \\
&\leq \frac{8-4\gamma}{\gamma-1}(1+t) \int_0^{a(t)} \rho^\gamma r dr + 4\mu(1+t) \frac{d}{dt} a^2(t) + C,
\end{aligned} \tag{3.9}$$

where we have used  $1 < \beta \leq \gamma$  and  $a'(t) = u(a(t), t)$ . If  $\gamma \geq 2$ ,

$$\begin{aligned}
H'(t) &\leq 4\mu(1+t) \frac{d}{dt} a^2(t) + C = 4\mu \frac{d}{dt} ((1+t)a^2(t)) - 4\mu a^2(t) + C \\
&\leq 4\mu \frac{d}{dt} ((1+t)a^2(t)) + C.
\end{aligned}$$

Integrating the above inequality with respect to  $t$ , we have

$$\int_0^{a(t)} \rho^\gamma r dr \leq C(1+t)^{-1} a^2(t). \tag{3.10}$$

Combining (3.10) with (3.6) gives that

$$a(t) \geq C(1+t)^{\frac{1}{2\gamma}}, \quad \gamma \geq 2. \tag{3.11}$$

For  $\gamma < 2$ ,

$$H'(t) \leq \frac{4-2\gamma}{1+t} H(t) + 4\mu(1+t) \frac{d}{dt} a^2(t) + C. \tag{3.12}$$

In the case  $\gamma = \frac{3}{2}$ , the inequality (3.12) can be rewritten as

$$\left(\frac{H(t)}{1+t}\right)' \leq 4\mu \frac{d}{dt} a^2(t) + \frac{C}{1+t}. \tag{3.13}$$

Integrating the above inequality with respect to  $t$ , and using the definition of  $H(t)$ , we have

$$\int_0^{a(t)} \rho^\gamma r dr \leq C a^2(t) (1+t)^{-1} (1 + \ln(1+t)), \tag{3.14}$$

which combined with (3.6) leads to

$$a(t) \geq C(1+t)^{\frac{1}{2\gamma}} [1 + \ln(1+t)]^{-\frac{1}{2\gamma}}, \quad \gamma = \frac{3}{2}. \quad (3.15)$$

If  $\frac{3}{2} < \gamma < 2$ , the inequality (3.12) can be rewritten as

$$\begin{aligned} (H(t)(1+t)^{2\gamma-4})' &\leq 4\mu(1+t)^{2\gamma-3} \frac{d}{dt} a^2(t) + C(1+t)^{2\gamma-4} \\ &\leq 4\mu \frac{d}{dt} \left( (1+t)^{2\gamma-3} a^2(t) \right) - 4\mu(2\gamma-3)a^2(t) + C(1+t)^{2\gamma-4} \\ &\leq 4\mu \frac{d}{dt} \left( (1+t)^{2\gamma-3} a^2(t) \right) + C(1+t)^{2\gamma-4}. \end{aligned} \quad (3.16)$$

Integrating the above inequality with respect to time  $t$ , we obtain

$$\int_0^{a(t)} \rho^\gamma r dr \leq C a^2(t) (1+t)^{-1} + (1+t)^{2-2\gamma} \leq C a^2(t) (1+t)^{-1}, \quad (3.17)$$

which with (3.6) implies

$$a(t) \geq C(1+t)^{\frac{1}{2\gamma}}, \quad \frac{3}{2} < \gamma < 2. \quad (3.18)$$

Combining (3.11), (3.15) and (3.18), we can obtain (1.12). The proof of Theorem 1.1 is completed.  $\square$

## Acknowledgments

The authors would like to thank Professor Zhouping Xin for many helpful suggestions and discussions.

