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Existence theorems for a general 2×2 non-Abelian Chern–Simons–Higgs system over a torus

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Abstract

In this paper we study a general 2×2 non-Abelian Chern–Simons–Higgs system of the form

$$\Delta u_i + \frac{1}{\varepsilon^2}\left(\sum_{j=1}^{2} K_{ji}\mathrm{e}^{u_j} - \sum_{j=1}^{2}\sum_{k=1}^{2} K_{kj}K_{ji}\mathrm{e}^{u_j}\mathrm{e}^{u_k}\right) = 4\pi\sum_{j=1}^{N_i}\delta_{p_{ij}}(x), \quad i=1,2$$

over a flat 2-torus $\mathbb{T}^2$, where $\varepsilon > 0$, δ_p is the Dirac measure at p, $N_i \in \mathbb{N}$ $(i=1,2)$, K is a non-degenerate 2×2 matrix of the form $K = \begin{pmatrix} 1+a & -a \\ -b & 1+b \end{pmatrix}$, which may cover the physically interesting case when K is a Cartan matrix (of a rank 2 semisimple Lie algebra). Concerning the existence results of this type system over $\mathbb{T}^2$, usually in the literature there is a requirement that $a, b > 0$. However, it is an open problem so far for the solvability about such system with $a, b < 0$, which naturally appears in several Chern–Simons–Higgs models with some specific gauge groups. We partially solve this problem by showing that there exists a constant $\varepsilon_0 > 0$ such that this system admits a solution over the torus if $0 < \varepsilon < \varepsilon_0$ provided $|a|, |b|$ are suitably small. Furthermore, if $ab \geq 0$ in addition, with suitable condition on a, b, N_1, N_2, this system

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admits a mountain-pass solution. Our argument is based on a perturbation approach and the mountain-pass lemma.

Keywords: Chern–Simons vortices; Self-dual equations; Topological solutions; Doubly periodic solutions

1. Introduction

Since 1980's the Chern–Simons terms [12,13] have been used in $2+1$ dimensional gauge field models for the characterization of dually charged vortices [16–19,46], which have applications in many branches of modern physics such as high-temperature superconductivity [39,44], integer and fractional quantum Hall effects [23,48,49], and anyon physics [25,56,57]. For the full Chern–Simons–Higgs models usually it is hard to study the equations of motion due to their complicated structures, even for the radially symmetric case, which was just solved not long ago in [9]. Thanks to the seminal works [32,36], self-dual equations [5,47] have been found in various Abelian and non-Abelian Chern–Simons–Higgs models, relativistic or non-relativistic [20–22,35–37,40–42], which lead to considerable progress for understanding of these equations both physically and mathematically. See [33] for a recent review about Chern–Simons models.

For the Abelian Chern–Simons–Higgs model, the self-dual equations found in [32,36] can be formulated into

$$\Delta u + \frac{1}{\varepsilon^2} \mathrm{e}^u (1 - \mathrm{e}^u) = 4\pi \sum_{s=1}^{N} \delta_{p_s}, \tag{1.1}$$

where $\varepsilon > 0$ is a coupling parameter, δ_p denotes the Dirac measure concentrated at point p, and $N \in \mathbb{N}$. Finite-energy condition implies two kinds of admissible boundary conditions on $\mathbb{R}^2$: $u \to 0$ and $u \to -\infty$ at infinity, which are separately called topological and non-topological [59]. The existence of topological solutions for (1.1) was established in [55] by a variational argument, and in [50] via a monotone iteration approach, respectively. For non-topological solutions of (1.1), the first result is due to [51], dealing with the radially symmetric solutions by a shooting argument, which was refined in [11] to tackle more general problems. Concerning the existence of non-radially non-topological solutions, [7] established the first existence result by a perturbation argument, which was extended by [8] and [15] to get more general existence results. Another type physically interesting solutions for (1.1) is called vortex condensates [1] modelling the lattice structure, that is, to construct solutions for (1.1) over a doubly periodic domain (a flat torus), which were first constructed by [6] and later generalized by [52] to get a multiple existence result. More complete existence results concerning (1.1) can be found in the monographs [54,59].

As for the non-Abelian Chern–Simons–Higgs model, the self-dual equations in [20–22] can be reduced into the following nonlinear elliptic system [58,59]

$$\Delta u_i + \frac{1}{\varepsilon^2} \left(\sum_{j=1}^{r} K_{ji} \mathrm{e}^{u_j} - \sum_{j=1}^{r} \sum_{k=1}^{r} K_{kj} K_{ji} \mathrm{e}^{u_j} \mathrm{e}^{u_k} \right) = 4\pi \sum_{j=1}^{N_i} \delta_{p_{ij}}, \quad i = 1, \cdots, r, \tag{1.2}$$

where $K = (K_{ij})$ is the Cartan matrix of a finite-dimensional semisimple Lie algebra L, r the rank of L, $\varepsilon > 0$ the coupling parameter, and $N_i \in \mathbb{N}$ $(i = 1, \dots, r)$. Formally the equation (1.1) can be viewed as a limiting case of (1.2) with the matrix K reducing to the number 1. In view of finite-energy condition, it is interesting to study the system (1.2) in $\mathbb{R}^2$ [20–22,58,59] with the topological boundary conditions

$$u_i \to \ln \sum_{j=1}^{r} (K^{-1})_{ji}, \quad |x| \to \infty, \quad i = 1, \cdots, r$$

or the non-topological conditions

$$u_i \to -\infty, \quad |x| \to \infty, \quad i = 1, \cdots, r.$$

Based on a Cholesky decomposition technique for the positive definite matrices, Yang [58] used a variational approach to establish the first existence result concerning topological solutions for system (1.2) with K being a general matrix such that the Cartan matrices [38] are included.

Due to the well-known constraints' difficulty, as witnessed by [6,52], it is difficult to solve (1.2) when $r \geq 2$ over a doubly periodic domain. In this context the first existence result was established by Nolasco and Tarantello [45] when K is the Cartan matrix of $SU(3)$, i.e.

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

by refining the constrained minimization approach in [6,52]. Subsequently, Han and Tanratello [30] extended the results [45] to a general case when K is a non-degenerate matrix of the form $K = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$ with $\alpha, \beta, \delta, \gamma > 0$ and $\alpha\delta - \beta\gamma > 0$ such that Cartan matrix of a rank 2 semisimple Lie algebra is contained. The case with K being the Cartan matrix of $SU(n+1)$ of (1.2) was solved by [31], where the constraints' difficulty was overcome by an iteration trick with the help of implicit function theorem. The most recent progress on this issue is due to [29], where a new approach based on a degree-theory argument has been developed to resolve the constraints, and the existence results for (1.2) are established when K assumes a very general form such that the Cartan matrices of all semisimple Lie algebras are included.

On the other hand, in the works [29–31,45], for dealing with the solutions over a torus, the common feature shared by the corresponding matrix K is that its off-diagonal entries are non-positive. In fact, this feature is crucial for resolution of the constraints' difficulty involved in these studies. A natural question is whether there are any solutions of system (1.2) with K having non-negative off-diagonal entries, or more general, no sign assumption of signs being imposed on them. This motivates us to study the solvability of (1.2) over a flat 2-torus when K is a general 2×2 non-degenerate matrix with no restrictions on signs of the off-diagonal entries.

For this purpose, we first make a transformation of system (1.2) with a general matrix $K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha, \delta > 0, \alpha\delta - \beta\gamma > 0$. In order to guarantee the existence of topological solution, one needs

$$\sum_{j=1}^{2} (K^{-1})_{ji} > 0, \quad \text{for } i = 1, 2, \quad \text{i.e.} \quad \frac{\delta - \beta}{\alpha\delta - \beta\gamma}, \ \frac{\alpha - \gamma}{\alpha\delta - \beta\gamma} > 0.$$

Define a transformation as follows:

$$(u_1, u_2) \to \left(u_1 + \ln \frac{\delta - \beta}{\alpha\delta - \beta\gamma},\ u_2 + \ln \frac{\alpha - \gamma}{\alpha\delta - \beta\gamma} \right).$$

Then system (1.2) can be changed to

$$\begin{cases} \Delta u_1 = \dfrac{1}{\varepsilon^2} \left\{ a\mathrm{e}^{u_2}(1 - (1+b)\mathrm{e}^{u_2} + b\mathrm{e}^{u_1}) \right. \\ \qquad \left. - (1+a)\mathrm{e}^{u_1}(1 - (1+a)\mathrm{e}^{u_1} + a\mathrm{e}^{u_2}) \right\} + 4\pi \displaystyle\sum_{i=1}^{N_1} \delta_{p_{i1}} \\ \Delta u_2 = \dfrac{1}{\varepsilon^2} \left\{ b\mathrm{e}^{u_1}(1 - (1+a)\mathrm{e}^{u_1} + a\mathrm{e}^{u_2}) \right. \\ \qquad \left. - (1+b)\mathrm{e}^{u_2}(1 - (1+b)\mathrm{e}^{u_2} + b\mathrm{e}^{u_1}) \right\} + 4\pi \displaystyle\sum_{i=1}^{N_2} \delta_{p_{i2}} \end{cases} \quad \text{in} \quad \mathbb{T}^2 \tag{1.3}$$

with $K = \begin{pmatrix} 1+a & -a \\ -b & 1+b \end{pmatrix}$, where $a = \frac{\beta(\gamma-\alpha)}{\alpha\delta-\beta\gamma}$, $b = \frac{\gamma(\beta-\delta)}{\alpha\delta-\beta\gamma}$. In all the cases considered in [29–31,45], one always has $a, b > 0$, which is crucial for the constrained minimization problems involved in these studies. However, some corresponding systems with $a, b < 0$ in (1.3) arise naturally from specific field-theoretical models.

A first example is the following system, which originates in the Gudnason model [26,27] whose gauge groups are $SO(2M)$ and $USp(2M)$:

$$\begin{cases} \Delta U = \dfrac{\alpha}{M^2} \left[\displaystyle\sum_{i=1}^{M} (\mathrm{e}^{U+V_i} + \mathrm{e}^{U-V_i} - 2) \right] \left[\displaystyle\sum_{j=1}^{M} (\mathrm{e}^{U+V_j} + \mathrm{e}^{U-V_j}) \right] \\ \qquad + \dfrac{\alpha\beta}{M} \displaystyle\sum_{i=1}^{M} (\mathrm{e}^{U+V_i} - \mathrm{e}^{U-V_i}) + 4\pi \sum_{i=1}^{M} \sum_{s=1}^{n_i} \delta_{p_{is}} \\ \Delta V_j = \dfrac{\alpha\beta}{M} \left[\displaystyle\sum_{i=1}^{M} (\mathrm{e}^{U+V_i} + \mathrm{e}^{U-V_i} - 2) \right] (\mathrm{e}^{U+V_j} - \mathrm{e}^{U-V_j}) \\ \qquad + \beta^2 (\mathrm{e}^{2U+2V_j} - \mathrm{e}^{2U-2V_j}) + 4\pi \displaystyle\sum_{s=1}^{n_j} \delta_{p_{js}}, \quad j = 1, \cdots, M \end{cases} \quad \text{in} \quad \mathbb{T}^2, \tag{1.4}$$

where $\alpha, \beta > 0$ are coupling constants. The derivation and more physical motivation of this system can be found in [26,27]. For $M = 1$, by setting $u_1 = U + V_1$, $u_2 = U - V_1$, we see that system (1.4) can be reduced to a specific case system (1.3) with $a = b = \frac{\beta-\alpha}{2\alpha}$, $\varepsilon = \frac{1}{2\alpha}$. We refer readers to [28] for the details of the transformation. Therefore when $\beta < \alpha$ it is necessary to study system (1.3) with $a, b < 0$.

Another typical example comes from a non-Abelian Chern–Simons–Higgs model with gauge group $SU(N) \times U(1)$ and flavor $SU(N)$:

$$\begin{cases} \Delta u_1 = \dfrac{1}{\varepsilon^2}\left\{-\dfrac{N-1+\kappa}{N}\left(\mathrm{e}^{u_1} - \dfrac{N-1+\kappa}{N}\mathrm{e}^{2u_1} + \dfrac{\kappa-1}{N}\mathrm{e}^{u_1+u_2}\right)\right. \\ \qquad \left. + \dfrac{\kappa-1}{N}\left(\mathrm{e}^{u_2} - \dfrac{1+(N-1)\kappa}{N}\mathrm{e}^{2u_2} + \dfrac{(N-1)(\kappa-1)}{N}\mathrm{e}^{u_1+u_2}\right)\right\} + 4\pi\displaystyle\sum_{i=1}^{N_1}\delta_{p_{i1}} \\ \Delta u_2 = \dfrac{1}{\varepsilon^2}\left\{\dfrac{(N-1)(\kappa-1)}{N}\left(\mathrm{e}^{u_1} - \dfrac{N-1+\kappa}{N}\mathrm{e}^{2u_1} + \dfrac{\kappa-1}{N}\mathrm{e}^{u_1+u_2}\right)\right. \\ \qquad \left. - \dfrac{1+(N-1)\kappa}{N}\left(\mathrm{e}^{u_2} - \dfrac{1+(N-1)\kappa}{N}\mathrm{e}^{2u_2} + \dfrac{(N-1)(\kappa-1)}{N}\mathrm{e}^{u_1+u_2}\right)\right\} \\ \qquad + 4\pi\displaystyle\sum_{i=1}^{N_2}\delta_{p_{i2}} \end{cases} \tag{1.5}$$

in $\mathbb{T}^2$, where $N \geq 2$, κ is a coupling constant. System (1.5) was first derived by Lozano, Marqués, Moreno and Schaposnik in [43]. In fact, we can see that system (1.5) is a special case of system (1.3) with $a = \frac{\kappa-1}{N}$, $b = \frac{(N-1)(\kappa-1)}{N}$. Then as $\kappa < 1$ we need to consider system (1.3) with $a, b < 0$.

Due to the difficulty of the constraints, system (1.4) and system (1.5) are solved by [28] and [10] respectively for the case $a, b > 0$ (i.e. $\beta > \alpha$ in (1.4) and $\kappa > 1$ in (1.5)) in $\mathbb{T}^2$. The existence of solutions over $\mathbb{T}^2$ for system (1.4) and (1.5) remains open for $a, b < 0$ (i.e. $\beta < \alpha$ in (1.4) and $\kappa < 1$ in (1.5)). This unsolved case for the above two systems concerning the specific models provides us more physical motivation to study the solvability of the general system (1.3) without the requirement $a, b > 0$. We will establish the existence results for system (1.3) over $\mathbb{T}^2$ when $|a|, |b|$ are small enough, which partially solves this open problem. Specifically, we have the following existence theorem on solutions of (1.3) over a flat 2-torus $\mathbb{T}^2$.

Theorem 1.1. *Consider system (1.3) over a flat 2-torus $\mathbb{T}^2$. Let $p_{11}, \cdots, p_{1N_1}, p_{21}, \cdots, p_{2N_2} \in \mathbb{T}^2$ be given. Then, there exist two constants, $\delta_0 = \delta_0(N_1, N_2)$, $\varepsilon_0 = \varepsilon_0(p_{ij}, a, b)$ such that for $|a|, |b| < \delta_0$, $0 < \varepsilon < \varepsilon_0$, system (1.3) admits a solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$ which has the following property:*

$$\varepsilon^k |u_{i,\varepsilon(x)}| + \varepsilon^k |\nabla u_{i,\varepsilon}(x)| \to 0, \quad i = 1, 2, \quad as \quad \varepsilon \to 0, \tag{1.6}$$

locally uniformly in $\mathbb{T}^2 \setminus \cup_{ij} p_{ij}$, for any fixed $k \in \mathbb{N}^+ \cup \{0\}$.

