ARTICLE IN PRESS YJDEQ:8912

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Journal of Differential Equations

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Wavefronts for a nonlinear nonlocal bistable reaction–diffusion equation in population dynamics ☆

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Received 24 January 2017; revised 27 June 2017

Abstract

The wavefronts of a nonlinear nonlocal bistable reaction–diffusion equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2(1 - J_\sigma * u) - du, \quad (t, x) \in (0, \infty) \times \mathbb{R},$$

with $J_\sigma(x) = (1/\sigma)J(x/\sigma)$ and $\int_{\mathbb{R}} J(x)dx = 1$ are investigated in this article. It is proven that there exists a $c_*(\sigma)$ such that for all $c \geq c_*(\sigma)$, a monotone wavefront (c, ω) can be connected by the two positive equilibrium points. On the other hand, there exists a $c^*(\sigma)$ such that the model admits a semi-wavefront $(c^*(\sigma), \omega)$ with $\omega(-\infty) = 0$. Furthermore, it is shown that for sufficiently small σ, the semi-wavefronts are in fact wavefronts connecting 0 to the largest equilibrium. In addition, the wavefronts converge to those of the local problem as $\sigma \to 0$.

MSC: 35K65; 35K40

☆ Jing Li is supported by the Beijing Natural Science Foundation (Grant no. 9152013) and the Research Grant Funds of Minzu University of China. This work is partially supported by DFG Project CH 955/3-1 and the DAAD project "DAAD-PPP VR China" (Project-ID: 57215936).

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http://dx.doi.org/10.1016/j.jde.2017.07.019
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Keywords: Wavefronts; Nonlocal; Bistable; Reaction–diffusion equation

1. Introduction

In this work we study the nonlinear nonlocal reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2(1 - J_\sigma * u) - du \quad \text{in } (0, \infty) \times \mathbb{R}, \tag{1.1}$$

where $0 \le d < \frac{2}{9}$, $J_\sigma(x) = \frac{1}{\sigma} J(\frac{x}{\sigma})$ is a σ-parameterized nonnegative kernel with

$$J \in L^1(\mathbb{R}), \qquad \int_{\mathbb{R}} J(x)dx = 1$$

and

$$J_\sigma * u(x) = \int_{\mathbb{R}} J_\sigma(x-y)u(y)dy.$$

This equation has three constant solutions,

$$0, \quad a = \frac{1}{2}(1 - \sqrt{1-4d}), \quad A = \frac{1}{2}(1 + \sqrt{1-4d}).$$

The problem arises in population dynamics with nonlocal consumption of resources, for example in [7,19]. It is used to model the behavior of various biological phenomena such as emergence and evolution of biological species and the process of speciation. Actually, similar nonlocal structure in the reaction term appears also in describing the behavior of cancer cells with therapy as well as polychemotherapy and chemotherapy [16,17].

The reaction term $u^2(1 - J_\sigma * u) - du$ consists of the reproduction which is proportional to the square of the density, the available resources and the mortality. The nonlocal consumption of the resources $J_\sigma * u(x)$ describes the phenomenon that consumption at the space point x is determined by the individuals located in some area around this point, where J_σ represents the probability density function that describes the distribution of individuals.

For $J(x) = 1$, with a general nonlinearity, $u^\alpha(1 - \int u(x,t)dx)$ in the multi-dimensional case, the problem has been studied [9,10] in terms of the existence of the classical solutions both in bounded and unbounded domains correspondingly.

In the case of $J(x) = \delta(x)$, where $\delta(x)$ is the Dirac function, equation (1.1) becomes the so called Huxley equation, which is a classical reaction–diffusion equation. It has the same constant solutions, 0, a and A to the nonlocal problem. The existence of traveling waves has been studied extensively in the literature (see [15,4,5,8,12,20] among others). It's proved that there exists a minimum speed such that the traveling waves connecting a and A exist for all values of the speed greater than or equal to this minimum speed. While the traveling waves connecting 0 and A exist only for a single value of the speed.

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Compared to the rich results for the local version of the Fisher-KPP reaction diffusion equation, very limited theoretical results exist for its nonlocal version. In the last few years, there has been several works on wavefronts for some typical nonlocal reaction diffusion equations. In the research of wavefronts, in order to get a priori bounds for the existence and monotonicity properties of the fronts, the classical methods substantially depend on the application of comparison principle. However, for the equation with nonlocal competition term, the most challenging point arises from the lack of the comparison principle. One first example is the following nonlocal Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - J_\sigma * u) \quad \text{in } (0, \infty) \times \mathbb{R}. \tag{1.2}$$

Berestycki et al. [7] proved that (1.2) admits a semi-wavefront connecting 0 to an unknown positive state for all $c \geq c^* = 2$ and there is no such kind of wavefront with wave speed $c < 2$. In [18], Nadin et al. numerically verified the existence of monotone wavefronts. After that, Alfaro et al. [1] rigorously proved that (1.2) admits the rapid wavefront connecting 0 and 1. Furthermore, Fang et al. [11] gave a sufficient and necessary condition for the existence of monotone wavefronts of (1.2) that connect the two equilibrium points 0 and 1. In a recent paper by Hasik et al. [14], for nonsymmetric interaction kernel J_σ, the different roles of the right and the left interactions are investigated. Nonlocal equations with bistable reactions have been investigated in [21,2,3]. In [21], Wang et al. studied

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u, J * S(u)), \tag{1.3}$$

where $g(u, J * S(u))$ satisfies some bistable assumptions. Although it is a nonlocal problem, due to their special assumptions, the comparison principle still holds. Therefore, by constructing various pairs of super- and sub-solutions, employing the comparison principle and the squeezing technique, the authors proved the existence of monotone traveling wavefronts.

There are further results on equations with other bistable reactions, where comparison principle can not be applied. In [2], Alfaro et al. considered the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u - \theta)(1 - J_\sigma * u) \quad \text{in } (0, \infty) \times \mathbb{R} \tag{1.4}$$

with $0 < \theta < 1$. The Leray–Schauder degree method is used to indicate that (1.4) admits semi-wavefronts connecting 0 to an unknown positive steady state, which is above and away from the intermediate equilibrium. For focusing kernel, it is proved that the wave connects 0 and 1.

The wavefront solution $\omega(x - ct)$ for Equation (1.1) has been investigated, for small σ, in [3] by Apreutesei et al. It satisfies

$$\omega''(\xi) - c\omega'(\xi) + \omega^2(\xi)(1 - J_\sigma * \omega(\xi)) - d\omega(\xi) = 0. \tag{1.5}$$

They proved the existence of wavefronts of (1.1) that connect 0 and A. In fact, for small σ, the nonlocal operator is a perturbation of the corresponding local operator, thus the implicit function theorem can be applied. More precisely, under the assumptions

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$$\int_{\mathbb{R}} |z|J(z)dz < \infty, \quad \int_{\mathbb{R}} |z|^2 J(z)dz < \infty,$$

they obtained that there exists $\sigma_0 > 0$ such that, for any $|\sigma| < \sigma_0$, equation (1.6) has a solution $(c, \omega) \in C^{2+\alpha}(\mathbb{R}) \times \mathbb{R}$ with $\omega(-\infty) = 0$ and $\omega(+\infty) = A$. Furthermore, the solution is of the class C^1 with respect to σ.

In this paper, we study the existence of wavefronts of (1.1) which connect a to A and 0 to A respectively by using a totally different method from [3]. The main results we obtained in this paper are as follows.

The first result shows the existence of wavefronts connecting a to A for any σ with big enough wave speed c.

Theorem 1.1. *Suppose* $0 \le d < \frac{2}{9}$, *then it holds that*

(i) *for any* $\sigma > 0$, *there exists a* $c_*(\sigma) > 0$ *such that when* $c \ge \max\{2\sqrt{2A - d},\ c_*(\sigma)\}$, (1.1) *admits a monotone wavefront* $\omega \in C^2(\mathbb{R})$, *i.e.,* (c, ω) *is the solution of the following problem*

$$\begin{aligned} &\omega'' - c\omega' + \omega^2(1 - J_\sigma * \omega) - d\omega = 0 \qquad \text{in } \mathbb{R}, \\ &\omega(-\infty) = a, \quad \omega(+\infty) = A. \end{aligned} \tag{1.6}$$

(ii) *As* $\sigma \to 0$, $c_*(\sigma)$ *converges to* c_*. *Moreover, for any* $c \ge \max\{2\sqrt{2A - d}, c_*\}$, *by fixing* $\omega(0) = \frac{1}{2}$, ω *has a subsequence converging to* ω_0 *in* $C^{1,\alpha}_{loc}(\mathbb{R})$, *where* (ω_0, c) *is the solution of the following problem*

$$\begin{aligned} &\omega_0'' - c\omega_0' + \omega_0^2(1 - \omega_0) - d\omega_0 = 0 \qquad \text{in } \mathbb{R}, \\ &\omega_0(-\infty) = a, \quad \omega_0(0) = \frac{1}{2}, \quad \omega_0(+\infty) = A. \end{aligned} \tag{1.7}$$

The second result demonstrates the existence of a semi-wavefronts connecting 0 to an intermediate state d_0 for any σ; and furthermore this semi-wavefront can be extended to A as x goes to $+\infty$ in the case of small σ.

Theorem 1.2. *Suppose* $0 < d < \frac{2}{9}$, *then it holds that*

(i) *there exists an* $M > 0$ *such that for any* $\sigma > 0$ *and* $0 < d_0 < d$, (1.1) *admits a semi-wavefront* $(\omega, c^*(\sigma))$ *with* $\max\{|c^*(\sigma)|, \|\omega\|_{C^2(\mathbb{R})}\} \le M$, *i.e.* ω *is the solution of the following problem*

$$\begin{aligned} &\omega'' - c^*(\sigma)\omega' + \omega^2(1 - J_\sigma * \omega) - d\omega = 0 \qquad \text{in } \mathbb{R}, \\ &\omega(-\infty) = 0, \quad \omega(0) = d_0, \end{aligned} \tag{1.8}$$

and $0 \le \omega \le A$ *on* $\mathbb{R}$, $\omega' \ge 0$ *in* $(-\infty, 0]$.

