ARTICLE IN PRESS YJDEQ:8507

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term

Xinru Cao [1]

Institute for Mathematical Sciences, Renmin University of China, Zhongguancun Str. 59, 100872 Beijing, China

Received 15 April 2016; revised 24 June 2016

Abstract

The coupled chemotaxis fluid system

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(x,n,c)\cdot \nabla c) - u\cdot \nabla n, & (x,t)\in \Omega\times(0,T), \\ c_t = \Delta c - nc - u\cdot\nabla c, & (x,t)\in \Omega\times(0,T), \\ u_t = \Delta u - \kappa(u\cdot\nabla)u + \nabla P + n\nabla\phi, & (x,t)\in \Omega\times(0,T), \\ \nabla\cdot u = 0, & (x,t)\in \Omega\times(0,T), \end{cases} \tag{$\star$}$$

is considered under the no-flux boundary conditions for n, c and the Dirichlet boundary condition for u on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ $(N=2,3)$, $\kappa \in \{0,1\}$. We assume that $S(x,n,c)$ is a matrix-valued sensitivity under a mild assumption such that $|S(x,n,c)| < S_0(c_0)$ with some non-decreasing function $S_0 \in C^2((0,\infty))$. It contrasts with the related scalar sensitivity case that $(\star)$ does not possess the natural *gradient-like* functional structure. Associated estimates based on the natural functional seem no longer available. In the present work, a global classical solution is constructed under a smallness assumption on $\|c_0\|_{L^\infty(\Omega)}$ and moreover we obtain boundedness and large time convergence for the solution, meaning that small initial concentration of chemical forces stabilization.

E-mail address: caoxinru@gmail.com.

[1] Supported by the Fundamental Research Funds for the Central Universities, and the Research Funds of Renmin University of China (15XNLF21), and by China Postdoctoral Science Foundation (2016M591319).

http://dx.doi.org/10.1016/j.jde.2016.09.007

0022-0396/© 2016 Elsevier Inc. All rights reserved.

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

MSC: 35D05; 35K45

Keywords: Chemotaxis; Navier–Stokes; Global existence; Large time behavior

1. Introduction

In this paper, we study the chemotaxis-Navier–Stokes system

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(x,n,c)\cdot \nabla c) - u\cdot \nabla n, & (x,t)\in \Omega\times(0,T), \\ c_t = \Delta c - nc - u\cdot\nabla c, & (x,t)\in \Omega\times(0,T), \\ u_t = \Delta u - \kappa(u\cdot\nabla)u + \nabla P + n\nabla\phi, & (x,t)\in \Omega\times(0,T), \\ \nabla\cdot u = 0, & (x,t)\in \Omega\times(0,T), \\ \nabla c\cdot\nu = (\nabla n - S(x,n,c)\nabla c)\cdot \nu = 0, u = 0, & (x,t)\in \partial\Omega\times(0,T), \\ n(x,0) = n_0(x), v(x,0) = v_0(x), u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $T \in (0, \infty]$, $\kappa \in \{0, 1\}$, $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ is a bounded domain with smooth boundary and ν denotes the outward normal vector on $\partial\Omega$. Here $S(x, n, c) = (s_{ij}(x, n, c))_{i,j\in\{1,2\}}$ is a matrix-valued function and $\phi \in W^{1,\infty}(\Omega)$.

The PDE system of type (1.1) has been proposed by Tuval [28] to describe the motion of oxygen consumed by bacteria in a drop of water. Here n and c denote the density of Bacteria and concentration of oxygen, respectively. We also write the fluid velocity by u and the associated pressure by P. In addition to random diffusion, the bacteria bias their movement to the favorable direction which is determined by the environment and distribution of oxygen consumed by the bacteria themselves. Meanwhile, both the oxygen and bacteria are supposed to be transported by the surrounding fluid. Let ϕ be a potential function; the fluid motion is described by incompressible Navier–Stokes equation and also influenced by external force $n\nabla\phi$, which can be understood as buoyant, electric or magnetic force of bacterial mass. This mechanism is an important variation of chemotaxis model, which has been extensively studied in the past 40 years; we refer to surveys [10,11,1] for a broad view.

In the paper [28], $S = \chi \cdot \mathbb{I}$ with $\chi \in \mathbb{R}$, thus the cross diffusion term reduces to $\nabla \cdot (n\chi\nabla c)$, which indicates that the bacteria always move towards the higher concentration of oxygen. Therefore, a coupled chemotaxis fluid model reads as

$$\begin{cases} n_t = \Delta n - \nabla\cdot(\chi(c)n\nabla c) - u\cdot\nabla n, \\ c_t = \Delta c - nf(c) - u\cdot\nabla c, \\ u_t = \Delta u - (u\cdot\nabla)u + \nabla P + n\nabla\phi, \\ \nabla\cdot u = 0. \end{cases} \tag{1.2}$$

Now let us mention some work [4,6,18,17,7,26,25,23] on the above system. Actually, under suitable assumptions on χ and f, which are mild enough such that the prototypical choice $\chi(c) \equiv 1$ and $f(c) \equiv c$ is allowed, and a natural gradient-like functional for (1.2) is expressed as

$$\frac{d}{dt}\left(\int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c}\right) + \int_\Omega \left(\frac{|\nabla n|^2}{n} + c|D^2\ln c|^2\right) \le C\int_\Omega |u|^4 \tag{1.3}$$

with some constant $C > 0$. A crucial point to identify the above functional is that the term $\int_\Omega \chi(c)\nabla n \cdot \nabla c$ from a natural Lyapunov functional for the first equation can be canceled by a suitable testing procedure on the second equation. Starting from (1.3), a large number of articles have gained considerable results. Global classical solutions are demonstrated for two-dimensional bounded domains [33]. Beyond this, a deeper understanding of the functional leads to boundedness of solutions for large initial data, and furthermore the solutions approach the spatially homogeneous equilibrium [34]:

$$(n, c, u) \to (\bar{n}_0, 0, 0) \text{ as } t \to \infty,$$

where $\bar{n}_0 = \frac{1}{|\Omega|}\int_\Omega n_0$. The convergence rates are studied later in [42] for the convergent solutions. Concerning the case $N = 3$, (1.3) is still crucial, [33] asserts the existence of global weak solutions for the Stokes-governed system based on it. Recently, a global weak solution was constructed for the full Navier–Stokes system for large initial data [36]. More recently, with a concept of *eventual energy solution*, [37] showed that such solutions become smooth after finite time and uniformly converge to the constant steady state in the large time limit. For more results depending on the natural functional like (1.3), see, e.g., [18,17,6,5,27,36,33] and the references therein.

However, in [41], the authors suggest a wider choice of S due to some complicated interaction neighborhood environment around cells. A kind of interactions between the cell motion speed and directional effects stemming from the action of gravity may result in abnormal mechanism – they do not move directly to the direction of higher density of oxygen but with some rotation; so this requires S to be a general matrix. Apart from the complexity as it stands, this tensor-valued chemotactic sensitivity also gives rise to some difficulty in mathematical analysis. Upon the aforementioned reasoning of (1.3), we may see that it heavily relies on the structure of the cross diffusion term. Here the term from the Lyapunov functional reads as $\int_\Omega \chi(c)nS \cdot \nabla c \cdot \nabla n$ and it is no longer cancelable for arbitrary choices of S. Thus when (1.3) is absent, it is much more difficult to study (1.1) from a mathematical point of view.

Generally, stronger assumptions seem necessary for the existence of classical solutions. For instance, considering the two-dimensional fluid-free system, that is, $u \equiv 0$, it is shown that the system admits global classical solutions which converge to the constant steady state if $\|c_0\|_{L^\infty(\Omega)}$ is sufficiently small [15]. This is in sharp contrast to the case that S is a scalar-valued sensitivity [25], where (1.3) is still applicable, and finally it leads to boundedness of solutions and large time convergence without any smallness condition on the initial data. On the other hand, the same problem for large data are studied in [39,38] with or without fluid effect. In this case, it is shown that a certain generalized solution exists, and converges to the constant steady state in the large time limit. However, the results do not exclude singularity on intermediate time scales.

Considering porous medium type cell diffusion, that is, when the first equation in (1.1) is replaced by $n_t = \Delta n^m - \nabla \cdot (nS \cdot \nabla c)$ with $m > 1$, the existence of global weak solution is derived for any reasonable regular initial data, moreover, the solution is actually bounded [2]. Taking the fluid into account, the coupled Stokes and Navier–Stokes counterpart to the same problem is studied in [26] and [12], where the authors prove boundedness and global existence of weak solutions.

Due to the difficulty arising from the three-dimensional Navier–Stokes equation, only until very recently, the full chemotaxis-Navier–Stokes system (1.1) with scalar sensitivity is known to admit considerably weak solutions. Accordingly, many works have focused on the simplified Stokes coupled system and shown more progress. Assume that $|S| \le C(1+n)^{-\alpha}$ with

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

$C > 0$ and $\alpha > \frac{1}{6}$, the global existence of locally bounded classical solution is constructed in [29]. Considering porous medium variant of (1.1), that is, when the first equation becomes $n_t = \Delta n^m - \nabla \cdot (n\nabla c) - u \cdot \nabla n$, it is proved that locally bounded solutions exist and are locally bounded under the hypothesis that $m > \frac{8}{7}$ [27]. If we assume in addition that $m > \frac{7}{6}$, upon a more robust approach, the solutions become globally bounded and converge to $(\overline{n_0}, 0, 0)$ [40]. Without assuming superlinear diffusion, the question of boundedness of classical solution is actually more delicate and solved recently in [3] that even matrix-valued sensitivity is allowed. It is shown that if $\|n - \bar{n}_0\|_{L^p(\Omega)}$, $\|\nabla c_0\|_{L^q(\Omega)}$ and $\|u_0\|_{L^N(\Omega)}$ (with any $p > \frac{N}{2}$ and $q > N$) are small, the system admits a unique global classical solution which converges to the homogeneous equilibrium. Also there have been a few works considering the classical Keller–Segel coupled fluid system where the second equation in (1.1) is replaced by $c_t = \Delta c - c + n - u \cdot \nabla c$. Progress in this direction can be seen in [13,30] and the references therein.

The purpose of the present work is to study the full chemotaxis-Navier–Stokes system with tensor-valued sensitivity in dimension 2 and the corresponding chemotaxis-Stokes system in dimension 3. When the natural Lyapunov functional is lacking, we impose a smallness assumption on the initial data to get some uniform bound for the solution. Using this tool, we can prove global existence of classical solution and its large time behavior. Compared with [3], the smallness condition here is only on $\|c_0\|_{L^\infty(\Omega)}$, meaning that small concentration of oxygen can force stability. This result coincides with the fluid-free system in [15]. The convexity of the physical domain is unnecessary in this paper since we use a different approach from many previous works [33].

Before stating our main result, let us briefly introduce some elementary background of functional spaces, Stokes operator as well as their applications and some notations.

Let $L^p_\sigma(\Omega)$ $(1 < p < \infty)$ denote solenoidal space equipped with $\|\cdot\|_{L^p(\Omega)}$ norm:

$$L^p_\sigma(\Omega) := \{\varphi \in C_0^\infty(\Omega; \mathbb{R}^N) | \nabla \cdot \varphi = 0\}$$

The so-called Helmholtz-projection is defined as $\mathscr{P} : L^p(\Omega, R^N) \to L^p_\sigma(\Omega)$, which is a bounded operator. Let $A_p = -\mathscr{P}\Delta$ denote the Stokes operator in $D(A_p) = L^p_\sigma(\Omega) \cap W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. From [8], we know that A is sectorial and generates analytical semigroup $(e^{-tA})_{t>0}$ in $L^p_\sigma(\Omega)$. We refer to [31, Chapter 6] and [3, Lemma 2.3] for fundamental L^p-L^q estimates for the semigroup. Moreover, since $\mathrm{Re}\,\sigma(A) > 0$, we can define $A^{-\alpha}$ with $\alpha > 0$ and easily check that it is one-to-one. Thus A^α is defined as the inverse of $A^{-\alpha}$ and $D(A^\alpha) = R(A^{-\alpha})$ [19, Chapter 2.6]. The following estimate is fundamental:

$$\|A^\alpha e^{tA}\| \le C_\alpha t^{-\alpha} e^{-\mu t} \text{ for } t > 0 \text{ and for some } \mu > 0. \tag{1.4}$$

Throughout the paper, we denote the first eigenvalue of A by λ_1', and by λ_1 the first nonzero eigenvalue of $-\Delta$ on Ω under Neumann boundary conditions. Moreover, we assume that

$$s_{ij} \in C^2(\overline{\Omega} \times [0,\infty) \times [0,\infty)), \tag{1.5}$$

$$|S(x,n,c)| := \max_{i,j\in\{1,2\}} \{s_{ij}(x,n,c)\} \le S_0(c) \text{ for all } (x,n,c) \in \overline{\Omega} \times [0,\infty) \times [0,\infty), \tag{1.6}$$

where S_0 is a non-decreasing function on $[0, \infty)$. The initial data are chosen as

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

$$\begin{cases} n_0 \in L^\infty(\Omega), \\ c_0 \in W^{1,q}(\Omega), \ q > N, \\ u_0 \in D(A^\alpha), \ \alpha \in (\frac{N}{4}, 1), \end{cases} \tag{1.7}$$

and particularly

$$n_0 \geq 0, \quad c_0 \geq 0 \text{ on } \Omega. \tag{1.8}$$

Under the above assumptions and notations, our main result is as follows:

Theorem 1. *Let $N \in \{2,3\}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that S fulfills (1.5)–(1.6). Either of the following conditions holds,*

(i) $N = 2$, $\kappa = 1$;
(ii) $N = 3$, $\kappa = 0$.