Moreover, system (1.3) admits at least two solutions, provided one of the following case happens in addition:

(1) $a, b > 0$;
(2) $a, b < 0$, $\frac{1}{\lambda} \leq \frac{a}{b} \leq \lambda$, for some constant $\lambda > 1$, $|a|, |b| \leq \delta_0(N_1, N_2, \lambda)$ and $\left[(1+b)N_1 + aN_2\right]\left[(1+a)N_2 + bN_1\right] \neq 0$;
(3) $a = 0$, $N_2 \neq 0$ or $b = 0$, $N_1 \neq 0$.

On the other hand, for the solutions obtained, there hold the following quantized integrals

$$\frac{1}{\varepsilon^2}\int\limits_{\mathbb{T}^2}\left\{(1+a)\mathrm{e}^{u_1}(1-(1+a)\mathrm{e}^{u_1}+a\mathrm{e}^{u_2})-a\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2}+b\mathrm{e}^{u_1})\right\}\mathrm{d}x=4\pi N_1,$$

$$\frac{1}{\varepsilon^2}\int\limits_{\mathbb{T}^2}\left\{(1+b)\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2}+b\mathrm{e}^{u_1})-b\mathrm{e}^{u_1}(1-(1+a)\mathrm{e}^{u_1}+a\mathrm{e}^{u_2})\right\}\mathrm{d}x=4\pi N_2.$$

Remark 1.1. By our theorem, in particular, we obtain the existence of solutions of system (1.4) and (1.5) over a flat 2-torus $\mathbb{T}^2$ for (corresponding) $a, b < 0$ and $|a|$, $|b|$ are small enough.

Since a, b are not necessarily positive, the constrained minimization for dealing with the solution of (1.2) over $\mathbb{T}^2$ does not apply for (1.3). We need to seek other ideas to tackle this problem. Observe that for $a, b = 0$ system (1.3) decouples into two Abelian Chern–Simons–Higgs equations of the form (1.1), which are well understood as we mentioned previously. This sheds some light on the possibility to solve (1.3) when $|a|$, $|b|$ is small. In fact, to get the existence a first solution, we use a perturbation approach which is inspired by the work of [14] and a generalization of [34].

To this end, we need to investigate the existence of topological solutions of the system

$$\begin{cases}\Delta u_1=a\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2}+b\mathrm{e}^{u_1})\\ \qquad\quad -(1+a)\mathrm{e}^{u_1}(1-(1+a)e^{u_1}+a\mathrm{e}^{u_2})+4\pi N_1\delta_0\\ \Delta u_2=b\mathrm{e}^{u_1}(1-(1+a)\mathrm{e}^{u_1}+a\mathrm{e}^{u_2})\\ \qquad\quad -(1+b)\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2}+b\mathrm{e}^{u_1})+4\pi N_2\delta_0\end{cases}\quad\text{in}\quad\mathbb{R}^2.\tag{1.7}$$

Formally, system (1.7) can be seen as a perturbation of the following decoupled system of two Abelian Chern–Simons–Higgs equations

$$\begin{cases}\Delta u_1+\mathrm{e}^{u_1}(1-\mathrm{e}^{u_1})=4\pi N_1\delta_0\\ \Delta u_2+\mathrm{e}^{u_2}(1-\mathrm{e}^{u_2})=4\pi N_2\delta_0\end{cases}\quad\text{in}\quad\mathbb{R}^2,\tag{1.8}$$

which corresponds to (1.7) with $a = b = 0$. For simplicity, throughout this paper we use the following notation

$$f(u_1,u_2)\triangleq\mathrm{e}^{u_1}(1-(1+a)\mathrm{e}^{u_1}+a\mathrm{e}^{u_2}),\quad g(u_1,u_2)\triangleq\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2}+b\mathrm{e}^{u_1}).\tag{1.9}$$

Then the linearized operator $L(u_1, u_2) \triangleq (L_1(u_1, u_2),\ L_2(u_1, u_2))$ of (1.7) at a solution (u_1, u_2) is as follows:

$$\begin{cases}L_1(u_1,u_2)(h_1,h_2)\triangleq\Delta h_1+((1+a)f_{u_1}-ag_{u_1})h_1+((1+a)f_{u_2}-ag_{u_2})h_2,\\ L_2(u_1,u_2)(h_1,h_2)\triangleq\Delta h_2+((1+b)g_{u_2}-bf_{u_2})h_2+((1+b)g_{u_1}-bf_{u_1})h_1.\end{cases}\tag{1.10}$$

Thanks to the uniqueness and strictly stable property of the topological solution of (1.8) which is proved in [14], we have the following result concerning the existence of topological solutions of (1.7) and the invertibility of the corresponding linearized operator, which is important for the proof of Theorem 1.1.

Theorem 1.2. *For any $N_1, N_2 \geq 0$, system (1.7) admits a topological solution (u_1, u_2), whose linearized operator defined by (1.10) is a one to one map from $H^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$, and there is a constant $c_0 = c_0(N_1, N_2) > 0$ such that*

$$\|L(u_1, u_2)(h_1, h_2)\|_{L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} \geq c_0\|(h_1, h_2)\|_{H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)}$$

provided $|a|, |b| < \delta_0(N_1, N_2)$ for some constant $\delta_0(N_1, N_2) > 0$. Moreover, this topological solution satisfies the following properties:

(1). There exists $R_\varepsilon(N_1, N_2) > 0$ such that $|u_i(x)| + |\nabla u_i(x)| \leq C_1 \mathrm{e}^{-\frac{1}{2}(1-\varepsilon)|x|}$, $i = 1, 2$ for any $|x| > R_\varepsilon(N_1, N_2)$;

(2). The following estimate holds:

$$\|u_1 - \phi_1\|_{H^2(\mathbb{R}^2)} + \|u_2 - \phi_2\|_{H^2(\mathbb{R}^2)} \leq C_2(\sqrt{|a|} + \sqrt{|b|}), \tag{1.11}$$

where (ϕ_1, ϕ_2) is the unique topological solution of (1.8);

(3). If a, b is non-negative, this topological solution is the unique topological solution of (1.7) and it is radial symmetric;

(4). For the solutions obtained, there hold the following quantized integrals:

$$\int_{\mathbb{R}^2} \Big\{(1+a)\mathrm{e}^{u_1}(1-(1+a)\mathrm{e}^{u_1} + a\mathrm{e}^{u_2}) - a\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2} + b\mathrm{e}^{u_1})\Big\}\mathrm{d}x = 4\pi N_1,$$

$$\int_{\mathbb{R}^2} \Big\{(1+b)\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2} + b\mathrm{e}^{u_1}) - b\mathrm{e}^{u_1}(1-(1+a)\mathrm{e}^{u_1} + a\mathrm{e}^{u_2})\Big\}\mathrm{d}x = 4\pi N_2.$$

The rest of our paper is organized as follows. In Section 2, we prove the existence part of Theorem 1.2 via a perturbation argument and the uniqueness part by a Pohozaev's type identity. Section 3 is devoted to the proof of Theorem 1.1. We establish the existence of a first solution of (1.3) by a contradiction argument and contraction mapping principle which borrows the idea from [14]. And the existence of a second solution of (1.3) is proved with the help of mountain-pass lemma [2].

2. Topological solution in $\mathbb{R}^2$

This section is mainly devoted to proving Theorem 1.2. Our proof is based on a perturbation argument and Pohozaev's identity.

2.1. Preliminaries

In this subsection we present some preliminaries for our later use. To this end, we need to analyze system (1.7) with $|a|, |b|$ being small constants. Let (ϕ_1, ϕ_2) be the unique topological solution of the decoupled system (1.8).

Before starting our proof, we first recall some facts without proof.
Consider the following equation,

$$\Delta u + e^u(1 - e^u) = 4\pi N \delta_0, \quad \text{in} \quad \mathbb{R}^2. \tag{2.1}$$

If we consider the radial symmetric solutions of (2.1), then

$$(ru')' + re^u(1 - e^u) = 0,\ r > 0; \quad u(r) = 2N \ln r + s_0,\ r \to 0. \tag{2.2}$$

(I). For $N = 0$, after a translation of coordinate, both topological solution and non-topological solutions are radial symmetric and unique. Set $\beta(s_0) \triangleq \int_0^\infty re^u(1 - e^u)dr$. Then $\beta(s_0)$ is strictly increasing on $(-\infty, 0)$ and

$$\lim_{s_0 \to -\infty} \beta(s_0) = 4; \quad \lim_{s_0 \to 0} \beta(s_0) = +\infty. \tag{2.3}$$

We refer the readers to [8] for the details of the proof.

(II). For $N \geq 0$, all the topological solutions of (2.1) are radial symmetric. Its linearized operator $\mathcal{L} : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$:

$$\mathcal{L}(h) \triangleq \Delta h + e^u(1 - 2e^u)h$$

is a one to one map and there is a constant $c(N) > 0$ such that

$$\|\mathcal{L}h\|_{L^2(\mathbb{R}^2)} \geq c(N)\|h\|_{H^2(\mathbb{R}^2)} \tag{2.4}$$

for all $h \in H^2(\mathbb{R}^2)$. The details of the proof can be found in [14].

We also recall the following form of Harnack type inequality from [4].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and u satisfy:*

$$-\Delta u = f \quad in \quad \Omega,$$

with $f \in L^p(\Omega)$, $p > 1$. For any subdomain $\Omega' \subset\subset \Omega$, there exist two positive constants $\tau \in (0, 1)$ and $M > 0$, depending only on Ω', such that

(1). if $\sup_{\partial\Omega} u \leq C$, then $\sup_{\Omega'} u \leq \tau \inf_{\Omega'} u + M\|f\|_{L^p(\Omega)} + (1 - \tau)C$;
(2). if $\inf_{\partial\Omega} u \geq -C$, then $\tau \sup_{\Omega'} u \leq \inf_{\Omega'} u + M\|f\|_{L^p(\Omega)} + (1 - \tau)C$.

We first derive a decay estimate of the topological solutions for system (1.7) which will be used later. Although the proof is standard as in [58], we present it here for completeness.

Lemma 2.2. *Suppose (u_1, u_2) is a topological solution of system (1.7). Then there exists $R_\varepsilon > 0$ such that*

$$|u_i(x)| + |\nabla u_i(x)| \leq C_\varepsilon e^{-\frac{1}{2}(1-\varepsilon)|x|},\ i = 1, 2, \quad for\ |x| > R_\varepsilon, \tag{2.5}$$

provided that $|a|, |b| \leq \delta_0$ for some $\delta_0 > 0$ small enough.

Proof. Denote

$$K=\begin{pmatrix}1+a & -a\\ -b & 1+b\end{pmatrix},\ \mathbf{u}=\begin{pmatrix}u_1\\ u_2\end{pmatrix},\ U=\begin{pmatrix}\mathrm{e}^{u_1} & 0\\ 0 & \mathrm{e}^{u_2}\end{pmatrix},\ \mathbf{U}=\begin{pmatrix}\mathrm{e}^{u_1}\\ \mathrm{e}^{u_2}\end{pmatrix},\ \mathbf{1}=\begin{pmatrix}1\\ 1\end{pmatrix}.$$

Then system (1.7) in $\mathbb{R}^2\backslash B_R(0)$ is equivalent to

$$\Delta\mathbf{u}=KUK(\mathbf{U}-\mathbf{1})=KUKU_\xi\mathbf{u} \tag{2.6}$$

where $U_\xi=\begin{pmatrix}\mathrm{e}^{\xi_1} & 0\\ 0 & \mathrm{e}^{\xi_2}\end{pmatrix}$ for some ξ_i between 0 and u_i, $i=1,2$. Since K can be regarded as a perturbation of the identity matrix I, we can choose $|a|,|b|$ small enough such that $\frac{1}{2}\big(K^2+(K^T)^2\big)\geq\frac{1}{2}I$. From the theory of linear algebra, we can find an orthogonal matrix O such that $\frac{1}{2}O\big(K^2+(K^T)^2\big)O^T=\mathrm{diag}\big(\lambda_1,\lambda_2\big)=\Lambda$, $\lambda_1,\lambda_2\geq\frac{1}{2}$. Set $\mathbf{w}=O\mathbf{u}$. Then we have

$$\begin{aligned}\Delta\mathbf{w}^2&=\mathbf{w}^TO(K^2+(K^T)^2)O^T\mathbf{w}+\mathbf{w}^TO(K^2-(K^T)^2)O^T\mathbf{w}+\mathbf{w}^TO(KUKU_\xi-K^2)O^{-1}\mathbf{w}\\ &\geq 2\min(\lambda_1,\lambda_2)\mathbf{w}^2-o(1)\mathbf{w}^2\geq(1-\varepsilon^2)\mathbf{w}^2.\end{aligned} \tag{2.7}$$

Since $u_1,u_2\to 0$ as $x\to\infty$, we can choose $R_\varepsilon>0$ large enough to guarantee the last inequality in (2.7). Now by a standard barrier function argument, we can deduce from (2.7) that

$$|\mathbf{w}|^2\leq C_\varepsilon e^{-(1-\varepsilon)|x|},$$

which proves the present lemma. □

With the help of the decay estimate as above, we can derive a Pohozaev's identity for system (1.7).

Lemma 2.3. *Suppose* (u_1,u_2) *is a topological solution of system* (1.7) *with* $1+a+b\neq 0$. *Then there hold the following identities:*

$$\begin{aligned}&\int\limits_{\mathbb{R}^2}\mathrm{e}^{u_1}(1-(1+a)\mathrm{e}^{u_1}+a\mathrm{e}^{u_2})=\frac{4\pi((1+b)N_1+aN_2)}{1+a+b},\\ &\int\limits_{\mathbb{R}^2}\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2}+b\mathrm{e}^{u_1})=\frac{4\pi((1+a)N_2+bN_1)}{1+a+b},\\ &b\int\limits_{\mathbb{R}^2}(1-\mathrm{e}^{u_1})+a\int\limits_{\mathbb{R}^2}(1-\mathrm{e}^{u_2})=\frac{4\pi(1+b)b}{1+a+b}N_1^2+\frac{4\pi(1+a)a}{1+a+b}N_2^2\\ &-\frac{8\pi ab}{1+a+b}N_1N_2+\frac{4\pi b((1+b)N_1+aN_2)}{1+a+b}+\frac{4\pi a((1+a)N_2+bN_1)}{1+a+b}.\end{aligned} \tag{2.8}$$

Proof. In fact, by a straightforward computation, we can change system (1.7) to

$$\begin{cases}\Delta((1+b)u_1+au_2)+(1+a+b)\mathrm{e}^{u_1}(1-(1+a)\mathrm{e}^{u_1}+a\mathrm{e}^{u_2})=0\\ \Delta((1+a)u_2+bu_1)+(1+a+b)\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2}+b\mathrm{e}^{u_1})=0\end{cases} \quad \text{in} \quad \mathbb{R}^2\backslash\{0\}. \tag{2.9}$$

A direct integration of (2.9) on $\mathbb{R}^2$ will yield that

$$\begin{aligned}&\int_{\mathbb{R}^2}\mathrm{e}^{u_1}(1-(1+a)\mathrm{e}^{u_1}+a\mathrm{e}^{u_2})=\frac{4\pi((1+b)N_1+aN_2)}{1+a+b},\\ &\int_{\mathbb{R}^2}\mathrm{e}^{u_2}(1-(1+b)\mathrm{e}^{u_2}+b\mathrm{e}^{u_1})=\frac{4\pi((1+a)N_2+bN_1)}{1+a+b}.\end{aligned} \tag{2.10}$$

Multiplying the first equation of (2.9) with $x\cdot\nabla u_1$ and integrating on $\mathbb{R}^2\backslash B_\delta(0)$, one gets

$$\begin{aligned}&-\int_{\mathbb{R}^2\backslash B_\delta(0)} x\cdot\nabla u_1\big[(1+b)\Delta u_1+a\Delta u_2\big]\\ &=(1+a+b)\int_{\mathbb{R}^2\backslash B_\delta(0)}\left[x\cdot\nabla(\mathrm{e}^{u_1}-1)-\frac{1+a}{2}x\cdot\nabla(\mathrm{e}^{2u_1}-1)+a\mathrm{e}^{u_2}x\cdot\nabla\mathrm{e}^{u_1}\right].\end{aligned} \tag{2.11}$$