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(ii) *If furthermore* $m_i = \int\limits_{\mathbb{R}} |z|^i J(z)dz < +\infty$ *for* $i = 1, 2$, *then there exists* $\sigma^* > 0$ *such that for* $\sigma < \sigma^*$, $c^*(\sigma)$ *is positive and the semi-wavefronts are in fact wavefronts with* $\omega(+\infty) = A$.

(iii) *As* $\sigma \to 0$, $c^*(\sigma)$ *converges to* c^*. *Moreover, the solution* ω *has a subsequence converging to* ω_0 *in* $C^{1,\alpha}_{loc}(\mathbb{R})$, *which satisfies*

$$\begin{aligned} &\omega_0'' - c^*\omega_0' + \omega_0^2(1-\omega_0) - d\omega_0 = 0 \qquad \text{in } \mathbb{R} \\ &\omega_0(-\infty) = 0, \quad \omega_0(0) = d_0, \quad \omega_0(+\infty) = A. \end{aligned} \tag{1.9}$$

Next we summarize the main methods used in this paper. To study the existence of monotone traveling waves, we use the classical method of sub- and super-solutions for an appropriate monotone operator, which is motivated by [13] on the time-delay Fisher-KPP equation and [11] on the nonlocal Fisher-KPP equation. In our case, it's verified that the obtained monotone wavefronts connect the two positive states a and A. The proof of the existence of wavefronts connecting 0 and A is more delicate. We start from a cut-off approximation, in a bounded domain $[-L, L]$, of the original problem and show that the solutions are between 0 and A. Furthermore we can obtain the uniform C^2-bound of the solutions independent of L and the scale of the cut-off. By removing the cut-off and letting L tend to infinity we derive the existence of semi-wavefronts which connect 0 to d_0. To show the semi-wavefronts are in fact wavefronts with $\omega(+\infty) = A$, the main difficulty is to exclude the case that $\omega(+\infty) = 0$ and $\omega(+\infty) = a$. Such a difficulty also arises in the construction of bistable wavefronts in [6,2]. Instead of using the energy methods as in [6], we adopt a rather direct method by comparing the semi-wavefronts that has been obtained from the nonlocal problem with those of the corresponding local problem.

The paper is organized as follows. In Section 2, by monotone iteration method, we establish the existence of monotone wavefronts connecting the two positive equilibrium a and A. In Section 3, we prove the existence of semi-wavefronts by a limiting process. Moreover, for σ sufficiently small, we prove that the semi-wavefronts are wavefronts connecting 0 and A. Furthermore, as $\sigma \to 0$, in both of Section 2 and Section 3, we prove that the wavefronts converge to those of the corresponding local problems.

2. Monotone wavefronts connecting a and A

To prove the existence of monotone wavefronts, we adopt the method of the sub- and super-solution. The main task is to define a monotone operator and to construct a pair of ordered lower and upper fixed points. To this end, we prove the following lemmas.

Lemma 2.1. *Denote*

$$F(\omega)(\xi) = 2A\omega(\xi) - \omega^2(\xi)(1 - J_\sigma * \omega(\xi)),$$

then for any $0 \le \omega \le A$, *we have*

(i) $F(\omega)(\xi) > 0$;
(ii) $F(\omega_1)(\xi) \ge F(\omega_2)(\xi)$ *if* $\omega_1(\xi) \ge \omega_2(\xi)$.

Proof. (i) It can be easily checked that

$$\begin{aligned}F(\omega) =&2A\omega - \omega^2(1 - J_\sigma * \omega)\\ =&\omega[2A - \omega(1 - J_\sigma * \omega)]\\ \geq&\omega(2A - A) > 0.\end{aligned}$$

(ii) Denote $g(\omega) = 2A\omega - \omega^2$, then $g'(\omega) = 2A - 2\omega \geq 0$, which together with the monotonicity of $h(\omega) = \omega^2 J_\sigma * \omega$ with ω imply the monotonicity of $F(\omega)$ in ω. □

Note that if $\omega(\xi)$ satisfies (1.6), then we have

$$\omega''(\xi) - c\omega'(\xi) + (2A - d)\omega(\xi) = F(\omega)(\xi).$$

Define

$$L[\omega] = \omega''(\xi) - c\omega'(\xi) + (2A - d)\omega(\xi) - F(\omega)(\xi),$$

then it is clear that finding a solution of (1.6) is equivalent to searching a function ω satisfying $L[\omega] = 0$, which is equivalent to

$$\omega(\xi) = \frac{1}{\mu_2 - \mu_1} \int_\xi^{+\infty} \left(e^{\mu_1(\xi-y)} - e^{\mu_2(\xi-y)}\right) F(\omega(y))dy,$$

where $0 < \mu_1 \leq \mu_2$ are the two different real and positive roots of $\mu^2 - c\mu + 2A - d = 0$ as $c > 2\sqrt{2A - d}$.

Lemma 2.2. *For $c > 2\sqrt{2A - d}$, let*

$$T[\omega](\xi) = \frac{1}{\mu_2 - \mu_1} \int_\xi^{+\infty} \left(e^{\mu_1(\xi-y)} - e^{\mu_2(\xi-y)}\right) F(\omega(y))dy,$$

then

(i) *if $\overline{\omega}(\xi)$ is a super-solution of (1.6), then $T[\overline{\omega}](\xi) \leq \overline{\omega}(\xi)$ and $T[\overline{\omega}](\xi)$ is also a super-solution. Moreover, for any sub-solution $\underline{\omega}(\xi)$ of (1.6) that satisfies $\underline{\omega}(\xi) \leq \overline{\omega}(\xi)$, we have $\underline{\omega}(\xi) \leq T[\overline{\omega}](\xi)$.*
(ii) *If $\omega(\xi)$ is increasing, then $T[\omega](\xi)$ is also increasing.*

Proof. (i) If $\overline{\omega}(\xi)$ is a super-solution of (1.6), then

$$L[\overline{\omega}] = \overline{\omega}''(\xi) - c\overline{\omega}'(\xi) + (2A - d)\overline{\omega}(\xi) - F(\overline{\omega})(\xi) \geq 0. \tag{2.1}$$

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Let $\omega_1 = T[\overline{\omega}]$, then

$$\omega_1''(\xi) - c\omega_1'(\xi) + (2A - d)\omega_1(\xi) - F(\overline{\omega}(\xi)) = 0. \tag{2.2}$$

Let $\varphi(\xi) = \overline{\omega}(\xi) - \omega_1(\xi)$, $r(\xi) = \varphi''(\xi) - c\varphi'(\xi) + (2A - d)\varphi(\xi)$, then from (2.1) and (2.2), we obtain $r(\xi) \geq 0$ and

$$\varphi(\xi) = \frac{1}{\mu_2 - \mu_1} \int_{\xi}^{+\infty} \left(e^{\mu_1(\xi - y)} - e^{\mu_2(\xi - y)}\right) r(\xi) dy \geq 0,$$

which means that $T[\overline{\omega}](\xi) \leq \overline{\omega}(\xi)$. Similarly, we can get $\underline{\omega} \leq T[\overline{\omega}](\xi)$. Furthermore, noticing

$$\overline{\omega} \geq T[\overline{\omega}] = \omega_1,$$

from (ii) of Lemma 2.1 we derive

$$\omega_1'' - c\omega_1' + (2A - d)\omega_1 = F(\overline{\omega}) \geq F(\omega_1).$$

It follows that $L[\omega_1] \geq 0$ and ω_1 is also a super-solution.

(ii) If $\omega(\xi)$ is increasing, then from (ii) of Lemma 2.1 we obtain $F(\omega)(\xi)$ is also increasing, therefore

$$F(\omega)(\xi + t) - F(\omega)(\xi) \geq 0, \quad \forall t > 0.$$

Furthermore, we have

$$\begin{aligned} &T[\omega](\xi + t) - T[\omega](\xi) \\ =&\frac{1}{\mu_2 - \mu_1} \int_{\xi}^{+\infty} \left(e^{\mu_1(\xi - y)} - e^{\mu_2(\xi - y)}\right) (F(\omega)(\xi + t) - F(\omega)(\xi)) dy \geq 0, \end{aligned}$$

which implies that $T[\omega](\xi)$ is also increasing in ξ. $\square$

Define

$$\Phi_1(c, \sigma, \lambda) := \lambda^2 - c\lambda - d - A^2 \int_{\mathbb{R}} J_\sigma(s) e^{-\lambda s} ds = 0$$

and

$$\Phi_2(c, \sigma, \lambda) := \lambda^2 - c\lambda + d - A^2 \int_{\mathbb{R}} J_\sigma(s) e^{-\lambda s} ds = 0,$$

which, by a change of variable, are equivalent to

$$\frac{1}{\sigma^2}\lambda^2 - \frac{c}{\sigma}\lambda - d - A^2 \int_{\mathbb{R}} J(s)e^{-\lambda s} ds = 0$$

and

$$\frac{1}{\sigma^2}\lambda^2 - \frac{c}{\sigma}\lambda + d - A^2 \int_{\mathbb{R}} J(s)e^{-\lambda s} ds = 0$$

respectively. Similar to Proposition 2.1 in [11], we have the following result

Proposition 2.1. *For any $\sigma > 0$, there exists $c_*(\sigma) \in (0, +\infty]$, which is increasing in σ, such that if $c \geq c_*(\sigma)$, then $\Phi_i(c, \sigma, \lambda) = 0$, $i = 1, 2$, admit the largest negative roots λ_i, and there exists $\varepsilon_i = \varepsilon_i(c, \sigma) > 0$ such that $\Phi_i(c, \sigma, \lambda_i - \varepsilon_i) > 0$. While if $c < c_*(\sigma)$, there exists $i \in \{1, 2\}$ such that $\Phi_i(c, \sigma, \lambda) = 0$ admits no negative root.*

