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On the existence of local strong solutions to chemotaxis–shallow water system with large data and vacuum

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Abstract

In this paper, motivated by the chemotaxis–Navier–Stokes system arising from mathematical biology [43], a modified shallow water type chemotactic model is derived. For large initial data allowing vacuum, the local existence of strong solutions together with the blow-up criterion is established.

MSC: primary 35Q35, 35Q92, 76N10, 92C17; secondary 35M10, 35Q30

Keywords: Strong solution; Chemotaxis–fluid interaction; Shallow water system; Vacuum; A priori estimate

1. Introduction and main results

A mathematical model was proposed in [43] for bacteria cells living in a viscous fluid, where the process is under the influence of convective fluid transportation, the gravitational force and the

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chemotactic movement driven by biological signals. In this paper, we consider the chemotaxis–shallow water system

$$\begin{cases} n_t + \mathrm{div}(n\boldsymbol{u}) = D_n \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t + \mathrm{div}(c\boldsymbol{u}) = D_c \Delta c - nf(c), \\ h_t + \mathrm{div}(h\boldsymbol{u}) = 0, \\ h\boldsymbol{u}_t + h\boldsymbol{u}\cdot\nabla\boldsymbol{u} + h^2\nabla n + \dfrac{1}{2}(1+n)\nabla h^2 = \mu\Delta\boldsymbol{u} + (\mu+\lambda)\nabla(\mathrm{div}\boldsymbol{u}), \end{cases} \tag{1.1}$$

which is derived from the chemotaxis–Navier–Stokes equations in [43]. Here, the unknowns are n, c, h, $\boldsymbol{u}$ presenting bacterial density, substrate concentration, the fluid height and the fluid velocity field, respectively. $\Omega \subset \mathbb{R}^2$ is the physical domain where the cells and fluid move and interact. Constants D_n and D_c are the corresponding diffusion coefficients for the cells and substrate. The chemotactic sensitivity $\chi(c)$ and the consumption rate of the substrate by the cells $f(c)$ are supposed to be given smooth functions. The constants μ and λ are the shear viscosity and the bulk viscosity coefficients respectively with the following physical restrictions:

$$\mu > 0, \quad \mu + \lambda \geq 0.$$

Before getting into details on the derivation and mathematical analysis of (1.1), related mathematical results on chemotactic models in biomathematics and shallow water system in fluid dynamics will be outlined.

Chemotaxis is a well-known biological phenomenon describing the collective motion of cells or the evolution of density of bacteria driven by chemicals, such as cell migration, formation of organs, cancer progress (and etc.). In the last few decades, scientists developed mathematical models for chemotaxis, among which the best-studied one is the Patlak–Keller–Segel system [27,28,38]. After the first existence and blow-up results in [25], mathematical analysis on chemotactic models attracted many mathematicians to work in this field. The reason why this type of system is interesting is that it induced two different mechanisms, namely diffusion and aggregation. A large series of results have been obtained for a phenomenon called chemotactic collapse, which was originally conjectured in [11,37], i.e., there exists a threshold of critical mass for global existence and finite-time blow-up. One may refer to [5,21,39,41,44] for more details. For multi-dimensional Patlak–Keller–Segel system with degenerate diffusion, the threshold was established in [1,7,8,24,45]. For the parabolic–parabolic Keller–Segel model and kinetic models for chemotaxis, interested readers can refer to [2,6,10] and the references therein.

The evolution of an incompressible fluid in three space dimensions in response to gravitational and rotational accelerations can be simulated by the non-linear shallow water equations. The solutions were studied in [20,40,42] with the initial data close to a constant equilibrium state away from vacuum. The local solutions for general initial data and global solutions for small initial data in various spaces are achieved in [14,46]. For arbitrary large initial data and the case that the height of fluid surface may vanish, the global weak entropy solution was obtained in [3,4,19,29]. Later on, for initial data allowing vacuum, the local existence of classical solution was obtained in [15], and the case of the degenerate viscosity was treated in [30].

The shallow water system is also regarded as an important extension of the two dimensional isentropic compressible Navier–Stokes equations with rotating force. There are numerous literatures on the existence and behavior of solutions to compressible Navier–Stokes equations with constant viscosity. For initial data close to a non-vacuum equilibrium, the existence of classical solutions is known in [36]. For arbitrary data, under suitable compatibility condition, a local theory was established successfully, see [12,13,35]. The major breakthrough is due to Lions [32] where global existence theory of global-in-time weak solutions is achieved, see also the generalizations in [17,26]. However, little is known on the structure of such weak solutions, in particular, the regularity and the uniqueness of such weak solutions remain open. Recently, for the case that the initial density is allowed to vanish and even has compact support, the quite surprising global existence and uniqueness of classical solutions are established; one may refer to [23,31] and the references therein for more details.

Recently, a coupled system of chemotaxis and viscous incompressible fluid proposed in [43] has been investigated by many mathematicians. In [34], local-in-time weak solutions were constructed. In [9,16], some existence results and blowup criteria of classical solutions to Cauchy problem were obtained. In [33], global existence of a weak solution was obtained in two dimensions, see also [18]. For initial boundary value problem, the global existence of two dimensional regular solutions and three dimensional weak solution were obtained in [47,48].

Motivated by the experiment report [43] and the fact that the surface of the fluid is a free boundary, we propose the following two hypotheses for a modified model. Firstly, the cells and substrate both stay at the surface of the fluid. Secondly, the vertical acceleration of the fluid can be neglected comparing to the horizontal scales of the fluid. Both of the assumptions are based on the observation in [43], and the rationality of them will be discussed in the next section. With the aid of these two hypotheses, the idea of deriving shallow water system from three-dimensional incompressible Navier–Stokes equations will be implemented in order to obtain (1.1) in section 2. Therefore, compared to the model that has been studied in [9,47], the shallow water type chemotactic model has its own advantages since it keeps the essential ingredients of the three dimensional fluid mechanics with free surface. The derivation of this modified system stems from physical backgrounds: conservation laws and diffusion mechanism.

For simplicity, let

$$D_n = D_c = 1, \quad \chi(c) \equiv 1, \quad f(c) = c,$$

and the results obtained in the current paper can be easily modified for general χ and f, as the choices in [47,48]. As usual, the system will be studied with initial conditions

$$(n, c, h, \boldsymbol{u})(x, 0) = (n_0, c_0, h_0, \boldsymbol{u}_0)(x), \tag{1.2}$$

where $n_0 \geq 0$, $c_0 \geq 0$, $h_0 \geq 0$ satisfying the compatibility conditions

$$-\mu \Delta \boldsymbol{u}_0 - (\mu + \lambda)\nabla(\operatorname{div} \boldsymbol{u}_0) + h_0^2 \nabla n_0 + \frac{1}{2}(1 + n_0)\nabla h_0^2 = \sqrt{h_0}\, g \tag{1.3}$$

for some $g \in L^2(\Omega)$. We look for solutions in the following two cases for (1.1):

Cauchy problem: $\Omega = \mathbb{R}^2$, and

$$(n, c, \boldsymbol{u})(x, t) \to (0, 0, \boldsymbol{0}), \ \ h(x, t) \to \tilde{h} > 0, \ \text{ as } |x| \to \infty, \tag{1.4}$$

where $\tilde{h}$ is a positive constant;

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initial boundary value problem: Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, and

$$(\frac{\partial n}{\partial \nu}, \frac{\partial c}{\partial \nu}, \boldsymbol{u}) = (0, 0, \mathbf{0}) \quad \text{on } (0, T) \times \partial\Omega, \tag{1.5}$$

where ν is the outer unit normal to the boundary.

For $1 < r < \infty$, we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{cases} L^r = L^r(\Omega), \quad D^{k,r} = D^{k,r}(\Omega) = \left\{\boldsymbol{u} \in L^1_{loc}(\Omega) \,\middle|\, \nabla^k \boldsymbol{u} \in L^r(\Omega)\right\}, \quad \|\boldsymbol{u}\|_{D^{k,r}} := \left\|\nabla^k \boldsymbol{u}\right\|_{L^r}, \\ W^{k,r} = W^{k,r}(\Omega), \quad H^k = W^{k,2}, \quad D^k = D^{k,2}, D_0^k = \left\{\boldsymbol{u} \in D^k; \,\middle|\, \text{(1.4) or (1.5) holds}\right\}, \\ H_0^k = L^2 \cap D_0^k, \quad \int f \,\mathrm{d}x = \int_\Omega f \,\mathrm{d}x. \end{cases}$$

Moreover, the material derivative is denoted by

$$\dot{f} := f_t + \boldsymbol{u} \cdot \nabla f.$$

The aim of this paper is to establish strong solutions without restriction on the smallness of the initial data, which allows the vacuum for both the density of the bacteria cells and the fluid, i.e., the concentration of bacteria cells and the height of the fluid are allowed to vanish if necessary. Furthermore, a blow-up criterion is obtained.

Theorem 1. *Assume that the initial data (1.2) satisfy that n_0, c_0, h_0 are nonnegative and*

$$n_0 \in H_0^2, \quad c_0 \in H_0^3, \quad \boldsymbol{u}_0 \in D_0^1 \cap D^2, \quad h_0 - \tilde{h} \in L^1 \cap W^{1,q}, \tag{1.6}$$

for some $q > 2$. Furthermore, the compatibility conditions (1.3) hold. Then there exist a $T^* > 0$ and a unique strong solution $(n, c, h, \boldsymbol{u})$ to the initial boundary value problem (1.1)–(1.2) together with (1.4) or (1.5) such that

$$\begin{cases} n \in C([0, T^*]; H_0^2), \quad c \in C([0, T^*]; H_0^3), \\ n_t \in L^\infty(0, T^*; L^2) \cap L^2(0, T^*; H^1), \quad c_t \in L^\infty(0, T^*; H^1) \cap L^2(0, T^*; H^2), \\ (h - \tilde{h}) \in C([0, T^*]; W^{1,q}), \quad h_t \in C(0, T^*; L^q), \\ \boldsymbol{u} \in C([0, T^*]; D_0^1 \cap D^2) \cap L^2(0, T^*; D^{2,q}), \\ \boldsymbol{u}_t \in L^\infty(0, T^*; L^2) \cap L^2(0, T^*; D^1), \quad \sqrt{h}\boldsymbol{u}_t \in L^\infty(0, T^*; L^2), \end{cases} \tag{1.7}$$

where $\tilde{h} \equiv 0$ in the case of Ω being a bounded domain in $\mathbb{R}^2$. Furthermore, the following blow-up criterion holds: if $\widetilde{T}$ is the maximal time of existence of the strong solution $(n, c, h, \boldsymbol{u})$ and $\widetilde{T} < +\infty$ then

$$\lim_{T \to \widetilde{T}} \left(\int_0^T \|\mathcal{D}(\boldsymbol{u})\|_{L^\infty} \mathrm{d}t + \sup_{0 \le t \le T} \|n\|_{L^\infty} \right) = \infty, \tag{1.8}$$

where $\mathcal{D}(\boldsymbol{u})$ is the deformation tensor with $\mathcal{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t)$.

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Remark 1. For chemotaxis–Navier–Stokes system, some known results concerning the existence of local-in-time solutions and blow-up criteria were obtained in the case of positive bacterial density, or in the case of absence of fluid, or to Cauchy problem. In this paper, we consider the chemotaxis–shallow water system containing vacuum states, that is, the bacterial density and the fluid surface height are allowed to vanish.

Remark 2. The higher order estimates of the unique strong solution obtained in Theorem 1 can be easily obtained if $n_0 \in H_0^3$, $c_0 \in H_0^3$, $\boldsymbol{u}_0 \in D_0^1 \cap D^3$, $h_0 - \tilde{h} \in H^3$, and thus the unique strong solution becomes a classical one for positive time, please refer to [12,15] for details. Our result for the chemotaxis–shallow water system (1.1) is the very first step by using a chemotaxis–compressible fluid model to investigate the dynamics of swimming bacteria. Based on the blow-up criterion (1.8), the next expected work is on the global existence of classical solutions.

Remark 3. Notice that the chemotaxis term in the equation for n $(1.1)_1$ leads to a cross-diffusion term $n\chi(c)\Delta c$, thus it is reasonable to have higher regularity of c than that the regularity of n in (1.7).

Remark 4. We remark here that our model is a chemotaxis–compressible fluid model, and thus the fluid velocity field is not divergence free. As a consequence, some estimates in the existence theory for the chemotaxis–incompressible fluid model cannot be applied here. Moreover, compared with the compressible fluid models, there are strong non-linear terms $(1+n)\nabla h^2$ and $h^2\nabla n$ in the conservation of momentum, which are new ingredients in the model of chemotaxis–fluid. One of the main technical difficulties in this paper is to deal with the two terms for arbitrary large initial data allowing vacuum.

The rest of this paper is organized as follows. In section 2, two general hypotheses are stated based on the experimental observations in [43] and then a chemotaxis–shallow water model is derived coupling with the chemotaxis equations and the viscous incompressible fluid. In section 3, the existence and uniqueness of a local strong solution are proved based on the corresponding *a priori* estimates obtained for the linearized system. Finally, section 4 is dedicated to derive a blow-up criterion.

2. Derivation of chemotaxis–shallow water system

2.1. Background and hypotheses

Idan Tuval and collaborators reported a detailed experimental and theoretical study of an interesting mechanism called the chemotactic Boycott effect in [43], where the following chemotaxis–Navier–Stokes system was proposed

$$\begin{cases} n_t + \boldsymbol{U}\cdot\nabla n = D_n\Delta n - \nabla\cdot(n\chi(c)\nabla c), \\ c_t + \boldsymbol{U}\cdot\nabla c = D_c\Delta c - nf(c), \\ \mathrm{div}\boldsymbol{U} = 0, \\ \rho(\boldsymbol{U}_t + \boldsymbol{U}\cdot\nabla\boldsymbol{U}) + \nabla p = \Delta\boldsymbol{U} - n\nabla\phi, \end{cases}$$

where ρ is the constant density of the incompressible fluid, $\boldsymbol{U}$ is the vector field in three dimension, p is the pressure and ϕ represents the gravitational potential. Recalling the original experiment in [43], we notice the following facts. Very tiny amount of cells is slowly dropped into the steady fluid by a syringe. Compared to the diffusion effect, the advection in the Boycott-like flows dominates the mechanism. Therefore, the buoyant force is significant until the cells move to the surface of the fluid. As a consequence, after the bacteria cells have been injected into the fluid, they move vertically to the surface of the fluid in a short time. Afterwards, they stay on the surface. Furthermore, the horizontal scales of the fluid are much larger than the vertical scale. Based on the above concerns and mathematical interests we propose two fundamental hypotheses for the modified model, motivated by the experiment report [43] and the fact that the surface of the fluid is a free boundary, namely,

1. the cells and substrate both stay at the surface of the fluid, which was observed in the experiment;
2. the vertical acceleration of the fluid can be neglected comparing to the horizontal scales of the fluid.