Now we estimate the left-hand side term of (2.11). Integrating by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^2\backslash B_\delta(0)} x\cdot\nabla u_1\Delta u_1\\ &=\int_{\mathbb{R}^2\backslash B_\delta(0)}\partial_i(x_j\partial_j u_1\partial_i u_1)-\int_{\mathbb{R}^2\backslash B_\delta(0)}|\nabla u_1|^2-\frac{1}{2}\int_{\mathbb{R}^2\backslash B_\delta(0)}x\cdot\nabla|\nabla u_1|^2\\ &=-\int_{\partial B_\delta(0)}(\nabla u_1\cdot\nu)(x\cdot\nabla u_1)+\frac{1}{2}\int_{\partial B_\delta(0)}(x\cdot\nu)|\nabla u_1|^2\\ &=-4\pi N_1^2+O(\delta).\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we get

$$\begin{aligned}&2\int_{\mathbb{R}^2}(1-\mathrm{e}^{u_1})-(1+a)\int_{\mathbb{R}^2}(1-\mathrm{e}^{2u_1})+a\int_{\mathbb{R}^2\backslash B_\delta(0)}\mathrm{e}^{u_2}x\cdot\nabla\mathrm{e}^{u_1}\\ &=\frac{4\pi(1+b)}{1+a+b}N_1^2-\frac{a}{1+a+b}\int_{\mathbb{R}^2\backslash B_\delta(0)}(x\cdot\nabla u_1)\cdot\Delta u_2+O(\delta).\end{aligned} \tag{2.13}$$

A similar argument also yields that

$$\begin{aligned}&2\int_{\mathbb{R}^2}(1-e^{u_2})-(1+b)\int_{\mathbb{R}^2}(1-e^{2u_2})+b\int_{\mathbb{R}^2\setminus B_\delta(0)}e^{u_1}x\cdot\nabla e^{u_2}\\&=\frac{4\pi(1+a)}{1+a+b}N_2^2-\frac{b}{1+a+b}\int_{\mathbb{R}^2\setminus B_\delta(0)}(x\cdot\nabla u_2)\cdot\Delta u_1+O(\delta).\end{aligned}\tag{2.14}$$

Combining (2.13) and (2.14), we have

$$\begin{aligned}&2b\int_{\mathbb{R}^2}(1-e^{u_1})+2a\int_{\mathbb{R}^2}(1-e^{u_2})-b(1+a)\int_{\mathbb{R}^2}(1-e^{2u_1})-(1+b)a\int_{\mathbb{R}^2}(1-e^{2u_2})\\&+2ab\int_{\mathbb{R}^2}(1-e^{u_1+u_2})=\frac{4\pi(1+b)b}{1+a+b}N_1^2-\frac{8\pi ab}{1+a+b}N_1N_2+\frac{4\pi(1+a)a}{1+a+b}N_2^2.\end{aligned}\tag{2.15}$$

By (2.15) and (2.10), we get the last equality of (2.8). □

Lemma 2.4. *Suppose (u_1,u_2) is a topological solution of (1.7) with $a,b\ge 0$. Then we have $u_1,u_2\le 0$ in $\mathbb{R}^2$.*

Proof. Suppose not, without loss of generality, we may assume $0<\max_{\mathbb{R}^2}u_1=u_1(x_0)\ge\max_{\mathbb{R}^2}u_2$. At x_0, one has

$$\begin{aligned}0\le&-\Delta u_1(x_0)=(1+a)e^{u_1}(1-(1+a)e^{u_1}+ae^{u_2})(x_0)-ae^{u_2}(1-(1+b)e^{u_2}+be^{u_1})(x_0)\\&\le(1+a)e^{u_1}(1-e^{u_1})(x_0)-ae^{u_2}(1-e^{u_2})(x_0)<a(e^{u_1}-e^{u_2})(1-e^{u_2}-e^{u_1})(x_0)\le 0.\end{aligned}$$

This yields a contradiction and proves the present lemma. □

Lemma 2.5. *Suppose $(u_{1,n},u_{2,n})$ is a sequence of topological solutions of system (1.7) with $(a,b)=(a_n,b_n)$ and $a_n,b_n\ge 0$. Then by taking a subsequence, one can get $(u_{1,n},u_{2,n})$ converges to $(\bar u_1,\bar u_2)$ in $C^2_{loc}(\mathbb{R}^2\setminus\{0\})$ such that $(\bar u_1,\bar u_2)$ is a topological solution of system (1.7) with $(a,b)=\left(\lim_{n\to\infty}a_n,\lim_{n\to\infty}b_n\right)=(\bar a,\bar b)$.*

Proof. We distinguish into two cases:

1. $\bar a,\bar b>0$. In this case, by Lemma 2.4, we see that

$$\int_{\mathbb{R}^2}(1-e^{u_{1,n}})+\int_{\mathbb{R}^2}(1-e^{u_{2,n}})\le C<+\infty\tag{2.16}$$

uniformly in n. By Lemma 2.4, $u_{1,n},u_{2,n}\le 0$. This allows us to apply the Harnack inequality of Lemma 2.1 to show that $u_{1,n}$ either goes $-\infty$ locally uniformly in $\mathbb{R}^2\setminus\{0\}$ or remains locally uniformly bounded in $\mathbb{R}^2\setminus\{0\}$. Due to (2.16), we can conclude that $u_{1,n}$ remains locally uniformly bounded in $\mathbb{R}^2\backslash\{0\}$. Hence, by a subsequence, we may assume $u_{1,n}, u_{2,n}$ converge to $\bar{u}_1, \bar{u}_2 \le 0$ in $C^2_{loc}(\mathbb{R}^2\backslash\{0\})$, respectively. By Fatou's Lemma and (2.16), we also have

$$\int_{\mathbb{R}^2}(1-\mathrm{e}^{\bar{u}_1}) + \int_{\mathbb{R}^2}(1-\mathrm{e}^{\bar{u}_2}) \le C < +\infty, \tag{2.17}$$

which implies $(\bar{u}_1, \bar{u}_2)$ is a topological solution. Since if not, we may find $x_n \to \infty$ such that $\bar{u}_1(x_n) + \bar{u}_2(x_n) \le -\gamma_0 < 0$ for some constant $\gamma_0 > 0$. Then by intermediate value theorem, we can find $y_n \to \infty$ such that $\bar{u}_1(y_n) + \bar{u}_2(y_n) = -\gamma_0$. By the equation for $\bar{u}_1 + \bar{u}_2$ and Harnack inequality, there exists $M_0 > 0$ such that $\bar{u}_1 + \bar{u}_2 \le -\frac{\gamma_0}{2}$ in $B_{M_0}(y_n)$ which contradicts (2.17).

2. $\bar{a}\bar{b} = 0$. Without loss of generality, we may assume $0 < a_n, b_n \to 0$ as $n \to \infty$, otherwise it is the trivial case. By (2.8) in Lemma 2.3, we have

$$\int_{\mathbb{R}^2}(1-\mathrm{e}^{u_{2,n}}) \le C < \infty. \tag{2.18}$$

As in Case 1, we can apply Harnack inequality in Lemma 2.1 to $u_{1,n}$ to get $u_{1,n}$ either goes to $-\infty$ or remains bounded locally uniformly in $\mathbb{R}^2\backslash\{0\}$. In fact, we have the following three different situations:

(2.a) $(u_{1,n}, u_{2,n})$ converges to $(\bar{u}_1, \bar{u}_2)$ with $\int_{\mathbb{R}^2}(1-\mathrm{e}^{\bar{u}_1}) + \int_{\mathbb{R}^2}(1-\mathrm{e}^{\bar{u}_2}) \le C < +\infty$. This is just the Case 1 and proves the present lemma.

(2.b) $(u_{1,n}, u_{2,n})$ converges to $(\bar{u}_1, \bar{u}_2)$ with $\int_{\mathbb{R}^2}(1-\mathrm{e}^{\bar{u}_1}) = +\infty$, $\int_{\mathbb{R}^2}(1-\mathrm{e}^{\bar{u}_2}) \le C < +\infty$. In fact, this implies $(\bar{u}_1, \bar{u}_2)$ is not a topological solution of (1.7). In other words, there exists $\gamma_0 < 0$ and $x_n \to \infty$ such that $\bar{u}_1(x_n) \le \gamma_0$.

(2.c) $u_{1,n}$ goes to $-\infty$ locally uniformly in $\mathbb{R}^2\backslash\{0\}$.

We need to show that Case (2.b) and (2.c) will never happen. In fact, if Case (2.b) or (2.c) happens, by $u_{1,n}(x) \to 0$ as $|x| \to +\infty$ and intermediate value theorem, we can find $y_n \to \infty$ such that $u_{1,n}(y_n) = s_0 < 0$ and $\beta(s_0) >> 1$ with $\beta(s)$ as in (2.3). Set

$$\tilde{u}_{1,n}(x) \triangleq u_{1,n}(x+y_n),\ \tilde{u}_{2,n}(x) \triangleq u_{2,n}(x+y_n). \tag{2.19}$$

Then by the Harnack inequality of Lemma 2.1, we know that $\tilde{u}_{1,n}$ is locally uniformly bounded and $\tilde{u}_{2,n}$ either goes to $-\infty$ or remains uniformly bounded locally in $\mathbb{R}^2$. From (2.18), we know $\tilde{u}_{2,n}$ is locally uniformly bounded. By standard elliptic estimates, we may assume $\tilde{u}_{1,n}, \tilde{u}_{2,n}$ converge to $\tilde{u}_1, \tilde{u}_2$ in $C^2_{loc}(\mathbb{R}^2)$ which solve the following system

$$\begin{cases} \Delta\tilde{u}_1 + (1+\bar{a})\mathrm{e}^{\tilde{u}_1}(1-(1+\bar{a})\mathrm{e}^{\tilde{u}_1} + \bar{a}\mathrm{e}^{\tilde{u}_2}) - \bar{a}\mathrm{e}^{\tilde{u}_2}(1-\mathrm{e}^{\tilde{u}_2}) = 0, \\ \Delta\tilde{u}_2 + \mathrm{e}^{\tilde{u}_2}(1-\mathrm{e}^{\tilde{u}_2}) = 0, \quad \text{in} \quad \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2}(1-\mathrm{e}^{\tilde{u}_2}) \le C < \infty, \quad \tilde{u}_1(0) = s_0. \end{cases} \tag{2.20}$$

By the first and third equality in (2.8), one can get $(1+\bar{a})^2\int_{\mathbb{R}^2}\mathrm{e}^{u_{1,n}}(1-\mathrm{e}^{u_{1,n}}) \le C_0 < \infty$ uniformly. Since $\tilde{u}_2$ is a topological solution, we know $\tilde{u}_2 \equiv 0$. In fact, (2.20) is reduced to (2.1) with $N = 0$. By our choice of s_0, we get a contradiction. This completes the proof of the present lemma. □

Remark 2.1. In fact, Lemma 2.5 is also true for system (1.7) with more general distribution of vortices.

2.2. The proof of Theorem 1.2

In this subsection we carry out the proof of Theorem 1.2. For any $z_1, z_2 \in H^2(\mathbb{R}^2)$, we first introduce the following operator:

$$\begin{cases} F_1(z_1,z_2) \triangleq \Delta z_1 + \dfrac{1}{\kappa}(e^{\kappa z_1+\phi_1} - e^{\phi_1})(1 - e^{\kappa z_1+\phi_1} - e^{\phi_1}) + \dfrac{a}{\kappa}e^{\kappa z_1+\phi_1}(1-(2+a)e^{\kappa z_1+\phi_1}) \\ \qquad\qquad - \dfrac{a}{\kappa}e^{\kappa z_2+\phi_2}(1-(1+b)e^{\kappa z_2+\phi_2} - (1+a-b)e^{\kappa z_1+\phi_1}), \\ F_2(z_1,z_2) \triangleq \Delta z_2 + \dfrac{1}{\kappa}(e^{\kappa z_2+\phi_2} - e^{\phi_2})(1 - e^{\kappa z_2+\phi_2} - e^{\phi_2}) + \dfrac{b}{\kappa}e^{\kappa z_2+\phi_2}(1-(2+b)e^{\kappa z_2+\phi_2}) \\ \qquad\qquad - \dfrac{b}{\kappa}e^{\kappa z_1+\phi_1}(1-(1+a)e^{\kappa z_1+\phi_1} - (1+b-a)e^{\kappa z_2+\phi_2}). \end{cases}$$

By a direct computation, we can see that $u_1 = \kappa z_1 + \phi_1, u_2 = \kappa z_2 + \phi_2$ is a solution of (1.7) if and only if $F(z_1, z_2) \triangleq (F_1(z_1, z_2), F_2(z_1, z_2)) = 0$. Here (ϕ_1, ϕ_2) is the unique topological solution of system (1.8).

Lemma 2.6. *Let $F(z_1, z_2) : H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ be defined as above and $\kappa^2 \triangleq \max(|a|, |b|)$. Then there exists $\delta_0 \triangleq \delta_0(N_1, N_2) > 0$ such that*

(1). $\|F(0,0)\|_{L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} \le c_1(N_1, N_2)\kappa$;
(2). For any $\psi_1, \psi_2 \in H^2(\mathbb{R}^2)$, we have

$$\|DF(0,0)(\psi_1,\psi_2)\|_{L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} \ge c_2(N_1,N_2)\big(\|\psi_1\|_{H^2(\mathbb{R}^2)} + \|\psi_2\|_{H^2(\mathbb{R}^2)}\big);$$

(3). For any $z_1, z_2 \in H^2(\mathbb{R}^2)$ with $\|z_i\|_{H^2(\mathbb{R}^2)} \le 1$, we have

$$\begin{aligned} &\|(DF(z_1,z_2) - DF(0,0))(\psi_1,\psi_2)\|_{L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} \\ &\quad \le c_3(N_1,N_2)\kappa\big(\|\psi_1\|_{H^2(\mathbb{R}^2)} + \|\psi_2\|_{H^2(\mathbb{R}^2)}\big) \end{aligned}$$

provided $|a|, |b| \le \delta_0$.

Proof. First we prove (1). By the definition of $F(z_1, z_2)$, we get

$$\begin{aligned} \|F_1(0,0)\|_{L^2(\mathbb{R}^2)} &= \left\| \frac{a}{\kappa}e^{\phi_1}(1-(2+a)e^{\phi_1}) - \frac{a}{\kappa}e^{\phi_2}(1-(1+b)e^{\phi_2} - (1+a-b)e^{\phi_1}) \right\|_{L^2(\mathbb{R}^2)} \\ &\le C\kappa\Big(\|1-e^{\phi_1}\|_{L^2(\mathbb{R}^2)} + \|1-e^{\phi_2}\|_{L^2(\mathbb{R}^2)}\Big) \le c_1\kappa. \end{aligned}$$

Here we have used (ϕ_1, ϕ_2) is the unique topological solution of (1.8). The proof for $F_2(0, 0)$ is the same.