Proof. Due to the fact that for any fixed $\lambda < 0$, $\frac{1}{\sigma^2}\lambda^2 - \frac{c}{\sigma}\lambda$ is increasing in $c \geq 0$ and decreasing in $\sigma > 0$, one has that for any $\sigma > 0$, there exists $c_*^i(\sigma) \geq 0$ such that $c_*^i(\sigma)$ is increasing in σ and $\Phi_i(c, \sigma, \lambda) = 0$, $i = 1, 2$, have at least one negative root if and only if $c \geq c_*^i(\sigma)$. Choosing $c_*(\sigma) = \max\{c_*^1(\sigma), c_*^2(\sigma)\}$, the proposition follows. □

Next, we will construct a pair of sub- and super-solutions in order to obtain a wavefront ω. For fixed $c > \max\{2\sqrt{2A - d}, c_*(\sigma)\}$, let

$$\underline{\omega}(\xi) = \begin{cases} \alpha e^{\mu\xi} + d, & \xi \leq \xi_-, \\ A(1 - e^{\lambda_1 \xi}), & \xi > \xi_-, \end{cases}$$

where $\mu > 0$ is a solution of $\mu^2 - c\mu + 1 = 0$, $\lambda_1 < 0$ is a solution of $\Phi_1(c, \sigma, \lambda) = 0$, α and ξ_- are uniquely determined by

$$\begin{cases} \alpha e^{\mu\xi_-} + d = A(1 - e^{\lambda_1 \xi_-}), \\ \alpha\mu e^{\mu\xi_-} = -\lambda_1 A e^{\lambda_1 \xi_-}, \end{cases}$$

so that

$$\alpha e^{\mu\xi} + d \geq A(1 - e^{\lambda_1 \xi}), \quad \forall \xi \geq \xi_-.$$

Proposition 2.2. *For $c \geq c_*(\sigma)$, $\underline{\omega}$ is a sub-solution of (1.6), i.e., $L[\underline{\omega}] \leq 0$.*

Proof. For $\xi \leq \xi_-$, due to the fact that

$$d \leq \underline{\omega} \leq A \leq 1,$$

we have

$$\begin{aligned}L[\underline{\omega}] &= \alpha(\mu^2 - c\mu)e^{\mu\xi} + \underline{\omega}^2(1 - J_\sigma * \underline{\omega}) - d\underline{\omega} \\ &= (\mu^2 - c\mu + 1)(\underline{\omega} - d) - (\underline{\omega} - d) + \underline{\omega}^2(1 - J_\sigma * \underline{\omega}) - d\underline{\omega} \\ &= -(1 - \underline{\omega})(\underline{\omega} - d) - \underline{\omega}^2 J_\sigma * \underline{\omega} < 0,\end{aligned}$$

where we have used the fact that $\mu^2 - c\mu + 1 = 0$.

For $\xi > \xi_-$, noticing that

$$\alpha e^{\mu\xi} + d \geq A(1 - e^{\lambda_1\xi}),$$

we have

$$\begin{aligned}L[\underline{\omega}] =& A(-\lambda_1^2 + c\lambda_1)e^{\lambda_1\xi} + \underline{\omega}^2(1 - J_\sigma * \underline{\omega}) - dA(1 - e^{\lambda_1\xi}) \\ =& A(-\lambda_1^2 + c\lambda_1 + d)e^{\lambda_1\xi} \\ & + \underline{\omega}^2\left(1 - \int_{-\infty}^{\xi_-}(\alpha e^{\mu s} + d)J_\sigma(\xi - s)ds - \int_{\xi_-}^{+\infty} A(1 - e^{\lambda_1 s})J_\sigma(\xi - s)ds\right) - dA \\ \leq& - A^3\int_{\mathbb{R}} J_\sigma(s)e^{-\lambda_1(s-\xi)}ds + \underline{\omega}^2(1 - \int_{\mathbb{R}} A(1 - e^{\lambda_1 s})J_\sigma(\xi - s)ds) - A^2(1 - A) \\ =& - A^3\int_{\mathbb{R}} J_\sigma(s)e^{-\lambda_1(s-\xi)}ds + \underline{\omega}^2(1 - A) + \underline{\omega}^2 A\int_{\mathbb{R}} e^{\lambda_1 s}J_\sigma(\xi - s)ds - A^2(1 - A) \\ =& A(\underline{\omega}^2 - A^2)\int_{\mathbb{R}} J_\sigma(s)e^{-\lambda_1(s-\xi)}ds + (\underline{\omega}^2 - A^2)(1 - A) < 0,\end{aligned}$$

where we have used that $d = A(1 - A)$ and

$$\Phi_1(c, \sigma, \lambda_1) = \lambda_1^2 - c\lambda_1 - d - A^2\int_{\mathbb{R}} J_\sigma(s)e^{-\lambda_1 s}ds = 0. \quad \square$$

Denote

$$\overline{\omega}(b, \xi) = \begin{cases} A(1 - e^{\lambda_2\xi} + be^{(\lambda_2-\varepsilon_2)\xi}), & \xi \geq \xi_b, \\ \mu_b, & \xi < \xi_b, \end{cases}$$

where $\lambda_2 < 0$ is the largest negative root of $\Phi_2(c, \sigma, \lambda_2) = 0$, $\varepsilon_2 > 0$ is the constant such that $\Phi_2(c, \sigma, \lambda_2 - \varepsilon_2) > 0$, $b > 0$ is a constant to be determined later, and $A(1 - e^{\lambda_2\xi} + be^{(\lambda_2-\varepsilon_2)\xi})$ achieves its minimum μ_b at the point

$$\xi = \xi_b = \frac{1}{\varepsilon_2}\ln\frac{b(\lambda_2 - \varepsilon_2)}{\lambda_2}$$

with

$$\mu_b = A(1 - e^{\lambda_2 \xi_b} + b e^{(\lambda_2 - \varepsilon_2)\xi_b}) = A + \frac{\varepsilon_2 A}{\lambda_2 - \varepsilon_2}\left(\frac{b(\lambda_2 - \varepsilon_2)}{\lambda_2}\right)^{\lambda_2/\varepsilon_2}.$$

Since $\lambda_2 < 0$, it is easy to verify that for sufficiently large b, $\xi_b > 0$ and $a < \mu_b < A$. Moreover, $\overline{\omega}$ is a C^1 function and is increasing with respect to $b > 0$.

Proposition 2.3. *For $c \geq c_*(\sigma)$, $\overline{\omega}$ is a super-solution of (1.6) for $b \gg 1$, i.e., $L[\overline{\omega}] \geq 0$.*

Proof. For $\xi < \xi_b$,

$$\begin{aligned} L[\overline{\omega}] &= \overline{\omega}^2(1 - J_\sigma * \overline{\omega}) - d\overline{\omega} \\ &= \mu_b^2(1 - A) + \mu_b^2(A - J_\sigma * \overline{\omega}) - d\mu_b \\ &= \mu_b^2(1 - A) + \mu_b^2 A \int_{\xi_b}^{+\infty} (e^{\lambda_2 s} - b e^{(\lambda_2 - \varepsilon_2)s}) J_\sigma(\xi - s) ds \\ &\quad + \mu_b^2 \int_{-\infty}^{\xi_b} (A - \mu_b) J_\sigma(\xi - s) ds - d\mu_b \\ &\geq \mu_b^2(1 - A) + \mu_b^2 \int_{-\infty}^{\xi} (A - \mu_b) J_\sigma(\xi - s) ds - d\mu_b \\ &= \mu_b^2(1 - A) + \frac{1}{2}\mu_b^2(A - \mu_b) - d\mu_b \\ &= \mu_b(A - \mu_b)(\frac{1}{2}\mu_b - (1 - A)). \end{aligned}$$

For fixed $0 \leq d < \frac{2}{9}$, we have

$$A = \frac{1 + \sqrt{1 - 4d}}{2} > \frac{2}{3}.$$

For any

$$0 < \varepsilon < \min\{1 - \frac{9d}{2}, 3A - 2\},$$

noticing that $\lim_{b \to \infty} \mu_b = A$, we have

$$2(1 - A) < A - \varepsilon < \mu_b < A$$

for sufficiently large b, which implies

$$L[\overline{\omega}] > 0.$$

For $\xi \geq \xi_b$, noticing that

$$\Phi_2(c, \sigma, \lambda_2) = \lambda_2^2 - c\lambda_2 + d - A^2 \int_{\mathbb{R}} J_\sigma(s) e^{-\lambda_2 s} ds = 0$$

and

$$\Phi_2(c, \sigma, \lambda_2 - \varepsilon_2) = (\lambda_2 - \varepsilon_2)^2 - c(\lambda_2 - \varepsilon_2) + d - A^2 \int_{\mathbb{R}} J_\sigma(s) e^{-(\lambda_2 - \varepsilon_2)s} ds > 0,$$