2.2. Formulation

This subsection is devoted to derive chemotaxis–shallow water system (1.1). Denote $h = h(t, x, y)$ the height of the flow at time t and position $(x, y) \in \Omega$. Under the first hypothesis, the cell density and substrate concentration are independent of the vertical variable z, i.e., $n = n(t, x, y)$ and $c = c(t, x, y)$. Moreover, from the second hypothesis we have the following equations for the velocity field $\boldsymbol{U} = (u, v, w)(t, x, y, z)$, for $(x, y) \in \Omega$ and $z \in (0, h)$, where (u, v) and w are the horizontal velocity and the vertical velocity respectively.

$$\begin{aligned} &\frac{\mathrm{d}w}{\mathrm{d}t} := w_t + uw_x + vw_y + ww_z = 0, \\ &\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = 0, \\ &u(t, x, y, z) = u(t, x, y), v(t, x, y, z) = v(t, x, y). \end{aligned} \tag{2.1}$$

We first derive the equation for $n(t, x, y)$. Let $D \subset \Omega$ be any bounded domain with smooth boundary, $\boldsymbol{u}(t, \mathrm{x}) = (u, v)(t, \mathrm{x})$ with $\mathrm{x} := (x, y) \in D$ and ν be the outer unit normal to ∂D. The change of total cells in D from time t_1 to t_2 $(t_1 < t_2)$ comes from the transportation flux $n\boldsymbol{u}$, the diffusion flux $J_n = -D_n \nabla n$ and the chemotactic flux $n\chi(c)\nabla c$ across the boundary ∂D. The mass conservation of the bacteria cells in D is

$$\begin{aligned} &\int_D n(t_2, \mathrm{x}) - n(t_1, \mathrm{x}) \mathrm{d}\mathrm{x} \\ &= -\int_{t_1}^{t_2} \int_{\partial D} n\boldsymbol{u} \cdot \nu \mathrm{d}S \mathrm{d}t - \int_{t_1}^{t_2} \int_{\partial D} J_n \cdot \nu \mathrm{d}S \mathrm{d}t - \int_{t_1}^{t_2} \int_{\partial D} (n\chi(c)\nabla c) \cdot \nu \mathrm{d}S \mathrm{d}t \\ &= -\int_{t_1}^{t_2} \int_D \operatorname{div}(n\boldsymbol{u}) \mathrm{d}\mathrm{x}\mathrm{d}t + \int_{t_1}^{t_2} \int_D D_n \Delta n \mathrm{d}\mathrm{x}\mathrm{d}t - \int_{t_1}^{t_2} \int_D \nabla \cdot (n\chi(c)\nabla c) \mathrm{d}\mathrm{x}\mathrm{d}t. \end{aligned}$$

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It leads to the differential equation for n,

$$n_t + \operatorname{div}(n\boldsymbol{u}) = D_n \Delta n - \nabla \cdot (n\chi(c)\nabla c).$$

As to the substrate distribution $c(t, x, y)$, one can follow the same argument to arrive at

$$c_t + \operatorname{div}(c\boldsymbol{u}) = D_c \Delta c - nf(c).$$

Next, the differential equation for the height of the fluid $h(t, \mathrm{x})$ will be derived. Suppose $z \in [0, h(t, \mathrm{x})]$, thus

$$\begin{aligned} 0 &= \int_0^{h(t,\mathrm{x})} \operatorname{div}\boldsymbol{U} \mathrm{d}z \\ &= \frac{\partial}{\partial x} \int_0^{h(t,\mathrm{x})} u \mathrm{d}z + \frac{\partial}{\partial y} \int_0^{h(t,\mathrm{x})} v \mathrm{d}z + \big(- u h_x(t, \mathrm{x}) - v h_y(t, \mathrm{x}) + w \big)|_{z=h(t,\mathrm{x})} - w|_{z=0}. \end{aligned}$$

Note that $w|_{z=0} = 0$, and on the surface $z = h(t, \mathrm{x})$, the normal velocity

$$w = \frac{\mathrm{d}h(t, \mathrm{x})}{\mathrm{d}t} = h_t(t, \mathrm{x}) + u h_x(t, \mathrm{x}) + v h_y(t, \mathrm{x}).$$

Therefore, the equation for $h(t, \mathrm{x})$ is given by

$$h_t + \operatorname{div}(h\boldsymbol{u}) = 0.$$

Finally we turn to the equations for the conservation of momentum. As to the general three dimensional inviscid fluid, the equations for momentum are given by

$$\begin{aligned} \rho \left(u_t + u u_x + v u_y + w u_z \right) &= -p_x, \\ \rho \left(v_t + u v_x + v v_y + w v_z \right) &= -p_y, \\ \rho \left(w_t + u w_x + v w_y + w w_z \right) &= -p_z - (\rho + n) g, \end{aligned}$$

where g is the gravity constant. With the aid of (2.1), the system is thus reduced to

$$\begin{aligned} \rho \left(u_t + u u_x + v u_y \right) &= -p_x, \\ \rho \left(v_t + u v_x + v v_y \right) &= -p_y, \\ -p_z - (\rho + n) g = \rho \frac{\mathrm{d}w}{\mathrm{d}t} &= 0. \end{aligned} \tag{2.2}$$

Integrating the third equation in (2.2) along z direction from the surface $z = h(t, \mathrm{x})$ to the bottom $z = 0$ yields

$$
\begin{aligned}
p(t, \mathrm{x}, 0) &= \int\limits_{h(t,\mathrm{x})}^{0} p_z(t, \mathrm{x}, z)\mathrm{d}z + p(t, \mathrm{x}, h(t, \mathrm{x})) \\
&= \int\limits_{h(t,\mathrm{x})}^{0} -(\rho + n(t, \mathrm{x}))g\mathrm{d}z + p_a \\
&= g(\rho + n(t, \mathrm{x}))h(t, \mathrm{x}) + p_a,
\end{aligned}
$$

where $p_a = 1Pa$ is the unit pressure of the air. Therefore, the gradient of p is

$$
p_x = gn_x h + g(\rho + n)h_x, \quad p_y = gn_y h + g(\rho + n)h_y. \tag{2.3}
$$

Substitute (2.3) into (2.2), the momentum equations can be rewritten as

$$
\rho(\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + gh\nabla n + g(\rho + n)\nabla h = 0.
$$

In the end, we integrate the above equation from $z = 0$ to $z = h(t, \mathrm{x})$ to discover

$$
\rho h\boldsymbol{u}_t + \rho h\boldsymbol{u} \cdot \nabla \boldsymbol{u} + gh^2\nabla n + g\frac{1}{2}(\rho + n)\nabla h^2 = 0.
$$

By adding the viscosity term as in [15], taking the physical constants ρ and g to be one for simplicity, the chemotaxis–shallow water system (1.1) is established.

3. Local existence of a strong solution

In this section, we aim to prove the local existence of strong solutions to the initial boundary value problem (1.1)–(1.2) together with (1.4) or (1.5). In order to do this, we first focus on the following linearized system

$$
\begin{cases}
c_t + \mathrm{div}(c\boldsymbol{v}) = \Delta c - mc, \\
n_t + \mathrm{div}(n\boldsymbol{v}) = \Delta n - \nabla \cdot (n\nabla c), \\
h_t + \mathrm{div}(h\boldsymbol{v}) = 0, \\
h\boldsymbol{u}_t + h\boldsymbol{v} \cdot \nabla \boldsymbol{u} + L\boldsymbol{u} + h^2\nabla n + \frac{1}{2}(1 + n)\nabla h^2 = 0, \\
(c, n, h, \boldsymbol{u})|_{t=0} = (n_0, c_0, h_0, \boldsymbol{u}_0) \quad \text{in } \Omega, \\
\text{(1.4) or (1.5)}
\end{cases} \tag{3.1}
$$

and intend to obtain *a priori* estimates for the linearized system. Here $L := -\mu\Delta - (\lambda + \mu)\nabla\mathrm{div}$ is a strongly elliptic operator and $\boldsymbol{v}$ is a given vector satisfying

$$
\boldsymbol{v} \in C([0, T]; H_0^1 \cap H^2) \cap L^2([0, T]; W^{2,q}), \boldsymbol{v}_t \in L^2([0, T]; H^1) \tag{3.2}
$$

for $q > 2$. Moreover, m is nonnegative and

$$
m \in C([0, T]; H_0^1) \cap L^2([0, T]; H^2), \; m_t \in C([0, T]; L^2) \cap L^2([0, T]; H^1). \tag{3.3}
$$

Throughout this section we suppose that $t < 1$.

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Lemma 1. *Assume that the initial data (n_0, c_0, h_0, u_0) satisfy (1.3), (1.6) and $h_0 \geq \delta$ for some constant $\delta > 0$. $\tilde{h}$ and q are as in the assumption of Theorem 1. $\boldsymbol{v}$ and m are given functions which satisfy (3.2) and (3.3). Then there exists a unique strong solution (n, c, h, u) to (3.1) such that for any $T > 0$,*

$$\begin{cases} n \in C([0, T^*]; H_0^2), \quad c \in C([0, T^*]; H_0^3), \\ n_t \in L^\infty(0, T^*; L^2) \cap L^2(0, T^*; H^1), \quad c_t \in L^\infty(0, T^*; H^1) \cap L^2(0, T^*; H^2), \\ (h - \tilde{h}) \in C([0, T^*]; W^{1,q}), \quad h_t \in C(0, T^*; L^q), \\ \boldsymbol{u} \in C([0, T^*]; D_0^1 \cap D^2) \cap L^2(0, T^*; D^{2,q}), \\ \boldsymbol{u}_t \in L^\infty(0, T^*; L^2) \cap L^2(0, T^*; D^1), \quad \sqrt{h}\boldsymbol{u}_t \in L^\infty(0, T^*; L^2). \end{cases} \tag{3.4}$$

Proof. It is well known (see Lemma 1 and Lemma 9 in [12]) that the existence and regularity of unique solution to the equation $(3.1)_3$ can be obtained by the characteristic method and the solution is represented as

$$h(\mathrm{x}, t) = h_0\Big(U(0; \mathrm{x}, t)\Big) \exp\Big\{ - \int_0^t \mathrm{div}\boldsymbol{v}(U(s; \mathrm{x}, t), s)\mathrm{d}s \Big\}, \tag{3.5}$$

where $U \in C([0, T]; [0, T] \times \Omega)$ is the solution of the following backward ordinary differential equation

$$\begin{cases} \dfrac{d}{ds} U(s; \mathrm{x}, t) = \boldsymbol{v}(U(s; \mathrm{x}, t), s), \\ U(t; \mathrm{x}, t) = \mathrm{x}. \end{cases}$$

Next, with (3.2)–(3.3), the unique solution (c, n) satisfying the regularities in (3.4) can be achieved by solving the linear parabolic equation $(3.1)_1$, and then $(3.1)_2$.

Finally, recall that $L \triangleq -\mu\Delta - (\mu + \lambda)\nabla\mathrm{div}$ is a strongly elliptic operator (see [12] for instance). With the regularity properties of h and n, the existence and regularity results on solutions to the linear parabolic equation $(3.1)_4$ can be obtained. We omit the details. □

Now, we are devoting to establish *a priori* estimates on the solutions obtained in Lemma 1, uniformly in $\inf_\Omega h_0 = \delta$. Before giving *a priori* estimates, we define

$$\Phi(\boldsymbol{v}, m, t) := 1 + \sup_{0 \leq s \leq t} \left\{ \|\nabla \boldsymbol{v}\|_{H^1}^2 + \|m\|_{H^1}^2 + \|m_t\|_{L^2}^2 \right\} + \int_0^t \Big(\|\boldsymbol{v}_t\|_{H^1}^2 + \|\nabla \boldsymbol{v}\|_{W^{1,q}}^2 + \|\nabla m\|_{H^1}^2 + \|\nabla m_t\|_{L^2}^2 \Big)\mathrm{d}s, \tag{3.6}$$

and

$$C_0 = 1 + \|c_0\|_{H^3}^2 + \|n_0\|_{H^2}^2 + \|\sqrt{h_0}\boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{u}_0\|_{D_0^1 \cap D^2}^2 + \|\nabla h_0\|_{L^q}^2 + \|h_0 - \tilde{h}\|_{L^2 \cap L^\infty}^2 + \|g\|_{L^2}^2. \tag{3.7}$$

A priori estimates for the linearized system are obtained in a specific order: starting from the equation for mass conservation to have *a priori* estimate for h, we then investigate the equation of c. Based on these estimates, the equation for n is the next and eventually the estimate for $\boldsymbol{u}$ is able to be obtained. The results will be listed in the following subsection.

3.1. A priori estimates

Lemma 2. *Suppose that $h_0 - \tilde{h} \in L^2(\Omega) \cap W^{1,q}(\Omega)$ for $\tilde{h} > 0$, where $\tilde{h} \equiv 0$ in the case of Ω being a bounded domain in $\mathbb{R}^2$. With $\boldsymbol{v}$ given in (3.2), the solution of the following linearized equation*

$$\begin{cases} h_t + \operatorname{div}(h\boldsymbol{v}) = 0, & \text{in } (0,\mathrm{T}) \times \Omega, \\ h|_{t=0} = h_0, & (1.4) \text{ or } (1.5) \end{cases} \tag{3.8}$$

yields the estimates

$$\sup_{0\le s\le t} \left(\|\left(h-\tilde{h}, h^2-\tilde{h}^2\right)\|^2_{L^2\cap L^\infty} \le CC_0 \exp\left\{C\Phi(\boldsymbol{v},m,t)t^{\frac{1}{2}}\right\}, \tag{3.9}$$

$$\sup_{0\le s\le t} \left(\|\nabla h\|^2_{L^p} + \|\nabla h^2\|^2_{L^p} + \|h_t\|^2_{L^p}\right) \le CC_0\Phi(\boldsymbol{v},m,t) \exp\left\{C\Phi(\boldsymbol{v},m,t)t^{\frac{1}{2}}\right\}, \tag{3.10}$$

for all $p \in [2,q]$, where Φ and C_0 are defined in (3.6) and (3.7).