There is $\delta_0 > 0$ with the following property: If the initial data fulfill (1.7)–(1.8), and

$$\|c_0\|_{L^\infty(\Omega)} < \delta_0, \tag{1.9}$$

then (1.1) admits a global classical solution (n, c, u, P) which is bounded, and satisfies

$$\begin{cases} n \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap C^0_{loc}(\overline{\Omega} \times (0,\infty)), \\ c \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap C^0_{loc}(\overline{\Omega} \times (0,\infty)) \cap L^\infty((0,\infty); W^{1,q}(\Omega)), \\ u \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap L^\infty((0,\infty); D(A^\alpha)) \cap C^0_{loc}([0,\infty); L^2(\Omega)), \\ P \in L^1((0,\infty); W^{1,2}(\Omega)). \end{cases} \tag{1.10}$$

Remark 1.1. The uniqueness of classical solutions in the indicated class can be proved similarly as in [33].

Apart from boundedness and global existence, we can also show each component converges to the homogeneous equilibrium with *optimal* rates.

Corollary 1.1. *Under the assumptions of Theorem 1, let $0 < \alpha < \min\{\bar{n}_0, \lambda_1\}$ and $0 < \alpha' < \min\{\alpha, \lambda_1'\}$. The solution of (1.1) has the property that there is $C > 0$ fulfilling*

$$\|n(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} \leq Ce^{-\alpha t}, \qquad \|c(\cdot,t)\|_{W^{1,q_0}(\Omega)} \leq Ce^{-\alpha t},$$
$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq Ce^{-\alpha' t} \text{ for all } t > 0.$$

We note that compared with the result in [34], Theorem 1 furthermore has restrictions on the size of initial data (1.9), which seems necessary for the existence of classical solutions. As a subcase of (1.1), results on the corresponding fluid-free version are not yet rich: Without assuming small data, the global generalized solutions constructed in [39] still possibly become unbounded in the intermediate time; Only additionally assuming $\|c_0\|_{L^\infty(\Omega)}$ small, global classical solutions are known to exist and blow-up is entirely ruled out [15]. When the system is coupled with fluid component, our results give the same condition which guarantee the global existence of smooth solution.

The plan of the paper is as follows:

In Section 2, we approximate the problem by a well-posed system (see (2.4) later). Sections 3–5 are devoted to study the boundedness of regularized problem, we will see the bounds are independent of the way we regularize the problem. Thus upon appropriate estimates, we can let $\varepsilon \to 0$ to obtain limit functions of the regularized solutions. This procedure is done in Section 6, and also these limit functions are shown to be smooth enough and solve (1.1) classically for any positive time. In Section 7, we prove stabilization of the solution by applying the result from [3].

2. Approximation

Since it is convenient to deal with the Neumann boundary conditions for both n and c, we follow the same approximation procedure as in [15]. Let $\varepsilon \in (0, 1)$, we find a family of functions $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ satisfying

$$\rho_\varepsilon \in C_0^\infty(\Omega) \quad \text{with} \quad 0 \le \rho_\varepsilon \le 1 \text{ in } \Omega \text{ and } \rho_\varepsilon \nearrow 1 \text{ in } \Omega \text{ as } \varepsilon \searrow 0, \tag{2.1}$$

and define

$$S_\varepsilon(x, n_\varepsilon, c_\varepsilon) = \rho_\varepsilon(x) S(x, n, c), \quad x \in \bar{\Omega}. \tag{2.2}$$

Then we have $S_\varepsilon(x, n, c) = 0$ on $\partial\Omega$ and

$$|S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \le S_0(\|c_0\|_{L^\infty(\Omega)}) \quad \text{for all } x \in \Omega,\ n_\varepsilon > 0,\ c_\varepsilon > 0. \tag{2.3}$$

Now we consider the following regularized problem

$$\begin{cases} n_{\varepsilon t} = \Delta n_\varepsilon - \nabla\cdot(n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\cdot\nabla c_\varepsilon) + u_\varepsilon\cdot\nabla n_\varepsilon, & (x,t) \in \Omega\times(0,T),\\ c_{\varepsilon t} = \Delta c_\varepsilon - n_\varepsilon c_\varepsilon + u_\varepsilon\cdot\nabla c_\varepsilon, & (x,t) \in \Omega\times(0,T),\\ u_{\varepsilon t} = \Delta u_\varepsilon - \kappa(u_\varepsilon\cdot\nabla)u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon\nabla\phi,\ \nabla\cdot u_\varepsilon = 0, & (x,t) \in \Omega\times(0,T),\\ \nabla n_\varepsilon\cdot\nu = \nabla c_\varepsilon\cdot\nu = 0,\ u_\varepsilon = 0, & (x,t) \in \partial\Omega\times(0,T),\\ n_\varepsilon(x,0) = n_0(x), c_\varepsilon(x,0) = c_0(x), u_\varepsilon(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.4}$$

Without essential difficulty, the above system is locally solvable in the classical sense by an adaption of well-established fixed point argument [33, Lemma 2.1]. We give the following lemma without proof.

Lemma 2.1. *Let $N \in \{2, 3\}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and $\kappa \in \mathbb{R}$. Assume initial data (n_0, c_0, u_0) satisfy (1.7) and (1.8), and S fulfills (1.5)–(1.6). Then there exist $T_{\max} \in (0, \infty]$ and a unique classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ to (2.4) in $\Omega \times [0, T_{\max})$ with $n_\varepsilon, c_\varepsilon > 0$. Moreover, if $T_{\max} < \infty$, then*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty \quad \text{as } t \nearrow T_{max}.$$

In order to see the global existence and qualitative behavior of the regularized problem, it is sufficient to show boundedness for each criterion in the above lemma. The following lemma is immediately obtained upon observation.

Lemma 2.2. *Let $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ be a classical solution of (2.4). It follows that*

$$\|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)}, \tag{2.5}$$

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \ \text{for all } t \in (0, T_{max}). \tag{2.6}$$

Proof. The mass conservation (2.5) is obtained by integrating the first equation of (1.1) on Ω and using the Neumann boundary condition. Since n_ε and c_ε are nonnegative, an application of the maximum principle to the second equation yields (2.6). □

We then obtain boundedness and global existence for the regularized problem (2.4).

Proposition 2.1. *Let $N \in \{2,3\}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that S fulfills (1.5)–(1.6). Either of the following conditions holds*

(i) $N = 2$, $\kappa = 1$;
(ii) $N = 3$, $\kappa = 0$.

Then there exists $\delta_0 > 0$ with the following property: If the initial data fulfill (1.7)–(1.8), and

$$\|c_0\|_{L^\infty(\Omega)} < \delta_0, \tag{2.7}$$

then (2.4) admits a global classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$. And there is $C > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C, \quad \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} \le C, \quad \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \tag{2.8}$$

for all $t \in (0,\infty)$ and all $\varepsilon \in (0,1)$.

We will prove boundedness for the 2-dimensional and 3-dimensional cases in Section 4 and Section 5, respectively. However, the $L^p(\Omega)$ estimate for n_ε derived in the next section will be applied to both.

3. A priori estimate for n_ε

In this section, we obtain boundedness of n_ε in $L^p(\Omega)$ under the assumption that $\|c_0\|_{L^\infty(\Omega)}$ is suitably small. The approach is based on the weighted estimate of $\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon)$ with appropriate choice of φ which has been developed in [35] and adapted to the consumed type signal in [24,34].

Lemma 3.1. *Let $p > 1$, there are $\delta_0 := \delta_0(p) > 0$ and $C > 0$ with the property: If the initial data satisfy (1.7)–(1.8) and*

$$\|c_0\|_{L^\infty(\Omega)} < \delta_0, \tag{3.1}$$

then for all $\varepsilon \in (0,1)$, we have

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \ \textit{ for all } t \in (0, T_{\max}), \tag{3.2}$$

$$\textit{and} \int_0^{T_{\max}} \int_\Omega n_\varepsilon^{p-2} |\nabla n_\varepsilon|^2 \le C. \tag{3.3}$$

Remark 3.2. The argument does not depend on dimension N or the value of κ.

Proof. Let $p > 1$, $0 < h < \frac{1}{48}$. We can find δ_0 satisfying

$$3p(p-1)\delta_0^2 S_0^2(\delta_0) \le h(h+1), \tag{3.4}$$

$$3p\delta_0 S_0(\delta_0) \le h+1, \tag{3.5}$$

where S_0 is non-decreasing function as introduced in (1.6). Under the assumption of (3.1), we can define $\varphi(c_\varepsilon) = (\delta_0 - c_\varepsilon)^{-h}$ according to (2.6), thus $\varphi(c_\varepsilon) > 0$. Elementary calculus shows that

$$\varphi'(c_\varepsilon) = h(\delta_0 - c_\varepsilon)^{-h-1} > 0, \tag{3.6}$$

$$\varphi''(c_\varepsilon) = h(h+1)(\delta_0 - c_\varepsilon)^{-h-2} > 0. \tag{3.7}$$

Using the first two equations in (2.4), upon integrating by part we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon) \\
&= \int_\Omega p n_\varepsilon^{p-1} \varphi(c_\varepsilon)(\Delta n_\varepsilon - \nabla\cdot(n_\varepsilon S_\varepsilon \cdot \nabla c_\varepsilon) - u_\varepsilon \cdot \nabla n_\varepsilon) + \int_\Omega n_\varepsilon^p \varphi'(c_\varepsilon)(\Delta c_\varepsilon - n_\varepsilon c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon) \\
&= -\int_\Omega \nabla n_\varepsilon \cdot (p(p-1) n_\varepsilon^{p-2} \varphi(c_\varepsilon) \nabla n_\varepsilon + p n_\varepsilon^{p-1} \varphi'(c_\varepsilon) \nabla c_\varepsilon) \\
&\quad + \int_\Omega n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \cdot \Big(p(p-1)\varphi(c_\varepsilon) n_\varepsilon^{p-2} \nabla n_\varepsilon + p n_\varepsilon^{p-1} \varphi'(c_\varepsilon) \nabla c_\varepsilon \Big) \\
&\quad - \int_\Omega p n_\varepsilon^{p-1} \varphi(c_\varepsilon) u_\varepsilon \cdot \nabla n_\varepsilon - \int_\Omega \nabla c_\varepsilon \cdot (p n_\varepsilon^{p-1} \varphi'(c_\varepsilon) \nabla n_\varepsilon + n_\varepsilon^p \varphi''(c_\varepsilon) \nabla c_\varepsilon) \\
&\quad - \int_\Omega n_\varepsilon^p \varphi'(c_\varepsilon) u_\varepsilon \cdot \nabla c_\varepsilon - \int_\Omega n_\varepsilon^{p+1} c_\varepsilon \varphi'(c_\varepsilon) \\
&= -p(p-1)\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2 - p\int_\Omega n_\varepsilon^{p-1}\varphi'(c_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
&\quad + p(p-1)\int_\Omega n_\varepsilon^{p-1}\varphi(c_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\cdot\nabla c_\varepsilon \cdot \nabla n_\varepsilon + p\int_\Omega n_\varepsilon^p \varphi'(c_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \cdot \nabla c_\varepsilon
\end{aligned}$$

$$-p\int_\Omega n_\varepsilon^{p-1}\varphi'(c_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon-\int_\Omega n_\varepsilon^p\varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2-\int_\Omega n_\varepsilon^{p+1}\varphi'(c_\varepsilon)c \tag{3.8}$$

for all $t\in(0,T_{\max})$, where we have used the identity

$$\begin{aligned}-p\int_\Omega n_\varepsilon^{p-1}\varphi(c_\varepsilon)u_\varepsilon\cdot\nabla n_\varepsilon-\int_\Omega n_\varepsilon^p\varphi'(c_\varepsilon)u_\varepsilon\cdot\nabla c_\varepsilon&=-\int_\Omega\varphi(c_\varepsilon)u_\varepsilon\cdot\nabla n_\varepsilon^p-\int_\Omega n_\varepsilon^p u_\varepsilon\cdot\nabla\varphi(c_\varepsilon)\\&=\int_\Omega n_\varepsilon^p\varphi(c_\varepsilon)(\nabla\cdot u_\varepsilon)=0.\end{aligned}$$

