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Effective nonlinear Neumann boundary conditions for 1D nonconvex Hamilton–Jacobi equations

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Abstract

We study Hamilton–Jacobi equations in $[0, +\infty)$ of evolution type with nonlinear Neumann boundary conditions in the case where the Hamiltonian is not necessarily convex with respect to the gradient variable. In this paper, we give two main results. First, we prove for a nonconvex and coercive Hamiltonian that general boundary conditions in a relaxed sense are equivalent to *effective ones* in a strong sense. Here, we exhibit the *effective* boundary conditions while for a quasi-convex Hamiltonian, we already know them (Imbert and Monneau, 2016). Second, we give a comparison principle for a nonconvex and nonnecessarily coercive Hamiltonian where the boundary condition can have constant parts.

MSC: 49L25; 35B51; 35F30; 35F21

Keywords: Hamilton–Jacobi equations; Nonconvex Hamiltonians; Discontinuous Hamiltonians; Viscosity solutions; Comparison principle; Effective boundary conditions

1. Introduction

Let us consider the following Hamilton–Jacobi equation in $(0, T) \times [0, +\infty)$, where $T > 0$

$$\begin{cases} u_t + H(u_x) = 0 & \text{for} \quad t \in (0, T) \quad \text{and} \quad x > 0 \\ u_t + F(u_x) = 0 & \text{for} \quad t \in (0, T) \quad \text{and} \quad x = 0 \end{cases} \tag{1}$$

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subject to the initial condition

$$u(0,x) = u_0(x) \quad \text{for} \quad x \geq 0. \tag{2}$$

1.1. Main theorems

In order to state our first main theorem, we first define strong boundary nonlinearities associated to a Hamiltonian.

Definition 1.1 *(Strong boundary nonlinearity).* Let $H : \mathbb{R} \to \mathbb{R}$ be continuous and coercive, i.e.,

$$\lim_{|p|\to+\infty} H(p) = +\infty. \tag{3}$$

A function $F : \mathbb{R} \to \mathbb{R}$ is called a *strong boundary nonlinearity* for H, if F is continuous, non-increasing and semi-coercive, i.e.,

$$\lim_{p\to-\infty} F(p) = +\infty, \tag{4}$$

and satisfies

$$F(p_0) \neq H(p_0) \Longrightarrow F = F(p_0) \quad \text{on a neighborhood of } p_0.$$

Remark 1.2. The hypothesis "F is non-increasing" is necessary for the maximum principle to hold true.

Our first main theorem is about exhibiting *equivalent classes* of boundary conditions and a representative for each class, the *effective* boundary condition. These *effective* boundary conditions are exactly the strong boundary nonlinearities and are the only one to be satisfied in a strong sense.

Theorem 1.3 *(Effective boundary conditions). Assume that the Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ is continuous and coercive (3) and the function $F : \mathbb{R} \to \mathbb{R}$ is continuous, non-increasing and semi-coercive (4). Then there exists a unique strong boundary nonlinearity F_{eff} such that a function u is a viscosity solution of (1) with F if and only if u is a strong viscosity solution of (1) with F_{eff}.*

Remark 1.4. The definition of viscosity solutions and strong viscosity solutions are given in section 2. Precisely in the theorem, for u a viscosity solution of

$$u_t + H(u_x) = 0 \quad \text{in } (0,T) \times (0,+\infty),$$

u satisfies at the boundary (i.e., at $x = 0$) the following inequalities in the viscosity sense (relaxed sense)

$$u_t + \min(H(u_x), F(u_x)) \leq 0,$$

and

$$u_t + \max(H(u_x), F(u_x)) \geq 0,$$

if and only if u satisfies at the boundary the following equality in the viscosity sense (strong sense)

$$u_t + F_{\text{eff}}(u_x) = 0.$$

Remark 1.5. Many functions F are associated to the same F_{eff}. Precisely, only the Hamiltonian H and few points of the function F characterize the effective F_{eff}. The set of these points is referred to as the *set of effective points* and we define it in section 3.

Remark 1.6. This theorem is the nonconvex counterpart of [16]. Monneau [23] is developing independently a different approach for multi-dimensional junctions [17].

To understand what the set of effective points is, we comment it on an example, see Fig. 1. In the general case, F_{eff} is a non-increasing function which is "almost" the function H where each non-decreasing part is replaced by the "right constant". In the particular case of Fig. 1, the "right constants" are given by the intersections of F and the non-decreasing parts of H. So here the set of effective points associated to F is $A_F = \{p_1, p_2, p_3\}$ and the constants are $A_i = H(p_i)$ for $i = 1, 2, 3$. Theorem 1.3 implies here that taking another function $\tilde{F}$ instead of F having the same intersection with the non-decreasing parts of H gives the same viscosity solutions of (1). In other words, all these boundary conditions with F, $\tilde{F}$ and F_{eff} are in the same equivalence class.

Remark 1.7. In fact, we prove a more general result in section 3 (see Proposition 3.23). Theorem 1.3 is true if F is only continuous and non-increasing i.e., not necessarily semi-coercive (4), providing that the solution satisfies the "weak continuity" condition (12). Assuming that F is semi-coercive (4) implies that the solution satisfies (12), see Lemma 2.5. Without the "weak continuity" condition, Theorem 1.3 does not hold true. Indeed, let

$$u(t,x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{elsewhere.} \end{cases}$$

The function u does not satisfy (12). It is a viscosity solution of (1) with $H(p) = |p|$ and $F(p) = 0$ but it is not a strong viscosity solution of the corresponding (1) with F_{eff} (see section 3 for the construction of F_{eff}), here

$$F_{\text{eff}}(p) = \begin{cases} -p & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x \geq 0. \end{cases}$$

Our second main result deals with comparison principles.

Theorem 1.8 *(Comparison principles). Assume that the Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ is continuous, the function $F : \mathbb{R} \to \mathbb{R}$ is continuous, non-increasing and semi-coercive (4) and the initial datum u_0 is uniformly continuous. Moreover, assume that we have one of the following assumptions,*

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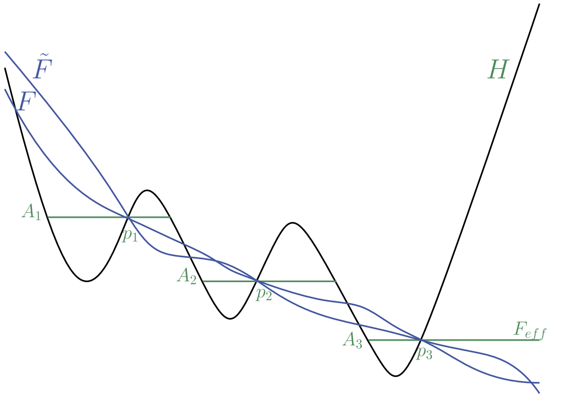

Fig. 1. Illustration of a function F_{eff} associated to F and $\tilde{F}$ in Theorem 1.3.

1. *(a noncoercive H and a "coercive" F)*

$$\lim_{p\to+\infty} F(p) = -\infty, \tag{5}$$

2. *(a coercive H and a semi-coercive F)*

$$\lim_{|p|\to+\infty} H(p) = +\infty.$$

Then for all viscosity sub-solution u and viscosity super-solution v of (1)–(2) satisfying for some $T > 0$ and $C_T > 0$,

$$u(t,x) \le C_T(1+x), \quad v(t,x) \ge -C_T(1+x), \quad \forall (t,x) \in (0,T) \times [0,+\infty),$$

we have

$$u \le v \quad \textit{in} \quad [0,T) \times [0,+\infty).$$

Remark 1.9. In fact, we have $u^* \le v_*$ (see section 2) but since $u \le u^*$ and $v_* \le v$, we get $u \le v$. So in all the following proofs, we assume that u is upper semi-continuous and v is lower semi-continuous.

Remark 1.10. As for Theorem 1.3, we can prove a more general result for the second part of the theorem. The second part is true if F is only continuous and non-increasing i.e., not necessarily semi-coercive (4) providing that sub-solutions satisfy (12).

As far as existence results are concerned, the proof of [16, Theorem 2.14] prove also the existence of a solution in our case, for a nonconvex and noncoercive Hamiltonian.

1.2. Comparison with known results

First we review known results about comparison principles. There exist many results for Hamilton–Jacobi equations with boundary conditions of Neumann type. In [21], the author studies the case of linear Neumann boundary condition. For first-order Hamilton–Jacobi equations, Barles and Lions prove a comparison principle result in [7] under a nondegeneracy condition on the boundary nonlinearity (see (6) below). The second-order case was treated by Ishii and Barles in [19,6,8]. More precisely, Barles proves in [8] a comparison principle for fully nonlinear second order, degenerate, parabolic equations, in a smooth subset Ω of $\mathbb{R}^N$, i.e.,

$$u_t + H(x, u, Du, D^2u) = 0 \text{ in } \Omega,$$

with a nonlinear Neumann boundary condition satisfying the same nondegeneracy as in [7] where it is studied for,

$$u_t + F(x, u, Du) = 0 \text{ in } \partial\Omega.$$

In this paper, we restrict ourselves to the case where H and F only depends on the gradient variable. In [8,7], considering only the gradient variable dependence, the boundary condition satisfies

$$F(p - \lambda) - F(p) \geq C\lambda, \qquad \text{for } \lambda > 0. \tag{6}$$

Here we assume a more general boundary condition, F is non-increasing, possibly with constant parts, and satisfies

$$\lim_{p\to-\infty} F(p) = +\infty \quad \text{and} \quad \lim_{p\to+\infty} F(p) = -\infty. \tag{7}$$

For example, the function $F(p) = -\text{argsh}(p)$ does not satisfy the first condition but satisfies the second one. Moreover, condition (6) is too restrictive to modify F to make it non-increasing by a density argument as in Theorem 4.1.

Dealing with convex Hamiltonians, Soner [24] and Ishii and Koike [20] prove a comparison principle for state constraint problems. For a quasi-convex Hamiltonian H, in [16] the authors prove that the following state constraint problem,

$$\begin{cases} u_t + H(u_x) &= 0 \quad \text{in } (0, T) \times (0, +\infty) \\ u_t + H(u_x) &\geq 0 \quad \text{in } (0, T) \times \{0\}, \end{cases} \tag{8}$$

is equivalent to

$$\begin{cases} u_t + H(u_x) &= 0 \quad \text{in } (0, T) \times (0, +\infty) \\ u_t + H^-(u_x) &= 0 \quad \text{in } (0, T) \times \{0\}, \end{cases} \tag{9}$$

where H^- is the decreasing part of the Hamiltonian defined by

$$H^-(p) = \inf_{q \leq p} H(q) \tag{10}$$

(this definition is also valid for H nonconvex), see also [13] for the multidimensional case, and they prove a comparison principle for (9). More generally, they give a comparison principle for (1) with a quasi-convex H and F a non-increasing function. In [22], the authors deal with nonconvex coercive Hamiltonians on junctions. In particular, they prove a comparison principle for (8) with H nonconvex. One can prove the equivalence between (8) and (9) for H nonconvex using the same methods as in [16,13] and results of this paper (see Appendix A). For a junction with many branches, one can get the same kind of equivalence of equations with the same tools. In this paper, still in the nonconvex case, we get a comparison principle for (9) which is equivalent to (8), and also more generally for (1) with F a non-increasing function (i.e., not only for $F = H^-$).

As far as effective boundary conditions are concerned, in a pioneer work Andreianov and Sbihi [3,2,4] are able to describe effective boundary conditions for scalar conservation laws. Concerning the Hamilton–Jacobi framework, first results were obtained for quasi-convex Hamiltonians by Imbert and Monneau. They treat the problem on a junction with several branches in 1D [16] and in the multi-dimensional case [17]. They prove that the effective boundary conditions are flux-limited ones

$$F_A(p) = \max(A, H^-(p)), \tag{11}$$

where H^- is the non-increasing part of the Hamiltonian H defined in (10). Still in a quasi-convex framework, the authors in [18] exhibit effective boundary conditions for degenerate parabolic equations. The nonconvex case has been out of reach so far. In this paper, we describe effective boundary conditions for a nonconvex Hamiltonian in 1D on the half-line. Monneau [23] mentioned to us that he developed a different approach dealing with N branches in the multi-dimensional case.

After Imbert and Monneau [16,17], many papers deal with the flux-limited formulation (i.e., we take $F = F_A$ in (1)) and results associated to the reduction of the set of test functions. These problems show the relevance of considering a more general class of boundary conditions than the classical state constraint problem [24,20] (i.e. considering F_A that is more general than H^-). Homogenization results using the flux-limited formulation have been recently obtained in [12,11]. Moreover, there have been numerical results for a quasi-convex Hamiltonian and a flux-limited function at the junction point. There is a convergence result for a flux-limited function at the junction point in [9]. In [15], the authors find an error estimate of order $\Delta x^{\frac{1}{3}}$ of the same scheme as in [9], and prove a convergence result for a general junction function at the junction point. This error estimate has been improved in [14] to order $\Delta x^{\frac{1}{2}}$. There are also applications in optimal control, for example in [1] where the authors study problem related to flux-limited functions.