we have

$$\begin{aligned}
L[\overline{\omega}] =& A(1 - e^{\lambda_2 \xi} + b e^{(\lambda_2 - \varepsilon_2)\xi})'' - cA(1 - e^{\lambda_2 \xi} + b e^{(\lambda_2 - \varepsilon_2)\xi})' + \overline{\omega}^2(1 - J_\sigma * \overline{\omega}) - d\overline{\omega} \\
\geq& A(-\lambda_2^2 + c\lambda_2) e^{\lambda_2 \xi} + Ab([(\lambda_2 - \varepsilon_2)^2 - c(\lambda_2 - \varepsilon_2)] e^{(\lambda_2 - \varepsilon_2)\xi} + \overline{\omega}^2(1 - A) \\
& \quad + \overline{\omega}^2(A - J_\sigma * \overline{\omega}) - dA(1 - e^{\lambda_2 \xi} + b e^{(\lambda_2 - \varepsilon_2)\xi}) \\
=& A(-\lambda_2^2 + c\lambda_2 + d) e^{\lambda_2 \xi} + Ab[(\lambda_2 - \varepsilon_2)^2 - c(\lambda_2 - \varepsilon_2) - d] e^{(\lambda_2 - \varepsilon_2)\xi} + \overline{\omega}^2(1 - A) \\
& \quad + \overline{\omega}^2 A \int_{\mathbb{R}} (e^{\lambda_2(\xi - s)} - b e^{(\lambda_2 - \varepsilon_2)(\xi - s)}) J_\sigma(s) ds - dA \\
=& \frac{1}{A}(A^2 - \overline{\omega}^2)(-\lambda_2^2 + c\lambda_2 - d) e^{\lambda_2 \xi} + \frac{b}{A}(A^2 - \overline{\omega}^2)[(\lambda_2 - \varepsilon_2)^2 - c(\lambda_2 - \varepsilon_2) + d] e^{(\lambda_2 - \varepsilon_2)\xi} \\
& \quad + \frac{b}{A}\overline{\omega}^2 \Phi_2(c, \sigma, \lambda_2 - \varepsilon_2) e^{(\lambda_2 - \varepsilon_2)\xi} - (A^2 - \overline{\omega}^2)(1 - A) + 2dA(e^{\lambda_2 \xi} - b e^{(\lambda_2 - \varepsilon_2)\xi}) \\
\geq& \frac{b}{A}\overline{\omega}^2 \Phi_2(c, \sigma, \lambda_2 - \varepsilon_2) e^{(\lambda_2 - \varepsilon_2)\xi} - \frac{1}{A}(A^2 - \overline{\omega}^2)(\lambda_2^2 - c\lambda_2 + d) e^{\lambda_2 \xi} + (1 - A)(A - \overline{\omega})^2 \\
\geq& \frac{b}{A}\overline{\omega}^2 e^{(\lambda_2 - \varepsilon_2)\xi} \left[\Phi_2(c, \sigma, \lambda_2 - \varepsilon_2) - \frac{A^2}{b\overline{\omega}^2}(2e^{(\lambda_2 + \varepsilon_2)\xi} + 2b e^{2\lambda_2 \xi})(\lambda_2^2 - c\lambda_2 + d) \right] \\
\geq& \frac{b}{A}\overline{\omega}^2 e^{(\lambda_2 - \varepsilon_2)\xi} \left[\Phi_2(c, \sigma, \lambda_2 - \varepsilon_2) - \frac{A^2}{b\overline{\omega}^2}(2e^{(\lambda_2 + \varepsilon_2)\xi_b} + 2b e^{2\lambda_2 \xi_b})(\lambda_2^2 - c\lambda_2 + d) \right] \\
\geq& \frac{b}{A}\overline{\omega}^2 e^{(\lambda_2 - \varepsilon_2)\xi} \cdot \left[\Phi_2(c, \sigma, \lambda_2 - \varepsilon_2) - \frac{2A^2}{a^2} \left(\frac{1}{b} \left(\frac{\lambda_2}{b(\lambda_2 - \varepsilon_2)} \right)^{-\lambda_2/\varepsilon_2 - 1} \right. \right. \\
& \quad \left. \left. + \left(\frac{\lambda_2}{b(\lambda_2 - \varepsilon_2)} \right)^{-2\lambda_2/\varepsilon_2} \right) (\lambda_2^2 - c\lambda_2 + d) \right],
\end{aligned}$$

where we have used the fact that $\xi_b = \frac{1}{\varepsilon_2} \ln \frac{b(\lambda_2 - \varepsilon_2)}{\lambda_2}$, $a < \overline{\omega} < A$ and

$$(A^2 - \overline{\omega}^2) \leq A^2(2e^{\lambda_2 \xi} + 2b e^{(2\lambda_2 - \varepsilon_2)\xi}).$$

For b sufficiently large, since $\lambda_2 < 0$, it is easy to see that $L[\overline{\omega}] > 0$. $\square$

Lemma 2.3. *Any solution $\omega \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to (1.5) with $\lim_{\xi\to-\infty} \omega(\xi) = \alpha_0$ and $\lim_{\xi\to+\infty} \omega(\xi) = \beta_0$ has the property that $\alpha_0, \beta_0 \in \{0, a, A\}$.*

Proof. Let $x_n \to \infty$, then the sequence of functions $v_n(x) = \omega(x + x_n)$ solve

$$v_n'' - cv_n' + v_n^2(1 - J_\sigma * v_n) - dv_n = 0, \quad \text{in } \mathbb{R}.$$

Since ω is bounded, v_n is uniformly bounded with respect to n. From the classical $W^{2,p}$ theory for second order linear elliptic equations, we obtain that for all $1 < p < \infty$,

$$\|v_n\|_{W^{2,p}_{loc}(\mathbb{R})} \le C.$$

From Sobolev embedding theorem, there is a subsequence of v_n, still denoted by v_n itself, such that $v_n \to v$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R})$ and weakly in $W^{2,p}_{loc}(\mathbb{R})$. Then $v(x) \equiv \beta_0$ and

$$v'' - cv' + v^2(1 - J_\sigma * v) - dv = 0,$$

which implies $\beta_0^2(1 - \beta_0) - d\beta_0 = 0$ and $\beta_0 \in \{0, a, A\}$. Similarly, we can prove that $\alpha_0 \in \{0, a, A\}$. $\square$

Proof of Theorem 1.1. The proof consists of the following two parts.

(i) First we consider the case

$$c > \max\{2\sqrt{2A - d}, c_*(\sigma)\}.$$

Let $\overline{\omega}_0 = \overline{\omega}$ and define the bounded continuous function sequence $\overline{\omega}_m$ by the following iteration scheme

$$\overline{\omega}_m''(\xi) - c\overline{\omega}_m'(\xi) + (2A - d)\overline{\omega}_m(\xi) = F(\overline{\omega}_{m-1})(\xi).$$

Then from Lemma 2.1 and Lemma 2.2, we can obtain that for any m,

$$\overline{\omega}_m(\xi) = T[\overline{\omega}_{m-1}](\xi)$$

is increasing and satisfies

$$\underline{\omega} \le \cdots \le \overline{\omega}_m \le \cdots \le \overline{\omega}_1 \le \overline{\omega}_0 = \overline{\omega}. \tag{2.3}$$

Hence, there exists a increasing function $\omega(\xi)$ such that $\overline{\omega}_m(\xi) \to \omega(\xi)$ a.e. for $\xi \in \mathbb{R}$. Therefore, we have

$$\omega(\xi) = T[\omega](\xi),$$

which implies that ω is a solution of (1.5). Since $0 \le \omega(\xi) \le A$ is increasing, there exist two non-negative constants α_0, β_0 such that

$$\lim_{\xi\to-\infty}\omega(\xi)=\alpha_0,\qquad \lim_{\xi\to+\infty}\omega(\xi)=\beta_0.$$

By Lemma 2.3, we have $\alpha_0, \beta_0 \in \{0, a, A\}$. Noticing that

$$\lim_{\xi\to+\infty}\underline{\omega}(\xi)=\lim_{\xi\to+\infty}\overline{\omega}(\xi)=A,$$

we have $\beta_0 = A$. Furthermore,

$$\lim_{\xi\to-\infty}\underline{\omega}(\xi)=d,\qquad \lim_{\xi\to+\infty}\overline{\omega}(\xi)=\mu_b<A,$$

imply $d < \alpha_0 < A$, then $\alpha_0 = a$, which means that (c, ω) is a solution of (1.6).
Since $a \le \omega \le A$ and $\omega' \ge 0$, we claim that $\omega'(\xi) \le \mu_1 A$ for $\xi \in \mathbb{R}$. In order to prove this, a direct computation from

$$\omega(\xi)=\frac{1}{\mu_2-\mu_1}\int_{\xi}^{+\infty}\left(e^{\mu_1(\xi-y)}-e^{\mu_2(\xi-y)}\right)F(\omega(y))dy$$

gives that

$$\omega'(\xi)=\frac{1}{\mu_2-\mu_1}\int_{\xi}^{+\infty}\left(\mu_1e^{\mu_1(\xi-y)}-\mu_2e^{\mu_2(\xi-y)}\right)F(\omega(y))dy.$$

Therefore,

$$\omega'(\xi)-\mu_1A\le\omega'(\xi)-\mu_1\omega(\xi)=-\int_{\xi}^{+\infty}e^{\mu_2(\xi-y)}F(\omega(y))dy\le0,$$

by noticing that

$$F(\omega)=2A\omega-\omega^2(1-J_\sigma*\omega)>0,$$

and $0 < \mu_1 \le \mu_2$ are the two positive roots of $\mu^2 - c\mu + 2A - d = 0$. Thus we have $\omega'(\xi) \le \mu_1 A$ for $\xi \in \mathbb{R}$. Furthermore, $\|\omega\|_{C^2(\mathbb{R})} \le M$ can be obtained directly from (1.6).
We are left to consider the case

$$c=\max\{2\sqrt{2A-d},c_*(\sigma)\}.$$

Choosing $\{c_n\}$ such that $c_n > \max\{2\sqrt{2A-d}, c_*(\sigma)\}$ and $c_n \to \max\{2\sqrt{2A-d}, c_*(\sigma)\}$, then for each n, the above discussion gives a monotone traveling wavefront ω_n with speed c_n, such that

$$\|\omega_n\|_{C^2(\mathbb{R})}\le M.$$

By appropriate translations, we fix

$$\omega_n(0) = \frac{1}{2} \quad \text{for all } n.$$

By Arzelà–Ascoli theorem, ω_n and ω_n' have a locally uniformly convergent subsequence with limit ω, ω' and

$$a \le \omega \le A, \quad 0 \le \omega' \le \mu_1 A$$

together with

$$\omega(0) = \frac{1}{2}, \quad \omega(-\infty) = a, \quad \omega(+\infty) = A,$$

such that (c, ω) is the solution of (1.6).