In light of (2.3), we find that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega n_\varepsilon^p\varphi(c_\varepsilon)+p(p-1)\int_\Omega\varphi(c_\varepsilon)n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2+\int_\Omega n_\varepsilon^p\varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2\\&=p(p-1)S_0(\|c_0\|_{L^\infty(\Omega)})\int_\Omega n_\varepsilon^{p-1}\varphi(c_\varepsilon)|\nabla n_\varepsilon||\nabla c_\varepsilon|+2p\int_\Omega n_\varepsilon^{p-1}\varphi'(c_\varepsilon)|\nabla n_\varepsilon||\nabla c_\varepsilon|\\&\quad+pS_0(\|c_0\|_{L^\infty(\Omega)})\int_\Omega n_\varepsilon^p\varphi'(c_\varepsilon)|\nabla c_\varepsilon|^2\end{aligned}\tag{3.9}$$

for all $t\in(0,T_{\max})$. Here Young's inequality yields that

$$\begin{aligned}&p(p-1)S_0(\|c_0\|_{L^\infty(\Omega)})\int_\Omega n_\varepsilon^{p-1}\varphi(c_\varepsilon)|\nabla n_\varepsilon||\nabla c_\varepsilon|\\&\le\frac{p(p-1)}{4}\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2+p(p-1)S_0^2(\|c_0\|_{L^\infty(\Omega)})\int_\Omega n_\varepsilon^p\varphi(c_\varepsilon)|\nabla c_\varepsilon|^2,\end{aligned}\tag{3.10}$$

$$2p\int_\Omega n_\varepsilon^{p-1}\varphi'(c_\varepsilon)|\nabla n_\varepsilon||\nabla c_\varepsilon|\le\frac{p(p-1)}{4}\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2+16\int_\Omega n_\varepsilon^p\frac{\varphi'^2(c_\varepsilon)}{\varphi(c_\varepsilon)}|\nabla c_\varepsilon|^2.\tag{3.11}$$

We see that (3.9)–(3.10) imply

$$\begin{aligned}&\frac{d}{dt}\int_\Omega n_\varepsilon^p\varphi(c_\varepsilon)+\frac{p(p-1)}{2}\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2\\&+\int_\Omega n_\varepsilon^p|\nabla c_\varepsilon|^2\Big(\varphi''(c_\varepsilon)-16\frac{\varphi'^2(c_\varepsilon)}{\varphi(c_\varepsilon)}-p(p-1)S_0^2(\|c_0\|_{L^\infty(\Omega)})\varphi(c_\varepsilon)\\&-pS_0(\|c_0\|_{L^\infty(\Omega)})\varphi'(c_\varepsilon)\Big)\le 0\end{aligned}\tag{3.12}$$

for all $t \in (0, T_{\max})$. Now using (3.4)–(3.5), and in view of the fact that $S_0(\delta)$ is non-decreasing, we see that

$$16\frac{\varphi'^2(c_\varepsilon)}{\varphi(c_\varepsilon)} = 16h^2(\delta_0 - c_\varepsilon)^{-h-2} \le \frac{1}{3}\varphi''(c_\varepsilon),$$
$$p(p-1)S_0^2(\delta_0)\varphi(c_\varepsilon) = p(p-1)S_0^2(\delta_0)(\delta_0 - c_\varepsilon)^{-h} \le \frac{1}{3}\varphi''(c_\varepsilon),$$
$$pS_0(\delta_0)\varphi'(c_\varepsilon) = hpS_0(\delta_0)(\delta_0 - c_\varepsilon)^{-h-1} \le \frac{1}{3}\varphi''(c_\varepsilon).$$

Thus the term $\int_\Omega n_\varepsilon^p |\nabla c_\varepsilon|^2 \left(\varphi''(c_\varepsilon) - 16\frac{\varphi'^2(c_\varepsilon)}{\varphi(c_\varepsilon)} - p(p-1)S_0^2(\delta_0)\varphi(c_\varepsilon) - pS_0(\delta_0)\varphi'(c_\varepsilon)\right)$ on the right hand side of (3.12) is nonnegative, we immediately deduce that

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon) + \frac{p(p-1)}{2}\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2 \le 0, \text{ for all } t \in (0, T_{\max}). \tag{3.13}$$

Since $\varphi(c_\varepsilon)$ is bounded from above and below, (3.2) and (3.3) result from the above inequality upon integrating on $(0, T_{\max})$. □

4. Boundedness in two-dimensional case ($N = 2$, $\kappa = 1$)

We expect that the $L^p(\Omega)$ estimate obtained in the last section guarantees boundedness of n_ε in $L^\infty(\Omega)$ as in the fluid-free system. However, the iteration procedure is much more delicate due to the appearance of the transport terms in the current case. Since the regularity of ∇c_ε is crucial, which is also associated to the regularity of u_ε, we will first get the suitable regularity of u_ε. More precisely, the $L^2(\Omega)$ norm of ∇u_ε implies boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ for any $p > 1$. This is sufficient to prove boundedness of $\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$.

4.1. Boundedness of $\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$

Lemma 4.1. *Let $N \in \{2,3\}$. Suppose that*

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty. \tag{4.1}$$

Then there exists $C > 0$ such that for any $\varepsilon > 0$

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < C \quad \textit{for all } t \in (0, T_{\max}), \tag{4.2}$$
$$\int_k^{\min\{k+1,T_{\max}\}}\int_\Omega |\nabla u_\varepsilon|^2 < C \quad \textit{for all } k \in \mathbb{T} := \{s \in \mathbb{N}, s \le [T_{\max}]\}. \tag{4.3}$$

Proof. Testing the third equation with u_ε, integrating by parts and Young's inequality yield that

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon \nabla\phi \cdot u_\varepsilon$$
$$\leq \frac{\lambda_1'}{2}\int_\Omega |u_\varepsilon|^2 + \frac{1}{2\lambda_1'}\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2 \tag{4.4}$$

for all $t \in (0, T_{\max})$. The Poincaré inequality combined with (4.1) implies the existence of $c_1 > 0$ such that

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \lambda_1'\int_\Omega |u_\varepsilon|^2 \leq c_1 \tag{4.5}$$

for all $t \in (0, T_{\max})$. Thus (4.2) is obtained by the comparison theorem. Now we integrate (4.4) on $(k, k+1)$ $(k \in \mathbb{T})$ to find that (4.3) holds due to (4.2). □

Remark 4.2. We note that the lemma does not depend on the dimensions, thus we are able to use the same reasoning in other situations, e.g. Lemma 7.5.

Based on (4.17) in [33], we can prove $\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ is bounded with the aid of (4.3). The assumption $N = 2$ is crucial here.

Lemma 4.3. *Let $N = 2$. Suppose that*

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty. \tag{4.6}$$

There is $C > 0$ fulfilling for any $\varepsilon > 0$

$$\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C \ \textit{ for all } t \in (0, T_{\max}). \tag{4.7}$$

Proof. First we apply Lemma 4.1 to obtain (4.2) and (4.3). Let $A = -\mathscr{P}\Delta$ and hence $\|A^{\frac{1}{2}}u_\varepsilon\|_{L^2(\Omega)} = \|\nabla u_\varepsilon\|_{L^2(\Omega)}$. Testing the third equation by Au_ε implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}}u_\varepsilon|^2 + \int_\Omega |Au_\varepsilon|^2 = \int_\Omega Au_\varepsilon(u_\varepsilon\cdot\nabla)u_\varepsilon + \int_\Omega n_\varepsilon\nabla\phi Au_\varepsilon$$
$$\leq \frac{1}{4}\int_\Omega |Au_\varepsilon|^2 + \int_\Omega |u_\varepsilon|^2|\nabla u_\varepsilon|^2 + \frac{1}{4}\int_\Omega |Au_\varepsilon|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2$$
$$\leq \int_\Omega |u_\varepsilon|^2|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega |Au_\varepsilon|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2 \tag{4.8}$$

for all $t \in (0, T_{\max})$. By Young's inequality, an interpolation inequality for $\|u_\varepsilon\|_{L^4(\Omega)}$ and $\|\nabla u_\varepsilon\|_{L^4(\Omega)}$ (see also in [33, Proof of Theorem 1.1]), and the equivalence between the norms $\|A(\cdot)\|_{L^2(\Omega)}$ and $\|\cdot\|_{W^{2,2}(\Omega)}$

$$\begin{aligned}\int_\Omega |u_\varepsilon|^2|\nabla u_\varepsilon|^2 &\le (\int_\Omega |u_\varepsilon|^4)^{\frac12}(\int_\Omega |\nabla u_\varepsilon|^4)^{\frac12}\\ &\le (\int_\Omega |\nabla u_\varepsilon|^2)^{\frac12}(\int_\Omega |u_\varepsilon|^2)^{\frac12}(\int_\Omega |Au_\varepsilon|^2)^{\frac12}(\int_\Omega |\nabla u_\varepsilon|^2)^{\frac12}\\ &\le \frac12\int_\Omega |Au_\varepsilon|^2 + \frac12(\int_\Omega |u_\varepsilon|^2)(\int_\Omega |\nabla u_\varepsilon|^2)^2. \end{aligned}\tag{4.9}$$

We see that (4.8) and (4.9) in conjunction with our assumption and (4.2) imply that there is $c_1 > 0$ fulfilling

$$\frac{d}{dt}\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega |Au_\varepsilon|^2 \le c_1\left(\int_\Omega |\nabla u_\varepsilon|^2 + 1\right)^2 \tag{4.10}$$

for all $t \in (0, T_{\max})$. Let $y(t) := \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 + 1$, thus $y(t)$ satisfies

$$y'(t) \le c_1 y^2(t) \tag{4.11}$$

for all $t \in [k, \min\{k+1, T_{\max}\})$.

If $T_{\max} > 1$, for all $k \in \mathbb{T}$, Lemma 4.1 warrants the existences of $c_2 > 0$ and $s_k \in [k, k+1]$ such that

$$y(s_k) \le c_2, \quad \int_k^{k+1} y(s)ds \le c_2. \tag{4.12}$$

We deduce from (4.11)–(4.12) that

$$y(t) \le e^{c_1\int_{s_k}^t y(s)ds}y(s_k) \le e^{c_1\int_k^{\min\{k+2,T_{\max}\}} y(s)ds}y(s_k) \le e^{2c_1c_2}c_2 \tag{4.13}$$

for all $t \in [k+1, \min\{k+2, T_{\max}\}] \subset [s_k, \min\{k+2, T_{\max}\})$ $(k \in \mathbb{T})$. Thus (4.13) holds for all $t \in [1, T_{\max})$. A similar reasoning gives

$$y(t) \le e^{c_1\int_0^1 y(s)ds}y(0) \le e^{c_1c_2}y(0) \text{ for all } t \in [0,1]. \tag{4.14}$$

If $T_{\max} < 1$, it is easy to see the above estimate still holds for $t \in [0, T_{\max})$. Thus the proof is complete by letting $C := \max\{e^{2c_1c_2}c_2, e^{c_1c_2}\|\nabla u_0\|_{L^2(\Omega)}\}$. □

The following lemma is an immediate consequence of Sobolev embedding theorem for dimension 2.