Now we turn to prove (2). By a direct computation, one gets

$$
\begin{aligned}
& DF_1(0,0)(h_1,h_2) \\
&= \Delta h_1 + \mathrm{e}^{\phi_1}(1-2\mathrm{e}^{\phi_1})h_1 + a\mathrm{e}^{\phi_1}(1-2(2+a)\mathrm{e}^{\phi_1})h_1 \\
&- a\mathrm{e}^{\phi_2}(1-2(1+b)\mathrm{e}^{\phi_2} - (1+a-b)\mathrm{e}^{\phi_1})h_2 + a(1+a-b)\mathrm{e}^{\phi_1+\phi_2}h_1.
\end{aligned}
$$

By a rearrangement, we get that h_1 satisfies

$$
\Delta h_1 + \mathrm{e}^{\phi_1}(1-2\mathrm{e}^{\phi_1})h_1 = aI_1h_1 + aI_2h_2 + DF_1(0,0)(h_1,h_2),
$$

where $|I_1|, |I_2|$ are some functions with universal L^∞-bounds. Now by the property (2.4) of ϕ_1, we have

$$
\|h_1\|_{H^2(\mathbb{R}^2)} \le C\Big\{|a|(\|h_1\|_{L^2(\mathbb{R}^2)} + \|h_2\|_{L^2(\mathbb{R}^2)}) + \|DF_1(0,0)(h_1,h_2)\|_{L^2(\mathbb{R}^2)}\Big\}. \tag{2.21}
$$

A similar argument yields that

$$
\|h_2\|_{H^2(\mathbb{R}^2)} \le C\Big\{|b|(\|h_1\|_{L^2(\mathbb{R}^2)} + \|h_2\|_{L^2(\mathbb{R}^2)}) + \|DF_2(0,0)(h_1,h_2)\|_{L^2(\mathbb{R}^2)}\Big\}. \tag{2.22}
$$

If we take $|a|, |b|$ small enough such that $C(|a|+|b|) < \frac{1}{2}$, (2) follows from inequalities (2.21) and (2.22) immediately.

It remains to prove (3). In fact, by a direct computation, one gets

$$
\big(DF_1(z_1,z_2) - DF_1(0,0)\big)(h_1,h_2) = I_3h_1 + I_4h_2, \tag{2.23}
$$

where I_3, I_4 are functions which are controlled by $|1-\mathrm{e}^{\kappa z_1}| + |1-\mathrm{e}^{\kappa z_2}|$. By Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and mean value theorem, one can show (3) is true. □

Proof of Theorem 1.2. Set

$$
G(z_1,z_2) \triangleq (z_1,z_2) - (DF)^{-1}(0,0)F(z_1,z_2) : H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2). \tag{2.24}
$$

It is easy to see that $F(z_1,z_2)=0$ is equivalent to (z_1,z_2) is a fixed point of G. In the following, we will show $G(z_1,z_2)$ is a contraction mapping from $\mathcal{B}_1\times\mathcal{B}_1$ to $\mathcal{B}_1\times\mathcal{B}_1$, where $\mathcal{B}_1 \triangleq \{z \in H^2(\mathbb{R}^2) | \|z\|_{H^2(\mathbb{R}^2)} \le 1\}$. Also set $\kappa^2 \triangleq \max(|a|,|b|)$.

By the definition of $G(z_1,z_2)$ and (1), (2) of Lemma 2.6, we see that

$$
\|G(0,0)\|_{H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)} = \Big\|(DF)^{-1}(0,0)F(0,0)\Big\|_{H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)} \le C(N_1,N_2)\kappa. \tag{2.25}
$$

Also by (2), (3) of Lemma 2.6, we have

$$
\|DG(z_1,z_2)\| \le \Big\|(DF)^{-1}(0,0)\Big\| \, \|DF(z_1,z_2) - DF(0,0)\| \le C(N_1,N_2)\kappa. \tag{2.26}
$$

From inequalities (2.25) and (2.26), one gets

$$
\begin{aligned}
&\|G(z_1, z_2)\|_{H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)} \\
&= \|G(0,0)\|_{H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)} + \sup_{\tilde{z}_1,\tilde{z}_2\in\mathcal{B}_1} \|DG(\tilde{z}_1,\tilde{z}_2)\|\big(\|z_1\|_{H^2(\mathbb{R}^2)} + \|z_2\|_{H^2(\mathbb{R}^2)}\big) \\
&\le C(N_1, N_2)\kappa
\end{aligned}
\tag{2.27}
$$

for any $z_1, z_2 \in \mathcal{B}_1$. Moreover,

$$
\begin{aligned}
&\|G(z_1, z_2) - G(\tilde{z}_1, \tilde{z}_2)\|_{H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)} \\
\le& \sup_{\bar{z}_1,\bar{z}_2\in\mathcal{B}_1} \|DG(\bar{z}_1,\bar{z}_2)\|(\|z_1-\tilde{z}_1\|_{H^2(\mathbb{R}^2)} + \|z_2-\tilde{z}_2\|_{H^2(\mathbb{R}^2)}) \\
\le& C\kappa\big(\|z_1-\tilde{z}_1\|_{H^2(\mathbb{R}^2)} + \|z_2-\tilde{z}_2\|_{H^2(\mathbb{R}^2)}\big) \le \frac{1}{2}\big(\|z_1-\tilde{z}_1\|_{H^2(\mathbb{R}^2)} + \|z_2-\tilde{z}_2\|_{H^2(\mathbb{R}^2)}\big)
\end{aligned}
\tag{2.28}
$$

for any $z_1, z_2 \in \mathcal{B}_1$ and κ small enough. In view of (2.27) and (2.28), we see that $G(z_1, z_2)$ is a well-defined contraction mapping from $\mathcal{B}_1 \times \mathcal{B}_1$ to $\mathcal{B}_1 \times \mathcal{B}_1$ provided $|a|, |b|$ is small enough. Therefore, $G(z_1, z_2)$ has a unique fixed point $(z_{1,a,b}, z_{2,a,b})$ in $\mathcal{B}_1 \times \mathcal{B}_1$. By the definition of $G(z_1, z_2)$ and $F(z_1, z_2)$, we have found a solution $(\varphi_{1,a,b}, \varphi_{2,a,b})$ of (1.7) as follows:

$$
\varphi_{1,a,b}(x) = \kappa z_{1,a,b}(x) + \phi_1(x), \quad \varphi_{2,a,b}(x) = \kappa z_{2,a,b}(x) + \phi_2(x).
$$

From the construction, by Lemma 2.6, we can see the non-degeneracy of the linearized operator of (1.7) at $(\varphi_{1,a,b}, \varphi_{2,a,b})$. Since $z_{1,a,b}, z_{2,a,b} \in \mathcal{B}_1$, by Sobolev embedding $H^2(D) \hookrightarrow L^\infty(D)$, $D \subset \mathbb{R}^2$, it is easy to see that $z_{i,a,b}(x) \to 0$ as $|x| \to \infty$, $i = 1, 2$. This implies $(\varphi_{1,a,b}, \varphi_{2,a,b})$ is a topological solution of (1.7).

By Lemma 2.2, we know that there exists $R_\varepsilon(N_1, N_2, a, b) > 0$ such that property (1) in Theorem 1.1 holds. Then the only thing left is to show $R_\varepsilon(N_1, N_2, a, b)$ is independent of a, b provided a, b are small. Suppose that there exist two sequences $a_n, b_n \to 0$ such that $R_\varepsilon(N_1, N_2, a_n, b_n) \to +\infty$. Without loss of generality, we may assume $\varphi_{1,n}(x_n) = \varphi_{1,a_n,b_n}(x_n) \le \gamma_0 < 0$ and $x_n \to \infty$. By our construction of $(\varphi_{1,n}, \varphi_{2,n})$, we have

$$
\int_{\mathbb{R}^2} |1 - e^{\varphi_{i,n}}| \le \int_{\mathbb{R}^2} |1 - e^{\kappa_n z_{i,n}}| + e^{\kappa_n z_{i,n}}|1 - e^{\phi_i}| \le C < \infty, \tag{2.29}
$$

since we have $\|z_{i,n}\|_{H^2(\mathbb{R}^2)} \le 1$. Set $\tilde{\varphi}_{i,n}(x) \triangleq \varphi_{i,n}(x + x_n)$. By a similar argument as in Lemma 2.4, we have $(\tilde{\varphi}_{1,n}, \tilde{\varphi}_{2,n})$ converges to $(0, 0)$ in $C^2_{loc}(\mathbb{R}^2)$. This yields a contradiction as $\tilde{\varphi}_{1,n}(0) \le \gamma_0 < 0$. This proves that property (1) that $R_\varepsilon(N_1, N_2)$ depends only on N_1, N_2. Property (2) is a direct conclusion of our procession of construction.

Now we prove the conclusion (3) in Theorem 1.2. Suppose not, we may assume for $0 \le a_n, b_n \to 0$, $(\varphi_{1,n}, \varphi_{1,n})$ and $(\hat{\varphi}_{2,n}, \hat{\varphi}_{2,n})$ are two different topological solution of system (1.7). Denote

$$
h_{1,n}(x) \triangleq \frac{\varphi_{1,n} - \hat{\varphi}_{1,n}}{|\varphi_{1,n} - \hat{\varphi}_{1,n}|_\infty + |\varphi_{2,n} - \hat{\varphi}_{2,n}|_\infty}, \quad h_{2,n}(x) \triangleq \frac{\varphi_{2,n} - \hat{\varphi}_{2,n}}{|\varphi_{1,n} - \hat{\varphi}_{1,n}|_\infty + |\varphi_{2,n} - \hat{\varphi}_{2,n}|_\infty}.
$$

By assumption, without loss of generality, we may assume $h_{1,n}(x_n) \ge \frac{1}{2}$ for some $x_n \in \mathbb{R}^2$. Set $\hat{h}_{i,n}(x) \triangleq h_{i,n}(x + x_n)$. By Lemma 2.4, we know that $\hat{h}_{1,n}$ converges to $\hat{h}_1$ which solves

$$\Delta \hat{h}_1 + e^{\hat{u}_1}(1-2e^{\hat{u}_1})\hat{h}_1 = 0 \quad \text{in} \quad \mathbb{R}^2, \quad \hat{h}_1(0) \geq \frac{1}{2}, \quad |\hat{h}_1|_\infty \leq 1. \tag{2.30}$$

Here $\hat{u}_1$ is the unique topological solution of

$$\Delta \hat{u}_1 + e^{\hat{u}_1}(1-e^{\hat{u}_1}) = 4\pi \hat{N}\delta_{\hat{x}},$$

where $\hat{N} = 0$ if $\lim x_n \to \infty$ and $\hat{N} = N_1$ if $\lim x_n = \hat{x}$. By a result from [8], we know that the equation in (2.30) only admits trivial bounded solution. This contradicts to $\hat{h}_1(0) \geq \frac{1}{2}$. This proves the uniqueness for $a, b \geq 0$ and small enough.

In view of the decay property of the topological solutions stated in (1) of Theorem 1.2, the last conclusion in Theorem 1.2 concerning the quantized integrals follows from a direct integration. Then we complete the proof of Theorem 1.2. □

3. Solutions on torus

This section is devoted to the proof of Theorem 1.1. On the basis of Theorem 1.2, we prove the existence of a first solution of (1.3) over $\mathbb{T}^2$ by using a perturbation argument. The existence of a second solution is obtained by using the mountain-pass lemma [2].

3.1. Existence of a first solutions

To carry out our proof, we just need to consider the following elliptic system over $\mathbb{T}^2$:

$$\begin{cases} \Delta u_1 = \dfrac{1}{\varepsilon^2}\Big\{a e^{u_2}(1-(1+b)e^{u_2}+b e^{u_1}) \\ \qquad\qquad -(1+a)e^{u_1}(1-(1+a)e^{u_1}+a e^{u_2})\Big\} + 4\pi \displaystyle\sum_{j=1}^{N} \alpha_{1,j}\delta_{p_j}, \\ \Delta u_2 = \dfrac{1}{\varepsilon^2}\Big\{b e^{u_1}(1-(1+a)e^{u_1}+a e^{u_2}) \\ \qquad\qquad -(1+b)e^{u_2}(1-(1+b)e^{u_2}+b e^{u_1})\Big\} + 4\pi \displaystyle\sum_{j=1}^{N} \alpha_{2,j}\delta_{p_j}, \end{cases} \tag{3.1}$$

where $\sum_{j=1}^{N} \alpha_{i,j} = N_i$, $i = 1, 2$. Here we relabel the configuration $\{p_{11}, \cdots, p_{1N_1}, p_{21}, \cdots, p_{2N_2}\}$ in Theorem 1.1 by $\{p_1, \cdots, p_N\}$.

Let $(\varphi_{1i}, \varphi_{2i})$ be the topological solution of the following system which is constructed in Section 2:

$$\begin{cases} \Delta \varphi_{1i} = a e^{\varphi_{2i}}(1-(1+b)e^{\varphi_{2i}}+b e^{\varphi_{1i}}) \\ \qquad\qquad -(1+a)e^{\varphi_{1i}}(1-(1+a)e^{\varphi_{1i}}+a e^{\varphi_{2i}}) + 4\pi\alpha_{1,i}\delta_0 \\ \Delta \varphi_{2i} = b e^{\varphi_{1i}}(1-(1+a)e^{\varphi_{1i}}+a e^{\varphi_{2i}}) \\ \qquad\qquad -(1+b)e^{\varphi_{2i}}(1-(1+b)e^{\varphi_{2i}}+b e^{\varphi_{1i}}) + 4\pi\alpha_{2,i}\delta_0 \end{cases} \quad \text{in } \mathbb{R}^2. \tag{3.2}$$

Now we assume $\min_{i\neq j}|p_i - p_j| \geq 4d > 0$. Take a cut-off function $\xi(x) \in C_c^\infty(B_{2d})$ with $\xi(x) = 1$ for $x \in B_d$ and $0 \leq \xi \leq 1$. For $3 \leq k \in \mathbb{N}^+$, set

$$\begin{aligned} u_1(x) &= \varepsilon^k z_1(x) + \sum_{i=1}^N \xi(x-p_i)\varphi_{1i}\left(\frac{x-p_i}{\varepsilon}\right) \triangleq \varepsilon^k z_1(x) + \sum_{i=1}^N \xi_i \varphi_{1i,\varepsilon}(x), \\ u_2(x) &= \varepsilon^k z_2(x) + \sum_{i=1}^N \xi(x-p_i)\varphi_{2i}\left(\frac{x-p_i}{\varepsilon}\right) \triangleq \varepsilon^k z_2(x) + \sum_{i=1}^N \xi_i \varphi_{2i,\varepsilon}(x). \end{aligned} \tag{3.3}$$

Then (u_1, u_2) solves (3.1) is equivalent to (z_1, z_2) solves

$$\begin{cases} \Delta z_1 + \dfrac{1}{\varepsilon^{k+2}}\Bigg\{(1+a)\left[f(u_1,u_2) - \displaystyle\sum_{i=1}^N \xi_i f\big(\varphi_{1i,\varepsilon}(x), \varphi_{2i,\varepsilon}(x)\big)\right] \\ \quad - a\left[g(u_1,u_2) - \displaystyle\sum_{i=1}^N \xi_i g\big(\varphi_{1i,\varepsilon}(x), \varphi_{2i,\varepsilon}(x)\big)\right]\Bigg\} \\ \quad + \dfrac{1}{\varepsilon^k}\displaystyle\sum_{i=1}^N \varphi_{1i,\varepsilon}\Delta\xi_i + \dfrac{2}{\varepsilon^k}\displaystyle\sum_{i=1}^N \nabla\varphi_{1i,\varepsilon}\cdot\nabla\xi_i = 0, \\ \Delta z_2 + \dfrac{1}{\varepsilon^{k+2}}\Bigg\{(1+b)\left[g(u_1,u_2) - \displaystyle\sum_{i=1}^N \xi_i g\big(\varphi_{1i,\varepsilon}(x), \varphi_{2i,\varepsilon}(x)\big)\right] \\ \quad - b\left[f(u_1,u_2) - \displaystyle\sum_{i=1}^N \xi_i f\big(\varphi_{1i,\varepsilon}(x), \varphi_{2i,\varepsilon}(x)\big)\right]\Bigg\} \\ \quad + \dfrac{1}{\varepsilon^k}\displaystyle\sum_{i=1}^N \varphi_{2i,\varepsilon}\Delta\xi_i + \dfrac{2}{\varepsilon^k}\displaystyle\sum_{i=1}^N \nabla\varphi_{2i,\varepsilon}\cdot\nabla\xi_i = 0, \end{cases} \tag{3.4}$$

where

$$f(s,t) \triangleq \mathrm{e}^s(1-(1+a)\mathrm{e}^s + a\mathrm{e}^t), \quad g(s,t) \triangleq \mathrm{e}^t(1-(1+b)\mathrm{e}^t + b\mathrm{e}^s). \tag{3.5}$$

For simplicity, we write (3.4) in the following form

$$F_\varepsilon(z_1,z_2) \triangleq \big(F_{\varepsilon,1}(z_1,z_2),\ F_{\varepsilon,2}(z_1,z_2)\big) = (0,0). \tag{3.6}$$

It is easy to see that $F_\varepsilon(z_1,z_2)$ is a well-defined map from $H^2(\mathbb{T}^2)\times H^2(\mathbb{T}^2)$ to $L^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2)$. Moreover, we can prove $F_\varepsilon(z_1,z_2)$ admits the following properties.