(ii) By Proposition 2.1, we have that $c_*(\sigma) \ge 0$ and $c_*(\sigma)$ is decreasing as $\sigma \to 0$. Thus there exists $c_* \ge 0$ such that $c_*(\sigma) \to c_*$. Next we take the limit $\sigma \to 0$. Let (ω_σ, c) be the solution of (1.6) that has been obtained in the previous step, where $c \ge \max\{2\sqrt{2A - d}, c_*(\sigma)\}$, and by appropriate translations, fix

$$\omega_\sigma(0) = \frac{1}{2} \quad \text{for all } \sigma$$

and

$$\|\omega_\sigma\|_{C^2(\mathbb{R})} \le M.$$

Therefore, ω_σ has a subsequence which converges to ω_0 locally uniformly in $C^{1,\alpha}(\mathbb{R})$ as $\sigma \to 0$, where $\omega_0 \in C^2(\mathbb{R})$ is the solution of (1.7), that is,

$$\omega_0'' - c\omega_0' + \omega_0^2(1 - \omega_0) - d\omega_0 = 0 \qquad \text{in } \mathbb{R},$$
$$\omega_0(-\infty) = a, \quad \omega_0(0) = 1/2, \quad \omega_0(+\infty) = A. \quad \square$$

3. Semi-wavefronts with $\omega(-\infty) = 0$ and wavefronts connecting 0 and A

In this section, we study the existence of wavefronts connecting 0 and A. We construct the wavefronts connecting 0 and A by considering a sequence of approximating problems on intervals $[-L, L]$, and then pass to the limit $L \to \infty$. In particular, two difficulties arise in the proof. One comes in proving that the speed c and the C^1 norm of ω are controlled by a constant independent of L, while the other one lies in establishing that the two equilibriums 0 and A are indeed reached at infinity.

For $L > 0$, we introduce the homotopy parameter $0 \le \tau \le 1$ and a smooth cut-off function $g_\varepsilon(s) \in C_0^\infty(0, A)$ with $\varepsilon \in (0, A/6)$ such that $0 \le g_\varepsilon(s) \le 1$ for $0 \le s \le A$ and

$$g_\varepsilon(s) \equiv 1 \quad \text{for } s \in (3\varepsilon, A - 3\varepsilon).$$

We consider the following problem with a cut-off both in space variable and in the nonlinear reaction,

$$\omega'' - c\omega' + \tau g_\varepsilon(\omega)[\omega^2(1 - J_\sigma * \tilde{\omega}) - d\omega] = 0 \qquad \text{in } (-L, L) \tag{3.1}$$

with

$$\omega(-L) = 0, \quad \omega(L) = A, \tag{3.2}$$

where $\tilde{\omega}$ is the extension of ω with $\omega = 0$ on $(-\infty, -L)$ and $\omega = A$ on $(L, +\infty)$.

If

$$\max_{t\in[-L,L]} \omega(t) = \omega(t_0) > A \text{ or } \min_{t\in[-L,L]} \omega(t) = \omega(t_0) < 0,$$

then $t_0 \in (-L, L)$ and $g_\varepsilon(\omega) = 0$ in a neighborhood of t_0, which together with (3.1) implies $\omega'' - c\omega' = 0$ in the same neighborhood. The maximum principle implies that $\omega \equiv \omega(t_0)$, which is a contraction. Thus,

$$0 \le \omega(t) \le A \text{ for all } t \in [-L, L].$$

For fixed $d_0 \in (0, d)$, we normalize the wavefront ω such that

$$\max_{-L\le t\le 0} \omega(t) = d_0. \tag{3.3}$$

This constraint indirectly fixes the speed c.

We claim that ω is increasing in $[-L, 0]$. In fact, if there exists a local maximal point $t_0 \in [-L, 0)$ such that $\omega''(t_0) \le 0$, $\omega'(t_0) = 0$, then from (3.1), we obtain

$$\omega(t_0)(1 - J_\sigma * \omega(t_0)) \ge d,$$

which contradicts to (3.3). Therefore, ω is increasing in $[-L, 0]$ and $\omega(0) = d_0$.

The following lemma provides a priori bounds for $\|\omega\|_{C^2(-L,L)}$.

Lemma 3.1. *There exist C and L_0 such that, for all $\tau \in [0, 1]$, $L \ge L_0$, $\varepsilon \in (0, \frac{1}{6}A)$ and $\sigma > 0$, any solution (c, ω) of (3.1)–(3.3) satisfies*

$$\|\omega\|_{C^2(-L,L)} \le C.$$

Proof. Noticing $0 \le \omega(t) \le A$ for all $t \in [-L, L]$, which together with the fact that $\omega(-L) = 0$ and $\omega(L) = A$ imply that

$$\omega'(-L) \ge 0 \text{ and } \omega'(L) \ge 0.$$

Let

$$H(t) = \tau g_\varepsilon(\omega)[\omega^2(1 - J_\sigma * \omega) - d\omega],$$

then $|H(t)| \le A^2$ for all $t \in [-L, L]$. From

$$(e^{-ct}\omega')' = -e^{-ct}H(t),$$

we obtain that for $-L \le t_1 \le t_2 \le L$,

$$\omega'(t_1)e^{c(t_2-t_1)} + \frac{A^2}{c}(1 - e^{c(t_2-t_1)}) \le \omega'(t_2) \le \omega'(t_1)e^{c(t_2-t_1)} - \frac{A^2}{c}(1 - e^{c(t_2-t_1)}) \tag{3.4}$$

and

$$\omega'(t_2)e^{c(t_1-t_2)} + \frac{A^2}{c}(e^{c(t_1-t_2)} - 1) \le \omega'(t_1) \le \omega'(t_2)e^{c(t_1-t_2)} - \frac{A^2}{c}(e^{c(t_1-t_2)} - 1). \tag{3.5}$$

We claim that

$$0 \le \omega'(-L) \le \frac{A^2}{c}, \quad \text{for } c > 0, \tag{3.6}$$

$$0 \le \omega'(L) \le -\frac{A^2}{c}, \quad \text{for } c < 0. \tag{3.7}$$

We first prove (3.6). Assuming the contrary, from (3.4), by choosing $t_1 = -L$, we have

$$\omega'(t_2) \ge \frac{A^2}{c} + (\omega'(-L) - \frac{A^2}{c})e^{c(t_2-t_1)}) \ge \frac{A^2}{c}, \qquad \text{for all } t_2 \in [-L, L].$$

This cannot hold for a bounded function $0 \le \omega(t) \le A$ and $\omega(L) = A$ for $L \ge L_0 = \frac{2c}{A}$. Similarly from (3.5) and choosing $t_2 = L$, we can verify (3.7).

Next we prove the boundedness of $\omega'(t)$ on $[-L, L]$ uniformly in τ, L, ε and σ.
For $c > 0$, with the change of variables

$$\omega(t) = e^{x(t)} - 1,$$

(3.1) is transformed into

$$x'' - cx' + (x')^2 + \tau g_\varepsilon(\omega)\frac{\omega}{\omega+1}[\omega(1 - J_\sigma * \omega) - d] = 0.$$

Denote $y(t) = x'(t)$, we obtain

$$y' - cy + y^2 + f(t) = 0, \tag{3.8}$$

where

$$f(t) = \tau g_\varepsilon(\omega)\frac{\omega}{\omega+1}[\omega(1 - J_\sigma * \omega) - d].$$

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We have that $|f(t)| \leq A$, which is a direct consequence of $0 \leq \omega(t) \leq A$. $\omega \in C^2[-L, L]$ shows that $y(t) \in C^1[-L, L]$. Let

$$\beta = \min_{t\in[-L,L]} y(t), \quad \gamma = \max_{t\in[-L,L]} y(t).$$

Next we will give a lower bound for β and an upper bound for γ uniformly in τ, ε, L and σ. Denote

$$\lambda_1(t) = \frac{c - \sqrt{c^2 - 4f(t)}}{2}, \quad \lambda_2(t) = \frac{c + \sqrt{c^2 - 4f(t)}}{2},$$

which are the roots of $y^2 - cy + f(t) = 0$. Suppose that $y(t)$ achieves its minimum at t_1, i.e.,

$$\beta = \min_{t\in[-L,L]} y(t) = y(t_1).$$

If $t_1 = -L$, then

$$\beta = y(-L) = x'(-L) = \omega'(-L) \geq 0.$$

If $t_1 = L$, then

$$\beta = y(L) = x'(L) = \frac{\omega'(L)}{A+1} \geq 0.$$

While if $t_1 \in (-L, L)$, then $y'(t_1) = 0$. From (3.8), we obtain

$$y^2(t_1) - cy(t_1) + f(t_1) = 0,$$

thus

$$\beta = y(t_1) \in \{\lambda_1(t_1), \lambda_2(t_1)\} \geq \frac{c - \sqrt{c^2 + 4A}}{2}.$$

On the other hand, suppose that $y(t)$ achieves its maximum at t_2, i.e.,

$$\gamma = \max_{t\in[-L,L]} y(t) = y(t_2).$$

If $t_2 = -L$, then from (3.6),

$$\gamma = y(-L) = x'(-L) = \omega'(-L) \leq \frac{A^2}{c}.$$

If $t_2 = L$, then $y'(L) \geq 0$, from (3.8), we obtain

$$y^2(L) - cy(L) + f(L) \leq 0,$$

thus

$$\gamma = y(L) \in (\lambda_1(L), \lambda_2(L)) \leq \frac{c + \sqrt{c^2 + 4A}}{2}.$$

If $t_2 \in (-L, L)$, then $y'(t_2) = 0$. From (3.8), we obtain

$$y^2(t_2) - cy(t_2) + f(t_2) = 0,$$

then

$$\gamma = y(t_2) \in \{\lambda_1(t_2), \lambda_2(t_2)\} \leq \frac{c + \sqrt{c^2 + 4A}}{2}.$$

From the above discussion, we obtain that

$$\beta \geq \frac{c - \sqrt{c^2 + 4A}}{2}$$

and

$$\gamma \leq \max\{\frac{A^2}{c}, \frac{c + \sqrt{c^2 + 4A}}{2}\}.$$

Furthermore, noticing $\omega(t) = e^{x(t)} - 1$ and $\omega'(t) = (\omega(t) + 1)y(t)$, the uniform boundedness of ω' can be obtained.

For $c < 0$, with the change of variables $\omega(t) = e^{-x(t)} - 1$, since

$$\frac{A^2}{c(A+1)} \leq x'(L) = -\frac{\omega'(L)}{A+1} \leq 0$$

and

$$x'(-L) = -\omega'(-L) \leq 0,$$

by similar analysis, the uniform boundedness of ω' achieves.