Lemma 4.4. *Let $N = 2$. Suppose that*

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty. \tag{4.15}$$

Then for any $1 < p < \infty$, *there is* $C > 0$ *such that for any* $\varepsilon > 0$

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}). \tag{4.16}$$

4.2. Boundedness of $\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$

Now we are in a position to get higher regularity of ∇c_ε, the approach is carried out by fixed-point argument involving L^p-L^q estimates for semigroups combined with a typical integral estimate, which is borrowed from [32,3].

Lemma 4.5. *For all* $\eta > 0$ *there is* $C = C(\eta) > 0$ *such that for all* $\alpha, \beta \in [0, 1-\eta)$, *and* $\gamma, \delta \in \mathbb{R}$ *satisfying* $\frac{1}{\eta} \ge \gamma - \delta \ge \eta$, *we have*

$$\int_0^t (1+s^{-\alpha})(1+(t-s)^{-\beta})e^{-\gamma s}e^{-\delta(t-s)}ds \le C(\eta)e^{-\min\{\gamma,\delta\}t}(1+t^{\min\{0,1-\alpha-\beta\}}) \quad \text{for all } t > 0.$$

Lemma 4.6. *Let* $N = 2$, $p_0 > 2$. *Suppose that*

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)} < \infty.$$

Then there is $C > 0$ *such that for any* $\varepsilon > 0$

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}). \tag{4.17}$$

Proof. The variation of constants formula associated to c_ε implies

$$\begin{aligned}\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|\nabla e^{t\Delta}c_0\|_{L^\infty(\Omega)} &+ \int_0^t \|\nabla e^{(t-s)\Delta}n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds \\ &+ \int_0^t \|\nabla e^{(t-s)\Delta}(u_\varepsilon(\cdot,s)\cdot\nabla c_\varepsilon(\cdot,s))\|_{L^\infty(\Omega)}ds\end{aligned} \tag{4.18}$$

for all $t \in (0, T_{\max})$. Recall that by the classical L^p-L^q estimates for Neumann semigroup, there is $c_1 > 0$ such that

$$\|\nabla e^{t\Delta}c_0\|_{L^\infty(\Omega)} \le c_1\|\nabla c_0\|_{L^\infty(\Omega)} \tag{4.19}$$

for all $t \in (0, T_{\max})$ and for all $c_0 \in W^{1,\infty}(\Omega)$. Since $p_0 > 2$, L^p-L^q estimates yield that

$$\int_0^t \|\nabla e^{(t-s)\Delta}n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds$$

$$\leq \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{1}{p_0}})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)\|_{L^{p_0}(\Omega)}ds$$

$$\leq \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{1}{p_0}})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot,s)\|_{L^{p_0}(\Omega)}\|c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds, \tag{4.20}$$

for all $t \in (0, T_{\max})$, which is bounded by (4.1) and (2.6). Next we fix $p > 2$ and moreover $p_1, p_2 \in (p, \infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $\theta = 1 - \frac{2}{p_2} \in (0, 1)$, we thereby obtain

$$\int_0^t \|\nabla e^{(t-s)\Delta}(u_\varepsilon(\cdot,t)\cdot\nabla c_\varepsilon(\cdot,t))\|_{L^\infty(\Omega)}ds$$

$$\leq \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{1}{p}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,t)\cdot\nabla c_\varepsilon(\cdot,t)\|_{L^p(\Omega)}ds$$

$$\leq \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{1}{p}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,t)\|_{L^{p_1}(\Omega)}\|\nabla c_\varepsilon(\cdot,t)\|_{L^{p_2}(\Omega)}ds$$

$$\leq \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{1}{p}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,t)\|_{L^{p_1}(\Omega)}(\|\nabla c_\varepsilon(\cdot,t)\|^\theta_{L^\infty(\Omega)}\|c_\varepsilon(\cdot,t)\|^{1-\theta}_{L^\infty(\Omega)}$$

$$+\|c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)})ds \tag{4.21}$$

for all $t \in (0, T_{\max})$. Let $T \in (0, T_{\max})$, and $M := \sup_{t\in(0,T)} \|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$. We see from (4.18)–(4.21) that

$$M \leq c_2 + c_2 M^\theta,$$

with some $c_2 > 0$. Since $\theta < 1$, (4.17) is obtained by Young's inequality. □

4.3. Boundedness of n_ε

Lemma 4.7. *Let $N = 2$, $p_0 > 2$. Suppose that*

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)} < \infty.$$

Then there is $C > 0$ such that for any $\varepsilon > 0$

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{\max}). \tag{4.22}$$

Proof. Following the variation-of-constants formula, we see that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{t\Delta}n_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)\Delta}\nabla\cdot(n_\varepsilon S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)(\cdot,s)\|_{L^\infty(\Omega)}ds$$
$$+\int_0^t \|e^{(t-s)\Delta}u_\varepsilon(\cdot,s)\cdot\nabla n_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds \tag{4.23}$$

for all $t\in(0,T_{\max})$. The first term can be estimated as

$$\|e^{t\Delta}n_0\|_{L^\infty(\Omega)} \le c_1\|n_0\|_{L^\infty(\Omega)} \quad \text{for all } t\in(0,T_{\max}) \tag{4.24}$$

with some $c_1>0$. Moreover, applying L^p-L^q for Neumann semigroup, we obtain $c_2>0$ such that

$$\begin{aligned}&\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(n_\varepsilon S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)(\cdot,s)\|_{L^\infty(\Omega)}ds\\ &\le c_2\int_0^t (1+(t-s)^{-\frac12-\frac{1}{p_0}})e^{-\lambda_1(t-s)}\|(n_\varepsilon S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)(\cdot,s)\|_{L^{p_0}(\Omega)}ds\\ &\le c_2S_0(\|c_0\|_{L^\infty(\Omega)})\int_0^t (1+(t-s)^{-\frac12-\frac{1}{p_0}})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot,s)\|_{L^{p_0}(\Omega)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds\end{aligned} \tag{4.25}$$

for all $t\in(0,T_{\max})$. By (4.1) and (4.17), we know the right hand side of (4.25) is bounded. Noting that $u_\varepsilon\cdot\nabla n_\varepsilon=\nabla\cdot(n_\varepsilon u_\varepsilon)$, we pick $p>2$ and $p'>p$ such that $\frac1p=\frac{1}{p_0}+\frac{1}{p'}$, a similar reasoning as the above inequality shows that

$$\begin{aligned}&\int_0^t \|e^{(t-s)\Delta}u_\varepsilon(\cdot,s)\cdot\nabla n_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds\\ &=\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s))\|_{L^\infty(\Omega)}ds\\ &\le c_2\int_0^t (1+(t-s)^{-\frac12-\frac1p})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s)\|_{L^p(\Omega)}ds\\ &\le c_2\int_0^t (1+(t-s)^{-\frac12-\frac1p})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot,s)\|_{L^{p_0}(\Omega)}\|u(\cdot,s)\|_{L^{p'}(\Omega)}ds\end{aligned}$$

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

for all $t \in (0, T_{\max})$ due to (4.1) and (4.16), is bounded by Lemma 4.5. Thus we complete the proof by collecting the above estimates. □

4.4. Proof of (i) in Proposition 2.1

In order to prove global existence of the solution, it is left to show boundedness of $\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ due to the extensive criterion.

Lemma 4.8. *Suppose that*

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty, \quad \sup_{t\in(0,T_{\max})} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty, \quad \sup_{t\in(0,T_{\max})} \|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty.$$

Then there is $C > 0$ such that for every $\varepsilon > 0$

$$\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}). \tag{4.26}$$

Proof. Let $T > 0$, we first define $M(t) := \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ for $t \in (0,T)$. Let $a = \frac{N}{4\alpha}$, from the Gagliardo–Nirenberg inequality and [3, Lemma 2.3(iv)] we know that there is constant $c_1 > 0$ such that

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le c_1 \|A^\alpha u_\varepsilon\|^a_{L^2(\Omega)} \|u_\varepsilon\|^{1-a}_{L^2(\Omega)}. \tag{4.27}$$

We apply A^α to both sides of the third equation in (2.4), a triangle-inequality implies that

$$\begin{aligned} \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le \|A^\alpha e^{-tA} u_0\|_{L^2(\Omega)} &+ \int_0^t \|A^\alpha e^{-(t-s)A} \mathscr{P}(u_\varepsilon \cdot \nabla) u_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds \\ &+ \int_0^t \|A^\alpha e^{-(t-s)A} \mathscr{P} n_\varepsilon(\cdot,s) \nabla\phi\|_{L^2(\Omega)} ds. \end{aligned} \tag{4.28}$$

First we have

$$\|A^\alpha e^{-tA} u_0\|_{L^2(\Omega)} \le c_\alpha \|e^{-(t-1)A} u_0\|_{L^2(\Omega)} \le c_2 e^{-\lambda_1'(t-1)} \|u_0\|_{L^2(\Omega)} \quad \text{for all } t > 0. \tag{4.29}$$

Thanks to Lemma 4.3, we know that $\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le c_2$ with some $c_2 > 0$, which together with (1.4), (4.27) and Lemma 4.5 yields the existence of $c_\alpha > 0$ and $c_3 > 0$ such that

$$\begin{aligned} &\int_0^t \|A^\alpha e^{-(t-s)A} \mathscr{P}(u_\varepsilon \cdot \nabla) u_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds \\ &\le \int_0^t C_\alpha (t-s)^{-\alpha} e^{-\lambda_1'(t-s)} \|(u_\varepsilon \cdot \nabla) u_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds \end{aligned}$$

$$\leq \int_0^t C_\alpha (t-s)^{-\alpha} e^{-\lambda_1'(t-s)} \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} \|\nabla u_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds$$

$$\leq \int_0^t C_\alpha c_1 c_2 (t-s)^{-\alpha} e^{-\lambda_1'(t-s)} \|A^\alpha u_\varepsilon(\cdot,s)\|^a_{L^2(\Omega)} \|u_\varepsilon(\cdot,s)\|^{1-a}_{L^2(\Omega)} ds$$

$$\leq \sup_{t\in(0,T_{\max})} \|u_\varepsilon(\cdot,t)\|^{1-a}_{L^2(\Omega)} \int_0^t C_\alpha c_1 c_2 (t-s)^{-\alpha} e^{-\lambda_1'(t-s)} M^a(s) ds$$

$$\leq C_\alpha c_1 c_2 c_3 \sup_{t\in(0,T_{\max})} \|u_\varepsilon(\cdot,t)\|^{1-a}_{L^2(\Omega)} \sup_{t\in(0,T)} M^a(t) \tag{4.30}$$

for all $t \in (0, T_{\max})$. Furthermore, by (1.4) and Lemma 4.5 we can find and $c_4 > 0$ such that

$$\int_0^t \|A^\alpha e^{-(t-s)A} \mathscr{P} n_\varepsilon(\cdot,s)\nabla\phi\|_{L^2(\Omega)} ds$$

$$\leq \int_0^t C_\alpha \|\nabla\phi\|_{L^\infty(\Omega)} (t-s)^{-\alpha} e^{-\lambda_1'(t-s)} \|n_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds$$

$$\leq C_\alpha \|\nabla\phi\|_{L^\infty(\Omega)} \sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \int_0^t (t-s)^{-\alpha} e^{-\lambda_1'(t-s)} ds$$

$$\leq C_\alpha c_4 \|\nabla\phi\|_{L^\infty(\Omega)} \sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \tag{4.31}$$

for all $t \in (0, T_{\max})$. Taking supremum on both sides of (4.28) on $(0, T)$ with $T \in (0, T_{\max})$, we use (4.30) and (4.31) to find $c_5 > 0$ such that

$$\tilde{M} \leq c_5 + c_5 \tilde{M}^a, \tag{4.32}$$

where we have used the notation $\tilde{M} := \sup_{t\in(0,T)} M(t)$. An application of Young's inequality to the above inequality leads to the assertion. □

Proof of Proposition 2.1 (i). Let $p_0 > 2$ and let $\delta_0 := \delta(p_0)$ as defined in Lemma 3.1. We immediately see from Lemmata 4.1, 4.3–4.7 that $\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. The boundedness of $\|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)}$ is obvious from Lemma 2.2 and Lemma 4.6. Also Lemma 4.26 implies that $\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ is bounded. According to Lemma 2.1, we deduce $T_{\max} = \infty$, thus the solution is global. □

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

5. Boundedness in three-dimensional case ($N = 3$, $\kappa = 0$)

In this section, we deal with the chemotaxis-Stokes system in the three-dimensional setting. Since for the Navier–Stokes system, it is impossible to have global classical solutions without any restrictions on u_0, we only consider the case $\kappa = 0$ and only assume $\|c_0\|_{L^\infty(\Omega)}$ to be small.