Proof. In view of the formula (3.5), it is evident that

$$\begin{aligned} &\sup_{0\le s\le t} \|h-\tilde{h}\|^2_{L^2\cap L^\infty} \\ &\le 2 \sup_{0\le s\le t} \left(\|h_0\big(U(0;\mathrm{x},t)\big) - \tilde{h}\|^2_{L^2\cap L^\infty} + \tilde{h}^2\right) \exp\left\{C\int_0^t \|\operatorname{div}\boldsymbol{v}(U(s;\mathrm{x},t),s)\|_{L^\infty} \mathrm{d}s\right\} \\ &\le CC_0 \exp\left\{C\Phi(\boldsymbol{v},m,t)t^{\frac{1}{2}}\right\}. \end{aligned} \tag{3.11}$$

And we further observe from (3.11) that

$$\sup_{0\le s\le t} \|h^2-\tilde{h}^2\|^2_{L^2\cap L^\infty} \le \sup_{0\le s\le t} \|h+\tilde{h}\|^2_{L^\infty} \cdot \|h-\tilde{h}\|^2_{L^2\cap L^\infty} \le CC_0 \exp\left\{C\Phi(\boldsymbol{v},m,t)t^{\frac{1}{2}}\right\}. \tag{3.12}$$

Hence the first estimate (3.9) is obtained from (3.11)–(3.12). Next we will prove the second estimate (3.10). For any $p \in [2,q]$, operating ∇ on each term of the equation (3.8), multiplying the resulting equation by $p|\nabla h|^{p-2}\nabla h$ and integrating over Ω give

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |\nabla h|^p \mathrm{dx} &= -\int p\nabla h \cdot \nabla(\nabla h \cdot \boldsymbol{v})|\nabla h|^{p-2}\mathrm{dx} - \int p\nabla h \cdot \nabla(h \operatorname{div}\boldsymbol{v})|\nabla h|^{p-2}\mathrm{dx} \\ &\le C\|\nabla \boldsymbol{v}\|_{L^\infty}\|\nabla h\|^p_{L^p} + C\|h\|_{L^\infty}\|\nabla^2\boldsymbol{v}\|_{L^p}\|\nabla h\|^{p-1}_{L^p}. \end{aligned}$$

We thus deduce from Gronwall's inequality that

$$\sup_{0\le s\le t}\|\nabla h\|_{L^p}^2 \le CC_0 \exp\left\{C\Phi(\boldsymbol{v},m,t)t^{\frac{1}{2}}\right\}, \quad \forall p\in[2,q], \tag{3.13}$$

which implies for any $p\in[2,q]$,

$$\begin{aligned}\sup_{0\le s\le t}\|\nabla h^2\|_{L^p}^2 &= \sup_{0\le s\le t}\|2h\nabla h\|_{L^p}^2\\ &\le C\sup_{0\le s\le t}\|h\|_{L^\infty}^2\|\nabla h\|_{L^p}^2 \le CC_0\exp\left\{C\Phi(\boldsymbol{v},m,t)t^{\frac{1}{2}}\right\}.\end{aligned} \tag{3.14}$$

Finally, the equation (3.8) together with (3.13) guarantees for all $p\in[2,q]$,

$$\sup_{0\le s\le t}\|h_t\|_{L^p}^2 = \sup_{0\le s\le t}\|-\operatorname{div}(h\boldsymbol{v})\|_{L^p}^2 \le CC_0\Phi(\boldsymbol{v},m,t)\exp\left\{C\Phi(\boldsymbol{v},m,t)t^{\frac{1}{2}}\right\}. \tag{3.15}$$

Therefore the second estimate (3.10) holds from (3.13)–(3.15). □

Lemma 3. *Let c be a solution of the following linearized equation*

$$\begin{cases} c_t + \operatorname{div}(c\boldsymbol{v}) = \Delta c - mc \quad \text{in } (0,\mathrm{T})\times\Omega,\\ c|_{t=0}=c_0 \quad \text{in } \Omega, \quad (1.4) \text{ or } (1.5).\end{cases} \tag{3.16}$$

Then we have

$$\sup_{0\le s\le t}\left(\|c\|_{H^1}^2+\|c_t\|_{L^2}^2\right)+\int_0^t\left(\|\nabla c_t\|_{L^2}^2+\|\nabla^2 c\|_{H^1}^2\right)\mathrm{d}s \le CC_0\exp\left\{C\Phi^2(\boldsymbol{v},m,t)t^{\frac{1}{4}}\right\}, \tag{3.17}$$

$$\begin{aligned}&\sup_{0\le s\le t}(\|\nabla^2 c\|_{H^1}^2+\|\nabla c_t\|_{L^2}^2)+\int_0^t\left(\|\nabla^2 c_t\|_{L^2}^2+\|c_{tt}\|_{L^2}^2\right)\mathrm{d}s\\ &\qquad \le CC_0\Phi^2(\boldsymbol{v},m,t)\exp\left\{C\Phi^2(\boldsymbol{v},m,t)t^{\frac{1}{4}}\right\}.\end{aligned} \tag{3.18}$$

Proof. Multiplying the equation (3.16) by c and integrating the resulting equation over Ω yield that

$$\frac{1}{2}\frac{d}{dt}\int|c|^2\mathrm{d}x+\int|\nabla c|^2\mathrm{d}x = -\int c\operatorname{div}(c\boldsymbol{v})\mathrm{d}x-\int mc^2\mathrm{d}x \le \frac{1}{2}(\|\operatorname{div}\boldsymbol{v}\|_{L^\infty}+\|m\|_{L^\infty})\|c\|_{L^2}^2.$$

Gronwall's inequality leads to

$$\sup_{0\le s\le t}\|c\|_{L^2}^2+\int_0^t\|\nabla c\|_{L^2}^2\mathrm{d}s \le CC_0\exp\left\{C\Phi(\boldsymbol{v},m,t)t^{\frac{1}{2}}\right\}. \tag{3.19}$$

In order to estimate $\sup_{0\le s\le t}\|\nabla c\|_{L^2}^2$, we then multiply the equation (3.16) by c_t, integrate by parts, and apply Gagliardo–Nirenberg's inequality to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla c|^2\mathrm{dx}+\int|c_t|^2\mathrm{dx}&=-\int\nabla c\cdot\boldsymbol{v}c_t\mathrm{dx}-\int\mathrm{div}\boldsymbol{v}\cdot c\cdot c_t\mathrm{dx}-\int mc\cdot c_t\mathrm{dx}\\&\le\varepsilon\|c_t\|_{L^2}^2+C(\varepsilon)[\|\boldsymbol{v}\|_{L^\infty}^2\|\nabla c\|_{L^2}^2+\|c\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^\infty}^2+\|m\|_{H^1}^2\|c\|_{L^2}^2+\|\nabla c\|_{L^2}^2].\end{aligned}$$

By choosing $\varepsilon=1/2$ and using Gronwall's inequality, we have

$$\sup_{0\le s\le t}\|\nabla c\|_{L^2}^2+\int_0^t\|c_t\|_{L^2}^2\mathrm{d}s\le CC_0\exp\left\{C\Phi^2(\boldsymbol{v},m,t)t^{\frac{1}{2}}\right\}.\tag{3.20}$$

For the estimate of $\sup_{0\le s\le t}\|c_t\|_{L^2}^2$, we differentiate the equation (3.16) with respect to t,

$$c_{tt}+\nabla c_t\cdot\boldsymbol{v}+\nabla c\cdot\boldsymbol{v}_t+c_t\mathrm{div}\boldsymbol{v}+c\mathrm{div}\boldsymbol{v}_t=\Delta c_t-m_tc-mc_t,\tag{3.21}$$

multiply the resulting equation by c_t and integrate the resulting equation to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|c_t|^2\mathrm{dx}+\int|\nabla c_t|^2\mathrm{dx}&=-\int c_t\nabla c_t\cdot\boldsymbol{v}\mathrm{dx}-\int c_t\nabla c\cdot\boldsymbol{v}_t\mathrm{dx}-\int c_t^2\mathrm{div}\boldsymbol{v}\mathrm{dx}\\&\quad-\int cc_t\mathrm{div}\boldsymbol{v}_t\mathrm{dx}-\int m_tcc_t\mathrm{dx}-\int mc_t^2\mathrm{dx}=:\sum_{i=1}^6 I_i.\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}I_1&=-\int c_t\nabla c_t\cdot\boldsymbol{v}\mathrm{dx}=\frac{1}{2}\int c_t^2\mathrm{div}\boldsymbol{v}dx\le\frac{1}{2}\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\|c_t\|_{L^2}^2,\\I_3&=-\int c_t^2\mathrm{div}\boldsymbol{v}\mathrm{dx}\le\|\nabla\boldsymbol{v}\|_{L^\infty}\|c_t\|_{L^2}^2,\\I_5&=-\int m_tcc_t\mathrm{dx}\le 2\|c_t\|_{L^2}^2+2\|c\|_{L^2}\|\nabla c\|_{L^2}\|m_t\|_{L^2}\|\nabla m_t\|_{L^2},\\I_6&=-\int mc_t^2\mathrm{dx}\le\|m\|_{L^\infty}\|c_t\|_{L^2}^2.\end{aligned}$$

Additionally, by applying Hölder's inequality, Gagliardo–Nirenberg's inequality and Young's inequality, we observe

$$\begin{aligned}I_2=-\int c_t\nabla c\cdot\boldsymbol{v}_t\mathrm{dx}&\le C\|\nabla c\|_{L^2}\|c_t\|_{L^2}^{\frac{1}{2}}\|\nabla c_t\|_{L^2}^{\frac{1}{2}}\|\boldsymbol{v}_t\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{v}_t\|_{L^2}^{\frac{1}{2}}\\&\le\varepsilon\|\nabla c_t\|_{L^2}^2+C(\varepsilon)\|\nabla c\|_{L^2}^2\|\nabla\boldsymbol{v}_t\|_{L^2}^2\|c_t\|_{L^2}^2+C(\varepsilon)\|\nabla c\|_{L^2}\|\boldsymbol{v}_t\|_{L^2},\\I_4=-\int cc_t\mathrm{div}\boldsymbol{v}_t\mathrm{dx}&\le\|\mathrm{div}\boldsymbol{v}_t\|_{L^2}\|c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|c_t\|_{L^2}^{\frac{1}{2}}\|\nabla c_t\|_{L^2}^{\frac{1}{2}}\\&\le\varepsilon\|\nabla c_t\|_{L^2}^2+C(\varepsilon)\|\nabla\boldsymbol{v}_t\|_{L^2}\|c_t\|_{L^2}^2+C(\varepsilon)\|\nabla\boldsymbol{v}_t\|_{L^2}^{\frac{3}{2}}\|\nabla c\|_{L^2}\|c\|_{L^2}.\end{aligned}$$

Then combining all the estimates I_i, $i = 1, \cdots, 6$ and Gronwall's inequality leads to

$$\sup_{0\le s\le t} \|c_t\|_{L^2}^2 + \int_0^t \|\nabla c_t\|_{L^2}^2 \mathrm{d}s \le CC_0 \exp\left\{C\Phi^2(\boldsymbol{v}, m, t)t^{\frac{1}{4}}\right\}. \tag{3.22}$$

Moreover, the classical L^p-theory asserts

$$\begin{aligned}\|\nabla^2 c\|_{L^2} &\le C\Big(\|c_t\|_{L^2} + \|\mathrm{div}(c\boldsymbol{v})\|_{L^2} + \|mc\|_{L^2}\Big)\\ &\le C\Big(\|c_t\|_{L^2} + \|\boldsymbol{v}\|_{L^\infty}\|\nabla c\|_{L^2} + \|c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\nabla \boldsymbol{v}\|_{H^1} + \|c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|m\|_{H^1}\Big),\end{aligned} \tag{3.23}$$

which together with (3.19), (3.20) and (3.22) implies that

$$\int_0^t \|\nabla^2 c\|_{L^2} \mathrm{d}s \le CC_0 \exp\left\{C\Phi^2(\boldsymbol{v}, m, t)t^{\frac{1}{4}}\right\},$$

$$\sup_{0\le s\le t} \|\nabla^2 c\|_{L^2}^2 \le CC_0\Phi^2(\boldsymbol{v}, m, t) \exp\left\{C\Phi^2(\boldsymbol{v}, m, t)t^{\frac{1}{4}}\right\}.$$

The same argument shows that

$$\int_0^t \|\nabla^2 c\|_{H^1}^2 \mathrm{d}s \le CC_0 \exp\left\{C\Phi^2(\boldsymbol{v}, m, t)t^{\frac{1}{4}}\right\}, \tag{3.24}$$

$$\sup_{0\le s\le t} \|\nabla^2 c\|_{H^1}^2 \le CC_0\Phi^2(\boldsymbol{v}, m, t) \exp\left\{C\Phi^2(\boldsymbol{v}, m, t)t^{\frac{1}{4}}\right\}. \tag{3.25}$$

Therefore the first estimate (3.17) holds by virtue of (3.19)–(3.22) and (3.24). Now we turn our attention to prove (3.18). Starting from the estimate for $\sup\limits_{0\le t\le T} \|\nabla c_t\|_{L^2}^2$, we multiply the equation (3.21) by c_{tt} and integrate the resulting equation to discover

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |\nabla c_t|\mathrm{d}x + \int c_{tt}^2 \mathrm{d}x = &-\int \nabla c_t \cdot \boldsymbol{v} c_{tt} \mathrm{d}x - \int \nabla c \cdot \boldsymbol{v}_t c_{tt} \mathrm{d}x - \int c_t c_{tt} \mathrm{div}\boldsymbol{v} \mathrm{d}x\\ &- \int c c_{tt} \mathrm{div}\boldsymbol{v}_t \mathrm{d}x - \int m_t c c_{tt} \mathrm{d}x - \int m c_t c_{tt} \mathrm{d}x =: \sum_{i=1}^{6} J_i.\end{aligned}$$

We readily check that

$$J_1 = -\int \nabla c_t \cdot \boldsymbol{v} c_{tt} \mathrm{d}x \le \varepsilon\|c_{tt}\|_{L^2}^2 + C(\varepsilon)\|\boldsymbol{v}\|_{L^\infty}^2\|\nabla c_t\|_{L^2}^2,$$

$$J_2 = -\int \nabla c \cdot \boldsymbol{v}_t c_{tt} \mathrm{d}x \le \varepsilon\|c_{tt}\|_{L^2}^2 + C(\varepsilon)\|\nabla c\|_{L^2}\|\nabla^2 c\|_{L^2}\|\boldsymbol{v}_t\|_{L^2}\|\nabla \boldsymbol{v}_t\|_{L^2},$$

$$J_3 = -\int c_t c_{tt} \operatorname{div} \boldsymbol{v} \mathrm{dx} \le \varepsilon \|c_{tt}\|_{L^2}^2 + C(\varepsilon)\|\operatorname{div} \boldsymbol{v}\|_{L^\infty}^2 \|c_t\|_{L^2}^2,$$

$$J_4 = -\int c c_{tt} \operatorname{div} \boldsymbol{v}_t \mathrm{dx} \le \varepsilon \|c_{tt}\|_{L^2}^2 + C(\varepsilon)\|c\|_{L^\infty}^2 \|\boldsymbol{v}_t\|_{H^1}^2,$$

$$J_5 = -\int m_t c c_{tt} \mathrm{dx} \le \varepsilon \|c_{tt}\|_{L^2}^2 + C(\varepsilon)\|c\|_{L^\infty}^2 \|m_t\|_{L^2}^2,$$

$$\begin{aligned} J_6 = -\int m c_t c_{tt} \mathrm{dx} &\le C\|c_{tt}\|_{L^2}\|m\|_{L^2}^{\frac12}\|\nabla m\|_{L^2}^{\frac12}\|c_t\|_{L^2}^{\frac12}\|\nabla c_t\|_{L^2}^{\frac12} \\ &\le \varepsilon \|c_{tt}\|_{L^2}^2 + C(\varepsilon)\|m\|_{H^1}^2\|c_t\|_{L^2}\|\nabla c_t\|_{L^2}^2 + C(\varepsilon)\|m\|_{H^1}^2\|c_t\|_{L^2}. \end{aligned}$$

Consequently, we conclude from the estimates J_i, $i = 1, \cdots, 6$ and Gronwall's inequality that

$$\sup_{0\le s\le t} \|\nabla c_t\|_{L^2}^2 + \int_0^t \|c_{tt}\|_{L^2}^2 \mathrm{d}s \le CC_0 \Phi^2(\boldsymbol{v}, m, t)\exp\left\{C\Phi^2(\boldsymbol{v}, m, t)t^{\frac14}\right\}. \tag{3.26}$$