1.3. Comments and difficulties

For the effective boundary condition result, the main difficulty was to find the good definition of strong boundary nonlinearity F_{eff} for a nonconvex coercive Hamiltonian. In [16], for a quasi-convex Hamiltonian, Imbert and Monneau prove that the effective boundary conditions are the flux-limited functions of the following form (see Fig. 2)

$$F_A(p) = \max(A, H^-(p)),$$

which are also BLN flux functions (see [5]) defined as, for $p_0 \in \mathbb{R}$,

$$F_{p_0}(p) = \begin{cases} \sup\limits_{q\in[p,p_0]} H(q) & \text{if } p \le p_0 \\ \inf\limits_{q\in[p_0,p]} H(q) & \text{if } p \ge p_0. \end{cases}$$

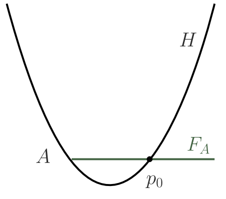

Fig. 2. Illustration of the function F_A in the convex case.

The BLN flux functions can be defined for nonconvex Hamiltonians. However, in the nonconvex case, BLN flux functions are effective boundary conditions but are not enough to cover all the effective boundary conditions as we see in section 3. For example, for an Hamiltonian with two minima (see Fig. 3), we need functions with two constant parts A_1 and A_2 like in Fig. 3, but this function is not a BLN flux function. However, it is locally a BLN function. In fact, it is exactly the "effective" boundary condition introduced in [3,2,4]. Since we only have a comparison result for the half line case, we only give the proof of the effective boundary condition result in this setting.

For the comparison principle, we tried to generalize the idea of Imbert and Monneau in [16] of the "vertex test function". In their comparison principle, they replaced the classical term $\frac{(x-y)^2}{2\varepsilon}$ by a function G called the "vertex test function" which satisfies (almost) the following condition

$$H(y, -G_y) \leq H(x, G_x),$$

which gives a contradiction combining the two viscosity inequalities. But for nonconvex Hamiltonians even for a junction with only one branch, it is very difficult to find such a "vertex test function". However, we follow the idea of coupling time and space in the doubling variable method in [10]. For example for the boundary condition $F(p) = H(0, p) = -p$, taking

$$\frac{(t-s)^2}{2\delta} + \frac{(t-s)}{\delta}(x-y) + \frac{(x-y)^2}{2\delta},$$

instead of the classical term

$$\frac{(t-s)^2}{2\delta} + \frac{(x-y)^2}{2\delta},$$

allows to get rid of the case $x = 0$ or $y = 0$ in the viscosity inequalities. In this paper, we give an example of such a function coupling time and space of the form $\delta\varphi\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right)$ which solves the problem for all boundary conditions satisfying, F is non-increasing and satisfies (7). This approach does not seem to be adaptable for a junction with several branches, that is why, this paper is written only for a half-line domain.

1.4. Organization of the paper

In section 2, we give the definition of viscosity and strong viscosity solutions. In section 3, as in [16] for quasi-convex Hamiltonians, we prove first that boundary conditions can be reduced to the *effective ones* for a nonconvex coercive Hamiltonian. Precisely, we exhibit *equivalent*

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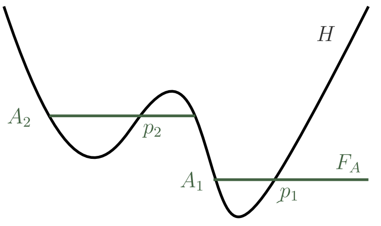

Fig. 3. Illustration of a function F_A in the nonconvex case.

classes for boundary conditions where the representative of the class is a strong boundary nonlinearity F_{eff}. Moreover, we prove that for these *effective boundary conditions* (i.e., with F_{eff}), viscosity solutions of (1) are solutions in a stronger sense: they are also strong viscosity solutions (see Definition 2.3), and this property is only true for these strong boundary nonlinearity F_{eff} (see Lemmas 3.24 and 3.25). At the end of the section, we prove the associated comparison principle. In section 4, we prove a comparison principle for a nonconvex and noncoercive Hamiltonian where the boundary condition can have constant parts.

2. Viscosity solutions

In this section, we give the definitions of viscosity solutions and strong viscosity solutions and we recall that we have a weak continuity condition for sub-solutions when F is semi-coercive (4).

A test function is a $\mathcal{C}^1$ function $\phi : (0,T) \times [0,+\infty) \to \mathbb{R}$ which touches a function u from below (resp. from above) at (t,x), i.e., $u-\phi$ reaches a local minimum (resp. maximum) at (t,x).

We recall the definition of upper and lower semi-continuous envelopes u^* and u_* of a (locally bounded) function u defined on $[0,T) \times [0,+\infty)$,

$$u^*(t,x) = \limsup_{(s,y)\to(t,x)} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{(s,y)\to(t,x)} u(s,y).$$

Definition 2.1 *(Viscosity solutions).* Let $u : [0,T) \times [0,+\infty) \to \mathbb{R}$.

i) We say that u is a *viscosity sub-solution* (resp. *viscosity super-solution*) of (1) in $(0,T) \times [0,+\infty)$ if for all test function $\phi \in \mathcal{C}^1$ touching u^* (resp. u_*) from above (resp. from below) at (t_0,x_0), we have if $x_0 > 0$,

$$\phi_t(t_0,x_0) + H(\phi_x(t_0,x_0)) \le 0 \quad (\text{resp. } \ge 0)$$

if $x_0 = 0$,

$$\begin{array}{lll} \text{either} & \phi_t(t_0,0) + H(\phi_x(t_0,0)) \le 0 & (\text{resp. } \ge 0) \\ \text{or} & \phi_t(t_0,0) + F(\phi_x(t_0,0)) \le 0 & (\text{resp. } \ge 0). \end{array}$$

ii) We say that u is a *viscosity sub-solution* (resp. *viscosity super-solution*) of (1)–(2) on $[0,T) \times [0,+\infty)$ if additionally

$$u^*(0,x) \le u_0(x) \quad (\text{resp.} \quad u_*(0,x) \ge u_0(x)) \quad \forall x \in [0,+\infty).$$

iii) We say that u is a *viscosity solution* if u is both a viscosity sub-solution and super-solution.

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Remark 2.2. It is well-known that the boundary condition has to be considered in a relaxed sense, see [21,8].

Let us give the definition of strong viscosity solutions.

Definition 2.3 *(Strong viscosity solutions).* Let $u : [0, T) \times [0, +\infty) \to \mathbb{R}$.

i) We say that u is a *strong viscosity sub-solution* (resp. *strong viscosity super-solution*) of (1) in $(0, T) \times [0, +\infty)$ if for all test function $\phi \in \mathcal{C}^1$ touching u^* (resp. u_*) from above (resp. from below) at (t_0, x_0), we have if $x_0 > 0$,

$$\phi_t(t_0, x_0) + H(\phi_x(t_0, x_0)) \leq 0 \quad (\text{resp. } \geq 0)$$

if $x_0 = 0$,

$$\phi_t(t_0, 0) + F(\phi_x(t_0, 0)) \leq 0 \quad (\text{resp. } \geq 0).$$

ii) We say that u is a *strong viscosity sub-solution* (resp. *strong viscosity super-solution*) of (1)–(2) on $[0, T) \times [0, +\infty)$ if additionally

$$u^*(0, x) \leq u_0(x) \quad (\text{resp.} \quad u_*(0, x) \geq u_0(x)) \quad \forall x \in [0, +\infty).$$

iii) We say that u is a *strong viscosity solution* if u is both a strong viscosity sub-solution and a strong viscosity super-solution.

Remark 2.4. A strong viscosity sub-solution (resp. super-solution) is obviously a viscosity sub-solution (resp. super-solution).

For the same reason as in [16], we need a weak continuity condition for sub-solutions to obtain the effective boundary condition result in section 3. More precisely, let us recall that any viscosity sub-solution satisfies automatically the “weak continuity” condition if the function F is semi-coercive (4). In fact, we recall [16, Lemma 2.3] without proving it since the proof is the same in our case.

Lemma 2.5 *(“Weak continuity” condition). Assume that the Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ is continuous and coercive (3), the function $F : \mathbb{R} \to \mathbb{R}$ is continuous, non-increasing and semi-coercive (4). Then any viscosity sub-solution u of (1) satisfies for all $t \in (0, T)$*

$$u(t, 0) = \limsup_{(s,y)\to(t,0),\, y>0} u(s, y). \tag{12}$$

3. Effective boundary conditions

In this section, we see that only the Hamiltonian H and few points of the function F characterize the boundary conditions. First, we characterize strong boundary nonlinearities by exhibiting this important set of points, the set of effective points A. We obtain a result of reduction of the set of test functions as [16] for the strong boundary nonlinearities. Then we prove the effective

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boundary condition theorem. Using the result of the fourth section, we prove that the viscosity solution of the problem (1)–(2) is unique.

In this section, the Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and coercive (3).

3.1. *Set of effective points*

In this subsection, we exhibit a set of points which characterizes the strong boundary nonlinearities. This characterization is more practical for the following proofs. First, let us give some definitions and lemmas which are used to define this set.

3.1.1. *Numbers p^- and p^+*

Definition 3.1 *(Numbers p^- and p^+).* Let $p \in \mathbb{R}$. We define

$$p^- = \sup\{q < p \mid H(q) \geq H(p)\},$$

and

$$p^+ = \inf\{q > p \mid H(q) \leq H(p)\},$$

with the convention $\inf \emptyset = +\infty$.

Remark 3.2. Since the Hamiltonian H is coercive, p^- is the supremum of a nonempty set.

Remark 3.3. Notice that if H is non-increasing on $[a, b]$, then for all $p \in [a, b]$, $p^- = p = p^+$.

We deduce the following lemma from the definition.

Lemma 3.4. *For all $p \in \mathbb{R}$, we have*

$$H(p^-) = H(p) = H(p^+).$$

Moreover, we have

$$\forall q \in]p^-, p[, \quad H(q) < H(p), \tag{13}$$

and

$$\forall q \in]p, p^+[, \quad H(q) > H(p). \tag{14}$$

Proof of Lemma 3.4. The second part of the lemma is a consequence of the definition of p^- and p^+. Let us prove the first part. By definition, we have $H(p^-) \geq H(p)$ and for all $q \in]p^-, p[$, $H(q) < H(p)$. Sending $q \to p^-$ and by continuity of H, we deduce $H(p^-) \leq H(p)$ so $H(p^-) = H(p)$. By the same arguments, we have $H(p) = H(p^+)$. $\square$

In Fig. 4, the position of H compared to $H(p)$ is illustrated.

Let us give the following useful lemma.

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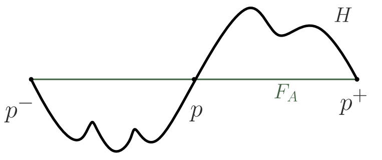

Fig. 4. Illustration of p^- and p^+ in Definition 3.1.

Lemma 3.5. *We have the following properties.*

1. *Assume* $]p^-, p[\cap]q^-, q[\neq\emptyset$. *We have* $H(p)\leq H(q)$ *if and only if* $[p^-, p]\subset[q^-, q]$ *i.e.,* $q^-\leq p^- < p\leq q$.
2. *Assume* $]p, p^+[\cap]q, q^+[\neq\emptyset$. *We have* $H(p)\leq H(q)$ *if and only if* $[q, q^+]\subset[p, p^+]$ *i.e.,* $p\leq q < q^+\leq p^+$.
3. *If* $]p^-, p[\cap]q, q^+[\neq\emptyset$, *then* $H(p) > H(q)$.

Proof of Lemma 3.5. Let us prove the first point. The second point is very similar to the first one so we skip the proof. Assume that $H(p)\leq H(q)$. If by contradiction $p > q$, then since $]p^-, p[\cap]q^-, q[\neq\emptyset$, we have $p^- < q < p$. We deduce that

$$H(q) < H(p)\leq H(q)$$

which gives a contradiction. So we deduce that $p\leq q$. Moreover, since $]p^-, p[\cap]q^-, q[\neq\emptyset$, we have $q^- < p\leq q$. Assume by contradiction that $p^- < q^-$, then

$$H(p^-) = H(p)\leq H(q) = H(q^-),$$

but $q^-\in]p^-, p[$, which gives a contradiction with Lemma 3.4. So we deduce that $[p^-, p]\subset[q^-, q]$. Assume now that $[p^-, p]\subset[q^-, q]$. In particular we have $p\in[q^-, q]$, hence $H(p)\leq H(q)$.