We first give a sufficient condition for boundedness which in conjunction with Lemma 2.1 proves Theorem 1. In fact, since Lemma 3.1 provides L^p estimate for any $p > 1$, we can of course choose p sufficiently large to get boundedness in $L^\infty(\Omega)$. However, we would like to give an optimal condition in the following for our own interest.

Proposition 5.1. *Let $N = 3$, $p > \frac{N}{2}$, suppose that*

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} < \infty \quad \textit{for all } t \in (0, T_{\max}). \tag{5.1}$$

Then we have for any $\varepsilon > 0$

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} < \infty \quad \textit{for all } t \in (0, T_{\max}). \tag{5.2}$$

We will prove the proposition by several lemmata, which improve regularity for u_ε and ∇c_ε in suitable way.

Lemma 5.1. *Let $p > \frac{N}{2}$, suppose that*

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} < \infty. \tag{5.3}$$

There are $\alpha \in (\frac{N}{4}, 1)$ and $C > 0$ such that

$$\begin{aligned} &\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C,\\ &\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \textit{ for all } t \in (0, T_{\max}). \end{aligned}$$

Proof. The proof is very similar to Lemma 4.3, we only need to deal with the term with less regularity of n_ε, say, the case $p < 2$. For $p \in (\frac{N}{2}, 2)$ given in Lemma 5.1, we can find $\alpha \in (\frac{N}{4}, \min\{1 - \frac{N}{2p} + \frac{N}{4}, 1\})$. We apply Lemma 4.5 to obtain some $c_2 > 0$ such that

$$\begin{aligned} &\int_0^t \|A^\alpha e^{(t-s)A} n_\varepsilon \nabla\phi\|_{L^2(\Omega)} ds\\ &\quad \le \int_0^t \|A^\alpha e^{\frac{(t-s)}{2}A}(e^{\frac{(t-s)}{2}A} n_\varepsilon(\cdot,s)\nabla\phi)\|_{L^2(\Omega)} ds\\ &\quad \le \int_0^t C_\alpha\Big(\frac{t-s}{2}\Big)^{-\alpha} e^{-\frac{\lambda_1'}{2}(t-s)} \|e^{\frac{(t-s)}{2}A} n_\varepsilon(\cdot,s)\nabla\phi\|_{L^2(\Omega)} ds \end{aligned}$$

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

$$\leq \int_0^t C_\alpha (\frac{t-s}{2})^{-\alpha} e^{-\frac{\lambda_1'}{2}(t-s)} (\frac{t-s}{2})^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{2})} \|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \|\nabla\phi\|_{L^\infty(\Omega)} ds$$

$$\leq C_\alpha \int_0^t (t-s)^{-\alpha-\frac{N}{2p}+\frac{N}{4}} e^{-\frac{\lambda_1'}{2}(t-s)} \|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \|\nabla\phi\|_{L^\infty(\Omega)} ds$$

$$\leq C_\alpha \|\nabla\phi\|_{L^\infty(\Omega)} \sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \int_0^t (t-s)^{-\alpha-\frac{N}{2p}+\frac{N}{4}} e^{-\frac{\lambda_1'}{2}(t-s)} ds$$

$$\leq C_\alpha c_2 \|\nabla\phi\|_{L^\infty(\Omega)} \sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)},$$

for all $t \in (0, T_{\max})$. Combine this fact with the proof of Lemma 4.8 we see that $\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ is bounded for all $t \in (0, T_{\max})$. Since $\alpha > \frac{N}{4}$, embedding theorem implies the boundedness of $\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$. Thus the proof is complete. □

Lemma 5.2. *Let $p > \frac{N}{2}$, suppose that*

$$\sup_{t\in(0,T_{\max})} \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} < \infty. \tag{5.4}$$

For $N < q_0 < \frac{Np}{N-p}$, there is $C > 0$ such that for any $\varepsilon > 0$,

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{5.5}$$

Proof. Let $t \in (0, T_{\max})$ and $M := \sup_{t\in(0,T)} \|\nabla c_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)}$. The variation of constants formula implies

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)} \leq \|\nabla e^{t\Delta} c_0\|_{L^{q_0}(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}(n_\varepsilon c_\varepsilon)(\cdot,s)\|_{L^{q_0}(\Omega)} ds$$
$$+ \int_0^t \|\nabla e^{(t-s)\Delta}(u_\varepsilon \cdot \nabla c_\varepsilon)(\cdot,s)\|_{L^{q_0}(\Omega)} ds$$

for all $t \in (0, T_{\max})$. By the L^p-L^q estimates we see that

$$\|\nabla e^{t\Delta} c_0\|_{L^{q_0}(\Omega)} \leq c_1 t^{-\frac{1}{2}} \|c_0\|_{L^{q_0}(\Omega)}$$

for some $c_1 > 0$ and for all $t > 0$. Since $1 < q_0 < \frac{Np}{N-p}$, we know that $-\frac{1}{2} - \frac{N}{2}(\frac{1}{p} - \frac{1}{q_0}) > -1$, thus we estimate the second term by L^p-L^q estimate for Neumann semigroup with $c_2 > 0$ such that

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

$$\begin{aligned}
&\int_0^t \|\nabla e^{(t-s)\Delta} n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)\|_{L^{q_0}(\Omega)}ds\\
&\le \int_0^t c_2(1+(t-s)^{-\frac12-\frac N2(\frac1p-\frac1{q_0})})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)\|_{L^p(\Omega)}ds\\
&\le \int_0^t c_2(1+(t-s)^{-\frac12-\frac N2(\frac1p-\frac1{q_0})})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)}\|c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds
\end{aligned}$$

for all $t \in (0, T_{\max})$. It is bounded due to the choice of q_0 and our assumption on $\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}$. Now we fix $q < q_0$ satisfying $\frac1q - \frac1{q_0} < \frac1N$, and let $a = \frac{1-\frac Nq}{1-\frac N{q_0}} \in (0,1)$. Hölder's inequality as well as interpolation inequality yield the existence of $c_3 > 0$,

$$\begin{aligned}
&\int_0^t \|\nabla e^{(t-s)\Delta} u_\varepsilon(\cdot,s)\cdot\nabla c_\varepsilon(\cdot,s)\|_{L^{q_0}(\Omega)}ds\\
&\le \int_0^t c_2(1+(t-s)^{-\frac12-\frac N2(\frac1q-\frac1{q_0})})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\nabla c_\varepsilon(\cdot,s)\|_{L^q(\Omega)}ds\\
&\le \int_0^t c_2(1+(t-s)^{-\frac12-\frac N2(\frac1q-\frac1{q_0})})e^{-\lambda_1(t-s)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^q(\Omega)}\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds\\
&\le \int_0^t c_2(1+(t-s)^{-\frac12-\frac N2(\frac1q-\frac1{q_0})})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}(c_3\|\nabla c_\varepsilon(\cdot,s)\|^a_{L^{q_0}(\Omega)}\|c_\varepsilon(\cdot,s)\|^{1-a}_{L^\infty(\Omega)}\\
&\qquad\qquad + c_3\|c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)})ds\\
&\le c_4M^a + c_4
\end{aligned}$$

for all $t \in (0, T_{\max})$ with some $c_4 > 0$. The assertion can be seen by combining the above estimates and the fact that $a < 1$. □

Having enough regularity for both u_ε and ∇c_ε, we are ready to prove boundedness for n_ε.

Proof of Proposition 5.1. Let $T \in (0, T_{\max})$ and $M := \sup_{t\in(0,T)} \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$. The representation formula for n_ε yields that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{t\Delta}n_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)\Delta}\nabla\cdot(n_\varepsilon S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon)(\cdot,s)\|_{L^\infty(\Omega)}ds \tag{5.6}$$

$$+\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s))\|_{L^\infty(\Omega)}ds$$

for all $t\in(0,T_{\max})$. Since $q_0>N$, we can find $N<p_0<q_0$ and $q_1>1$ such that $\frac{1}{p_0}=\frac{1}{q_0}+\frac{1}{q_1}$. Let $a=1-\frac{1}{q_1}$. The L^p-L^q estimates for the Neumann heat semigroup and Hölder inequality imply $c_1>0$ and $c_2>0$

$$\begin{aligned}
&\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(n_\varepsilon S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)(\cdot,s)\|_{L^\infty(\Omega)}ds\\
&\le \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_0}})e^{-\lambda_1(t-s)}\|(n_\varepsilon S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)(\cdot,s)\|_{L^{p_0}(\Omega)}ds\\
&\le \int_0^t c_1S_0(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_0}})e^{-\lambda_1(t-s)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^{q_0}(\Omega)}\|n_\varepsilon(\cdot,s)\|_{L^{q_1}(\Omega)}ds\\
&\le \int_0^t c_1S_0(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_0}})e^{-\lambda_1(t-s)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^{q_0}(\Omega)}\|n_\varepsilon(\cdot,s)\|^a_{L^\infty(\Omega)}\|n_\varepsilon(\cdot,s)\|_{L^1(\Omega)}ds\\
&\le c_2+c_2M^a,\ \text{ for all } t\in(0,T_{\max}).
\end{aligned}$$

Now we pick $p_1>N$, and let $b=1-\frac{1}{p_1}$. The L^p-L^q estimates and the interpolation inequality imply

$$\begin{aligned}
&\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s))\|_{L^\infty(\Omega)}ds\\
&\le \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_1}})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s)\|_{L^{p_1}(\Omega)}ds\\
&\le \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_1}})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot,s)\|_{L^{p_1}(\Omega)}\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds\\
&\le \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_1}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\|n_\varepsilon(\cdot,s)\|^b_{L^\infty(\Omega)}\|n_\varepsilon(\cdot,s)\|^{1-b}_{L^1(\Omega)}ds.
\end{aligned}\tag{5.7}$$

Finally, collecting the above estimates, we conclude the assertion by a similar reasoning as in Lemma 5.2. □

5.1. Proof of Proposition 2.1 (ii)

Combining Proposition 5.1 and Lemma 3.1 proves Proposition 2.1.

Proof of Proposition 2.1 (ii). Let $p > 2$ and let $\delta_0 := \delta_0(p)$ as defined in Lemma 3.1. We see that (2.7) implies $\|n(\cdot,t)\|_{L^p(\Omega)}$ is bounded, which combined with Proposition 5.1 yields the boundedness of $\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$. The boundedness of $\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ has been shown in Lemma 5.2. Together with Lemma 5.1, we see that the solution is global by Lemma 2.1. □

6. Passing to the limit

We now wish to obtain the solution of (1.1) by sending $\varepsilon \to 0$ for the approximated solution. In order to achieve this, we shall first prepare some estimates for $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ which are independent of ε. Since we cannot expect the regularity in $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times(0,\infty))$ to be uniform in ε due to the presence of S_ε, we will first show the triple of limit functions solves (1.1) in the sense of distributions, then apply standard parabolic regularity to show that it is actually a classical solution. The procedure is quite similar to that in [3].

Let us first define a weak solution.

Definition 6.1. We say that (n, c, u, P) is a global weak solution of (1.1) associated to initial data (n_0, c_0, u_0) if

$$\begin{cases} n \in L^\infty((0,\infty)\times\Omega) \cap L^2_{loc}((0,\infty); W^{1,2}(\Omega)), \\ c \in L^\infty((0,\infty)\times\Omega) \cap L^2_{loc}((0,\infty); W^{1,2}(\Omega)), \\ u \in L^\infty((0,\infty)\times\Omega) \cap L^2_{loc}((0,\infty); W^{1,2}_{0,\sigma}(\Omega)), \\ P \in L^2((0,T); W^{1,2}(\Omega)), \end{cases} \tag{6.1}$$

and for all $\psi \in C_0^\infty(\overline{\Omega}\times[0,\infty);\mathbb{R})$ and all $\zeta \in C^\infty_{0,\sigma}(\Omega\times[0,\infty);\mathbb{R}^N)$ the following identities hold:

$$-\int_0^\infty\int_\Omega n\psi_t - \int_\Omega n_0\psi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla n\cdot\nabla\psi + \int_0^\infty\int_\Omega nS(x,n,c)\cdot\nabla c\cdot\nabla\psi + \int_0^\infty\int_\Omega nu\cdot\nabla\psi, \tag{6.2}$$

$$-\int_0^\infty\int_\Omega c\psi_t - \int_\Omega c_0\psi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla c\cdot\nabla\psi - \int_0^\infty\int_\Omega nc\psi + \int_0^\infty\int_\Omega cu\cdot\nabla\psi, \tag{6.3}$$

$$-\int_0^\infty\int_\Omega u\cdot\zeta_t - \int_\Omega u_0\cdot\zeta(\cdot,0) = -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\zeta + \int_0^\infty\int_\Omega (u\cdot\nabla)u\cdot\zeta + \int_0^\infty\int_\Omega n\nabla\phi\cdot\zeta. \tag{6.4}$$

The required estimates are very close to those in [3]. We will state the results here and only give a sketch of the proofs.