Furthermore, we proceed the similar argument as in (3.23) to find

$$\begin{aligned} &\|\nabla^2 c_t\|_{L^2}^2 \\ &\le C\Big(\|c_{tt}\|_{L^2}^2 + \|\nabla c_t \cdot \boldsymbol{v}\|_{L^2}^2 + \|\nabla c \cdot \boldsymbol{v}_t\|_{L^2}^2 + \|c \operatorname{div} \boldsymbol{v}_t\|_{L^2}^2 + \|c_t \operatorname{div} \boldsymbol{v}\|_{L^2}^2 + \|c m_t\|_{L^2}^2 + \|m c_t\|_{L^2}^2\Big) \\ &\le CC_0\Phi^2(\boldsymbol{v}, m, t)\exp\left\{C\Phi^2(\boldsymbol{v}, m, t)t^{\frac14}\right\}, \end{aligned}$$

which along with (3.25) and (3.26) gives the second estimate (3.18). □

Lemma 4. *Suppose that n satisfies the following linearized equation*

$$\begin{cases} n_t + \operatorname{div}(n\boldsymbol{v}) = \Delta n - \nabla\cdot(n\nabla c) \quad \text{in } (0, \mathrm{T}) \times \Omega, \\ n|_{t=0} = n_0 \quad \text{in } \Omega, \quad (1.4) \text{ or } (1.5), \end{cases} \tag{3.27}$$

where $\boldsymbol{v}$ is given in (3.2). Then,

$$\sup_{0\le s\le t}\left(\|n\|_{H^1}^2 + \|n_t\|_{L^2}^2\right) + \int_0^t \|\nabla n_t\|_{L^2}^2 \mathrm{d}s \le CC_0\exp\left\{CC_0^2\exp\{C\Phi^3(\boldsymbol{v}, m, t)t^{\frac14}\}\right\}, \tag{3.28}$$

$$\int_0^t \|\nabla^2 n\|_{L^2}^2 \mathrm{d}s \le CC_0^2\exp\left\{CC_0^2\exp\{C\Phi^3(\boldsymbol{v}, m, t)t^{\frac14}\}\right\}, \tag{3.29}$$

$$\sup_{0\le s\le t}\|\nabla^2 n\|_{L^2}^2 \le CC_0^2\Phi^2(\boldsymbol{v}, m, t)\exp\left\{CC_0^2\exp\{C\Phi^3(\boldsymbol{v}, m, t)t^{\frac14}\}\right\}. \tag{3.30}$$

Proof. To prove the estimate (3.28), we can follow the argument in Lemma 3 step by step and it remains to check the terms $\int \nabla\cdot(n\nabla c) n \mathrm{dx}$, $\int \nabla\cdot(n\nabla c) n_t \mathrm{dx}$ and $\int (n\nabla c)_t \cdot \nabla n_t \mathrm{dx}$. A straightforward computation leads that

$$\int \nabla\cdot(n\nabla c)n\mathrm{d}x \le \|\nabla n\|_{L^2}\|n\|_{L^4}\|\nabla c\|_{L^4} \le 2\varepsilon\|\nabla n\|_{L^2}^2 + C(\varepsilon)\|n\|_{L^2}^2\|\nabla c\|_{L^2}^2\|\nabla^2 c\|_{L^2}^2,$$

$$\int \nabla\cdot(n\nabla c)n_t\mathrm{d}x \le \varepsilon\|n_t\|_{L^2}^2 + C(\varepsilon)\big(\|\Delta c\|_{H^1}^2 + \|\nabla c\|_{L^\infty}^2\big)\big(\|\nabla n\|_{L^2}^2 + \|n\|_{L^2}^2\big),$$

$$\int (n\nabla c)_t\cdot\nabla n_t\mathrm{d}x \le \varepsilon\|\nabla n_t\|_{L^2}^2 + C(\varepsilon)\big(\|\nabla c\|_{L^2}\|\nabla^3 c\|_{L^2}\|n_t\|_{L^2}^2 + \|n\|_{L^2}\|\nabla n\|_{L^2}\|\nabla c_t\|_{L^2}\|\nabla^2 c_t\|_{L^2}\big).$$

The above estimates and (3.17)–(3.18) imply the first estimate (3.28). Eventually, the standard L^p theory guarantees

$$\begin{aligned}\|\nabla^2 n\|_{L^2}^2 &\le C\Big(\|n_t\|_{L^2}^2 + \|\mathrm{div}(n\boldsymbol{v})\|_{L^2}^2 + \|\nabla\cdot(n\nabla c)\|_{L^2}^2\Big)\\ &\le C\|n_t\|_{L^2}^2 + C\|\boldsymbol{v}\|_{L^\infty}^2\|\nabla n\|_{L^2}^2 + \varepsilon\|\nabla^2 n\|_{L^2}^2 + C(\varepsilon)\|\nabla n\|_{L^2}^2\|\nabla c\|_{L^2}^2\|\nabla^2 c\|_{L^2}^2\\ &\quad + C\|n\|_{L^2}\|\nabla n\|_{L^2}(\|\mathrm{div}\boldsymbol{v}\|_{L^2}\|\nabla^2\boldsymbol{v}\|_{L^2} + \|\nabla^2 c\|_{H^1}^2),\end{aligned}$$

which provides us the last two estimates (3.29), (3.30) in view of (3.17), (3.18), (3.28) together with the assumptions on $\boldsymbol{v}$. □

Lemma 5 is devoted to obtain *a priori* estimate for $\boldsymbol{u}$ in Lemma 6.

Lemma 5 *([15]). If $\tilde h > 0$ is a constant, $h(x)$ and $\boldsymbol{w}(x)$ are two functions satisfying $h - \tilde h \in L^2$, $\boldsymbol{w} \in D^1$, $h^{1/2}\boldsymbol{w} \in L^2$. Then there exists some constant $C(\tilde h) > 0$ such that the following estimate holds*

$$\|\boldsymbol{w}\|_{L^2}^2 \le C\left(\int h|\boldsymbol{w}|^2dx + \|\tilde h - h\|_{L^2}^2\|\nabla\boldsymbol{w}\|_{L^2}^2\right).$$

Remark 5. For the case of $\Omega = \mathbb{R}^2$, this lemma is used to control $\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}$ and $\|(\boldsymbol{u}_t, \dot{\boldsymbol{u}})\|_{L^2(0,T;L^2(\mathbb{R}^2))}$.

Lemma 6. *Let $\boldsymbol{u}$ be the solution of the following linearized equation*

$$\begin{cases} h\boldsymbol{u}_t + h\boldsymbol{v}\cdot\nabla\boldsymbol{u} + L\boldsymbol{u} + h^2\nabla n + \frac{1}{2}(1+n)\nabla h^2 = 0 & \text{in } (0,T)\times\Omega,\\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad (1.4) \text{ or } (1.5), \end{cases} \tag{3.31}$$

where $L := -\mu\Delta - (\lambda+\mu)\nabla\mathrm{div}$ is a strongly elliptic operator and $\boldsymbol{v}$, m are given in (3.2), (3.3). Then,

$$\sup_{0\le s\le t}\big(\|\sqrt{h}\boldsymbol{u}\|_{L^2}^2 + \|\nabla\boldsymbol{u}\|_{L^2}^2\big) + \int_0^t \|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2\mathrm{d}s \le CC_0\exp\Big\{CC_0^2\exp\{C\Phi^2(\boldsymbol{v},m,t)t^{\frac14}\}\Big\}, \tag{3.32}$$

$$\sup_{0\le s\le t}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2 + \int_0^t \|\nabla\dot{\boldsymbol{u}}\|_{L^2}^2\mathrm{d}s \le CC_0\exp\Big\{CC_0^2\exp\{C\Phi^3(\boldsymbol{v},m,t)t^{\alpha}\}\Big\}, \tag{3.33}$$

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$$\sup_{0\le s\le t}\|\nabla^2\boldsymbol{u}\|_{L^2}^2+\int_0^t\|\nabla^2\boldsymbol{u}\|_{L^q}^2\mathrm{d}s\le CC_0\exp\left\{CC_0^2\exp\{C\Phi^3(\boldsymbol{v},m,t)t^\alpha\}\right\},\tag{3.34}$$

where the material derivative $\dot{\boldsymbol{u}}=\boldsymbol{u}_t+\boldsymbol{v}\cdot\nabla\boldsymbol{u}$ *and* $\alpha=\min\left\{\frac{3q-4}{4(q-1)},\frac{1}{4}\right\}$.

Proof. We multiply the equation (3.31) by $\boldsymbol{u}$ and integrate the resulting equation to discover

$$\begin{aligned}&\frac{d}{dt}\int|\sqrt{h}\boldsymbol{u}|^2\mathrm{dx}+2\mu\int|\nabla\boldsymbol{u}|^2\mathrm{dx}+2(\mu+\lambda)\int|\mathrm{div}\boldsymbol{u}|^2\mathrm{dx}\\&=-\int(1+n)\nabla h^2\cdot\boldsymbol{u}\mathrm{dx}-\int 2h^2\nabla n\cdot\boldsymbol{u}\mathrm{dx}\\&\le C(\|h^3\|_{L^\infty}+\|\nabla h\|_{L^2}^2)\|\sqrt{h}\boldsymbol{u}\|_{L^2}^2+C\|h\|_{L^\infty}+C\|\nabla n\|_{L^2}^2\\&\quad+\varepsilon\|\mathrm{div}\boldsymbol{u}\|_{L^2}^2+C(\varepsilon)\|h^4\|_{L^\infty}\|n\|_{L^2}^2.\end{aligned}$$

An application of Gronwall's inequality yields

$$\sup_{0\le s\le t}\|\sqrt{h}\boldsymbol{u}\|_{L^2}^2+\int_0^t\|\nabla\boldsymbol{u}\|_{L^2}^2\mathrm{d}s\le CC_0\exp\left\{CC_0^2\exp\{C\Phi^2(\boldsymbol{v},m,t)t^{\frac{1}{4}}\}\right\}.\tag{3.35}$$

Additionally, we multiply the equation (3.31) by $\dot{\boldsymbol{u}}$ and integrate the resulting equation

$$\int h|\dot{\boldsymbol{u}}|^2\mathrm{dx}=-\int h^2\nabla n\cdot\dot{\boldsymbol{u}}\mathrm{dx}-\frac{1}{2}\int(1+n)\nabla h^2\cdot\dot{\boldsymbol{u}}\mathrm{dx}-\int L\boldsymbol{u}\cdot\dot{\boldsymbol{u}}\mathrm{dx}=:\sum_{i=1}^3 M_i.$$

Now we will estimate M_i for each $i=1,2,3$. It follows from Lemma 2–Lemma 4, Hölder's inequality and Gagliardo–Nirenberg's inequality that

$$\begin{aligned}M_1&=-\int h^2\nabla n\cdot\dot{\boldsymbol{u}}\mathrm{dx}\le\varepsilon\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}+C(\varepsilon)\|h\|_{L^\infty}^3\|\nabla n\|_{L^2},\\M_2&=-\frac{1}{2}\int(1+n)\nabla h^2\cdot\dot{\boldsymbol{u}}\mathrm{dx}=-\int(1+n)h\nabla h\cdot\dot{\boldsymbol{u}}\mathrm{dx}\\&\le\varepsilon\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2+C(\varepsilon)\|h\|_{L^\infty}\left(\|\nabla h\|_{L^2}^2+\|\nabla h\|_{L^q}\|n\|_{L^2}^{\frac{q-2}{q}}\|\nabla n\|_{L^2}^{\frac{2}{q}}\right),\\M_3&=\int\left[\mu\Delta\boldsymbol{u}+(\lambda+\mu)\nabla\mathrm{div}\boldsymbol{u}\right]\dot{\boldsymbol{u}}\mathrm{dx}\\&\le-\frac{\mu}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|_{L^2}^2-\frac{\lambda+\mu}{2}\frac{d}{dt}\|\mathrm{div}\boldsymbol{u}\|_{L^2}^2+C\|\nabla\boldsymbol{v}\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{L^2}^2.\end{aligned}$$

Combining the estimates of M_i, $i=1,2,3$ with Lemma 2–Lemma 4 and Gronwall's inequality, we derive

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$$\sup_{0\le s\le t}\|\nabla \boldsymbol{u}\|_{L^2}^2+\int_0^t\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2\mathrm{d}s\le CC_0\exp\left\{CC_0^2\exp\{C\Phi^2(\boldsymbol{v},m,t)t^{\frac14}\}\right\}.$$

This together with (3.35) gives the first estimate (3.32). Next we intend to prove the second estimate (3.33). Let $\boldsymbol{v}^k$ be the k-th component of a vector $\boldsymbol{v}$ and denote the operator $\frac{D}{Dt}:=\partial_t+\partial_k(\boldsymbol{v}^k\cdot)$. By taking the operator $\frac{D}{Dt}$ on the equation (3.31), we get

$$\begin{aligned}0=&\,h\dot{\boldsymbol{u}}_t+h\boldsymbol{v}\nabla\dot{\boldsymbol{u}}+h\mathrm{div}\boldsymbol{v}\;\dot{\boldsymbol{u}}+h^2\nabla n_t+\frac12\nabla h^2n_t-(1+n)\nabla(h^2\mathrm{div}\boldsymbol{v})\\&-\frac12(1+n)\nabla h^2\nabla\boldsymbol{v}+\frac12(1+n)\mathrm{div}\boldsymbol{v}\;\nabla h^2+\frac12\boldsymbol{v}\cdot\nabla n\nabla h^2+h^2\boldsymbol{v}\cdot\nabla(\nabla n)\\&-h^2\mathrm{div}\boldsymbol{v}\;\nabla n+\mu\Delta\dot{\boldsymbol{u}}_t+\mu\big[\partial_i(\partial_k\boldsymbol{v}^k\partial_i\boldsymbol{u})-\partial_i(\partial_i\boldsymbol{v}^k\partial_k\boldsymbol{u})-\partial_k(\partial_i\boldsymbol{v}^k\partial_i\boldsymbol{u})\big]\\&+(\mu+\lambda)\big[\nabla\mathrm{div}\dot{\boldsymbol{u}}+\nabla(\mathrm{div}\boldsymbol{v}\mathrm{div}\boldsymbol{u})-\nabla(\partial_i\boldsymbol{v}^k\partial_k\boldsymbol{u}^i)-\partial_k(\nabla\boldsymbol{v}^k\mathrm{div}\boldsymbol{u})\big].\end{aligned}$$

Multiplying the equation above by $\dot{\boldsymbol{u}}$ and integrating over Ω give

$$\frac12\frac{d}{dt}\int|\sqrt{h}\dot{\boldsymbol{u}}|^2\mathrm{dx}+\mu\int|\nabla\dot{\boldsymbol{u}}|^2\mathrm{dx}+(\mu+\lambda)\int|\mathrm{div}\dot{\boldsymbol{u}}|^2\mathrm{dx}=:\sum_{k=1}^{19}N_k.$$