Lemma 3.1. *Let $F_\varepsilon(z_1,z_2)$ be defined as above. Then for $0 < \varepsilon < \varepsilon_0(a,b,p_i,N_1,N_2,k)$, $|a|, |b| < \delta_0(N_1,N_2)$, F_ε satisfies:*

(i) $\|F_\varepsilon(0,0)\|_{L^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2)} \leq c_1\mathrm{e}^{-c_2/\varepsilon}$;
(ii) $\|DF_\varepsilon(0,0)(h_1,h_2)\|_{L^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2)} \geq c_3\big(\|h_1\|_{H^2(\mathbb{T}^2)} + \|h_2\|_{H^2(\mathbb{T}^2)}\big)$;

(iii) $\|(DF_\varepsilon(z_1,z_2) - DF_\varepsilon(0,0))(h_1,h_2)\|_{L^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2)} \le c_4\varepsilon\big(\|h_1\|_{H^2(\mathbb{T}^2)} + \|h_2\|_{H^2(\mathbb{T}^2)}\big)$, *for any* $z_1, z_2 \in \mathcal{B}_1 \triangleq \big\{z \in H^2(\mathbb{T}^2) \big| \, \|z\|_{H^2(\mathbb{T}^2)} \le 1\big\}$.

Here, all the constants c_1, c_2, c_3, c_4 *depend only on* N_1, N_2, p_i, k.

Proof. For the first component of $F_\varepsilon(z_1, z_2)$, by definition, we have

$$\begin{aligned}F_{\varepsilon,1}(0,0) = \frac{1}{\varepsilon^{k+2}}\Bigg\{&(1+a)\Bigg[f\Bigg(\sum_{i=1}^N \xi_i\varphi_{1i,\varepsilon}(x),\ \sum_{i=1}^N \xi_i\varphi_{2i,\varepsilon}(x)\Bigg) - \sum_{i=1}^N \xi_i f\big(\varphi_{1i,\varepsilon}(x),\ \varphi_{2i,\varepsilon}(x)\big)\Bigg]\\ &- a\Bigg[g\Bigg(\sum_{i=1}^N \xi_i\varphi_{1i,\varepsilon}(x),\ \sum_{i=1}^N \xi_i\varphi_{2i,\varepsilon}(x)\Bigg) - \sum_{i=1}^N \xi_i g\big(\varphi_{1i,\varepsilon}(x),\ \varphi_{2i,\varepsilon}(x)\big)\Bigg]\Bigg\}\\ &+ \frac{1}{\varepsilon^k}\sum_{i=1}^N \varphi_{1i,\varepsilon}\Delta\xi_i + \frac{2}{\varepsilon^k}\sum_{i=1}^N \nabla\varphi_{1i,\varepsilon}\cdot\nabla\xi_i.\end{aligned}$$

By the property (1) in Theorem 1.2, we have

$$\|\varphi_{ij,\varepsilon}\|_{L^\infty(B_d^c(p_j))} + \|\nabla\varphi_{ij,\varepsilon}\|_{L^\infty(B_d^c(p_j))} \le c_1 \mathrm{e}^{-c_2/\varepsilon},\ i = 1,2, \quad \text{as} \quad \varepsilon \to 0. \tag{3.7}$$

Since $\xi_i \equiv 1$ in $B_d(p_i)$ and $f(s,t),\ g(s,t) = O(t+s)$ as $t, s \to 0$, we get that

$$\|F_{\varepsilon,1}(0,0)\|_{L^2(\mathbb{T}^2)} \le c_1\mathrm{e}^{-c_2/\varepsilon}.$$

A similar estimate also holds for $F_{\varepsilon,2}(0,0)$. This proves (i).

Now we assume (ii) is not true. This implies that there are two sequences of $h_{1,n}, h_{2,n} \in H^2(\mathbb{T}^2)$ satisfying the following property:

$$\|h_{1,n}\|_{H^2(\mathbb{T}^2)} + \|h_{2,n}\|_{H^2(\mathbb{T}^2)} = 1, \quad \|DF_{\varepsilon_n}(0,0)(h_{1,n}, h_{2,n})\|_{L^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2)} = o(1)$$

as $n \to +\infty$. For simplicity, we will denote $DF_{\varepsilon_n}(0,0)(h_{1,n}, h_{2,n})$ by $DF_n(0,0)(h_{1,n}, h_{2,n})$. By definition, we have

$$\begin{aligned}&DF_{n,1}(0,0)(h_{1,n}, h_{2,n})\\ &= \Delta h_{1,n} + \frac{1}{\varepsilon_n^2}\Bigg\{(1+a)\mathrm{e}^{\sum_{i=1}^N \xi_i\varphi_{1i,n}}\Bigg(1 - 2(1+a)\mathrm{e}^{\sum_{i=1}^N \xi_i\varphi_{1i,n}} + a\mathrm{e}^{\sum_{i=1}^N \xi_i\varphi_{2i,n}}\Bigg)h_{1,n}\\ &\quad - a\mathrm{e}^{\sum_{i=1}^N \xi_i\varphi_{2i,n}}\Bigg(1 - 2(1+b)\mathrm{e}^{\sum_{i=1}^N \xi_i\varphi_{2i,n}} + b\mathrm{e}^{\sum_{i=1}^N \xi_i\varphi_{1i,n}}\Bigg)h_{2,n}\\ &\quad + a(1+a)\mathrm{e}^{\sum_{i=1}^N \xi_i\varphi_{1i,n}+\xi_i\varphi_{2i,n}} h_{2,n} - ab\mathrm{e}^{\sum_{i=1}^N \xi_i\varphi_{1i,n}+\xi_i\varphi_{2i,n}} h_{1,n}\Bigg\},\end{aligned} \tag{3.8}$$

where $\varphi_{1i,n} \triangleq \varphi_{1i,\varepsilon_n}$, $\varphi_{2i,n} \triangleq \varphi_{2i,\varepsilon_n}$. Consider a cut-off function $0 \le \sigma(x) \le 1$ which satisfies that $\sigma(x) \equiv 0$ in $\cup_i B_{d/4}(p_i)$; $\sigma(x) \equiv 1$ in $\mathbb{T}^2 \setminus \cup_i B_{d/2}(p_i)$. Set $\bar{h}_{i,n} \triangleq \sigma h_{i,n}$ for $i = 1, 2$. Then $\bar{h}_{1,n}$ satisfies

$$\begin{aligned}
-\Delta \bar{h}_{1,n} = \frac{1}{\varepsilon_n^2} \Bigg\{ &(1+a)\mathrm{e}^{\sum\limits_{i=1}^{N} \xi_i \varphi_{1i,n}} \left(1 - 2(1+a)\mathrm{e}^{\sum\limits_{i=1}^{N} \xi_i \varphi_{1i,n}} + a\mathrm{e}^{\sum\limits_{i=1}^{N} \xi_i \varphi_{2i,n}} \right) \bar{h}_{1,n} \\
&- a\mathrm{e}^{\sum\limits_{i=1}^{N} \xi_i \varphi_{2i,n}} \left(1 - 2(1+b)\mathrm{e}^{\sum\limits_{i=1}^{N} \xi_i \varphi_{2i,n}} + b\mathrm{e}^{\sum\limits_{i=1}^{N} \xi_i \varphi_{1i,n}} \right) \bar{h}_{2,n} \\
&+ a(1+a)\mathrm{e}^{\sum\limits_{i=1}^{N} \xi_i \varphi_{1i,n} + \xi_i \varphi_{2i,n}} \bar{h}_{2,n} - ab\mathrm{e}^{\sum\limits_{i=1}^{N} \xi_i \varphi_{1i,n} + \xi_i \varphi_{2i,n}} \bar{h}_{1,n} \Bigg\} \\
&- 2\nabla\sigma \cdot \nabla h_{1n} - h_{1,n}\Delta\sigma - \sigma DF_{n,1}(0,0)(h_{1,n}, h_{2,n}).
\end{aligned} \tag{3.9}$$

Multiplying (3.9) with $\bar{h}_{1,n}$ and integrating by parts, we get,

$$\begin{aligned}
&\|\nabla \bar{h}_{1,n}\|^2_{L^2(\mathbb{T}^2)} + \frac{(1+a)^2 + ab}{\varepsilon_n^2(1 + C\varepsilon_n)} \|\bar{h}_{1,n}\|^2_{L^2(\mathbb{T}^2)} - \frac{|a|(2+a+b)}{\varepsilon_n^2(1 - C\varepsilon_n)} \int\limits_{\mathbb{T}^2} |\bar{h}_{1,n}\bar{h}_{2,n}| \\
&\le C(\|h_{1,n}\|_{H^1(\mathbb{T}^2)} + \|h_{2,n}\|_{H^1(\mathbb{T}^2)} + \|DF_{n,1}(h_{1,n}, h_{2,n})\|_{L^2(\mathbb{T}^2)})\|\bar{h}_{1,n}\|_{L^2(\mathbb{T}^2)}.
\end{aligned} \tag{3.10}$$

In getting (3.10), we use the decay property of (3.7). A similar argument yields that

$$\begin{aligned}
&\|\nabla \bar{h}_{2,n}\|^2_{L^2(\mathbb{T}^2)} + \frac{(1+b)^2 + ab}{\varepsilon_n^2(1 + C\varepsilon_n)} \|\bar{h}_{2,n}\|^2_{L^2(\mathbb{T}^2)} - \frac{|b|(2+a+b)}{\varepsilon_n^2(1 - C\varepsilon_n)} \int\limits_{\mathbb{T}^2} |\bar{h}_{1,n}\bar{h}_{2,n}| \\
&\le C(\|h_{1,n}\|_{H^1(\mathbb{T}^2)} + \|h_{2,n}\|_{H^1(\mathbb{T}^2)} + \|DF_{n,2}(h_{1,n}, h_{2,n})\|_{L^2(\mathbb{T}^2)})\|\bar{h}_{2,n}\|_{L^2(\mathbb{T}^2)}.
\end{aligned} \tag{3.11}$$

From the assumption that $|a|$, $|b|$ is small enough, (3.10) and (3.11) tell us that

$$\begin{aligned}
&\|\nabla \bar{h}_{1,n}\|^2_{L^2(\mathbb{T}^2)} + \|\nabla \bar{h}_{2,n}\|^2_{L^2(\mathbb{T}^2)} + \frac{c}{\varepsilon_n^2}\|\bar{h}_{1,n}\|^2_{L^2(\mathbb{T}^2)} + \frac{c}{\varepsilon_n^2}\|\bar{h}_{2,n}\|^2_{L^2(\mathbb{T}^2)} \\
&\le C\left(\sum_{i=1}^{2} \|h_{i,n}\|_{H^1(\mathbb{T}^2)} + \|DF_n(h_{1,n}, h_{2,n})\|_{L^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2)} \right) \left(\|\bar{h}_{1,n}\|_{L^2(\mathbb{T}^2)} + \|\bar{h}_{2,n}\|_{L^2(\mathbb{T}^2)} \right) \\
&\le C\left(\|\bar{h}_{1,n}\|_{L^2(\mathbb{T}^2)} + \|\bar{h}_{2,n}\|_{L^2(\mathbb{T}^2)} \right).
\end{aligned} \tag{3.12}$$

By (3.12), one gets $\|\bar{h}_{1,n}\|_{L^2(\mathbb{T}^2)} + \|\bar{h}_{2,n}\|_{L^2(\mathbb{T}^2)} = O(\varepsilon_n^2)$. This again implies that

$$\|\nabla \bar{h}_{1,n}\|_{L^2(\mathbb{T}^2)} + \|\nabla \bar{h}_{2,n}\|_{L^2(\mathbb{T}^2)} = O(\varepsilon_n).$$

Now we take another cut-off function $0 \le \tilde{\sigma}(x) \le 1$ such that $\tilde{\sigma}(x) \equiv 0$ in $\cup B_{\frac{d}{2}}(p_i)$; $\tilde{\sigma}(x) \equiv 1$ in $\mathbb{T}^2 \setminus \cup_i B_d(p_i)$. Set $\tilde{h}_{i,n}(x) \triangleq \tilde{\sigma} h_{i,n}(x)$. Repeating the arguments for $\bar{h}_{i,n}$, $i = 1, 2$, we can get

$$\begin{aligned}
&\|\nabla \tilde{h}_{1n}\|^2_{L^2(\mathbb{T}^2)} + \|\nabla \tilde{h}_{2n}\|^2_{L^2(\mathbb{T}^2)} + \frac{c}{\varepsilon_n^2}\|\tilde{h}_{1,n}\|^2_{L^2(\mathbb{T}^2)} + \frac{c}{\varepsilon_n^2}\|\tilde{h}_{2,n}\|^2_{L^2(\mathbb{T}^2)} \\
&\le C\left(\sum_{i=1}^{2}\|\bar{h}_{i,n}\|_{H^1(\mathbb{T}^2)} + \|DF_n(h_{1,n}, h_{2,n})\|_{L^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2)}\right)\left(\|\tilde{h}_{1,n}\|_{L^2(\mathbb{T}^2)} + \|\tilde{h}_{2,n}\|_{L^2(\mathbb{T}^2)}\right) \\
&= o(1)\left(\|\tilde{h}_{1,n}\|_{L^2(\mathbb{T}^2)} + \|\tilde{h}_{2,n}\|_{L^2(\mathbb{T}^2)}\right). \qquad (3.13)
\end{aligned}$$

By (3.13), we have $\|\tilde{h}_{1,n}\|_{L^2(\mathbb{T}^2)} + \|\tilde{h}_{2,n}\|_{L^2(\mathbb{T}^2)} = o(\varepsilon_n^2)$. From standard $W^{2,2}$-estimates of uniformly elliptic partial differential equations, we get

$$\|h_{1,n}\|_{H^2(\mathbb{T}^2\setminus\cup_i B_d(p_i))} + \|h_{2,n}\|_{H^2(\mathbb{T}^2\setminus\cup_i B_d(p_i))} = o(1). \qquad (3.14)$$

By (3.14), without loss of generality, we may assume that

$$\|h_{1,n}\|_{H^2(B_d(p_1))} + \|h_{2,n}\|_{H^2(B_d(p_1))} \ge c_0 > 0. \qquad (3.15)$$