Let us prove the third point. Assume that

$$]p^-, p[\cap]q, q^+[\neq\emptyset, \tag{15}$$

then we have $q\leq p$. Necessarily by Lemma 3.4, we have $H(p)\geq H(q)$. If by contradiction, we have $H(p) = H(q)$, then either $q = p$ so $q^- = p^-$ or $q\leq p^-$ so $q^+\leq p^-$. But these two cases gives a contradiction with (15). So we deduce that $H(p) > H(q)$. □

3.1.2. Set of effective points and A-strong boundary nonlinearity

Definition 3.6 *(Set of effective points A).* The set A is called a *set of effective points* if A is a set of points of $\mathbb{R}$ indexed by I, $A = (p_\alpha)_{\alpha\in I}$, such that

1. $\forall\alpha\in I$, $p_\alpha^-\neq p_\alpha^+$,
2. for $\alpha_1, \alpha_2\in I$, if $p_{\alpha_1} < p_{\alpha_2}$ then $H(p_{\alpha_1})\geq H(p_{\alpha_2})$,
3. • $\forall p\in\mathbb{R}$ such that $p^- < p$, $\exists\alpha\in I$ such that $]p^-, p[\cap]p_\alpha^-, p_\alpha^+[\neq\emptyset$,
   • $\forall p\in\mathbb{R}$ such that $p < p^+$, $\exists\alpha\in I$ such that $]p, p^+[\cap]p_\alpha^-, p_\alpha^+[\neq\emptyset$.

Remark 3.7. A is not empty since the Hamiltonian H is coercive.

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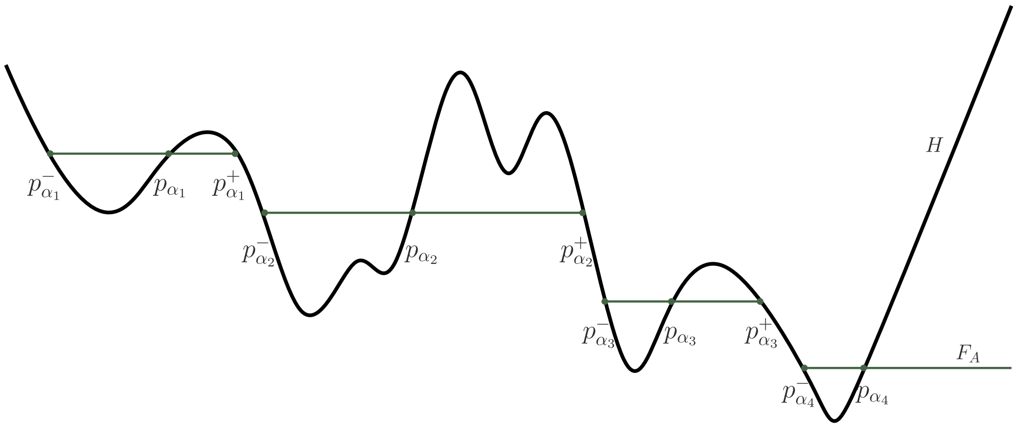

Fig. 5. Illustration of a function F_A in Definition 3.9.

We deduce the following lemma which allows to define the A-strong boundary nonlinearity.

Lemma 3.8. *If $p_1 < p_2$ and $H(p_1) \geq H(p_2)$ then we have $]p_1^-, p_1^+[\cap]p_2^-, p_2^+[= \emptyset$. In particular, the intervals $]p_\alpha^-, p_\alpha^+[$ for $\alpha \in I$ are disjoint and A is countable.*

Proof of Lemma 3.8. This lemma is a direct consequence of Lemma 3.5. □

Now we can define the A-strong boundary nonlinearity which is a characterization of strong boundary nonlinearities associated to the parameter A.

Definition 3.9 *(A-strong boundary nonlinearity F_A).* Let A be a set of effective points. The function $F_A : \mathbb{R} \to \mathbb{R}$ defined by

$$F_A(p) = \begin{cases} H(p_\alpha) & \text{if } p \in [p_\alpha^-, p_\alpha^+], \text{ for } \alpha \in I \\ H(p) & \text{elsewhere} \end{cases}$$

is called a *A-strong boundary nonlinearity*.

Proposition 3.10. *The function F_A is well-defined, continuous, non-increasing and semi-coercive (4). Moreover, F_A is a strong boundary nonlinearity.*

We give an example of a A-strong boundary nonlinearity in Fig. 5.

Proof of Proposition 3.10. Lemma 3.8 ensures that the function F_A is well-defined and Lemma 3.4 ensures that F_A is continuous. Let us prove that F_A is non-increasing. Assume by contradiction that there exists $p < q$ such that $F_A(p) < F_A(q)$. Without loss of generality, we assume that $p < q$ such that $H(p) = F_A(p) < F_A(q) = H(q)$. Indeed, if we have $p \in [p_\alpha^-, p_\alpha^+]$ for $\alpha \in I$, we also have $p_\alpha < q$ and $H(p_\alpha) = F_A(p_\alpha) = F_A(p) < F_A(q)$. We can use the same argument for q, if $q \in [p_{\alpha'}^-, p_{\alpha'}^+]$ for $\alpha' \in I$.

Let $p_1 = \inf\left\{r \geq p \mid H(r) = \frac{H(p)+H(q)}{2}\right\}$ and $q_1 = \sup\left\{r \leq q \mid H(r) = \frac{H(p)+H(q)}{2}\right\}$. We have

$$p_1^- < p < p_1 \leq q_1 < q < q_1^+,$$

and

$$H(p) < H(p_1) = H(q_1) < H(q). \tag{16}$$

Using 3. of Definition 3.6, there exists $\alpha \in I$ such that

$$]p_1^-, p_1[\, \cap\,]p_\alpha^-, p_\alpha^+[\, \neq \emptyset.$$

We distinguish two cases.

If $]p_1^-, p_1[\, \cap\,]p_\alpha^-, p_\alpha[\, \neq \emptyset$, then using 1. of Lemma 3.5, we deduce $H(p_\alpha) < H(p_1)$ and $p_\alpha < p_1$. Indeed, if by contradiction we have $H(p_\alpha) \geq H(p_1)$, then by 1. of Lemma 3.5, we deduce that $p \in [p_1^-, p_1] \subset [p_\alpha^-, p_\alpha]$. Hence, we have

$$H(p) = F_A(p) = F_A(p_\alpha) = H(p_\alpha) \geq H(p_1),$$

which gives a contradiction with (16). We deduce that

$$H(p_\alpha) = F_A(p_\alpha) < H(p_1)$$

and $[p_\alpha^-, p_\alpha] \subset [p_1^-, p_1]$ with 1. of Lemma 3.5, hence $p_\alpha < p_1$.

If $]p_1^-, p_1[\, \cap\,]p_\alpha, p_\alpha^+[\, \neq \emptyset$, then $p_\alpha < p_1$ and using 3. of Lemma 3.5, we deduce that

$$H(p_\alpha) = F_A(p_\alpha) < H(p_1).$$

By symmetric arguments, we also have $\alpha' \in I$ such that

$$H(p_{\alpha'}) = F_A(p_{\alpha'}) > H(q_1),$$

and $q_1 < p_{\alpha'}$.

Combining these conclusions, we deduce that

$$p_\alpha < p_1 < q_1 < p_{\alpha'},$$

and

$$H(p_\alpha) < H(p_1) = H(q_1) < H(p_{\alpha'}),$$

which gives a contradiction with 2. of Definition 3.6. We deduce that F_A is non-increasing.

Assume by contradiction that F_A is not semi-coercive. Since H is semi-coercive, it exists p such that for all $q < p$, $F_A(q) < H(q)$. Let $q_1 < p$ then by definition of F_A, it exists $\alpha \in I$ such that $q_1 \in [p_\alpha, p_\alpha^+]$ (since H is above F_A in this set). But by definition of F_A, we have $F_A(p_\alpha) = H(p_\alpha)$ and since $p_\alpha < p$ we get a contradiction. So F_A is semi-coercive.

Now by definition, F_A is clearly a strong boundary nonlinearity. □

Lemma 3.11. *Let A_1 and A_2 be two sets of effective points. If $A_1 \neq A_2$ then $F_{A_1} \neq F_{A_2}$.*

Proof of Lemma 3.11. Assume that $p_{\alpha_1} \in A_1$ but $p_{\alpha_1} \notin A_2$. Then $p^-_{\alpha_1} \neq p^+_{\alpha_1}$. By symmetry, assume that $p^-_{\alpha_1} < p_{\alpha_1}$. By 3. of Definition 3.6, it exists $p_{\alpha_2} \in A_2$ such that $]p^-_{\alpha_1}, p_{\alpha_1}[\cap]p^-_{\alpha_2}, p^+_{\alpha_2}[\neq \emptyset$. So we have two cases either $]p^-_{\alpha_1}, p_{\alpha_1}[\cap]p_{\alpha_2}, p^+_{\alpha_2}[\neq \emptyset$ or $]p^-_{\alpha_1}, p_{\alpha_1}[\cap]p^-_{\alpha_2}, p_{\alpha_2}[\neq \emptyset$. In the first case, let $p \in]p^-_{\alpha_1}, p_{\alpha_1}[\cap]p_{\alpha_2}, p^+_{\alpha_2}[$. By 3. of Lemma 3.5, we deduce

$$F_{A_1}(p) = H(p_{\alpha_1}) < H(p_{\alpha_2}) = F_{A_2}(p).$$

So $F_{A_1} \neq F_{A_2}$. In the second case, let $p \in]p^-_{\alpha_1}, p_{\alpha_1}[\cap]p^-_{\alpha_2}, p_{\alpha_2}[$. Assume by contradiction that $F_{A_1}(p) = F_{A_2}(p)$. Then

$$H(p_{\alpha_1}) = F_{A_1}(p_{\alpha_1}) = F_{A_1}(p) = F_{A_2}(p) = F_{A_2}(p_{\alpha_2}) = H(p_{\alpha_2}).$$

So by 1. of Lemma 3.5, necessarily $p_{\alpha_1} = p_{\alpha_2}$ which gives a contradiction. We deduce that $F_{A_1} \neq F_{A_2}$. □

Now let us prove the characterization of strong boundary nonlinearity with the set A: a strong boundary nonlinearity is in fact a A-strong boundary nonlinearity for some A.

Proposition 3.12. *Let F be a strong boundary nonlinearity. There exists a unique set of effective points A such that, $F = F_A$, i.e., F is a A-strong boundary nonlinearity.*

Proof. The uniqueness is a direct consequence of Lemma 3.11. Let us define the following set

$$A = \left\{p \in \mathbb{R} \mid \quad F(p) = H(p) \quad \text{and} \quad p^- \neq p^+\right\}.$$

Let us prove that

$$F(q) = \begin{cases} H(p) & \text{if } q \in [p^-, p^+], \text{ for } p \in A \\ H(q) & \text{elsewhere} \end{cases} \tag{17}$$

before showing that A is a set of effective points.

First, we prove that for all $p \in A$, F is constant on $[p^-, p^+]$. Let p be in A. Since F is non-increasing and $H(p^-) = H(p^+) = H(p) = F(p)$, we only have to prove that $F(p^-) = H(p^-)$ and $F(p^+) = H(p^+)$. By symmetry we only prove the first equality. Assume by contradiction that $F(p^-) \neq H(p^-)$. Then since F is non-increasing, necessarily $F(p^-) > F(p) = H(p^-)$. Let $p_1 = \inf\left\{s > p^- \mid F(s) < F(p^-)\right\}$, then by continuity of F,

$$F(p_1) = F(p^-) > F(p).$$

So $p^- < p_1 < p$ and

$$F(p_1) > F(p) = H(p) > H(p_1).$$

But F is nonconstant on a neighborhood of p_1, which gives a contradiction with Definition 1.1. We deduce that F is constant on $[p^-, p^+]$.

For $q \notin \bigcup_{p\in A}]p^-, p^+[$, let us prove that $F(q) = H(q)$. Let $q \notin \bigcup_{p\in A}]p^-, p^+[$. Assume by contradiction that $F(q) \neq H(q)$. By symmetry we assume that $F(q) < H(q)$. We define

$$a = \sup \{s < q \mid F(s) = H(s)\},$$

and

$$b = \inf \{s > q \mid F(s) = H(s)\}.$$

Notice that a is finite but b can be $+\infty$. Indeed, since F is semi-coercive (4), it cannot be constant on $(-\infty, q)$, so it exists $q_1 \in (-\infty, q)$ such that F is not constant on a neighborhood of q_1 and $H(q_1) = F(q_1)$ by Definition 1.1 so a must be finite. Necessarily, $F = F(q) < H$ on (a, b) with $F(a) = H(a)$ and $a < q < a^+ = b$. So we have $a \in A$ and $q \in \bigcup_{p\in A}]p^-, p^+[$ which gives a contradiction. So we deduce (17).