Then we will estimate N_k for each k. From Hölder's inequality and Gagliardo–Nirenberg's inequality, it follows that

$$\begin{aligned}N_1=&\int h\mathrm{div}\boldsymbol{v}\;\dot{\boldsymbol{u}}^2\mathrm{dx}\le\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2,\\N_2=&-\int h^2\nabla n_t\cdot\dot{\boldsymbol{u}}\mathrm{dx}\le\frac12\|\nabla n_t\|_{L^2}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2+\frac12\|h\|_{L^\infty}^3\|\nabla n_t\|_{L^2},\\N_3=&\,\frac12\int\nabla h^2n_t\cdot\dot{\boldsymbol{u}}\mathrm{dx}\le C\|h\|_{L^\infty}^{\frac12}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2+C\|h\|_{L^\infty}^{\frac12}\|\nabla h\|_{L^q}^2\|n_t\|_{L^2}^{\frac{2(q-2)}{q}}\|\nabla n_t\|_{L^2}^{\frac4q},\\N_4=&\int-(1+n)\nabla(h^2\mathrm{div}\boldsymbol{v})\cdot\dot{\boldsymbol{u}}\mathrm{dx}\le2\varepsilon\|\mathrm{div}\dot{\boldsymbol{u}}\|_{L^2}^2+\|h^{\frac32}\mathrm{div}\boldsymbol{v}\|_{L^\infty}\big(\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2+\|\nabla n\|_{L^2}^2\big)\\&+C\|h\|_{L^\infty}^4\|\mathrm{div}\boldsymbol{v}\|_{L^2}^2(\|n\|_{L^2}^2+1),\\N_5=&-\frac12\int(1+n)\nabla h^2\nabla\boldsymbol{v}\cdot\dot{\boldsymbol{u}}\mathrm{dx}\le C\|\nabla\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac12}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2+C\|\nabla\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac12}\|\nabla h\|_{L^2}^2\\&+C\|\nabla\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac12}\|\nabla h\|_{L^q}^2\|n\|_{L^2}^{\frac{2(q-2)}{q}}\|\nabla n\|_{L^2}^{\frac4q},\\N_6=&\,\frac12\int\nabla h^2(1+n)\mathrm{div}\boldsymbol{v}\cdot\dot{\boldsymbol{u}}\mathrm{dx}\le C\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac12}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2+C\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac12}\|\nabla h\|_{L^2}^2\\&+C\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac12}\|\nabla h\|_{L^q}^2\|n\|_{L^2}^{\frac{2(q-2)}{q}}\|\nabla n\|_{L^2}^{\frac4q},\\N_7=&\,\frac12\int\boldsymbol{v}\cdot\nabla n\nabla h^2\cdot\dot{\boldsymbol{u}}\mathrm{dx}\le2\|\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac12}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2\end{aligned}$$

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$$+ C\|\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac{1}{2}}\|\nabla h\|_{L^q}^2\|\nabla n\|_{L^2}^{\frac{2(q-2)}{q}}\|\nabla^2 n\|_{L^2}^{\frac{4}{q}},$$

$$N_8 = \int h^2\boldsymbol{v}\nabla(\nabla n)\cdot\dot{\boldsymbol{u}}\mathrm{dx} \le \varepsilon\|\mathrm{div}\dot{\boldsymbol{u}}\|_{L^2}^2 + 2\Big(\|\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac{1}{2}} + \|h^{\frac{3}{2}}\|_{L^\infty}\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\Big)\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2$$
$$+ C(\varepsilon)\Big(\|h^2\boldsymbol{v}\|_{L^\infty} + \|h^{\frac{3}{2}}\|_{L^\infty}\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\Big)\|\nabla n\|_{L^2}^2$$
$$+ C\|\boldsymbol{v}\|_{L^\infty}\|h\|_{L^\infty}^{\frac{1}{2}}\|\nabla h\|_{L^q}^2\|\nabla n\|_{L^2}^{\frac{2(q-2)}{q}}\|\nabla^2 n\|_{L^2}^{\frac{4}{q}},$$

$$N_9 = -\int h^2\mathrm{div}\boldsymbol{v}\nabla n\cdot\dot{\boldsymbol{u}}\mathrm{dx} \le \frac{1}{2}\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\|h^{\frac{3}{2}}\|_{L^\infty}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2 + \frac{1}{2}\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\|h^{\frac{3}{2}}\|_{L^\infty}\|\nabla n\|_{L^2}^2.$$

Since all the remaining integrals can be handled in the same way, we shall only evaluate N_{10} for an instance

$$N_{10} = (\mu+\lambda)\int\nabla(\mathrm{div}\boldsymbol{v}\ \mathrm{div}\boldsymbol{u})\cdot\dot{\boldsymbol{u}}\mathrm{dx} \le \varepsilon\|\mathrm{div}\dot{\boldsymbol{u}}\|_{L^2}^2 + C(\varepsilon)\|\mathrm{div}\boldsymbol{v}\|_{L^\infty}\|\mathrm{div}\boldsymbol{u}\|_{L^2}^2.$$

We thus achieve the second estimate (3.33) by collecting all the estimates N_i, $i = 1, \cdots, 19$ and applying Gronwall's inequality,

$$\sup_{0\le s\le t}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2 + \int_0^t\|\nabla\dot{\boldsymbol{u}}\|_{L^2}^2\mathrm{d}s \le CC_0\exp\Big\{CC_0^2\exp\{C\Phi^3(\boldsymbol{v}, m, t)t^\alpha\}\Big\}$$

with $\alpha = \min\left\{\frac{3q-4}{4(q-1)}, \frac{1}{4}\right\}$, where the inequality

$$\int_0^t\|\nabla\boldsymbol{v}\|_{L^\infty}\mathrm{d}s \le t^{\frac{1}{2}}\Big(\sup_{0\le s\le t}\|\nabla\boldsymbol{v}\|_{L^2}^{\frac{q-2}{q-1}}\Big)^{\frac{1}{2}}\Big(\int_0^t\|\nabla^2\boldsymbol{v}\|_{L^q}^{\frac{q}{q-1}}ds\Big)^{\frac{1}{2}} \le \Phi^{\frac{1}{2}+\frac{q}{4(q-1)}}t^{\frac{3(q-4)}{4(q-1)}}$$

has been used. It remains to prove the last estimate (3.34). The standard L^p estimate for elliptic equation guarantees

$$\begin{aligned}\|\nabla^2\boldsymbol{u}\|_{L^2} &\le C\Big(\|h\dot{\boldsymbol{u}}\|_{L^2} + \|h^2\nabla n\|_{L^2} + \|\frac{1}{2}(1+n)\nabla h^2\|_{L^2}\Big)\\ &\le C\|h\|_{L^\infty}\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2} + \|h^2\|_{L^\infty}\|\nabla n\|_{L^2} + \frac{1}{2}\|\nabla h^2\|_{L^2} + \|h\|_{L^\infty}\|\nabla h\|_{L^q}\|n\|_{L^{\frac{2q}{q-2}}}.\end{aligned} \tag{3.36}$$

Note that $\|n\|_{L^{\frac{2q}{q-2}}} \le C\|n\|_{L^2}^{\frac{q-2}{q}}\|\nabla n\|_{L^2}^{\frac{2}{q}}$. So

$$\sup_{0\le s\le t}\|\nabla^2\boldsymbol{u}\|_{L^2} \le CC_0\exp\Big\{CC_0^2\exp\{C\Phi^3(\boldsymbol{v}, m, t)t^\alpha\}\Big\}. \tag{3.37}$$

Moreover, we proceed as previously in (3.36) to obtain

$$\int_0^t \|\nabla^2 \boldsymbol{u}\|_{L^q}^2 \mathrm{d}s \le C \int_0^t \Big(\|h^2\|_{L^\infty} \|\nabla \dot{\boldsymbol{u}}\|_{L^2}^2 + \|h^4\|_{L^\infty} \|\nabla n\|_{L^2}^{\frac{4}{q}} \|\nabla^2 n\|_{L^2}^{\frac{2(q-2)}{q}}$$
$$+ \frac{1}{2}\|\nabla h^2\|_{L^q}^2 + \frac{1}{2}\|n\|_{L^\infty}\|\nabla h^2\|_{L^q}^2 \Big) \mathrm{d}s$$
$$\le C C_0 \exp\left\{ C C_0 \exp\{C \Phi^3(\boldsymbol{v}, m, t) t^\alpha\} \right\}, \tag{3.38}$$

where α is the same as in (3.33) and the estimates (3.11), (3.14), (3.28), (3.29), (3.33) have been used. Consequently, the last estimate (3.34) is thus proved from (3.37) and (3.38). □

Lemma 7. *There exists $T^* \in (0, 1)$ depending on C_0, μ, λ, such that*

$$\Phi(\boldsymbol{u}, n, T^*) \le M$$

provided $\Phi(\boldsymbol{v}, m, T^*) < M$ for some $M > 1$ depending on C_0, μ, λ.

Proof. According to Lemma 2–Lemma 4, Lemma 6, we conclude that

$$\Phi(\boldsymbol{u}, n, t) \le C C_0 \exp\left\{ C C_0^2 \exp\{C \Phi^3(\boldsymbol{v}, m, t) t^\alpha\} \right\},$$

which yields

$$\Phi(\boldsymbol{u}, n, T^*) \le M$$

by choosing

$$M = C C_0 \exp\left\{ C C_0^2 \exp\{C\} \right\}, \quad T^* = \min\left\{ M^{-3/\alpha}, 1 \right\}$$

with the same α as in (3.33), $\alpha = \min\left\{ \frac{3q-4}{4(q-1)}, \frac{1}{4} \right\}$. □

3.2. *Compact mapping fixed point theory*

In this subsection, based on the uniform estimates for the solutions of the linearized system (3.1), we apply the Schauder's fixed point theory to show that the chemotaxis–shallow water system (1.1)–(1.2) together with (1.4) or (1.5) has a unique local strong solution $(n, c, h, \boldsymbol{u})$. The proof is analogous to the discussion in [13,15], we only sketch the proof for completeness.

Proposition 1. *Assume that the initial data $(n_0, c_0, h_0 \ge 0, \boldsymbol{u}_0)$ satisfy the conditions in Theorem 1. Then there exists a unique strong solution $(n, c, h, \boldsymbol{u})$ to the initial boundary value problem (1.1)–(1.2) together with (1.4) or (1.5) on $\Omega \times [0, T^*]$ with T^* obtained in Lemma 7, such that*

$$\begin{cases} n \in C([0,T^*];H_0^2), \quad n_t \in L^\infty(0,T^*;L^2)\cap L^2(0,T^*;H^1), \\ c \in C([0,T^*];H_0^3), \quad c_t \in L^\infty(0,T^*;H^1)\cap L^2(0,T^*;H^2), \\ (h-\tilde{h}) \in C([0,T^*];W^{1,p}), \quad h_t \in C(0,T^*;L^p), \\ \boldsymbol{u} \in C([0,T^*];D_0^1\cap D^2)\cap L^2(0,T^*;D^{2,q}), \\ \boldsymbol{u}_t \in L^\infty(0,T^*;L^2)\cap L^2(0,T^*;D^1), \quad \sqrt{h}\boldsymbol{u}_t \in L^\infty(0,T^*;L^2) \end{cases} \tag{3.39}$$

for q, $\tilde{h}$ defined as in Theorem 1 and all $p \in [2,q]$. Moreover, the following estimate holds:

$$\begin{aligned} &\sup_{0\le t\le T^*}\left(\|\nabla c\|_{H^2(\Omega)}^2+\|c_t\|_{H^1(\Omega)}^2+\|n\|_{H^2(\Omega)}^2+\|n_t\|_{L^2(\Omega)}^2+\|\nabla^2\boldsymbol{u}\|_{L^2(\Omega)}^2\right) \\ &+\sup_{0\le t\le T^*}\left(\|h-\tilde{h}\|_{L^2\cap W^{1,p}(\Omega)}^2+\|h_t\|_{L^p(\Omega)}^2+\int_\Omega h(|\boldsymbol{u}|^2+|\dot{\boldsymbol{u}}|^2)\mathrm{d}x\right) \\ &+\int_0^{T^*}\left(\|\nabla c_t\|_{H^1(\Omega)}^2+\|\nabla n_t\|_{L^2(\Omega)}^2+\|\nabla\dot{\boldsymbol{u}}\|_{L^2(\Omega)}^2+\|\nabla^2\boldsymbol{u}\|_{L^q(\Omega)}^2\right)\mathrm{d}t \\ &\le C(\mu,\lambda,C_0). \end{aligned}$$

Proof. Step 1 In this step, we assume $h_0 \ge \delta > 0$. For a bounded domain $\Omega \subset \mathbb{R}^2$, with the same T^* and M as in Lemma 7, we denote $\mathcal{B} = L^2(0,T^*;H_0^1)\times L^2(0,T^*;H_0^1)$ and

$$\begin{aligned} \mathcal{R} := \{(n,\boldsymbol{u})|\ &(n,\boldsymbol{u}) \in L^\infty(0,T^*;H_0^1)\cap L^2(0,T^*;D^2) \\ &\times L^\infty(0,T^*;H_0^1\cap D^2)\cap L^2(0,T^*;D^{2,q}), \\ &(n_t,\boldsymbol{u}_t) \in L^\infty(0,T^*;L^2)\cap L^2(0,T^*;H_0^1)\times L^2(0,T^*;H_0^1),\ \Phi(\boldsymbol{u},n,T^*)<M\}. \end{aligned}$$

It is easily seen that $\mathcal{R}$ is a convex and compact subset of Banach space $\mathcal{B}$. For any $(\boldsymbol{v},m)\in\mathcal{R}$, there exists a unique solution $h=h(\boldsymbol{v})$ of the linearized equation (3.8) on $\Omega\times[0,T^*]$ with $h\in C([0,T^*];H^1\cap W^{1,q})$, $h_t\in C([0,T^*];L^q)$. Besides, the linearized equation (3.16) admits a unique solution $c=c(\boldsymbol{v},m)$ on $\Omega\times[0,T^*]$ as well. Moreover, a unique $n=\mathcal{T}_1(\boldsymbol{v},c(\boldsymbol{v},m))$ solves the linearized equation (3.27) on $\Omega\times[0,T^*]$ and thus the linearized equation (3.31) has a unique solution $\boldsymbol{u}=\mathcal{T}_2(\boldsymbol{v},h(\boldsymbol{v}),\mathcal{T}_1(\boldsymbol{v},c(\boldsymbol{v},m)))$ on $\Omega\times[0,T^*]$. Therefore, we may write $(n,\boldsymbol{u}):=\mathcal{T}(m,\boldsymbol{v})=(\mathcal{T}_1,\mathcal{T}_2)$ with $\mathcal{T}$ mapping from $\mathcal{R}$ to $\mathcal{R}$. Next we will show that $\mathcal{T}$ is a continuous operator from $\mathcal{B}$ into itself. First of all, according to Lemma 2, we have

$$\sup_{0\le s\le T^*}\left(\|h(\boldsymbol{v})\|_{W^{1,p}}+\|(h(\boldsymbol{v}))_t\|_{L^p}\right)\le C(\mu,\lambda,C_0).$$

Suppose $\{\boldsymbol{v}_k\}_{k=1}^\infty\in\mathcal{R}$ and $\boldsymbol{v}_k\to\boldsymbol{v}$, in $L^2(0,T^*;H_0^1)$, as $k\to\infty$. This implies that

$$\boldsymbol{v}_k \rightharpoonup \boldsymbol{v},\ w^* \text{ in } L^\infty(0,T^*;H_0^1\cap D^2)\cap L^2(0,T^*;D^{2,q}), \text{ as } k\to\infty.$$