Lemma 6.1. *There exists $C > 0$ such that for all $\varepsilon \in (0, 1)$*

$$\int_0^\infty \int_\Omega |\nabla c_\varepsilon|^2 < C, \tag{6.5}$$

$$\int_0^\infty \int_\Omega |\nabla n_\varepsilon|^2 < C. \tag{6.6}$$

Proof. Multiply the second equation by c_ε, using the fact $\nabla \cdot u_\varepsilon = 0$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \int_\Omega |\nabla c_\varepsilon|^2 \le 0, \tag{6.7}$$

this implies (6.5) by direct integration. Since $T_{\max} = \infty$, letting $p = 2$ in (3.3), we see that (6.6) holds. □

Lemma 6.2. *All bounded solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ of (2.4) satisfy*

$$n_\varepsilon \in C_{loc}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega} \times (0,\infty)),\ c_\varepsilon \in C_{loc}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega} \times (0,\infty)),\ u_\varepsilon \in C_{loc}^{1+\gamma,\gamma}(\overline{\Omega} \times (0,\infty)). \tag{6.8}$$

More precisely, there is $C > 0$ such that for all $\varepsilon \in (0, 1)$, and all $s \in [1, \infty)$ we have

$$\|n_\varepsilon\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \le C, \tag{6.9}$$

$$\|c_\varepsilon\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \le C, \tag{6.10}$$

$$\|u_\varepsilon\|_{C^{1+\gamma,\gamma}(\overline{\Omega}\times[s,s+1])} \le C. \tag{6.11}$$

Proof. Let $s \ge 1$. We define $\tilde{n}_\varepsilon(\cdot, t) = n_\varepsilon(\cdot, t + s - 1)$, $\tilde{c}_\varepsilon(\cdot, t) = c_\varepsilon(\cdot, t + s - 1)$ and $\tilde{u}_\varepsilon(\cdot, t) = u_\varepsilon(\cdot, t + s - 1)$. Let $\xi \in C^\infty((0, \infty))$ satisfy $\xi = 0$ on $(0, \frac{1}{2}) \cup (\frac{5}{2}, \infty)$ and $\xi = 1$ on $[1, 2]$. We see that $\xi\tilde{n}_\varepsilon$ is a weak solution of

$$(\xi\tilde{n}_\varepsilon)_t - \nabla \cdot (\nabla(\xi\tilde{n}_\varepsilon) - \xi\tilde{n}_\varepsilon S_\varepsilon(x, \tilde{n}_\varepsilon, \tilde{c}_\varepsilon)\nabla\tilde{c}_\varepsilon) = \xi'\tilde{n}_\varepsilon,\ \ t \in [0, \infty),$$

associated with Neumann boundary condition and $\xi\tilde{n}_\varepsilon(\cdot, \frac{1}{2}) = 0$. Since $(\nabla(\xi\tilde{n}_\varepsilon) - \xi\tilde{n}_\varepsilon S_\varepsilon(x, \tilde{n}_\varepsilon, \tilde{c}_\varepsilon)\nabla\tilde{c}_\varepsilon - \tilde{n}_\varepsilon\tilde{u}_\varepsilon) \cdot \nabla(\xi\tilde{n}_\varepsilon) > \frac{1}{2}|\nabla(\xi\tilde{n}_\varepsilon)|^2 - \tilde{n}_\varepsilon^2|S_\varepsilon|^2|\nabla\tilde{c}_\varepsilon|^2 - \tilde{n}_\varepsilon^2|\tilde{u}_\varepsilon|^2$, we see together with the fact guaranteed in Lemma 4.6 and Lemma 5.2, that the norms of $\tilde{n}_\varepsilon^2|S_\varepsilon|^2|\nabla\tilde{c}_\varepsilon|^2 + \tilde{n}_\varepsilon^2|\tilde{u}_\varepsilon|^2$ and $\xi'\tilde{n}_\varepsilon$ are bounded in $L^p((\frac{1}{2}, \frac{5}{2}); L^q(\Omega))$ for suitably large p or q and independent of s, thus Theorem 1.3 in [20] implies there are $\gamma_1 \in (0, 1)$ and $c_1 > 0$ such that

$$\|n_\varepsilon\|_{C^{\gamma_1,\frac{\gamma_1}{2}}(\overline{\Omega}\times[s,s+1])} = \|\tilde{n}_\varepsilon\|_{C^{\gamma_1,\frac{\gamma_1}{2}}(\overline{\Omega}\times[1,2])} \le \|\xi\tilde{n}_\varepsilon\|_{C^{\gamma_1,\frac{\gamma_1}{2}}(\overline{\Omega}\times[\frac{1}{2},\frac{5}{2}])} \le c_1,$$

and c_1 depends on $\|\xi \tilde{n}_\varepsilon\|_{L^\infty(\Omega\times(\frac{1}{2},\frac{2}{5}))}$ and the norms of $\tilde{n}_\varepsilon^2|S_\varepsilon|^2|\nabla\tilde{c}_\varepsilon|^2+\tilde{n}_\varepsilon^2|\tilde{u}_\varepsilon|^2$ in appropriate spaces only. A similar reasoning yields some $\gamma_2\in(0,1)$ and $c_2>0$ such that

$$\|c_\varepsilon\|_{C^{\gamma_2,\frac{\gamma_2}{2}}(\overline{\Omega}\times[s,s+1])}\le c_2.$$

The derivation of the regularity of u_ε is similar to [3, Lemma 5.3]. Let $s\ge 1$, and ξ_s be a smooth function: $(0,\infty)\to[0,1]$ satisfying $\xi_s(t)=0$ on $(0,s-\frac{1}{2})\cup(s+\frac{3}{2},\infty)$ and $\xi_s(t)=1$ on $[s,s+1]$. We consider $\xi\cdot u_\varepsilon$, it satisfies

$$(\xi u_\varepsilon)_t=\xi_t u_\varepsilon+\xi u_{\varepsilon t}=\Delta(\xi u_\varepsilon)-\xi(u_\varepsilon\cdot\nabla)u_\varepsilon+\xi n_\varepsilon\nabla\phi+\xi' u_\varepsilon,\ \text{on }(s-\frac{1}{2},s+\frac{3}{2}),$$

with $\xi u_\varepsilon(\cdot,0)=0$ and $\xi u_\varepsilon=0$ on $\partial\Omega$. Thus by an application of [9, Thm. 2.8], for any $r\in(1,\infty)$, we deduce the existence of constant $C_r>0$ fulfilling

$$\begin{aligned}&\int_0^\infty\|(\xi u_\varepsilon)_t\|^r_{L^r(\Omega)}+\int_0^\infty\|D^2(\xi u_\varepsilon)\|^r_{L^r(\Omega)}\\&\quad\le C_r\left(0+\int_0^\infty\|\mathscr{P}((\xi u_\varepsilon\cdot\nabla)u_\varepsilon)+\mathscr{P}(\xi n_\varepsilon\nabla\phi)+\mathscr{P}\left(\xi' u_\varepsilon\right)\|^r_{L^r(\Omega)}ds\right),\end{aligned}$$

which, due to the definition of ξ and boundedness of $u_\varepsilon, n_\varepsilon, \xi'$ reads as

$$\int_{s-\frac{1}{2}}^{s+\frac{3}{2}}\|(\xi u_\varepsilon)_t\|^r_{L^r(\Omega)}+\int_{s-\frac{1}{2}}^{s+\frac{3}{2}}\|D^2(\xi u_\varepsilon)\|^r_{L^r(\Omega)}\le C_1\int_{s-\frac{1}{2}}^{s+\frac{3}{2}}\|\nabla(\xi u_\varepsilon)\|^r_{L^r(\Omega)}+C_2$$

for all $s\in(1,\infty)$ and for some $C_1>0$, $C_2>0$. Let $a=\frac{1-\frac{N}{r}}{2-\frac{N}{r}}\in(0,1)$, the Gargliardo–Nirenberg inequality shows

$$\|\nabla(\xi u_\varepsilon)\|^r_{L^r(\Omega)}\le C_3\|D^2(\xi u_\varepsilon)\|^{ar}_{L^r(\Omega)}\|(\xi u_\varepsilon)\|^{(1-a)r}_{L^\infty(\Omega)}$$

for some $C_3>0$. Integrating the above inequality on $(s-\frac{1}{2},s+\frac{3}{2})$ and using Young's inequality yields that

$$\int_{s-\frac{1}{2}}^{s+\frac{3}{2}}\|\nabla(\xi u_\varepsilon)\|^r_{L^r(\Omega)}\le C_4\int_{s-\frac{1}{2}}^{s+\frac{3}{2}}\|D^2(\xi u_\varepsilon)\|^{ar}_{L^r(\Omega)}\le\int_{s-\frac{1}{2}}^{s+\frac{3}{2}}\left(\frac{1}{2}\|D^2(\xi u_\varepsilon)\|^r_{L^r(\Omega)}+C_5\right)$$

with some $C_4>0$, $C_5>0$ and for all $s\in[1,\infty)$. Combining the above estimates we see that there is $C_6>0$ such that for all $s\ge 1$ and all $\varepsilon\in(0,1)$,

$$\int_s^{s+1} \|(u_\varepsilon)_t\|^r_{L^r(\Omega)} + \int_s^{s+1} \|D^2 u_\varepsilon\|^r_{L^r(\Omega)} \le C_6.$$

Let $r \in (1,\infty)$ be sufficiently large, the embedding theorem implies the existence of $\gamma_3 \in (0,1)$, $C > 0$ such that

$$\|u_\varepsilon\|_{C^{1+\gamma_3,\gamma_3}(\overline{\Omega}\times[s,s+1])} \le C.$$

Choosing $\gamma = \min\{\gamma_1,\gamma_2,\gamma_3\}$ we have proved (6.9)–(6.11). For all $\tau > 0$, we can choose $\xi_\tau \in C_0^\infty((0,\infty))$ in such way that $\xi_\tau = 0$ on $(0,\tau)$ and $(\max\{3\tau,1\},\infty)$, and $\xi_\tau = 1$ on $[\tau,\max\{2\tau,1\}]$. We consider the equations for $\xi_\tau n_\varepsilon$, $\xi_\tau c_\varepsilon$ and $\xi_\tau u_\varepsilon$, respectively. Then (6.8) is obtained by the same reasoning as above. Actually, the way of γ_i $(i = 1,2)$ depending on τ is through the non-decreasing dependence of the norms $\|\xi n_\varepsilon\|_{L^\infty(\Omega\times[\tau,\max\{3\tau,1\}])}$, $\|\xi c_\varepsilon\|_{L^\infty(\Omega\times[\tau,\max\{3\tau,1\}])}$ [20, Theorem 1.3], which are independent of τ, thus we can choose the same γ_i $(i = 1,2)$ as before. Moreover, γ_3 can be chosen in the same manner upon choosing the same r. □

Lemma 6.3. *Let $\gamma \in (0,1)$ be chosen as in Lemma 6.2. There exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that as $\varepsilon = \varepsilon_j \searrow 0$, it holds that*

$$n_\varepsilon \to n \text{ in } C^\gamma_{loc}(\overline{\Omega}\times(0,\infty)), \tag{6.12}$$

$$\nabla n_\varepsilon \rightharpoonup \nabla n \text{ in } L^2(\Omega\times(0,\infty)), \tag{6.13}$$

$$c_\varepsilon \to c \text{ in } C^\gamma_{loc}(\overline{\Omega}\times(0,\infty)), \tag{6.14}$$

$$\nabla c_\varepsilon \rightharpoonup \nabla c \text{ in } L^2(\Omega\times(0,\infty)), \tag{6.15}$$

$$u_\varepsilon \to u \text{ in } C^\gamma_{loc}(\overline{\Omega}\times(0,\infty)), \tag{6.16}$$

$$\nabla u_\varepsilon \to \nabla u \text{ in } C^\gamma_{loc}(\Omega\times(0,\infty)), \tag{6.17}$$

$$u_\varepsilon \overset{\star}{\rightharpoonup} u \text{ in } L^\infty((0,\infty);D(A^\alpha)), \tag{6.18}$$

$$S_\varepsilon(x,n_\varepsilon(x,t),c_\varepsilon(x,t)) \to S(x,n(x,t),c(x,t)) \text{ a.e. in } \Omega\times(0,\infty). \tag{6.19}$$

Proof. First (6.12), (6.14) and (6.16), (6.17) are obtained from Lemma 6.2. Lemma 6.1 implies (6.13) and (6.15). Due to the obtained convergence (6.12) and (6.14) and the continuity of S, we conclude that (6.19) holds. □

Lemma 6.4. *The functions n, c, u from Lemma 6.3 form a weak solution to (1.1) in the sense of Definition 6.1.*

Proof. Take ψ and ζ as specified in Definition 6.1 and test them to (2.4). Lemma 6.3 allows us to take limit in each integral, thus we obtain the weak formulation. □

Lemma 6.5. *The functions n, c, u from the previous lemma satisfy*

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

$$n \in C_{loc}^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega} \times (0,\infty)),\ c \in C_{loc}^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega} \times (0,\infty)),$$
$$u \in C_{loc}^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega} \times (0,\infty)) \tag{6.20}$$

for some $\gamma \in (0,1)$. Moreover, let $s \geq 1$. There is a constant $C > 0$ such that

$$\|n\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \leq C,\ \|c\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \leq C,$$
$$\|u\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \leq C \tag{6.21}$$

for all $t \geq 1$.