Now, let us prove that A is a set of effective points. It is clear that A satisfies 1. and 2. of Definition 3.6 since F is non-increasing. Assume by contradiction that A does not satisfy 3. of Definition 3.6. Then it exists q such that $]q^-, q[\cap \bigcup_{p\in A}]p^-, p^+[= \emptyset$ (or $]q, q^+[\cap \bigcup_{p\in A}]p^-, p^+[= \emptyset$, by symmetry we only treat the first case). So $F = H$ on $]q^-, q[$ by (17). For $p \in]q^-, q[$ we have $F(p) = H(p) < H(q) = F(q)$, by continuity of F. But F is non-increasing which gives a contradiction. We deduce that A is a set of effective points. □

Remark 3.13. Since strong boundary nonlinearities and A-strong boundary nonlinearities are the same, we use A-strong boundary nonlinearities in the following subsections.

We give the following lemma which is useful for the next subsection.

Lemma 3.14. *The function F_A satisfies the following properties,*

1. *for $\alpha \in I$, $\forall p \in]p_\alpha^-, p_\alpha[$, $F_A(p) > H(p)$,*
2. *for $\alpha \in I$, $\forall p \in]p_\alpha, p_\alpha^+[$, $F_A(p) < H(p)$,*
3. *if $p \notin \bigcup_{\alpha\in I}]p_\alpha^-, p_\alpha[\cup]p_\alpha, p_\alpha^+[$, then $F_A(p) = H(p)$.*

Proof. This result is a direct consequence of Lemma 3.4 and Definition 3.9. □

3.2. Reducing the set of test functions

With this definition of F_A, as in [16,17,13], we can prove a theorem for reducing the set of test functions for the A-strong boundary nonlinearity. We consider functions satisfying a Hamilton–Jacobi equation in $(0, +\infty)$, solution of

$$u_t + H(u_x) = 0 \quad \text{for} \quad (t, x) \in (0, T) \times (0, +\infty). \tag{18}$$

Theorem 3.15 *(Reduced set of test functions). Assume that the Hamiltonian H is continuous and coercive (3). Let A be a set of effective points. For all $\alpha \in A$, let us fix any time independent test function $\phi_\alpha(x)$ satisfying*

$$\phi'_\alpha(0) = p_\alpha.$$

Given a function $u : (0, T) \times J \to \mathbb{R}$, the following properties hold true.

i) If, for $t_0 \in (0, T)$, u is an upper semi-continuous sub-solution of (18) and satisfies (12) at $t = t_0$ and if for any test function φ touching u from above at $(t_0, 0)$ with

$$\varphi(t, x) = \psi(t) + \phi_\alpha(x) \tag{19}$$

where $\psi \in C^1(0, +\infty)$ and where $\alpha \in I$ is such that $p^-_\alpha \neq p_\alpha$, we have

$$\varphi_t + F_A(\varphi_x) \le 0 \quad at \quad (t_0, 0),$$

then u is a strong viscosity sub-solution at $(t_0, 0)$ for $F = F_A$.

ii) If for $t_0 \in (0, T)$, u is a lower semi-continuous super-solution of (18) and if for any test function φ touching u from below at $(t_0, 0)$ with

$$\varphi(t, x) = \psi(t) + \phi_\alpha(x)$$

where $\psi \in C^1(0, +\infty)$ and where $\alpha \in I$ is such that $p_\alpha \neq p^+_\alpha$, we have

$$\varphi_t + F_A(\varphi_x) \ge 0 \quad at \quad (t_0, 0),$$

then u is a strong viscosity super-solution at $(t_0, 0)$ for $F = F_A$.

Remark 3.16. We only need to consider test functions satisfying $p^-_\alpha \neq p_\alpha$ (resp. $p_\alpha \neq p^+_\alpha$) for the sub-solution (resp. super-solution) case. Indeed in $[p_\alpha, p^+_\alpha]$ (resp. $[p^-_\alpha, p_\alpha]$), the function F_A is lower (resp. upper) than H that gives directly the result, using the following Lemmas (see the following proof). We recover the result [16, Theorem 2.7 i)] for a quasi-convex Hamiltonian and for $F = F_{A_0} = H^-$ the decreasing part of the Hamiltonian. In [16], $\pi^+(A)$ is the supremum of intersection points between A and the nondecreasing part of H. In this case, the set of effective points is $A = \{\pi^+(A_0)\}$ where $H(\pi^+(A_0)) = A_0$ the minimum of H, we have $(\pi^+(A_0))^- = \pi^+(A_0)$. That is why the author doesn't need any test function for this case in [16, Theorem 2.7 i)].

To prove this result, we need the two following lemmas already proven in [16,17,13]. Here we skip the proof on these lemmas.

Lemma 3.17 *(Critical slope for sub-solution [16]). Let u be an upper semi-continuous sub-solution of (18) which satisfies (12) and let φ be a test function touching u from above at some point $(t_0, 0)$ where $t_0 \in (0, T)$. Then the critical slope given by*

$$\bar{p} = \inf\{p \in \mathbb{R} : \exists r > 0, \quad \varphi(t, x) + px \ge u(t, x), \quad \forall (t, x) \in (t_0 - r, t_0 + r) \times [0, r)\}$$

is finite, satisfies $\bar{p} \le 0$ and

$$\varphi_t(t_0, 0) + H(\varphi_x(t_0, 0) + \bar{p}) \le 0.$$

Remark 3.18. We need the "weak continuity" of sub-solutions to prove that $\bar{p}$ is finite. And we need $\bar{p}$ to be finite for the proof of Theorem 3.15.

Lemma 3.19 *(Critical slope for super-solution [16]). Let u be a lower semi-continuous super-solution of (18) and let φ be a test function touching u from below at some point $(t_0, 0)$ where $t_0 \in (0, T)$. If the critical slope given by*

$$\bar{p} = \sup\{p \in \mathbb{R} : \exists r > 0, \quad \varphi(t, x) + px \le u(t, x), \forall (t, x) \in (t_0 - r, t_0 + r) \times [0, r)\}$$

is finite, then it satisfies $\bar{p} \ge 0$ and we have

$$\varphi_t(t_0, 0) + H(\varphi_x(t_0, 0) + \bar{p}) \ge 0.$$

Proof of Proposition 3.15. We first prove the results concerning sub-solutions.

Sub-solution. Let ϕ be a test function touching u from above at $(t_0, 0)$ and let $\lambda = -\phi_t(t_0, 0)$. Let $p = \phi_x(t_0, 0)$. We want to show that

$$F_A(p) \le \lambda. \tag{20}$$

Notice that by Lemma 3.17, there exists $\bar{p} \le 0$ such that

$$H(p + \bar{p}) \le \lambda.$$

Since F_A is non-increasing, we have

$$F_A(p) \le F_A(p + \bar{p})$$

and using Lemma 3.14, if $p + \bar{p} \notin \bigcup_{\alpha \in I}]p_\alpha^-, p_\alpha[$ we have

$$F_A(p) \le F_A(p + \bar{p}) \le H(p + \bar{p}) \le \lambda,$$

which proves the result.

Now if $p + \bar{p} \in]p_\alpha^-, p_\alpha[$ for some $\alpha \in I$ such that $p_\alpha^- \ne p_\alpha$, then

$$p + \bar{p} < p_\alpha = \phi_\alpha'(0).$$

Let us consider the modified test function

$$\varphi(t, x) = \phi(t, 0) + \phi_\alpha(x) - \phi_\alpha(0).$$

We have

$$\varphi(t_0, 0) = \phi(t_0, 0) = u(t_0, 0).$$

Let us show that

$$\varphi(t, x) \ge u(t, x), \tag{21}$$

on a neighborhood of $(t_0, 0)$. We have

$$p + \bar{p} = \phi_x(t_0, 0) + \bar{p} < \phi'_\alpha(0),$$

so there exist p_1 and p_2 such that $\bar{p} < p_1 < p_2$ and which satisfy

$$p + p_i = \phi_x(t_0, 0) + p_i < \phi'_\alpha(0), \quad \forall i \in \{1, 2\}.$$

Since ϕ_x and ϕ'_α are continuous, on a neighborhood of $(t_0, 0)$, we have

$$\phi_x(t, x) + p_i < \phi'_\alpha(x), \quad \forall i \in \{1, 2\}.$$

So we have on a neighborhood of $(t_0, 0)$,

$$\begin{aligned} \phi(t, x) &= \phi(t, 0) + \int_0^x \phi_x(t, y)\mathrm{d}y \\ &= \varphi(t, x) + \phi_\alpha(0) - \phi_\alpha(x) + \int_0^x \phi_x(t, y)\mathrm{d}y \\ &= \varphi(t, x) + \int_0^x (\phi_x(t, y) - \phi'_\alpha(y))\mathrm{d}y \\ &\leq \varphi(t, x) - p_2 x, \end{aligned}$$

and by definition of $\bar{p}$, there exists a neighborhood $(t_0 - r, t_0 + r) \times [0, r)$ of $(t_0, 0)$, for some $r > 0$ such that

$$\begin{aligned} u(t, x) &\leq \phi(t, x) + p_1 x \\ &\leq \varphi(t, x) + (p_1 - p_2)x, \\ &\leq \varphi(t, x) \end{aligned}$$

so we get (21).

This test function satisfies in particular (19) so we deduce that

$$-\lambda + F_A(p_\alpha) \leq 0,$$

so we have since $p + \bar{p} \in]p_\alpha^-, p_\alpha[$ and F_A is constant in this interval,

$$F_A(p) \leq F_A(p + \bar{p}) = F_A(p_\alpha) \leq \lambda.$$

Therefore (20) holds true.

Let us prove now the super-solution case.

Super-solution. Let ϕ be a test function touching u from below at $(t_0, 0)$. Let $\lambda = -\phi_t(t_0, 0)$, and $p = \phi_x(t_0, 0)$. We want to show that

$$F_A(p) \geq \lambda. \tag{22}$$

By Lemma 3.19, if $\bar{p}$ is finite, then $\bar{p} \geq 0$ and

$$H(p+\bar{p}) \geq \lambda. \tag{23}$$

If $\bar{p} = +\infty$ then since H is coercive, the inequality (23) is true replacing $\bar{p}$ with some large $\tilde{p}$. To simplify the notations, $\bar{p}$ will denote the real number satisfying the inequality (23) in the first or the second case.

Since F_A is non-increasing, we have

$$F_A(p) \geq F_A(p+\bar{p})$$

and using Lemma 3.14, if $p+\bar{p} \notin \bigcup_{\alpha\in I}]p_\alpha, p_\alpha^+[$ we have

$$F_A(p) \geq F_A(p+\bar{p}) \geq H(p+\bar{p}) \geq \lambda,$$

which prove the result. Now if $p+\bar{p} \in]p_\alpha, p_\alpha^+[$ for some $\alpha \in I$ such that $p_\alpha \neq p_\alpha^+$, then

$$p+\bar{p} > p_\alpha = \phi'_\alpha(0).$$

As for the sub-solution case, let us consider the modified test function

$$\varphi(t,x) = \phi(t,0) + \phi_\alpha(x) - \phi_\alpha(0).$$

Arguing as in the subsolution case, we can show that φ touches u from below at $(t_0, 0)$.

This test function satisfies in particular (19) so we deduce that

$$-\lambda + F_A(p_\alpha) \geq 0,$$

so we have

$$F_A(p+\bar{p}) = F_A(p_\alpha) \geq \lambda.$$

Therefore (22) holds true. □

3.3. *Proof of the effective boundary condition result*

To prove Theorem 1.3, we first have to define the set of effective points A_F associated to the function $F : \mathbb{R} \to \mathbb{R}$ continuous, non-increasing and semi-coercive (4). In fact, we don't need F to be semi-coercive, if we assume that all sub-solutions satisfy (12), see Remark 1.7. So for Definition 3.20 and Proposition 3.21, which don't involve solutions, we don't assume that F is semi-coercive.