Then it follows from Aubin–Lions Lemma that there exists a convergent subsequence, denoted by $\{\boldsymbol{v}_{k_j}\}_{j=1}^\infty$, such that

$$h(\boldsymbol{v}_{k_j}) \to h, \quad \text{in } C([0,T^*]\times\mathbb{R}^2), \text{ as } k_j \to \infty.$$

We replace h and $\boldsymbol{v}$ with h_{k_j} and $\boldsymbol{v}_{k_j}$ in (3.8) and pass the limit to obtain that $h = h(\boldsymbol{v})$ is a weak solution of (3.8). The uniqueness of the strong solution yields

$$h(\boldsymbol{v}_k) \to h, \text{in } C([0,T^*]\times\bar{\Omega}), \ h(\boldsymbol{v}_k) \rightharpoonup h, \text{in } L^\infty(0,T^*;W^{1,q}), \text{as } k\to\infty.$$

Secondly, let $\{(m_k,\boldsymbol{v}_k)\}_{n=1}^\infty \in \mathcal{R}$ and $(\boldsymbol{v}_k,m_k) \to (\boldsymbol{v},m)$ in $(L^2(0,T^*;H_0^1))^2$ as $k\to\infty$. By Aubin–Lions lemma, we have

$$\begin{aligned}
c(\boldsymbol{v}_{k_j},m_{k_j}) &\to c, \text{in } C([0,T^*];H^1)\cap L^2(0,T^*;H^3),\\
m_k &\rightharpoonup m, \text{in } L^\infty(0,T^*;H^1)\cap L^2(0,T^*;D^2)
\end{aligned}$$

as $k\to\infty$. Taking the limit in the equation (3.16), where m, c, $\boldsymbol{v}$ are replaced by m_{k_j}, $c(m_{k_j},\boldsymbol{v}_{k_j})$, $\boldsymbol{v}_{k_j}$, we obtain that $c = c(m,\boldsymbol{v})$ is a weak solution of (3.16). In analogy with the previous argument, we have

$$\begin{aligned}
c(m_k,\boldsymbol{v}_k) &\to c, \text{in } C([0,T^*];H^1)\cap L^2(0,T^*;H^3),\\
c(m_k,\boldsymbol{v}_k) &\rightharpoonup c, \text{in } \ L^\infty(0,T^*;H^1)\cap L^2(0,T^*;H^2)
\end{aligned}$$

as $k\to\infty$. After that, let $n_{k_j} = \mathcal{T}_1(\boldsymbol{v}_{k_j}, c(m_{k_j},\boldsymbol{v}_{k_j})) = \mathcal{T}_1(\boldsymbol{v}_{k_j},c_{k_j})$. In view of Aubin–Lions lemma, we obtain up to a subsequence,

$$n_{k_j} \to n, \text{in } \ C([0,T^*];H_0^1)\cap L^2(0,T^*;H^2), \text{as } k_j\to\infty.$$

Passing the limit in (3.27), where $\boldsymbol{v}$, c, n are replaced by $\boldsymbol{v}_{k_j}$, c_{k_j}, n_{k_j}, we obtain that $n = \mathcal{T}_1(\boldsymbol{v},c)$ is a weak solution of (3.27) with

$$\begin{aligned}
\mathcal{T}_1(\boldsymbol{v}_k, c(m_k,c_k)) &\to n, \text{in } \ C([0,T^*];H_0^1)\cap L^2(0,T^*;H^2) \text{ as } k\to\infty.\\
\mathcal{T}_1(\boldsymbol{v}_k, c(m_k,c_k)) &\rightharpoonup n, \text{in } \ L^\infty(0,T^*;H_0^1)\cap L^2(0,T^*;H^2) \text{ as } k\to\infty.
\end{aligned}$$

At last, let $\boldsymbol{u}_k = \mathcal{T}_2(\boldsymbol{v}_k,h(\boldsymbol{v}_k),n_k) = \mathcal{T}_2(\boldsymbol{v}_k,h(\boldsymbol{v}_k),\mathcal{T}_1(\boldsymbol{v}_k,c(m_k,\boldsymbol{v}_k)))$. It follows from Aubin–Lions lemma that

$$\boldsymbol{u}_{k_j} \to \boldsymbol{u}, \text{in } \ C([0,T^*];H^2), \ \boldsymbol{u}_{k_j} \rightharpoonup \boldsymbol{u}, \text{in } \ L^\infty(0,T^*;H_0^1\cap D^2)\cap L^2(0,T^*;D^{2,q}) \text{ as } k_j\to\infty.$$

Taking the limit in (3.31), where h, $\boldsymbol{u}$, n, $\boldsymbol{v}$ are replaced by $h(\boldsymbol{v}_{k_j})$, $\boldsymbol{u}_{k_j}$, n_{k_j}, $\boldsymbol{v}_{k_j}$, we obtain that $\boldsymbol{u} = \mathcal{T}_2(\boldsymbol{v},h(\boldsymbol{u}),n) = \mathcal{T}_2(\boldsymbol{v},h(\boldsymbol{v}),\mathcal{T}_1(\boldsymbol{v},c(\boldsymbol{v},m)))$ is a weak solution of (3.31) with

$$\boldsymbol{u}_k \to \boldsymbol{u}, \text{in } \ C([0,T^*];H_0^2).$$

Therefore $(\boldsymbol{u},n) = \mathcal{T}(\boldsymbol{v},n)$ is a continuous operator in $\mathcal{B} = (L^2(0,T^*;H_0^1))^2$. By Schauder fixed point theorem, there exists $(\boldsymbol{u},n)\in\mathcal{R}$ such that

$$\big(\mathcal{T}_2(\boldsymbol{u},h(\boldsymbol{u}),\mathcal{T}_1(\boldsymbol{u},c(\boldsymbol{u},m)),\mathcal{T}_1(\boldsymbol{u},c(\boldsymbol{u},n))\big) = (\boldsymbol{u},n).$$

Moreover, for all $p \in [2, q]$,

$$\begin{aligned}
&\sup_{0\le s\le T^*}\Big(\|c\|_{H^1}^2+\|c_t\|_{L^2}^2+\|h\|_{L^2\cap L^\infty}^2+\|\nabla h\|_{L^p}^2+\|h_t\|_{L^p}^2+\|n\|_{H^1}^2+\|n_t\|_{L^2}^2\\
&\quad+\|\sqrt{h}\boldsymbol{u}\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{H^1}^2+\|\sqrt{h}\dot{\boldsymbol{u}}\|_{L^2}^2\Big)\\
&\quad+\int_0^{T^*}\Big(\|\nabla c\|_{H^2}^2+\|\nabla c_t\|_{L^2}^2+\|\nabla n\|_{H^1}^2+\|\nabla n_t\|_{L^2}^2+\|\nabla^2\boldsymbol{u}\|_{L^q}^2+\|\nabla\dot{\boldsymbol{u}}\|_{L^2}^2\Big)\mathrm{d}t \qquad (3.40)\\
&\le C(\mu,\lambda,C_0).
\end{aligned}$$

And by (3.18) and (3.30), we have the higher regularity for n, c

$$\sup_{0\le s\le T^*}\Big(\|\nabla^2 c\|_{H^1}^2+\|\nabla c_t\|_{L^2}^2+\|\nabla^2 n\|_{L^2}^2\Big)+\int_0^{T^*}\Big(\|c_{tt}\|_{L^2}^2+\|\nabla^2 c_t\|_{L^2}^2\Big)\mathrm{d}t\le C(\mu,\lambda,C_0). \tag{3.41}$$

Step 2 For general $h_0\ge 0$, for each $\delta\in(0,1)$, choose $h_0^\delta=h_0+\delta$, and let $\boldsymbol{u}_0^\delta\in D_0^1\cap D^2$ be the unique solution to the problem

$$-\mu\Delta\boldsymbol{u}_0^\delta-(\mu+\lambda)\nabla(\operatorname{div}\boldsymbol{u}_0^\delta)+(h_0^\delta)^2\nabla n_0+\frac{1}{2}(1+n_0)\nabla(h_0^\delta)^2=\sqrt{h_0^\delta}g.$$

We denote by $(n^\delta, c^\delta, h^\delta, \boldsymbol{u}^\delta)$ the unique local strong solution in $[0, T^*]$ to the problems (1.1)–(1.2) with the initial data replaced by $(n_0, c_0, h_0^\delta, \boldsymbol{u}_0^\delta)$. Obviously, $(n^\delta, c^\delta, h^\delta, \boldsymbol{u}^\delta)$ satisfy the uniformly bounds (3.40)–(3.41) and T^* is independent of δ. Then letting $\delta\to 0^+$, we obtain a strong solution $(n, c, h, \boldsymbol{u})$ to problem (1.1)–(1.2) satisfying the estimates (3.40)–(3.41). Please refer to [12] for details.

Step 3 Finally, the uniqueness of the strong solution follows directly from the above two estimates (3.40)–(3.41). For the case $\Omega=\mathbb{R}^2$ and $\widetilde{h}>0$, we can bound $\|\boldsymbol{u}\|_{L^\infty(0,T^*;L^2(\mathbb{R}^2))}$ and $\|(\boldsymbol{u}_t,\dot{\boldsymbol{u}})\|_{L^2(0,T^*;L^2(\mathbb{R}^2))}$ according to Lemma 5. As similar in [13], it is easy to get the unique strong solution $(n, c, h, \boldsymbol{u})$ satisfying (3.39). The proof of Proposition 1 is thus completed. □

4. Blow up criterion and proof of Theorem 1

In this section, we aim to complete the proof of Theorem 1. It remains to show the blow-up criterion. We use the idea in [22] and argue by contradiction. Let $(n, c, h, \boldsymbol{u})$ be a strong solution of the chemotaxis–shallow water system (1.1)–(1.2) together with (1.4) or (1.5). And suppose $\widetilde{T}$ is the maximal time so that the strong solution exists. Assume further that

$$\lim_{T\to\widetilde{T}}\Big(\int_0^T\|\mathcal{D}(\boldsymbol{u})\|_{L^\infty}\mathrm{d}t+\sup_{0\le t\le T}\|n\|_{L^\infty}\Big)<M<+\infty, \tag{4.1}$$

where $\mathcal{D}(\boldsymbol{u})$ is the deformation tensor with $\mathcal{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t)$. First, (3.5) and (4.1) yield immediately

$$\sup_{0\le t\le T} \|h\|_{L^\infty} \le C, \quad 0 \le T < \widetilde{T}. \tag{4.2}$$

Here and hereafter, C denotes a generic constant depending only on C_0, M, T, the initial data and the domain Ω. Then the standard energy estimate and (4.1), (4.2) lead to

Lemma 8. *Let $(n, c, h, \boldsymbol{u})$ be a strong solution in Proposition 1. Under the condition (4.1), it holds that*

$$\sup_{0\le t\le T} \left(\|n\|_{L^2}^2 + \|c\|_{L^2}^2 + \|\sqrt{h}\boldsymbol{u}\|_{L^2}^2 + \|h - \tilde{h}\|_{L^2}^2\right) + \int_0^T \int \left(|\nabla n|^2 + |\nabla c|^2 + |\nabla \boldsymbol{u}|^2\right) \mathrm{d}x\mathrm{d}t \le C,$$

for $0 \le T < \widetilde{T}$. Moreover, for all $p \in [2, +\infty)$,

$$\sup_{0\le t\le T} \left(\|n\|_{L^p}^p + \|c\|_{L^p}^p\right) \le C, \quad 0 \le T < \widetilde{T}.$$

Proof. We multiply the equation $(1.1)_2$ by c and integrate the resulting equation to get

$$\frac{1}{2}\frac{d}{dt}\int |c|^2 \mathrm{d}x + \int |\nabla c|^2 \mathrm{d}x \le \left(\|\mathrm{div}\boldsymbol{u}\|_{L^\infty} + \|n\|_{L^\infty}\right)\|c\|_{L^2}^2, \tag{4.3}$$

which together with Gronwall's inequality and (4.1) gives

$$\sup_{0\le t\le T} \|c\|_{L^2}^2 + \int_0^T \int |\nabla c|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{4.4}$$

The same procedure as in (4.3) leads to

$$\frac{1}{2}\frac{d}{dt}\int |n|^2 \mathrm{d}x + \int |\nabla n|^2 \mathrm{d}x \le \|\mathrm{div}\boldsymbol{u}\|_{L^\infty}\|n\|_{L^2}^2 + \varepsilon\|\nabla n\|_{L^2}^2 + C(\varepsilon)\|n\|_{L^\infty}^2\|\nabla c\|_{L^2}^2.$$

Thus,

$$\sup_{0\le t\le T} \|n\|_{L^2}^2 + \int_0^T \int |\nabla n|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{4.5}$$

In order to get the estimate for $(h, \boldsymbol{u})$, recalling that h satisfies the equation

$$(h - \tilde{h})_t + \boldsymbol{u} \cdot \nabla(h - \tilde{h}) = -h \,\mathrm{div}\boldsymbol{u}, \tag{4.6}$$

we multiply (4.6) and $(1.1)_4$ by $h - \tilde{h}$ and $\boldsymbol{u}$ respectively, add them up, integrate over Ω and apply Gronwall's inequality together with (4.1), (4.2) and (4.5) to deduce that

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$$\sup_{0\le t\le T}\left(\|h-\tilde{h}\|_{L^2}^2+\|\sqrt{h}\boldsymbol{u}\|_{L^2}^2\right)+\int_0^T\int|\nabla\boldsymbol{u}|^2\mathrm{d}x\mathrm{d}t\le C. \tag{4.7}$$

Hence the first estimate is proved by virtue of (4.4), (4.5) and (4.7). Our next task is to show the second estimate. It follows from (4.1), (4.5) and the interpolation theorem that

$$\sup_{0\le t\le T}\|n\|_{L^p}^p\le C,\ \text{ for any } p\in[2,+\infty). \tag{4.8}$$

In addition, multiplying $(1.1)_1$ by c^{p-1}, $p\in(2,+\infty)$ and integrating over Ω yield

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int c^p\mathrm{d}x+\int(p-1)c^{p-2}|\nabla c|^2\mathrm{d}x&=\int\left((p-1)c^{p-1}\nabla c\cdot\boldsymbol{u}-nc^p\right)\mathrm{d}x\\&\le\left(\frac{p-1}{p}\|\mathrm{div}\boldsymbol{u}\|_{L^\infty}+\|n\|_{L^\infty}\right)\|c\|_{L^p}^p,\end{aligned}$$

which leads to

$$\sup_{0\le t\le T}\|c\|_{L^p}^p+\int_0^T\|\nabla c^{\frac{p}{2}}\|_{L^2}^2\mathrm{d}t\le C(p).$$

This along with (4.8) gives us the second estimate. □

The higher regularity estimates for $(n,c,h,\boldsymbol{u})$ will be given in the following lemma.