Now we have proved that the bounds of ω and ω' are independent of τ, L, ε and σ. Then from (3.1), for $c \neq 0$, the uniform boundedness of ω'' can be obtained. While for the case $c = 0$, the uniform boundedness of ω'' follows directly from (3.1). Now we obtain that for any $c \in \mathbb{R}$, there exists a constant C independent of τ, L, ε and σ such that $\|\omega\|_{C^2(-L,L)} \leq C$. □

The next lemma provides an a priori bound for the speed c. The proof follows the ideas from [7,2].

Lemma 3.2. *There exists $L_0 > 0$, for any $L > L_0$, there exists $K(L) > 0$ such that for all $\tau \in [0, 1]$, $\varepsilon \in (0, \frac{1}{6}A)$, any solution (c, ω) of (3.1)–(3.3) satisfies $-K(L) \leq c \leq c_{max} = 2\sqrt{A}$. Moreover, for $\tau = 1$, there exists $c_{min} < 0$ such that for all $L \geq L_0$, $\varepsilon > 0$ and $\sigma > 0$, we have $c \geq c_{min}$.*

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Proof. Since $0 \le \omega \le A \le 1$, the solution ω of (3.1) satisfies the inequality

$$\omega'' - c\omega' + A\omega \ge 0. \tag{3.9}$$

We will prove $c \le 2\sqrt{A}$ for big enough L by a contradiction argument. If $c > 2\sqrt{A}$, let

$$h_\mu(t) = \mu e^{\sqrt{A}(t-L)},$$

then

$$ch'_\mu(t) > h''_\mu(t) + Ah_\mu(t). \tag{3.10}$$

Noticing that $\omega(t) \in L^\infty(-L, L)$, when $\mu > 0$ is sufficiently large, we have that $\omega(t) \le h_\mu(t)$ in $(-L, L)$. While for $\mu < 0$ we have $\omega(t) > h_\mu(t)$. Therefore, define

$$\mu_0 = \inf\{\mu : h_\mu(t) > \omega(t) \text{ for all } t \in [-L, L]\}.$$

It follows that there exists $t_0 \in [-L, L]$ so that $h_{\mu_0}(t_0) = \omega(t_0)$ and $\mu_0 > 0$. However, (3.9) and (3.10) imply that $t_0 \notin (-L, L)$. As $\mu_0 > 0$, it is impossible that $t_0 = -L$, hence $t_0 = L$. As a consequence, $h_{\mu_0}(L) = A$, thus $\mu_0 = A$. However,

$$d_0 = \omega(0) \le h_{\mu_0}(0) = Ae^{-L\sqrt{A}},$$

which is impossible for

$$L > L_0 = (\ln A - \ln d_0)/\sqrt{A}.$$

Hence, $c > 2\sqrt{A}$ is impossible for L sufficiently large.

Next we prove a lower bound for c with given $L > 0$. We consider a solution (c, ω) of (3.1)–(3.3). It satisfies

$$\omega'' - c\omega' \le d\omega,$$

as well as $\omega(-L) = 0$, $\omega(L) = A$. If v is the solution of $v'' - cv' = dv$ with $v(-L) = 0$ and $v(L) = A$, then by comparison principle, we obtain $\omega \ge v$. As v can be computed explicitly and

$$v(0) = \frac{A}{e^{\lambda_+ L} + e^{\lambda_- L}}, \quad \lambda_\pm := \frac{c \pm \sqrt{c^2 + 4d}}{2}.$$

We see that $v(0) \to A$ as $c \to -\infty$. It follows that, for any $L > 0$, there exists $K(L) > 0$ such that $c < -K(L)$ implies $v(0) > d_0$, which contradict with the fact that $\omega \ge v$ and $\omega(0) = d_0$. Therefore, if (c, ω) is a solution of (3.1)–(3.3), then $c \ge -K(L)$.

In the end, we obtain a lower bound for the speed c with $\tau = 1$. Suppose that $c < -1$. We start by proving that the derivative ω' is bounded by $-A^2/c$ on an interval $[-L + K_0, L]$ with the constants K_0 independent of L. Choosing $t_1 = -L$ in (3.4) and noticing that $\omega'(-L) \ge 0$, we obtain

$$\omega'(t_2) \geq \frac{A^2}{c} \text{ for all } t_2 \in [-L, L] \tag{3.11}$$

and for some constant $K_0 > -\frac{c}{A}$ independent of L, we have

$$\omega'(t_2) \leq -\frac{A^2}{c} \text{ for any } t_2 \in [-L + K_0, L]. \tag{3.12}$$

Otherwise there exists $t_2 \in [-L + K_0, L]$ such that $\omega'(t_2) > -\frac{A^2}{c}$, the from (3.5), we have $\omega'(t_1) \geq -\frac{A^2}{c}$ for all $t_1 \in [-L, t_2]$. Integrating this from $-L$ to t_2 we see that

$$A \geq \omega(t_2) \geq -\frac{A^2}{c} K_0,$$

which contradicts the definition of K_0. This proves (3.12).

Now we consider the case $\tau = 1$. For a fixed $\varepsilon_0 = \frac{1-4d}{36}$, there exists $R > 1$ such that

$$A \int_{[-R,R]^c} J_\sigma(x)dx \leq \varepsilon_0.$$

We are going to prove that

$$c \geq c_{min} = -\frac{2A^2 R}{\varepsilon_0}.$$

If this is not true, assume $c \leq c_{min}$. Thanks to the conditions $\omega(L) = A$ and $\omega(0) = d_0 < d < \frac{2}{9}$, we can define $t_0 > 0$ as the smallest positive real such that $\omega(t_0) = \frac{1}{2}$. From (3.12), we obtain for $t \in [t_0 - R, t_0 + 2R] \cap [-L, L]$, we have

$$d_0 < \frac{1}{2} - \varepsilon_0 \leq \frac{1}{2} + \frac{2A^2 R}{c} \leq \omega(t) \leq \frac{1}{2} - \frac{2A^2 R}{c} \leq \frac{1}{2} + \varepsilon_0 < A \tag{3.13}$$

and $[t_0 - R, t_0 + 2R] \subset [0, L]$ as soon as $c \leq -\frac{2A^2 R}{\varepsilon_0} = c_{min}$. Furthermore, for $t \in [t_0, t_0 + R]$, we have

$$\begin{aligned} J_\sigma * \omega(t) &= \int_{[-R,R]} J_\sigma(x)\omega(t-x)dx + \int_{[-R,R]^c} J_\sigma(x)\omega(t-x)dx \\ &\leq \frac{1}{2} + \varepsilon_0 + A \int_{[-R,R]^c} J_\sigma(x)dx \leq \frac{1}{2} + 2\varepsilon_0 \end{aligned} \tag{3.14}$$

as soon as $c \le c_{min}$. We claim that for $c \le c_{min}$, ω is increasing on $(t_0, t_0 + R)$. If not, the definition of t_0 implies the existence of a local minimum $\bar{t} \in (t_0, t_0 + R)$. Noticing $d < \frac{2}{9}$, $\omega'(\bar{t}) = 0$, $\omega''(\bar{t}) \ge 0$, from (3.1), we have $J_\sigma * \omega(\bar{t}) \ge 1 - d/\omega(\bar{t})$, which together with (3.13) and (3.14) implies

$$\frac{1}{2} + 2\varepsilon_0 \ge J_\sigma * \omega(\bar{t}) \ge 1 - d/\omega(\bar{t}) \ge 1 - \frac{2d}{1 - 2\varepsilon_0} > \frac{1}{2} + 2\varepsilon_0,$$

which is a contraction. Therefore, for $c \le c_{min}$, $t \in (t_0, t_0 + R)$, we have $\omega'(t) \ge 0$ and thus

$$\begin{aligned}
\omega'' &\le \omega'' - c\omega' = g_\varepsilon(\omega)[d\omega - \omega^2(1 - J_\sigma * \omega)] \\
&\le \omega(d + \omega(2\varepsilon_0 - \frac{1}{2})) \\
&\le d - (\frac{1}{2} - 2\varepsilon_0)(\frac{1}{2} - \varepsilon_0) \\
&< d - \frac{1}{4} + \frac{3}{2}\varepsilon_0 = -\frac{15}{2}\varepsilon_0.
\end{aligned}$$

It follows that $\omega'(t_0) - \omega'(t_0 + R) \ge \frac{15}{2}\varepsilon_0 R$, which together with (3.11) and (3.12) implies $c \ge -\frac{4A^2}{15\varepsilon_0 R} > c_{min}$, which is a contraction.

Finally, it is proved that $c_{min} = -\frac{2A^2 R}{\varepsilon_0}$ is an explicit lower bound for c. □

Now we begin the homotopy argument. The a priori bounds obtained in Lemma 3.1 and 3.2 allow us to use the Leray–Schauder topological degree argument to prove existence of solutions to the problem (3.1)–(3.3) with $\tau = 1$ on the bounded domain $[-L, L]$.