## References

- [1] D. Bresch, B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, *Comm. Math. Phys.* 238 (2003) 211–223.
- [2] D. Bresch, B. Desjardins, On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier–Stokes models, *J. Math. Pures Appl.* 86 (2006) 362–368.
- [3] D. Bresch, B. Desjardins, D. Gérard-Varet, On compressible Navier–Stokes equations with density dependent viscosities in bounded domains, *J. Math. Pures Appl.* 87 (2007) 227–235.
- [4] D. Bresch, B. Desjardins, C.-K. Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems, *Comm. Partial Differential Equations* 28 (2003) 843–868.
- [5] G.-Q. Chen, M. Kratka, Global solutions to the Navier–Stokes equations for compressible heat-conducting flow with symmetry and free boundary, *Comm. Partial Differential Equations* 27 (2002) 907–943.
- [6] P. Chen, T. Zhang, A vacuum problem for multidimensional compressible Navier–Stokes equations with degenerate viscosity coefficients, *Commun. Pure Appl. Anal.* 7 (2008) 987–1016.
- [7] R. Danchin, Global existence in critical spaces for compressible Navier–Stokes equations, *Invent. Math.* 141 (2000) 579–614.
- [8] E. Feireisl, *Dynamic of Viscous Compressible Flow*, Oxford University Press, 2004.

- [9] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier–Stokes equations of isentropic compressible fluids, *J. Math. Fluid Mech.* 3 (2001) 358–392.
- [10] J.F. Gerbeau, B. Perthame, Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation, *Discrete Contin. Dyn. Syst. Ser. B* 1 (2001) 89–102.
- [11] Z.-H. Guo, Q.-S. Jiu, Z. Xin, Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, *SIAM J. Math. Anal.* 39 (2008) 1402–1427.
- [12] Z.-H. Guo, H.-L. Li, Z. Xin, Lagrange structure and dynamics for spherically symmetric compressible Navier–Stokes equations, *Comm. Math. Phys.* 309 (2012) 371–412.
- [13] D. Hoff, Global existence of the Navier–Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations* 120 (1995) 215–254.
- [14] D. Hoff, D. Serre, The failure of continuous dependence on initial data for the Navier–Stokes equations of compressible flow, *SIAM J. Appl. Math.* 51 (1991) 887–898.
- [15] X.D. Huang, J. Li, Z. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations, *Comm. Pure Appl. Math.* 65 (4) (2012) 549–585.
- [16] X.D. Huang, J. Li, Existence and blowup behavior of global strong solutions to the two-dimensional barotropic compressible Navier–Stokes system with vacuum and large initial data, preprint, arXiv:1205.5342.
- [17] X.D. Huang, J. Li, Global well-posedness of classical solutions to the Cauchy problem of two-dimensional barotropic compressible Navier–Stokes system with vacuum and large initial data, preprint, arXiv:1207.3746.
- [18] S. Jiang, Z.P. Xin, P. Zhang, Global weak solutions to 1D compressible isentropic Navier–Stokes with density-dependent viscosity, *Methods Appl. Anal.* 12 (2005) 239–251.
- [19] S. Jiang, P. Zhang, Global spherically symmetric solutions of the compressible isentropic Navier–Stokes equations, *Comm. Math. Phys.* 215 (2001) 559–581.
- [20] Q.S. Jiu, Y. Wang, Z.P. Xin, Global well-posedness of the Cauchy problem of two-dimensional compressible Navier–Stokes equations in weighted spaces, *J. Differential Equations* 255 (2013) 351–404.
- [21] A.V. Kazhikhov, V.V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.* 41 (2) (1977) 273–282.
- [22] H.-L. Li, J. Li, Z. Xin, Vanishing of vacuum states and blow-up phenomena of the compressible Navier–Stokes equations, *Comm. Math. Phys.* 281 (2008) 401–444.
- [23] J. Li, Z. Xin, Global existence of weak solutions to the barotropic compressible Navier–Stokes flows with degenerate viscosities, <http://arxiv.org/abs/1504.06826>.
- [24] R. Lian, Z. Guo, H.-L. Li, Dynamical behavior of vacuum states for 1D compressible Navier–Stokes equations, *J. Differential Equations* 248 (2010) 1915–1930.
- [25] P.L. Lions, *Mathematical Topics in Fluid Dynamics 2, Compressible Models*, Oxford Science Publication, Oxford, 1998.
- [26] T. Liu, J. Smoller, On the vacuum state for the isentropic gas dynamics equations, *Adv. in Appl. Math.* 4 (1) (1980) 345–359.
- [27] T. Luo, Z. Xin, T. Yang, Interface behavior of compressible Navier–Stokes equations with vacuum, *SIAM J. Math. Anal.* 31 (2000) 1175–1191.
- [28] F. Marche, Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects, *Eur. J. Mech. B Fluids* 26 (2007) 49–63.
- [29] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* 20 (1980) 67–104.
- [30] A. Mellet, A. Vasseur, On the isentropic compressible Navier–Stokes equation, *Comm. Partial Differential Equations* 32 (3) (2007) 431–452.
- [31] T. Nishida, Equations of fluid dynamics-free surface problems, *Comm. Pure Appl. Math.* 39 (1986) 221–238.
- [32] M. Okada, S. Matuso-Necasova, T. Makino, Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity, *Ann. Univ. Ferrara Sez. VII (N.S.)* 48 (2002) 1–20.
- [33] M. Perepelitsa, On the global existence of weak solutions for the Navier–Stokes equations of compressible fluid flows, *SIAM J. Math. Anal.* 38 (4) (2006) 1126–1153.
- [34] J. Pedlosky, *Geophysical Fluid Dynamics*, 2nd edition, Springer, 1992.
- [35] R. Salvi, I. Straškraba, Global existence for viscous compressible fluids and their behavior as  $t \rightarrow \infty$ , *J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math.* 40 (1993) 17–51.
- [36] J. Serrin, Mathematical principles of classical fluid mechanics, in: *Handbuch der Physik*, vol. 8/1, Springer-Verlag, 1959, pp. 125–263.