Definition 3.20 *(Set of effective points A_F).* The set of effective points A_F is the set of points $p \in \mathbb{R}$ such that either

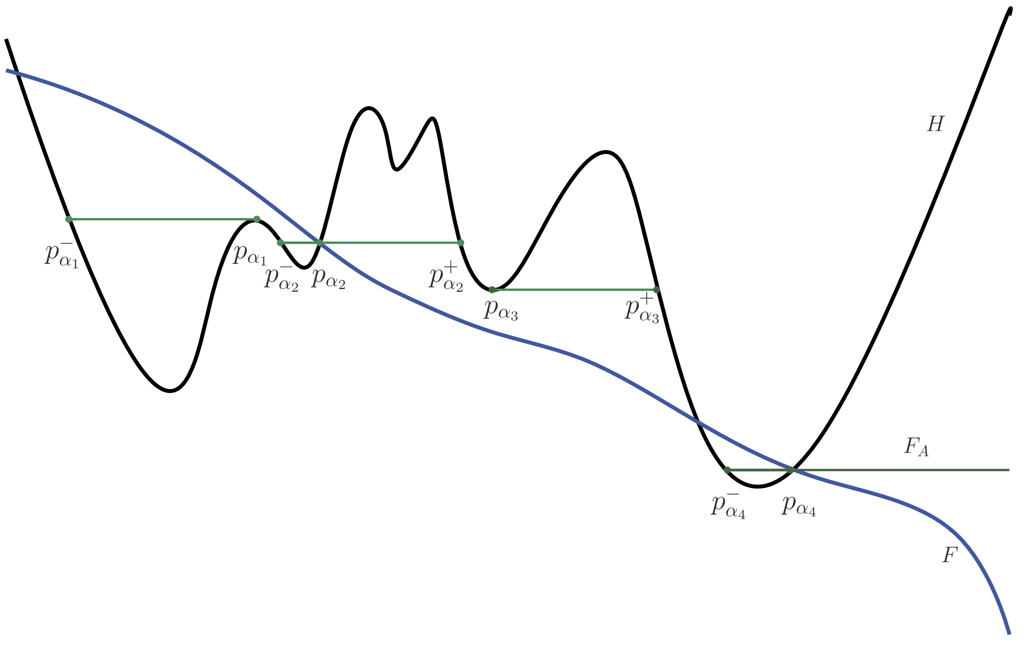

Fig. 6. Illustration of a function F_{A_F} in Definition 3.20.

$$\begin{cases} (i) & p^- \neq p, \\ (ii) & F(p) \geq H(p), \\ (iii) & \forall q \in \mathbb{R} \text{ such that } F(q) \geq H(q) \text{ and }]q^-, q^+[\, \cap\,]p^-, p[\, \neq \emptyset, \\ & \text{ we have } H(q) \leq H(p), \end{cases} \tag{24}$$

or

$$\begin{cases} (i) & p^+ \neq p, \\ (ii) & F(p) \leq H(p), \\ (iii) & \forall q \in \mathbb{R} \text{ such that } F(q) \leq H(q) \text{ and }]q^-, q^+[\, \cap\,]p, p^+[\, \neq \emptyset, \\ & \text{ we have } H(q) \geq H(p). \end{cases} \tag{25}$$

Notice that since H is coercive, A_F is not empty. We give an example of a A_F-strong boundary nonlinearity in Fig. 6. To illustrate the set A_F, one can see that in the sets where $F \geq H$, the points of A_F satisfying (24) are local maximas of H. In the sets where $F \leq H$, the points of A_F satisfying (25) are local minimas of H. The points of A_F satisfying (24) and (25) are intersection points of F with non-decreasing part of H if H has a finite number of minimas (see Fig. 6). We now show that $p^- \neq p$ or $p^+ \neq p$ for $p \in A_F$ characterizes the fact that p satisfies (24) or (25).

Proposition 3.21. *Let $F : \mathbb{R} \to \mathbb{R}$ be continuous, non-increasing, and A_F be defined as in Definition 3.20, then A_F is a set of effective points. Moreover A_F satisfies the following property. If $p \in A_F$ and $p^- \neq p$ (resp. $p^+ \neq p$) then p satisfies (24) (resp. (25)). In particular, if $p^- < p < p^+$, then $F(p) = H(p)$.*

Proof. Let us prove that A_F is a set of effective points. The set A_F satisfies 1. of Definition 3.6 since either $p^- \neq p$ or $p^+ \neq p$. Let us prove that it satisfies 2. and 3. of Definition 3.6.

Step 1: A_F satisfies 2. of Definition 3.6.
Assume by contradiction that there exist $p_1, p_2 \in A_F$ such that $p_1 < p_2$ and $H(p_1) < H(p_2)$. We distinguish four cases.

Case 1: p_1 satisfies (25), p_2 satisfies (24)
We have

$$F(p_1) \leq H(p_1) < H(p_2) \leq F(p_2).$$

But F is non-increasing, so we get a contradiction and we have $H(p_1) \geq H(p_2)$.

Case 2: p_1, p_2 satisfy (24)
Let $p = \inf\{q > p_1 \mid H(q) \geq H(p_2)\}$. We have

$$p^- < p_1^- < p_1 < p \leq p_2$$

and

$$F(p) \geq F(p_2) \geq H(p_2) = H(p) > H(p_1).$$

So p_1 does not satisfy (24) (iii) with p, that gives a contradiction.

Case 3: p_1, p_2 satisfy (25)
Let $p = \sup\{q < p_2 \mid H(q) \leq H(p_1)\}$. By symmetry with case 2, we prove that p_2 does not satisfy (25) (iii) and get a contradiction.

Case 4: p_1 satisfies (24), p_2 satisfies (25)
We have $F(p_1) \geq H(p_1)$ and $F(p_2) \leq H(p_2)$. Let us define

$$q_1 = \inf\{q \geq p_1 \mid H(q) = F(q)\},$$
$$r_1 = \inf\{q \geq p_1 \mid H(q) = H(q_1)\},$$

and

$$q_2 = \sup\{q \leq p_2 \mid H(q) = F(q)\},$$
$$r_2 = \sup\{q \leq p_2 \mid H(q) = H(q_2)\}.$$

Then if $H(r_1) = H(q_1) > H(p_1)$, we have

$$r_1^- < p_1^- < p_1 < r_1$$

and $F(r_1) \geq F(q_1) = H(q_1) = H(r_1)$. So p_1 does not satisfy (24) (iii) with r_1 that gives a contradiction. We deduce that $H(q_1) \leq H(p_1)$, so

$$H(r_2) = H(q_2) = F(q_2) \leq F(q_1) = H(q_1) \leq H(p_1) < H(p_2)$$

and we have

$$r_2 < p_2 < p_2^+ < r_2^+,$$

and $F(r_2) \leq F(q_2) = H(q_2) = H(r_2)$. So p_2 does not satisfy (25) (iii) with r_2 that gives a contradiction.

Step 2: A_F satisfies 3. of Definition 3.6.
Let $p \in \mathbb{R}$ such that $p^- \neq p^+$. We distinguish four cases.

Case 1: $p^- \neq p$ and $F(p) < H(p)$.
Let $p_1 = \sup\{q \leq p \mid H(q) = F(q)\}$ and $p_2 = \sup\left\{q \in [p_1, p] \mid H(q) = \min_{s\in[p_1,p]} H(s)\right\}$.
The number p_1 could be $-\infty$ but since H is coercive, $p_2 < +\infty$.

We are going to prove that $p_2 \in A_F$ and $]p^-, p^+[\, \cap\,]p_2^-, p_2^+[\, \neq \emptyset$. Observe first that p_2 satisfies (25) (i), (ii). Let us prove that it satisfies (25) (iii). Assume by contradiction that there exists $q \in \mathbb{R}$ such that

$$F(q) \leq H(q), \tag{26}$$

$$]q^-, q^+[\, \cap\,]p_2, p_2^+[\, \neq \emptyset \tag{27}$$

and

$$H(q) < H(p_2). \tag{28}$$

We distinguish three possibilities for q. If $q < p_1$ then using (26) and (28), we have $F(q) < H(p_2) \leq H(p_1) \leq F(p_1)$, that gives a contradiction with the fact that F is non-increasing. If $q \in [p_1, p]$ then by definition of p_2, $H(p_2) \leq H(q)$ that gives a contradiction with (28). If $q > p$ then using (28), we deduce that $q^- \geq p_2^+$ that gives a contradiction with (27). We deduce that $p_2 \in A_F$. Moreover, p_2 satisfies

$$]p^-, p[\, \cap\,]p_2^-, p_2^+[\, \neq \emptyset. \tag{29}$$

Indeed, we have for $r \in\,]p^-, p[$, $H(r) < H(p)$ by Lemma 3.4, so $H(p_2) < H(p)$ and $p_2 < p < p_2^+$.

Case 2: $p^- \neq p$ and $F(p) \geq H(p)$.
Let $p_1 = \inf\{q \geq p \mid H(q) = F(q)\}$ and $p_2 = \inf\left\{q \in [p, p_1] \mid H(q) = \max_{s\in[p,p_1]} H(s)\right\}$. We are going to prove that $p_2 \in A_F$ and satisfies (29). We have

$$p_2^- \leq p^- < p \leq p_2 \leq p_1,$$

so we deduce that p_2 satisfies (24) (i) and by definition, we deduce that p_2 satisfies (24) (ii). Let us prove that it satisfies (24) (iii). Assume by contradiction that there exists $q \in \mathbb{R}$ such that

$$F(q) \geq H(q), \tag{30}$$

q satisfies

$$]q^-, q^+[\cap]p_2^-, p_2[\neq \emptyset \tag{31}$$

and

$$H(q) > H(p_2). \tag{32}$$

We distinguish three possibilities for q. If $q > p_1$ then using (30) and (32), we have $F(q) > F(p_1)$, that gives a contradiction with the fact that F is non-increasing. If $q \in [p^-, p_1]$ then $H(p_2) \geq H(q)$ that gives a contradiction with (32). If $q < p^-$ then $q^+ \leq p_2^-$ that gives a contradiction with (31). We deduce that $p_2 \in A_F$ and satisfies (29).

Case 3: $p \neq p^+$ and $F(p) \leq H(p)$.

Using the same arguments as in cases 1 and 2 with $p_1 = \sup\{q \leq p \mid H(q) = F(q)\}$ and $p_2 = \sup\left\{q \in [p_1, p] \mid H(q) = \min_{s\in[p_1,p]} H(s)\right\}$, we deduce that $p_2 \in A_F$ and satisfies

$$]p, p^+[\cap]p_2^-, p_2^+[\neq \emptyset. \tag{33}$$

Case 4: $p \neq p^+$ and $F(p) > H(p)$.

Using the same arguments as in cases 1 and 2 with $p_1 = \inf\{q \geq p \mid H(q) = F(q)\}$ and $p_2 = \inf\left\{q \in [p, p_1] \mid H(q) = \max_{s\in[p,p_1]} H(s)\right\}$, we deduce that $p_2 \in A_F$ and satisfies (33).

Now let us prove the property of A_F. We only prove the result for $p^+ \neq p$ since it is very similar for $p^- \neq p$. If p satisfies (25), we are done. If p satisfies (24), let us prove that it also satisfies (25) in this case. By hypothesis, it satisfies (25) (i). Let us prove that it satisfies (25) (ii). Assume by contradiction that $F(p) > H(p)$. Consider p_2 defined in Step 2 Case 2. Then p_2 gives a contradiction with (24) (iii), so p satisfies (25) (ii) and $F(p) = H(p)$.

Now let us prove that p satisfies (25) (iii). Assume by contradiction that there exists $q \in \mathbb{R}$ such that

$$]q^-, q^+[\cup]p, p^+[\neq \emptyset, \tag{34}$$

$$F(q) \leq H(q) \tag{35}$$

and

$$H(q) < H(p). \tag{36}$$

We have that (35), (36) implies $H(p) = F(p) > H(q) \geq F(q)$. So since F is non-increasing, we have $q > p$ and Lemma 3.8 gives a contradiction with (34). We deduce the result. □

The next lemma shows that the set A_F associated to the function F is uniquely determined.