Proof. First taking ξ_s as chosen in Lemma 6.2, we see that $\xi \cdot c$ is a weak solution of $(\xi c)_t - \Delta(\xi c) + n(\xi c) + u \cdot \nabla(\xi c) - \xi' c = 0$ on $t \in (s - \frac{1}{2}, s + \frac{3}{2})$ associated with Neumann boundary condition and $\xi c(\cdot, s - \frac{1}{2}) = 0$. First [14, Thm. 5.3] guarantees that $\xi c \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega} \times [s - \frac{1}{2}, s + \frac{3}{2}])$, therefore, [16, Thm. 4.9] shows that the norm $\|\xi c\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s-\frac{1}{2},s+\frac{3}{2}])}$ is controlled by the corresponding Hölder norms of n and u, which is bounded by Lemma 6.2 and Lemma 6.3.

For the regularity of n, we improve it similarly as c but more carefully since its boundary condition also involves c. We first estimate its $C^{1+\gamma,\frac{1+\gamma}{2}}$ norm, then its $C^{2+\gamma,1+\frac{\gamma}{2}}$ norm, by [16, Thm. 4.8] and [16, Thm. 4.9], respectively.

For the regularity of u, we again consider $\xi \cdot u$, which satisfies $(\xi u)_t = \Delta(\xi u) - \xi(u \cdot \nabla)u + \xi n \nabla\phi + \xi' u$, with Dirichlet boundary condition. Lemma 6.1 already ensures the $C^{\gamma,\frac{\gamma}{2}}$ bound on the right hand side. Thus [22, Thm. 1.1] together with the uniqueness guaranteed in [21, Thm. V.1.5.1] implies the (6.21).

For any fixed $\tau > 0$, by choosing $\xi_\tau \in C_0^\infty((0,\infty))$ such that $\xi_\tau = 0$ on $(0,\tau)$ and $(3\tau,\infty)$, and $\xi_\tau = 1$ on $[\tau, 2\tau]$, we can see (6.20) holds by the same reasoning as introduced above. □

Having in hand the regularity for the weak solution (n, c, u) of (1.1), we have shown that it is actually classical solution.

7. Stabilization

In this section, we prove large time convergence for each component of the solution but not the approximated one. Since we have already derived that the solution is globally bounded, then it has uniform in time regularity in Hölder space. In order to make the convergence property from the previous section applicable, we have to gain some uniform smooth regularity.

The first obtained convergence of n is crucial, it will imply convergence for c and u later.

Lemma 7.1. *Let (n, c, u, P) be the classical bounded solution of (1.1). We have*

$$\int_0^\infty \int_\Omega |\nabla n|^2 < \infty.$$

Proof. The statement holds due to (6.6) and (6.13). □

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

Lemma 7.2. *Let (n, c, u, P) be the classical solution of (1.1), we have*

$$\|(n(\cdot, t) - \bar{n}_0)\|_{L^\infty(\Omega)} \to 0, \text{ as } t \to \infty. \tag{7.1}$$

Proof. Suppose on contrary that there are $c_1 > 0$ and a sequence $t_k \to \infty$ such that

$$\|n(\cdot, t_k) - \bar{n}_0\|_{L^\infty(\Omega)} > c_1 \quad \text{for } k \in \mathbb{N}. \tag{7.2}$$

Now we define

$$g_k(x, s) = n(x, s + t_k), \ (x, s) \in \overline{\Omega} \times [0, 1].$$

By the regularity guaranteed in Lemma 6.5, we see that for all $k \in \mathbb{N}$, there is $c_2 > 0$ such that

$$\|g_k\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [0,1])} \le c_2.$$

The Arzelá–Ascoli theorem implies that g_k is relatively compact in $C^1(\overline{\Omega} \times [0, 1])$. Thus we can find a subsequence $\{g_{k_j}\}_j$ and $n_\infty \in C^1(\overline{\Omega} \times [s, s + 1])$ such that

$$g_{k_j} \to n_\infty \in C^1(\overline{\Omega} \times [0, 1]), \text{ as } j \to \infty. \tag{7.3}$$

It is left to show $n_\infty = \bar{n}_0$. We see from Lemma 7.1 that

$$\int_0^1 \int_\Omega |\nabla g_{k_j}|^2 \to 0 \text{ as } j \to \infty,$$

which combined with (7.3) implies

$$\int_0^1 \int_\Omega |\nabla n_\infty|^2 = 0.$$

Since $n_\infty \in C^1(\overline{\Omega} \times [0, 1])$, we deduce that $n_\infty \equiv L$ with $L \in \mathbb{R}$. Moreover, we have

$$|\Omega| \cdot L = \int_0^1 \int_\Omega n_\infty = \lim_{j \to \infty} \int_0^1 \int_\Omega f_{k_j} = \int_0^1 \int_\Omega n_0.$$

Thus we conclude $n_\infty \equiv \bar{n}_0$. This contradicts (7.2) by the definition of n_∞. $\square$

Lemma 7.3. *Let (n, c, u, P) be the classical bounded solution of (1.1). For any $0 < \eta < \bar{n}_0$, there is $C > 0$ such that*

$$\|c(\cdot, t)\|_{L^\infty(\Omega)} \le C e^{-\eta t} \quad \textit{for all } t \ge 0. \tag{7.4}$$

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

Proof. For all $0 < \eta < \bar{n}_0$, we can find $T > 0$ such that

$$n \geq \eta \quad \text{for all } t \geq T.$$

Thus the second equation of (1.1) can be written as

$$c_t \leq \Delta c - \eta c - u \cdot \nabla c \quad \text{for all } t \geq T.$$

The maximum principle yields that

$$\|c(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c(\cdot, T)\|_{L^\infty(\Omega)} e^{-\eta t} \quad \text{for all } t \geq T.$$

An obvious choice of C completes the proof. □

Lemma 7.4. *Let (n, c, u, P) be the classical bounded solution of (1.1), and let $0 < \eta < \min\{\bar{n}_0, \lambda_1\}$, then we can find some $C > 0$ such that*

$$\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq C e^{-\eta t} \quad \text{for all } t \geq 0. \tag{7.5}$$

Proof. The proof is based on the constants variation formula, $L^p - L^q$ estimates as well as Lemma 4.5. Recall (4.18), the first term can be estimated easily

$$\begin{aligned} \|\nabla e^{t\Delta} c_0\|_{L^\infty(\Omega)} &\leq c_1 t^{-\frac{1}{2}} e^{-\lambda_1 t} \|c_0\|_{L^\infty(\Omega)} \\ &\leq c_1 \|c_0\|_{L^\infty(\Omega)} e^{-\lambda_1 t} \end{aligned}$$

for all $t \geq 1$. With local existence theory, $\|\nabla e^{t\Delta} c_0\|_{L^\infty(\Omega)}$ is bounded for $t > 0$. Similarly to (4.20), with $\|n\|_{L^p(\Omega)} \leq c_2$ and $\|c\|_{L^\infty(\Omega)} \leq c_3$, if we choose $p > \frac{1}{N}$, Lemma 4.5 implies the existence of $c_4 > 0$ such that

$$\begin{aligned} &\int_0^t \|\nabla e^{(t-s)\Delta} n(\cdot, s) c(\cdot, s)\|_{L^\infty(\Omega)} ds \\ &\leq \int_0^t c_1 (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2p}}) e^{-\lambda_1 (t-s)} \|n(\cdot, s)\|_{L^p(\Omega)} \|c(\cdot, s)\|_{L^\infty(\Omega)} ds \\ &\leq \int_0^t c_1 (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2p}}) e^{-\lambda_1 (t-s)} c_2 c_3 e^{-\eta s} ds \\ &\leq c_1 c_2 c_3 c_4 e^{-\eta t} \end{aligned}$$

for all $t > 0$. Moreover, let $\theta = 1 - \frac{N}{p} \in (0, 1)$ and $M(t) = e^{\eta t} \|\nabla c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}$,

$$
\begin{aligned}
&\int_0^t \|\nabla e^{(t-s)\Delta}(u(\cdot,t)\cdot\nabla c(\cdot,t))\|_{L^\infty(\Omega)}ds \\
&\leq \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p}})e^{-\lambda_1(t-s)}\|u(\cdot,s)\cdot\nabla c\|_{L^p(\Omega)}ds \\
&\leq \int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p}})e^{-\lambda_1(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}(\|\nabla c(\cdot,t)\|^{\theta}_{L^\infty(\Omega)}\|c(\cdot,t)\|^{1-\theta}_{L^\infty(\Omega)} \\
&\quad + \|c(\cdot,s)\|_{L^\infty(\Omega)})ds \\
&\leq c_1\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2p}})e^{-\lambda_1(t-s)}c_2M^{\theta}(s)e^{-\theta\eta s}\|c_0\|^{1-\theta}_{L^\infty(\Omega)}e^{-(1-\theta)\eta s}ds \\
&\qquad + c_1\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2p}})e^{-\lambda_1(t-s)}c_2\|c_0\|_{L^\infty(\Omega)}e^{-\eta s}ds
\end{aligned}
$$

for all $t>0$. If we multiply $e^{\eta t}$ on both sides of (4.18) and let $\tilde{M} := \sup_{t\in(0,T)} M(t)$ for any $T\in(0,\infty)$, we find that

$$\tilde{M} \leq c_5\tilde{M}^{\theta} + c_6,$$

for some $c_5, c_6 > 0$, thus $\tilde{M}$ is bounded, which leads to the assertion. □

Lemma 7.5. *Let (n,c,u,P) be the classical bounded solution of (1.1). There is $C>0$, such that*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq C, \tag{7.6}$$

$$\|A^{\alpha}u(\cdot,t)\|_{L^2(\Omega)} \leq C \quad \text{for all } t>0, \tag{7.7}$$

$$\|u(\cdot,t)\|_{L^2(\Omega)} \to 0, \ \text{as } t\to\infty, \tag{7.8}$$

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0, \ \text{as } t\to\infty. \tag{7.9}$$

Proof. First, (7.6) and (7.7) are immediately obtained by (6.16) and (6.18), respectively. Since (4.27) together with (7.7) and (7.8) immediately implies (7.9), it is left to prove (7.8). Testing the third equation in (1.1) with u and integrating by part, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 &= \int_\Omega n\nabla\phi\cdot u = \int_\Omega (n-\bar{n}_0)\nabla\phi\cdot u \\
&\leq \frac{\lambda_1'}{2}\int_\Omega |u|^2 + \frac{1}{2\lambda_1'}\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega |n-\bar{n}_0|^2
\end{aligned}
$$

for all $t \in (0, \infty)$. Using the Poincaré inequality

$$\frac{d}{dt}\int_\Omega |u|^2 + \lambda_1' \int_\Omega |u|^2 \le \frac{1}{\lambda_1'}\|\nabla\phi\|_{L^\infty(\Omega)}\|n - \bar{n}_0\|_{L^2(\Omega)} \quad \text{for all } t > 0. \tag{7.10}$$