Consider a cut-off function $0 \le \hat{\sigma}(x) \le 1$ such that $\hat{\sigma}(x) \equiv 1$ in $B_d(p_1)$; $\hat{\sigma}(x) \equiv 0$ in $B^c_{2d}(p_1)$. Set $\hat{\sigma}_n(x) \triangleq \hat{\sigma}(\varepsilon_n x + p_1)$, $\hat{h}_{i,n}(x) \triangleq h_{i,n}(\varepsilon_n x + p_1)\hat{\sigma}_n$, $\Sigma_1 \triangleq B_{2d}(p_1)\setminus B_d(p_1)$, and $\hat{\Sigma}_1 \triangleq (\Sigma_1 - p_1)/\varepsilon_n$. Then $(\hat{h}_{1,n}, \hat{h}_{2,n})$ should solve

$$\begin{aligned}
&L(\varphi_{11}, \varphi_{21})(\hat{h}_{1,n}, \hat{h}_{2,n}) \\
&= A_n(x)\begin{pmatrix}\hat{h}_{1,n}\\ \hat{h}_{2,n}\end{pmatrix} + \varepsilon_n^2\hat{\sigma}_n DF_n(0,0)(h_{1,n}, h_{2,n}) - 2\varepsilon_n^2\nabla\hat{\sigma}_n \cdot \begin{pmatrix}\nabla\hat{h}_{1,n}\\ \nabla\hat{h}_{2,n}\end{pmatrix} - \varepsilon_n^2\Delta\hat{\sigma}_n\begin{pmatrix}\hat{h}_{1,n}\\ \hat{h}_{2,n}\end{pmatrix},
\end{aligned}$$

where $L(\varphi_{11}, \varphi_{21})$ is the linearized operator of (3.2) and $A_n(x)$ is a 2×2 matrix function with its entries $|A_{n,ij}(x)| = O(\mathrm{e}^{-c/\varepsilon_n})$. By Theorem 1.2, we have

$$\begin{aligned}
&\|\hat{h}_{1,n}\|_{H^2(\mathbb{R}^2)} + \|\hat{h}_{2,n}\|_{H^2(\mathbb{R}^2)} \\
&\le C\left(\varepsilon_n\sum_{i=1}^{2}\|\nabla\hat{h}_{i,n}\|_{L^2(\hat{\Sigma}_1)} + \sum_{i=1}^{2}\|\hat{h}_{i,n}\|_{L^2(\hat{\Sigma}_1)} + \varepsilon_n\|DF_n(0,0)(h_{1,n}, h_{2,n})\|_{L^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2}\right) \\
&\le C\left(\varepsilon_n\sum_{i=1}^{2}\|\nabla h_{i,n}\|_{L^2(\Sigma_1)} + \varepsilon_n^{-1}\sum_{i=1}^{2}\|h_{i,n}\|_{L^2(\Sigma_1)} + o(\varepsilon_n)\right) = o(\varepsilon_n). \qquad (3.16)
\end{aligned}$$

From inequality (3.16), we obtain $\|h_{i,n}\|_{H^2(B_d(p_1))} \le \varepsilon_n^{-1}\|\hat{h}_{i,n}\|_{H^2(\mathbb{R}^2)} = o(1)$. This contradicts to our choice of $B_d(p_1)$. This proves (ii) of Lemma 3.1.

Now we are in a position to prove (iii). In fact, by a direct computation, we have

$$\big(DF_{\varepsilon,1}(z_1, z_2) - DF_{\varepsilon,1}(0,0)\big)(h_1, h_2) = I_1h_1 + I_2h_2, \qquad (3.17)$$

where I_1, I_2 are functions whose absolute values are controlled by $\frac{1}{\varepsilon^2}|1-e^{\varepsilon^k z_1}|$ and $\frac{1}{\varepsilon^2}|1-e^{\varepsilon^k z_2}|$. By the Sobolev embedding $H^2(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$, (iii) follows easily from (3.17). □

With the previous preparation, we now can prove the first part of Theorem 1.1.

Proof of Theorem 1.1 (the existence of a first solution). Define a map $G_\varepsilon(z_1, z_2) : H^2(\mathbb{T}^2) \times H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2) \times H^2(\mathbb{T}^2)$ by

$$G_\varepsilon(z_1, z_2) \triangleq (z_1, z_2) - (DF_\varepsilon)^{-1}(0,0)F_\varepsilon(z_1, z_2). \tag{3.18}$$

By Lemma 3.1, one gets that

$$\|G_\varepsilon(0,0)\|_{H^2(\mathbb{T}^2)\times H^2(\mathbb{T}^2)} \le \left\|(DF_\varepsilon)^{-1}(0,0)F_\varepsilon(0,0)\right\|_{H^2(\mathbb{T}^2)\times H^2(\mathbb{T}^2)} \le Ce^{-c/\varepsilon}; \tag{3.19}$$

and $\forall\, z_1, z_2 \in \mathcal{B}_1$,

$$\|DG_\varepsilon(z_1, z_2)\| \le \left\|(DF_\varepsilon)^{-1}(0,0)\right\| \|DF_\varepsilon(z_1, z_2) - DF_\varepsilon(0,0)\| \le C\varepsilon. \tag{3.20}$$

From (3.19) and (3.20), for $\forall\, z_1, z_2 \in \mathcal{B}_1$, we have

$$\begin{aligned} &\|G_\varepsilon(z_1, z_2)\|_{H^2(\mathbb{T}^2)\times H^2(\mathbb{T}^2)} \\ &\le \|G_\varepsilon(0,0)\|_{H^2(\mathbb{T}^2)\times H^2(\mathbb{T}^2)} + \sup_{\tilde z_1, \tilde z_2 \in \mathcal{B}_1} \|DG_\varepsilon(\tilde z_1, \tilde z_2)\| \big(\|z_1\|_{H^2(\mathbb{T}^2)} + \|z_1\|_{H^2(\mathbb{T}^2)}\big) \\ &\le C\varepsilon. \end{aligned} \tag{3.21}$$

For $z_i, \tilde z_i \in \mathcal{B}_1$, $i = 1, 2$, we also have,

$$\begin{aligned} &\|G_\varepsilon(z_1, z_2) - G_\varepsilon(\tilde z_1, \tilde z_2)\|_{H^2(\mathbb{T}^2)\times H^2(\mathbb{T}^2)} \\ &\le \sup_{\bar z_1, \bar z_2 \in \mathcal{B}_1} \|DG_\varepsilon(\bar z_1, \bar z_2)\| \sum_{i=1}^{2} \|z_i - \tilde z_i\|_{H^2(\mathbb{T}^2)} \le C\varepsilon \sum_{i=1}^{2} \|z_i - \tilde z_i\|_{H^2(\mathbb{T}^2)}. \end{aligned} \tag{3.22}$$

Hence, inequalities (3.21) and (3.22) imply that $G_\varepsilon(z_1, z_2)$ is a well-defined contraction mapping from $\mathcal{B}_1 \times \mathcal{B}_1$ to itself provided ε is small enough. So $G_\varepsilon(z_1, z_2)$ has a unique fixed point $(z_{1,\varepsilon}, z_{2,\varepsilon})$ in $\mathcal{B}_1 \times \mathcal{B}_1$, i.e. $F_\varepsilon(z_{1,\varepsilon}, z_{2,\varepsilon}) = 0$. As a consequence, we obtain a solution of (3.1) with the following form:

$$u_{1,\varepsilon}(x) = \varepsilon^k z_{1,\varepsilon} + \sum_{i=1}^{N} \xi_i \varphi_{1i,\varepsilon}(x), \quad u_{2,\varepsilon}(x) = \varepsilon^k z_{2,\varepsilon} + \sum_{i=1}^{N} \xi_i \varphi_{2i,\varepsilon}(x). \tag{3.23}$$

By the arbitrariness of k and standard $W^{2,p}$ estimates, one gets (1.6) in Theorem 1.1. This proves the existence of a first solution of system (3.1). □

3.2. Mountain-pass solution on torus

In this subsection, we will prove the existence of mountain-pass solution for system (3.1). For simplicity we set $|\mathbb{T}^2| = 1$ in our discussion. Since the case for $a, b > 0$ is already solved by Han–Tarantello [30], we just need to focus on the case $a, b < 0$ or $ab = 0$. Here and in what follows, we always assume (u_0^1, u_0^2) is the unique solution of the following decoupled system [3]:

$$\begin{cases} \Delta u_0^1 = 4\pi \sum\limits_{i=1}^{N} \alpha_{1,i}\delta_{p_i} - 4\pi N_1, & \int\limits_{\mathbb{T}^2} u_0^1 = 0, \quad \sum\limits_{i=1}^{N} \alpha_{1,i} = N_1, \\ \Delta u_0^2 = 4\pi \sum\limits_{i=1}^{N} \alpha_{2,i}\delta_{p_i} - 4\pi N_2, & \int\limits_{\mathbb{T}^2} u_0^2 = 0, \quad \sum\limits_{i=1}^{N} \alpha_{2,i} = N_2. \end{cases} \tag{3.24}$$

Also, from [53], one can get for ε small enough, $\tilde{u}_{i,\varepsilon}$ is the unique strictly stable solution of the following equation:

$$\Delta \tilde{u}_{i,\varepsilon} + \frac{1}{\varepsilon^2} e^{\tilde{u}_{i,\varepsilon}}(1 - e^{\tilde{u}_{i,\varepsilon}}) = 4\pi \sum_{j=1}^{N} \alpha_{i,j}\delta_{p_j}. \tag{3.25}$$

Set

$$\tilde{I}_{\varepsilon,i}(v) = \int\limits_{\mathbb{T}^2} \left[|\nabla v|^2 + \frac{1}{\varepsilon^2}(e^{v+u_0^i} - 1)^2 \right] + 4\pi N_i \int\limits_{\mathbb{T}^2} v, \quad i = 1, 2.$$

Then $\tilde{v}_{i,\varepsilon} = \tilde{u}_{i,\varepsilon} - u_0^i$ is a strictly local minimizer of $\tilde{I}_{\varepsilon,i}(v)$. In other words,

$$\tilde{J}_{\varepsilon,i}(\psi) = \int\limits_{\mathbb{T}^2} \left[|\nabla \psi|^2 + \frac{1}{\varepsilon^2}(2e^{v_{i,\varepsilon}+u_0^i} - 1)e^{v_{i,\varepsilon}+u_0^i}\psi^2 \right] \geq c_0 \|\psi\|^2_{H^1(\mathbb{T}^2)}, \ \forall \psi \in H^1(\mathbb{T}^2), \tag{3.26}$$

for some constant $c_0 > 0$.

3.2.1. Case $a, b < 0$, $\frac{1}{\lambda} \leq \frac{a}{b} \leq \lambda$, $\lambda > 1$

As a first step, we need to show the solution constructed in Section 3.1 is actually a strictly local minimizer. By a direct computation, we can derive the following functional for $a, b < 0$.

$$\begin{aligned} &I_\varepsilon(v_1, v_2) \\ &= \frac{|b|(1+b)}{2} \int\limits_{\mathbb{T}^2} |\nabla v_1|^2 + \frac{|a|(1+a)}{2} \int\limits_{\mathbb{T}^2} |\nabla v_2|^2 - ab \int\limits_{\mathbb{T}^2} \nabla v_1 \cdot \nabla v_2 \\ &\quad + \frac{1+a+b}{\varepsilon^2} \left[|b| \int\limits_{\mathbb{T}^2} \left(\frac{1+a}{2} e^{2v_1+2u_0^1} - e^{v_1+u_0^1} \right) + |a| \int\limits_{\mathbb{T}^2} \left(\frac{1+b}{2} e^{2v_2+2u_0^2} - e^{v_2+u_0^2} \right) \right. \end{aligned}$$

$$+ ab \int_{\mathbb{T}^2} \mathrm{e}^{v_1+v_2+u_0^1+u_0^2} \Bigg] + 4\pi|b|[(1+b)N_1 + aN_2] \int_{\mathbb{T}^2} v_1 + 4\pi|a|[(1+a)N_2 + bN_1] \int_{\mathbb{T}^2} v_2.$$

It is easy to check that (v_1, v_2) is a critical point of the functional $I_\varepsilon(v_1, v_2)$ if and only if $(u_1, u_2) = (v_1 + u_0^1,\ v_2 + u_0^2)$ is a solution of system (3.1).

Lemma 3.2. *Suppose the assumptions in Theorem 1.1 are fulfilled and $a, b < 0$ and $\frac{1}{\lambda} \le \frac{a}{b} \le \lambda$, $\lambda > 1$. Then $(v_{1,\varepsilon},\ v_{2,\varepsilon}) = (u_{1,\varepsilon} - u_0^1,\ u_{2,\varepsilon} - u_0^2)$ is a strictly local minimizer of $I_\varepsilon(v_1, v_2)$, where $(u_{1,\varepsilon},\ u_{2,\varepsilon})$ is the topological solution of system (3.1) constructed in Section 3.1 provided $|a|, |b| \le \delta_0(N_1, N_2, \lambda)$.*

Proof. Since $(v_{1,\varepsilon}, v_{2,\varepsilon})$ is a critical point of $I_\varepsilon(v_1, v_2)$, we only need to show that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_\varepsilon(v_{1,\varepsilon} + t\psi_1, v_{2,\varepsilon} + t\psi_2)|_{t=0} \triangleq J_\varepsilon(\psi_1, \psi_2) \ge c_0' \sum_{i=1}^{2} \|\psi_i\|_{H^1(\mathbb{T}^2)}, \quad \text{for some } c_0' > 0. \tag{3.27}$$

By a direct computation, one gets

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_\varepsilon(v_{1,\varepsilon} + t\psi_1,\ v_{2,\varepsilon} + t\psi_2)|_{t=0} \\ &= |b|(1+b)\int_{\mathbb{T}^2} |\nabla\psi_1|^2 + |a|(1+a)\int |\nabla\psi_2|^2 - 2ab\int_{\mathbb{T}^2} \nabla\psi_1\cdot\nabla\psi_2 + \\ &\quad \frac{1+a+b}{\varepsilon^2}\Bigg\{ ab\int \mathrm{e}^{u_{1,\varepsilon}+u_{2,\varepsilon}}(\psi_1+\psi_2)^2 - b\int_{\mathbb{T}^2} (2(1+a)\mathrm{e}^{2u_{1,\varepsilon}} - \mathrm{e}^{u_{1,\varepsilon}})\psi_1^2 \\ &\quad - a\int_{\mathbb{T}^2} (2(1+b)\mathrm{e}^{2u_{2,\varepsilon}} - \mathrm{e}^{u_{2,\varepsilon}})\psi_2^2 \Bigg\}. \end{aligned} \tag{3.28}$$

Suppose Lemma 3.2 is not true. Then there exist $\varepsilon_n \to 0$, $\|\psi_{1,n}\|_{H^1(\mathbb{T}^2)} + \|\psi_{2,n}\|_{H^1(\mathbb{T}^2)} = 1$ such that

$$J_n(\psi_{1,n},\ \psi_{2,n}) \triangleq J_{\varepsilon_n}(\psi_{1,n},\ \psi_{2,n}) \le o(1)(\|\psi_{1,n}\|_{H^1(\mathbb{T}^2)} + \|\psi_{2,n}\|_{H^1(\mathbb{T}^2)}).$$