Proposition 3.1. *There exist $K > 0$ and L_0 such that, for all $L \ge L_0$, $\varepsilon \in (0, A/6)$ and $\sigma > 0$, (3.1)–(3.3) with $\tau = 1$ has a solution (c, ω), i.e., (c, ω) satisfies*

$$\begin{cases} \omega'' - c\omega' + g_\varepsilon(\omega)[\omega^2(1 - J_\sigma * \tilde{\omega}) - d\omega] = 0 & \text{in } (-L, L), \\ \omega(-L) = 0, \quad \omega(0) = d_0, \quad \omega(L) = A \end{cases} \tag{3.15}$$

with

$$\|\omega\|_{C^2(-L,L)} \le K, \quad -c_{min} \le c \le c_{max}.$$

Proof. We introduce a map K_τ which is defined from the Banach space $X = \mathbb{R} \times C^1[-L, L]$, equipped with the norm $\|(c, v)\|_X = \max\{|c|, \|v\|_{C^1[-L,L]}\}$, onto itself, i.e.,

$$K_\tau : (c, v) \to (d_0 - v(0) + c, \omega),$$

where ω is the solution of the linear system

$$P_\tau \begin{cases} \omega'' - c\omega' + \tau g_\varepsilon(v)[v^2(1 - J_\sigma * \tilde{v}) - dv] = 0 & \text{in } (-L, L), \\ \omega(-L) = 0, \quad \omega(L) = A. \end{cases} \tag{3.16}$$

A solution (c_τ, ω_τ) of the finite interval problem (3.1)–(3.3) is a fixed point of K_τ and satisfies $K_\tau(c_\tau, \omega_\tau) = (c_\tau, \omega_\tau)$ and vice versa. Hence, in order to show that (3.15) has a wavefront, it suffices to show that the kernel of the operator $Id - K_1$ is nontrivial. The classical regularity theory implies that the operator K_τ is compact and continuous in $\tau \in [0, 1]$. Let $B_M = \{(c, v) : \|(c, v)\|_X \leq M\}$. Then Lemma 3.1 and 3.2 show that the operator $Id - K_\tau$ does not vanish on the boundary ∂B_M with M sufficiently large for any $\tau \in [0, 1]$. It remains only to show that $deg(Id - K_1, B_M, 0) \neq 0$ in $\overline{B}_M$. The homotopy invariance property of the degree implies that $deg(Id - K_1, B_M, 0) = deg(Id - K_0, B_M, 0)$. Moreover, for $\tau = 0$, the operator $F_0 = Id - K_0$ is given by

$$F_0(c, v) = (v(0) - d_0, v - \omega_0^c).$$

Here $\omega_0^c(t)$ solves

$$(\omega_0^c)'' - c(\omega_0^c)' = 0,$$
$$\omega_0^c(-L) = 0, \quad \omega_0^c(L) = A$$

and is given by

$$\omega_0^c(t) = \begin{cases} A\dfrac{e^{ct} - e^{-cL}}{e^{cL} - e^{-cL}}, & c \neq 0, \\ \dfrac{At}{2L} + \dfrac{A}{2}, & c = 0. \end{cases}$$

In particular, since $\omega_0^c(0)$ is decreasing in c, $\omega_0^0(0) = \dfrac{A}{2} > \dfrac{d}{2}$ and $\lim\limits_{c \to +\infty} \omega_0^c(0) = 0$, there exists a unique c_0 such that $\omega_0^{c_0}(0) = d_0$. The mapping $F_0 = Id - K_0$ is homotopic to

$$\Phi(c, v) = (\omega_0^c(0) - d_0, v - \omega_0^{c_0}).$$

The degree of the mapping Φ is the product of the degrees of each component. As $\omega_0^c(0)$ is decreasing in c, $deg(\omega_0^c(0) - d_0, B_M, 0) = -1$. While $deg(v - \omega_0^{c_0}, B_M, 0) = 1$. Thus

$$deg(Id - K_1, B_M, 0) = deg(Id - K_0, B_M, 0) = -1,$$

and thereafter a solution $(\omega, c) \in B_M$ of P_1 exists. $\square$

The following lemma is used as a preparation in passing to the limit $L \to \infty$ and $\varepsilon \to 0$. The proof is inspired by [1].

Lemma 3.3. *For any solution (c, ω) of (1.8) with $\omega \in C^2(\mathbb{R})$ and*

$$|c| > 2\sigma A^2 \sqrt{m_2}, \tag{3.17}$$

where $m_i = \int_{\mathbb{R}} |z|^i J(z) dz$, $i = 1, 2$, it holds that $\lim\limits_{t \to +\infty} \omega \in \{0, a, A\}$.

Proof. Rewrite the first equation of (1.8) as

$$\omega'' - c\omega' + \omega^2(1-\omega) + \omega^2(\omega - J_\sigma * \omega) - d\omega = 0,$$

then multiply it by ω' and integrate from $-R$ to R for arbitrary R, we get

$$c\int_{-R}^{R} |\omega'|^2 dt = [\frac{1}{2}(\omega')^2 + \frac{1}{3}\omega^3 - \frac{1}{4}\omega^4 - \frac{d}{2}\omega^2]|_{-R}^{R} + \int_{-R}^{R} \omega'\omega^2(\omega - J_\sigma * \omega)dt. \tag{3.18}$$

Denote the last term by I, Cauchy's inequality implies

$$I^2 \le \int_{-R}^{R} (\omega'\omega^2)^2 dt \int_{-R}^{R} (\omega - J_\sigma * \omega)^2 dt \le A^4 \int_{-R}^{R} (\omega')^2 dt \int_{-R}^{R} (\omega - J_\sigma * \omega)^2 dt. \tag{3.19}$$

Noticing

$$\omega(t) - J_\sigma * \omega(t) = \int_{\mathbb{R}} J_\sigma(t-y)(\omega(t) - \omega(y))dy = \int_{\mathbb{R}}\int_0^1 J_\sigma(-z)\omega'(t+\theta z))(-z)d\theta dz,$$

again by Cauchy's inequality we obtain

$$(\omega(t) - J_\sigma * \omega(t))^2 \le m_2\sigma^2 \int_{\mathbb{R}}\int_0^1 J_\sigma(-z)\omega'^2(t+\theta z))d\theta dz.$$

Integrating the above inequality, we have

$$\begin{aligned}\int_{-R}^{R} (\omega(t) - J_\sigma * \omega(t))^2 dt &\le m_2\sigma^2 \int_0^1\int_{\mathbb{R}} J_\sigma(-z) \int_{-R+\theta z}^{R+\theta z} \omega'^2(t) dt dz d\theta \\ &\le m_2\sigma^2 \int_{-R}^{R} \omega'(t)^2 dt + m_2\sigma^2 \int_0^1\int_{\mathbb{R}} J_\sigma(-z) 2\theta |z| C^2 dz d\theta \\ &\le m_2\sigma^2 \int_{-R}^{R} \omega'(t)^2 dt + m_2 m_1 \sigma^3 C^2.\end{aligned} \tag{3.20}$$

A combination of (3.18), (3.19) and (3.20) gives us

$$|c|\int_{-R}^{R} \omega'(t)^2 dt \le \left[\frac{1}{2}(\omega')^2 + \frac{1}{3}\omega^3 - \frac{1}{4}\omega^4 - \frac{d}{2}\omega^2\right]|_{-R}^{R}$$

$$+A^2\sqrt{\int_{-R}^{R}\omega'(t)^2dt\left(m_2\sigma^2\int_{-R}^{R}\omega'(t)^2dt+m_1m_2\sigma^3C^2\right)}$$

$$\leq C+A^2\sqrt{m_2}\sigma\left(2\int_{-R}^{R}\omega'(t)^2dt+m_1\sigma C^2\right). \tag{3.21}$$

If $|c|>2\sigma A^2\sqrt{m_2}$, then $\omega'\in L^2(\mathbb{R})$. Furthermore, $\omega\in C^2(\mathbb{R})$ implies that $\lim_{t\to\pm\infty}\omega'=0$, thus $\lim_{t\to+\infty}\omega$ exists. Then from Lemma 2.3, we have $\lim_{t\to+\infty}\omega\in\{0,a,A\}$. □

Proof of Theorem 1.2. From Proposition 3.1, for each $L\gg 1$ and $0<\varepsilon\ll 1$, the problem (3.15) does have at least one solution $(c_{L,\varepsilon},\omega_{L,\varepsilon})$. Next we will show that, for $\varepsilon\to 0$ and then $L\to+\infty$, the sequence $(c_{L,\varepsilon},\omega_{L,\varepsilon})$ (or an extracted subsequence) converges to a solution of (1.8).

(i) Having constructed a solution $(c_{L,\varepsilon},\omega_{L,\varepsilon})$ of (3.15) with

$$-c_{min}\leq c_{L,\varepsilon}\leq c_{max},\quad \|\omega_{L,\varepsilon}\|_{C^2(-L,L)}\leq K$$

and noticing that K, c_{min} and c_{max} are uniform in $L\geq L_0$ and $\varepsilon\in(0,A/6)$. We can take the limit $\varepsilon\to 0$ and $L\to+\infty$ in the approximating problem, and show that the limit (ω,c) is the wavefront that connects 0 and A. Namely, with fixed L, for $\varepsilon\to 0$, there exists a subsequence of $c_{L,\varepsilon}$ and $\omega_{L,\varepsilon}$, denoted by itself, such that $c_{L,\varepsilon}\to c_L$ and $\omega_{L,\varepsilon}\to\omega_L$ in $C^1_{loc}(\mathbb{R})$. Then

$$-c_{min}\leq c_L\leq c_{max},\quad \|\omega_L\|_{C^2(-L,L)}\leq K.$$

Moreover, from the definition of g_ε, we have $g_\varepsilon\to 1$ in $\mathbb{R}$ as $\varepsilon\to 0$. Then (c_L,ω_L) is the solution of

$$\begin{cases}\omega''-c\omega'+\omega^2(1-J_\sigma*\tilde{\omega})-d\omega=0 & \text{in } (-L,L),\\ \omega(-L)=0,\quad \omega(0)=d_0,\quad \omega(L)=A. & \end{cases}$$

Again, there exists a subsequence $L_n\to\infty$, such that $c_{L_n}\to c^*(\sigma)$ and $\omega_{L_n}\to\omega$ in $C^1_{loc}(\mathbb{R})$, and

$$-c_{min}\leq c^*(\sigma)\leq c_{max},\quad \|\omega\|_{C^2(\mathbb{R})}\leq M,$$

together with

$$0\leq\omega\leq A,\ \text{in } \mathbb{R},\ \text{and } 0\leq\omega\leq d_0 \text{ in } (-\infty,0].$$

Furthermore, the limit $(c^*(\sigma),\omega)$ is a solution of (1.8), with $\omega'(t)\geq 0$ for $t\in(-\infty,0]$. Hence, as $t\to-\infty$, $\omega(t)\to\alpha_0$ and $0\leq\alpha_0\leq d_0$. By Lemma 2.3, we have $\alpha_0=0$.