- [37] V.A. Solonnikov, A. Tani, Evolution free boundary problem for equations of motion of viscous compressible barotropic liquid, in: *The Navier–Stokes Equations II—Theory and Numerical Methods*, Oberwolfach, 1991, in: *Lecture Notes in Math.*, vol. 1530, Springer, Berlin, 1992, pp. 30–55.
- [38] V.A. Vaigant, A.V. Kazhikhov, On the existence of global solutions of two-dimensional Navier–Stokes equations of a compressible viscous fluid, *Sibirsk. Mat. Zh.* 36 (6) (1995) 1283–1316 (in Russian); translation in *Sib. Math. J.* 36 (6) (1995) 1108–1141.
- [39] A. Vasseur, C. Yu, Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations, <http://arxiv.org/abs/1501.06803>.
- [40] W. Wang, C.J. Xu, The Cauchy problem for viscous shallow water equations, *Rev. Mat. Iberoam.* 21 (3) (2005) 1–24.
- [41] Z.P. Xin, Blow-up of smooth solution to the compressible Navier–Stokes equations with compact density, *Comm. Pure Appl. Math.* 51 (1998) 229–240.
- [42] T. Yang, Z.A. Yao, C.J. Zhu, Compressible Navier–Stokes equations with density-dependent viscosity and vacuum, *Comm. Partial Differential Equations* 26 (2001) 965–981.
- [43] T. Yang, C.J. Zhu, Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum, *Comm. Math. Phys.* 230 (2002) 329–363.
- [44] W.M. Zajączkowski, On nonstationary motion of a compressible barotropic viscous fluid bounded by a free surface, *Dissertationes Math.* 324 (1993), 101 pp.
- [45] W.M. Zajączkowski, On nonstationary motion of a compressible barotropic viscous capillary fluid bounded by a free surface, *SIAM J. Math. Anal.* 25 (1) (1994) 1–84.
- [46] T. Zhang, D. Fang, Global behavior of spherically symmetric Navier–Stokes–Poisson system with degenerate viscosity coefficients, *Arch. Ration. Mech. Anal.* 191 (2) (2009) 195–243.
- [47] C.-J. Zhu, Asymptotic behavior of compressible Navier–Stokes equations with density-dependent viscosity and vacuum, *Comm. Math. Phys.* 293 (1) (2010) 279–299.