Lemma 3.22. *Let A_1 and A_2 be two sets of effective points. If*

$$\{u \mid u \text{ solution of (1) with } F = F_{A_1}\} = \{u \mid u \text{ solution of (1) with } F = F_{A_2}\},$$

then

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$$A_1 = A_2.$$

Proof. Assume by contradiction that $A_1 \neq A_2$. Then let $p_{\alpha_1} \in A_1$ such that $p_{\alpha_1} \notin A_2$. By 3. of Definition 3.6, there exists $p_{\alpha_2} \in A_2$ such that

$$]p_{\alpha_1}^-, p_{\alpha_1}^+[\cap]p_{\alpha_2}^-, p_{\alpha_2}^+[\neq \emptyset. \tag{37}$$

We have two cases, either $p_{\alpha_1}^- \in]p_{\alpha_2}^-, p_{\alpha_2}^+[$ or $p_{\alpha_2}^- \in]p_{\alpha_1}^-, p_{\alpha_1}^+[$. By symmetry, one can suppose that $p_{\alpha_1}^- \in]p_{\alpha_2}^-, p_{\alpha_2}^+[$. Consider $u(t,x) = -H(p)t + px$, with $p = p_{\alpha_1}^-$. For $x > 0$, the first equation in (1) is clearly satisfied. For $x = 0$, we notice that $F_{A_1}(p) = H(p_{\alpha_1})$ by Definition 3.9 of F_{A_1}. It follows that the second equation in (1) holds with $F = F_{A_1}$. Therefore u is solution of (1) with $F = F_{A_1}$. So by assumption, it is also solution of (1) with $F = F_{A_2}$. Writing the second equation in (1) with $F = F_{A_2}$, we deduce $F_{A_2}(p) = H(p)$. And by Definition 3.9, $F_{A_2}(p) = H(p_{\alpha_2})$ since $p \in]p_{\alpha_2}^-, p_{\alpha_2}^+[$. It follows $H(p_{\alpha_2}) = H(p) = H(p_{\alpha_1})$. Necessarily, since $p_{\alpha_1} \neq p_{\alpha_2}$, Lemma 3.8 gives a contradiction with (37). We deduce that $A_1 = A_2$. □

Now we can deduce the main Theorem 1.3 from the following proposition.

Proposition 3.23 *(General Neumann boundaries reduce to strong boundary nonlinearities). Assume that the Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ is continuous and coercive, the function $F : \mathbb{R} \to \mathbb{R}$ is continuous, non-increasing. Then there exists a set of effective points A_F such that*

- *any viscosity super-solution of (1) with F is a strong viscosity super-solution of (1) with F_{A_F};*
- *any viscosity sub-solution of (1) with F such that for all $T \in (0, T)$ we have (12), is a strong viscosity sub-solution with F_{A_F};*
- *any strong viscosity sub-solution (resp. super-solution) of (1) with F_{A_F} is a viscosity sub-solution (resp. super-solution) of (1) with F.*

Proof of Theorem 3.23. We first prove that viscosity sub-solutions satisfying (12) are strong viscosity sub-solutions. We only do the proof for sub-solutions since it is very similar for super-solutions. Let u be a viscosity sub-solution. Thanks to Theorem 3.15, it is enough to show that for all φ touching u^* from above at $(t, 0)$ such that $\varphi_x(t,0) = p \in A_F$, and $p^- \neq p$, we have

$$\varphi_t(t,0) + H(p) \leq 0.$$

Let φ be such a test function. Since u is a viscosity sub-solution, we have

$$\varphi_t + \min(F(p), H(p)) \leq 0.$$

Since $p^- \neq p$, Proposition 3.21 implies $F(p) \geq H(p)$ so we deduce the result.

The third point of the theorem is a direct consequence of the inequality

$$\min(F, H) \leq F_{A_F} \leq \max(F, H).$$

Indeed, if $p \in [p_\alpha^-, p_\alpha]$ where $p_\alpha \in A_F$ and $p_\alpha^- \neq p_\alpha$, using Proposition 3.21, and (24) (ii), we have

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$$F(p) \geq F(p_\alpha) \geq H(p_\alpha) = F_{A_F}(p) \geq H(p).$$

If $p \in [p_\alpha, p_\alpha^+]$ where $p_\alpha \in A_F$ and $p_\alpha^+ \neq p_\alpha$, using Proposition 3.21, and (25) (ii), we have

$$F(p) \leq F(p_\alpha) \leq H(p_\alpha) = F_{A_F}(p) \leq H(p).$$

If $p \notin \bigcup_{\alpha \in I} [p_\alpha^-, p_\alpha^+]$, then $H(p) = F_{A_F}(p)$. □

Proof of Theorem 1.3. Apply Proposition 3.23 and Lemma 3.22. □

Let us prove in two lemmas that for a A-strong boundary nonlinearity F_A, viscosity solutions and strong viscosity solutions are the same and this property is only true for A-strong boundary nonlinearities.

Lemma 3.24. *Let A be a set of effective points. The set of effective points A_{F_A} associated to the strong boundary nonlinearity F_A is the set A. In particular, a viscosity sub-solution (resp. super-solution) of (1) for $F = F_A$ is a strong viscosity sub-solution (resp. super-solution) for $F = F_A$.*

Proof of Lemma 3.24. Let us prove that $A \subset A_{F_A}$. Let $p \in A$. Without loss of generality, assume that $p^- \neq p$, so p satisfies (i) of Definition 3.20. By definition of F_A, we have $F_A(p) = H(p)$, so p satisfies (ii) of Definition 3.20. Let us prove that p satisfies (iii) of Definition 3.20. Assume by contradiction that there exists q such that $F_A(q) \geq H(q)$ and

$$]q^-, q^+[\, \cap \,]p^-, p[\, \neq \emptyset, \tag{38}$$

and

$$H(p) < H(q). \tag{39}$$

Then we deduce that

$$F_A(p) = H(p) < H(q) \leq F_A(q),$$

so $q < p$. We distinguish two cases, either $q \in \,]p^-, p[$, or $q < p^-$. The first case is not possible since q satisfies (39) which gives a contradiction with Lemma 3.4. So we have $q < p^-$. But (39) and Lemma 3.4 imply that $q^+ < p^-$, that gives a contradiction with (38). So we have $A \subset A_{F_A}$. Using Proposition 3.21, A_{F_A} is a set of effective points. Notice that if we add (resp. remove) an element to (resp. from) a set of effective points, this new set is not a set of effective points anymore. So necessarily, $A = A_{F_A}$ and we get the result. □

Now we prove that A-strong boundary nonlinearities F_A are the only continuous and non-increasing functions F such that any viscosity solutions of (1) with F satisfying (12) is in fact a strong viscosity solution of (1) with F.

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Lemma 3.25. *Assume that F is continuous, non-increasing and semi-coercive (4), and assume F is not a A-strong boundary nonlinearity, then*

$$\{u \mid u \text{ strong visc. solution of (1) with } F\} \subsetneq \{u \mid u \text{ visc. solution of (1) with } F\}.$$

Proof of Lemma 3.25. The inclusion is obvious. Let us prove that the two sets are not equal. The function F is not a A-strong boundary nonlinearity, so it exists $q \in \mathbb{R}$ such that $F(q) \neq F_{A_F}(q)$. Either $F(q) \neq F_{A_F}(q) = H(q)$ or $q \in [p_\alpha^-, p_\alpha^+]$ where $p_\alpha \in A_F$. In the first case, consider $u(t,x) = -H(q)t + qx$. In the second case, using Definition 3.20, at least one of the following inequalities holds true,

$$F(p_\alpha^-) > F_{A_F}(p_\alpha^-),$$
$$F(p_\alpha^+) < F_{A_F}(p_\alpha^+).$$

By symmetry, assume that the first one holds true. So consider $u(t,x) = -H(p)t + px$ with $p = p_\alpha^-$. The function u is a viscosity solution of (1) with F_{A_F}, so by Proposition 3.23, u is a viscosity solution of (1) with F. But u is not a strong viscosity solution of (1) with F. So we deduce the result. □

Remark 3.26. In a framework where we have uniqueness of viscosity solution of (1), for example assuming the hypothesis of Part 2 of Theorem 1.8, we see that we can have no existence of strong viscosity solution of (1) if F is not a A-strong boundary nonlinearity.

3.4. Comparison principle for a coercive Hamiltonian

Using 1. of Theorem 1.8 and Proposition 3.23, we can deduce a comparison principle for a coercive Hamiltonian, but for F only semi-coercive. Although the proof of Part 1 of Theorem 1.8 is in the next section, we prove Part 2 in this section because it is the comparison principle associated to Theorem 1.3. It implies that it exists a unique viscosity solution u of (1)–(2) satisfying $|u(t,x)| \le C_T(1+x)$ which is also the unique strong viscosity solution of (1)–(2) with $F = F_{A_F}$.

Proof of 2. of Theorem 1.8. We assume here that F is semi-coercive (4). If $F < H$ let p be any real number else we define

$$p = \sup\{q \in \mathbb{R} \mid H(q) = F(q)\},$$

and $G : \mathbb{R} \to \mathbb{R}$ a continuous function such that $G(x) \to -\infty$ when $x \to +\infty$, G satisfies $G \le F$ on $[p, +\infty[$ and $G(p) = F(p)$. We define the function $\tilde{F} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{F} = \begin{cases} F & \text{on }]-\infty, p] \\ G & \text{on } [p, +\infty[. \end{cases}$$

We have $A_F = A_{\tilde{F}}$. Indeed, notice that we have the following equivalences for F and $\tilde{F}$,

$$H(p) \le F(p) \iff H(p) \le \tilde{F}(p)$$

and

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$$H(p) \geq F(p) \iff H(p) \geq \tilde{F}(p).$$

Since in the definition of A_F, only the relative position between F and H takes the function F into account, the previous equivalences give the result. So we deduce using Proposition 3.23 that a function u is a viscosity sub-solution (resp. super-solution) for F if and only if u is a strong viscosity sub-solution (resp. super-solution) for F_{A_F}, if and only if u is a viscosity sub-solution (resp. super-solution) for $\tilde{F}$. We deduce the comparison principle for F using the comparison principle for $\tilde{F}$ (1. of Theorem 1.8). □

Remark 3.27. In Remark 1.10, we say that we don't need F to be semi-coercive (4), providing that sub-solutions satisfy (12). Using the same arguments as in the previous proof, if F is not semi-coercive, we define $\bar{p} \leq p$ such that if $F < H$, $\bar{p}$ is any real number satisfying $\bar{p} \leq p$, else

$$\bar{p} = \inf\{q \in \mathbb{R} \mid H(q) = F(q)\}.$$

We define $\tilde{F} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{F} = \begin{cases} \bar{G} & \text{on }]-\infty, \bar{p}] \\ F & \text{on } [\bar{p}, p] \\ G & \text{on } [p, +\infty[, \end{cases}$$

where $\bar{G} : \mathbb{R} \to \mathbb{R}$ a continuous function such that $\bar{G}(x) \to +\infty$ when $x \to -\infty$, $\bar{G}$ satisfies $F \leq \bar{G} \leq H$ on $[p, +\infty[$ and $\bar{G}(\bar{p}) = F(\bar{p})$. Then the following of the proof is the same as the previous proof.

4. Comparison principle for nonconvex and noncoercive Hamilton–Jacobi equations allowing constant parts

In this section, we prove the first main comparison principle 1. of Theorem 1.8 for a nonconvex and noncoercive Hamiltonian where the boundary condition allows constant parts. The proof follows the idea of coupling time and space in the doubling variable method in [10]. First, we give a restricted version of the theorem which easily implies the main theorem. Then we prove the theorem assuming the existence of a class of test function which satisfies some properties. Finally, we give an example of such a test function so that the theorem is proven.

4.1. Simplification of the theorem

Let us prove a restricted version of 1. of Theorem 1.8 where the function F satisfies more hypotheses.