Lemma 9. *Let $(n,c,h,\boldsymbol{u})$ be a strong solution in Proposition 1. Under the condition (4.1), it holds that for $0\le T<\widetilde{T}$,*

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\nabla n\|_{L^2}^2+\|\nabla c\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla h\|_{L^2}^2\right)\\&\quad+\int_0^T\int\left(|\nabla^2n|^2+|\nabla^2c|^2+|\nabla^2\boldsymbol{u}|^2+|n_t|^2+|c_t|^2\right)\mathrm{d}x\mathrm{d}t\le C.\end{aligned}$$

Proof. First of all, we evaluate the integrals involving $(h,\boldsymbol{u})$. We operate ∇ to each term of equation $(1.1)_3$, multiply the resulting equation by $2\nabla h$ and integrate over Ω to find

$$\begin{aligned}\frac{d}{dt}\int|\nabla h|^2\mathrm{d}x&=-\int\left(\nabla\left(|\nabla h|^2\right)\cdot\boldsymbol{u}+2|\nabla h|^2\mathrm{div}\boldsymbol{u}+\nabla h^2\cdot\nabla\mathrm{div}\boldsymbol{u}\right)\mathrm{d}x\\&\le C(\varepsilon)\|\nabla h\|_{L^2}^2\left(\|\mathcal{D}(\boldsymbol{u})\|_{L^\infty}+\|h\|_{L^\infty}^2\right)+\varepsilon\|\nabla^2\boldsymbol{u}\|_{L^2}^2.\end{aligned} \tag{4.9}$$

With the boundary condition (1.4) or (1.5), the last term of the inequality above is controlled from standard L^2-theory of elliptic system by

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$$
\begin{aligned}
\left\|\nabla^2 \boldsymbol{u}\right\|_{L^2}^2 &\leq C\left\|\mu\Delta\boldsymbol{u}+(\mu+\lambda)\nabla\mathrm{div}\boldsymbol{u}\right\|_{L^2}^2+C\|\nabla\boldsymbol{u}\|_{L^2}^2\\
&\leq C\int h^{-1}\left|\mu\Delta\boldsymbol{u}+(\mu+\lambda)\nabla\mathrm{div}\boldsymbol{u}\right|^2\mathrm{dx}+C\|\nabla\boldsymbol{u}\|_{L^2}^2
\end{aligned}
$$

for $\|h\|_{L^\infty}^{-1}\geq C^{-1}>0$. Furthermore, multiplying (1.1)$_4$ by $h^{-1}\left[\mu\Delta\boldsymbol{u}+(\mu+\lambda)\nabla\mathrm{div}\boldsymbol{u}\right]$ and integrating over Ω, we have

$$
\begin{aligned}
&\frac{d}{dt}\int\left(\frac{\mu}{2}|\nabla\boldsymbol{u}|^2+\frac{\mu+\lambda}{2}(\mathrm{div}\boldsymbol{u})^2\right)\mathrm{dx}+\int h^{-1}|\mu\Delta\boldsymbol{u}+(\mu+\lambda)\nabla(\mathrm{div}\boldsymbol{u})|^2\mathrm{dx}\\
&=-\mu\int(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla\times\mathrm{curl}\boldsymbol{u})\mathrm{dx}+(2\mu+\lambda)\int\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla(\mathrm{div}\boldsymbol{u})\mathrm{dx}\\
&\quad+\int\frac{1+n}{2}\nabla(h^2)\cdot[\mu\Delta\boldsymbol{u}+(\lambda+\mu)\nabla(\mathrm{div}\boldsymbol{u})]h^{-1}\mathrm{dx}\\
&\quad+\int h\nabla n\cdot(\mu\Delta\boldsymbol{u}+(\lambda+\mu)\nabla(\mathrm{div}\boldsymbol{u}))\mathrm{dx}\\
&=:\sum_{i=1}^{4}\mathrm{I}_i,
\end{aligned}
\tag{4.10}
$$

where $\Delta\boldsymbol{u}=\nabla(\mathrm{div}\boldsymbol{u})-\nabla\times\mathrm{curl}\boldsymbol{u}$ has been used. The estimates for I_1, I_2 are similar as in [22], we only sketch them here. Since $\boldsymbol{u}\times\mathrm{curl}\boldsymbol{u}=\frac{1}{2}\nabla(|\boldsymbol{u}|^2)-\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ and $\nabla\times(a\times b)=(b\cdot\nabla)a-(a\cdot\nabla)b+(\mathrm{div}b)a-(\mathrm{div}a)b$, we observe that

$$
\begin{aligned}
|\mathrm{I}_1|&=\left|\mu\int(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla\times\mathrm{curl}\boldsymbol{u})\mathrm{dx}\right|=\mu\left|\frac{1}{2}\int|\mathrm{curl}\boldsymbol{u}|^2\mathrm{div}\boldsymbol{u}\mathrm{dx}-\int\mathrm{curl}\boldsymbol{u}\cdot\mathcal{D}(\boldsymbol{u})\mathrm{curl}\boldsymbol{u}\mathrm{dx}\right|\\
&\leq C\|\nabla\boldsymbol{u}\|_{L^2}^2\|\mathcal{D}(\boldsymbol{u})\|_{L^\infty},\\
|\mathrm{I}_2|&=(2\mu+\lambda)\left|\int\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla(\mathrm{div}\boldsymbol{u})\mathrm{dx}\right|=(2\mu+\lambda)\left|\int\nabla\boldsymbol{u}:\nabla\boldsymbol{u}^t\mathrm{div}\boldsymbol{u}\mathrm{dx}+\frac{1}{2}\int(\mathrm{div}\boldsymbol{u})^3\mathrm{dx}\right|\\
&\leq C\|\nabla\boldsymbol{u}\|_{L^2}^2\|\mathcal{D}(\boldsymbol{u})\|_{L^\infty},\\
|\mathrm{I}_3|&=\left|\int\frac{1+n}{2}(\nabla h)\cdot[\mu\Delta\boldsymbol{u}+(\mu+\lambda)\nabla(\mathrm{div}\boldsymbol{u})]\,\mathrm{dx}\right|\leq\varepsilon\|\nabla^2\boldsymbol{u}\|_{L^2}^2+C(\varepsilon)\|1+n\|_{L^\infty}^2\|\nabla h\|_{L^2}^2,\\
|\mathrm{I}_4|&=\left|\int(h\nabla n)\cdot(\mu\Delta\boldsymbol{u}+(\mu+\lambda)\nabla(\mathrm{div}\boldsymbol{u}))\mathrm{dx}\right|\leq\varepsilon\left\|\nabla^2\boldsymbol{u}\right\|_{L^2}^2+C(\varepsilon)\|h\|_{L^\infty}^2\|\nabla n\|_{L^2}^2.
\end{aligned}
$$

We add (4.9), (4.10) up and combine all the estimates I_i, $i=1,\cdots,4$ to find

$$
\begin{aligned}
&\frac{d}{dt}\int\left(\frac{\mu}{2}|\nabla\boldsymbol{u}|^2+\frac{\mu+\lambda}{2}(\mathrm{div}\boldsymbol{u})^2+\|\nabla h\|_{L^2}^2\right)\mathrm{dx}+C_0\left\|\nabla^2\boldsymbol{u}\right\|_{L^2}\\
&\leq C\|\nabla h\|_{L^2}^2\left(1+\|n\|_{L^\infty}^2\right)+\varepsilon\left\|\nabla^2\boldsymbol{u}\right\|_{L^2}^2\\
&\quad+C\|\nabla\boldsymbol{u}\|_{L^2}^2\|\mathcal{D}(\boldsymbol{u})\|_{L^\infty}+C\|h\|_{L^\infty}^2\|\nabla n\|_{L^2}^2.
\end{aligned}
$$

An application of Lemma 8 and Gronwall's inequality gives

$$\sup_{0\le t<T}\|\nabla \boldsymbol{u}\|_{L^2}^2+\sup_{0\le t<T}\|\nabla h\|_{L^2}^2+\int_0^T\int\left|\nabla^2\boldsymbol{u}\right|^2\mathrm{d}x\mathrm{d}t\le C. \tag{4.11}$$

This implies the following bound

$$\int_0^T\|\boldsymbol{u}\|_{L^\infty}^2\mathrm{d}t\le\int_0^T(\|\boldsymbol{u}\|_{L^2}^2+\|\nabla^2\boldsymbol{u}\|_{L^2}^2)\mathrm{d}t\le C \tag{4.12}$$

due to Lemma 5. Next, we aim to deduce the higher regularity of (n,c). Operating ∇ to $(1.1)_2$, multiplying the resulting equation by ∇c and integrating over Ω lead to

$$\frac{1}{2}\frac{d}{dt}\int|\nabla c|^2\mathrm{d}x+\int\left|\nabla^2 c\right|^2\mathrm{d}x=-\int \mathrm{div}(\nabla(c\boldsymbol{u}))\cdot\nabla c\mathrm{d}x-\int\nabla(nc)\cdot\nabla c\mathrm{d}x=:\bar{\mathrm{I}}_1+\bar{\mathrm{I}}_2.$$

Direct computation shows that

$$\begin{aligned}|\bar{\mathrm{I}}_1|&\le\left\|\nabla^2c\right\|_{L^2}(\|\nabla c\|_{L^2}\|\boldsymbol{u}\|_{L^\infty}+\|c\|_{L^4}\|\nabla\boldsymbol{u}\|_{L^4})\\&\le\varepsilon\left\|\nabla^2c\right\|_{L^2}^2+C(\varepsilon)\left(\|\nabla c\|_{L^2}^2\|\boldsymbol{u}\|_{L^\infty}^2+\|c\|_{L^2}^2\|\nabla c\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{L^2}^2\|\nabla^2\boldsymbol{u}\|_{L^2}^2\right),\\|\bar{\mathrm{I}}_2|&\le\varepsilon\left\|\nabla^2c\right\|_{L^2}^2+C(\varepsilon)\|n\|_{L^\infty}^2\|c\|_{L^2}^2.\end{aligned}$$

Thus, one sees that

$$\sup_{0\le t\le T}\|\nabla c\|_{L^2}^2+\int_0^T\left\|\nabla^2c\right\|_{L^2}^2\mathrm{d}t\le C. \tag{4.13}$$

For the regularity of n, we proceed as previously in (4.13) and we only need to check the term $\int\nabla\mathrm{div}(n\nabla c)\cdot\nabla n d\mathrm{x}$. It is verified that

$$\begin{aligned}\int\nabla\mathrm{div}(n\nabla c)\cdot\nabla n d\mathrm{x}&\le\varepsilon\left\|\nabla^2n\right\|_{L^2}^2+C\left(\|\nabla n\|_{L^4}^2\|\nabla c\|_{L^4}^2+\|n\|_{L^\infty}^2\|\nabla^2c\|_{L^2}^2\right)\\&\le\varepsilon\left\|\nabla^2n\right\|_{L^2}^2+C\|n\|_{L^\infty}^2\left\|\nabla^2c\right\|_{L^2}^2+C\|\nabla n\|_{L^2}^2\|\nabla c\|_{L^2}^2\|\nabla^2c\|_{L^2}^2.\end{aligned}$$

This, together with Lemma 8, (4.12) and (4.13), implies

$$\sup_{0\le t\le T}\|\nabla n\|_{L^2}^2+\int_0^T\left\|\nabla^2n\right\|_{L^2}^2\mathrm{d}t\le C. \tag{4.14}$$

Moreover, we observe that

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$$
\begin{aligned}
\|n_t\|_{L^2} &\le \|-\operatorname{div}(n\boldsymbol{u})\|_{L^2} + \|\Delta n\|_{L^2} + \|\operatorname{div}(n\nabla c)\|_{L^2} \\
&\le \|\nabla n\|_{L^2}\|\boldsymbol{u}\|_{L^\infty} + \|n\|_{L^\infty}\|\nabla \boldsymbol{u}\|_{L^2} + \left\|\nabla^2 n\right\|_{L^2} + \|\nabla n\|_{L^4}\|\nabla c\|_{L^4} + \|n\|_{L^\infty}\|\nabla^2 c\|_{L^2}, \\
\|c_t\|_{L^2} &\le \|\nabla c\|_{L^2}\|\boldsymbol{u}\|_{L^\infty} + \|c\|_{L^4}\|\nabla \boldsymbol{u}\|_{L^4} + \left\|\nabla^2 c\right\|_{L^2} + \|n\|_{L^\infty}\|c\|_{L^2}.
\end{aligned}
$$

These two inequalities give us the bound

$$
\int_0^T \left(\|n_t\|_{L^2}^2 + \|c_t\|_{L^2}^2\right)\mathrm{d}t \le C. \tag{4.15}
$$

Consequently the conclusion follows from (4.11) and (4.13)–(4.15). □

We next improve the regularity of c and n.

Lemma 10. *Let $(n, c, h, \boldsymbol{u})$ be a strong solution in Proposition 1. Under the condition (4.1), it holds that for $0 \le T < \widetilde{T}$,*

$$
\sup_{0\le t\le T}\left(\|\nabla^2 c\|_{L^2}^2 + \|\nabla^2 n\|_{L^2}^2\right) + \int_0^T \left(\|\nabla c\|_{H^2}^2 + \|\nabla n\|_{H^2}^2 + \|\nabla n_t\|_{L^2}^2 + \|\nabla c_t\|_{L^2}^2\right)\mathrm{d}t \le C.
$$

Proof. Operating $\partial_i\partial_j$ to the equation $(1.1)_2$, multiplying the resulting equation by $\partial_i\partial_j c$ and integrating over Ω give

$$
\frac{d}{dt}\int |\nabla^2 c|^2 \mathrm{dx} + \|\nabla c\|_{H^2}^2 = \int \left[-\nabla^2(\operatorname{div}(cu)) - \nabla^2(nc)\right]\nabla^2 c\mathrm{dx} =: \mathrm{K}_1 + \mathrm{K}_2.
$$

Straightforward computation shows that

$$
\begin{aligned}
|\mathrm{K}_1| &\le \varepsilon\|\nabla c\|_{H^2}^2 + C(\varepsilon)\left(\left\|\nabla^2 c\right\|_{L^2}^2\|\boldsymbol{u}\|_{L^\infty}^2 + \|\nabla c\|_{L^4}^2\|\nabla \boldsymbol{u}\|_{L^4}^2 + \|c\|_{L^\infty}^2\|\nabla^2\boldsymbol{u}\|_{L^2}^2\right) \\
&\le \varepsilon\|\nabla c\|_{H^2}^2 + C(\varepsilon)\left\|\nabla^2 c\right\|_{L^2}^2\left(\|\boldsymbol{u}\|_{L^\infty}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla^2\boldsymbol{u}\|_{L^2}^2\right) \\
&\quad + C(\varepsilon)\|\nabla^2\boldsymbol{u}\|_{L^2}^2(\|c\|_{L^2}^2 + \|\nabla\boldsymbol{u}\|_{L^2}^2), \\
|\mathrm{K}_2| &\le \varepsilon\|\nabla c\|_{H^2}^2 + C(\varepsilon)\,\|\nabla(nc)\|_{L^2}^2 \le \varepsilon\|\nabla c\|_{H^2}^2 + C(\varepsilon)\|n\|_{L^\infty}^2\|\nabla c\|_{L^2}^2 \\
&\quad + C(\varepsilon)\|\nabla n\|_{L^2}^2\|c\|_{L^2}\left\|\nabla^2 c\right\|_{L^2}.
\end{aligned}
$$

We thus conclude that

$$
\sup_{0\le t\le T}\left\|\nabla^2 c\right\|_{L^2}^2 + \int_0^T \|\nabla c\|_{H^2}^2\mathrm{d}t \le C.
$$