By the boundedness of $\|n(\cdot,t)\|_{L^\infty(\Omega)}$ and ODE comparison lemma, we conclude that $\|u(\cdot,t)\|_{L^2(\Omega)} < c_1$ for some $c_1 > 0$ and for all $t > 0$. If we apply Lemma 7.2, for any $\epsilon > 0$, we can find $t_0 > 0$ large enough satisfying

$$\|n(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} < \frac{\lambda_1'\epsilon}{\sqrt{2\|\nabla\phi\|_{L^\infty(\Omega)}|\Omega|}} \text{ for all } t > t_0.$$

Again, (7.10) with Gronwall's inequality implies that

$$\begin{aligned}\int_\Omega |u(\cdot,t)|^2 &\le e^{-\lambda_1'(t-t_0)}\int |u(\cdot,t_0)|^2 + \int_{t_0}^t e^{-\lambda_1'(t-s)}\frac{1}{\lambda_1'}\|\nabla\phi\|_{L^\infty(\Omega)}|\Omega|\|n(\cdot,t) - \overline{n_0}\|^2_{L^\infty(\Omega)}ds \\ &\le e^{-\lambda_1'(t-t_0)}c_1^2 + \frac{1}{(\lambda_1')^2}\|\nabla\phi\|_{L^\infty(\Omega)}|\Omega|\|n(\cdot,t) - \overline{n_0}\|^2_{L^\infty(\Omega)} \\ &\le \epsilon^2\end{aligned}$$

for all $t > \max\{t_0, \frac{1}{\lambda_1'}\ln e^{\frac{t_0}{\lambda_1'}}\frac{2c_1^2}{\epsilon^2}\}$. Thus we have shown (7.8). □

Now we are ready to prove the main result.

Proof of Theorem 1. The function (n, c, u) obtained as the limit of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ is a weak solution of (1.1) by Lemma 6.4. Moreover, its smooth regularity is guaranteed by Lemma 6.5. Hence we know that it solves (1.1) classically. The boundedness of $\|n(\cdot,t)\|_{L^\infty(\Omega)}$ can be seen from Proposition 5.1 and Lemma 6.12. Lemmata 2.2, 7.4 imply that $\|c(\cdot,t)\|_{W^{1,q}(\Omega)}$ is bounded. Moreover, $u \in L^\infty((0,T); D(A^\beta))$ is asserted by (7.7). The continuity up to the initial time can be proven similarly as in [3, Lemma 5.8]; first we prove that for $T > 0$, n_t, c_t and u_t are in $L^2((0,T); (W^{1,2}(\Omega))^*)$ and $L^2((0,T); (W^{1,2}_{0,\sigma}(\Omega))^*)$, respectively. Then we can conclude the assertions for c and u by embedding. Using the continuity of n_ε and the uniform convergence (6.12), the continuity of n can be done similarly as in [3, Lemma 5.8]. Hence we have proved Theorem 1. □

Now we have already shown the convergence of the solution (see Lemma 7.2, Lemma 7.4 and Lemma 7.5). In order to show the convergence rates are exponential, we only need to apply known result from [3], where the Navier–Stokes is considered. However, the arguments there obviously work for the Stokes case which can be seen by dropping the convective term.

Proof of Corollary 1.1. From [3, Thm. 1], we know that for all $m > 0$, and $\alpha, \alpha' > 0$ as chosen in Corollary 1.1, there are p_0, q_0 and ε_0 such that if initial data satisfies

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

$$\bar{\tilde{n}}_0 = \frac{1}{|\Omega|}\int_\Omega \tilde{n}_0 = m, \quad \|\tilde{n}_0 - \bar{\tilde{n}}_0\|_{L^{p_0}(\Omega)} \le \varepsilon_0, \quad \|\tilde{c}_0\|_{L^\infty(\Omega)} \le \varepsilon_0, \quad \|\tilde{u}_0\|_{L^N(\Omega)} \le \varepsilon_0, \tag{7.11}$$

the solution fulfilling

$$\|n(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} \le Ce^{-\alpha t}, \qquad \|c(\cdot,t)\|_{L^{q_0}(\Omega)} \le Ce^{-\alpha t},$$
$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\alpha' t} \quad \text{for all } t > 0.$$

According to Lemmata 7.2 7.4 and 7.5, for the aforementioned $\varepsilon_0 > 0$, there is $T > 0$ such that

$$\|n(\cdot,t) - \bar{n}_0\|_{L^{p_0}(\Omega)} \le \varepsilon_0, \quad \|c(\cdot,t)\|_{L^\infty(\Omega)} \le \varepsilon_0, \quad \|u(\cdot,t)\|_{L^N(\Omega)} \le \varepsilon_0 \quad \text{for all } t \ge T.$$

Let $\tilde{n}(\cdot,t) = n(\cdot,t+T)$, $\tilde{c}(\cdot,t) = c(\cdot,t+T)$, $\tilde{u}(\cdot,t) = u(\cdot,t+T)$, it is easy to see that $(\tilde{n}_0, \tilde{c}_0, \tilde{u}_0) = (\tilde{n}(\cdot,0), \tilde{c}(\cdot,0), \tilde{u}(\cdot,0))$ fulfills (7.11), we immediately obtain the convergence rate by substituting $(\tilde{n}_0, \tilde{c}_0, \tilde{u}_0)$ into (7.11) and uniqueness of solution for (1.1). □

References

[1] N. Bellomo, A. Belloquid, Y. Tao, M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci. 25 (2015).

[2] X. Cao, S. Ishida, Global-in-time bounded weak solutions to a degenerate quasilinear Keller–Segel system with rotation, Nonlinearity 27 (2014) 1899–1913.

[3] X. Cao, J. Lankeit, Global classical solutions in three-dimensional chemotaxis-Navier–Stokes system involving matrix-valued sensitivity, Calc. Var. Partial Differential Equations 55 (4) (2016) 55–107.

[4] M. Chae, K. Kang, J. Lee, Global existence and temporal decay in Keller–Segel models coupled to fluid equations, Comm. Partial Differential Equations 39 (7) (2014) 1205–1235.

[5] M. Chae, K. Kang, J. Lee, Existence of smooth solutions to coupled chemotaxis-fluid equations, Discrete Contin. Dyn. Syst. 33 (6) (2013) 2271–2297.

[6] R. Duan, A. Lorz, P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, Comm. Partial Differential Equations 35 (9) (2010) 1635–1673.

[7] R. Duan, Z. Xiang, A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion, Int. Math. Res. Not. IMRN (7) (2014) 1833–1852.

[8] Y. Giga, Analyticity of the semigroup generated by the Stokes operator in L_r spaces, Math. Z. 178 (3) (1981) 297–329.

[9] Y. Giga, H. Sohr, Abstract L^p estimates for the cauchy problem with applications to the Navier–Stokes equations in exterior domains, J. Funct. Anal. 102 (1) (1991) 72–94.

[10] T. Hillen, K.J. Painter, A user's guide for PDE models for chemotaxis, J. Math. Biol. 58 (1–2) (2009) 183–217.

[11] D. Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences I, Jahresber. Dtsch. Math.-Ver. 105 (3) (2003) 103–165.

[12] S. Ishida, Global existence and boundedness for chemotaxis-Navier–Stokes systems with position-dependent sensitivity in 2D bounded domains, Discrete Contin. Dyn. Syst. 35 (8) (2015) 3463–3482.

[13] H. Kozono, M. Miura, Y. Sugiyama, Existence and uniqueness theorem on mild solutions to the Keller–Segel system coupled with the Navier–Stokes fluid, J. Funct. Anal. 270 (5) (2016) 1663–1683.

[14] O. Ladyzhenskaya, V. Solonnikov, N. Uratseva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monogr., vol. 23, American Mathematical Society, Providence, RI, 1968.

[15] T. Li, A. Suen, M. Winkler, C. Xue, Global small-data solutions of a two-dimensional chemotaxis system with rotational flux terms, Math. Models Methods Appl. Sci. 25 (4) (2015) 721–746.

[16] G.M. Lieberman, Second order parabolic differential equations, 1996.

[17] J. Liu, A. Lorz, A coupled chemotaxis-fluid model: global existence, Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (5) (2011) 643–652.

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007

[18] A. Lorz, Coupled chemotaxis-fuidmodel, Math. Models Methods Appl. Sci. 20 (6) (2010) 987–1004.
[19] A. Pazy, Semigroups of Linear Operators and Its Applications to Partial Differential Equations, Springer Science Business Media, 2012.
[20] M.M. Porzio, V. Verpri, Hölder estimates for local solutions of some doubly nonlinear degenerate equations, J. Differential Equations 103 (1993) 146–178.
[21] H. Sohr, The Navier–Stokes Equations: An Elementary Functional Analytic Approach, Birkhäuser Verlag, Besel, 2001.
[22] V.A. Solonnnikov, Schauder Estimates for the Evolutionary Generalized Stokes Problem, Amer. Math. Soc. Transl, 2007, pp. 197–205.
[23] Z. Tan, X. Zhang, Decay estimates of the coupled chemotaxis-fluid equations in $\mathbb{R}^3$, J. Math. Anal. Appl. 410 (1) (2014) 27–38.
[24] Y. Tao, Boundedness in a chemotaxis model with oxygen consumption by bacteria, J. Math. Anal. Appl. 381 (2) (2011) 521–529.
[25] Y. Tao, M. Winkler, Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant, J. Differential Equations 252 (2012) 2520–2543.
[26] Y. Tao, M. Winkler, Global existence and boundedness in a Keller–Segel–Stokes model with arbitrary porous medium diffusion, Discrete Contin. Dyn. Syst. 32 (5) (2012) 1901–1914.
[27] Y. Tao, M. Winkler, Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion, Ann. Inst. H. Poincaré Anal. Non Linéaire 30 (1) (2013) 157–178.
[28] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler, R.E. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. USA 102 (2005) 2277–2282.
[29] Y. Wang, X. Cao, Global solutions of a 3D chemotaxis-Stokes system with rotation, Discrete Contin. Dyn. Syst. Ser. B 20 (2015) 3235–3254.
[30] Y. Wang, Z. Xiang, Global existence and boundedness in a Keller–Segel–Stokes system involving a tensor-valued sensitivity with saturation, J. Differential Equations 259 (12) (2015) 7578–7609.
[31] M. Wiegner, The Navier–Stokes equations – a never ending challenge?, Jahresber. Dtsch. Math.-Ver. 101 (1) (1999) 1–25.
[32] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, J. Differential Equations 248 (12) (2010) 2889–2905.
[33] M. Winkler, Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, Comm. Partial Differential Equations 37 (2012) 319–351.
[34] M. Winkler, Stabilization in a two-dimensional chemotaxis-Navier–Stokes system, Arch. Ration. Mech. Anal. 211 (2) (2014) 455–487.
[35] M. Winkler, Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity, Math. Nachr. 283 (11) (2010) 1664–1673.
[36] M. Winkler, Global weak solutions in a three-dimensional chemotaxis-Navier–Stokes system, Ann. Inst. H. Poincaré Anal. Non Linéaire 33 (5) (2016) 1329–1352.
[37] M. Winkler, How far do chemotaxis-driven forces influence regularity in the Navier–Stokes system?, Trans. Amer. Math. Soc. (2016), http://dx.doi.org/10.1090/tran/6733.
[38] M. Winkler, A two-dimensional chemotaxis-Stokes system with rotational flux: global solvability, eventual smoothness and stabilization, preprint, 2014.
[39] M. Winkler, Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities, SIAM J. Math. Anal. 47 (4) (2015) 3092–3115.
[40] M. Winkler, Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity, Calc. Var. Partial Differential Equations 54 (4) (2015) 3789–3828.
[41] C. Xue, H.G. Othmer, Multiscale models of taxis-driven patterning in bacterial populations, SIAM J. Appl. Math. 70 (1) (2009) 133–167.
[42] Q. Zhang, Y. Li, Convergence rates of solutions for a two-dimensional chemotaxis-Navier–Stokes system, Discrete Contin. Dyn. Syst. Ser. B 20 (2015) 2751–2759.

Please cite this article in press as: X. Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.09.007