Theorem 4.1 *(Restricted comparison principle). Assume that the Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ is continuous, the function $F : \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^1$ and satisfies $F' < 0$, $F(0) = 0$ and (4)–(5), and the initial datum u_0 is uniformly continuous. Then for all viscosity sub-solution u and viscosity super-solution v of (1)–(2) satisfying for some $T > 0$ and $C_T > 0$,*

$$u(t, x) \leq C_T(1 + x), \quad v(t, x) \geq -C_T(1 + x), \quad \forall (t, x) \in (0, T) \times [0, +\infty),$$

we have

$$u \le v \quad in \quad [0,T)\times[0,+\infty).$$

Proof of 1. of Theorem 1.8 using Theorem 4.1. It is enough to assume $F(0)=0$ as in [16, Lemma 3.1], by defining

$$u(t,x)=\tilde{u}(t,x)-tF(0) \quad \text{and} \quad v(t,x)=\tilde{v}(t,x)-tF(0)$$

and $\tilde{F}=F-F(0)$, $\tilde{H}=H-F(0)$. The function u (resp. v) is a sub-solution (resp. super-solution) of (1) if and only if $\tilde{u}$ (resp. $\tilde{v}$) is a sub-solution (resp. super-solution) of (1) replacing H by $\tilde{H}$ and F by $\tilde{F}$. Let the function F be such that $F(0)=0$ and satisfy the hypothesis of 1. of Theorem 1.8, i.e. a continuous and non-increasing function which satisfies (4)–(5). By density, one can approximate F by a sequence F_n satisfying

$$\|F_n-F\|_\infty \le \frac{1}{n} \quad \forall n\in\mathbb{N}^*,$$

with the hypothesis of Theorem 4.1, i.e. of class $\mathcal{C}^1$ and decreasing such that $F'<0$ which satisfies (4)–(5). Let u be a sub-solution of (1) with the function F. Let us define $u_n=u(x)-\frac{t}{n}$ which is a sub-solution of (1) with the function F_n and $v_n=v(x)+\frac{t}{n}$ which is a super-solution of (1) with the function F_n. Using Theorem 4.1, we deduce

$$u(t,x)-\frac{t}{n}\le v(t,x)+\frac{t}{n} \quad \forall (t,x)\in[0,T)\times[0,+\infty).$$

Sending n to $+\infty$, we deduce the result. $\square$

4.2. The coupling time and space test function

Let us define the norm $|(.,.)|:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$|(t,x)|=\sqrt{t^2+x^2}.$$

Theorem 4.2 *(Coupling time and space test function). Assume the function $F:\mathbb{R}\to\mathbb{R}$ is of class $\mathcal{C}^1$ and satisfies $F'<0$, $F(0)=0$ and (4)–(5). Then there exists a function $\varphi:\mathbb{R}^2\to\mathbb{R}$ of class $\mathcal{C}^1$ which satisfies the following properties.*

1. *(Superlinearity)*

$$\lim_{|(t,x)|\to+\infty}\frac{\varphi(t,x)}{1+|(t,x)|}=+\infty. \tag{40}$$

2. *(Bounded from below)*

$$\forall (t,x)\neq(0,0), \quad \varphi(t,x)>\varphi(0,0)=0. \tag{41}$$

3. *(Differential inequalities) For all $t\in\mathbb{R}$,*

$$\begin{cases} \varphi_t(t,x)+F(\varphi_x(t,x)) & \ge 0 \quad \text{if } x\le 0,\\ \varphi_t(t,x)+F(\varphi_x(t,x)) & \le 0 \quad \text{if } x\ge 0. \end{cases} \tag{42}$$

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Remark 4.3. We first admit this theorem to prove the comparison principle and we show it in the next subsection. The idea of the proof is to replace in the doubling variable method, the usual term $\frac{(t-s)^2}{2\delta}+\frac{(x-y)^2}{2\varepsilon}$ by $\delta\varphi\left(\frac{t-s}{\delta},\frac{x-y}{\delta}\right)$ which prevents the following supremum to be reached at the boundary.

4.3. Proof of the comparison principle

Let us recall [16, Lemma 3.4] since we use it in the proof. The proof of this lemma is exactly the same as in [16] so we skip it.

Lemma 4.4 *(A priori control). Let $T>0$ and let u be a sub-solution and v be a super-solution as in Theorem 4.1. Then there exists a constant $C=C(T)>0$ such that for all $(t,x),(s,y)\in[0,T)\times[0,+\infty)$, we have*

$$u(t,x)\le v(s,y)+C(1+|x-y|).$$

Proof of Theorem 4.1. The proof proceeds in several steps.

Step 1: Penalization procedure. As explain in Remark 1.9, we assume that u is upper semi-continuous and v is lower semi-continuous. We want to prove that

$$M=\sup_{(t,x)\in[0,T)\times[0,+\infty)}(u(t,x)-v(t,x))\le 0.$$

Assume by contradiction that $M>0$. Let us define

$$M_{\delta,\alpha}=\sup_{(t,x),(s,y)\in[0,T)\times[0,+\infty)}\left\{u(t,x)-v(s,y)-\delta\varphi\left(\frac{t-s}{\delta},\frac{x-y}{\delta}\right)\right.$$
$$\left.-\frac{\eta}{T-t}-\frac{\eta}{T-s}-\frac{\alpha x^2}{2}\right\}$$

where δ,η,α are positive constants. Then for α,η small enough, we have $M_{\delta,\alpha}\ge\frac{M}{2}>0$. Indeed, by definition of the supremum M, there exists $(t_0,x_0)\in[0,T)\times[0,+\infty)$ such that

$$u(t_0,x_0)-v(t_0,x_0)\ge\frac{3M}{4},$$

so

$$M_{\delta,\alpha}\ge u(t_0,x_0)-v(t_0,x_0)-\frac{2\eta}{T-t_0}-\alpha\frac{x_0^2}{2}\ge\frac{M}{2},$$

for α,η small enough. We want to show that this supremum is reached. For all x,y,t,s such that

$$0<\frac{M}{2}\le u(t,x)-v(s,y)-\delta\varphi\left(\frac{t-s}{\delta},\frac{x-y}{\delta}\right)-\frac{\eta}{T-t}-\frac{\eta}{T-s}-\alpha\frac{x^2}{2},\tag{43}$$

by Lemma 4.4, we have

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$$0 < \frac{M}{2} \le C_T(1+|x-y|) - \delta\varphi\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) - \frac{\eta}{T-t} - \frac{\eta}{T-s} - \alpha\frac{x^2}{2}, \tag{44}$$

so we deduce that

$$\delta\varphi\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) \le C_T(1+|x-y|), \tag{45}$$

and that

$$\frac{(\alpha x)^2}{2} \le \alpha C_T(1+|x-y|) \tag{46}$$

By dividing (45) by $1+|(t-s, x-y)|$, the property (40) of φ implies that $x-y$ and $t-s$ are bounded, independently of α, for x, y, t, s satisfying (43). So using (46), x, y, t, s are in a compact set so the supremum $M_{\delta,\alpha}$ is reached at some point $(t, x, s, y) = (t_\delta, x_\delta, s_\delta, y_\delta)$. Moreover, for $\delta \to 0$, using any converging subsequence and (45) dividing by $1+|(t-s, x-y)|$, using the property (40) and (41), we deduce that, $t_\delta - s_\delta$ and $x_\delta - y_\delta$ go to 0.

Step 2: Use of the initial condition. If t_δ or $s_\delta = 0$ along a subsequence then $t_\delta, s_\delta \to 0$ as $\delta \to 0$ by the previous step so, up to extract once again, $(x_\delta, y_\delta) \to (x_0, x_0)$. So we get from (43),

$$0 < \frac{M}{2} \le u(t_\delta, x_\delta) - v(s_\delta, y_\delta).$$

So letting $\delta \to 0$, the limit superior of the right hand side is smaller than $u_0(x_0) - u_0(x_0) = 0$ and we get a contradiction.

Step 3: Use of viscosity inequalities. We can now assume that $t_\delta > 0$ and $s_\delta > 0$ and write the viscosity inequalities at $(t, x, s, y) = (t_\delta, x_\delta, s_\delta, y_\delta)$.

Case 1: If $x = 0$ and $\min(H, F) = F$ at $\varphi_x\left(\frac{t-s}{\delta}, \frac{-y}{\delta}\right)$.

The inequality for the sub-solution is

$$\frac{\eta}{(T-t)^2} + \varphi_t\left(\frac{t-s}{\delta}, \frac{-y}{\delta}\right) + F\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{-y}{\delta}\right)\right) \le 0.$$

Using property (42), we get a positive left-hand side which gives a contradiction.

Case 2: If $y = 0$ and $\max(H, F) = F$ at $\varphi_x\left(\frac{t-s}{\delta}, \frac{x}{\delta}\right)$.

The inequality for the super-solution is

$$-\frac{\eta}{(T-s)^2} + \varphi_t\left(\frac{t-s}{\delta}, \frac{x}{\delta}\right) + F\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{x}{\delta}\right)\right) \ge 0.$$

Using property (42), we get a negatif left-hand side which gives a contradiction.

Case 3: Other cases

The inequality for the sub-solution is

$$\frac{\eta}{(T-t)^2} + \varphi_t\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) + H\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) + \alpha x\right) \le 0,$$

and the inequality for the super-solution is

$$-\frac{\eta}{(T-s)^2} + \varphi_t\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) + H\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right)\right) \ge 0.$$

Substracting these inequalities, we get

$$\frac{2\eta}{T^2} \le H\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right)\right) - H\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) + \alpha x\right). \tag{47}$$

Since $t-s$ and $x-y$ are bounded independently of α and since αx goes to 0 when $\alpha \to 0$, thanks to (46), using the fact that H is uniformly continuous in compact subsets, the right hand side of (47) goes to 0 when $\alpha \to 0$, we get a contradiction. The proof is now complete. □

4.4. Construction of the test function

The idea is to construct a test function coupling time and space, of the form

$$\varphi(t,x) = f(t) + g(x) + xE(t),$$

where the functions $f, g, E : \mathbb{R} \to \mathbb{R}$ are of class $\mathcal{C}^1$. In this section, the function F satisfies the hypothesis of Theorem 4.1. Let us first define a function G, we will next use it to define the function E.

Definition 4.5 *(Function G).* Let G be a continuous function such that

- $G \ge \max((-F^{-1})', (-2F)^{-1}) > 0$,
- G is even i.e. $\forall t \in \mathbb{R}, \quad G(-t) = G(t)$,
- G is non-increasing in $(-\infty, 0]$ and non-decreasing on $[0, +\infty)$.

Remark 4.6. The function G exists since $\max((-F^{-1})', (-2F)^{-1})$ is continuous and $(-F^{-1})'$ is positive. Moreover, we have

$$\lim_{x \to \pm\infty} G(x) = +\infty,$$

since $(-2F)^{-1}$ is increasing and goes to $+\infty$ at $+\infty$.

Proposition 4.7 *(Function E). Assume F is of class $\mathcal{C}^1$ and satisfies $F' < 0$, $F(0) = 0$ and (4)–(5). Then there exists a function E of class $\mathcal{C}^1$ solution of the ODE*

$$\begin{cases} E' = \frac{1}{G(-2F(E))} \\ E(0) = 0, \end{cases} \tag{48}$$

which satisfies the same properties as $-F$, *i.e.,* $E' > 0$, $E(0) = 0$ *and*

$$\lim_{x\to-\infty} E(x) = -\infty \quad \text{and} \quad \lim_{x\to+\infty} E(x) = +\infty. \tag{49}$$

Moreover, we have

$$\lim_{x\to\pm\infty} E'(x) = 0. \tag{50}$$

Proof of Proposition 4.7. The existence of a solution for (48) is given by Cauchy–Peano–Arzela global existence theorem. Indeed, since $0 < (-F^{-1})'(0) \le G$, we have $0 < \frac{1}{G} \le \frac{1}{(-F^{-1})'(0)}$ so the function

$$\frac{1}{G(-2F)}$$

is bounded and continuous. Moreover, since $G \ge (-F^{-1})' > 0$, we have $E' > 0$. Let us prove that E satisfies (49) by contradiction. If E has a finite limit then using (48), E' has a finite limit $L > 0$ so

$$E(t) \sim Lt$$

and E has an infinite limit which is a contradiction. We deduce (50) using (48). □

Let us define the function f.

Definition 4.8 *(Function f).* Let f be the function of class $\mathcal{C}^1$ such that $f'(t) = -F(E(t))$ and $f(0) = 0$.

Let us define the function g. First, we define some functions ψ, ψ_1 and ψ_2,

$$\psi(t,x) = -F^{-1}(xE'(t) - F(E(t)) - E(t),$$
$$\psi_1(x) = \sup_{t\in\mathbb{R}} \psi(t,x),$$
$$\psi_2(x) = \inf_{t\in\mathbb{R}} \psi(t,x).$$

Proposition 4.9. *The function ψ_1 is lower semi-continuous and locally bounded in* $[0,+\infty)$, *continuous at* 0 *and satisfies* $\psi_1(0) = 0$. *The function ψ_2 is upper semi-continuous and locally bounded in* $(-\infty, 0]$, *continuous at* 0 *and satisfies* $\psi_2(0) = 0$.

Proof of Proposition 4.9. The function ψ_1 (resp. ψ_2) is lower (resp. upper) semi-continuous because it is a supremum (resp. infimum) of continuous functions.