Then we proceed the same argument with respect to the equation $(1.1)_1$,

$$\begin{aligned}
&\frac{d}{dt}\int \left|\nabla^2 n\right|^2 \mathrm{dx} + \|\nabla n\|_{H^2}^2 = \int \left[-\nabla^2(\mathrm{div}(n\boldsymbol{u})) - \nabla^2 \mathrm{div}(n\cdot\nabla c)\right]\nabla^2 n \mathrm{dx} \\
&\quad \le \varepsilon\|\nabla n\|_{H^2}^2 + C(\varepsilon)\|\nabla^2 n\|_{L^2}^2\left(\|\boldsymbol{u}\|_{L^\infty}^2 + \|\nabla n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2\|\nabla^2 c\|_{L^2}^2\right) \\
&\qquad + C(\varepsilon)\|\nabla^2 \boldsymbol{u}\|_{L^2}^2(\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|n\|_{L^\infty}^2) + C(\varepsilon)\|\nabla c\|_{H^2}^2(\|\nabla^2 c\|_{L^2}^2 + \|n\|_{L^\infty}^2),
\end{aligned}$$

and deduce that

$$\sup_{0\le t\le T}\left\|\nabla^2 n\right\|_{L^2} + \int_0^T \|\nabla n\|_{H^2}^2 \mathrm{d}t \le C.$$

Finally, operating ∇ to the equation (1.1)$_1$, multiplying the resulting equation by ∇n_t and integrating over Ω, we find that

$$\begin{aligned}
\int |\nabla n_t|^2 \mathrm{dx} + \frac{d}{dt}\int |\Delta n|^2 \mathrm{dx} &\le \varepsilon\|\nabla n_t\|_{L^2}^2 + C(\varepsilon)\left\|\nabla^2 n\right\|_{L^2}^2 \|\boldsymbol{u}\|_{L^\infty}^2 \\
&\quad + C(\varepsilon)\|\nabla n\|_{L^4}^2\big(\|\nabla \boldsymbol{u}\|_{L^4}^2 + \left\|\nabla^2 c\right\|_{L^4}^2\big) \\
&\quad + C(\varepsilon)\|\nabla^2 n\|_{L^4}^2 \|\nabla c\|_{L^4}^2 + C(\varepsilon)\|n\|_{L^\infty}^2\big(\left\|\nabla^2 \boldsymbol{u}\right\|_{L^2}^2 + \|\nabla c\|_{H^2}^2\big).
\end{aligned}$$

Therefore, Gronwall's inequality yields $\int_0^T \|\nabla n_t\|_{L^2}^2 \mathrm{d}t \le C$. Similarly, $\int_0^T \|\nabla c_t\|_{L^2}^2 \mathrm{d}t \le C$. These two estimates along with (4.11), (4.13), (4.14) give us the desired inequality. □

We next improve the regularity of h and $\boldsymbol{u}$.

Lemma 11. *Let $(n, c, h, \boldsymbol{u})$ be a strong solution in Proposition 1. Under the condition (4.1), it holds that for $0 \le T < \widetilde{T}$ and $q \in [2, +\infty)$,*

$$\sup_{0\le t\le T}\left(\|\sqrt{h}\boldsymbol{u}_t\|_{L^2}^2 + \|\nabla^2 \boldsymbol{u}\|_{L^2}^2 + \|\nabla h\|_{L^q}^2\right) + \int_0^T \left(\|\nabla \boldsymbol{u}_t\|_{L^2}^2 + \|\nabla^2 \boldsymbol{u}\|_{L^q}^2\right) \mathrm{d}t \le C.$$

Proof. We first differentiate the equation (1.1)$_4$ with respect to t, multiply the resulting equation by $\boldsymbol{u}_t$ and integrate over Ω to discover

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int h|\boldsymbol{u}_t|^2 \mathrm{dx} + \int (\mu|\nabla \boldsymbol{u}_t|^2 + (\mu+\lambda)|\mathrm{div}\boldsymbol{u}_t|^2)\mathrm{dx} \\
&\quad = \int \Big[-h(\boldsymbol{u}_t\cdot\nabla \boldsymbol{u})\cdot \boldsymbol{u}_t - h\boldsymbol{u}\cdot\nabla((\boldsymbol{u}\cdot\nabla \boldsymbol{u})\boldsymbol{u}_t) - \frac{1}{2}n_t(\nabla h^2)\cdot \boldsymbol{u}_t - \frac{1+n}{2}(\nabla(h^2)_t)\cdot \boldsymbol{u}_t \\
&\qquad - (h^2)_t \boldsymbol{u}_t\cdot\nabla n - h^2(\nabla n_t)\cdot \boldsymbol{u}_t\Big]\mathrm{dx} =: \sum_{i=1}^6 \mathrm{L}_i.
\end{aligned}$$

It is verified that

$$
\begin{aligned}
|\mathrm{L}_1| &= \left|\int h(\boldsymbol{u}_t\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{u}_t\mathrm{dx}\right| \le \|\sqrt{h}\boldsymbol{u}_t\|_{L^2}\|h\|_{L^\infty}^{\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{L^4}\|\boldsymbol{u}_t\|_{L^4}\\
&\le \varepsilon\|\nabla\boldsymbol{u}_t\|_{L^2}^2 + C\|\sqrt{h}\boldsymbol{u}_t\|_{L^2}^2\left(\|h\|_{L^\infty}^2\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|h-\tilde{h}\|_{L^2}^2\|\nabla^2\boldsymbol{u}\|_{L^2}^2\right)\\
&\quad + C\|h\|_{L^\infty}^{\frac{2}{3}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{2}{3}}\|\nabla^2\boldsymbol{u}\|_{L^2}^{\frac{2}{3}},\\
|\mathrm{L}_2| &= \left|\int h\boldsymbol{u}\cdot\nabla((\boldsymbol{u}\cdot\nabla\boldsymbol{u})\boldsymbol{u}_t)\mathrm{dx}\right|\\
&\le C\|\sqrt{h}\boldsymbol{u}_t\|_{L^2}\||\nabla\boldsymbol{u}|^2\|_{L^2}\|\boldsymbol{u}\|_{L^\infty} + C\|\boldsymbol{u}_t\|_{L^4}\|\boldsymbol{u}^2\|_{L^4}\|\nabla^2\boldsymbol{u}\|_{L^2} + \varepsilon\|\nabla\boldsymbol{u}_t\|_{L^2}^2\\
&\quad + C(\varepsilon)\|h\|_{L^\infty}^2\|\boldsymbol{u}^2\|_{L^4}\|\nabla\boldsymbol{u}\|_{L^4}^2\\
&\le \varepsilon\|\nabla\boldsymbol{u}_t\|_{L^2}^2 + C(\varepsilon)\|\sqrt{h}\boldsymbol{u}_t\|_{L^2}^2\|\boldsymbol{u}\|_{L^\infty}^2 + C(\varepsilon)\|\nabla\boldsymbol{u}\|_{L^2}^2\|\nabla^2\boldsymbol{u}\|_{L^2}^2\\
&\quad + C(\varepsilon)\|\boldsymbol{u}_t\|_{L^2}^{\frac{2}{3}}\|\boldsymbol{u}\|_{L^2}^{\frac{4}{3}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{4}{3}}\|\nabla^2\boldsymbol{u}\|_{L^2}^{\frac{4}{3}} + C(\varepsilon)\|\boldsymbol{u}\|_{L^2}^2\|\nabla\boldsymbol{u}\|_{L^2}^3\|\nabla^2\boldsymbol{u}\|_{L^2},\\
|\mathrm{L}_3| &= \left|-\frac{1}{2}\int n_t(\nabla h^2)\boldsymbol{u}_t\mathrm{dx}\right| \le C\|\nabla n_t\|_{L^2}^2\|h\|_{L^\infty}^{\frac{3}{2}}\\
&\quad + C\|\sqrt{h}\boldsymbol{u}_t\|_{L^2}^2 + C(\varepsilon)\|n_t\|_{L^2}^2\|h\|_{L^\infty}^4 + \varepsilon\|\nabla\boldsymbol{u}_t\|_{L^2}^2,\\
|\mathrm{L}_4| &= \left|-\int\frac{1+n}{2}\nabla(h^2)_t\boldsymbol{u}_t\mathrm{dx}\right| \le \left|\int\nabla n(h^2)_t\boldsymbol{u}_t\mathrm{dx}\right| + \left|\int\frac{1+n}{2}(h^2)_t\nabla\boldsymbol{u}_t\mathrm{dx}\right|\\
&\le \|\sqrt{h}\boldsymbol{u}_t\|_{L^2}^2\|\nabla n\|_{L^\infty}^2 + C(\varepsilon)\left(\|n\|_{L^\infty}^2+1\right)\|h\|_{L^\infty}^2\left(\|\nabla h\|_{L^2}^2\|\boldsymbol{u}\|_{L^\infty}^2 + \|h\|_{L^\infty}^2\|\nabla\boldsymbol{u}\|_{L^2}^2\right)\\
&\quad + \varepsilon\|\nabla\boldsymbol{u}_t\|_{L^2}^2,\\
|\mathrm{L}_5| &= \left|-\int(h^2)_t\boldsymbol{u}_t\nabla n\mathrm{dx}\right|\\
&\le C\|\sqrt{h}\boldsymbol{u}_t\|_{L^2}\|\sqrt{h}\|_{L^\infty}\left(\|\nabla h\|_{L^2}\|\boldsymbol{u}\|_{L^\infty} + \|h\|_{L^\infty}\|\mathrm{div}\boldsymbol{u}\|_{L^2}\right)\|\nabla n\|_{L^\infty}\\
&\le C\|\sqrt{h}\boldsymbol{u}_t\|_{L^2}^2\|\nabla n\|_{H^2}^2 + C\|\sqrt{h}\|_{L^\infty}^2\left(\|\nabla h\|_{L^2}^2\|\boldsymbol{u}\|_{L^\infty}^2 + \|h\|_{L^\infty}^2\|\mathrm{div}\boldsymbol{u}\|_{L^2}^2\right),\\
|\mathrm{L}_6| &= \left|\int h^2(\nabla n_t)\cdot\boldsymbol{u}_t\mathrm{dx}\right| \le C\|\nabla n_t\|_{L^2}^2 + C\|\sqrt{h}\boldsymbol{u}_t\|_{L^2}^2\|h^{\frac{3}{2}}\|_{L^\infty}^2.
\end{aligned}
$$

Combining all the estimates of L_i, $i=1,\cdots,6$, we conclude that

$$
\sup_{0\le t\le T}\|\sqrt{h}\boldsymbol{u}_t\|_{L^2}^2 + \int_0^T\|\nabla\boldsymbol{u}_t\|_{L^2}^2\mathrm{d}t \le C. \tag{4.16}
$$

Moreover, according to the equation $(1.1)_4$, we observe that

$$
\begin{aligned}
\left\|\nabla^2\boldsymbol{u}\right\|_{L^2} &\le C\left(\|h^{\frac{1}{2}}\boldsymbol{u}_t\|_{L^2} + \|h\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^2} + \|h\nabla(hn)\|_{L^2} + \left\|\nabla(h^2)\right\|_{L^2}\right)\\
&\le C\Big(\|h^{\frac{1}{2}}\boldsymbol{u}_t\|_{L^2} + \|\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|h\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{L^2}
\end{aligned}
$$

$$+ \|h\|_{L^\infty}\|\nabla h\|_{L^2}\|n\|_{L^\infty} + \|h\|_{L^\infty}^2\|\nabla n\|_{L^2} + \left\|\nabla(h^2)\right\|_{L^2}\Big),$$

which implies from Lemma 9 that

$$\sup_{0\le t\le T}\left\|\nabla^2 \boldsymbol{u}\right\|_{L^2} \le C. \tag{4.17}$$

This, together with the estimate for $\sup_{0\le s\le t}\|\boldsymbol{u}\|_{L^2}$, gives us the boundedness of $\sup_{0\le s\le t}\|\boldsymbol{u}\|_{L^\infty}$. Thus,

$$\begin{aligned}
\left\|\nabla^2\boldsymbol{u}\right\|_{L^q} &\le C\left(\|h\|_{L^\infty}\|\boldsymbol{u}_t\|_{L^q} + \|h\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^q} + \|h\nabla(hn)\|_{L^q} + \left\|\nabla(h^2)\right\|_{L^q}\right)\\
&\le C\|\boldsymbol{u}_t\|_{L^2}^{\frac{2}{q}}\|\nabla\boldsymbol{u}_t\|_{L^2}^{\frac{q-2}{q}} + C\|\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{L^q}\|h\|_{L^\infty}\\
&\quad + C\|\nabla h\|_{L^q}(1+\|h\|_{L^\infty}\|n\|_{L^\infty}) + C\|h\|_{L^\infty}^2\|\nabla n\|_{L^2}^{\frac{q}{2}}\|\nabla^2 n\|_{L^2}^{\frac{q-2}{q}}\\
&\le C\left(\|\nabla\boldsymbol{u}_t\|_{L^2} + \|\nabla h\|_{L^q} + 1\right) \qquad \text{for } q>2,
\end{aligned} \tag{4.18}$$

where Lemma 5, Lemma 8–10 and (4.16)–(4.17) have been used. Finally, operating ∇ on equation $(1.1)_3$, multiplying the resulting equation by $q|\nabla h|^{q-2}\nabla h$ and integrating over Ω lead to

$$\frac{d}{dt}\|\nabla h\|_{L^q} \le C\left(\|\mathcal{D}(\boldsymbol{u})\|_{L^\infty}+1\right)\|\nabla h\|_{L^q} + \|\nabla^2\boldsymbol{u}\|_{L^q} \le C\left(\|\mathcal{D}(\boldsymbol{u})\|_{L^\infty}+1\right)\|\nabla h\|_{L^q}$$

due to (4.18). Therefore, $\sup_{0\le t\le T}\|\nabla h\|_{L^q} \le C$. The conclusion is thus obtained from this estimate and (4.16)–(4.18). □

It suffices from Lemma 8 to Lemma 11 to extend the strong solutions $(n,c,h,\boldsymbol{u})$ in Proposition 1 beyond $t\ge\widetilde{T}$. Indeed, in view of the estimates in Lemma 8 to Lemma 11, $(n,c,h,\boldsymbol{u})|_{t=\widetilde{T}} = \lim_{t\to\widetilde{T}}(n,c,h,\boldsymbol{u})$ satisfies the initial condition (1.6) at the time $t=\widetilde{T}$. Moreover,

$$-\mu\Delta\boldsymbol{u} - (\mu+\lambda)\nabla\mathrm{div}\,\boldsymbol{u} + h^2\nabla n + \frac{1}{2}(1+n)\nabla h^2|_{t=\widetilde{T}} = \lim_{t\to\widetilde{T}}(h\boldsymbol{u}_t + h\boldsymbol{u}\cdot\nabla\boldsymbol{u}) =: \sqrt{h}g|_{t=\widetilde{T}},$$

with $g|_{t=\widetilde{T}}$ in L^2. Thus $(n,c,h,\boldsymbol{u})|_{t=\widetilde{T}}$ satisfies the compatibility condition (1.3) as well. Therefore, we may take $(n,c,h,\boldsymbol{u})|_{t=\widetilde{T}}$ as the initial data and apply Proposition 1 to extend the local strong solution beyond $\widetilde{T}$. This contradicts the assumption on $\widetilde{T}$.

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