(ii) In order to show that $\omega(+\infty) = A$, we have to start from the approximation $\omega_L(t)$ in $[-L, L]$ with $\omega_L(L) = A$ and then take the limit $L \to +\infty$. In other words, we need to prove $\lim\limits_{L\to+\infty} \omega_L(L) = A$. The uniform bound of $|\omega_L'|_\infty \le C$ provides that

$$\begin{aligned}|\omega_L - J_\sigma * \omega_L|(t) &\le \int_{\mathbb{R}} J_\sigma(s)|\omega_L(t) - \omega_L(t-s)|ds \\ &\le \|\omega_L'\|_\infty \int_{\mathbb{R}} sJ_\sigma(s)ds \\ &\le Cm_1\sigma.\end{aligned}$$

Thus

$$\omega_L'' - c_L\omega_L' + \omega_L^2(1-\omega_L) - d\omega_L = \omega_L^2(J_\sigma * \omega_L - \omega_L) \le A^2 Cm_1\sigma.$$

Denote $f(s) = s^2(1-s) - ds$, the equation can be rewritten as

$$\omega_L'' - c_L\omega_L' + f(\omega_L) - C_0\sigma \le 0,$$

where $C_0 = A^2Cm_1$. Let $\alpha < \gamma < \beta$ be the three solutions of $f(s) - C_0\sigma = 0$. There exists $\sigma_0 > 0$ such that for $\sigma < \sigma_0$, it holds $\alpha < 0 < a < \gamma < \beta < A$. Let ψ_L be the solution of

$$\psi_L'' - c_L\psi_L' + f(\psi_L) - C_0\sigma = 0 \ \text{ in } [-L, L], \tag{3.22}$$

$$\psi_L(-L) = \alpha, \psi_L(L) = \beta. \tag{3.23}$$

By maximum principle, we have that $\alpha \le \psi_L \le \beta$. By comparison principle as in [6], we get that $\psi_L(t) \le \omega_L(t)$ for $t \in [-L, L]$. Then by the classical theory of elliptic equations, there exists a subsequence L_n, denoted by itself, such that as $n \to \infty$, $L_n \to \infty$, $c_{L_n} \to c^*(\sigma)$ and $\psi_{L_n} \to \psi$ in $C^1_{loc}(\mathbb{R})$, and the limit satisfies the following local problem

$$\psi'' - c^*(\sigma)\psi' + f(\psi) - C_0\sigma = 0 \ \text{ in } \ \mathbb{R}. \tag{3.24}$$

It can be easily verified that $\psi \le \omega$ in $\mathbb{R}$, from which we obtain for $t \in (-\infty, 0]$, $\alpha \le \psi \le d_0 < \gamma$ and then $f(\psi) - C_0\sigma \le 0$, from which and (3.24) we obtain $\psi'(t) \ge 0$ for $t \in (-\infty, 0]$. By the same arguments as that in Lemma 2.3, we obtain $\lim\limits_{t\to-\infty} \psi(t) = \alpha$.

Now we claim that there exists a constant C such that $|\psi'(t)| \le C$, for any $t \in \mathbb{R}$. We reformulate (3.24) into its integral version

$$\psi - \alpha = \frac{1}{z_2 - z_1}\left(\int_{-\infty}^{t} e^{z_1(t-s)} r(\psi)(s)ds + \int_{t}^{+\infty} e^{z_2(t-s)} r(\psi)(s)ds\right),$$

where

$$r(\psi) = \psi^2(1-\psi) - d\psi + d(\psi - \alpha) - C_0\sigma > 0, \ \text{for } \sigma < -\frac{d\alpha}{C_0},$$

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and

$$z_1 = \frac{c^*(\sigma) - \sqrt{(c^*(\sigma))^2 + 4d}}{2} < 0, \quad z_2 = \frac{c^*(\sigma) + \sqrt{(c^*(\sigma))^2 + 4d}}{2} > 0.$$

The first order derivative of ψ is

$$\psi' = \frac{1}{z_2 - z_1}\left(z_1 \int_{-\infty}^{t} e^{z_1(t-s)} r(\psi)(s)ds + z_2 \int_{t}^{+\infty} e^{z_2(t-s)} r(\psi)(s)ds \right).$$

Thus we have

$$\psi' - z_1(\psi - \alpha) = \int_{t}^{+\infty} e^{z_2(t-s)} r(\psi)(s)ds \geq 0$$

and

$$\psi' - z_2(\psi - \alpha) = -\int_{-\infty}^{t} e^{z_1(t-s)} r(\psi)(s)ds \leq 0,$$

which imply that

$$z_1(\psi - \alpha) \leq \psi' \leq z_2(\psi - \alpha).$$

Then the boundedness of ψ' follows from the boundedness of ψ. Moreover, noticing $\psi(-\infty) = \alpha$, we have $\psi'(-\infty) = 0$. For any $R > 0$, multipling (3.24) by ψ' and integrating from $-R$ to R, we get

$$c^*(\sigma) \int_{-R}^{R} |\psi'|^2 dt = [\frac{1}{2}(\psi')^2 + \frac{1}{3}\psi^3 - \frac{1}{4}\psi^4 - \frac{d}{2}\psi^2 - C_0\sigma\psi]\Big|_{-R}^{R} \leq C. \quad (3.25)$$

Next, we need to prove that $c^*(\sigma)$ is strictly positive in order to show that ψ' is bounded in L^2. Noticing that

$$\beta^3 = \beta^2 - d\beta - C_0\sigma, \quad \alpha^3 = \alpha^2 - d\alpha - C_0\sigma,$$

we have that

$$\alpha \to 0, \beta \to A \text{ as } \sigma \to 0.$$

Together with the fact that

$$\lim_{L\to\infty} \psi_L'(-L) = \psi'(-\infty) = 0,$$

and

$$\left(\frac{1}{3}-\frac{1}{4}A^2-\frac{1}{2}d\right)A-\frac{1}{3}d>0, \quad \text{for } d<\frac{2}{9},$$

it is easy to verify that there exists $\sigma_1>0$ such that for $\sigma<\sigma_1$ and big enough L,

$$\begin{aligned} & c_L\int_{-L}^{L}|\psi_L'|^2dt \\ &= \left[\frac{1}{2}(\psi_L')^2+\frac{1}{3}\psi_L^3-\frac{1}{4}\psi_L^4-\frac{d}{2}\psi_L^2-C\sigma\psi_L\right]\Big|_{-L}^{L} \\ &\geq \frac{1}{3}(\beta^3-\alpha^3)-\frac{1}{4}(\beta^4-\alpha^4)-\frac{d}{2}(\beta^2-\alpha^2)-C\sigma(\beta-\alpha)-\frac{1}{2}(\psi_L'(-L))^2 \qquad (3.26) \\ &= \frac{1}{3}(\beta^2-\alpha^2-d(\beta-\alpha))-(\frac{1}{4}(\beta^2+\alpha^2)+\frac{d}{2})(\beta^2-\alpha^2)-C\sigma(\beta-\alpha)-\frac{1}{2}(\psi_L'(-L))^2 \\ &= \left(\left[\frac{1}{3}-\frac{1}{4}(\beta^2+\alpha^2)-\frac{1}{2}d\right](\beta+\alpha)-\frac{1}{3}d-C\sigma\right)(\beta-\alpha)-\frac{1}{2}(\psi_L'(-L))^2>0. \end{aligned}$$

From (3.25) and (3.26), we obtain

$$0<c^*(\sigma)\int_{\mathbb{R}}|\psi'|^2dt\leq C.$$

Thus

$$c^*(\sigma)>0, \quad 0<\int_{\mathbb{R}}|\psi'|^2dt<\frac{C}{c^*(\sigma)},$$

which implies that $\psi'\in L^2$. By the same arguments as that in Lemma 2.3, we obtain $\lim\limits_{t\to+\infty}\psi(t)$ exists and belong to $\{\alpha,\gamma,\beta\}$. Now we claim that $\psi(+\infty)>\alpha$. If $\psi(+\infty)=\alpha$, noticing that $c^*(\sigma)>0$, then from the theory of traveling waves for local problem, there must holds $\psi\equiv\alpha$, which contradict with the fact that $\int_{\mathbb{R}}|\psi'|^2dt>0$. Thus $\psi(+\infty)>\alpha$, which means that either $\psi(+\infty)=\beta$ or $\psi(+\infty)=\gamma$. Again from the theory of traveling waves for local problem, we obtain there exists c_0 independent of σ such that $c^*(\sigma)>c_0>0$. Then Lemma 3.3 implies that for

$$\sigma<\sigma^*=\{\sigma_0,\sigma_1,\frac{c_0}{\sqrt{m_2}A^2},\frac{-d\alpha}{C_0}\},$$

$\lim\limits_{t\to+\infty}\omega$ exists and belongs to $\{0,a,A\}$. Noticing that $\beta>\gamma>a$, we have

$$\omega(+\infty)\geq\psi(+\infty)>a.$$

Therefore, $\omega(+\infty)=A$.

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(iii) In the end, as a byproduct, we can also take the limit $\sigma \to 0$. Let ω_σ be the solution of (1.8) with $\omega_\sigma(+\infty) = A$. Noticing that

$$\|\omega_\sigma\|_{C^2(\mathbb{R})} \le K, \quad -c_{min} \le c^*(\sigma) \le c_{max}$$

with K, c_{min} and c_{max} independent of σ for $\sigma < \sigma^*$, we get a subsequence of $(c^*(\sigma), \omega_\sigma)$, denoted by itself, such that $c^*(\sigma) \to c^*$ and $\omega_\sigma \to \omega_0$ locally uniformly in $C^{1,\alpha}(\mathbb{R})$ as $\sigma \to 0$, where (c^*, ω_0) is the solution of (1.9), i.e.,

$$\omega_0'' - c^*\omega_0' + \omega_0^2(1-\omega_0) - d\omega_0 = 0 \qquad \text{in } \mathbb{R}$$
$$\omega_0(-\infty) = 0, \quad \omega_0(0) = d_0, \quad \omega_0(+\infty) = A. \quad \square$$

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