Set $u_{i,n}(x) = u_{i,\varepsilon_n}(x)$, $i = 1, 2$. Recall that

$$u_{i,n}(x) = \varepsilon_n^3 z_{i,n} + \sum_{j=1}^{N} \xi_j \varphi_{ij,n}, \quad \varphi_{ij,n}(x) = \varphi_{ij}\left(\frac{x - p_j}{\varepsilon_n}\right), \ i = 1, 2, \tag{3.29}$$

where $(\varphi_{1j}(x), \varphi_{2j}(x))$ are the topological solutions of (1.7) constructed in Section 2 with $N_1 = \alpha_{1,j}$, $N_2 = \alpha_{2,j}$ and $\|z_{i,n}\|_{H^2(\mathbb{T}^2)} \le 1$. Also by Theorem 1.2, we have

$$\varphi_{ij}(x) = \kappa \tilde{z}_{ij}(x) + \phi_{ij}(x), \quad \kappa = \max\left\{\sqrt{|a|}, \sqrt{|b|}\right\}, \tag{3.30}$$

where $(\phi_{1j}(x), \phi_{2j}(x))$ are the unique topological solutions of (1.8) with $N_1 = \alpha_{1,j}$, $N_2 = \alpha_{2,j}$ and $\|\tilde{z}_{ij}\|_{H^2(\mathbb{R}^2)} \le 1$. Hence we have

$$\begin{aligned}
\frac{1}{\varepsilon_n^2}\int\limits_{\mathbb{T}^2} \left|1 - \mathrm{e}^{u_{i,n}}\right|^2 &\le \frac{2}{\varepsilon_n^2}\int\limits_{\mathbb{T}^2} \left|1 - \mathrm{e}^{\varepsilon_n^3 z_{i,n}}\right|^2 + \mathrm{e}^{2\varepsilon_n^3 z_{i,n}} \sum_{j=1}^N \left|1 - \mathrm{e}^{\xi_j \varphi_{ij,n}}\right|^2 \\
&\le C\varepsilon_n + C\sum_{j=1}^N \int\limits_{\mathbb{R}^2} \left|1 - \mathrm{e}^{\varphi_{ij}}\right|^2 \\
&\le C\varepsilon_n + C\sum_{j=1}^N \int\limits_{\mathbb{R}^2} \left(\left|1 - \mathrm{e}^{\kappa \tilde{z}_{ij}}\right|^2 + \mathrm{e}^{\kappa \tilde{z}_{ij}} \left|1 - \mathrm{e}^{\phi_{ij}}\right|^2\right) \\
&\le C'.
\end{aligned} \tag{3.31}$$

In getting (3.31), we have used the embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. By definition, we have

$$\begin{aligned}
&J_n(\psi_{1n}, \psi_{2n}) - (1 + a + b)\left(|b|\int\limits_{\mathbb{T}^2} |\nabla \psi_1|^2 + |a|\int\limits_{\mathbb{T}^2} |\nabla \psi_2|^2\right) \\
&\ge \frac{1 + a + b}{\varepsilon_n^2}\Bigg\{2(1 + a)|b|\int\limits_{\mathbb{T}^2} \mathrm{e}^{u_{1,n}}(\mathrm{e}^{u_{1,n}} - 1)\psi_{1,n}^2 + |b|(1 + 2a)\int\limits_{\mathbb{T}^2} (\mathrm{e}^{u_{1,n}} - 1)\psi_{1,n}^2 \\
&\quad + |b|(1 + 2a)\int\limits_{\mathbb{T}^2} \psi_{1,n}^2 + 2(1 + b)|a|\int\limits_{\mathbb{T}^2} \mathrm{e}^{u_{2,n}}(\mathrm{e}^{u_{2,n}} - 1)\psi_{2,n}^2 \\
&\quad + |a|(1 + 2b)\int\limits_{\mathbb{T}^2} (\mathrm{e}^{u_{2,n}} - 1)\psi_{2,n}^2 + |a|(1 + 2b)\int\limits_{\mathbb{T}^2} \psi_{2,n}^2\Bigg\} \\
&\ge \frac{|b|}{4\varepsilon_n^2}\int\limits_{\mathbb{T}^2} \psi_{1,n}^2 + \frac{|a|}{4\varepsilon_n^2}\int\limits_{\mathbb{T}^2} \psi_{2,n}^2 - C_1|b|\left(\int\limits_{\mathbb{T}^2} \frac{|1 - \mathrm{e}^{u_{1,n}}|^2}{\varepsilon_n^2}\right)^{\frac{1}{2}} \left(\int\limits_{\mathbb{T}^2} \frac{\psi_{1,n}^4}{\varepsilon_n^2}\right)^{\frac{1}{2}} \\
&\quad - C_1|a|\left(\int\limits_{\mathbb{T}^2} \frac{|1 - \mathrm{e}^{u_{2,n}}|^2}{\varepsilon_n^2}\right)^{\frac{1}{2}} \left(\int\limits_{\mathbb{T}^2} \frac{\psi_{2,n}^4}{\varepsilon_n^2}\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.32}$$

By Sobolev inequality, one gets

$$\begin{aligned}\|\psi_{i,n}\|^2_{L^4(\mathbb{T}^2)} &\le \|\psi_{i,n}\|_{L^6(\mathbb{T}^2)}\|\psi_{i,n}\|_{L^2(\mathbb{T}^2)} \\ &\le C\|\psi_{i,n}\|_{H^1(\mathbb{T}^2)}\|\psi_{i,n}\|_{L^2(\mathbb{T}^2)} \le C\|\psi_{i,n}\|_{L^2(\mathbb{T}^2)}.\end{aligned} \tag{3.33}$$

From assumption $J_n(\psi_{1,n},\psi_{2,n}) \le o(1)\sum_{i=1}^{2}\|\psi_i\|_{H^1(\mathbb{T}^2)}$ and (3.31), (3.33), we get

$$\frac{1}{\varepsilon_n^2}\int_{\mathbb{T}^2}\psi_{1,n}^2 + \frac{1}{\varepsilon_n^2}\int_{\mathbb{T}^2}\psi_{2,n}^2 \le C(\lambda) < \infty. \tag{3.34}$$

Recall the definition of $u_{i,n}$ and $\tilde{u}_{i,n}$, $i=1,2$. One has

$$\begin{aligned}\left|\mathrm{e}^{u_{1,n}} - \mathrm{e}^{\tilde{u}_{1,n}}\right| &= \left|\mathrm{e}^{\varepsilon_n^3 z_{1,n}+\sum_{i=1}^N \xi_i(\kappa z_{1i,n}+\phi_{1i,n})} - \mathrm{e}^{\varepsilon_n^3\tilde{z}_{1,n}+\sum_{i=1}^N \xi_i\phi_{1i,n}}\right| \\ &\le \mathrm{e}^{\sum_{i=1}^N \xi_i\phi_{1i,n}}\left(\left|1-\mathrm{e}^{\varepsilon_n^3 z_{1,n}+\sum_{i=1}^N \xi_i\kappa z_{1i,n}}\right| + \left|1-\mathrm{e}^{\varepsilon_n^3\tilde{z}_{1,n}}\right|\right) \\ &\le C(\kappa+\varepsilon_n^3).\end{aligned} \tag{3.35}$$

Here $\|z_{1,n}\|_{H^2(\mathbb{T}^2)}$, $\|z_{1i}\|_{H^2(\mathbb{R}^2)}$, $\|\tilde{z}_{1,n}\|_{H^2(\mathbb{T}^2)} \le 1$ and $z_{1i,n}(x) \triangleq z_{1i}((x-p_i)/\varepsilon_n)$, $\phi_{1i,n}(x) \triangleq \phi_{1i}((x-p_i)/\varepsilon_n)$, where $\phi_{1i}(x)$ is the unique topological solution of (2.1) with $N=\alpha_{1,i}$. Now we rewrite $J_n(\psi_{1,n},\psi_{2,n})$ as

$$\begin{aligned}J_n(\psi_{1,n},\psi_{2,n}) &= (1+a+b)\left(|b|\tilde{J}_{1,n}(\psi_{1,n})+|a|\tilde{J}_{2,n}(\psi_{2,n})\right) + ab\int_{\mathbb{T}^2}|\nabla\psi_{1,n}-\nabla\psi_{2,n}|^2 \\ &\quad - \frac{b(1+a+b)}{\varepsilon_n^2}\int_{\mathbb{T}^2}\left[2(1+a)(\mathrm{e}^{2u_{1,n}}-\mathrm{e}^{2\tilde{u}_{1,n}}) - (\mathrm{e}^{u_{1,n}}-\mathrm{e}^{\tilde{u}_{1,n}})\right]\psi_{1,n}^2 \\ &\quad - \frac{a(1+a+b)}{\varepsilon_n^2}\int_{\mathbb{T}^2}\left[2(1+b)(\mathrm{e}^{2u_{2,n}}-\mathrm{e}^{2\tilde{u}_{2,n}}) - (\mathrm{e}^{u_{2,n}}-\mathrm{e}^{\tilde{u}_{2,n}})\right]\psi_{2,n}^2 \\ &\quad - \frac{2ab(1+a+b)}{\varepsilon_n^2}\int_{\mathbb{T}^2}\left[\mathrm{e}^{2\tilde{u}_{1,n}}\psi_{1,n}^2 + \mathrm{e}^{2\tilde{u}_{2,n}}\psi_{2,n}^2 - \frac{1}{2}\mathrm{e}^{u_{1,n}+u_{2,n}}(\psi_{1,n}+\psi_{2,n})^2\right] \\ &\ge \tilde{c}_0|b|\|\psi_1\|_{H^1(\mathbb{T}^2)} + \tilde{c}_0|a|\|\psi_2\|_{H^1(\mathbb{T}^2)} - C(\lambda)(|a|+|b|)\kappa - Cab \\ &\ge c_0'\min(|a|,\,|b|)\end{aligned} \tag{3.36}$$

provided $|a|$, $|b|$ are small enough which yields a contradiction. In getting (3.36), we have used (3.34) and that $(\tilde{u}_{1,n},\tilde{u}_{2,n})$ is the unique topological solution of (3.25) and the strictly stable property (3.26). □

The next step is to show that the functional I_ε satisfies the compactness condition of Palais–Smale.

Lemma 3.3. *Suppose the assumptions in Lemma 3.2 are fulfilled. Let $v_{1,n}, v_{2,n} \in H^1(\mathbb{T}^2)$ be two sequences of functions satisfying*

(i) $I_\varepsilon(v_{1,n}, v_{2,n}) \to \gamma_0$ *as* $n \to \infty$*;*
(ii) $\|I'_\varepsilon(v_{1,n}, v_{2,n})\| \to 0$ *as* $n \to \infty$*.*

Then $\{(v_{1,n}, v_{2,n})\}$ admits a convergent subsequence in $H^1(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ provided $[(1+b)N_1 + aN_2][(1+a)N_2 + bN_1] \neq 0$.

Proof. By a direct computation, one gets

$$\begin{aligned}
&I'_\varepsilon(v_{1,n}, v_{2,n})(\psi_1, \psi_2)\\
&= |b|(1+b)\int_{\mathbb{T}^2} \nabla v_{1,n}\cdot\nabla\psi_1 + |a|(1+a)\int_{\mathbb{T}^2} \nabla v_{2,n}\cdot\nabla\psi_2 - ab\int_{\mathbb{T}^2} \nabla v_{1,n}\cdot\nabla\psi_2\\
&\quad - ab\int_{\mathbb{T}^2} \nabla v_{2,n}\cdot\nabla\psi_1 - \frac{(1+a+b)}{\varepsilon^2}\Bigg[|b|\int_{\mathbb{T}^2} (\mathrm{e}^{v_{1,n}+u_0^1} - (1+a)\mathrm{e}^{2v_{1,n}+2u_0^1})\psi_1\\
&\quad + |a|\int_{\mathbb{T}^2} (\mathrm{e}^{v_{2,n}+u_0^2} - (1+b)\mathrm{e}^{2v_{2,n}+2u_0^2})\psi_2 + ab\int_{\mathbb{T}^2} \mathrm{e}^{v_{1,n}+u_0^1+v_{2,n}+u_0^2}(\psi_1+\psi_2)\Bigg]\\
&\quad + 4\pi|b|[(1+b)N_1 + aN_2]\int_{\mathbb{T}^2} \psi_1 + 4\pi|a|[(1+a)N_2 + bN_1]\int_{\mathbb{T}^2} \psi_2.
\end{aligned} \tag{3.37}$$

Set $\psi_1 = \psi_2 = 1$, we get from (3.37) that

$$\int_{\mathbb{T}^2} \mathrm{e}^{2v_{1,n}+2u_0^1},\ \int_{\mathbb{T}^2} \mathrm{e}^{2v_{2,n}+2u_0^2},\ \int_{\mathbb{T}^2} \mathrm{e}^{v_{1,n}+u_0^1},\ \int_{\mathbb{T}^2} \mathrm{e}^{v_{2,n}+u_0^2},\ \int_{\mathbb{T}^2} \mathrm{e}^{v_{1,n}+u_0^1+v_{2,n}+u_0^2} \le C. \tag{3.38}$$

In what follow we use the decomposition $v_{i,n} = w_{i,n} + c_{i,n}$ with $c_{i,n} = \fint_{\mathbb{T}^2} v_{i,n}$, $i = 1, 2$. By Jessen's inequality and the convexity of function e^t, (3.38) implies that $\mathrm{e}^{c_{1,n}}, \mathrm{e}^{c_{2,n}} \le C$. Set $\psi_i = w_{i,n}$, $i = 1, 2$. Then

$$\begin{aligned}
&I'_\varepsilon(v_{1,n}, v_{2,n})(w_{1,n}, w_{2,n})\\
&= |b|(1+b)\|\nabla w_{1,n}\|_2^2 + |a|(1+a)\|\nabla w_{2,n}\|_2^2 - 2ab\int_{\mathbb{T}^2} \nabla w_{1,n}\cdot\nabla w_{2,n}\\
&\quad + \frac{1+a+b}{\varepsilon^2}\Bigg\{(1+a)|b|\int_{\mathbb{T}^2} \mathrm{e}^{2v_{1,n}+2u_0^1} w_{1,n} + (1+b)|a|\int_{\mathbb{T}^2} \mathrm{e}^{2v_{2,n}+2u_0^2} w_{2,n}
\end{aligned}$$

$$
\begin{aligned}
&\quad + ab\int_{\mathbb{T}^2} \mathrm{e}^{v_{1,n}+u_0^1+v_{2,n}+u_0^2}(w_{1,n}+w_{2,n}) + b\int_{\mathbb{T}^2} \mathrm{e}^{v_{1,n}+u_0^1} w_{1,n} + a\int_{\mathbb{T}^2} \mathrm{e}^{v_{2,n}+u_0^2} w_{2,n} \Bigg\} \\
&\geq (1+a+b)(|b|\|\nabla w_{1,n}\|_2^2 + |a|\|\nabla w_{2,n}\|_2^2) - C(\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2) \\
&\quad + \frac{1+a+b}{\varepsilon^2}\Bigg[(1+a)|b|\int_{\mathbb{T}^2} \mathrm{e}^{2(c_{1,n}+u_0^1)}(\mathrm{e}^{2w_{1,n}}-1)w_{1,n} \\
&\quad + (1+b)|a|\int_{\mathbb{T}^2} \mathrm{e}^{2c_{2,n}+2u_0^2}(\mathrm{e}^{2w_{2,n}}-1)w_{2,n} \\
&\quad + |a|\int_{\mathbb{T}^2} \mathrm{e}^{c_{1,n}+c_{2,n}+u_0^1+u_0^2}(\mathrm{e}^{w_{1,n}+w_{2,n}}-1)(w_{1,n}+w_{2,n}) + (1+a)|b|\int_{\mathbb{T}^2} \mathrm{e}^{2c_{1,n}+2u_0^1} w_{1,n} \\
&\quad + (1+b)|a|\int_{\mathbb{T}^2} \mathrm{e}^{2c_{2,n}+2u_0^2} w_{2,n} + |a|\int_{\mathbb{T}^2} \mathrm{e}^{c_{1,n}+c_{2,n}+u_0^1+u_0^2}(w_{1,n}+w_{2,n})\Bigg] \\
&\geq |b|(1+a+b)\|\nabla w_{1,n}\|_2^2 + |a|(1+a+b)\|\nabla w_{2,n}\|_2^2 - C(\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2).
\end{aligned}
$$

This implies that $\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2 \leq C$. Then from assumption (i), we have

$$
4\pi|b|[(1+b)N_1 + aN_2]c_{1,n} + 4\pi|a|[(1+a)N_2 + bN_1]c_{2,n} = O(1). \tag{3.39}
$$

Since $(1+b)(1+a) > ab$, without loss of generality, we may assume $(1+b)N_1 + aN_2 > 0$. There are two cases:

(1) $(1+a)N_2 + bN_1 > 0$. Then by (3.39), we see that $c_{1,n}, c_{2,n}$ are uniformly bounded.
(2) $(1+a)N_2 + bN_1 < 0$. Then by (3.39) and $c_{1,n}, c_{2,n} \leq C$, we have $c_{1,n}, c_{2,n}$ either go to $-\infty$ or remain uniformly bounded simultaneously. If $c_{1,n}, c_{2,n} \to -\infty$, by taking $\psi_i = \gamma_i$, $i = 1, 2$ in (3.37), we get

$$
\begin{aligned}
o(1) &= I'_\varepsilon(v_{1,n}, v_{2,n})(\gamma_1, \gamma_2) \\
&= 4\pi|b|[(1+b)N_1 + aN_2]\gamma_1 + 4\pi|a|[(1+a)N_2 + bN_1]\gamma_2 + o(1).
\end{aligned} \tag{3.40}
$$

Since γ_i are arbitrary constants, (3.40) yields a contradiction. This proves that $c_{1,n}, c_{2,n}$ are uniformly bounded.