Let us prove that ψ_1 and ψ_2 are locally bounded and continuous at 0. By using the Taylor expansion of the function $-F^{-1}$ of class $\mathcal{C}^1$, there exists $\theta : \mathbb{R}^2 \to [0, 1]$ such that

$$\psi(t,x) = xE'(t)(-F^{-1})'(-F(E(t)) + \theta(t,x)xE'(t)).$$

If $0 \le x \le R$, for $R > 0$, since $G \ge (-F^{-1})' > 0$, we have

$$\begin{aligned} 0 \le \psi(t,x) &\le xE'(t)G(-F(E(t)) + \theta(t,x)xE'(t)) \\ &\le xE'(t)G(-F(E(t)) + RE'(t)). \end{aligned} \tag{51}$$

Let us prove that the continuous function $h : t \to E'(t)G(-F(E(t)) + RE'(t))$ is bounded in $\mathbb{R}$. Since h is continuous, we only need to prove that h is bounded for $|t|$ big enough. Using (50), for $t \ge 0$ big enough, we have $RE'(t) \le 1$ and $-F(E(t)) + 1 \le -2F(E(t))$. Using that G is non-decreasing in $[0, +\infty)$, we deduce from (48) that

$$0 \le h(t) \le E'(t)G(-F(E(t)) + 1) \le \frac{G(-F(E(t)) + 1)}{G(-2F(E(t)))} \le 1.$$

By the same argument, for $t \le 0$ small enough, we have $RE'(t) \ge -1$ and $-F(E(t)) - 1 \ge -2F(E(t))$. So since G is non-increasing in $(-\infty, 0]$, we deduce with (48) that

$$0 \le h(t) \le E'(t)G(-F(E(t)) - 1) \le 1.$$

We deduce from (51) that ψ_1 is locally bounded in $[0, +\infty)$ and that $\psi_1(0) = 0$. By the same arguments, we also deduce that ψ_2 is locally bounded in $(-\infty, 0]$ and that $\psi_2(0) = 0$. The proof is now complete. □

Lemma 4.10 *(Function g). Let g be a function of class $\mathcal{C}^1$ such that $g(0) = 0$ and such that g' satisfies $g'(0) = 0$ and*

$$g'(x) \ge \max(2x, \psi_1(x)) \quad \textit{for } x \ge 0,$$

and

$$g'(x) \le \min(2x, \psi_2(x)) \quad \textit{for } x \le 0.$$

Proof. The construction of the function g' is a consequence of the fact that ψ_1 and ψ_2 are locally bounded and continuous at 0. □

Now, we can prove that the function φ defined by $\varphi(t,x) = f(t) + g(x) + xE(t)$ satisfies (42).

Proposition 4.11. *The function $\varphi(t,x) = f(t) + g(x) + xE(t)$ satisfies (42).*

Proof of Proposition 4.11. Since the function g satisfies for all $t \in \mathbb{R}$,

$$g'(x) \ge \psi_1(x) \ge \psi(t,x) = (-F^{-1})(xE'(t) - F(E(t)) - E(t) \quad \text{for } x \ge 0,$$

and

$$g'(x) \le \psi_2(x) \le \psi(t,x) = (-F^{-1})(xE'(t) - F(E(t)) - E(t) \quad \text{for } x \le 0,$$

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and since $-F^{-1}$ is increasing, we deduce that

$$-F(E(t)) + xE'(t) + F(g'(x) + E(t)) \leq 0 \quad \text{for } x \geq 0,$$

and

$$-F(E(t)) + xE'(t) + F(g'(x) + E(t)) \geq 0 \quad \text{for } x \leq 0.$$

These inequalities are exactly (42). □

Let us prove that the function φ satisfies (40) and (41).

Proposition 4.12. *The function φ is of class $\mathcal{C}^1$ and superlinear (40).*

Proof of Proposition 4.12. By construction, the function φ is of class $\mathcal{C}^1$. With the definition of g in hand, we deduce that $g(x) \geq x^2$. Using that

$$|xE(t)| \leq \frac{x^2}{2} + \frac{E(t)^2}{2},$$

we deduce that

$$\begin{aligned} \varphi(t,x) &\geq f(t) + x^2 - \tfrac{E(t)^2}{2} - \tfrac{x^2}{2}, \\ &\geq f(t) - \tfrac{E(t)^2}{2} + \tfrac{x^2}{2}. \end{aligned} \tag{52}$$

Let us prove that $\frac{E^2}{2f}$ goes to 0 when $|t| \to +\infty$. We first compare their derivative which are simpler. We have

$$\begin{aligned} \frac{2f'(t)}{(E^2)'(t)} = \frac{-F(E(t))}{E'(t)E(t)} &= \frac{-F(E(t))G(-2F(E(t)))}{E(t)}, \\ &\geq \frac{-F(E(t))(-2F)^{-1}(-2F(E(t)))}{E(t)} \\ &\geq -F(E(t)). \end{aligned} \tag{53}$$

where the last term goes to $+\infty$ as t goes to $+\infty$. We have the same result for $t \leq 0$ using the same argument and the fact that G is even,

$$\frac{2f'(t)}{(E^2)'(t)} \geq F(E(t)),$$

where the last term goes to $+\infty$ as t goes to $-\infty$. We deduce that

$$\frac{(E^2)'(t)}{f'(t)} \to 0 \quad \text{for } t \to \pm\infty.$$

Since $\int_0^t E^{2'}(s)\,\mathrm{d}s = E^2(t)$ diverges when $t \to \pm\infty$, we have

$$\frac{\int_0^t (E^2)'(s)\,\mathrm{d}s}{\int_0^t f'(s)\,\mathrm{d}s} \to 0,$$

so

$$\frac{E(t)^2}{f(t)} \to 0 \quad \text{for } t \to \pm\infty.$$

And since f is superlinear (40), $t \to f(t) - \frac{E(t)^2}{2}$ is superlinear. We deduce, from (52) that φ satisfies (40).

Proposition 4.13. *The function φ satisfies (41).*

Proof of Proposition 4.13. The function φ is of class $\mathcal{C}^1$, satisfies $\varphi(0,0) = 0$ and is superlinear (40) in (t, x). Let us prove that its local extremum is reached only at the point $(0, 0)$ and this implies (41). Let $(t, x) \in \mathbb{R}^2$ satisfy,

$$\begin{cases} \varphi_t(t,x) & = & -F(E(t)) + xE'(t) & = & 0 \\ \varphi_x(t,x) & = & g'(x) + E(t) & = & 0. \end{cases} \tag{54}$$

First, we notice that for (t, x) satisfying (54), $t = 0$ if and only if $x = 0$. Let us prove that $t = 0$ as soon as $x > 0$ and (t, x) satisfies (54). If $x > 0$, we have taking $s = 0$

$$\begin{aligned} -E(t) = g'(x) & \geq \sup_{s\in\mathbb{R}} \left\{(-F^{-1}(xE'(s) - F(E(s))) - E(s)\right\} \\ & \geq -F^{-1}(xE'(0)), \end{aligned}$$

so we have

$$E(t) \leq F^{-1}(xE'(0)).$$

And we also have, since F is decreasing,

$$xE'(t) = F(E(t)) \geq F(F^{-1}(xE'(0))) = xE'(0).$$

If $t \geq 0$, since E' is non-increasing in $[0, +\infty)$, we deduce that $t \leq 0$ so $t = 0$ and $x = 0$, which gives a contradiction. If $t \leq 0$, since E' is non-decreasing, we deduce that $t \geq 0$ so $t = 0$ and $x = 0$, which also gives a contradiction. The case $x < 0$ is similar so we skip it. This ends the proof. □

Proof of Theorem 4.2. Combine Propositions 4.11, 4.12 and 4.13. □

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Appendix A. Reformulation of state constraints

Let us prove the reformulation of state constraint result in the case where the Hamiltonian is not necessarily convex.

Theorem A.1 *(Reformulation of state constraints). Assume $H : \mathbb{R} \to \mathbb{R}$ is continuous and coercive (3) and $u : (0,T) \times [0,+\infty) \to \mathbb{R}$ satisfies (12) then u is a viscosity solution of*

$$\begin{cases} u_t + H(u_x) = 0 & \text{in} \quad (0,T) \times (0,+\infty) \\ u_t + H(u_x) \geq 0 & \text{in} \quad (0,T) \times \{0\}, \end{cases} \tag{55}$$

if and only if u is a viscosity solution of

$$\begin{cases} u_t + H(u_x) = 0 & \text{in} \quad (0,T) \times (0,+\infty) \\ u_t + H^-(u_x) = 0 & \text{on} \quad (0,T) \times \{0\}, \end{cases} \tag{56}$$

where H^- is the decreasing part of the Hamiltonian defined by

$$H^-(p) = \inf_{q \leq p} H(q).$$

First we prove that $F_{A_{H^-}} = H^-$ that allows us to use Theorem 3.15 of reduction of the set of test functions.

Definition A.2 *(Set of effective points A_0).* Let $H : \mathbb{R} \to \mathbb{R}$ be continuous and coercive (3). The set of effective points A_0 is the set of points $p \in \mathbb{R}$ such that

- $p^- = p < p^+$,
- $\forall q \in \mathbb{R}$ such that $]q^-, q^+[\cap]p, p^+[\neq \emptyset$, we have $H(q) \geq H(p)$.

Lemma A.3. *We have $A_{H^-} = A_0$.*

Proof of Lemma A.3. Notice first that $H^- \leq H$ and that H^- is non-increasing. Using Definition 3.20, it only remains to prove that for all $p \in A_{H^-}$ we have $p^- = p$. Assume by contradiction that there exists $p \in A_{H^-}$ such that $p^- < p$. Then using Proposition 3.21 we deduce that p satisfies (ii) of (24) so $H(p) = H^-(p)$. We deduce from Lemma 3.4 that

$$\forall q \in]p^-, p[\quad H^-(q) \leq H(q) < H(p) = H^-(p),$$

but H^- is non-increasing which gives a contradiction. So we have $p^- = p$. We deduce that $A_{H^-} = A_0$. $\square$

Lemma A.4. *We have $F_{A_{H^-}} = F_{A_0} = H^-$.*

Proof of Lemma A.4. From Lemma A.3, we deduce that $F_{H^-} = F_{A_0}$. Let us prove that $F_{A_0} = H^-$. Notice first that

$$F_{A_0} \leq H. \tag{57}$$

Let $p \in \mathbb{R}$.

If there exists $p_\alpha \in A_0$ such that $p \in]p_\alpha, p_\alpha^+[$ then we have

$$H^-(p) \le F_{A_0}(p) = H(p_\alpha).$$

Moreover, from Lemma 3.4 we have

$$\forall q \in]p_\alpha, p[\quad H(p_\alpha) < H(q)$$

and since F_{A_0} is non-increasing and by (57), we have also

$$\forall q \le p_\alpha \quad H(p_\alpha) = F_{A_0}(p_\alpha) \le F_{A_0}(q) \le H(q).$$

So we have

$$H^-(p) = \inf_{q \le p} H(q) = H(p_\alpha) = F_{A_0}(p).$$

If $p \notin \bigcup_{p_\alpha \in A_0}]p_\alpha, p_\alpha^+[$, then

$$F_{A_0}(p) = H(p) \ge H^-(p).$$

Moreover, since F_{A_0} is non-increasing and by (57), we have

$$\forall q \le p \quad H(p) = F_{A_0}(p) \le F_{A_0}(q) \le H(q).$$

So $F_{A_0}(p) = H(p) = H^-(p)$. We deduce that $F_{A_0} = H^-$. □

The proof is exactly the same as in [13,16].

Proof of Theorem A.1. We do the proof in three steps.

1st step: Let us prove that

$$u_t + H(u_x) \le 0 \quad \text{in } (0,T) \times (0,+\infty),$$

implies

$$u_t + H^-(u_x) \le 0 \quad \text{on } (0,T) \times \{0\}.$$

Since $\forall p_\alpha \in A_0,\ p_\alpha^- = p_\alpha$, using Theorem 3.15, we deduce that u is a strong viscosity subsolution with F_{A_0}, so

$$u_t + F_{A_0}(u_x) \le 0 \quad \text{on } (0,T) \times \{0\}.$$

Since $F_{A_0}(u_x) = H^-(u_x)$, we have

$$u_t + H^-(u_x) \le 0 \quad \text{on } (0,T) \times \{0\}.$$

2nd step: Let us prove that

$$u_t + H(u_x) \geq 0 \quad \text{in } (0,T) \times [0,+\infty),$$

implies

$$u_t + H^-(u_x) \geq 0 \quad \text{on } (0,T) \times \{0\}.$$

Let φ be a test function touching u_* from below at $(t_0, 0)$. Using Theorem 3.15, we assume that

$$\varphi(t,x) = \psi(t) + \phi_\alpha(x),$$

where $\psi \in \mathcal{C}^1((0,T))$ and

$$\phi_\alpha \in \mathcal{C}^1([0,+\infty)), \quad \phi'_\alpha(0) = p_\alpha.$$

We have $\varphi_x(t_0, 0) = p_\alpha$ and

$$H(\varphi_x(t_0,0)) = H(p_\alpha) = F_{A_0}(p_\alpha) = H^-(p_\alpha) = H^-(\varphi_x(t_0,0)),$$

so by hypothesis, we have $\varphi_t + H(\varphi_x(t_0,0)) \geq 0$. We deduce that

$$\varphi_t + H^-(\varphi_x(t_0,0)) \geq 0.$$

3rd step: The reverse comes from the fact that $H^- \leq H$. $\square$

Remark A.5. In [13], the author gives simpler proofs without using Theorem of reduction of the set of test functions.

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