So, by taking a subsequence, we get $v_{i,n} \rightharpoonup \bar{v}_i$, $i = 1, 2$, weakly in $H^1(\mathbb{T}^2)$, strongly in $L^p(\mathbb{T}^2)$, $p \geq 1$. By Moser–Trudinger inequality [24], we also have $\mathrm{e}^{v_{i,n}} \to \mathrm{e}^{\bar{v}_i}$ strongly in $L^p(\mathbb{T}^2)$. This yields that, $\forall\, \psi_1, \psi_2 \in H^1(\mathbb{T}^2)$, there holds

$$
I'_\varepsilon(v_{1,n}, v_{2,n})(\psi_1, \psi_2) \to I'_\varepsilon(\bar{v}_1, \bar{v}_2)(\psi_1, \psi_2) = 0.
$$

Then we see that $(\bar{v}_1, \bar{v}_2)$ is a critical point of I_ε. In fact, by taking $\psi_i = v_{i,n} - \bar{v}_i$, $i = 1, 2$, we have

$$\left|(I'_\varepsilon(v_{1,n}, v_{2,n}) - I'_\varepsilon(\bar{v}_1, \bar{v}_2))(v_{1,n} - \bar{v}_1, v_{2,n} - \bar{v}_2)\right| = o(1).$$

This will imply $(v_{1,n}, v_{2,n}) \to (\bar{v}_1, \bar{v}_2)$ strongly in $H^1(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$. $\square$

Now we are in a position to show the existence of mountain-pass solution of (3.1) for $a, b < 0$ small. Since $(1+a)(1+b) > ab$, without loss of generality, we may assume $(1+b)N_1 + aN_2 > 0$. Let $(v_{1,\varepsilon}, v_{2,\varepsilon})$ be the strictly local minimizer as in Lemma 3.2. Then $I_\varepsilon(v_{1,\varepsilon} - c, v_{2,\varepsilon}) \to -\infty$ as $c \to +\infty$. By Lemma 3.2, there exists $\rho_0 > 0$ such that $I_\varepsilon(v_1, v_2) \geq \gamma_1 > I_\varepsilon(v_{1,\varepsilon}, v_{2,\varepsilon}) = \gamma_0$, for any $\|v_1 - v_{1,\varepsilon}\|_{H^1(\mathbb{T}^2)} + \|v_2 - v_{2,\varepsilon}\|_{H^1(\mathbb{T}^2)} = \rho_0$. Also we choose c_0 large enough such that $I_\varepsilon(v_{1,\varepsilon} - c_0, v_{2,\varepsilon}) < \gamma_0$. Set

$$\mathcal{P} \triangleq \left\{ l(t) : [0,1] \to H^1(\mathbb{T}^2) \times H^1(\mathbb{T}^2) \,\middle|\, l(0) = (v_{1,\varepsilon}, v_{2,\varepsilon}),\ l(1) = (v_{1,\varepsilon} - c_0, v_{2,\varepsilon}) \right\}.$$

Then, we obtain

$$\gamma'_0 \triangleq \inf_{l \in \mathcal{P}} \sup_{(v_1, v_2) \in l} I_\varepsilon(v_1, v_2) > \gamma_0.$$

Therefore, by Lemma 3.3 and the mountain-pass lemma [2], we conclude that there exists $(v'_{1,\varepsilon}, v'_{2,\varepsilon}) \in H^1(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ such that $(v'_{1,\varepsilon}, v'_{2,\varepsilon})$ is a critical point of $I_\varepsilon(v_1, v_2)$ and $I_\varepsilon(v'_{1,\varepsilon}, v'_{2,\varepsilon}) = \gamma'_0$.

3.2.2. Case $ab = 0$

Without loss of generality, we may assume in this case $a = 0$, $b \neq 0$. In this case, we need to consider the following functional

$$I_\varepsilon(v) = \int\limits_{\mathbb{T}^2} |\nabla v|^2 + \frac{1+b}{\varepsilon^2} \int\limits_{\mathbb{T}^2} \left(\frac{1+b}{2} \mathrm{e}^{2v + 2u_0^2} - \mathrm{e}^{v + u_0^2} \right) + 4\pi N_2 \int\limits_{\mathbb{T}^2} v + \frac{b}{\varepsilon^2} \int\limits_{\mathbb{T}^2} \mathrm{e}^{u_{1,\varepsilon}}(1 - \mathrm{e}^{u_{1,\varepsilon}}) v,$$

where $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is the solution constructed in Section 3.1. The proof for this case is similar as in Section 3.2.1. Then, for this case, it is sufficient to complete the proof by observing the following two lemmas, which can be verified as previously.

Lemma 3.4. *Let the assumptions in Theorem 1.1 be fulfilled and $a = 0, b \neq 0$. Then $v_{2,\varepsilon} = u_{2,\varepsilon} - u_0^2$ is a strictly local minimizer of $I_\varepsilon(v)$.*

Lemma 3.5. *Let the assumptions in Lemma 3.4 be fulfilled. Suppose $v_n \in H^1(\mathbb{T}^2)$ is a sequence of functions satisfying:*

(i) $I_\varepsilon(v_n) \to \gamma_1$, *as* $n \to \infty$;
(ii) $\|I'_\varepsilon(v_n)\| \to 0$ *as* $n \to \infty$.

Then v_n admits a convergent subsequence in $H^1(\mathbb{T}^2)$ provided $N_2 \neq 0$.

Hence we get the existence part of Theorem 1.1.

Finally, after an integration we obtain the quantized integrals stated in Theorem 1.1, which completes our proof.

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References

[1] A.A. Abrikosov, On the magnetic properties of superconductors of the second group, Sov. Phys. JETP 5 (1957) 1174–1182.
[2] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[3] T. Aubin, Nonlinear Analysis on Manifolds: Monge–Ampére Equations, Springer, Berlin, New York, 1982.
[4] D. Bartolucci, G. Tarantello, Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory, Comm. Math. Phys. 229 (2002) 3–47.
[5] E.B. Bogomol'nyi, The stability of classical solutions, Sov. J. Nucl. Phys. 24 (1976) 449–454.
[6] L.A. Caffarelli, Y. Yang, Vortex condensation in the Chern–Simons Higgs model: an existence theorem, Comm. Math. Phys. 168 (1995) 321–336.
[7] D. Chae, O.Y. Imanuvilov, The existence of non-topological multivortex solutions in the relativistic self-dual Chern–Simons theory, Comm. Math. Phys. 215 (2000) 119–142.
[8] H. Chan, C.C. Fu, C.-S. Lin, Non-topological multi-vortex solutions to the self-dual Chern–Simons–Higgs equation, Comm. Math. Phys. 231 (2002) 189–221.
[9] R. Chen, Y. Guo, D. Spirn, Y. Yang, Electrically and magnetically charged vortices in the Chern–Simons–Higgs theory, Proc. R. Soc. Ser. A Math. Phys. Eng. Sci. 465 (2009) 3489–3516.
[10] S. Chen, X. Han, G. Lozano, F.A. Schaposnik, Existence theorems for non-Abelian Chern–Simons–Higgs vortices with flavor, J. Differential Equations 259 (2015) 2458–2498.
[11] X. Chen, S. Hastings, J.B. McLeod, Y. Yang, A nonlinear elliptic equation arising from gauge field theory and cosmology, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 446 (1994) 453–478.
[12] S.S. Chern, J. Simons, Some cohomology classes in principal fiber bundles and their application to Riemannian geometry, Proc. Natl. Acad. Sci. USA 68 (1971) 791–794.
[13] S.S. Chern, J. Simons, Characteristic forms and geometric invariants, Ann. of Math. 99 (1974) 48–69.
[14] K. Choe, Uniqueness of the topological multivortex solution in the self-dual Chern–Simons theory, J. Math. Phys. 46 (2005) 012305.
[15] K. Choe, N. Kim, C.-S. Lin, Existence of self-dual non-topological solutions in the Chern–Simons Higgs model, Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011) 837–852.
[16] S. Deser, R. Jackiw, S. Templeton, Three-dimensional massive gauge theories, Phys. Rev. Lett. 48 (1982) 975–978.
[17] S. Deser, R. Jackiw, S. Templeton, Topologically massive gauge theories, Ann. Phys. 140 (1982) 372–411.
[18] H.J. De Vega, F.A. Schaposnik, Electrically charged vortices in non-Abelian gauge theories with Chern–Simons term, Phys. Rev. Lett. 56 (1986) 2564–2566.
[19] H.J. De Vega, F.A. Schaposnik, Vortices and electrically charged vortices in non-Abelian gauge theories, Phys. Rev. D 34 (1986) 3206–3213.
[20] G. Dunne, Self-Dual Chern–Simons Theories, Lecture Notes in Physics, vol. 36, Springer, Berlin, 1995.
[21] G. Dunne, Mass degeneracies in self-dual models, Phys. Lett. B 345 (1995) 452–457.
[22] G. Dunne, R. Jackiw, S.-Y. Pi, C. Trugenberger, Self-dual Chern–Simons solitons and two-dimensional nonlinear equations, Phys. Rev. D 43 (1991) 1332–1345.
[23] F. Ezawa, Quantum Hall Effects, World Scientific, 2000.
[24] L. Fontana, Sharp borderline Sobolev inequalities on compact Riemannian manifolds, Comment. Math. Helv. 68 (1993) 415–454.
[25] J. Fröhlich, P.A. Marchetti, Quantum field theories of vortices and anyons, Comm. Math. Phys. 121 (1989) 177–223.
[26] S.B. Gudnason, Non-Abelian Chern–Simons vortices with generic gauge groups, Nuclear Phys. B 821 (2009) 151–169.
[27] S.B. Gudnason, Fractional and semi-local non-Abelian Chern–Simons vortices, Nuclear Phys. B 840 (2010) 160–185.
[28] X. Han, C.-S. Lin, G. Tarantello, Y. Yang, Chern–Simons vortices in the Gudnason model, J. Funct. Anal. 267 (2014) 678–726.

[29] X. Han, C.-S. Lin, Y. Yang, Resolution of Chern–Simons–Higgs vortex equations, Comm. Math. Phys. 343 (2016) 701–724.
[30] X. Han, G. Tarantello, Doubly periodic self-dual vortices in a relativistic non-Abelian Chern–Simons model, Calc. Var. Partial Differential Equations 49 (2014) 1149–1176.
[31] X. Han, Y. Yang, Relativistic Chern–Simons–Higgs vortex equations, Trans. Amer. Math. Soc. 368 (2016) 3565–3590.
[32] J. Hong, Y. Kim, P.-Y. Pac, Multivortex solutions of the Abelian Chern–Simons–Higgs theory, Phys. Rev. Lett. 64 (1990) 2330–2333.
[33] P.A. Horvathy, P. Zhang, Vortices in (Abelian) Chern–Simons gauge theory, Phys. Rep. 481 (2009) 83–142.
[34] G. Huang, C.-S. Lin, Uniqueness of topological solutions of self-dual Chern–Simons equation with collapsing vortices, J. Differential Equations 259 (2015) 1819–1840.
[35] R. Jackiw, K.-M. Lee, E.J. Weinberg, Self-dual Chern–Simons solitons, Phys. Rev. D 42 (1990) 3488–3499.
[36] R. Jackiw, E.J. Weinberg, Self-dual Chern–Simons vortices, Phys. Rev. Lett. 64 (1990) 2334–2337.
[37] R. Jackiw, S.-Y. Pi, Classical and quantal nonrelativistic Chern–Simons theory, Phys. Rev. D 42 (1990) 3500–3513.
[38] V.G. Kac, Infinite-Dimensional Lie Algebras, third edition, Cambridge University Press, Cambridge, 1990.
[39] D.I. Khomskii, A. Freimuth, Charged vortices in high temperature superconductors, Phys. Rev. Lett. 75 (1995) 1384–1386.
[40] K.-M. Lee, Self-dual nonabelian Chern–Simons solitons, Phys. Rev. Lett. 66 (1991) 553–555.
[41] K.-M. Lee, Relativistic non-Abelian self-dual Chern–Simons systems, Phys. Lett. B 255 (1991) 381–384.
[42] C.-K. Lee, K.-M. Lee, E.J. Weinberg, Supersymmetry and self-dual Chern–Simons systems, Phys. Lett. B 243 (1990) 105–108.
[43] G.S. Lozano, D. Marqués, E.F. Moreno, F.A. Schaposnik, Non-Abelian Chern–Simons vortices, Phys. Lett. B 654 (2007) 27–34.
[44] Y. Matsuda, K. Nozakib, K. Kumagaib, Charged vortices in high temperature superconductors probed by nuclear magnetic resonance, J. Phys. Chem. Solids 63 (2002) 1061–1063.
[45] M. Nolasco, G. Tarantello, Vortex condensates for the $SU(3)$ Chern–Simons theory, Comm. Math. Phys. 213 (2000) 599–639.
[46] S.K. Paul, A. Khare, Charged vortices in an Abelian Higgs model with Chern–Simons term, Phys. Lett. B 174 (1986) 420–422.
[47] M.K. Prasad, C.M. Sommerfield, Exact classical solutions for the ’t Hooft monopole and the Julia–Zee dyon, Phys. Rev. Lett. 35 (1975) 760–762.
[48] J.B. Sokoloff, Charged vortex excitations in quantum Hall systems, Phys. Rev. B 31 (1985) 1924–1928.
[49] M. Stone, The Quantum Hall Effect, World Scientific, Singapore, 1992.
[50] J. Spruck, Y. Yang, Topological solutions in the self-dual Chern–Simons theory: existence and approximation, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (1995) 75–97.
[51] J. Spruck, Y. Yang, The existence of non-topological solitons in the self-dual Chern–Simons theory, Comm. Math. Phys. 149 (1992) 361–376.
[52] G. Tarantello, Multiple condensate solutions for the Chern–Simons–Higgs theory, J. Math. Phys. 37 (1996) 3769–3796.
[53] G. Tarantello, Uniqueness of selfdual periodic Chern–Simons vortices of topological type, Calc. Var. Partial Differential Equations 29 (2007) 191–217.
[54] G. Tarantello, Self-Dual Gauge Field Vortices, an Analytic Approach, Progress in Nonlinear Differential Equations and Their Applications, vol. 72, Birkhäuser, Boston, Basel, Berlin, 2008.
[55] R. Wang, The existence of Chern–Simons vortices, Comm. Math. Phys. 137 (1991) 587–597.
[56] F. Wilczek, Quantum mechanics of fractional-spin particles, Phys. Rev. Lett. 49 (1982) 957–959.
[57] F. Wilczek, Fractional Statistics and Anyon Superconductors, World Scientific, Singapore, 1990.
[58] Y. Yang, The relativistic non-Abelian Chern–Simons equations, Comm. Math. Phys. 186 (1997) 199–218.
[59] Y. Yang, Solitons in Field Theory and Nonlinear Analysis, Springer, New York, 2001.